Hölder regularity and convergence for a non-local model of nematic liquid crystals in the large-domain limit

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ABSTRACT

We consider a non-local free energy functional, modelling a competition between entropy and pairwise interactions reminiscent of the second order virial expansion, with applications to nematic liquid crystals as a particular case. We build on previous work on understanding the behaviour of such models within the large-domain limit, where minimisers converge to minimisers of a quadratic elastic energy with manifold-valued constraint, analogous to harmonic maps. We extend this work to establish Hölder bounds for (almost-)minimisers on bounded domains, and demonstrate stronger convergence of (almost-)minimisers away from the singular set of the limit solution. The proof techniques bear analogy with recent work of singularly perturbed energy functionals, in particular in the context of the Ginzburg–Landau and Landau–de Gennes models.

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1. Introduction

1.1. Variational models of liquid crystals

Liquid crystalline systems are those which sit outside of the classical solid–liquid–gas trichotomy. While there are a plethora of different systems classified as liquid crystals, they can be broadly described as fluid systems where molecules admit a long range order of certain degrees of freedom. This is in contrast to classical fluids, which lack long range correlations between molecules. The fluidity of the systems makes them “soft”, that is, easily susceptible to influence by external influences such as fields or stresses, whilst the long range ordering permits anisotropic electrostatic and optical behaviour. These two properties combined make them ideal for a variety of technological applications, as their anisotropy is exploitable whilst their softness makes them easy to manipulate.

The simplest liquid crystalline system is that of a nematic liquid crystal. These are systems of elongated molecules, often idealised as having axial symmetry, which form phases with no long range positional order,

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but where the long axes of molecules is generally well aligned over larger length scales. Even in the well studied case of nematics there are a variety of models that one may use to study their theoretical behaviour, where the choice of model is usually dependent on the length scales considered and the type of defects one wishes to observe.

One of the earliest and most studied free energy functionals one may consider in continuum modelling is the Oseen–Frank model [17]. In the simplest formulation, we consider a prescribed domain $\Omega \subseteq \mathbb{R}^3$ and a unit vector field as our continuum variable $n: \Omega \rightarrow \mathbb{S}^2$, interpreted as the local alignment axis of molecules. As molecules are assumed to be (statistically) head-to-tail symmetric, we interpret the configurations $n, -n$ as physically equivalent, that is, they are two mathematically distinct representations of the same physical state. In the simplified *one-constant approximation*, we look for minimisers of the free energy

$$\int_{\Omega} \frac{K}{2} |\nabla n(x)|^2 \, dx, \quad (1.1)$$

subject to certain boundary conditions, although more general formulations are possible. The problem has attracted interest not only from the liquid crystal community, but also from the mathematical community as the prototypical harmonic map problem. If a prescribed Dirichlet boundary condition admits non-zero degree, then by necessity any n satisfying it must admit discontinuities, meaning that defects/singularities are an unavoidable part of the model’s study.

More generally, one may consider an Oseen–Frank energy where different modes of deformation are penalised to different extents. Neglecting the saddle-splay null-Lagrangian term, this gives a free energy of the form

$$\int_{\Omega} \frac{K_1}{2} (\nabla \cdot n(x))^2 + \frac{K_2}{2} (n(x) \cdot \nabla \times n(x))^2 + \frac{K_3}{2} |n(x) \times \nabla \times n(x)|^2 \, dx. \quad (1.2)$$

The constants K_1, K_2, K_3 are known as the Frank constants, and represent the penalisations of splay, twist, and bend deformations respectively. In the case where $K_1 = K_2 = K_3 = K$, we reclaim the one-constant approximation (1.1).

It is natural to ask if such a free energy can be justified. While the original formulation was more phenomenological in nature and based solely on symmetry arguments and a small-deformation assumption, attempts have been made to identify the Oseen–Frank model as a large-domain limit of a more fundamental model, the Landau–de Gennes model [19,41]. In the Landau–de Gennes model, the continuum variable is the *Q-tensor*, corresponding to the normalised second moment of a one-particle distribution function. Explicitly, if the distribution of the long axes of molecules in a small neighbourhood of a point $x \in \Omega$ are described by a probability distribution $f(x, \cdot): \mathbb{S}^2 \rightarrow [0, +\infty)$, we define the Q-tensor at the point x as

$$Q(x) = \int_{\mathbb{S}^2} f(x, p) \left(p \otimes p - \frac{1}{3} I \right) \, dp. \quad (1.3)$$

As molecules are assumed to be head-to-tail symmetric, a molecule is as likely to have orientation $p \in \mathbb{S}^2$ as $-p$, so that $f(x, p) = f(x, -p)$. For this reason the first moment of $f(x, \cdot)$ will always vanish, making the Q-tensor the first non-trivial moment, containing information on molecular alignment. Q-tensors are, following their definition, traceless, symmetric, 3×3 matrices. We denote this set as

$$\text{Sym}_0(3) = \{Q \in \mathbb{R}^3: Q = Q^T, \text{Trace}(Q) = 0\}. \quad (1.4)$$

The Q-tensor contains more information than the director field, namely that it does not force the interpretation of axially symmetric ordering about an axis (less symmetric configurations are permitted), and the degree of orientational ordering is permitted to vary.

Depending on their eigenvalues, they come in one of three varieties.

- If all eigenvalues are equal, $Q = 0$, and we say that Q is *isotropic*, and representative of a disordered system. In particular, if f is a uniform distribution on $\mathbb{S}^2$, $Q = 0$.

- If two eigenvalues are equal and the third is distinct, we say Q is *uniaxial*. A uniaxial Q -tensor can be written as $Q = s \left(n \otimes n - \frac{1}{3} I \right)$, for a scalar s and unit vector n . We interpret n as the favoured direction of alignment, and s as a measure of the degree of ordering molecules about n .
- If all three eigenvalues are distinct, we say that Q is *biaxial*.

The corresponding free energy to be minimised is

$$\int_{\Omega} \psi_b(Q(x)) + W(Q(x), \nabla Q(x)) \, dx. \quad (1.5)$$

The function $\psi_b: \text{Sym}_0(3) \rightarrow \mathbb{R} \cup \{+\infty\}$ is a frame indifferent bulk potential, which may be taken as a polynomial or the Ball–Majumdar singular potential [2], and further discussion in (H₁)–(H₆). Its main characteristic is that, in the cases considered, it is minimised on the set

$$\mathcal{N} = \left\{ Q \in \text{Sym}_0(3) : \text{there exists } n \in \mathbb{S}^2 \text{ such that } Q = s_0 \left(n \otimes n - \frac{1}{3} I \right) \right\}, \quad (1.6)$$

with s_0 a temperature, concentration and material dependent constant. The elastic energy W is minimised when $\nabla Q = 0$. While many forms are possible, by symmetry the only frame-indifferent, quadratic energy that only depends on the gradient of Q is of the form

$$W(\nabla Q) = \frac{L_1}{2} Q_{ij,k} Q_{ij,k} + \frac{L_2}{2} Q_{ij,k} Q_{ik,j} + \frac{L_3}{2} Q_{ij,j} Q_{ik,k}, \quad (1.7)$$

where Einstein summation notation is used. While Oseen–Frank represents nematic defects as discontinuities in the continuum variables, the Landau de–Gennes approach admits a different description, where nematic defects point defects are typically described as a melting of nematic order, that is $Q = 0$. This permits smooth configurations to describe defects.

In an appropriate large-domain limit of a rescaled problem, the contributions of the bulk energy become overwhelming, and we expect the minimisers to converge to minimisers of a constrained problem, where we minimise the elastic energy

$$\int_{\Omega} W(\nabla Q) \, dx, \quad (1.8)$$

subject to the constraint that $Q(x) \in \mathcal{N}$ almost everywhere. In the case where $Q = s_0 \left(n \otimes n - \frac{1}{3} I \right)$ almost everywhere for some $n \in W^{1,2}(\Omega, \mathbb{S}^2)$, we say that Q is *orientable*, and the problem in the presence of Dirichlet boundary conditions that are $\mathcal{N}$ -valued almost everywhere becomes equivalent to that of the minimising the energy (1.2) for n . The constants L_i and K_i are related in the case of Dirichlet boundary conditions where null-Lagrangian terms may be neglected as

$$\begin{aligned} \frac{1}{s_0^2} K_1 &= 2L_1 + L_2 + L_3, \\ \frac{1}{s_0^2} K_2 &= 2L_1, \\ \frac{1}{s_0^2} K_3 &= 2L_1. \end{aligned} \quad (1.9)$$

An energy purely quadratic in ∇Q cannot give rise to three independent elastic constants in the Oseen–Frank model, with the so-called “cubic term” $Q_{ij} Q_{kl,i} Q_{kl,j}$ often being used to fill the degeneracy. Such a term does not arise from the model we will consider, although a more complex variant taking into account molecular length scales has been proposed to avoid this issue [13].

Studying the convergence of minimisers of Landau–de Gennes towards the Oseen–Frank limit has attracted interest, with Majumdar and Zarnescu showing global $W^{1,2}$ convergence and uniform convergence away from singular sets in the one-constant case [39], Nguyen and Zarnescu proving convergence results

in stronger topologies [42], Contreras, Lamy and Rodiac generalising the approach to other harmonic-map problems [12], and further extensions by Contreras and Lamy [11] and Canevari, Majumdar and Stroffolini [9] to more general elastic energies. In other settings, the $W^{1,2}$ -convergence does not hold globally but only locally, away from the singular sets, due to topological obstructions carried by the boundary data and/or the domain (see e.g. [3,8,22,27]). Recently, Di Fratta, Robbins, Slustikov and Zarnescu found higher-order Landau–de Gennes corrections to the Oseen–Frank functional, in two dimensional domains, by studying the Γ -expansion of the Landau–de Gennes functional in the large-domain limit [15]. The problem holds many parallels to the now-classical Ginzburg–Landau problem [6]. Other singular limits and qualitative features of Landau–de Gennes solutions have been studied too; see, for instance, [10,14,26,28–31,33] and the references therein.

While Landau–de Gennes has proven an effective model in many situations, there are still open questions as to how one may justify the model in a rigorous way. While one may use Landau–de Gennes, in appropriate situations, to justify Oseen–Frank, a rigorous justification of Landau–de Gennes itself is lacking. Historically it was justified on a phenomenological basis, but other work has been able to provide Landau–de Gennes as a gradient expansion of a non-local mean field model [18,23]. Justification by formal gradient expansions leaves open the question as to the consistency of minimisers of the original free energy with minimisers of its approximation, that is, are minimisers of the approximate model necessarily good approximations of the minimisers of the original problem? To this end, recent work has been focused on rigorous asymptotic analysis of non-local free energies, which similarly produce the Oseen–Frank model in a large-domain limit [35,36,48,49]. These approaches “bypass” the intermediate and non-rigorous derivation of Landau–de Gennes. This is analogous to recent investigations into peridynamics, a formulation of elasticity based on non-local interactions. These formulations of elasticity bear mathematical similarity with the mean-field theory approach, where stress–strain relations are described in terms of non-local operators on the deformation map, rather than derivatives as in the more classical formulations of elasticity [4,45]. The classical density functional theory we will consider in this work is based on a simplified competition between an entropic contribution to the energy, favouring disorder, and an interaction energy, favouring order. The models themselves are justified as a second order truncation of the virial expansion in the dilute regime based on long-range attractive interactions in the style of Maier and Saupe [38] and with mathematical similarity to the model of Onsager [43]. Explicitly, given the one-particle distribution function $f(x, \cdot)$ in a neighbourhood of x , we define a free energy functional

$$k_B T \rho_0 \int_{\Omega \times \mathbb{S}^2} f(x, p) \ln f(x, p) \, dx \, dp - \frac{\rho_0^2}{2} \int_{\Omega \times \mathbb{S}^2} \int_{\Omega \times \mathbb{S}^2} f(x, p) f(y, q) \mathcal{K}(x - y, p, q) \, dx \, dp \, dy \, dq. \quad (1.10)$$

$\rho_0 > 0$ is the number density of particles in space, k_B the Boltzmann constant, $T > 0$ temperature and $\mathcal{K}(z, p, q)$ denotes the interaction energy of particles with orientations p, q and with centres of mass separated by a vector z . The entropic term on the left is convex and readily shown to be minimised at a uniform distribution, that is, an isotropic disordered system. The nature of the pairwise interaction energy on the right is that nearby particles will prefer to be aligned with each other. We see that temperature and concentration mediate the competition between these opposing mechanisms. Recent work has established the Oseen–Frank energy (1.2) in terms of a large-domain limit of the energy (1.10) under certain assumptions, in which the elastic constants K_i can be related to second moments of the interaction kernel. Previous work has established weaker modes of convergence, while in this work we will establish stronger convergence of minimisers away from defect sets, analogous to the approach taken by Majumdar and Zarnescu for the Landau–de Gennes model [39]. The results however will require stronger assumptions on the regularity and decay of the interaction kernel than those of [48], owing to the need for more precise control on the decay of various integral quantities.

1.2. Simplification of the model and non-dimensionalisation

Here and throughout the sequel, we consider the more general case where molecules admit an internal degree of freedom p in a manifold $\mathcal{M}$. We will employ a macroscopic order parameter $u \in \mathbb{R}^m$ to emphasise the analysis is not limited to the concrete case of nematic liquid crystals.

Through most of the paper, we consider the case where f is prescribed on $(\mathbb{R}^3 \setminus \frac{1}{\varepsilon}\Omega) \times \mathcal{M}$, where Ω is a non-dimensional reference domain and $\varepsilon > 0$ is a small parameter, representative of the inverse of a large length scale of the domain. In Section 5, we relax this assumption and study a minimisation problem where f is prescribed only in a neighbourhood of the domain, of suitable thickness. We consider the free energy

$$\begin{aligned} \tilde{\mathcal{G}}_\varepsilon(f) = & k_B T \rho_0 \int_{\frac{1}{\varepsilon}\Omega \times \mathcal{M}} f(x, p) \ln f(x, p) \, dx \, dp \\ & - \frac{\rho_0^2}{2} \int_{\mathbb{R}^3 \times \mathcal{M}} \int_{\mathbb{R}^3 \times \mathcal{M}} f(x, p) f(y, q) \mathcal{K}(x - y, p, q) \, dx \, dp \, dy \, dq. \end{aligned} \quad (1.11)$$

For simplification of the problem, we take the interaction energy to be of the form

$$\mathcal{K}(z, p, q) = K(z) \sigma(p) \cdot \sigma(q), \quad (1.12)$$

where $\sigma \in L^\infty(\mathcal{M}, \mathbb{R}^m)$ is some “microscopic order parameter”, and $K: \mathbb{R}^3 \rightarrow \mathbb{R}^{m \times m}$ is a symmetric tensor field, which will satisfy certain technical conditions (see (K₁)–(K₆) in Section 2). By applying Fubini we may then introduce a “macroscopic order parameter”, $u \in L^\infty(\mathbb{R}^3, \mathbb{R}^m)$ by

$$u(x) = \int_{\mathcal{M}} f(x, p) \sigma(p) \, dp, \quad (1.13)$$

and re-write the interaction energy as

$$\begin{aligned} & - \frac{\rho_0^2}{2} \int_{\mathbb{R}^3 \times \mathcal{M}} \int_{\mathbb{R}^3 \times \mathcal{M}} f(x, p) f(y, p) \mathcal{K}(x - y, p, q) \, dx \, dp \, dy \, dq \\ & = - \frac{\rho_0^2}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x - y) u(x) \cdot u(y) \, dx \, dy. \end{aligned} \quad (1.14)$$

While it is not possible to write the entropic term explicitly in terms of u , we may provide a lower bound by means of the singular Ball–Majumdar/Katriel potential and its extensions [2,32,47] by

$$\int_{\frac{1}{\varepsilon}\Omega \times \mathcal{M}} f(x, p) \ln f(x, p) \, dx \, dp \geq \int_{\frac{1}{\varepsilon}\Omega} \psi_s(u(x)) \, dx, \quad (1.15)$$

where the function $\psi_s: \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\psi_s(u) = \min \left\{ \int_{\mathcal{M}} f(p) \ln f(p) \, dp : f \geq 0 \text{ a.e., } \int_{\mathcal{M}} f(p) \, dp = 1, \int_{\mathcal{M}} f(p) \sigma(p) \, dp = u \right\}, \quad (1.16)$$

where by convention $\psi_s(u) = +\infty$ when the constraint set is empty. Note that the minimisation problem (1.16) is strictly convex, thus solutions are necessarily unique, and we may define f_u to be the corresponding minimiser for $u \in \mathcal{Q} = \{u : \psi_s(u) < +\infty\}$. That is,

$$f_u = \arg \min \left\{ \int_{\mathcal{M}} f(p) \ln f(p) \, dp : f \geq 0 \text{ a.e., } \int_{\mathcal{M}} f(p) \, dp = 1, \int_{\mathcal{M}} f(p) \sigma(p) \, dp = u \right\}. \quad (1.17)$$

The precise definition of ψ_s will be unimportant in this work, and we employ any function ψ_s satisfying certain technical assumptions in the sequel (see (H₁)–(H₆) in Section 2)

We in fact have the result that f^* is a minimiser of $\tilde{\mathcal{G}}_\varepsilon$ if and only if, for $u^*(x) = \int_{\mathbb{S}^2} f^*(x, p) \sigma(p) dp$, u^* is a minimiser of

$$\tilde{\mathcal{F}}_\varepsilon(u) = k_B T \rho_0 \int_{\frac{1}{\varepsilon} \Omega} \psi_s(u(x)) dx - \frac{\rho_0^2}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x-y) u(x) \cdot u(y) dx dy, \quad (1.18)$$

with $f^* = f_{u^*}$. This is readily seen by writing the minimisation as a two-step process, first minimising over all f such that $f_u = f$, and later minimising over u and noting that the first minimisation may be performed pointwise almost-everywhere in $\mathbb{R}^3$, as in [48]. That is to say, we have a simpler, macroscopic energy with equivalent minimisers. By introducing a change of variables,

$$x = \frac{x'}{\varepsilon}, \quad y = \frac{y'}{\varepsilon}, \quad u'(x') = u(x), \quad \varepsilon' := \frac{\varepsilon}{\rho_0^{1/3}}, \quad K'(x') = \frac{1}{k_B T} K(\varepsilon' x'),$$

and a (non-dimensional) constant $C_{\varepsilon'}$ to be specified later, we rescale the domain and obtain the free energy we will consider for the remainder of this work, so that

$$\begin{aligned} E_{\varepsilon'}(u') &:= \frac{\varepsilon}{k_B T \rho_0^{1/3}} \tilde{\mathcal{F}}_\varepsilon(u) + C_{\varepsilon'} \\ &= \frac{1}{\varepsilon'^2} \int_{\Omega} \psi_s(u'(x')) dx' - \frac{1}{2\varepsilon'^5} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K' \left(\frac{x' - y'}{\varepsilon'} \right) u'(x') \cdot u'(y') dx' dy' + C_{\varepsilon'}. \end{aligned} \quad (1.19)$$

The additive constant $C_{\varepsilon'}$ is irrelevant for the purpose of minimisation; however, we will make a specific choice of $C_{\varepsilon'}$ (see Eq. (2.6) below) for analytical convenience. We will consider the regime as $\varepsilon' \rightarrow 0$ in this work. From the definition of ε' , this may be interpreted in two forms, one in which the characteristic length scale of the domain, $\frac{1}{\varepsilon}$, becomes large, and one in which the density ρ_0 becomes large. However, as the energy we consider is based on the second order virial expansion which is explicitly a model for dilute regimes, we interpret the limit $\varepsilon' \rightarrow 0$ as the former, that is, a large-domain limit. In the sequel we omit the primes and consider (1.19) as our free energy functional to be minimised at scale $\varepsilon > 0$.

2. Technical assumptions and main results

Let $\text{Sym}(m)$ be the space of $(m \times m)$ -symmetric matrices, with real coefficients. Given an interaction kernel $K: \mathbb{R}^3 \rightarrow \text{Sym}(m)$ and $\varepsilon > 0$, we define $K_\varepsilon(z) := \varepsilon^{-3} K(\varepsilon^{-1} z)$ for any $z \in \mathbb{R}^3$. Then, we may rewrite the functional (1.19) as

$$E_\varepsilon(u) := -\frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} K_\varepsilon(x-y) u(x) \cdot u(y) dx dy + \frac{1}{\varepsilon^2} \int_{\Omega} \psi_s(u(x)) dx + C_\varepsilon, \quad (2.1)$$

where $u: \mathbb{R}^3 \rightarrow \mathbb{R}^m$ is the macroscopic order parameter, $\Omega \subseteq \mathbb{R}^3$ is a bounded, smooth domain, and $\psi_s: \mathbb{R}^m \rightarrow [0, +\infty]$ is any convex potential that satisfies the assumptions (H₁)–(H₆) below (for instance, the Ball–Majumdar/Katriel potential defined by (1.16)).

Assumptions on the kernel K . Our assumptions on the kernel K are reminiscent of [48]. We define $g(z) := \lambda_{\min}(K(z))$ for any $z \in \mathbb{R}^3$, where $\lambda_{\min}(K)$ denotes the minimum eigenvalue of K .

(K₁) $K \in W^{1,1}(\mathbb{R}^3, \text{Sym}(m))$.

(K₂) K is even, that is $K(z) = K(-z)$ for a.e. $z \in \mathbb{R}^m$.

(K₃) $g \geq 0$ a.e. on $\mathbb{R}^3$, and there exist positive numbers $\rho_1 < \rho_2$, k_* such that $g \geq k_*$ a.e. on $B_{\rho_2} \setminus B_{\rho_1}$.

(K₄) $g \in L^1(\mathbb{R}^3)$ and there exists $q > 7/2$ such that $\int_{\mathbb{R}^3} g(z) |z|^q dz < +\infty$.

(K₅) There exists a positive constant C such that $\lambda_{\max}(K(z)) \leq Cg(z)$ for a.e. $z \in \mathbb{R}^3$ (where $\lambda_{\max}(K)$ denotes the maximum eigenvalue of K).

(K₆) There exists $\nu > 1$ such that

$$\int_{\mathbb{R}^3} \|\nabla K(z)\| |z|^\nu dz < +\infty,$$

where $\|\nabla K(z)\|^2 := \partial_\alpha K_{ij}(z) \partial_\alpha K_{ij}(z)$.

