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Fractional kinetics in random/complex media

Abstract: In this chapter, we consider a randomly-scaled Gaussian process and discuss a number of applications to model fractional diffusion. Actually, this approach can be understood as a Gaussian diffusion in a medium characterized by a population of scales. This interpretation supports the idea that fractional diffusion emerges from standard diffusion occurring in a complex medium.

Keywords: Fractional diffusion, fractional calculus, stochastic processes, randomly-scaled Gaussian processes, generalized grey Brownian motion

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1 Introduction

The mathematical description of diffusion has a long history with many different formulations including phenomenological models based on: conservation of mass and constitutive laws, probabilistic models based on random walks and central limit theorems, stochastic models based on Wiener process or on the Langevin equation, as well as models based on master equations or on the Fokker–Planck equation. On the other hand many processes in life sciences, soft condensed matter, geology, and ecology show a diffusive behavior that cannot be modeled by classical methods. These phenomena are generally labeled with the term anomalous diffusion in order to distinguish them from the normal diffusion, where the adjective normal highlights that a Gaussian-based process is considered, and the mean square displacement of diffusing particles scales linearly with time. Numerous experimental measurements in which the mean square displacement of diffusing particles scales with a nonlinear power law in time are successfully modeled through fractional calculus, such that the corresponding process is referred to be a fractional diffusion phenomenon. Fractional diffusion can be interpreted as the consequence of the complexity, or randomness, of the medium in which a classical diffusion takes place. More explicitly, fractional diffusion emerges from a population of scales that characterizes the medium.

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We consider here the motion of particles in such a random/complex medium. The randomness of the medium is described by a characteristic quantity, for example, the lengthscale, that depends on a parameter β . The role of β consists in tuning the degree of randomness of the medium by modulating the distribution of the lengthscale ℓ_β .

The considered approach is that of a randomly-scaled Gaussian process, that is, a Gaussian process multiplied times a non-negative independent random variable. This approach was inspired by the constitutive approach proposed by Mura [27] to built up processes meeting the characteristics of the so-called grey Brownian motion (gBm) [37, 38] and generalized grey Brownian motion (ggBm) [29, 28]. Actually, this approach is based on the product of the fractional Brownian motion (fBm) and an independent positive and constant random variable.

We can show that, in this scenario, a suitable choice of the length scale distribution ℓ_β allows to obtain a continuous transition from an ergodic to a nonergodic process through the parameter β . In particular, this is verified by the same condition that triggers the emergence of fractional kinetics, which is then straightforwardly associated with the occurrence of the ergodicity breaking.

The rest of the chapter is organized as follows: In Section 2, the modeling approach is described together with some properties. In Section 3, the formulation of stochastic processes with stationary increments is presented for the cases of Erdélyi–Kober fractional diffusion and space-time fractional diffusion. In Section 4, the formulation of stochastic processes with nonstationary increments is presented and the role of the dependence on time discussed. In Section 5, conclusions and final remarks are reported.

2 Modeling approach

2.1 Definition of the stochastic process

The proposed stochastic process $X(t)$ is expressed as

$$X(t) = \ell_\beta X_G(t), \quad (2.1)$$

where $X_G(t)$ is a Gaussian stochastic process and ℓ_β a non-negative random variable. The process (2.1) is a randomly-scaled Gaussian process. In the following, the process $X_G(t)$ is chosen to be the fBm $X_H(t)$ where $0 < H < 1$ is the Hurst exponent and the ggBm is recovered, for example, [29].

With and **abuse** of terminology, we call ggBm a processes defined by the product of a Gaussian process and an independent and constant non-negative random variable.

Notice that ℓ_β is a random length scale labeling the single trajectory. Each trajectory can be interpreted as a different sample path performed by a particle in different

experiments with the same particle initial conditions, but in different environmental conditions, which are parameterized by the random length scale ℓ_β .

2.2 Probability density function

It is well known that the probability density function (PDF) of the product of two independent random variables is given by an integral formula [34, 3, 21, 23].

Let Z_1 and Z_2 be two real independent random variables whose PDFs are $p_1(z_1)$ and $p_2(z_2)$, respectively, with $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}^+$. Let Z be the random variable obtained by the product of Z_1 and Z_2^γ , that is,

$$Z = Z_1 Z_2^\gamma. \quad (2.2)$$

Then, denoting with $p(z)$ the PDF of Z , it results

$$p(z) = \int_0^\infty p_1\left(\frac{z}{\lambda^\gamma}\right) p_2(\lambda) \frac{d\lambda}{\lambda^\gamma}. \quad (2.3)$$

From the PDF of the product of two variables (2.3), by applying the change of variables $z = \lambda t^{-H}$ the PDF of the process $X(t)$ results to be

$$\frac{1}{t^H} p\left(\frac{x}{t^H}\right) = \int_0^\infty \frac{1}{\lambda^\gamma t^H} p_1\left(\frac{x}{\lambda^\gamma t^H}\right) p_2(\lambda) d\lambda, \quad (2.4)$$

where p_1 is the Gaussian density of X_G with variance $\langle x^2 \rangle \sim t^{2H}$ and p_2 is the density of ℓ_β^2 .

2.3 Ergodicity breaking

The ergodicity of the process is evaluated through the limit for large T of the following quantity [1, 12, 8]

$$\text{EB}(T, \Delta) = \frac{\langle [\overline{\delta^2}(T, \Delta)]^2 \rangle}{\langle \overline{\delta^2}(T, \Delta) \rangle^2} - 1, \quad (2.5)$$

where $\langle \cdot \rangle$ denotes the ensemble average and $\overline{\delta^2}(T, \Delta)$ the Time-Averaged Mean-Square Displacement (TA-MSD)

$$\overline{\delta^2}(T, \Delta) = \frac{\int_0^{T-\Delta} [X(\xi + \Delta) - X(\xi)]^2 d\xi}{T - \Delta}, \quad (2.6)$$

being Δ the time lag and $T \gg \Delta$ the measurement time. For an ergodic process, it has been shown that $\text{EB}(T, \Delta)$ approaches 0 for large T . It is easy to see that, being computed as a time average on the single trajectory, the long-time ($T \rightarrow \infty$) TA-MSD

of the process $X(t)$, equation (2.1), is given by

$$\overline{\delta^2}(T, \Delta) \xrightarrow{T \rightarrow \infty} 2\ell_\beta^2 \Delta^{2H}, \quad (2.7)$$

so that the ensemble average of the long-time TA-MSD is simply given by [26]:

$$\langle \overline{\delta^2}(T, \Delta) \rangle \xrightarrow{T \rightarrow \infty} 2\langle \ell_\beta^2 \rangle \Delta^{2H}. \quad (2.8)$$

From the ratio of (2.7) and (2.8), we have

$$\xi = \frac{\overline{\delta^2}}{\langle \overline{\delta^2} \rangle} \xrightarrow{T \rightarrow \infty} \frac{\ell_\beta^2}{\langle \ell_\beta^2 \rangle}, \quad (2.9)$$

from which an indication about the distribution of ℓ_β can be obtained [26].

Indeed, setting a fixed and non-random length scale $\ell_\beta = 1$ in the stochastic process $X(t)$ defined in (2.1), it results

$$\text{EB}_{\ell_\beta=1}(T, \Delta) = \text{EB}_{\text{fBm}}(T, \Delta) \xrightarrow{T \rightarrow \infty} 0. \quad (2.10)$$

In contrast, if ℓ_β is chosen as a random variable, it holds

$$\text{EB}_{\ell_\beta}(T, \Delta) = \frac{\langle \ell_\beta^4 \rangle}{\langle \ell_\beta^2 \rangle^2} [\text{EB}_{\text{fBm}}(T, \Delta) + 1] - 1 \xrightarrow{T \rightarrow \infty} \frac{\langle \ell_\beta^4 \rangle}{\langle \ell_\beta^2 \rangle^2} - 1, \quad (2.11)$$

then the nonergodicity of the process is shown by the parameter EB if the random length scale ℓ_β verifies the inequality $\langle \ell_\beta^4 \rangle \neq \langle \ell_\beta^2 \rangle^2$. Since ℓ_β is an independent random variable, equation (2.11) remarks that, in our approach, the EB is solely due to the randomness of the medium, and it is not affected by the particular choice of the stochastic process $X_G(t)$ at the basis of the trajectories. It is worth noting that, being $X(t)$ a monoscaling stochastic process, the dependence of EB on Δ disappears in the long-time limit $T \rightarrow \infty$.

2.4 p-variation test

Beside the EB analysis, another feature characterizing the stochastic trajectories and applicable as a criterion for the selection of stochastic processes consists in the p-variation test, defined as [16, 26]

$$V^{(p)}(t) = \lim_{n \rightarrow \infty} V_n^{(p)}(t), \quad (2.12)$$

where, for $t \in [0, T]$,

$$V_n^{(p)}(t) = \sum_{j=0}^{2^n-1} \left| X\left(\frac{(j+1)T}{2^n} \wedge t\right) - X\left(\frac{jT}{2^n} \wedge t\right) \right|^p, \quad (2.13)$$

with $a \wedge b = \min\{a, b\}$.

It is easy to show that the stochastic process (2.1) maintains the same p-variation behavior as the fBm, in spite of the fact that EB occurs. Inserting (2.1) into (2.13) gives $V^{(p)}(t) = \ell_\beta^p V_{\text{fBm}}^{(p)}(t)$.

2.5 Fractional diffusion as a consequence of ergodicity breaking

Let $p_\beta(\ell)$ be the density of the random scale ℓ_β . For a proper choice of $p_\beta(\ell)$, the PDF of $X(t)$ solves a fractional diffusion equation.

From the properties reported above, we observe that if it holds

$$\lim_{\beta \rightarrow \beta_0} p_\beta(\ell) \rightarrow \delta(\ell - \ell_0), \quad (2.14)$$

then the PDF of the process $X(t)$ reduces to the Gaussian density of the process $X_G(t)$ and the ergodicity breaking parameter EB goes to 0, which means that the process is indeed ergodic.

This observation allows for relating the emergence of fractional diffusion to the emergence of ergodicity breaking providing a physical interpretation to fractional diffusion. Actually, parameter β describes the fluctuations of ℓ_β due to the complex/random medium where the diffusion occurs. The existence of these fluctuations causes the ergodicity breaking and a non-Gaussian diffusion. Since ℓ_β represents a scale of the process, we conclude that in a random/complex medium where a population of scales is expected, also ergodicity breaking is expected and for specific densities of the scales fractional diffusion emerges.