In the case of physically meaningful systems the tensor K will have to respect frame invariance. In the case of nematic liquid crystals, where the order parameter is a traceless symmetric matrix Q , frame indifference implies that the bilinear form must necessarily be of the form

$$K(z)Q_1 \cdot Q_2 = f_1(|z|)Q_1 \cdot Q_2 + f_2(|z|)Q_1 z \cdot Q_2 z + f_3(|z|)(Q_1 z \cdot z)(Q_2 z \cdot z), \quad (2.2)$$

for all $Q_1, Q_2 \in \text{Sym}_0(3)$, where f_1, f_2, f_3 are real-valued functions defined on $[0, +\infty)$ [48]. It is clear that by appropriate choices of f_1, f_2, f_3 which are C^1 and with sufficient decay at infinity the previous assumptions can be satisfied. This family of bilinear forms includes the simplified interaction energy

$$K(z)Q_1 \cdot Q_2 = \varphi(|z|)Q_1 \cdot Q_2 \quad (2.3)$$

for a suitable function φ , which includes the results of [35,36], where φ is taken to be rapidly decaying and C^∞ , which are stronger assumptions than we shall consider. Furthermore [7, Equation (3.43)] considers K to have the same structure, albeit with a slower decay of φ than our analysis would permit.

We remark that the integrability requirements in (K₄) and (K₆) and regularity requirement in (K₁), although weaker than the assumptions in the earlier work [35], are stronger than that of [48], and permit more delicate control of integral estimates needed to show convergence of minimisers in a stronger sense. (See also Remark 2.4.)

Assumptions on the singular potential ψ_s .

(H₁) $\psi_s: \mathbb{R}^m \rightarrow (-\infty, +\infty]$ is a convex function.

(H₂) The domain of ψ_s , $\mathcal{Q} := \psi_s^{-1}(\mathbb{R}) \subseteq \mathbb{R}^m$, is a non-empty, bounded open set that contains 0 and $\psi_s \in C^2(\mathcal{Q})$.

(H₃) There exists a constant $c > 0$ such that $\nabla^2 \psi_s(y) \chi \cdot \chi \geq c |\chi|^2$ for any $y \in \mathcal{Q}$ and any $\chi \in \mathbb{R}^m$.

(H₄) There holds $\psi_s(y) \rightarrow +\infty$ as $\text{dist}(y, \partial \mathcal{Q}) \rightarrow 0$.

We define the “bulk potential” $\psi_b: \mathcal{Q} \rightarrow \mathbb{R}$ in terms of K and ψ_s , as

$$\psi_b(y) := \psi_s(y) - \frac{1}{2} \left(\int_{\mathbb{R}^3} K(z) dz \right) y \cdot y + c_0 \quad \text{for any } y \in \mathcal{Q}, \quad (2.4)$$

where $c_0 \in \mathbb{R}$ is a constant, uniquely determined by imposing that $\inf \psi_b = 0$. We make the following assumptions on ψ_b :

(H₅) The set $\mathcal{N} := \psi_b^{-1}(0) \subseteq \mathcal{Q}$ is a compact, smooth, connected manifold without boundary.

(H₆) For any $y \in \mathcal{N}$ and any unit vector $\xi \in \mathbb{S}^{m-1}$ that is orthogonal to $\mathcal{N}$ at y , we have $\nabla^2 \psi_b(y) \xi \cdot \xi > 0$.

Remark 2.1. If the norm of $\int_{\mathbb{R}^3} K(z) dz$ is smaller than the constant c given by (H₃), then the function ψ_b is strictly convex and hence, its zero-set $\mathcal{N}$ reduces to a point. This happens, for example, in the sufficiently high temperature regime, independently of the precise form of K . Nevertheless, our arguments remain valid in this case, too.

Remark 2.2. The Ball–Majumdar/Katriel potential, defined by (1.16), satisfies the conditions (H₁)–(H₆). (H₁), (H₂), (H₄), and (H₅) follow from [2], apart from the C^2 smoothness of ψ_s which is implicitly proven in [32] via an inverse function theorem argument, although not stated. (H₃) is proven in [47]. With this choice of the potential, the set $\mathcal{N} := \psi_b^{-1}(0)$ is either a point or the manifold given by (1.6) (see [2, Section 4]). In both cases, (H₆) is satisfied (see [34, Proposition 4.2]).

The admissible class and an equivalent expression for the free energy. We complement the minimisation of the functional (2.1) by prescribing u on $\mathbb{R}^3 \setminus \Omega$. We take a map $u_{\text{bd}} \in H^1(\mathbb{R}^3, \mathbb{R}^m)$ such that

$$u_{\text{bd}}(x) \in \mathcal{Q} \quad \text{for a.e. } x \in \mathbb{R}^3 \setminus \Omega, \quad u_{\text{bd}}(x) \in \mathcal{N} \quad \text{for a.e. } x \in \Omega, \quad (\text{BD})$$

and we define the admissible class

$$\mathcal{A} := \{u \in L^\infty(\mathbb{R}^3, \mathcal{Q}) : \psi_s(u) \in L^1(\Omega), u = u_{\text{bd}} \text{ a.e. on } \mathbb{R}^3 \setminus \Omega\}. \quad (2.5)$$

Remark 2.3. In order for the assumption (BD) to be satisfied, it is necessary that the trace of u_{bd} on $\partial\Omega$ takes its values in the manifold $\mathcal{N}$. If $\mathcal{N}$ is simply connected, then any boundary value that belongs to $H^{1/2}(\partial\Omega, \mathcal{N})$ admits an extension in $H^1(\Omega, \mathcal{N})$; this follows from [25, Theorem 6.2]. However, when $\mathcal{N}$ is multiply connected (for instance, when $\mathcal{N}$ is the real projective plane, as in the applications to liquid crystals) there exist boundary values in $H^{1/2}(\partial\Omega, \mathcal{N})$ that do not have any extension in $H^1(\Omega, \mathcal{N})$. (See e.g. [5,40] for results on the extension problem for manifold-valued Sobolev maps.) On the other hand, $\mathcal{Q}$ is a convex set that contains $\mathcal{N}$ and 0, so any boundary value in $H^{1/2}(\partial\Omega, \mathcal{N})$ has an extension in $H^1(\mathbb{R}^3 \setminus \Omega, \mathcal{Q})$.

In the class $\mathcal{A}$, the functional E_ε has an alternative expression. For any $y \in \mathbb{R}^m$, we use the abbreviated notation $y^{\otimes 2} := y \otimes y$. We choose

$$C_\varepsilon := \frac{c_0}{\varepsilon^2} |\Omega| + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3 \setminus \Omega} \left(\int_{\mathbb{R}^3} K(z) dz \right) \cdot u_{\text{bd}}(x)^{\otimes 2} dx, \quad (2.6)$$

where $|\Omega|$ denotes the volume of Ω and $c_0 \in \mathbb{R}$ is the same number as in (2.4). The constant C_ε only depends on ε , Ω , K and u_{bd} , so it does not affect minimisers of the functional. By applying the algebraic identity

$$-2K(x-y)u(x) \cdot u(y) = K(x-y) \cdot (u(x) - u(y))^{\otimes 2} - K(x-y) \cdot u(x)^{\otimes 2} - K(x-y) \cdot u(y)^{\otimes 2}$$

and using (2.4), (2.6), we re-write (2.1) as

$$E_\varepsilon(u) = \frac{1}{4\varepsilon^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} K_\varepsilon(x-y) \cdot (u(x) - u(y))^{\otimes 2} dx dy + \frac{1}{\varepsilon^2} \int_\Omega \psi_b(u(x)) dx \quad (2.7)$$

for any $u \in \mathcal{A}$. We note that the free energy admits parallels to the Landau–de Gennes energy, with the right-hand term being a corresponding bulk energy and the left-hand term acting as a non-local analogue of the elastic energy, which we shall see is reclaimed in a precise way in the asymptotic limit as $\varepsilon \rightarrow 0$. Let L be the unique symmetric fourth-order tensor that satisfies

$$L\xi \cdot \xi := \frac{1}{4} \int_{\mathbb{R}^3} K(z) \cdot (\xi z)^{\otimes 2} dz \quad \text{for any } \xi \in \mathbb{R}^{m \times 3}. \quad (2.8)$$

The right-hand side of (2.8) is well-defined and finite for any $\xi \in \mathbb{R}^{m \times 3}$ because K has finite second moment, thanks to the assumption (K₄). Coordinate-wise, L is defined by

$$L_{ij\alpha\beta} = \frac{1}{4} \int_{\mathbb{R}^3} K_{\alpha\beta}(z) z_i z_j dz$$

for any $i, j \in \{1, 2, 3\}$ and $\alpha, \beta \in \{1, 2, \dots, m\}$. Let $E_0: \mathcal{A} \rightarrow [0, +\infty]$ be given as

$$E_0(u) := \begin{cases} \int_\Omega L \nabla u \cdot \nabla u & \text{if } u \in H^1(\Omega, \mathcal{N}) \cap \mathcal{A} \\ +\infty & \text{otherwise.} \end{cases} \quad (2.9)$$

By assumption (BD), the set $H^1(\Omega, \mathcal{N}) \cap \mathcal{A}$ is non-empty and hence, the functional E_0 is not identically equal to $+\infty$. Taylor [48] proved that, as $\varepsilon \rightarrow 0$, the functional E_ε Γ -converges to E_0 with respect to the L^2 -topology. In particular, up to subsequences, minimisers u_ε of E_ε in the class $\mathcal{A}$ converge L^2 -strongly to a minimiser u_0 of E_0 in $\mathcal{A}$. Our aim is to prove a convergence result for minimisers, in a stronger topology.

Main results. Given a Borel set $G \subseteq \mathbb{R}^3$ and $u \in L^\infty(G, \mathcal{Q})$, we define

$$F_\varepsilon(u, G) := \frac{1}{4\varepsilon^2} \int_{G \times G} K_\varepsilon(x - y) \cdot (u(x) - u(y))^{\otimes 2} dx dy + \frac{1}{\varepsilon^2} \int_G \psi_b(u(x)) dx. \quad (2.10)$$

For any $\mu \in (0, 1)$, we denote the μ -Hölder semi-norm of u on G as

$$[u]_{C^\mu(G)} := \sup_{x, y \in G, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\mu}$$

Theorem A (Uniform η -regularity). Assume that (K_1) – (K_6) , (H_1) – (H_6) and (BD) are satisfied. Then, there exist positive numbers η , ε_* , M and $\mu \in (0, 1)$ such that for any ball $B_{r_0}(x_0) \subseteq \Omega$, any $\varepsilon \in (0, \varepsilon_* r_0)$, and any minimiser u_ε of E_ε in $\mathcal{A}$ such that

$$r_0^{-1} F_\varepsilon(u_\varepsilon, B_{r_0}(x_0)) \leq \eta^2$$

there holds

$$r_0^\mu [u_\varepsilon]_{C^\mu(B_{r_0/2}(x_0))} \leq M.$$

As a corollary, we deduce a convergence result for minimisers of E_ε , in the locally uniform topology. We recall that any minimiser u_0 for the limit functional (2.9) in $\mathcal{A}$ is smooth in $\Omega \setminus S[u_0]$, where

$$S[u_0] := \left\{ x \in \Omega : \liminf_{\rho \rightarrow 0} \rho^{-1} \int_{B_\rho(x)} |\nabla u_0|^2 > 0 \right\}. \quad (2.11)$$

Moreover, $S[u_0]$ is a closed set of zero total length (see e.g. [24,37]).

Theorem B. Assume that the conditions (K_1) – (K_6) , (H_1) – (H_6) and (BD) are satisfied. Let u_ε be a minimiser of E_ε in $\mathcal{A}$. Then, up to extraction of a (non-relabelled) subsequence, we have

$$u_\varepsilon \rightarrow u_0 \quad \text{locally uniformly in } \Omega \setminus S[u_0],$$

where u_0 is a minimiser of the functional (2.9) in $\mathcal{A}$.

The strategy of the proof for Theorem A is inspired by [11]. Under the assumption $F_\varepsilon(u_\varepsilon, B_1) \leq \eta$, we obtain an algebraic decay for the mean oscillation of u_ε , that is

$$\left| \int_{B_\rho} u_\varepsilon - \int_{B_\rho} u_\varepsilon \right|^2 \leq C \rho^{2\mu} \quad (2.12)$$

for any $\rho \in (0, 1)$ and some positive constants C, μ that do not depend on ρ, ε . If the radius ρ is large enough, i.e. $\rho \geq \varepsilon^\gamma$ for some suitable $\gamma \in (0, 1)$, we obtain an algebraic decay for $F_\varepsilon(u_\varepsilon, B_\rho)$ as a function of ρ by adapting analogous arguments for the limit functional E_0 (cf. Luckhaus' partial regularity results in [37]); then, we deduce (2.12) via a suitable Poincaré inequality (Proposition 3.4). On the other hand, if $\rho \leq \varepsilon^\gamma$ we obtain (2.12) from the Euler–Lagrange equations for E_ε (Proposition 3.1). The inequality (2.12) immediately implies the desired bound on the Hölder norm of u_ε , by Campanato embedding. Once Theorem A is proven, Theorem B follows, via the Ascoli–Arzelà theorem.

Remark 2.4. As we observed before, if we are interested in weaker modes of convergence for the minimisers (e.g., L^2 -convergence), then we may replace (K_4) and (K_6) with the weaker condition that $g \in L^1(\mathbb{R}^3)$ and g has finite second moment, as in [48]. However, (K_4) and (K_6) play a very important rôle for us; both of them are used in the proof of the estimate (2.12) for small radii, $\rho \leq \varepsilon^\gamma$. We do not know whether Theorems A and B remain true under weaker assumptions.

3. Preliminary results

3.1. The Euler–Lagrange equations

Throughout the paper, we denote by C several constants that depend only on Ω , K , m , ψ_s and u_{bd} . We write $A \lesssim B$ as a short-hand for $A \leq CB$. We also define $g_\varepsilon(z) := \varepsilon^{-3}g(\varepsilon^{-1}z)$ for $z \in \mathbb{R}^3$ (where, we recall, $g(z)$ is the minimum eigenvalue of $K(z)$) and

$$\Lambda := \nabla \psi_s: \mathcal{Q} \rightarrow \mathbb{R}^m. \quad (3.1)$$

Proposition 3.1. *Consider the free energy E_ε , given by (2.1), with $u = u_{\text{bd}}$ on $\mathbb{R}^3 \setminus \Omega$. Then there exists a minimiser $u_\varepsilon \in L^\infty(\Omega, \mathcal{Q})$ (identified with its extension by u_{bd} to $\mathbb{R}^3$), and it satisfies the Euler–Lagrange equation,*

$$\Lambda(u_\varepsilon(x)) = \int_{\mathbb{R}^3} K_\varepsilon(x-y)u_\varepsilon(y) \, dy \quad (3.2)$$

for a.e. $x \in \Omega$.

Proof. By neglecting the additive constant in (2.1), and multiplying by ε^2 , without loss of generality we may consider the functional

$$\mathcal{F}(u) := \int_{\Omega} \psi_s(u(x)) \, dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_\varepsilon(x-y)u(x) \cdot u(y) \, dx \, dy$$

instead of E_ε . To show existence, we use a direct method argument. First we show that the bilinear form admits a global lower bound. As $u_{\text{bd}} \in L^2(\mathbb{R}^3, \mathcal{Q})$ and u admits uniform L^∞ -bounds on Ω , we have that $u \in L^2(\mathbb{R}^3, \overline{\mathcal{Q}})$, $\|u\|_{L^2(\mathbb{R}^3)}$ is bounded uniformly. We thus have the estimate that

$$\begin{aligned} & \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |K_\varepsilon(x-y)u(x) \cdot u(y)| \, dx \, dy \\ & \lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g_\varepsilon(x-y)|u(x)||u(y)| \, dx \, dy \\ & = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(g_\varepsilon(x-y)^{\frac{1}{2}}|u(x)| \right) \left(g_\varepsilon(x-y)^{\frac{1}{2}}|u(y)| \right) \, dx \, dy \\ & \lesssim \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g_\varepsilon(x-y)|u(x)|^2 \, dx \, dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g_\varepsilon(x-y)|u(y)|^2 \, dx \, dy \right)^{\frac{1}{2}} \\ & = \|g_\varepsilon\|_{L^1(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}^2 = \|g\|_{L^1(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}^2 \end{aligned} \quad (3.3)$$

The singular function ψ_s admits a lower bound pointwise, hence the functional $\mathcal{F}$ admits a global lower bound. To show the admissible set is non empty, simply take $u(x) = u_0 \in \mathcal{Q}$ for all $x \in \Omega$, so that $\psi_s(u(x))$ is a non-infinite constant.

The uniform L^∞ bounds on u imply that we have L^∞ weak-* compactness of a minimising sequence. As ψ_s is strictly convex, we have weak-* lower semicontinuity of the entropic term. It suffices to show weak-* lower semicontinuity of the bilinear term. First we split the bilinear term into the “boundary” and “bulk” contributions. That is, we write $u = u_{\text{bd}} \chi_{\mathbb{R}^3 \setminus \Omega} + u \chi_\Omega$, where $\chi_{\mathbb{R}^3 \setminus \Omega}$ and χ_Ω are the characteristic functions of $\mathbb{R}^3 \setminus \Omega$ and Ω respectively. As $K_\varepsilon * (u_{\text{bd}} \chi_{\mathbb{R}^3 \setminus \Omega}) \in L^1(\Omega)$, if $u_j \xrightarrow{*} u$,

$$\int_{\Omega} u_j(x) K_\varepsilon * (u_{\text{bd}} \chi_{\mathbb{R}^3 \setminus \Omega})(x) \, dx \rightarrow \int_{\Omega} u(x) K_\varepsilon * (u_{\text{bd}} \chi_{\mathbb{R}^3 \setminus \Omega})(x) \, dx. \quad (3.4)$$

The second term requires a little more care. Following [16, Corollary 4.1], the map $L^\infty(\Omega) \ni u \mapsto K_\varepsilon * (u \chi_\Omega)$ is L^∞ -to- L^1 compact if and only if the set $\{K_\varepsilon(x - \cdot) \chi_\Omega: x \in \Omega\}$ is relatively L^1 -compact. This is immediate

however as Ω is a bounded set and K_ε is integrable. Therefore the map

$$u \mapsto \int_{\Omega} \int_{\Omega} K_\varepsilon(x-y) u(x) \cdot u(y) \, dx \, dy \quad (3.5)$$

is in fact continuous with the weak- $*$ L^∞ topology, and therefore the entire bilinear term is continuous also. Therefore the energy functional is lower semicontinuous and minimisers exist by the direct method.

To show that minimisers satisfy the Euler–Lagrange equation, we note that if u has finite energy, then the measure of the set $\{x \in \Omega : u(x) \in \partial\mathcal{Q}\}$ is zero. In particular, we may define $U_\delta = \{x \in \Omega : \psi_s(u(x)) < 1/\delta\}$, and we have that

$$\Omega = \Gamma \cup \bigcup_{\delta>0} U_\delta, \quad (3.6)$$

where Γ is a null set. By Assumption (H₄), for every $\delta > 0$, there exists some $\gamma > 0$ so that if $\psi_s(\tilde{u}) < 1/\delta$, then $\text{dist}(\tilde{u}, \partial\mathcal{Q}) > \gamma$. In particular, for $\phi \in L^\infty(\mathbb{R}^3, \mathbb{R}^m)$ supported on U_δ and η sufficiently small, $u + \eta\phi$ is bounded away from $\partial\mathcal{Q}$ on U_δ . Therefore we may take variations without issue, as

$$\begin{aligned} \frac{1}{\eta} (\mathcal{F}(u + \eta\phi) - \mathcal{F}(u)) &= \int_{U_\delta} \frac{1}{\eta} (\psi_s(u(x) + \eta\phi(x)) - \psi_s(u(x))) \, dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_\varepsilon(x-y) \cdot (2\phi(x) \otimes u(y) + \eta\phi(x) \otimes \phi(y)) \, dx \, dy. \end{aligned}$$

Now we have no issue taking the limit as $\eta \rightarrow 0$, as ψ_s is C^2 away from $\partial\mathcal{Q}$, to give

$$\begin{aligned} \lim_{\eta \rightarrow 0} \frac{1}{\eta} (\mathcal{F}(u + \eta\phi) - \mathcal{F}(u)) &= \int_{U_\delta} \Lambda(u(x)) \cdot \phi(x) \, dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_\varepsilon(x-y) u(y) \, dy \cdot \phi(x) \, dx \\ &= \int_{U_\delta} \left(\Lambda(u(x)) - \int_{\mathbb{R}^3} K_\varepsilon(x-y) u(y) \, dy \right) \cdot \phi(x) \, dx, \end{aligned}$$

recalling that $\phi(x) = 0$ outside of U_δ . As ϕ was arbitrary, this implies that u satisfies

$$\Lambda(u(x)) = \int_{\mathbb{R}^3} K_\varepsilon(x-y) u(y) \, dy \quad (3.7)$$

on U_δ , and since δ was arbitrary, this implies that u satisfies the Euler–Lagrange equation outside of Γ , which is of measure zero. $\square$

The Euler–Lagrange equations are particularly useful when used in combination with the following property.

Lemma 3.2. *The map $\Lambda: \mathcal{Q} \rightarrow \mathbb{R}^m$ is invertible and its inverse is of class C^1 . Moreover,*

$$\sup_{z \in \mathbb{R}^m} \|\nabla(\Lambda^{-1})(z)\| \leq c^{-1}, \quad (3.8)$$

where c is the constant given by (H₃), and

$$|\Lambda(y)| \rightarrow +\infty \quad \text{as } \text{dist}(y, \partial\mathcal{Q}) \rightarrow 0. \quad (3.9)$$

Proof. To prove (3.9), it suffices to note that as ψ_s is a closed proper convex function which is C^1 on an open domain, so by applying classical results from convex analysis [44, Theorem 25.1, Theorem 26.1], we see that ψ_s satisfies the property of *essential smoothness*, which implies (3.9). More so, as ψ_s is also strictly convex on a bounded domain, this implies ψ_s is a Legendre-type function which provides the results that $\Lambda(\mathcal{Q}) = \mathbb{R}^m$ [44, Corollary 13.3.1], and that Λ is a C^0 bijection from $\mathcal{Q} \rightarrow \Lambda(\mathcal{Q})$ [44, Theorem 26.5]. The C^1 regularity of Λ^{-1} follows immediately from the inverse function theorem, as ψ_s is strongly convex. $\square$

The Euler–Lagrange equation (3.2) and Lemma 3.2 have important consequences in terms of regularity and “strict physicality” of minimisers — that is, the image of u_ε does not touch the boundary of the physically admissible set $\mathcal{Q}$.