3 Stochastic processes with stationary increments

3.1 Erdélyi–Kober fractional diffusion

3.1.1 The fundamental solution to the Erdélyi–Kober fractional diffusion

Normal diffusion, or Gaussian diffusion, is a Markovian stochastic process driven by the classical parabolic equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2}, \quad x \in \mathcal{R}, \quad t \in \mathcal{R}_0^+, \quad (3.1)$$

with initial condition $P(x, 0) = P_0(x)$. The fundamental solution of (3.1), which is named also the Green function, and corresponding to the case with initial condition $P(x, 0) = P_0(x) = \delta(x)$, is the Gaussian density,

$$f(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}, \quad (3.2)$$

whose variance grows linearly in time, that is, $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 f(x, t) dx = 2t$. The density function $P(x, t)$ with general initial condition $P(x, 0) = P_0(x)$ is related to the fundamental solution $f(x, t)$ by the following convolution integral:

$$P(x, t) = \int_{-\infty}^{+\infty} f(\xi, t) P_0(x - \xi) d\xi. \quad (3.3)$$

In order to generalize the classical Markovian setting to non-Markovian cases, the following integral equation has been introduced by A. Mura, M. S. Taqqu, and F. Mainardi [30]:

$$P(x, t) = P_0(x) + \int_0^t \frac{\partial g(s)}{\partial s} K[g(t) - g(s)] \frac{\partial^2 P(x, s)}{\partial x^2} ds, \quad (3.4)$$

where $K(t)$ is a memory kernel and $g(t)$, with $g(0) = 0$, is a smooth and increasing function describing a time stretching. The Green function of (3.4) $\mathcal{G}(x, t)$, which is the marginal one-point one-time PDF of the non-Markovian diffusion process, turns out to be

$$\mathcal{G}(x, t) = \int_0^\infty f(x, \tau) h(\tau, g(t)) d\tau, \quad (3.5)$$

where $f(x, t)$ is the Gaussian density (3.2) that is the fundamental solution of the Markovian diffusion process, that is, $K(t) = \delta(t)$, and $h(\tau, t)$ is the fundamental solution of the so-called *non-Markovian forward drift equation*

$$u(\tau, t) = u_0(\tau) - \int_0^t K(t - s) \frac{\partial u(\tau, s)}{\partial \tau} ds, \quad \tau, t \in \mathcal{R}_0^+, \quad (3.6)$$

where $u_0(\tau) = u(\tau, 0)$.

When the kernel and the time-stretching functions are stated as

$$K(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad g(t) = t^{\alpha/\beta}, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 1, \quad (3.7)$$

Equation (3.4) becomes

$$P(x, t) = P_0(x) + \frac{1}{\Gamma(\beta)} \frac{\alpha}{\beta} \int_0^t \tau^{\alpha/\beta-1} (t^{\alpha/\beta} - \tau^{\alpha/\beta})^{\beta-1} \frac{\partial^2 P(x, \tau)}{\partial x^2} d\tau, \quad (3.8)$$

that was originally introduced by A. Mura in his PhD thesis [27], and later discussed by him and collaborators in a number of papers [24, 28, 29, 30].

It is well known that there exists a relationship between the solutions of a certain class of integral equations that are used to model anomalous diffusion and stochastic processes. In this respect, the density function $P(x, t)$ which solves (3.8) is the marginal particle PDF, that is, the one-point one-time density function of particle dispersion, of the *generalized grey Brownian motion* (ggBm) [27, 28, 29].

The ggBm is a special class of H-sssi processes of order $H = \alpha$, or Hurst exponent $H = \alpha/2$, where, according to a common terminology, H-sssi means H-self-similar-stationary-increments. The ggBm provides non-Markovian stochastic models for anomalous diffusion, of both slow type $0 < \alpha < 1$ and fast type $1 < \alpha < 2$. The ggBm includes some well-known processes, so that it defines an interesting general theoretical framework. In fact, the fractional Brownian motion appears for $\beta = 1$, the grey Brownian motion, in the sense of W. R. Schneider [37, 38], corresponds to the choice $0 < \alpha = \beta < 1$, and finally the standard Brownian motion is recovered by setting $\alpha = \beta = 1$. It is worth noting to remark that only in the particular case of the Brownian motion the stochastic process is Markovian. Moreover, the ggBm is not an ergodic process [29].

The integral in the non-Markovian kinetic equation (3.8) can be expressed in terms of an Erdélyi–Kober fractional integral. In fact, let $\mu > 0$, $\eta > 0$, and $\gamma \in \mathcal{R}$. The Erdélyi–Kober fractional integral operator $I_\eta^{\gamma, \mu}$, for a sufficiently well-behaved function $\varphi(t)$, is defined as [9, formula (1.1.17)]

$$\begin{aligned} I_\eta^{\gamma, \mu} \varphi(t) &= \frac{t^{-\eta(\mu+\gamma)}}{\Gamma(\mu)} \int_0^t \tau^{\eta\gamma} (t^\eta - \tau^\eta)^{\mu-1} \varphi(\tau) d(\tau^\eta) \\ &= \frac{\eta}{\Gamma(\mu)} t^{-\eta(\mu+\gamma)} \int_0^t \tau^{\eta(\gamma+1)-1} (t^\eta - \tau^\eta)^{\mu-1} \varphi(\tau) d\tau, \end{aligned} \quad (3.9)$$

hence equation (3.8) can be rewritten as

$$P(x, t) = P_0(x) + t^\alpha \left[I_{\alpha/\beta}^{0, \beta} \frac{\partial^2 P}{\partial x^2} \right]. \quad (3.10)$$

The integral operator $I_\eta^{\gamma, \mu}$ was introduced by I. N. Sneddon (see, e. g., [40, 41, 42]) who studied its basic properties and emphasized its useful applications to the generalized axially symmetric potential theory (GASPT) and other physical problems (say in electrostatics, elasticity, etc). When $\eta = 1$, one obtains the operators of fractional integration as originally introduced by H. Kober [10] and A. Erdélyi [2] and, when $\eta = 2$, those introduced by I. N. Sneddon [40, 41, 42]. In the special case $\gamma = 0$ and $\eta = 1$, the Erdélyi–Kober fractional integral operator (3.9) and the Riemann–Liouville fractional integral of order μ , here noted by J^μ , are related by the formula

$$I_1^{0, \mu} \varphi(t) = \frac{t^{-\mu}}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} \varphi(\tau) d\tau = t^{-\mu} J^\mu \varphi(t). \quad (3.11)$$

The possibility to rewrite equation (3.8) as (3.10) was briefly noted by the author in [32] and largely discussed in [33]. This correspondence between the ggBm and the Erdélyi–Kober fractional integral operator is here stressed, because, since the ggBm serves as a stochastic model for the anomalous diffusion, this leads to define the family of diffusive processes governed by the ggBm as *Erdélyi–Kober fractional diffusion*.

In order to establish the diffusion-type equation corresponding to (3.8), the Erdélyi–Kober fractional differential operator is here introduced. Let $n - 1 < \mu \leq n$, $n \in \mathbb{N}$. The Erdélyi–Kober fractional derivative is defined as [9, formula (1.5.19)]

$$D_{\eta}^{\gamma, \mu} \varphi(t) = \prod_{j=1}^n \left(\gamma + j + \frac{1}{\eta} t \frac{d}{dt} \right) (I_{\eta}^{\gamma + \mu, n - \mu} \varphi(t)). \quad (3.12)$$

The Riemann–Liouville fractional derivative of order μ , $m - 1 < \mu \leq m$, $m \in \mathbb{N}$ is defined as $D_{RL}^{\mu} \varphi(t) = \frac{d^m}{dt^m} J^{m - \mu} \varphi(t)$, and it emerges that the Erdélyi–Kober and the Riemann–Liouville fractional derivatives are related through the formula

$$D_1^{-\mu, \mu} \varphi(t) = t^{\mu} D_{RL}^{\mu} \varphi(t). \quad (3.13)$$

A further important property of the Erdélyi–Kober fractional derivative is the reduction to the identity operator when $\mu = 0$, that is,

$$D_{\eta}^{\gamma, 0} \varphi(t) = \varphi(t). \quad (3.14)$$

Recently, the notions of Erdélyi–Kober fractional integrals and derivatives have been further extended by Yu. Luchko [13] and Yu. Luchko and J. Trujillo [14].

Equation (3.10) in diffusive form is obtained by deriving in time both sides and it results

$$\begin{aligned} \frac{\partial P}{\partial t} &= \alpha t^{\alpha - 1} I_{\alpha/\beta}^{0, \beta} \frac{\partial^2 P}{\partial x^2} + t^{\alpha} \frac{\partial}{\partial t} \left(I_{\alpha/\beta}^{0, \beta} \frac{\partial^2 P}{\partial x^2} \right) \\ &= t^{\alpha - 1} \left[\alpha + t \frac{\partial}{\partial t} \right] \left(I_{\alpha/\beta}^{0, \beta} \frac{\partial^2 P}{\partial x^2} \right), \end{aligned} \quad (3.15)$$

that can be recast as

$$\frac{\partial P}{\partial t} = \frac{\alpha}{\beta} t^{\alpha - 1} \left[(\beta - 1) + 1 + \frac{\beta}{\alpha} t \frac{\partial}{\partial t} \right] \left(I_{\alpha/\beta}^{0, \beta} \frac{\partial^2 P}{\partial x^2} \right), \quad (3.16)$$

and finally, by using (3.12),

$$\frac{\partial P}{\partial t} = \frac{\alpha}{\beta} t^{\alpha - 1} D_{\alpha/\beta}^{\beta - 1, 1 - \beta} \frac{\partial^2 P}{\partial x^2}. \quad (3.17)$$

A diffusion-type equation for the ggBm was previously proposed [24] but adopting, with an abuse of notation, the Riemann–Liouville fractional differential operator with

a stretched time variable. Then, since the Erdélyi–Kober fractional differential operator is taken into account, equation (3.17) follows to be the correct formulation.