Proposition 3.3. *Minimisers u_ε of the functional E_ε in the class $\mathcal{A}$ are Lipschitz-continuous on Ω , with $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \lesssim \varepsilon^{-1}$. Moreover, there exists a number $\delta > 0$ such that for any $\varepsilon > 0$ and any $x \in \Omega$,*

$$\text{dist}(u_\varepsilon(x), \partial\mathcal{Q}) \geq \delta. \quad (3.10)$$

Proof. The minimiser u_ε takes values in the bounded set $\mathcal{Q}$ and hence, $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq C$, where the constant C does not depend on ε . Moreover, $\|K_\varepsilon\|_{L^1(\mathbb{R}^3)} = \|K\|_{L^1(\mathbb{R}^3)} < +\infty$. Then, by applying Young’s inequality to (3.2), we obtain

$$\|A(u_\varepsilon)\|_{L^\infty(\Omega)} \leq \|K_\varepsilon\|_{L^1(\mathbb{R}^3)} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq C.$$

On the other hand, we have $|A(z)| \rightarrow +\infty$ as $z \rightarrow \partial\mathcal{Q}$ by (3.9) and hence, (3.10) follows. Since we have assumed that $K \in W^{1,1}(\mathbb{R}^3, \text{Sym}(m))$, from the Euler–Lagrange equation (3.2) we deduce

$$\|\nabla(A \circ u_\varepsilon)\|_{L^\infty(\Omega)} = \|\nabla K_\varepsilon * u_\varepsilon\|_{L^\infty(\Omega)} \leq \varepsilon^{-1} \|\nabla K\|_{L^1(\Omega)} \|u_\varepsilon\|_{L^\infty(\Omega)} < +\infty.$$

By Lemma 3.2, we conclude that $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \lesssim \varepsilon^{-1}$. $\square$

3.2. A Poincaré-type inequality for F_ε

The goal of this section is to prove the following inequality on F_ε . We recall that the functional F_ε is defined in (2.10).

Proposition 3.4. *There exists $\varepsilon_1 > 0$ such that, for any $u \in L^\infty(\mathbb{R}^3, \mathbb{R}^m)$, any $\rho > 0$, any $x_0 \in \mathbb{R}^3$ and any $\varepsilon \in (0, \varepsilon_1\rho]$, there holds*

$$\left| \int_{B_{\rho/2}(x_0)} u - \int_{B_{\rho/2}(x_0)} u \right|^2 \lesssim \rho^{-1} F_\varepsilon(u, B_\rho(x_0)).$$

To simplify the proof of Proposition 3.4, we will take advantage of the scaling properties of F_ε : if $u_\rho: B_1 \rightarrow \mathbb{R}^m$ is defined by $u_\rho(x) := u(\rho x + x_0)$ for $x \in B_1$, then a change of variables gives

$$\rho^{-1} F_\varepsilon(u, B_\rho(x_0)) = F_{\varepsilon/\rho}(u_\rho, B_1) \quad (3.11)$$

In the proof of Proposition 3.4, we will adapt arguments from [48]. By assumption (K₃), there exist positive numbers $\rho_1 < \rho_2$, k such that $g \geq k$ a.e. on $B_{\rho_2} \setminus B_{\rho_1}$. Let $\varphi \in C_c^\infty(B_{\rho_2} \setminus B_{\rho_1})$ be a non-negative, radial function (i.e. $\varphi(z) = \tilde{\varphi}(|z|)$ for $z \in \mathbb{R}^3$) such that $\int_{\mathbb{R}^3} \varphi(z) \, dz = 1$. Since g is bounded away from zero on the support of φ , there holds

$$\varphi + |\nabla \varphi| \leq Cg \quad \text{pointwise a.e. on } \mathbb{R}^3,$$

for some constant C that depends on g and φ ; however, φ is fixed once and for all, and so is C . We define $\varphi_\varepsilon(z) := \varepsilon^{-3} \varphi(\varepsilon^{-1}z)$ for any $z \in \mathbb{R}^3$ and $\varepsilon > 0$. Then, $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^3)$ is non-negative, even, satisfies $\int_{\mathbb{R}^3} \varphi_\varepsilon(z) \, dz = 1$ and

$$\varphi_\varepsilon + \varepsilon |\nabla \varphi_\varepsilon| \leq Cg_\varepsilon \quad \text{pointwise a.e. on } \mathbb{R}^3. \quad (3.12)$$

Lemma 3.5. *There exists $\varepsilon_2 > 0$ such that, for any $u \in L^\infty(B_1, \mathbb{R}^m)$ and any $\varepsilon \in (0, \varepsilon_2]$, there holds*

$$\int_{B_{1/2}} |\nabla(\varphi_\varepsilon * u)|^2 \lesssim \varepsilon^{-2} \int_{B_1 \times B_1} K_\varepsilon(x-y) \cdot (u(x) - u(y))^{\otimes 2} dx dy.$$

Proof. We adapt the arguments from [48, Lemma 2.1 and Proposition 2.1]. We define

$$I(y, z) := \int_{B_{1/2}} \nabla \varphi_\varepsilon(x-y) \cdot \nabla \varphi_\varepsilon(x-z) dx \quad \text{for } y, z \in \mathbb{R}^3.$$

We express the gradient of $\varphi_\varepsilon * u$ as $\nabla(\varphi_\varepsilon * u) = (\nabla \varphi_\varepsilon) * u$. By applying the identity $2a \cdot b = -|a-b|^2 + |a|^2 + |b|^2$, we obtain

$$\begin{aligned} \int_{B_{1/2}} |\nabla(\varphi_\varepsilon * u)(x)|^2 dx &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} u(y) \cdot u(z) I(y, z) dy dz \\ &= \underbrace{-\frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |u(y) - u(z)|^2 I(y, z) dy dz}_{=: I_1} \\ &\quad + \underbrace{\frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |u(y)|^2 I(y, z) dy dz}_{=: I_2} + \underbrace{\frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |u(z)|^2 I(y, z) dy dz}_{=: I_3} \end{aligned}$$

We first consider the term I_2 . Since φ_ε is compactly supported, we have $\int_{\mathbb{R}^3} \nabla \varphi_\varepsilon(z) dz = 0$. Therefore,

$$I_2 = \frac{1}{2} \int_{B_{1/2} \times \mathbb{R}^3} |u(y)|^2 \nabla \varphi_\varepsilon(x-y) \cdot \left(\int_{\mathbb{R}^3} \nabla \varphi_\varepsilon(x-z) dz \right) dx dy = 0,$$

and likewise $I_3 = 0$. Now, we consider I_1 . The gradient $\nabla \varphi_\varepsilon$ is supported in a ball of radius $C\varepsilon$, where C is an ε -independent constant. This implies

$$\begin{aligned} I_1 &= \frac{1}{2} \int_{B_{1/2} \times B_{1/2} + C\varepsilon \times B_{1/2} + C\varepsilon} |u(y) - u(z)|^2 \left(\int_{B_{1/2}} \nabla \varphi_\varepsilon(x-y) \cdot \nabla \varphi_\varepsilon(x-z) dx \right) dy dz \\ &\leq \int_{B_{1/2} \times B_{1/2} + C\varepsilon \times B_{1/2} + C\varepsilon} |u(y) - u(x)|^2 |\nabla \varphi_\varepsilon(x-y)| |\nabla \varphi_\varepsilon(x-z)| dx dy dz \\ &\quad + \int_{B_{1/2} \times B_{1/2} + C\varepsilon \times B_{1/2} + C\varepsilon} |u(x) - u(z)|^2 |\nabla \varphi_\varepsilon(x-y)| |\nabla \varphi_\varepsilon(x-z)| dx dy dz \\ &\leq 2 \|\nabla \varphi_\varepsilon\|_{L^1(\mathbb{R}^3)} \int_{B_{1/2} \times B_{1/2} + C\varepsilon} |u(y) - u(x)|^2 |\nabla \varphi_\varepsilon(y-x)| dx dy \end{aligned}$$

Thanks to (3.12), we obtain

$$I_1 \lesssim \varepsilon^{-2} \|g\|_{L^1(\mathbb{R}^3)} \int_{B_{1/2} \times B_{1/2} + C\varepsilon} |u(y) - u(x)|^2 g_\varepsilon(y-x) dx dy.$$

For ε sufficiently small we have $1/2 + C\varepsilon < 1$, and the lemma follows. $\square$

Given two sets $A \subseteq \mathbb{R}^3$, $A' \subseteq \mathbb{R}^3$, we write $A \subset\subset A'$ when the closure of A is contained in A' .

Lemma 3.6. *Let A, A' be open sets such that $A \subset\subset A' \subseteq \mathbb{R}^3$. Then, there exists $\varepsilon_3 = \varepsilon_3(A, A')$ such that, for any $u \in L^\infty(A', \mathbb{R}^m)$ and any $\varepsilon \in (0, \varepsilon_3]$, there holds*

$$\int_A |u - \varphi_\varepsilon * u|^2 \lesssim \int_{A' \times A'} K_\varepsilon(x-y) \cdot (u(x) - u(y))^{\otimes 2} dx dy.$$

Proof. Since $\int_{\mathbb{R}^3} \varphi_\varepsilon(z) \, dz = 1$, we have

$$I := \int_A |u(x) - (\varphi_\varepsilon * u)(x)|^2 \, dx = \int_A \left| \int_{\mathbb{R}^3} \varphi_\varepsilon(x-y) (u(x) - u(y)) \, dy \right|^2 \, dx.$$

We apply Jensen inequality with respect to the probability measure $\varphi_\varepsilon(x-y) \, dy$:

$$I \leq \int_A \left(\int_{\mathbb{R}^3} \varphi_\varepsilon(x-y) |u(x) - u(y)|^2 \, dy \right) \, dx.$$

Because the support of φ_ε is contained in a ball of radius $C\varepsilon$, where C is an ε -independent constant, the integrand is equal to zero if $x \in A$, $\text{dist}(y, A) > C\varepsilon$. By applying (3.12), we obtain

$$I \leq \int_{A \times \{y \in \mathbb{R}^3 : \text{dist}(y, A) \leq C\varepsilon\}} g_\varepsilon(x-y) |u(x) - u(y)|^2 \, dx \, dy$$

and, if $\varepsilon \leq C^{-1} \text{dist}(A, \partial A')$, the lemma follows. $\square$

Proof of Proposition 3.4. Due to the scaling property (3.11), it suffices to prove that

$$\left| \int_{B_{1/2}} u - \int_{B_{1/2}} u \right|^2 \lesssim F_{\varepsilon/\rho}(u, B_1) \quad (3.13)$$

for any $u \in L^\infty(\mathbb{R}^3, \mathbb{R}^m)$ and any ε, ρ with ε/ρ sufficiently small. The triangle inequality and the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ imply

$$\left| \int_{B_{1/2}} u - \int_{B_{1/2}} u \right|^2 \leq 6 \left| \int_{B_{1/2}} |u - \varphi_{\varepsilon/\rho} * u|^2 \right| + 3 \left| \int_{B_{1/2}} \varphi_{\varepsilon/\rho} * u - \int_{B_{1/2}} \varphi_{\varepsilon/\rho} * u \right|^2$$

Thanks to the Poincaré inequality, we obtain

$$\left| \int_{B_{1/2}} u - \int_{B_{1/2}} u \right|^2 \lesssim \int_{B_{1/2}} |u - \varphi_{\varepsilon/\rho} * u|^2 + \int_{B_{1/2}} |\nabla(\varphi_{\varepsilon/\rho} * u)|^2.$$

If ε/ρ is sufficiently small, Lemmas 3.5 and 3.6 give

$$\left| \int_{B_{1/2}} u - \int_{B_{1/2}} u \right|^2 \lesssim ((\varepsilon/\rho)^2 + 1) F_{\varepsilon/\rho}(u, B_1),$$

so (3.13) follows. $\square$

3.3. Localised Γ -convergence for the non-local term

The Γ -convergence of the functional E_ε , as $\varepsilon \rightarrow 0$, was studied in [48]. In this section, we adapt the arguments of [48] to prove a localised Γ -convergence result. We focus on the interaction part of the free energy only, since this is all we need in the proof of Theorem A. We denote by $F_\varepsilon^{\text{nl}}$ the non-local interaction part of F_ε , given by

$$\begin{aligned} F_\varepsilon^{\text{nl}}(u, G) &:= F_\varepsilon(u, G) - \frac{1}{\varepsilon^2} \int_G \psi_b(u(x)) \, dx \\ &= \frac{1}{4\varepsilon^2} \int_{G \times G} K_\varepsilon(x-y) \cdot (u(x) - u(y))^{\otimes 2} \, dx \, dy \end{aligned} \quad (3.14)$$

for any $u \in L^\infty(\mathbb{R}^3, \mathcal{Q})$ and any Borel set $G \subseteq \mathbb{R}^3$.

Proposition 3.7. Let $\rho > 0$, $x_0 \in \mathbb{R}^3$, and let $v_\varepsilon \in L^2(B_\rho(x_0), \mathbb{R}^m)$, $v_0 \in H^1(B_\rho(x_0), \mathbb{R}^m)$ be such that $v_\varepsilon \rightarrow v_0$ strongly in $L^2(B_\rho(x_0))$ as $\varepsilon \rightarrow 0$. Then, for any open set $G \subseteq B_\rho(x_0)$ we have

$$\int_G L \nabla v_0 \cdot \nabla v_0 \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon^{\text{nl}}(v_\varepsilon, G) \quad (3.15)$$

Proposition 3.8. Let $\rho > 0$. $x_0 \in \mathbb{R}^3$. Let $v_\varepsilon \in H^1(B_\rho(x_0), \mathbb{R}^m)$, $v_0 \in H^1(B_\rho(x_0), \mathbb{R}^m)$ be such that $v_\varepsilon \rightarrow v_0$ strongly in $H^1(B_\rho(x_0))$ as $\varepsilon \rightarrow 0$. Then

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon^{\text{nl}}(v_\varepsilon, B_\rho(x_0)) \leq \int_{B_\rho(x_0)} L \nabla v_0 \cdot \nabla v_0$$

In the proofs of [Propositions 3.7](#) and [3.8](#), we will use the following notation. Given a vector $w \in \mathbb{R}^3 \setminus \{0\}$ and a function u defined on a subset of $\mathbb{R}^3$, we define the difference quotient

$$D_w u(x) := \frac{u(x+w) - u(x)}{|w|}$$

for any x in the domain of u such that $x+w$ belongs to the domain of u . If $|w| \leq h$, $u \in H^1(B_{\rho+h})$, and $|\cdot|_*$ is any seminorm on $\mathbb{R}^m$, then

$$\int_{B_\rho} |D_{\varepsilon w} u(x)|_*^2 dx \leq \int_{B_{\rho+\varepsilon h}} |(\hat{w} \cdot \nabla) u(x)|_*^2 dx \quad (3.16)$$

where $\hat{w} := w/|w|$. This follows from the same technique as, e.g., [\[21, Lemma 7.23\]](#), for the case in which we have the standard Euclidean norm. However, we realise the proof only relies on the convexity of the seminorm, and no further structure. For convenience, we give the proof of [Proposition 3.8](#) first.

Proof of Proposition 3.8. We assume that $x_0 = 0$. Using a reflection across the boundary of B_ρ and a cut-off function, we define v_ε and v_0 on $\mathbb{R}^3 \setminus B_\rho$, in such a way that $v_\varepsilon \in H^1(\mathbb{R}^3, \mathbb{R}^m)$, $v_0 \in H^1(\mathbb{R}^3, \mathbb{R}^m)$ and $v_\varepsilon \rightarrow v_0$ strongly in $H^1(\mathbb{R}^3)$. Let $t > 0$ be a parameter. We have

$$\begin{aligned} & \frac{1}{4\varepsilon^2} \int_{B_\rho} \int_{B_\rho} K_\varepsilon(x-y) \cdot (v_\varepsilon(x) - v_\varepsilon(y))^{\otimes 2} dx dy \\ & \leq \frac{1}{4} \int_{B_\rho} \int_{\mathbb{R}^3} |z|^2 K(z) \cdot (D_{\varepsilon z} v_\varepsilon(x))^{\otimes 2} dz dx \\ & = \frac{1}{4} \int_{B_\rho} \int_{B_{\frac{t}{\varepsilon}}} |z|^2 K(z) \cdot (D_{\varepsilon z} v_\varepsilon(x))^{\otimes 2} dz dx + \frac{1}{4} \int_{B_\rho} \int_{\mathbb{R}^3 \setminus B_{\frac{t}{\varepsilon}}} |z|^2 K(z) \cdot (D_{\varepsilon z} v_\varepsilon(x))^{\otimes 2} dz dx. \end{aligned}$$

To estimate the first integral at the right-hand side, we exchange the order of integration and, for any z , we apply [\(3.16\)](#) to the seminorm $|\xi|_*^2 := |z|^2 K(z) \cdot \xi^{\otimes 2}$; for the second integral, we apply [\(K₅\)](#):

$$\begin{aligned} & \frac{1}{4\varepsilon^2} \int_{B_\rho} \int_{B_\rho} K_\varepsilon(x-y) \cdot (v_\varepsilon(x) - v_\varepsilon(y))^{\otimes 2} dx dy \\ & \leq \frac{1}{4} \int_{\mathbb{R}^3} \int_{B_{\rho+t}} K(z) \cdot ((z \cdot \nabla) v_\varepsilon(x))^{\otimes 2} dx dz + C \int_{B_\rho} \int_{\mathbb{R}^3 \setminus B_{\frac{t}{\varepsilon}}} g(z) |z|^2 |D_{\varepsilon z} v_\varepsilon(x)|^2 dz dx \\ & \stackrel{(2.8)}{=} \int_{B_{\rho+t}} L \nabla v_\varepsilon(x) \cdot \nabla v_\varepsilon(x) dx + C \int_{B_\rho} \int_{\mathbb{R}^3 \setminus B_{\frac{t}{\varepsilon}}} g(z) |z|^2 |D_{\varepsilon z} v_\varepsilon(x)|^2 dz dx. \end{aligned} \quad (3.17)$$

We now estimate the latter summand independently. For $z \in \mathbb{R}^3 \setminus B_{\frac{t}{\varepsilon}}$, $|\varepsilon z|^2 > t^2$, so

$$|D_{\varepsilon z} v_\varepsilon(x)|^2 \leq \frac{1}{t^2} |v_\varepsilon(x + \varepsilon z) - v_\varepsilon(x)|^2 \leq \frac{2}{t^2} (|v_\varepsilon(x + \varepsilon z)|^2 + |v_\varepsilon(x)|^2)$$

Therefore, by applying Fubini theorem, we may estimate

$$\int_{B_\rho} \int_{\mathbb{R}^3 \setminus B_{\frac{t}{\varepsilon}}} g(z) |z|^2 |D_{\varepsilon z} v_\varepsilon(x)|^2 dz dx \leq \frac{4 \|v_\varepsilon\|_{L^2(\mathbb{R}^3)}^2}{t^2} \int_{\mathbb{R}^3 \setminus B_{\frac{t}{\varepsilon}}} g(z) |z|^2 dz \quad (3.18)$$

As g has finite second moment and $\|v_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C$, for fixed t we must have that

$$\lim_{\varepsilon \rightarrow 0} \frac{2 \|v_\varepsilon\|_{L^2(\mathbb{R}^3)}^2}{t^2} \int_{\mathbb{R}^3 \setminus B_{\frac{t}{\varepsilon}}} g(z) |z|^2 dz = 0. \quad (3.19)$$

Combining (3.17), (3.18) and (3.19) gives

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon^2} \int_{B_\rho} \int_{B_\rho} K_\varepsilon(x-y) \cdot (v_\varepsilon(x) - v_\varepsilon(y))^{\otimes 2} dy dx \leq \limsup_{\varepsilon \rightarrow 0} \int_{B_{\rho+t}} L \nabla v_\varepsilon(x) \cdot \nabla v_\varepsilon(x) dx.$$

As $v_\varepsilon \rightarrow v_0$ in $H^1(\mathbb{R}^3)$, this implies

$$\lim_{\varepsilon \rightarrow 0} \int_{B_{\rho+t}} L \nabla v_\varepsilon(x) \cdot \nabla v_\varepsilon(x) dx = \int_{B_{\rho+t}} L \nabla v_0(x) \cdot \nabla v_0(x) dx.$$

Therefore we have

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon^2} \int_{B_\rho} \int_{B_\rho} K_\varepsilon(x-y) \cdot (v_\varepsilon(x) - v_\varepsilon(y))^{\otimes 2} dy dx \leq \int_{B_{\rho+t}} L \nabla v_0(x) \cdot \nabla v_0(x) dx,$$

and passing to the limit as $t \rightarrow 0$ on the right-hand side gives the desired result. $\square$

Proof of Proposition 3.7. Again, we assume that $x_0 = 0$. Without loss of generality, we may assume that

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon^2} \int_{G \times G} K_\varepsilon(x-y) \cdot (v_\varepsilon(x) - v_\varepsilon(y))^{\otimes 2} dx dy < +\infty, \quad (3.20)$$

otherwise there is nothing to prove. Up to extraction of a (non-relabelled) subsequence, we may also assume that the left-hand side of (3.20) is actually a limit. Let $G \subseteq B_\rho$ be open and $G' \subset\subset G$. Then we may write that

$$\begin{aligned} & \frac{1}{4\varepsilon^2} \int_G \int_G K_\varepsilon(x-y) \cdot (v_\varepsilon(x) - v_\varepsilon(y))^{\otimes 2} dy dx \\ & \geq \frac{1}{4\varepsilon^2} \int_{G'} \int_G K_\varepsilon(x-y) \cdot (v_\varepsilon(x) - v_\varepsilon(y))^{\otimes 2} dy dx \\ & = \frac{1}{4} \int_{G'} \int_{\frac{G-x}{\varepsilon}} |z|^2 K(z) \cdot (D_{\varepsilon z} v_\varepsilon(x))^{\otimes 2} dz dx. \end{aligned} \quad (3.21)$$

Let $G^c := \mathbb{R}^3 \setminus G$ and $\delta := \text{dist}(G', G^c) > 0$. We note that

$$\left| \int_{G'} \int_{\left(\frac{G-x}{\varepsilon}\right)^c} |z|^2 K(z) \cdot (D_{\varepsilon z} v_\varepsilon(x))^{\otimes 2} dy dx \right| \lesssim \int_{G'} \int_{B_{\frac{\delta}{\varepsilon}}} g(z) |z|^2 |D_{\varepsilon z} v_\varepsilon(x)|^2 dz dx,$$

which by previous estimates (see (3.18), (3.19)) we have seen converges to zero as $\varepsilon \rightarrow 0$. This means

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_{G'} \int_{\frac{G-x}{\varepsilon}} |z|^2 K(z) \cdot (D_{\varepsilon z} v_\varepsilon(x))^{\otimes 2} dz dx \\ & = \liminf_{\varepsilon \rightarrow 0} \int_{G'} \int_{\mathbb{R}^3} |z|^2 K(z) \cdot (D_{\varepsilon z} v_\varepsilon(x))^{\otimes 2} dz dx \end{aligned} \quad (3.22)$$

Furthermore, we note this can be written as an L^2 norm, by defining $w_\varepsilon : G' \times \mathbb{R}^3 \rightarrow \mathbb{R}^m$ by $w_\varepsilon(z, x) := |z|K^{\frac{1}{2}}(z)D_{\varepsilon z}v_\varepsilon(x)$. Thanks to (3.20), we immediately see that w_ε is L^2 -bounded, so must admit an L^2 -weakly converging subsequence $w_j := w_{\varepsilon_j}$ with $\varepsilon_j \rightarrow 0$ and w_j has weak- L^2 limit w_0 . Furthermore, we take

$$\liminf_{\varepsilon \rightarrow 0} \int_{G'} \int_{\mathbb{R}^3} |z|^2 K(z) \cdot (D_{\varepsilon z}v_\varepsilon(x))^{\otimes 2} dz dx = \liminf_{j \rightarrow \infty} \|w_j\|_{L^2(G' \times \mathbb{R}^3)}^2 \geq \|w_0\|_{L^2(G' \times \mathbb{R}^3)}^2. \quad (3.23)$$