The Green function which corresponds to (3.10, 3.17) is [27, 28, 29, 30]

$$\mathcal{G}(x, t) = \frac{1}{2} \frac{1}{t^{\alpha/2}} M_{\beta/2} \left(\frac{|x|}{t^{\alpha/2}} \right), \quad (3.18)$$

where $M_\nu(z)$ is the M -Wright function, often referred to as Mainardi function in the literature devoted to fractional diffusion [18, 36], and it is defined as [17]

$$\begin{aligned} M_\nu(z) &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma[-\nu n + (1 - \nu)]} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n), \quad 0 < \nu < 1; \end{aligned} \quad (3.19)$$

see [6, 7, 24] for a review. Here, it is reminded the noteworthy composition, or subordination-type, formula [21],

$$t^{-\nu} M_\nu \left(\frac{\xi}{t^\nu} \right) = t^{-\ell} \int_0^\infty M_\lambda \left(\frac{\xi}{\tau^\lambda} \right) M_\ell \left(\frac{\tau}{t^\ell} \right) \frac{d\tau}{\tau^\lambda}, \quad \text{with } \nu = \lambda \ell, \quad (3.20)$$

where $0 < \nu, \lambda, \ell < 1$ and $\xi, t, \tau \in \mathcal{R}_0^+$. By using (3.20) and the special case $M_{1/2}(z) = (1/\sqrt{\pi}) \exp(-z^2/4)$, Green function (3.18) can be expressed as [24, 29, 30]

$$\mathcal{G}(x, t) = \frac{1}{2} \frac{1}{t^{\alpha/2}} M_{\beta/2} \left(\frac{|x|}{t^{\alpha/2}} \right) \quad (3.21)$$

$$= \frac{1}{\sqrt{4t^\alpha}} \int_0^\infty M_{1/2} \left(\frac{|x| t^{-\alpha/2}}{\tau^{1/2}} \right) M_\beta(\tau) d\tau \quad (3.22)$$

$$= \int_0^\infty \frac{1}{\sqrt{4\pi\tau t^\alpha}} \exp \left\{ -\frac{x^2}{4\tau t^\alpha} \right\} M_\beta(\tau) d\tau, \quad (3.23)$$

so that, under the view point of statistical mechanics, the ggBm or the *Erdélyi–Kober fractional diffusion*, emerges to be the superposition of processes with stretched Gaussian density $\frac{1}{\sqrt{4\pi\tau t^\alpha}} \exp \left\{ -\frac{x^2}{4\tau t^\alpha} \right\}$, that is, fractional Brownian motions, whose variance is $\langle x^2 \rangle = 2\tau t^\alpha$ where τ is a random coefficient distributed according to $M_\beta(\tau)$.

However, equation (3.23) can be further remanaged to exhibit a subordination type representation. In fact, after the change of variable $t_* = \tau t^\alpha$, it follows that

$$\mathcal{G}(x, t) = \int_0^\infty \frac{1}{\sqrt{4\pi t_*}} \exp \left\{ -\frac{x^2}{4t_*} \right\} \frac{1}{t^\alpha} M_\beta \left(\frac{t_*}{t^\alpha} \right) dt_*, \quad (3.24)$$

which means that the random trajectory $x = x(t)$ can be obtained as a subordination process by $x = x(t) = y[t_*(t)]$, where $t_* = t_*(t)$ is a positive random variable

that evolves in the natural time t and is referred to as operational time [4, 5]. The process $t_* = t_*(t)$ is the directing process that realizes in the (t, t_*) -plane whose PDF is $t_*^{-\alpha} M_\beta(t_* t_*^{-\alpha})$. Please note that the PDF of the directing process belongs to the same family of the Green function $\mathcal{G}(x, t)$ and they differ for the parameter pair. The process $y = y(t_*)$ is the parent process that is a random trajectory in the (t_*, y) -plane with Gaussian PDF evolving in the operational time t_* . Geometrically, identifying the spatial coordinates y and x , the subordination structure $x = x(t) = y[t_*(t)]$ is obtained by concatenation.

The marginal PDF of the non-Markovian diffusion process ggBm emerges to be related to the Mainardi function M_β and it describes both slow and fast anomalous diffusion. In fact, the variance of the Green function (3.18) is $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \mathcal{G}(x, t) dx = (2/\Gamma(\beta + 1)) t^\alpha$, then the resulting process turns out to be self-similar with Hurst exponent $H = \alpha/2$ and the variance law is consistent with slow diffusion for $0 < \alpha < 1$ and fast diffusion for $1 < \alpha \leq 2$. However, it is worth noting to be remarked also that a linear variance growing is possible, but with non-Gaussian PDF, when $\beta \neq \alpha = 1$, and a Gaussian PDF with nonlinear variance growing when $\beta = 1$ and $\alpha \neq 1$.

3.1.2 The stochastic solution to the Erdélyi–Kober fractional diffusion

Let us consider the generalized grey Brownian motion (ggBm), which is obtained by setting $\ell_\beta = \sqrt{\Lambda_\beta}$,

$$X_{\beta,H}(t) = \sqrt{\Lambda_\beta} X_H(t), \quad (3.25)$$

where the positive random variable Λ_β is distributed according to the one-side M-Wright/Mainardi function $M_\beta(\lambda)$, with $\lambda \geq 0$ and $0 < \beta < 1$. Hence with reference to (2.9), we have that the PDF of $\xi = \delta^2 / \langle \delta^2 \rangle$ is

$$p(\xi) = M_\beta(\xi) = \frac{1}{\beta \xi^{1/\beta+1}} L_\beta^{-\beta} \left(\frac{1}{\xi^{1/\beta}} \right), \quad (3.26)$$

where $L_\beta^{-\beta}(z)$ is an extremal Lévy density with stable parameter $0 < \beta < 1$.

The one-time PDF $\mathcal{P}(x; t)$ results to be

$$\begin{aligned} \mathcal{P}(x; t) &= \frac{1}{\sqrt{4\pi\lambda t^{2H}}} \int_0^\infty \exp\left\{-\frac{x^2}{4\lambda t^{2H}}\right\} M_\beta(\lambda) d\lambda \\ &= \frac{1}{2t^H} M_{\beta/2} \left(\frac{|x|}{t^H} \right), \end{aligned} \quad (3.27)$$

where it emerges that the shape of the displacement PDF is affected by the randomness of the complex medium, here represented by $M_\beta(\lambda)$. $\mathcal{P}(x; t)$ can also be expressed in

terms of the H-function [22, 19]

$$\mathcal{P}(x; t) = \frac{1}{2t^H} H_{01}^{10} \left[\frac{|x|}{t^H} \left| \begin{array}{c} - \\ (0, 1) \end{array} \right. ; \begin{array}{c} (1 - \beta/2, \beta/2) \\ - \end{array} \right], \quad (3.28)$$

and the asymptotic decay is $M_{\beta/2}(|x| \rightarrow \infty) \sim |x|^{\frac{c}{2}(\beta-1)} e^{-b|x|^c}$, with $b = \frac{2^{1-c}}{c} \beta^{\beta c/2}$ and $c = \frac{2}{2-\beta}$ [20, 19].

The variance can be calculated as

$$\langle X_{\beta,H}^2 \rangle = \frac{2}{\Gamma(1+\beta)} t^{2H}, \quad (3.29)$$

showing that the power law dependence of the particle displacement variance is not affected by the medium properties. It is noteworthy to observe that the ggBm can show both subdiffusion ($0 < H < 1/2$) and superdiffusion ($1/2 < H < 1$). Moreover, a remarkable case is obtained for $H = 1/2$, where the particle variance results to be linear in time (3.29) but, according to (3.27), the density function is not Gaussian. Gaussian density can be obtained from (3.27) as a special case when $\beta = 1$.

We remark that fractional kinetics, that is, $\beta \neq 1$, is directly associated to the occurrence of EB through the randomness of $\ell_\beta = \sqrt{\Lambda_\beta}$, being $M_{\beta \neq 1}(\lambda) \neq \delta(\lambda - 1)$. Moreover, the fractional order related to β can be experimentally computed by means of the long-time limit of $EB(T)$. In fact, for large T , from (2.11) and $\ell_\beta = \sqrt{\Lambda_\beta}$ the limit of $EB_{\text{ggbm}}(t)$ results to be

$$EB_{\text{ggbm}}(T) \xrightarrow{T \rightarrow \infty} \frac{\langle \Lambda_\beta^2 \rangle}{\langle \Lambda_\beta \rangle^2} - 1 = \beta \frac{\Gamma(\beta)\Gamma(\beta)}{\Gamma(2\beta)} - 1. \quad (3.30)$$

Interestingly, formula (3.30) provides the same expression obtained considering a Continuous Time Random Walk (CTRW) with infinite average sojourn time and power-law distribution of waiting times, that is, $\psi(\tau) \propto \tau^{-(1+\beta)}$, $0 < \beta < 1$ [8, 31].

3.1.3 Some special cases of EK fractional diffusion

(i) Fractional Brownian motion

In the case $\beta = 1$, we have $M_1(\lambda) = \delta(\lambda - 1)$ and the Λ_β turns to be a non-random length scale, that is, $\Lambda_\beta = 1$.

(ii) Brownian motion

The Brownian motion is recovered as a special case of the fBm, by setting $\beta = 2H = 1$.

(iii) Time-fractional diffusion equation

In the special case $0 < \beta = 2H < 1$, we obtain the grey Brownian motion whose PDF is the solution of the time-fractional diffusion equation, that is,

$$\mathcal{P}(x; t) = \frac{1}{2t^{\beta/2}} M_{\beta/2} \left(\frac{|x|}{t^{\beta/2}} \right), \quad \langle x^2 \rangle \simeq t^\beta. \quad (3.31)$$

(iv) Brownian–non-Gaussian: M function

When $H = 1/2$, the linear dependence on time of the particle variance is obtained, but the PDF is not Gaussian, that is,

$$\mathcal{P}(x; t) = \frac{1}{2t^{1/2}} M_{\beta/2} \left(\frac{|x|}{t^{1/2}} \right), \quad \langle x^2 \rangle \simeq t. \quad (3.32)$$

(v) Brownian–non-Gaussian: exponential PDF

In particular, when $H = 1/2$ and $\beta = 0$, we get Brownian diffusion and the following exponential PDF:

$$\mathcal{P}(x; t) = \frac{1}{2t^{1/2}} e^{-|x|/t^{1/2}}, \quad \langle x^2 \rangle \simeq t, \quad (3.33)$$

as follows from the special case $M_0(z) = \exp(-|z|)$.