It remains to identify the limit w_0 . We may do this by integrating against test functions. Let $\phi \in C_c^\infty(G' \times \mathbb{R}^3)$. There exists some $R_0 > 0$ such that, for any $(y, z) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $|z| > R_0$, $\phi(y, z) = 0$. Furthermore, there exists some $\delta > 0$ so that if $\text{dist}(y, (G')^c) < \delta$, then $\phi(y, z) = 0$. In particular, if $\varepsilon_j < \frac{\delta}{R_0}$ and $(x - \varepsilon_j z, z) \in \text{supp}(\phi)$, then $x \in G'$. Therefore

$$\begin{aligned} \langle w_j, \phi \rangle &= \int_{G'} \int_{\mathbb{R}^3} \phi(x, z) |z| K^{\frac{1}{2}}(z) D_{\varepsilon_j z} v_{\varepsilon_j}(x) dz dx \\ &= \frac{1}{\varepsilon_j} \int_{G'} \int_{\mathbb{R}^3} \left(\phi(x - \varepsilon_j z, z) - \phi(x, z) \right) K^{\frac{1}{2}}(z) v_{\varepsilon_j}(x) dz dx, \end{aligned}$$

and we may exploit the fact that

$$\frac{1}{\varepsilon_j} \left(\phi(x - \varepsilon_j z, z) - \phi(x, z) \right) \rightarrow (-z \cdot \nabla_x) \phi(x, z) \quad \text{uniformly on } G' \times \mathbb{R}^3 \text{ as } j \rightarrow +\infty,$$

with the assumed L^2 convergence of $v_{\varepsilon_j} \rightarrow v_0$, to give that

$$\begin{aligned} \lim_{j \rightarrow \infty} \langle w_j, \phi \rangle &= \lim_{j \rightarrow \infty} \frac{1}{\varepsilon_j} \int_{G'} \int_{\mathbb{R}^3} \left(\phi(x - \varepsilon_j z, z) - \phi(x, z) \right) K^{\frac{1}{2}}(z) v_{\varepsilon_j}(x) dz dx \\ &= \int_{G'} \int_{\mathbb{R}^3} (-z \cdot \nabla_x) \phi(x, z) K^{\frac{1}{2}}(z) v_0(x) dz dx \\ &= \int_{G'} \int_{\mathbb{R}^3} \phi(x, z) K^{\frac{1}{2}}(z) (z \cdot \nabla) v_0(x) dz dx = \langle w_0, \phi \rangle. \end{aligned}$$

Therefore $w_0(x, z) = K^{\frac{1}{2}}(z)(z \cdot \nabla)v_0(x)$, and by (3.21), (3.22), (3.23) we have

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon^2} \int_G \int_G K_\varepsilon(x - y) \cdot (v_\varepsilon(x) - v_\varepsilon(y))^{\otimes 2} dy dx \\ \geq \frac{1}{4} \liminf_{j \rightarrow \infty} \|w_j\|_{L^2(G' \times \mathbb{R}^3)}^2 \\ \geq \frac{1}{4} \|w_0\|_{L^2(G' \times \mathbb{R}^3)}^2 \\ = \frac{1}{4} \int_{G'} \int_{\mathbb{R}^3} K(z) \cdot \left((z \cdot \nabla) v_0(x) \right)^{\otimes 2} dz dx \\ \stackrel{(2.8)}{=} \int_{G'} L \nabla v_0(x) \cdot \nabla v_0(x) dx. \end{aligned}$$

As the set $G' \subset\subset G$ was arbitrary, by monotonicity the lower bound (3.15) holds. $\square$

3.4. Other auxiliary results

In this section, we collect some auxiliary results that will be useful in the proof of Theorem A. Our first result is a remark on the kernel K , which will be used repeatedly. We will use the notation $a_\varepsilon = o(b_\varepsilon)$ if there exists a positive sequence c_ε , depending on K and $\mathcal{Q}$ only, such that $|a_\varepsilon| \leq c_\varepsilon |b_\varepsilon|$ and $c_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Lemma 3.9. For any $\sigma > 0$, there holds

$$\sup_{x \in \mathbb{R}^3} \left(\frac{1}{\varepsilon^2} \int_{\{y \in \mathbb{R}^3: |x-y| \geq \sigma\}} g_\varepsilon(x-y) dy \right) = o\left(\frac{\varepsilon^{q-2}}{\sigma^2}\right)$$

where $q > 7/2$ is the number given by Assumption (K₄).

Proof. By the change of variable $y = x + \varepsilon z$, we obtain

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\{y \in \mathbb{R}^3: |x-y| \geq \sigma\}} g_\varepsilon(x-y) dy &= \frac{1}{\varepsilon^2} \int_{\{z \in \mathbb{R}^3: |z| \geq \sigma/\varepsilon\}} g(z) dz \\ &\leq \frac{\varepsilon^{q-2}}{\sigma^q} \int_{\{z \in \mathbb{R}^3: |z| \geq \sigma/\varepsilon\}} g(z) |z|^q dz \end{aligned}$$

By assumption (K₄), g has finite moment of order q , so the lemma follows. $\square$

Given a function $u \in L^\infty(\mathbb{R}^3, \mathcal{Q})$ and Borel sets $G \subseteq \mathbb{R}^3$, $G' \subseteq \mathbb{R}^3$, we define

$$\Gamma_\varepsilon(u, G, G') := \frac{1}{2\varepsilon^2} \int_{G \times G'} K_\varepsilon(x-y) \cdot (u(x) - u(y))^{\otimes 2} dx dy \quad (3.24)$$

In the terminology introduced by Alberti and Bellettini [1], Γ_ε is called the ‘locality defect’. Indeed, if G and G' are disjoint and $F_\varepsilon^{\text{nl}}$ is defined as in (3.14), then

$$F_\varepsilon^{\text{nl}}(u, G \cup G') = F_\varepsilon^{\text{nl}}(u, G) + F_\varepsilon^{\text{nl}}(u, G') + \Gamma_\varepsilon(u, G, G') \quad (3.25)$$

because K is assumed to be an even function (by (K₂)). Moreover, for any $G \subseteq \mathbb{R}^3$, $G' \subseteq \mathbb{R}^3$ we have $\Gamma_\varepsilon(u, G, G') = \Gamma_\varepsilon(u, G', G)$. Given a set $G \subseteq \mathbb{R}^3$ and a number $\sigma > 0$, we define

$$\partial_\sigma G := \{x \in \Omega: \text{dist}(x, \partial G) < \sigma\}. \quad (3.26)$$

Our next result is an estimate for the ‘locality defect’ Γ_ε .

Lemma 3.10. Let $u \in L^\infty(\mathbb{R}^3, \mathcal{Q})$, let $G \subseteq G' \subseteq \mathbb{R}^3$ be bounded Borel sets, and let $\sigma > 0$. Then,

$$\Gamma_\varepsilon(u, G, G' \setminus G) \lesssim F_\varepsilon^{\text{nl}}(u, \partial_\sigma G) + o\left(\frac{\varepsilon^{q-2}}{\sigma^q}\right) \inf_{\zeta \in \mathbb{R}^m} \|u - \zeta\|_{L^2(G')}^2$$

where $q > 7/2$ is given by (K₄).

Proof. If two points $x \in G$, $y \in G' \setminus G$ satisfy $|x - y| \leq \sigma$, then necessarily $x \in \partial_\sigma G$, $y \in \partial_\sigma G$. Then,

$$\begin{aligned} \Gamma_\varepsilon(u, G, G' \setminus G) &\leq 2F_\varepsilon^{\text{nl}}(u, \partial_\sigma G) \\ &\quad + \underbrace{\frac{1}{2\varepsilon^2} \int_{\{x \in G', y \in G': |x-y| \geq \sigma\}} K_\varepsilon(x-y) \cdot (u(x) - u(y))^{\otimes 2} dx dy}_{=: I} \end{aligned}$$

We need to estimate the term I . Let $\zeta \in \mathbb{R}^m$ be a constant. The inequality $|u(x) - u(y)|^2 \leq 2|u(x) - \zeta|^2 + 2|u(y) - \zeta|^2$, and the assumption that K is even (see (K₂)), imply

$$\begin{aligned} I &\lesssim \frac{1}{2\varepsilon^2} \int_{\{x \in G', y \in G': |x-y| \geq \sigma\}} g_\varepsilon(x-y) |u(x) - u(y)|^2 dx dy \\ &\lesssim \frac{2}{\varepsilon^2} \int_{\{x \in G', y \in G': |x-y| \geq \sigma\}} g_\varepsilon(x-y) |u(x) - \zeta|^2 dx dy \end{aligned} \quad (3.27)$$

Then, by [Lemma 3.9](#), we obtain

$$I \lesssim o\left(\frac{\varepsilon^{q-2}}{\sigma^q}\right) \inf_{\zeta \in \mathbb{R}^m} \|u - \zeta\|_{L^2(G')}^2$$

and the lemma follows. $\square$

Lemma 3.11. *Let $u \in L^\infty(\mathbb{R}^3, \mathcal{Q})$, $\xi \in L^\infty(\mathbb{R}^3, \mathcal{Q})$, let $G \subseteq \mathbb{R}^3$ be a bounded Borel set such that $u = \xi$ a.e. in $\mathbb{R}^3 \setminus G$, and let $\sigma > 0$. Then,*

$$\Gamma_\varepsilon(\xi, G, \mathbb{R}^3 \setminus G) - \Gamma_\varepsilon(u, G, \mathbb{R}^3 \setminus G) \lesssim F_\varepsilon^{\text{nl}}(\xi, \partial_\sigma G) + o\left(\frac{\varepsilon^{q-2}}{\sigma^q}\right) |G|^{1/2} \|\xi - u\|_{L^2(G)}$$

where $q > 7/2$ is given by [\(K₄\)](#).

Remark 3.1. The right-hand side of [Lemma 3.11](#) contains the L^2 -norm of $\xi - u$, not the L^2 -norm squared. This loss of a power will be responsible for additional technicalities later on in the proof (see [Lemma 4.3](#)).

Proof of Lemma 3.11. Let $H_\sigma := \{(x, y) \in G \times (\mathbb{R}^3 \setminus G) : |x - y| \geq \sigma\}$. We have

$$G \times (\mathbb{R}^3 \setminus G) \subseteq (\partial_\sigma G \times \partial_\sigma G) \cup H_\sigma$$

and hence,

$$\begin{aligned} & \Gamma_\varepsilon(\xi, G, \mathbb{R}^3 \setminus G) - \Gamma_\varepsilon(u, G, \mathbb{R}^3 \setminus G) \\ & \leq 2F_\varepsilon^{\text{nl}}(\xi, \partial_\sigma G) + \underbrace{\frac{1}{2\varepsilon^2} \int_{H_\sigma} K_\varepsilon(x - y) \cdot \left((\xi(x) - u(y))^{\otimes 2} - (u(x) - u(y))^{\otimes 2} \right) dx dy}_{=: I} \end{aligned}$$

Using the identity $K_\varepsilon(x - y) \cdot (a^{\otimes 2} - b^{\otimes 2}) = K_\varepsilon(x - y)(a - b) \cdot (a + b)$, we obtain

$$I = \frac{1}{2\varepsilon^2} \int_{H_\sigma} K_\varepsilon(x - y) (\xi(x) - u(x)) \cdot (\xi(x) + u(x) - 2u(y)) dx dy$$

Since u, ξ take their values in the bounded set $\mathcal{Q}$, we deduce

$$\begin{aligned} I & \lesssim \frac{1}{\varepsilon^2} \int_{H_\sigma} g_\varepsilon(x - y) |\xi(x) - u(x)| dx dy \\ & \lesssim \sup_{x \in G} \left(\frac{1}{\varepsilon^2} \int_{\{y \in \mathbb{R}^3 : |x - y| \geq \sigma\}} g_\varepsilon(x - y) dy \right) \|\xi - u\|_{L^1(G)} \end{aligned}$$

By applying [Lemma 3.9](#) and the Hölder inequality at the right-hand side, the result follows. $\square$

Lemma 3.12. *Let $G \subseteq \mathbb{R}^3$ be a Borel set. Let $u_1 \in L^\infty(\mathbb{R}^3, \mathcal{Q})$, $u_2 \in L^\infty(\mathbb{R}^3, \mathcal{Q})$. Then,*

$$F_\varepsilon^{\text{nl}}(u_2, G) \lesssim F_\varepsilon^{\text{nl}}(u_1, G) + \frac{1}{\varepsilon^2} \|u_2 - u_1\|_{L^2(G)}^2$$

Proof. By writing $u_2(x) - u_2(y) = (u_1(x) - u_1(y)) + (u_2(x) - u_1(x)) + (u_1(y) - u_2(y))$, and using that K is even (by assumption [\(K₂\)](#)), we obtain

$$F_\varepsilon^{\text{nl}}(u_2, G) \lesssim F_\varepsilon^{\text{nl}}(u_1, G) + \frac{2}{\varepsilon^2} \int_{G \times G} K_\varepsilon(x - y) \cdot (u_2(x) - u_1(x))^{\otimes 2} dx dy$$

and the lemma follows. $\square$

Lemma 3.13. *Let $G \subset\subset G' \subset\subset \mathbb{R}^3$ be open sets and $\sigma \in (0, \text{dist}(G, \partial G'))$. Then, for any $u \in H^1(G', \mathcal{Q})$, we have*

$$F_\varepsilon^{\text{nl}}(u, G) \lesssim \int_{G \cup \partial_\sigma G} |\nabla u|^2 + o\left(\frac{\varepsilon^{q-2}}{\sigma^q}\right) \inf_{\zeta \in \mathbb{R}^m} \|u - \zeta\|_{L^2(G)}^2$$

where $q > 7/2$ is the number given by (K₄).

Proof. This lemma is a variant of Proposition 3.8. We have

$$\begin{aligned} F_\varepsilon^{\text{nl}}(u, G) &= \frac{1}{4\varepsilon^2} \int_{\{x \in G, y \in G: |x-y| \leq \sigma\}} K_\varepsilon(x-y) \cdot (u(x) - u(y))^{\otimes 2} dx dy \\ &\quad + \frac{1}{4\varepsilon^2} \int_{\{x \in G, y \in G: |x-y| \geq \sigma\}} K_\varepsilon(x-y) \cdot (u(x) - u(y))^{\otimes 2} dx dy =: I_1 + I_2 \end{aligned}$$

For the first term I_1 , we may repeat the very same argument from the proof of Proposition 3.8, which gives

$$I_1 \leq \frac{1}{4\varepsilon^2} \int_{G \times B_{\sigma/\varepsilon}} K(z) \cdot (u(x + \varepsilon z) - u(x))^{\otimes 2} dx dz \lesssim \int_{G \cup \partial_\sigma G} |\nabla u|^2$$

On the other hand, the estimate (3.27) in the proof of Lemma 3.10 shows that

$$I_2 \lesssim o\left(\frac{\varepsilon^{q-2}}{\sigma^q}\right) \inf_{\zeta \in \mathbb{R}^m} \|u - \zeta\|_{L^2(G)}^2 \quad \square$$

The next lemma is an estimate on the potential ψ_b .

Lemma 3.14. *For any $\delta > 0$, there exists a constant $C_\delta > 0$ such that, for any $y_1 \in \mathcal{Q}$, $y_2 \in \mathcal{Q}$ with $\text{dist}(y_2, \partial \mathcal{Q}) \geq \delta$, we have*

$$\psi_b(y_2) \leq C_\delta \left(\psi_b(y_1) + |y_1 - y_2|^2 \right). \quad (3.28)$$

Proof. The assumption (H₆) implies, via a Taylor expansion and a compactness argument, that there exist $\gamma > 0$, $\kappa_1 > 0$, $\kappa_2 > 0$ so that if $\text{dist}(y, \mathcal{N}) < \gamma$, then

$$\kappa_1 \text{dist}^2(y, \mathcal{N}) \leq \psi_b(y) \leq \kappa_2 \text{dist}^2(y, \mathcal{N}). \quad (3.29)$$

To prove the result we exhaust three cases,

1. $\text{dist}(y_1, \mathcal{N}) \geq \frac{1}{2}\gamma$.
2. $\text{dist}(y_1, \mathcal{N}) < \frac{1}{2}\gamma$, $\text{dist}(y_2, \mathcal{N}) \geq \gamma$.
3. $\text{dist}(y_1, \mathcal{N}) < \frac{1}{2}\gamma$, $\text{dist}(y_2, \mathcal{N}) < \gamma$.

In the case of (1), we have that such y_1 satisfy $\psi_b(y_1) > c_1$ for a constant $c_1 > 0$ (that depends on γ), as y_1 is bounded away from the minimising manifold. We furthermore have that $\psi_b(y_2) \leq c_2$ because $\text{dist}(y_2, \partial \mathcal{Q}) > \delta$ (and the constant c_2 will depend on δ). Therefore the inequality (3.28) holds trivially with $C_\delta = \frac{c_1}{c_2}$.

In the case of (2), since $\text{dist}(y_1, \mathcal{N}) < \frac{1}{2}\gamma$, $\text{dist}(y_2, \mathcal{N}) \geq \gamma$, we must have $|y_1 - y_2|^2 \geq \frac{1}{4}\gamma^2$, then we use the upper bound on $\psi_b(y_2)$ as before.

In the case of (3), we note that since y_1, y_2 are both sufficiently close to $\mathcal{N}$,

$$\psi_b(y_2) \stackrel{(3.29)}{\lesssim} \text{dist}^2(y_2, \mathcal{N}) \lesssim \text{dist}^2(y_1, \mathcal{N}) + |y_1 - y_2|^2 \stackrel{(3.29)}{\lesssim} \psi_b(y_1) + |y_1 - y_2|^2. \quad \square$$

Finally, we conclude this section with interpolation (or extension) results. The first one is a classical interpolation result for H^1 -maps; it constructs a suitable map in an annulus, with prescribed values on the boundary.

Lemma 3.15 ([11,37]). *For any $M > 0$, there exists $\eta = \eta(M) > 0$ such that the following statement holds. Let $\mathcal{Q}_0 \subset \subset \mathcal{Q}$ be a convex, open set that contains $\mathcal{N}$. Let ρ, λ be positive numbers with $\lambda < \rho$, and let $u \in H^1(\partial B_\rho, \mathcal{Q}_0)$, $v \in H^1(\partial B_\rho, \mathcal{N})$ be such that*

$$\int_{\partial B_\rho} (|\nabla u|^2 + |\nabla v|^2) d\mathcal{H}^2 \leq M, \quad \int_{\partial B_\rho} |u - v|^2 d\mathcal{H}^2 \leq \eta \lambda^2.$$

Then, there exists a map $w \in H^1(B_\rho \setminus B_{\rho-\lambda}, \mathcal{Q}_0)$ such that $w(x) = u(x)$ for $\mathcal{H}^2$ -a.e. $x \in \partial B_\rho$, $w(x) = v(\rho x/(\rho - \lambda))$ for $\mathcal{H}^2$ -a.e. $x \in \partial B_{\rho-\lambda}$, and

$$\begin{aligned} \int_{B_\rho \setminus B_{\rho-\lambda}} |\nabla w|^2 &\lesssim \lambda \int_{\partial B_\rho} \left(|\nabla u|^2 + |\nabla v|^2 + \frac{|u - v|^2}{\lambda^2} \right) d\mathcal{H}^2 \\ \int_{B_\rho \setminus B_{\rho-\lambda}} \psi_b(w) &\lesssim \lambda \int_{\partial B_\rho} \psi_b(u) d\mathcal{H}^2 \end{aligned}$$

Remark 3.2. Lemma 3.15, in case $\psi_b = 0$, was first proven by Luckhaus [37, Lemma 1]. Up to a scaling, the statement given here is essentially the same as [11, Lemma B.2]. However, in [11] the potential is assumed to be finite and smooth on the whole of $\mathbb{R}^m$, while our potential ψ_b is singular out of $\mathcal{Q}$. Nevertheless, the proof carries over to our setting. Indeed, the map w constructed in [11] takes values in a neighbourhood of $\mathcal{N}$, whose thickness can be made arbitrarily small by choosing η small (see also [37, Lemma 1]). In particular, we can make sure that the image of w is contained in the set $\mathcal{Q}_0$, where the function ψ_b is finite and smooth, and the arguments of [11] carry over. Incidentally, Lemma 3.15 crucially depends on the non-degeneracy assumption (H₆) for the bulk potential ψ_b .

We give a variant of Lemma 3.15 which is adapted to our non-local setting.

Lemma 3.16. *Let $\mathcal{Q}_0 \subset \subset \mathcal{Q}$ be an open convex set. Let $u_\varepsilon, u \in L^\infty(\mathbb{R}^3, \mathcal{Q}_0)$ and $u_\varepsilon^*, u^* \in H^1(B_{1/2}, \mathcal{N})$ satisfy the following conditions:*

$$M_\varepsilon := \int_{B_{1/2}} |\nabla u_\varepsilon^*|^2 + F_\varepsilon(u_\varepsilon, B_1) + \|u_\varepsilon^* - u_\varepsilon\|_{L^2(B_{1/2})}^2 \quad \text{is bounded} \quad (3.30)$$

$$u_\varepsilon \rightarrow u \quad \text{strongly in } L^2(B_{1/2}), \quad u_\varepsilon^* \rightarrow u^* \quad \text{strongly in } H^1(B_{1/2}) \quad \text{as } \varepsilon \rightarrow 0 \quad (3.31)$$

$$u^* = u \quad \text{a.e. in } B_1 \setminus B_s \quad \text{for some } s \in (1/4, 1/2) \quad (3.32)$$

Let $\sigma \in (0, 1/10)$. Then, up to extraction of a (non-relabelled) subsequence, there exist maps $\xi_\varepsilon \in L^\infty(\mathbb{R}^3, \mathcal{Q}_0)$ and radii r, t with $s < r < t < 1/2$ that satisfy the following conditions:

- (i) $\xi_\varepsilon = u_\varepsilon$ a.e. in $\mathbb{R}^3 \setminus B_t$;
- (ii) $\xi_\varepsilon|_{B_r} \in H^1(B_r, \mathcal{Q}_0)$ and $\xi_\varepsilon|_{B_r} \rightarrow u|_{B_r}$ strongly in $H^1(B_r)$;
- (iii) there holds

$$\begin{aligned} F_\varepsilon(\xi_\varepsilon, B_t) + \Gamma_\varepsilon(\xi_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) - \Gamma_\varepsilon(u_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) \\ \leq F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_r) + C_s \int_{B_r \setminus B_{r-\sigma}} |\nabla \xi_\varepsilon|^2 + C_s \sigma M_\varepsilon + o\left(\frac{C_s \varepsilon^{q-2} M_\varepsilon^{1/2}}{\sigma^q}\right) \end{aligned}$$

where $q > 7/2$ is given by (K₄) and C_s is a positive constant that depends on s , but not on ε, σ .

As we will see in the proof, the maps ξ_ε agree with u_ε^* on B_r , up to rescaling and interpolation near the boundary of B_r . We will make use of the following observation:

Lemma 3.17. *Let $(\rho_1, \rho_2) \subseteq \mathbb{R}$ be an interval. For $j \in \mathbb{N}$, let $f_j: (\rho_1, \rho_2) \rightarrow \mathbb{R}$ be a non-negative, integrable function. Then, there exists a subset $E \subseteq (\rho_1, \rho_2)$ of positive measure and, for any $\rho_* \in E$, a (non-relabelled) subsequence $j \rightarrow +\infty$ such that*

$$f_j(\rho_*) \leq \frac{2}{\rho_2 - \rho_1} \int_{\rho_1}^{\rho_2} f_j(\rho) \, d\rho \quad (3.33)$$

for any $j \in \mathbb{N}$.

Proof. If f_j is equal to zero almost everywhere for any j large enough, then the lemma follows easily. Otherwise, up to extraction of a subsequence, we may assume without loss of generality that $\|f_j\|_{L^1(\rho_1, \rho_2)} \neq 0$ for any j . By Fatou's lemma, we have

$$\int_{\rho_1}^{\rho_2} \liminf_{j \rightarrow +\infty} \frac{f_j(\rho)}{\|f_j\|_{L^1(\rho_1, \rho_2)}} \, d\rho \leq \liminf_{j \rightarrow +\infty} \int_{\rho_1}^{\rho_2} \frac{f_j(\rho)}{\|f_j\|_{L^1(\rho_1, \rho_2)}} \, d\rho = 1$$

Therefore, there exists a measurable subset $E \subseteq (\rho_1, \rho_2)$, of non-zero length, such that

$$\liminf_{j \rightarrow +\infty} \frac{f_{j,h}(\rho_*)}{\|f_{j,h}\|_{L^1(\rho_1, \rho_2)}} \leq \frac{1}{\rho_2 - \rho_1}$$

for any $\rho_* \in E$. The lemma follows. $\square$

Proof of Lemma 3.16. Throughout the proof, we treat s as a fixed parameter and omit the dependence on s from the notation. In particular, we write $A \lesssim B$ as a short-hand for $A \leq C_s B$, where the constant C_s may depend on s (but does *not* depend on ε or σ). If $M_\varepsilon = 0$, then $u_\varepsilon = u_\varepsilon^*$ is constant in $B_{1/2}$ and $\xi_\varepsilon := u_\varepsilon$ satisfies all the desired properties. Therefore, up to extraction of a subsequence, we assume without loss of generality that $M_\varepsilon > 0$ for any ε . We split the proof into several steps.

Step 1 (Construction of ξ_ε). Let $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^3)$ be a sequence of mollifiers, defined as in (3.12). Lemma 3.5 implies

$$\int_{B_{1/2}} |\nabla(\varphi_\varepsilon * u_\varepsilon)|^2 \lesssim F(u_\varepsilon, B_1) \stackrel{(3.30)}{\leq} M_\varepsilon \quad (3.34)$$

for ε small enough. Let $N \geq 1$ be an integer number such that

$$\frac{1}{18\sigma} \leq N \leq \frac{1}{6\sigma} \quad (3.35)$$

Such a number exists, because of the assumption that $0 < \sigma < 1/10$. We divide the annulus $B_{1/2} \setminus \bar{B}_s$ into N concentric sub-annuli:

$$A_i := B_{s+i\frac{1/2-s}{N}} \setminus \bar{B}_{s+(i-1)\frac{1/2-s}{N}} \quad \text{for } i = 1, 2, \dots, N.$$

By (3.30) and (3.34), we have

$$\sum_{i=1}^N \left(F_\varepsilon(u_\varepsilon, A_i) + \int_{A_i} |\nabla(\varphi_\varepsilon * u_\varepsilon)|^2 + \int_{A_i} |\nabla u_\varepsilon^*|^2 + \int_{A_i} |u_\varepsilon^* - u_\varepsilon|^2 \right) \lesssim M_\varepsilon$$

As a consequence, for any ε we can choose an index $i(\varepsilon)$ such that

$$F_\varepsilon(u_\varepsilon, A_{i(\varepsilon)}) + \int_{A_{i(\varepsilon)}} |\nabla(\varphi_\varepsilon * u_\varepsilon)|^2 + \int_{A_{i(\varepsilon)}} |\nabla u_\varepsilon^*|^2 + \int_{A_{i(\varepsilon)}} |u_\varepsilon^* - u_\varepsilon|^2 \lesssim \frac{M_\varepsilon}{N} \lesssim \sigma M_\varepsilon \quad (3.36)$$

Passing to a subsequence, we may also assume that all the indices $i(\varepsilon)$ are the same, so from now on, we write i instead of $i(\varepsilon)$. We take positive numbers $a < b$ such that $A' := B_b \setminus \bar{B}_a \subset\subset A_i$ and $b-a > 5\sigma(1/2-s)$. (Such numbers exist, thanks to (3.35).) Then, Lemma 3.6 gives

$$\frac{1}{\varepsilon^2} \int_{A'} |\varphi_\varepsilon * u_\varepsilon - u_\varepsilon|^2 \lesssim F_\varepsilon(u_\varepsilon, A_i) \stackrel{(3.36)}{\lesssim} \sigma M_\varepsilon \quad (3.37)$$

for ε small enough. We have assumed that u_ε takes its values in the convex set $\mathcal{Q}_0 \subset\subset \mathcal{Q}$; it follows that the image of $\varphi_\varepsilon * u_\varepsilon$ is contained in $\overline{\mathcal{Q}_0} \subset\subset \mathcal{Q}$. Thus, we may apply Lemma 3.14 to estimate the integral of $\psi_b(\varphi_\varepsilon * u_\varepsilon)$:

$$\frac{1}{\varepsilon^2} \int_{A'} \psi_b(\varphi_\varepsilon * u_\varepsilon) \lesssim \frac{1}{\varepsilon^2} \int_{A'} \left(\psi_b(u_\varepsilon) + |\varphi_\varepsilon * u_\varepsilon - u_\varepsilon|^2 \right) \stackrel{(3.37)}{\lesssim} \sigma M_\varepsilon \quad (3.38)$$

By applying (3.36), (3.37), (3.38) and Lemma 3.17, we find a radius r with

$$a + \sigma \left(\frac{1}{2} - s \right) < r < b - 3\sigma \left(\frac{1}{2} - s \right)$$

and a (non-relabelled, countable) subsequence $\varepsilon \rightarrow 0$ such that

$$\begin{aligned} \int_{\partial B_r} \left(|\nabla u_\varepsilon^*|^2 + |\nabla(\varphi_\varepsilon * u_\varepsilon)|^2 + \frac{1}{\varepsilon^2} \psi_b(\varphi_\varepsilon * u_\varepsilon) \right. \\ \left. + \frac{1}{\varepsilon^2} |\varphi_\varepsilon * u_\varepsilon - u_\varepsilon|^2 + |u_\varepsilon - u_\varepsilon^*|^2 \right) d\mathcal{H}^2 \lesssim M_\varepsilon \end{aligned} \quad (3.39)$$

The inequality (3.39) implies

$$\int_{\partial B_r} |\varphi_\varepsilon * u_\varepsilon - u_\varepsilon^*|^2 d\mathcal{H}^2 \lesssim \varepsilon^2 M_\varepsilon + \int_{\partial B_r} |u_\varepsilon^* - u_\varepsilon|^2 d\mathcal{H}^2 \quad (3.40)$$

Due to (3.31) and (3.32), we have $u_\varepsilon^* - u_\varepsilon \rightarrow u^* - u = 0$ in $L^2(A')$. Therefore, for a suitable choice of r , we can make sure that the right-hand side of (3.40) tends to zero. (Moreover, we have $\|u_\varepsilon^* - u_\varepsilon\|_{L^2(\partial B_r)} \lesssim M_\varepsilon$, thanks to (3.39).) Let

$$\lambda_\varepsilon := \left(\varepsilon^2 M_\varepsilon + \int_{\partial B_r} |u_\varepsilon^* - u_\varepsilon|^2 d\mathcal{H}^2 \right)^{1/4} > 0 \quad (3.41)$$

We have $\lambda_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover, for this choice of λ_ε , the assumptions of Lemma 3.15 are satisfied for ε small enough. By applying Lemma 3.15, we construct a map $w_\varepsilon \in H^1(B_r \setminus B_{r-\lambda_\varepsilon}, \mathcal{Q}_0)$ such that $w_\varepsilon(x) = (\varphi * u_\varepsilon)(x)$ for $x \in \partial B_r$, $w_\varepsilon(x) = u_\varepsilon^*(rx/(r-\lambda_\varepsilon))$ for $x \in \partial B_{r-\lambda_\varepsilon}$, and

$$\int_{B_r \setminus B_{r-\lambda_\varepsilon}} \left(|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon^2} \psi_b(w_\varepsilon) \right) \lesssim \lambda_\varepsilon M_\varepsilon + \frac{1}{\lambda_\varepsilon} \left(\varepsilon^2 M_\varepsilon + \int_{\partial B_r} |u_\varepsilon^* - u_\varepsilon|^2 d\mathcal{H}^2 \right) \quad (3.42)$$

and

$$\frac{1}{\varepsilon^2} \int_{B_r \setminus B_{r-\lambda_\varepsilon}} \psi_b(w_\varepsilon) \lesssim \frac{\lambda_\varepsilon}{\varepsilon^2} \int_{\partial B_r} \psi_b(w_\varepsilon) d\mathcal{H}^2 \lesssim \lambda_\varepsilon M_\varepsilon \quad (3.43)$$

The right-hand sides of (3.42), (3.43) converge to zero as $\varepsilon \rightarrow 0$. Finally, we take t with

$$r + 2\sigma \left(\frac{1}{2} - s \right) < t < b - \sigma \left(\frac{1}{2} - s \right)$$

and we define

$$\xi_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } x \in \mathbb{R}^3 \setminus B_t \\ (\varphi_\varepsilon * u_\varepsilon)(x) & \text{if } B_t \setminus B_r \\ w_\varepsilon(x) & \text{if } x \in B_r \setminus B_{r-\lambda_\varepsilon} \\ u_\varepsilon^* \left(\frac{rx}{r-\lambda_\varepsilon} \right) & \text{if } x \in B_{r-\lambda_\varepsilon} \end{cases} \quad (3.44)$$

By construction, $\xi_\varepsilon = u_\varepsilon$ out of B_t . Moreover, a routine computation, based on (3.42), shows that $\xi_\varepsilon \rightarrow u_\varepsilon^*$ strongly in $H^1(B_r)$

Step 2 (Bounds on $\|\xi_\varepsilon - u_\varepsilon\|_{L^2(B_t)}$). By construction, we have

$$\|\xi_\varepsilon - u_\varepsilon\|_{L^2(B_t \setminus B_r)} \leq \|\varphi_\varepsilon * u_\varepsilon - u_\varepsilon\|_{L^2(A')} \stackrel{(3.37)}{\lesssim} \varepsilon \sigma^{1/2} M_\varepsilon^{1/2}$$

On the other hand,

$$\|\xi_\varepsilon - u_\varepsilon\|_{L^2(B_r)} \lesssim \|w_\varepsilon - \tau_\varepsilon u_\varepsilon^*\|_{L^2(B_r \setminus B_{r-\lambda_\varepsilon})} + \|\tau_\varepsilon u_\varepsilon^* - u_\varepsilon^*\|_{L^2(B_r)} + \|u_\varepsilon^* - u_\varepsilon\|_{L^2(B_r)}$$

where $\tau_\varepsilon u_\varepsilon^*(x) := u_\varepsilon^*(rx/(r - \lambda_\varepsilon))$. We recall that $\|u_\varepsilon^* - u_\varepsilon\|_{L^2(B_{1/2})} \leq M_\varepsilon^{1/2}$. The norm of $w_\varepsilon - \tau_\varepsilon u_\varepsilon^*$ can be estimated using the Poincaré inequality and (3.42), because we know that $w_\varepsilon = \tau_\varepsilon u_\varepsilon^*$ on $\partial B_{r-\lambda_\varepsilon}$:

$$\|w_\varepsilon - \tau_\varepsilon u_\varepsilon^*\|_{L^2(B_r \setminus B_{r-\lambda_\varepsilon})} \lesssim \lambda_\varepsilon \|\nabla(w_\varepsilon - \tau_\varepsilon u_\varepsilon^*)\|_{L^2(B_r \setminus B_{r-\lambda_\varepsilon})} \lesssim (\lambda_\varepsilon + \lambda_\varepsilon^{1/2}) M_\varepsilon^{1/2}$$

A classical argument (see e.g. [21, Lemma 7.23] for an analogous result) gives

$$\|\tau_\varepsilon u_\varepsilon^* - u_\varepsilon^*\|_{L^2(B_r)} \lesssim \lambda_\varepsilon \|\nabla u_\varepsilon^*\|_{L^2(B_r)} \lesssim \lambda_\varepsilon M_\varepsilon^{1/2}$$

Combining the inequalities above, and recalling that $\lambda_\varepsilon \rightarrow 0$, we obtain

$$\|\xi_\varepsilon - u_\varepsilon\|_{L^2(B_t)} \lesssim M_\varepsilon^{1/2} \quad (3.45)$$

As a consequence, we deduce

$$\begin{aligned} \inf_{\zeta \in \mathbb{R}^m} \|\xi_\varepsilon - \zeta\|_{L^2(B_t)} &\leq \|\xi_\varepsilon - u_\varepsilon\|_{L^2(B_t)} + \inf_{\zeta \in \mathbb{R}^m} \|u_\varepsilon - \zeta\|_{L^2(B_t)} \\ &\lesssim M_\varepsilon^{1/2} + \inf_{\zeta \in \mathbb{R}^m} \|u_\varepsilon - \zeta\|_{L^2(B_t)} \end{aligned}$$

and hence, by Proposition 3.4,

$$\inf_{\zeta \in \mathbb{R}^m} \|\xi_\varepsilon - \zeta\|_{L^2(B_t)} \lesssim M_\varepsilon^{1/2} + F_\varepsilon(u_\varepsilon, B_1)^{1/2} \lesssim M_\varepsilon^{1/2} \quad (3.46)$$

Step 3 (Bounds on $F_\varepsilon(\xi_\varepsilon, B_t)$). Since K_ε is an even function (see (K₂)), we have

$$\begin{aligned} F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_t) &\leq F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_r) + F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_t \setminus B_r) + \Gamma_\varepsilon(\xi_\varepsilon, B_r, B_t \setminus B_r) \\ &\leq F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_r) + F_\varepsilon^{\text{nl}}(\varphi_\varepsilon * u_\varepsilon, A') + \Gamma_\varepsilon(\xi_\varepsilon, B_r, B_t \setminus B_r) \end{aligned} \quad (3.47)$$

where $\Gamma_\varepsilon(\xi_\varepsilon, B_r, B_t \setminus B_r)$ is defined as in (3.24). Lemma 3.12 implies

$$F_\varepsilon^{\text{nl}}(\varphi_\varepsilon * u_\varepsilon, A') \lesssim F_\varepsilon^{\text{nl}}(u_\varepsilon, A') + \frac{1}{\varepsilon^2} \|\varphi_\varepsilon * u_\varepsilon - u_\varepsilon\|_{L^2(A')}^2 \stackrel{(3.36), (3.38)}{\lesssim} \sigma M_\varepsilon \quad (3.48)$$

Now, we estimate $\Gamma_\varepsilon(\xi_\varepsilon, B_r, B_t \setminus B_r)$. By construction, we have

$$a + \sigma \left(\frac{1}{2} - s \right) < r < t < b - \sigma \left(\frac{1}{2} - s \right)$$

and hence $\partial_{\sigma(1/2-s)} B_r \subseteq A'$. We apply Lemmas 3.10, 3.13 and (3.46):

$$\begin{aligned} \Gamma_\varepsilon(\xi_\varepsilon, B_r, B_t \setminus B_r) &\lesssim F_\varepsilon^{\text{nl}}(\xi_\varepsilon, \partial_{\sigma(1/2-s)/2} B_r) + o\left(\frac{\varepsilon^{q-2}}{\sigma^q}\right) \inf_{\zeta \in \mathbb{R}^m} \|\xi_\varepsilon - \zeta\|_{L^2(B_t)}^2 \\ &\lesssim \int_{\partial_\sigma B_r} |\nabla \xi_\varepsilon|^2 + o\left(\frac{\varepsilon^{q-2}}{\sigma^q}\right) \inf_{\zeta \in \mathbb{R}^m} \|\xi_\varepsilon - \zeta\|_{L^2(B_t)}^2 \\ &\lesssim \int_{\partial_\sigma B_r} |\nabla \xi_\varepsilon|^2 + o\left(\frac{\varepsilon^{q-2} M_\varepsilon}{\sigma^q}\right) \end{aligned} \quad (3.49)$$

The gradient term at the right-hand side can be further estimated by (3.37):

$$\int_{\partial_\sigma B_r} |\nabla \xi_\varepsilon|^2 \lesssim \int_{B_r \setminus B_{r-\sigma}} |\nabla \xi_\varepsilon|^2 + \int_{A'} |\nabla(\varphi_\varepsilon * u_\varepsilon)|^2 \lesssim \int_{B_r \setminus B_{r-\sigma}} |\nabla \xi_\varepsilon|^2 + \sigma M_\varepsilon \quad (3.50)$$

Combining (3.47), (3.48), (3.49) and (3.50), we obtain

$$F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_t) \leq F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_r) + C \int_{B_r \setminus B_{r-\sigma}} |\nabla \xi_\varepsilon|^2 + C \sigma M_\varepsilon + o\left(\frac{C \varepsilon^{q-2} M_\varepsilon}{\sigma^q}\right) \quad (3.51)$$

The constant C here is independent of ε and σ , though it may depend on s . We estimate the local term of the energy, i.e. the integral of $\psi_b(\xi_\varepsilon)$ in B_t . By construction, ξ_ε restricted to $B_{r-\lambda_\varepsilon}$ takes its values in $\mathcal{N}$. As a consequence,

$$\frac{1}{\varepsilon^2} \int_{B_t} \psi_b(\xi_\varepsilon) = \frac{1}{\varepsilon^2} \int_{B_t \setminus B_r} \psi_b(\varphi_\varepsilon * u_\varepsilon) + \frac{1}{\varepsilon^2} \int_{B_r \setminus B_{r-\lambda_\varepsilon}} \psi_b(w_\varepsilon) \stackrel{(3.38), (3.43)}{\lesssim} \sigma M_\varepsilon \quad (3.52)$$

Step 4 (Bounds on $\Gamma_\varepsilon(\xi_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t)$). By construction, $\xi_\varepsilon = u_\varepsilon$ out of B_t . Then, Lemma 3.11 implies

$$\begin{aligned} \Gamma_\varepsilon(\xi_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) - \Gamma_\varepsilon(u_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) \\ \lesssim F_\varepsilon^{\text{nl}}(\xi_\varepsilon, \partial_{\sigma(1/2-s)} B_t) + o\left(\frac{\varepsilon^{q-2}}{\sigma^q}\right) \|\xi_\varepsilon - u_\varepsilon\|_{L^2(B_t)} \end{aligned}$$

Reasoning as in (3.48), and applying (3.36), we deduce

$$F_\varepsilon^{\text{nl}}(\xi_\varepsilon, \partial_{\sigma(1/2-s)} B_t) \lesssim \sigma M_\varepsilon$$

Then, due to (3.45), we conclude that

$$\Gamma_\varepsilon(\xi_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) - \Gamma_\varepsilon(u_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) \lesssim \sigma M_\varepsilon + o\left(\frac{\varepsilon^{q-2} M_\varepsilon^{1/2}}{\sigma^q}\right) \quad (3.53)$$

Combining (3.51), (3.52) and (3.53), we obtain the estimate (iii) in the statement of the lemma, and the proof is complete. $\square$

Remark 3.3. In addition to (3.30), (3.31) and (3.32), suppose there exist points $\zeta_\varepsilon \in \mathcal{N}$, a positive sequence $\eta_\varepsilon \rightarrow 0$ and maps $v \in H^1(B_{1/2}, \mathbb{R}^m)$, $v^* \in H^1(B_{1/2}, \mathbb{R}^m)$ such that $v = v^*$ out of B_s , $M_\varepsilon \lesssim \eta_\varepsilon^2$ and

$$\frac{u_\varepsilon - \zeta_\varepsilon}{\eta_\varepsilon} \rightarrow v \text{ strongly in } L^2(B_{1/2}), \quad \frac{u_\varepsilon^* - \zeta_\varepsilon}{\eta_\varepsilon} \rightarrow v^* \text{ strongly in } H^1(B_{1/2})$$

as $\varepsilon \rightarrow 0$. Then, the same sequence ξ_ε constructed above satisfies

$$\frac{\xi_\varepsilon - \zeta_\varepsilon}{\eta_\varepsilon} \rightarrow v^* \quad \text{strongly in } H^1(B_r),$$

so long as we choose λ_ε in a suitable way (see in particular Eqs. (3.42) and (3.44)). For instance, instead of (3.41), we may take

$$\lambda_\varepsilon := \left(\varepsilon^2 \frac{M_\varepsilon}{\eta_\varepsilon^2} + \frac{1}{\eta_\varepsilon^2} \int_{\partial B_r} |u_\varepsilon^* - u_\varepsilon|^2 \, d\mathcal{H}^2 \right)^{1/4}$$

4. Proof of the main results

4.1. A compactness result for ω -minimisers

The goal of this section is to prove a compactness result for minimisers of E_ε , subject to variable “boundary conditions”, as $\varepsilon \rightarrow 0$. For later convenience, we state our result in terms of “almost minimisers” — or, more precisely, ω -minimisers, as defined below. This will be useful to study variants of our original minimisation problem, as we will do in Section 5.