3.2 Space-time fractional diffusion

3.2.1 The fundamental solution to the space-time fractional diffusion equation

The space-time fractional diffusion equation reads

$${}_t D_*^\beta u(x, t) = {}_x D_\theta^\alpha u(x, t), \quad (3.34)$$

with

$$u(x, 0) = \delta(x), \quad u(\pm\infty, t) = 0, \quad -\infty < x < +\infty, \quad t \geq 0, \quad (3.35)$$

Equation (3.34) is obtained from the ordinary diffusion equation by replacing the first-order time derivative and the second-order space derivative with the Caputo time-fractional derivative ${}_t D_*^\beta$, of real order β , and the Riesz–Feller space-fractional derivative ${}_x D_\theta^\alpha$, of real order α and asymmetry parameter θ , respectively. The real parameters α , θ , and β are restricted as follows:

$$0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}, \quad 0 < \beta \leq 1 \text{ or } 1 < \beta \leq \alpha \leq 2. \quad (3.36)$$

The general solution of (3.34) can be represented as

$$u(x, t) = \int_{-\infty}^{+\infty} K_{\alpha, \beta}^\theta(x - \xi, t) u(\xi, 0) d\xi, \quad (3.37)$$

where $K_{\alpha, \beta}^\theta$ is the fundamental solution, or Green function, which is obtained by setting in (3.35) the initial condition $u(x, 0) = \delta(x)$. An important integral representation formula for the fundamental solution $K_{\alpha, \beta}^\theta(z)$ is [20, 21]:

$$K_{\alpha, \beta}^\theta(z) = \alpha \int_0^\infty \xi^{\alpha-1} M_\beta(\xi^\alpha) L_\alpha^\theta(z/\xi) \frac{d\xi}{\xi}, \quad 0 < \beta \leq 1, \quad (3.38)$$

where $L_\alpha^\theta(z)$ is the Lévy stable density and $M_\beta(\xi)$, $0 < \beta < 1$, is the M-Wright/Mainardi function. By replacing in (3.38) z with $x/t^{\beta/\alpha}$, and after the changes of variable $\xi = \tau^{1/\alpha}/t^{\beta/\alpha}$, it holds

$$t^{-\beta/\alpha} K_{\alpha,\beta}^\theta\left(\frac{x}{t^{\beta/\alpha}}\right) = K_{\alpha,\beta}^\theta(z) = \int_0^\infty \frac{1}{t^\beta} M_\beta\left(\frac{\tau}{t^\beta}\right) \frac{1}{\tau^{1/\alpha}} L_\alpha^\theta\left(\frac{x}{\tau^{1/\alpha}}\right) d\tau, \quad (3.39)$$

or

$$K_{\alpha,\beta}^\theta(x, t) = \int_0^\infty M_\beta(\tau, t) L_\alpha^\theta(x, \tau) d\tau, \quad 0 < \beta \leq 1. \quad (3.40)$$

By using formula (3.38), another integral representation formula for the space-time fractional diffusion can be derived [34]. Let $K_{\alpha,\beta}^\theta(x, t)$ be the fundamental solution of the space-time fractional diffusion equation (3.34) with initial and boundary conditions $u(x, 0) = \delta(x)$ and $u(\pm\infty, t) = 0$ and parameters α, θ, β such that $0 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2 - \alpha\}$, $0 < \beta \leq 1$, then the following integral representation formula holds true for $0 < x < \infty$:

$$K_{\alpha,\beta}^\theta(x, t) = \int_0^\infty L_\eta^\gamma(x, \xi) K_{\nu,\beta}^{-\nu}(\xi, t) d\xi, \quad (3.41)$$

with

$$\alpha = \eta\nu, \quad \theta = \gamma\nu,$$

and

$$0 < \eta \leq 2, \quad |\gamma| \leq \min\{\eta, 2 - \eta\}, \quad 0 < \nu \leq 1.$$

In the particular case $\eta = 2$ and $\gamma = 0$, so that $\nu = \alpha/2$ and $\theta = 0$, the spatial variable x emerges to be distributed according to a Gaussian probability density and the integral representation formula (3.41) becomes

$$K_{\alpha,\beta}^0(x, t) = \int_0^\infty G(x, \xi) K_{\alpha/2,\beta}^{-\alpha/2}(\xi, t) d\xi, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 1. \quad (3.42)$$

3.2.2 The stochastic solution to the space-time fractional diffusion

Then identifying functions and parameters as follows:

$$p(z) \equiv K_{\alpha,\beta}^0(z), \quad p_1(z_1) \equiv G(z_1), \quad p_2(z_2) \equiv K_{\alpha/2,\beta}^{-\alpha/2}(z_2), \quad (3.43)$$

$$\gamma = \frac{1}{2}, \quad \omega = \frac{2\beta}{\alpha}, \quad \gamma\omega = \frac{\beta}{\alpha}, \quad (3.44)$$

formula (2.3) reduces to the new integral representation formula (3.42) for the symmetric space-time fractional diffusion equation.