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. Let $\omega: [0, +\infty) \rightarrow [0, +\infty)$ be an increasing function such that $\omega(s) \rightarrow 0$ as $s \rightarrow 0$. We say that a function $u \in L^\infty(\mathbb{R}^3, \mathcal{Q})$ is an ω -minimiser of E_ε in Ω if, for any ball $B_\rho(x_0) \subseteq \Omega$ and any $v \in L^\infty(\mathbb{R}^3, \mathcal{Q})$ such that $v = u$ a.e. on $\mathbb{R}^3 \setminus B_\rho(x_0)$, there holds

$$E_\varepsilon(u) \leq E_\varepsilon(v) + \omega(\varepsilon) \rho.$$

By definition, a minimiser for E_ε in the class $\mathcal{A}$ defined by (2.5) is also a ω -minimiser in Ω , for any $\omega \geq 0$. ω -minimisers behave nicely with respect to scaling. Given $u \in L^\infty(\mathbb{R}^3, \mathcal{Q})$, an increasing function $\omega: [0, +\infty) \rightarrow [0, +\infty)$, $x_0 \in \mathbb{R}^3$ and $\rho > 0$, we define $u_\rho: \mathbb{R}^3 \rightarrow \mathcal{Q}$ and $\omega_\rho: [0, +\infty) \rightarrow [0, +\infty)$ as $u_\rho(y) := u(x_0 + \rho y)$ for $y \in \mathbb{R}^3$ and $\omega_\rho(s) := \omega(\rho s)$ for $s \geq 0$, respectively. A scaling argument implies

Lemma 4.1. If u is an ω -minimiser for E_ε in a bounded domain $\Omega \subseteq \mathbb{R}^3$, then u_ρ is an ω_ρ -minimiser for $E_{\varepsilon/\rho}$ in $(\Omega - x_0)/\rho$.

Proposition 4.2. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. Let $\omega: [0, +\infty) \rightarrow [0, +\infty)$ be an increasing function such that $\omega(s) \rightarrow 0$ as $s \rightarrow 0$. Let u_ε be a sequence of ω -minimisers of E_ε in Ω . Suppose that there exists an open, convex set $\mathcal{Q}_0 \subset \subset \mathcal{Q}$ such that $u_\varepsilon(x) \in \mathcal{Q}_0$ for any $\varepsilon > 0$ and a.e. $x \in \Omega$. Let $B_\rho(x_0) \subseteq \Omega$ be a ball such that $\sup_{\varepsilon > 0} F_\varepsilon(u_\varepsilon, B_\rho(x_0)) < +\infty$. Then, up to extraction of a non-relabelled subsequence, u_ε converge $L^2(B_{\rho/2}(x_0))$ -strongly to a map $u_0 \in H^1(B_{\rho/2}(x_0), \mathcal{N})$, which minimises the functional

$$w \in H^1(B_{\rho/2}(x_0), \mathcal{N}) \mapsto \int_{B_{\rho/2}(x_0)} L \nabla w \cdot \nabla w$$

subject to its own boundary conditions. Moreover, for any $s \in (0, \rho/2)$ there holds

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, B_s(x_0)) = \int_{B_s(x_0)} L \nabla u_0 \cdot \nabla u_0 \quad (4.1)$$

Proposition 4.2 differs from the results in [48] in that no “boundary condition” is prescribed: each u_ε minimises the functional E_ε (possibly up to a small error, which is quantified by the function ω) subject to its own “boundary condition”.

Proof of Proposition 4.2. If u is an ω -minimiser of E_ε in Ω and $B_\rho(x_0) \subseteq \Omega$, then u_ρ is an ω_ρ -minimiser for $E_{\varepsilon/\rho}$ in $(\Omega - x_0)/\rho$. Since we have assumed that Ω is bounded, the radius ρ is bounded too — say, $\rho \leq R_0$, where R_0 depends only on Ω . The function ω is increasing, so $\omega_\rho(s) = \omega(\rho s) \leq \omega(R_0 s) =: \omega_0(s)$. As a consequence, u_ρ is also an ω_0 -minimiser. Since ω_0 is independent of ρ , by a scaling argument (see Eq. (3.11)) we may assume without loss of generality that $\rho = 1$ and $x_0 = 0$.

Let $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^3)$ be defined as in Section 3.2. Lemmas 3.5 and 3.6 imply that

$$\int_{B_{1/2}} |\nabla(\varphi_\varepsilon * u_\varepsilon)|^2 \lesssim F(u_\varepsilon, B_1), \quad \int_{B_{1/2}} |\varphi_\varepsilon * u_\varepsilon - u_\varepsilon|^2 \lesssim \varepsilon^2 F(u_\varepsilon, B_1)$$

for ε small enough. Since $F(u_\varepsilon, B_1)$ is bounded, we can extract a (non-relabelled) subsequence so that $\varphi_\varepsilon * u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(B_{1/2})$, $u_\varepsilon \rightarrow u_0$ strongly in $L^2(B_{1/2})$. The map u_0 takes its values in $\mathcal{N}$, because

$$\int_{B_{1/2}} \psi_b(u_0) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \int_{B_{1/2}} \psi_b(u_\varepsilon) = 0$$

by Fatou lemma. We must show that

$$\int_{B_{1/2}} L \nabla u_0 \cdot \nabla u_0 \leq \int_{B_{1/2}} L \nabla v \cdot \nabla v \quad (4.2)$$

for any $v \in H^1(B_{1/2}, \mathcal{N})$ such that $v = u_0$ on $\partial B_{1/2}$. By an approximation argument, it suffices to prove (4.2) in case $v = u_0$ in a neighbourhood of $\partial B_{1/2}$. Therefore, we fix $s \in (0, 1/2)$ and we take a map $v \in H^1(B_{1/2}, \mathcal{N})$ such that $v = u_0$ on $B_{1/2} \setminus \bar{B}_s$. The map v is not an admissible competitor for u_ε , because in general $u_0 \neq u_\varepsilon$ on $\mathbb{R}^3 \setminus B_1$. However, we may construct suitable competitors by applying Lemma 3.16. We can indeed do so, because we have assumed that all the u_ε 's take their values in an open, convex set $\mathcal{Q}_0 \subset \subset \mathcal{Q}$.

Let $\sigma \in (0, 1/10)$. By applying Lemma 3.16 (with $u_\varepsilon^* = v$ for any ε), we find maps $\xi_\varepsilon \in L^\infty(\mathbb{R}^3, \mathcal{Q}_0)$ and radii r, t with $\max(s, 1/4) < r < t < 1/2$, so that $\xi_\varepsilon = u_\varepsilon$ a.e. in $\mathbb{R}^3 \setminus B_t$, $\xi_\varepsilon \rightarrow v$ strongly in $H^1(B_r)$ and

$$\begin{aligned} F_\varepsilon(\xi_\varepsilon, B_t) + \Gamma_\varepsilon(\xi_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) - \Gamma_\varepsilon(u_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) \\ \leq F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_r) + C_s \int_{B_r \setminus B_{r-\sigma}} |\nabla \xi_\varepsilon|^2 + C_s \sigma + o\left(\frac{C_s \varepsilon^{q-2}}{\sigma^q}\right) \end{aligned}$$

Here, $q > 7/2$ is given by (K₄) and C_s is a constant that depends on s , but not on ε, σ . Since u_ε is an ω -minimiser for E_ε and $\xi_\varepsilon = u_\varepsilon$ out of B_t , we have

$$F_\varepsilon(u_\varepsilon, B_t) + \Gamma_\varepsilon(u_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) \leq F_\varepsilon(\xi_\varepsilon, B_t) + \Gamma_\varepsilon(\xi_\varepsilon, B_t, \mathbb{R}^3 \setminus B_t) + \omega(\varepsilon)$$

and hence,

$$F_\varepsilon(u_\varepsilon, B_t) \leq F_\varepsilon^{\text{nl}}(\xi_\varepsilon, B_r) + C_s \int_{B_r \setminus B_{r-\sigma}} |\nabla \xi_\varepsilon|^2 + C_s \sigma + o\left(\frac{C_s \varepsilon^{q-2}}{\sigma^q}\right) + \omega(\varepsilon) \quad (4.3)$$

We apply Propositions 3.7 and 3.8 to pass to the limit in both sides of (4.3), first as $\varepsilon \rightarrow 0$, then as $\sigma \rightarrow 0$. We obtain

$$\int_{B_r} L \nabla u_0 \cdot \nabla u_0 \leq \int_{B_r} L \nabla v \cdot \nabla v$$

which implies (4.2). In case $v = u_0$, the same argument shows that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, B_r) \leq \int_{B_r} L \nabla u_0 \cdot \nabla u_0 \quad (4.4)$$

On the other hand, Proposition 3.7 implies

$$\int_G L \nabla u_0 \cdot \nabla u_0 \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, G) \quad \text{for any open set } G \subseteq B_{1/2} \quad (4.5)$$

Combining (4.4) with (4.5), we deduce (4.1). $\square$

4.2. A decay lemma for F_ε

The aim of this section is to prove a decay property for the localised energy F_ε :

Lemma 4.3. *There exist $\eta > 0$, $\theta \in (0, 1/2)$ and $\varepsilon_* > 0$ such that, for any ball $B_\rho(x_0) \subseteq \Omega$, any $\varepsilon \in (0, \varepsilon_*\rho)$ and any minimiser u_ε of E_ε in $\mathcal{A}$ such that*

$$F_\varepsilon(u_\varepsilon, B_\rho(x_0)) \leq \eta^2 \rho,$$

there holds

$$F_\varepsilon(u_\varepsilon, B_{\theta\rho}(x_0)) \leq \frac{\theta}{2} F_\varepsilon(u_\varepsilon, B_\rho(x_0)) + \left(\frac{\varepsilon}{\rho}\right)^{2q-4} \rho. \quad (4.6)$$

Compared with analogous results in the regularity theory for Oseen–Frank minimisers — see, for instance, Proposition 1 in [37] or Theorem 2.4 in [24] — the estimate (4.6) contains an extra term at the right-hand side. This additional term controls the contributions from the ‘locality defect’ Γ_ε , defined by (3.24) (cf. Lemma 3.11 and Remark 3.1). However, this term will introduce additional issues in the proof of Theorem A, which we are only able to resolve in case $q > 7/2$.

As a first step towards the proof of Lemma 4.3, we check that the limit tensor L (defined by (2.8)) is elliptic.

Proposition 4.4. *There exists a constant $\lambda > 1$ so that*

$$\lambda^{-1}|\xi|^2 \leq L\xi \cdot \xi \leq \lambda|\xi|^2$$

for all $\xi \in \mathbb{R}^{m \times 3}$.

Proof. The upper bound comes trivially, as

$$4L\xi \cdot \xi = \int_{\mathbb{R}^3} K(z)(\xi z) \cdot (\xi z) \, dz \lesssim \int_{\mathbb{R}^3} g(z)|\xi z|^2 \, dz \lesssim \left(\int_{\mathbb{R}^3} g(z)|z|^2 \, dz \right) |\xi|^2$$

and the constant at the right-hand side is finite, due to (K₄). For the lower bound, recall that g is non-negative and satisfies $g(z) \geq k$ for $\rho_1 < |z| < \rho_2$. Then we have that

$$\begin{aligned} 4L\xi \cdot \xi &= \int_{\mathbb{R}^3} K_{ij}(z) z_\alpha z_\beta \xi_{i,\alpha} \xi_{j,\beta} \, dz \geq \int_{\mathbb{R}^3} g(z) z_\alpha z_\beta \xi_{i,\alpha} \xi_{i,\beta} \, dz \\ &\geq k \int_{B_{\rho_2} \setminus B_{\rho_1}} z_\alpha z_\beta \, dz \xi_{i,\alpha} \xi_{i,\beta} \end{aligned}$$

We may evaluate the inner integral as

$$\begin{aligned} \int_{B_{\rho_2} \setminus B_{\rho_1}} z_\alpha z_\beta \, dz &= \int_{\rho_1}^{\rho_2} \int_{\mathbb{S}^2} r^2 p_\alpha p_\beta \, dp \, dr \\ &= \int_{\rho_1}^{\rho_2} r^2 \, dr \int_{\mathbb{S}^2} p_\alpha p_\beta \, dp \\ &= \left(\frac{\rho_2^3 - \rho_1^3}{3} \right) \frac{4\pi}{3} \delta_{\alpha\beta} \end{aligned}$$

This gives a lower bound on the bilinear form as

$$L\xi \cdot \xi \geq \frac{k\pi(\rho_2^3 - \rho_1^3)}{9} \delta_{\alpha\beta} \xi_{i\alpha} \xi_{i\beta} = \frac{k\pi(\rho_2^3 - \rho_1^3)}{9} |\xi|^2 \quad \square$$

We will prove [Lemma 4.3](#) by blow-up, adapting Luckhaus' arguments from [\[37\]](#). By a scaling argument (see [\(3.11\)](#)), we may assume without loss of generality that $x_0 = 0$ and $\rho = 1$. Suppose, towards a contradiction, that [Lemma 4.3](#) does not hold. Then, for any choice of the parameter $\theta \in (0, 1/2)$, we find a sequence $\varepsilon_j \rightarrow 0$ and minimisers u_j such that

$$\eta_j^2 := F_{\varepsilon_j}(u_j, B_1) \rightarrow 0 \quad \text{as } j \rightarrow +\infty, \quad (4.7)$$

$$F_{\varepsilon_j}(u_j, B_\theta) > \frac{\theta \eta_j^2}{2} + \varepsilon_j^{2q-4} \quad (4.8)$$

We denote by $T_\zeta \mathcal{N}$ the tangent space to the manifold $\mathcal{N}$ at a point $\zeta \in \mathcal{N}$ (regarded as a linear subspace of $\mathbb{R}^m$, i.e. $0 \in T_\zeta \mathcal{N}$). We recall that, for any point $y \in \mathbb{R}^m$ sufficiently close to the manifold $\mathcal{N}$, there exists a unique point $\pi(y) \in \mathcal{N}$ such that $\text{dist}(y, \mathcal{N}) = |y - \pi(y)|$. Moreover, there exists a constant $\delta_*(\mathcal{N}) > 0$ such that the map $y \mapsto \pi(y)$ is smooth (with bounded derivatives) in

$$\mathcal{U} := \{z \in \mathbb{R}^m : \text{dist}(z, \mathcal{N}) \leq \delta_*(\mathcal{N})\} \quad (4.9)$$

(see e.g. [\[46, Section 2.12.3\]](#)).

Lemma 4.5. *For any $j \in \mathbb{N}$ large enough, there exist a constant $\zeta_j \in \mathcal{N}$ and a measurable set $G_j \subseteq B_{1/2}$ such that*

$$\int_{B_{1/2}} |u_j(x) - \zeta_j|^2 \, dx \lesssim \eta_j^2 \quad (4.10)$$

$$\eta_j^{-1} \int_{B_{1/2} \setminus G_j} \text{dist}(u_j(x) - \zeta_j, T_{\zeta_j} \mathcal{N}) \, dx \rightarrow 0 \quad (4.11)$$

and $|G_j| \rightarrow 0$ as $j \rightarrow +\infty$.

Proof. Let $\tilde{\zeta}_j := \int_{B_{1/2}} u_j$. By the Poincaré-type inequality, [Proposition 3.4](#), we have

$$\int_{B_{1/2}} |u_j - \tilde{\zeta}_j|^2 \lesssim F_{\varepsilon_j}(u_j, B_1) \stackrel{(4.7)}{=} \eta_j^2. \quad (4.12)$$

On the other hand, [Proposition 3.3](#) and [Lemma 3.14](#) imply

$$\psi_b(\tilde{\zeta}_j) \lesssim \int_{B_{1/2}} \psi_b(u_j) + \int_{B_{1/2}} |u_j - \tilde{\zeta}_j|^2 \stackrel{(4.12)}{\lesssim} (\varepsilon_j^2 + 1) \eta_j^2 \rightarrow 0. \quad (4.13)$$

In particular, the distance between $\tilde{\zeta}_j$ and $\mathcal{N}$ tends to zero as $j \rightarrow +\infty$. Then, for j large enough, the projection $\zeta_j := \pi(\tilde{\zeta}_j) \in \mathcal{N}$ is well-defined. Moreover, recalling [\(3.29\)](#), we have

$$|\zeta_j - \tilde{\zeta}_j|^2 = \text{dist}^2(\tilde{\zeta}_j, \mathcal{N}) \lesssim \psi_b(\tilde{\zeta}_j) \stackrel{(4.13)}{\lesssim} \eta_j^2. \quad (4.14)$$

The estimate [\(4.10\)](#) follows from [\(4.12\)](#) and [\(4.14\)](#).

Let us consider the set

$$G_j := \{x \in B_{1/2} : \text{dist}(u_j(x), \mathcal{N}) \geq \delta_*(\mathcal{N})\}$$

where $\delta_*(\mathcal{N})$ is the constant from [\(4.9\)](#). On the set G_j , the function $\psi_b(u_j)$ is bounded from below by a strictly positive constant, which depends only on $\delta_*(\mathcal{N})$ and ψ_b . Then,

$$|G_j| \lesssim \int_{B_1} \psi_b(u_j) \lesssim \varepsilon_j^2 F_{\varepsilon_j}(u_j, B_1) = \varepsilon_j^2 \eta_j^2 \rightarrow 0.$$

Moreover, the projection $\pi \circ u_j$ is well-defined on $B_{1/2} \setminus G_j$ and

$$\begin{aligned} \int_{B_{1/2} \setminus G_j} \text{dist}(u_j - \zeta_j, T_{\zeta_j} \mathcal{N}) &\leq \int_{B_{1/2} \setminus G_j} \text{dist}(\pi \circ u_j - \zeta_j, T_{\zeta_j} \mathcal{N}) + \int_{B_{1/2} \setminus G_j} |u_j - \pi \circ u_j| \\ &\stackrel{(3.29)}{\lesssim} \int_{B_{1/2} \setminus G_j} \text{dist}(\pi \circ u_j - \zeta_j, T_{\zeta_j} \mathcal{N}) + \int_{B_1} \psi_b^{1/2}(u_j) \\ &\lesssim \int_{B_{1/2} \setminus G_j} \text{dist}(\pi \circ u_j - \zeta_j, T_{\zeta_j} \mathcal{N}) + \varepsilon_j \eta_j \end{aligned}$$

Therefore, in order to prove (4.11), it suffices to show that

$$\eta_j^{-1} \int_{B_{1/2} \setminus G_j} \text{dist}(\pi \circ u_j - \zeta_j, T_{\zeta_j} \mathcal{N}) \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \quad (4.15)$$

To this end, we fix a large number $M > 0$ and we consider the sets

$$A_j := \{x \in B_{1/2} \setminus G_j : |(\pi \circ u_j)(x) - \zeta_j| \leq M\eta_j\}, \quad B_j := B_{1/2} \setminus (G_j \cup A_j).$$

We first estimate the contribution from the set A_j . Since the manifold $\mathcal{N}$ is compact and smooth, we have

$$\text{dist}(y - z, T_z \mathcal{N}) \lesssim |y - z|^2 \quad \text{for any } y \in \mathcal{N}, z \in \mathcal{N} \quad (4.16)$$

(For y sufficiently close to z , say $|y - z| \leq \eta_0 = \eta_0(\mathcal{N})$, the inequality (4.16) can be obtained by writing $\mathcal{N}$ as the graph of a smooth function, locally around z , and using a Taylor expansion. If $|y - z| \geq \eta_0$, we remark that the left-hand side of (4.16) is bounded from above because $\mathcal{N}$ is compact.) Then,

$$\eta_j^{-1} \int_{A_j} \text{dist}(\pi \circ u_j - \zeta_j, T_{\zeta_j} \mathcal{N}) \lesssim \eta_j^{-1} \int_{A_j} |\pi \circ u_j - \zeta_j|^2 \leq M^2 \eta_j |A_j| \rightarrow 0 \quad (4.17)$$

as $j \rightarrow +\infty$. Now, we estimate the contribution from B_j . By construction, the image $u_j(B_{1/2} \setminus G_j)$ is contained in the neighbourhood $\mathcal{U}$ given by (4.9). Moreover, π is Lipschitz-continuous on $\mathcal{U}$ and $\pi(\zeta_j) = \zeta_j$, so

$$|(\pi \circ u_j)(x) - \zeta_j| \lesssim |u_j(x) - \zeta_j| \quad \text{for a.e. } x \in B_{1/2} \setminus G_j \quad (4.18)$$

By definition, we have $|\pi \circ u_j - \zeta_j| \geq M\eta_j$ on B_j and hence,

$$|B_j| \lesssim M^{-2} \eta_j^{-2} \int_{B_{1/2} \setminus G_j} |\pi \circ u_j - \zeta_j|^2 \stackrel{(4.18)}{\lesssim} M^{-2} \eta_j^{-2} \int_{B_{1/2} \setminus G_j} |u_j - \zeta_j|^2 \stackrel{(4.10)}{\lesssim} M^{-2} \quad (4.19)$$

By applying the inequality $\text{dist}(y - z, T_z \mathcal{N}) \leq |y - z|$, which holds for any $y \in \mathcal{N}, z \in \mathcal{N}$, we obtain

$$\eta_j^{-1} \int_{B_j} \text{dist}(\pi \circ u_j - \zeta_j, T_{\zeta_j} \mathcal{N}) \leq \eta_j^{-1} \int_{B_j} |\pi \circ u_j - \zeta_j| \stackrel{(4.18)}{\lesssim} \eta_j^{-1} \int_{B_j} |u_j - \zeta_j|$$

and hence,

$$\eta_j^{-1} \int_{B_j} \text{dist}(\pi \circ u_j - \zeta_j, T_{\zeta_j} \mathcal{N}) \lesssim \eta_j^{-1} |B_j|^{1/2} \left(\int_{B_{1/2}} |u_j - \zeta_j|^2 \right)^{1/2} \stackrel{(4.10), (4.19)}{\lesssim} M^{-1} \quad (4.20)$$

By combining (4.17) with (4.20), and passing to the limit as $M \rightarrow +\infty$, we deduce (4.15). $\square$

Let $\zeta_j \in \mathcal{N}$, $G_j \subseteq B_{1/2}$ be as in Lemma 4.5. We consider the maps $v_j: B_{1/2} \rightarrow \mathbb{R}^3$ given by

$$v_j := \frac{1}{\eta_j} (u_j - \zeta_j)$$

Thanks to (4.10), the sequence v_j is bounded in $L^2(B_{1/2})$.

Lemma 4.6. *There exist a (non-relabelled) subsequence, a point $\zeta \in \mathcal{N}$ and a map $v \in H^1(B_{1/2}, \mathbb{R}^m)$ such that $\zeta_j \rightarrow \zeta$, $v_j \rightarrow v$ strongly in $L^2(B_{1/2})$ and a.e. as $j \rightarrow +\infty$. Moreover,*

$$v(x) \in T_{\zeta} \mathcal{N} \quad \text{for a.e. } x \in B_{1/2}. \quad (4.21)$$

Proof. Let φ_{ε_j} be a sequence of mollifiers, as in (3.12). Since ζ_j is constant, we have $\varphi_{\varepsilon_j} * \zeta_j = \zeta_j \int_{\mathbb{R}^3} \varphi_{\varepsilon_j} = \zeta_j$ and hence, by Lemma 3.5,

$$\int_{B_{1/2}} |\nabla(\varphi_{\varepsilon_j} * v_j)|^2 = \frac{1}{\eta_j^2} \int_{B_{1/2}} |(\nabla \varphi_{\varepsilon_j}) * u_j|^2 \lesssim \frac{F_{\varepsilon_j}(u_j, B_1)}{\eta_j^2} \stackrel{(4.7)}{=} 1$$

Moreover, by Lemma 3.6,

$$\int_{B_{1/2}} |v_j - \varphi_{\varepsilon_j} * v_j|^2 = \frac{1}{\eta_j^2} \int_{B_{1/2}} |u_j - \varphi_{\varepsilon_j} * u_j|^2 \lesssim \frac{\varepsilon_j^2 F_{\varepsilon_j}(u_j, B_1)}{\eta_j^2} \stackrel{(4.7)}{=} \varepsilon_j^2 \rightarrow 0$$

Then, we may extract a subsequence in such a way that $\varphi_{\varepsilon_j} * v_j \rightharpoonup v$ weakly in $H^1(B_{1/2})$ and $v_j \rightarrow v$ strongly in $L^2(B_{1/2})$ and a.e. We may also assume that $\zeta_j \rightarrow \zeta \in \mathcal{N}$, because $\mathcal{N}$ is compact.

It remains to prove (4.21). Let $\delta > 0$ be a small parameter. By Lemma 4.5, we know that $|G_j| \rightarrow 0$, so we may extract a subsequence $j_k \rightarrow +\infty$ in such a way that $\sum_{k \in \mathbb{N}} |G_{j_k}| \leq \delta$. Let $G := \cup_{k \in \mathbb{N}} G_{j_k}$. The estimate (4.11) implies

$$\int_{B_{1/2} \setminus G} \text{dist}(v_{j_k}, T_{\zeta_{j_k}} \mathcal{N}) \leq \int_{B_{1/2} \setminus G_{j_k}} \text{dist}(v_{j_k}, T_{\zeta_{j_k}} \mathcal{N}) \rightarrow 0$$

as $k \rightarrow +\infty$. By Fatou lemma, we deduce that $v(x) \in T_{\zeta} \mathcal{N}$ for a.e. $x \in B_{1/2} \setminus G$. This implies

$$|\{x \in B_{1/2} : v(x) \notin T_{\zeta} \mathcal{N}\}| \leq |G| \leq \sum_{k \in \mathbb{N}} |G_{j_k}| \leq \delta.$$

Since $\delta > 0$ is arbitrary, (4.21) follows. $\square$

Next, we show that v minimises the gradient energy associated with the tensor L , subject to its own boundary conditions.

Lemma 4.7. *Let $v^* \in H^1(B_{1/2}, T_{\zeta} \mathcal{N})$ be such that $v^* = v$ on $\partial B_{1/2}$ (in the sense of traces). Then,*

$$\int_{B_{1/2}} L \nabla v \cdot \nabla v \leq \int_{B_{1/2}} L \nabla v^* \cdot \nabla v^* \quad (4.22)$$

Moreover, for any $s \in (0, 1/2)$ there holds

$$\lim_{\varepsilon \rightarrow 0} \eta_j^{-2} F_{\varepsilon_j}(u_j, B_s) = \int_{B_s} L \nabla v \cdot \nabla v \quad (4.23)$$

Proof. By a density argument, it suffices to prove (4.22) in case $v^* = v$ in a neighbourhood of $\partial B_{1/2}$. Let us fix $s \in (0, 1/2)$ and $v^* \in H^1(B_{1/2}, T_{\zeta} \mathcal{N})$ such that $v^* = v$ a.e. in $B_{1/2} \setminus \bar{B}_s$. For any $a > 0$, we let $\mathcal{U}_a := \{y \in \mathbb{R}^m : \text{dist}(y, \mathcal{N}) \leq a\}$. Let $z \in \mathcal{N}$, $R > 0$ and $w \in T_z \mathcal{N}$ be such that $|w| \leq R$. Since $\nabla \pi(z)$ is the orthogonal projection onto $T_z \mathcal{N}$ (see e.g. [46, Section 2.12.3]), we have

$$|z + \eta_j w - \pi(z + \eta_j w)| \lesssim \eta_j^2 R^2 \|\nabla^2 \pi\|_{L^\infty(\mathcal{U}_{\eta_j R})} \quad (4.24)$$

Moreover, for any $X \in T_z \mathcal{N}$ we have

$$|X - \nabla \pi(z + \eta_j w)X| \leq \|\nabla \pi(z) - \nabla \pi(z + \eta_j w)\| |X| \lesssim \eta_j R |X| \|\nabla^2 \pi\|_{L^\infty(\mathcal{U}_{\eta_j R})} \quad (4.25)$$

We choose a positive sequence $R_j \rightarrow +\infty$ such that $\eta_j R_j^2 \rightarrow 0$ and we define

$$v_j^* := \frac{R_j v^*}{\max(R_j, |v^*|)}, \quad u_j^* := \pi(\zeta_j + \eta_j v_j^*).$$

Using (4.24) and (4.25), a routine computation shows that

$$\frac{u_j^* - \zeta_j}{\eta_j} \rightarrow v^* \quad \text{strongly in } H^1(B_{1/2})$$

Now, we apply Lemma 3.16 (and Remark 3.3). For any $\sigma \in (0, 1/10)$, we find radii r, t with $\max(1/2, s) < r < t < 1/2$ and maps $\xi_j \in L^\infty(\mathbb{R}^3, \mathcal{Q})$ such that $\xi_j = u_j$ a.e. in $\mathbb{R}^3 \setminus B_t$,

$$\Xi_j := \frac{\xi_j - \zeta_j}{\eta_j} \rightarrow v^* \quad \text{strongly in } H^1(B_r) \quad (4.26)$$

and

$$\begin{aligned} F_{\varepsilon_j}(\xi_j, B_t) + \Gamma_{\varepsilon_j}(\xi_j, B_t, \mathbb{R}^3 \setminus B_t) - \Gamma_{\varepsilon_j}(u_j, B_t, \mathbb{R}^3 \setminus B_t) \\ \leq F_{\varepsilon_j}^{\text{nl}}(\xi_j, B_r) + C_s \int_{B_r \setminus B_{r-\sigma}} |\nabla \xi_j|^2 + C_s \sigma \eta_j^2 + o\left(\frac{C_s \varepsilon_j^{q-2} \eta_j}{\sigma^q}\right) \end{aligned}$$

The constant C_s here is independent on ε, σ , though it depends on s . Since u_j is a minimiser of E_{ε_j} , we have

$$F_{\varepsilon_j}(u_j, B_t) + \Gamma_{\varepsilon_j}(u_j, B_t, \mathbb{R}^3 \setminus B_t) \leq F_{\varepsilon_j}(\xi_j, B_t) + \Gamma_{\varepsilon_j}(\xi_j, B_t, \mathbb{R}^3 \setminus B_t)$$

and hence,

$$F_{\varepsilon_j}(u_j, B_t) \leq F_{\varepsilon_j}^{\text{nl}}(\xi_j, B_r) + C_s \int_{B_r \setminus B_{r-\sigma}} |\nabla \xi_j|^2 + C_s \sigma \eta_j^2 + o\left(\frac{C_s \varepsilon_j^{q-2} \eta_j}{\sigma^q}\right)$$

We divide both sides of this inequality by η_j^2 and obtain

$$\begin{aligned} F_{\varepsilon_j}^{\text{nl}}(v_j, B_t) &\leq \frac{1}{\eta_j^2} F_{\varepsilon_j}(u_j, B_t) \\ &\leq F_{\varepsilon_j}^{\text{nl}}(\Xi_j, B_r) + C_s \int_{B_r \setminus B_{r-\sigma}} |\nabla \Xi_j|^2 + C_s \sigma + o\left(\frac{C_s \varepsilon_j^{q-2} \eta_j}{\sigma^q}\right) \end{aligned} \quad (4.27)$$

The assumptions (4.7), (4.8) imply that $\eta_j^2 \geq \varepsilon_j^{2q-4}$, that is $\eta_j \geq \varepsilon_j^{q-2}$. Then, recalling (4.26), Propositions 3.7 and 3.8, we may pass to the limit in (4.27), first as $j \rightarrow +\infty$, then as $\sigma \rightarrow 0$. We obtain

$$\int_{B_t} L \nabla v \cdot \nabla v \leq \limsup_{j \rightarrow +\infty} \frac{1}{\eta_j^2} F_{\varepsilon_j}(u_j, B_t) \leq \int_{B_t} L \nabla v^* \cdot \nabla v^*$$

and (4.22) follows. By choosing $v^* = v$, we also deduce (4.23), exactly as in Proposition 4.2. $\square$

Now, we are in position to complete the proof of Lemma 4.3.

Proof of Lemma 4.3. The assumption (4.7), Lemma 4.6 and Proposition 3.7 imply that

$$\int_{B_{1/2}} L \nabla v \cdot \nabla v \leq \liminf_{j \rightarrow +\infty} \frac{1}{\eta_j^2} F_{\varepsilon_j}(u_j, B_{1/2}) \leq 1 \quad (4.28)$$

By Lemma 4.7, $v \in H^1(B_{1/2}, T_\zeta \mathcal{N})$ minimises a quadratic functional among maps with values in the linear space $T_\zeta \mathcal{N}$. In particular, v is a solution of the system

$$-\operatorname{div}(L \nabla v) = 0 \quad \text{in } B_{1/2}$$

This system has constant coefficients and is (strongly) elliptic, by [Proposition 4.4](#). Then, elliptic regularity theory (see e.g. [20, Section III.2, Theorem 2.1, Remarks 2.2 and 2.3]) implies that v is smooth and there exists a number $\theta \in (0, 1)$ (depending only on L) such that

$$\int_{B_\theta} L \nabla v \cdot \nabla v \leq \frac{\theta}{4} \int_{B_{1/2}} L \nabla v \cdot \nabla v \stackrel{(4.28)}{\leq} \frac{\theta}{4}$$

However, (4.8) and (4.23) imply

$$\int_{B_\theta} L \nabla v \cdot \nabla v = \lim_{j \rightarrow +\infty} \frac{1}{\eta_j^2} F_{\varepsilon_j}(u_j, B_{1/2}) \geq \frac{\theta}{2}$$

Thus, we have obtained a contradiction, and the lemma follows. $\square$

4.3. Proof of [Theorems A and B](#)

Proof of Theorem A. Let η , θ and ε_* be given by [Lemma 4.3](#). Let $B_{r_0}(x_0) \subset\subset \Omega$ be a ball, $\varepsilon \in (0, \varepsilon_* r_0)$, and let u_ε be a minimiser of E_ε in $\mathcal{A}$ such that

$$F_\varepsilon(u_\varepsilon, B_{r_0}(x_0)) \leq \eta^2 r_0. \quad (4.29)$$

By a scaling argument, using (3.11), we can assume without loss of generality that $x_0 = 0$ and $r_0 = 1$. We fix a parameter $\gamma \in (0, 1)$.

Step 1 (*Campanato Estimate for Radii $\rho \geq \varepsilon^\gamma$*). In this case, the result follows by [Lemma 4.3](#) combined with a classical iteration argument. Let $k \geq 1$ be an integer such that $\theta^k \geq \varepsilon^\gamma$. Thanks to (4.29), we can apply [Lemma 4.3](#) iteratively, first on the ball B_1 , then on B_θ , on B_{θ^2} , and so on. We obtain

$$F_\varepsilon(u_\varepsilon, B_{\theta^k}) \leq \frac{\theta^k}{2^k} F_\varepsilon(u_\varepsilon, B_1) + \varepsilon^{2q-4} \theta^{k-1} \sum_{j=0}^{k-1} 2^{-j} \theta^{(4-2q)(k-1-j)} \quad (4.30)$$

At each step of the iteration, the assumptions of [Lemma 4.3](#) remain satisfied, so long as $\theta^k \geq \varepsilon^\gamma$ and ε is small enough. Indeed, (4.30) implies

$$F_\varepsilon(u_\varepsilon, B_{\theta^k}) \leq \frac{\theta^k}{2^k} F_\varepsilon(u_\varepsilon, B_1) + \varepsilon^{2q-4} \theta^{(k-1)(5-2q)} \sum_{j=0}^{k-1} \left(\frac{\theta^{2q-4}}{2} \right)^j$$

The series at the right-hand side converges, because $\theta \leq 1/2$, and its sum is less than 2. The assumption (4.29) implies

$$F_\varepsilon(u_\varepsilon, B_{\theta^k}) \leq \theta^k \left(\frac{\eta^2}{2^k} + 2\theta^{2q-5} \left(\frac{\varepsilon}{\theta^k} \right)^{2q-4} \right) \quad (4.31)$$

Under the assumption that $\theta^k \geq \varepsilon^\gamma$, we can further estimate the right-hand side as

$$F_\varepsilon(u_\varepsilon, B_{\theta^k}) \leq \theta^k \left(\frac{\eta^2}{2^k} + 2\theta^{2q-5} \varepsilon^{(2q-4)(1-\gamma)} \right)$$

and we can make sure that $F_\varepsilon(u_\varepsilon, B_{\theta^k}) \leq \eta^2 \theta^k$ by taking ε small enough (depending on η , θ and γ only). Now, take a radius ρ such that $\varepsilon^\gamma \leq \rho < 1$. Let $k = k(\rho) \geq 0$ be the unique integer such that $\theta^{k+1} \leq \rho < \theta^k$. The inequality (4.31) implies

$$F_\varepsilon(u_\varepsilon, B_\rho) \leq \frac{\rho}{\theta} \left(\frac{\eta^2}{2^k} + 2\theta^{2q-5} \left(\frac{\rho^{1/\gamma}}{\rho} \right)^{2q-4} \right) \leq C \rho \left(\rho^\alpha + \rho^{(2q-4)(1/\gamma-1)} \right) \quad (4.32)$$

for $\alpha := \log 2 / |\log \theta| \in (0, 1)$ and some constant C that depends only on η, θ . We assume that

$$0 < \gamma < \frac{2q-4}{\alpha+2q-4} \quad (4.33)$$

which implies $\alpha < (2q-4)(1/\gamma-1)$. By applying [Proposition 3.4](#), and possibly modifying the value of C , from [\(4.32\)](#) we deduce

$$\left| \oint_{B_\rho} u_\varepsilon - \oint_{B_\rho} u_\varepsilon \right|^2 \leq C \rho^\alpha \quad \text{for } \varepsilon^\gamma \leq \rho < 1. \quad (4.34)$$

Step 2 (*Campanato Estimate for Radii $\rho \leq \varepsilon^\gamma$*). We need to show that an estimate similar to [\(4.34\)](#) holds for $\rho < \varepsilon^\gamma$ as well. To this end, we consider the number $\nu > 1$ given by Assumption [\(K₆\)](#) and we define

$$p := 3 + \alpha - \frac{\alpha}{\nu}, \quad \beta := 1 - \frac{\alpha}{(\alpha+3)\nu} \quad (4.35)$$

We have $p > 3$, $0 < \beta < 1$. We claim that we can choose the parameter γ so as to satisfy [\(4.33\)](#) and

$$\beta < \gamma \quad (4.36)$$

Indeed, a straightforward algebraic manipulation shows that

$$1 - \frac{\alpha}{(\alpha+3)\nu} < \frac{2q-4}{\alpha+2q-4} \quad \Longleftrightarrow \quad \nu < \frac{\alpha+2q-4}{\alpha+3} \quad (4.37)$$

The assumption that $q > 7/2$ guarantees that $(\alpha+2q-4)/(\alpha+3) > 1$. Moreover, we have assumed that ∇K is integrable (see [\(K₁\)](#)), so there is no loss of generality in taking a smaller value for ν (so long as $\nu > 1$). Therefore, we may assume that $\nu > 1$ satisfies [\(4.37\)](#) and hence, we can choose γ that satisfies [\(4.33\)](#) and [\(4.36\)](#).

Let

$$m_\varepsilon := \oint_{B_{2\varepsilon^\beta}} u_\varepsilon$$

and let χ_ε be the characteristic function of the ball B_{ε^β} . Since ∇K_ε has zero average, from the Euler–Lagrange equation ([Proposition 3.1](#)) we obtain

$$\begin{aligned} \nabla(\Lambda \circ u_\varepsilon) &= (\nabla K_\varepsilon) * (u_\varepsilon - m_\varepsilon) \\ &= (\chi_\varepsilon \nabla K_\varepsilon) * (u_\varepsilon - m_\varepsilon) + ((1 - \chi_\varepsilon) \nabla K_\varepsilon) * (u_\varepsilon - m_\varepsilon). \end{aligned}$$

Let $\tilde{\chi}_\varepsilon$ be the characteristic function of the ball $B_{2\varepsilon^\beta}$. Since $\chi_\varepsilon \nabla K_\varepsilon$ is supported on B_{ε^β} , we deduce

$$\nabla(\Lambda \circ u_\varepsilon) = (\chi_\varepsilon \nabla K_\varepsilon) * (\tilde{\chi}_\varepsilon(u_\varepsilon - m_\varepsilon)) + ((1 - \chi_\varepsilon) \nabla K_\varepsilon) * (u_\varepsilon - m_\varepsilon) \quad \text{in } B_{\varepsilon^\beta}.$$

We apply Hölder's inequality, and then Young's inequality for the convolution:

$$\begin{aligned} \|\nabla(\Lambda \circ u_\varepsilon)\|_{L^p(B_{\varepsilon^\beta})} &\lesssim \|(\chi_\varepsilon \nabla K_\varepsilon) * (\tilde{\chi}_\varepsilon(u_\varepsilon - m_\varepsilon))\|_{L^p(\mathbb{R}^3)} \\ &\quad + \varepsilon^{3\beta/p} \|((1 - \chi_\varepsilon) \nabla K_\varepsilon) * (u_\varepsilon - m_\varepsilon)\|_{L^\infty(\mathbb{R}^3)} \\ &\lesssim \|\nabla K_\varepsilon\|_{L^1(B_{\varepsilon^\beta})} \|u_\varepsilon - m_\varepsilon\|_{L^p(B_{2\varepsilon^\beta})} \\ &\quad + \varepsilon^{3\beta/p} \|\nabla K_\varepsilon\|_{L^1(\mathbb{R}^3 \setminus B_{\varepsilon^\beta})} \|u_\varepsilon - m_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \end{aligned} \quad (4.38)$$

We bound the terms at the right-hand side. We have $\beta < \gamma$, so $2\varepsilon^\beta \geq \varepsilon^\gamma$ and hence

$$\|u_\varepsilon - m_\varepsilon\|_{L^2(B_{2\varepsilon^\beta})}^2 \lesssim \varepsilon^{(3+\alpha)\beta}$$

due to (4.34). Since $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq C$, by interpolation we obtain

$$\|u_\varepsilon - m_\varepsilon\|_{L^p(B_{2\varepsilon\beta})} \lesssim \|u_\varepsilon - m_\varepsilon\|_{L^2(B_{2\varepsilon\beta})}^{2/p} \lesssim \varepsilon^{(3+\alpha)\beta/p}. \quad (4.39)$$

By a change of variable, we have

$$\|\nabla K_\varepsilon\|_{L^1(B_{\varepsilon\beta})} \leq \varepsilon^{-1} \|\nabla K\|_{L^1(\mathbb{R}^3)} \quad (4.40)$$

and

$$\begin{aligned} \|\nabla K_\varepsilon\|_{L^1(\mathbb{R}^3 \setminus B_{\varepsilon\beta})} &= \varepsilon^{-1} \int_{\mathbb{R}^3 \setminus B_{\varepsilon\beta-1}} \|\nabla K(z)\| \, dz \\ &\leq \varepsilon^{-1} \int_{\mathbb{R}^3 \setminus B_{\varepsilon\beta-1}} \|\nabla K(z)\| \frac{|z|^\nu}{\varepsilon^{\nu\beta-\nu}} \, dz \\ &\leq \varepsilon^{\nu-\nu\beta-1} \int_{\mathbb{R}^3} \|\nabla K(z)\| |z|^\nu \, dz, \end{aligned} \quad (4.41)$$

where the integral at the right-hand side is finite by Assumption (K₆). Combining (4.38), (4.39), (4.40) and (4.41), and using that u_ε is bounded in $L^\infty(\mathbb{R}^3)$, we obtain

$$\|\nabla(\Lambda \circ u_\varepsilon)\|_{L^p(B_{\varepsilon\beta})} \lesssim \varepsilon^{(3+\alpha)\beta/p-1} + \varepsilon^{3\beta/p+\nu-\nu\beta-1}$$

By simple algebra, from (4.35) we obtain

$$\frac{(3+\alpha)\beta}{p} - 1 = \frac{3\beta}{p} + \nu - \nu\beta - 1 = 0$$

so $\|\nabla(\Lambda \circ u_\varepsilon)\|_{L^p(B_{\varepsilon\beta})}$ is bounded. Thanks to (3.8), we deduce that $\|\nabla u_\varepsilon\|_{L^p(B_{\varepsilon\beta})}$ is bounded too. Since $p > 3$, we have the Sobolev embedding $W^{1,p}(B_{\varepsilon\beta}) \hookrightarrow C^\mu(B_{\varepsilon\beta})$, where

$$\mu := 1 - \frac{3}{p} = \frac{\alpha(\nu-1)}{\alpha(\nu-1) + 3\nu} \quad (4.42)$$

Moreover, the constant C in the Sobolev inequality $[u]_{C^\mu(B_r)} \leq C\|\nabla u\|_{L^p(B_r)}$ is independent of $r > 0$, as demonstrated by a scaling argument. Then, we obtain

$$[u_\varepsilon]_{C^\mu(B_{\varepsilon\beta})} \leq C$$

and hence,

$$\left| \oint_{B_\rho} u_\varepsilon - \oint_{B_\rho} u_\varepsilon \right|^2 \lesssim \rho^{2\mu} \quad \text{for any } \rho \in (0, \varepsilon^\beta]. \quad (4.43)$$

Step 3 (Conclusion). By combining (4.34), (4.42) and (4.43), we deduce that

$$\left| \oint_{B_\rho} u_\varepsilon - \oint_{B_\rho} u_\varepsilon \right|^2 \leq C\rho^{\min(\alpha, 2\mu)} = C\rho^{2\mu}$$

for any radius $\rho \in (0, 1)$ and for some constant $C > 0$ that does not depend on ε, ρ . Then, Campanato embedding gives an ε -independent bound on the μ -Hölder semi-norm of u_ε on $B_{1/2}$. This completes the proof. $\square$

Proof of Theorem B. Let u_ε be a minimiser of E_ε in $\mathcal{A}$. By the results of [48], there exists a (non-relabelled) subsequence such that $u_\varepsilon \rightarrow u_0$ strongly in $L^2(\Omega)$, where u_0 is a minimiser of the limit functional (2.9). Take a point $x_0 \in \Omega \setminus S[u_0]$, where $S[u_0]$ is defined by (2.11). By definition of $S[u_0]$, there exists a number $r_0 > 0$ such that

$$r_0^{-1} \int_{B_{r_0}(x_0)} L \nabla u_0 \cdot \nabla u_0 \leq \frac{\eta^2}{2},$$

where η is given by Theorem A. Proposition 4.2 implies

$$r_0^{-1} F_\varepsilon(u_\varepsilon, B_{r_0}(x_0)) \leq \eta^2$$

for any ε small enough and hence, by Theorem A, $[u_\varepsilon]_{C^\mu(B_{r_0}(x_0))}$ is uniformly bounded. Then, Ascoli–Arzelà’s theorem implies that $u_\varepsilon \rightarrow u_0$ uniformly in $B_{r_0}(x_0)$. $\square$

5. Generalisation to finite-thickness boundary conditions

In this section, we discuss a variant of the minimisation problem, where we prescribe u in a neighbourhood of $\partial\Omega$ only. Let $\Omega_\varepsilon \supset \Omega$ be a larger domain, possibly depending on ε . We consider the functional

$$\tilde{E}_\varepsilon(u) := -\frac{1}{2\varepsilon^2} \int_{\Omega_\varepsilon \times \Omega_\varepsilon} K_\varepsilon(x-y) u(x) \cdot u(y) \, dx \, dy + \frac{1}{\varepsilon^2} \int_\Omega \psi_s(u(x)) \, dx + \tilde{C}_\varepsilon, \quad (5.1)$$

where $\tilde{C}_\varepsilon$ is a constant. The value of $\tilde{C}_\varepsilon$ is irrelevant for the purposes of minimisation, but we will make a specific choice of $\tilde{C}_\varepsilon$ (see (5.3)) for convenience. As before, we take a map $u_{\text{bd}} \in H^1(\mathbb{R}^3, \mathcal{Q})$ that satisfies (BD) and define the admissible class

$$\tilde{\mathcal{A}}_\varepsilon := \{u \in L^\infty(\Omega_\varepsilon, \mathcal{Q}) : \psi_s(u) \in L^1(\Omega), \, u = u_{\text{bd}} \text{ a.e. on } \Omega_\varepsilon \setminus \Omega\}. \quad (5.2)$$

The thickness of the boundary layer $\Omega_\varepsilon \setminus \Omega$ must be related to the decay properties of the kernel K . More precisely, in addition to (K₁)–(K₆), (H₁)–(H₆), we assume that

(K') There exist numbers $q > 7/2$, $\beta \in (0, 1 - \frac{7}{2q})$ and $\tau > 0$ such that

$$\int_{\mathbb{R}^3} g(z) |z|^q \, dz < +\infty$$

and $\text{dist}(\Omega, \partial\Omega_\varepsilon) \geq \tau\varepsilon^\beta$ for any $\varepsilon > 0$.

Remark 5.1. In case the kernel K is compactly supported, we can allow for a boundary layer of thickness proportional to ε . More precisely, we can replace the assumption (K') with the following: there exists $R_0 > 0$ such that $\text{supp}(K) \subseteq B_{R_0}$ and $\text{dist}(\Omega, \partial\Omega_\varepsilon) \geq R_0\varepsilon$ for any $\varepsilon > 0$. The proofs in this case remain essentially unchanged.

Under these assumptions, we can prove the analogues of Theorems A and B.

Theorem 5.1. Assume that the conditions (K₁)–(K₆), (H₁)–(H₆), (BD) and (K') are satisfied. Then, there exist positive numbers η , ε_* , M and $\mu \in (0, 1)$ such that, for any ball $B_{r_0}(x_0) \subset \subset \Omega$, any $\varepsilon \in (0, \varepsilon_* r_0)$, and any minimiser $\tilde{u}_\varepsilon$ of $\tilde{E}_\varepsilon$ in $\tilde{\mathcal{A}}_\varepsilon$, there holds

$$r_0^{-1} F_\varepsilon(\tilde{u}_\varepsilon, B_{r_0}(x_0)) \leq \eta \quad \implies \quad r_0^\mu [\tilde{u}_\varepsilon]_{C^\mu(B_{r_0/2}(x_0))} \leq M.$$

Theorem 5.2. Assume that the conditions (K_1) – (K_6) , (H_1) – (H_6) , (BD) and (K') are satisfied. Let $\tilde{u}_\varepsilon$ be a minimiser of $\tilde{E}_\varepsilon$ in $\tilde{\mathcal{A}}_\varepsilon$. Then, up to extraction of a (non-relabelled) subsequence, we have

$$\tilde{u}_\varepsilon \rightarrow u_0 \quad \text{locally uniformly in } \Omega \setminus S[u_0],$$

where u_0 is a minimiser of the functional (2.9) in $\mathcal{A}$ and $S[u_0]$ is defined by (2.11).

The proofs of Theorems 5.1 and 5.2 are largely similar to those of Theorems A and B. First, we show that the energy $\tilde{E}_\varepsilon$ has an alternative expression, which is analogous to (2.7). We define

$$\tilde{C}_\varepsilon := \frac{c_0}{\varepsilon^2} |\Omega| + \frac{1}{2\varepsilon^2} \int_{\Omega_\varepsilon \setminus \Omega} \left(\int_{(\Omega_\varepsilon - x)/\varepsilon} K(z) \, dz \right) \cdot u_{\text{bd}}(x)^{\otimes 2} \, dx, \quad (5.3)$$

where c_0 is the same number as in (2.4), and

$$H_\varepsilon(x) := \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3 \setminus (\Omega_\varepsilon - x)/\varepsilon} K(z) \, dz \quad \text{for any } x \in \mathbb{R}^3. \quad (5.4)$$

Lemma 5.3. For any $u \in \tilde{\mathcal{A}}_\varepsilon$, we have

$$\begin{aligned} \tilde{E}_\varepsilon(u) &= \frac{1}{4\varepsilon^2} \int_{\Omega_\varepsilon \times \Omega_\varepsilon} K_\varepsilon(x - y) \cdot (u(x) - u(y))^{\otimes 2} \, dx \, dy \\ &\quad + \int_{\Omega} H_\varepsilon(x) \cdot u(x)^{\otimes 2} \, dx + \frac{1}{\varepsilon^2} \int_{\Omega} \psi_b(u(x)) \, dx \end{aligned}$$

Proof. We inject the algebraic identity

$$-2K(x - y)u(x) \cdot u(y) = K(x - y) \cdot (u(x) - u(y))^{\otimes 2} - K(x - y) \cdot u(x)^{\otimes 2} - K(x - y) \cdot u(y)^{\otimes 2}$$

in the expression for $\tilde{E}_\varepsilon$. Since K is even, we obtain

$$\begin{aligned} \tilde{E}_\varepsilon(u) &= F_\varepsilon^{\text{nl}}(u, \Omega_\varepsilon) - \frac{1}{2\varepsilon^2} \int_{\Omega_\varepsilon \times \Omega_\varepsilon} K_\varepsilon(x - y) \cdot u(x)^{\otimes 2} \, dx + \frac{1}{\varepsilon^2} \int_{\Omega} \psi_s(u(x)) \, dx + \tilde{C}_\varepsilon \\ &= F_\varepsilon^{\text{nl}}(u, \Omega_\varepsilon) - \frac{1}{2\varepsilon^2} \int_{\Omega} \left(\int_{(\Omega_\varepsilon - x)/\varepsilon} K(z) \, dz \right) \cdot u(x)^{\otimes 2} \, dx \\ &\quad - \frac{1}{2\varepsilon^2} \int_{\Omega_\varepsilon \setminus \Omega} \left(\int_{(\Omega_\varepsilon - x)/\varepsilon} K(z) \, dz \right) \cdot u_{\text{bd}}(x)^{\otimes 2} \, dx + \frac{1}{\varepsilon^2} \int_{\Omega} \psi_s(u(x)) \, dx + \tilde{C}_\varepsilon \end{aligned}$$

where $F_\varepsilon^{\text{nl}}$ is defined by (3.14). Due to (5.3), we deduce

$$\tilde{E}_\varepsilon(u) = F_\varepsilon^{\text{nl}}(u, \Omega_\varepsilon) - \frac{1}{2\varepsilon^2} \int_{\Omega} \left(\int_{(\Omega_\varepsilon - x)/\varepsilon} K(z) \, dz \right) \cdot u(x)^{\otimes 2} \, dx + \frac{1}{\varepsilon^2} \int_{\Omega} (\psi_s(u(x)) + c_0) \, dx$$

and, thanks to (2.4) and (5.4), the lemma follows. $\square$

Lemma 5.4. For any ε , there exists a minimiser $\tilde{u}_\varepsilon$ for $\tilde{E}_\varepsilon$ in $\tilde{\mathcal{A}}_\varepsilon$ and it satisfies the Euler–Lagrange equation,

$$\Lambda(\tilde{u}_\varepsilon(x)) = \int_{\Omega_\varepsilon} K_\varepsilon(x - y) \tilde{u}_\varepsilon(y) \, dy \quad (5.5)$$

for a.e. $x \in \Omega$.

The proof of [Lemma 5.4](#) is identical to that of [Proposition 3.1](#), so we skip it for brevity. We remark that Eq. (5.5) can be written as

$$\Lambda(\tilde{u}_\varepsilon) = K_\varepsilon * (\tilde{u}_\varepsilon \chi_\varepsilon) \quad \text{a.e. on } \Omega,$$

where χ_ε is the characteristic function of Ω_ε . In particular, the uniform strict physicality of $\tilde{u}_\varepsilon$ follows from (5.5), exactly as in [Proposition 3.3](#).

Lemma 5.5. *Under the assumption (K'), the tensor field H_ε defined by (5.4) satisfies*

$$\sup_{x \in \Omega} \|H_\varepsilon(x)\| = o(\varepsilon^{q-\beta q-2})$$

as $\varepsilon \rightarrow 0$.

Under the assumption (K'), we have $\beta < 1 - 7/(2q)$, which implies $q - \beta q - 2 > 3/2$. In particular, H_ε converges to zero uniformly in Ω as $\varepsilon \rightarrow 0$.

Proof of Lemma 5.5. For any $x \in \Omega$, we have $B_{\tau\varepsilon\beta}(x) \subseteq \Omega_\varepsilon$ by (K'), and hence $B_{\tau\varepsilon\beta-1} \subseteq (\Omega_\varepsilon - x)/\varepsilon$, $\mathbb{R}^3 \setminus (\Omega_\varepsilon - x)/\varepsilon \subseteq \mathbb{R}^3 \setminus B_{\tau\varepsilon\beta-1}$. Then, the definition (5.4) of H_ε gives

$$\begin{aligned} \|H_\varepsilon(x)\| &\leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3 \setminus (\Omega_\varepsilon - x)/\varepsilon} \|K(z)\| \, dz \\ &\leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3 \setminus B_{\tau\varepsilon\beta-1}} \|K(z)\| \, dz \\ &\leq \frac{\varepsilon^{q-\beta q-2}}{\tau^q} \int_{\mathbb{R}^3 \setminus B_{\tau\varepsilon\beta-1}} \|K(z)\| |z|^q \, dz \end{aligned}$$

and the lemma follows, due to (K'). $\square$

Lemma 5.6. *Let $\tilde{u}_\varepsilon$ be a minimiser for $\tilde{E}_\varepsilon$ in $\tilde{\mathcal{A}}_\varepsilon$, identified with its extension by u_{bd} to $\mathbb{R}^3$. Then, $\tilde{u}_\varepsilon$ is an ω -minimiser for E_ε in Ω , where $\omega(s) := Cs^{q-\beta q-2}$, $C > 0$ is a constant that depends only on Ω , K , $\mathcal{Q}$ and q , β are given by (K').*

[Lemma 5.6](#) guarantees that our compactness result, [Proposition 4.2](#), applies to minimisers of $\tilde{E}_\varepsilon$.

Proof of Lemma 5.6. We write $\Omega_\varepsilon^c := \mathbb{R}^3 \setminus \Omega_\varepsilon$. Let $\tilde{u}_\varepsilon$ be a minimiser for $\tilde{E}_\varepsilon$ in $\tilde{\mathcal{A}}_\varepsilon$. Let $B := B_\rho(x_0) \subseteq \Omega$ be a ball, and let $v \in L^\infty(\mathbb{R}^3, \mathcal{Q})$ be such that $v = \tilde{u}_\varepsilon$ a.e. on $\mathbb{R}^3 \setminus B$. By comparing (2.1) with (5.1), and using (K₂), we obtain

$$\begin{aligned} E_\varepsilon(v) - \tilde{E}_\varepsilon(v) - C_\varepsilon + \tilde{C}_\varepsilon &= -\frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon \times \Omega_\varepsilon^c} K_\varepsilon(x-y)v(x) \cdot v(y) \, dx \, dy - \frac{1}{2\varepsilon^2} \int_{\Omega_\varepsilon^c \times \Omega_\varepsilon^c} K_\varepsilon(x-y)v(x) \cdot v(y) \, dx \, dy \\ &= -\frac{1}{\varepsilon^2} \int_{B \times \Omega_\varepsilon^c} K_\varepsilon(x-y)v(x) \cdot u_{\text{bd}}(y) \, dx \, dy - \frac{1}{\varepsilon^2} \int_{(\Omega \setminus B) \times \Omega_\varepsilon^c} K_\varepsilon(x-y)\tilde{u}_\varepsilon(x) \cdot u_{\text{bd}}(y) \, dx \, dy \\ &\quad - \frac{1}{2\varepsilon^2} \int_{\Omega_\varepsilon^c \times \Omega_\varepsilon^c} K_\varepsilon(x-y)u_{\text{bd}}(x) \cdot u_{\text{bd}}(y) \, dx \, dy \end{aligned} \quad (5.6)$$

The second and third integral at the right-hand side are independent of v . We bound the first integral by making the change of variable $y = x + \varepsilon z$ and applying Fubini theorem:

$$\frac{1}{\varepsilon^2} \left| \int_{B \times \Omega_\varepsilon^c} K_\varepsilon(x-y)v(x) \cdot u_{\text{bd}}(y) \, dx \, dy \right| \leq \|u_{\text{bd}}\|_{L^\infty(\mathbb{R}^3)} \int_B \|H_\varepsilon(x)\| |v(x)| \, dx$$

where H_ε is defined by (5.4). Since u_{bd} and v both take values in the bounded set $\mathcal{Q}$, and since $|B| \lesssim \rho^3 \lesssim \rho$, by Lemma 5.5 we have

$$\frac{1}{\varepsilon^2} \left| \int_{B \times \Omega_\varepsilon^c} K_\varepsilon(x-y)v(x) \cdot u_{\text{bd}}(y) \, dx \, dy \right| \lesssim \varepsilon^{q-\beta q-2} \rho \quad (5.7)$$

From (5.6) and (5.7), we deduce

$$E_\varepsilon(\tilde{u}_\varepsilon) - \tilde{E}_\varepsilon(\tilde{u}_\varepsilon) \leq E_\varepsilon(v) - \tilde{E}_\varepsilon(v) + C\varepsilon^{q-\beta q-2} \rho \quad (5.8)$$

On the other hand,

$$\tilde{E}_\varepsilon(\tilde{u}_\varepsilon) \leq \tilde{E}_\varepsilon(v), \quad (5.9)$$

because $\tilde{u}_\varepsilon$ is a minimiser for $\tilde{E}_\varepsilon$ and $v \in \tilde{\mathcal{A}}_\varepsilon$. Combining (5.8) and (5.9), the lemma follows. $\square$

Finally, we prove the analogue of the decay lemma, Lemma 4.3.

Lemma 5.7. *There exist $\eta > 0$, $\theta \in (0, 1/2)$ and $\varepsilon_* > 0$ such that, for any ball $B_\rho(x_0) \subseteq \Omega$, any $\varepsilon \in (0, \varepsilon_*)$ and any minimiser $\tilde{u}_\varepsilon$ of $\tilde{E}_\varepsilon$ in $\tilde{\mathcal{A}}_\varepsilon$ such that*

$$F_\varepsilon(\tilde{u}_\varepsilon, B_\rho(x_0)) \leq \eta^2 \rho,$$

there holds

$$F_\varepsilon(\tilde{u}_\varepsilon, B_{\theta\rho}(x_0)) \leq \frac{\theta}{2} F_\varepsilon(\tilde{u}_\varepsilon, B_\rho(x_0)) + \left(\frac{\varepsilon}{\rho}\right)^{2q-2\beta q-4} \rho. \quad (5.10)$$

The estimate (5.10) is slightly worse than the corresponding estimate from Lemma 4.3, i.e. Eq. (4.6), because the exponent of ε/ρ at the right-hand side is strictly less than $2q - 4$. Nevertheless, (5.10) is still enough for our purposes. Indeed, the assumption $\beta < 1 - 7/(2q)$ implies that $\bar{q} := q - \beta q > 7/2$. Then, the proofs of Theorems A and B carry over; it suffices to replace all the occurrences of q with $\bar{q}$. In particular, once Lemma 5.7 is proved, Theorems 5.1 and 5.2 follow.

Proof of Lemma 5.7. Suppose that $B_\rho(x_0)$ is a ball contained in Ω and that $u: \Omega_\varepsilon \rightarrow \mathcal{Q}$. We consider $U_\rho := (\Omega - x_0)/\rho$, $U_{\varepsilon,\rho} := (\Omega_\varepsilon - x_0)/\rho$ and define $u_\rho: U_\rho \rightarrow \mathcal{Q}$ as $u_\rho(y) := u(x_0 + \rho y)$. Then,

$$\rho^{-1} \tilde{E}(u) = F_{\varepsilon/\rho}^{\text{nl}}(u_\rho, U_{\varepsilon,\rho}) + \frac{1}{(\varepsilon/\rho)^2} \int_{U_\rho} \psi_b(u_\rho(x)) \, dx + \int_{U_\rho} H_{\varepsilon/\rho}^{\rho, x_0}(x) \cdot u_\rho(x)^{\otimes 2} \, dx$$

where $F_\varepsilon^{\text{nl}}$ is defined by (3.14) and

$$H_{\varepsilon/\rho}^{\rho, x_0}(x) := \frac{1}{(\varepsilon/\rho)^2} \int_{\mathbb{R}^3 \setminus (\Omega_\varepsilon - x_0 - \rho x)/\varepsilon} K(z) \, dz = \rho^2 H_\varepsilon(x_0 + \rho x)$$

Lemma 5.5 implies that

$$\sup_{x \in U_\rho} \|H_{\varepsilon/\rho}^{\rho, x_0}(x)\| = o(\rho^2 \varepsilon^{q-\beta q-2}) = o((\varepsilon/\rho)^{q-\beta q-2})$$

As a consequence, if we prove the lemma in case $x_0 = 0$, $\rho = 1$, then the general statement will follow, by a scaling argument.

Suppose the lemma does not hold. Then, for any $\theta \in (0, 1/2)$, there exist a sequence $\varepsilon_j \rightarrow 0$ and minimisers $\tilde{u}_j$ of $\tilde{E}_{\varepsilon_j}$ such that

$$\eta_j^2 := F_{\varepsilon_j}(\tilde{u}_j, B_1) \rightarrow 0 \quad \text{as } j \rightarrow +\infty, \quad (5.11)$$

$$F_{\varepsilon_j}(\tilde{u}_j, B_\theta) > \frac{\theta \eta_j^2}{2} + \varepsilon_j^{2q-2\beta q-4} \geq \frac{\theta \eta_j^2}{2} + \varepsilon_j^{2q-4} \quad (5.12)$$

Lemmas 4.5 and 4.6 carry over, so there exists a sequence of points $\zeta_j \in \mathcal{N}$ such that, up to a non-relabelled subsequence, $\zeta_j \rightarrow \zeta$ and

$$\tilde{v}_j := \frac{\tilde{u}_j - \zeta_j}{\eta_j} \rightarrow \tilde{v} \quad \text{strongly in } L^2(B_{1/2}) \text{ and a.e.} \quad (5.13)$$

Moreover, the limit map $\tilde{v}$ belongs to $H^1(B_{1/2}, T_\zeta \mathcal{N})$. Let $s \in (0, 1/2)$ and let $\tilde{v}^* \in H^1(B_{1/2}, T_\zeta \mathcal{N})$ be such that $\tilde{v}^* = \tilde{v}$ a.e. in $B_{1/2} \setminus \bar{B}_s$. We construct admissible competitors $\tilde{\xi}_\varepsilon \in L^\infty(\Omega_\varepsilon, \mathcal{Q})$ exactly as before, by applying Lemma 3.16. We recall that there exist radii r, t with $\max(1/2, s) < r < t < 1/2$ such that $\tilde{\xi}_\varepsilon = \tilde{u}_\varepsilon$ out of B_t and

$$\frac{\tilde{\xi}_j - \zeta_j}{\eta_j} \rightarrow \tilde{v}^* \quad \text{strongly in } H^1(B_r) \quad (5.14)$$

However, $\tilde{u}_j$ is now a minimiser of $\tilde{E}_{\varepsilon_j}$, not of E_{ε_j} . As a result, we have the inequality

$$\begin{aligned} F_{\varepsilon_j}(\tilde{u}_j, B_t) + \Gamma_{\varepsilon_j}(\tilde{u}_j, B_t, \Omega_\varepsilon \setminus B_t) &\leq F_{\varepsilon_j}(\tilde{\xi}_j, B_t) + \Gamma_{\varepsilon_j}(\tilde{\xi}_j, B_t, \Omega_\varepsilon \setminus B_t) \\ &\quad + \int_{B_t} H_{\varepsilon_j}(x) \cdot (\tilde{\xi}_j(x)^{\otimes 2} - \tilde{u}_j(x)^{\otimes 2}) \, dx, \end{aligned}$$

with an additional term at the right-hand side. (We recall that Γ_ε is defined by (3.24).) We claim that

$$I_j := \frac{1}{\eta_j^2} \int_{B_t} H_{\varepsilon_j}(x) \cdot (\tilde{\xi}_j(x)^{\otimes 2} - \tilde{u}_j(x)^{\otimes 2}) \, dx \rightarrow 0 \quad (5.15)$$

as $j \rightarrow +\infty$. Once (5.15) is proved, the rest of the arguments carry over.

Eq. (5.4) implies that the matrix $H_\varepsilon(x)$ is symmetric, for any $x \in B_1$. Then,

$$I_j = \frac{1}{\eta_j^2} \int_{B_t} H_{\varepsilon_j}(x) (\tilde{\xi}_j(x) - \tilde{u}_j(x)) \cdot (\tilde{\xi}_j(x) + \tilde{u}_j(x)) \, dx$$

As both $\tilde{\xi}_j$ and $\tilde{u}_j$ take their values in the bounded set $\mathcal{Q}$, the Hölder inequality gives

$$I_j \lesssim \eta_j^{-2} \|H_{\varepsilon_j}\|_{L^\infty(B_t)} \left(\|\tilde{\xi}_j - \zeta_j\|_{L^2(B_{1/2})} + \|\tilde{u}_j - \zeta_j\|_{L^2(B_{1/2})} \right)$$

We can further estimate the right-hand side by applying (5.13), (5.14) and Lemma 5.5:

$$I_j = o\left(\frac{\varepsilon_j^{q-\beta q-2}}{\eta_j}\right)$$

However, (5.11) and (5.12) imply that $\eta_j^2 \geq \varepsilon^{2q-2\beta q-4}$, so $\eta_j \geq \varepsilon^{q-\beta q-2}$ and (5.15) follows. $\square$

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