In terms of random variables, it follows that [35, 34]

$$Z = Xt^{-\beta/\alpha} \quad \text{and} \quad Z = Z_1 Z_2^{1/2}, \quad (3.45)$$

hence it holds

$$X = Zt^{\beta/\alpha} = Z_1 t^{\beta/\alpha} Z_2^{1/2} = G_{2\beta/\alpha}(t) \sqrt{\Lambda_{\alpha/2,\beta}}. \quad (3.46)$$

Since $p_1(z_1) \equiv G(z_1)$, Z_1 is a Gaussian random variable. Consequently, the stochastic process $Z_1 t^{\beta/\alpha} = G_{2\beta/\alpha}(t)$ is a Gaussian process displaying anomalous diffusion. Further, the random variable $Z_2 = \Lambda_{\alpha/2,\beta}$ is distributed according to $p_2(z_2) \equiv K_{\alpha/2,\beta}^{-\alpha/2}(z_2)$.

In summary, we define the following class of processes: Let $X_{\alpha,\beta}(t)$, $t \geq 0$, be a process defined by

$$X_{\alpha,\beta}(t) = \sqrt{\Lambda_{\alpha/2,\beta}} G_{2\beta/\alpha}(t), \quad 0 < \beta \leq 1, \quad 0 < \alpha \leq 2, \quad (3.47)$$

where the stochastic process $G_{2\beta/\alpha}(t)$ is a Gaussian process with power law variance $t^{2\beta/\alpha}$ and $\Lambda_{\alpha/2,\beta}$ is an independent constant non-negative random variable distributed according to the PDF $K_{\alpha/2,\beta}^{-\alpha/2}(\lambda)$, $\lambda \geq 0$, that is a special case of (3.40). The parametric class of Gaussian-based stochastic processes $X_{\alpha,\beta}(t)$ defined in (3.47), and depending on the parameters $0 < \beta \leq 1$ and $0 < \alpha \leq 2$, is a class of stochastic solutions to the space-time fractional diffusion equation (3.34) in the symmetric case. This means that the one-time one-point PDF of $X_{\alpha,\beta}(t)$ is the fundamental solution of equation (3.34) in the symmetric case, namely the PDF $K_{\alpha,\beta}^0(x, t)$ defined in (3.42). The stochastic process $X_{\alpha,\beta}(t)$ stated in (3.47) generalizes the Gaussian processes, which are recovered when $\alpha = 2$ and $\beta = 1$. Similar to the Gaussian process, even this process is uniquely determined by the mean and the autocovariance structure. This property directly follows from the fact that $G_{2\beta/\alpha}(t)$ is a Gaussian stochastic process and $\Lambda_{\alpha/2,\beta}$ is an independent constant non-negative random variable. For example, if we chose the fractional Brownian motion as the Gaussian process, the process $X_{\alpha,\beta}$ has stationary increments [34], by constraints of parameters α and β in the intervals $0 < \beta \leq 1$ and $0 < \beta < \alpha \leq 2$.

The simulation of such a stochastic process with stationary increments whose PDF is the solution of the space-time fractional diffusion equation can be performed by implementing the following process:

$$X_{\alpha,\beta} = \sqrt{\mathcal{L}_{\alpha/2}^{\text{ext}}} \cdot [\mathcal{L}_{\beta}^{\text{ext}}]^{-\beta/\alpha} X_{\beta/\alpha}(t), \quad (3.48)$$

where β , such that $0 < \beta < 1$, is the fractional order of derivation in time and α , such that $0 < \beta < \alpha \leq 2$, is the fractional order of derivation in space. The process $X_{\beta/\alpha}(t)$ is a fBm with Hurst exponent $H = \beta/\alpha$ and $\mathcal{L}_{\mu}^{\text{ext}}$ is a positive random variable distributed according to an extremal one-side Lévy density $L_{\mu}^{-\mu}(z)$ with stable parameter $0 < \mu < 1$.

3.2.3 The symmetric space-fractional case

We know that $X_{\alpha,\beta}(t)$ defined in (3.47), and depending on the parameters $0 < \beta \leq 1$ and $0 < \alpha \leq 2$, is a class of stochastic solutions to the space-time fractional diffusion equation (3.34) in the symmetric case but we are interested to consider also the particular case of symmetric space-fractional diffusion equation. This means that the process $X_{\alpha,\beta}(t)$ becomes

$$X_{\alpha,1}(t) = \sqrt{\Lambda_{\alpha/2,1}} G_{2/\alpha}(t), \quad 0 < \alpha < 2. \quad (3.49)$$

From the equation (3.46), we know that the random variable $\Lambda_{\alpha/2,\beta}$ is distributed according to $K_{\alpha/2,\beta}^{-\alpha/2}(x, t)$. Moreover, by relation (3.40), that is,

$$K_{\alpha,\beta}^\theta(x, t) = \int_0^\infty M_\beta(\tau, t) L_\alpha^\theta(x, \tau) d\tau, \quad (3.50)$$

in the space-fractional case $\beta = 1$, we have

$$\begin{aligned} K_{\alpha/2,1}^{-\alpha/2}(x, t) &= \int_0^\infty M_1(\tau, t) L_{\alpha/2}^{-\alpha/2}(x, \tau) d\tau \\ &= \int_0^\infty \delta(t - \tau) L_{\alpha/2}^{-\alpha/2}(x, \tau) d\tau \\ &= L_{\alpha/2}^{-\alpha/2}(x, t). \end{aligned} \quad (3.51)$$

Then we have shown that $\Lambda_{\alpha/2,1}$ is distributed according $L_{\alpha/2}^{-\alpha/2}(x, t)$, but we are interested in the distribution of $\sqrt{\Lambda_{\alpha/2,1}}$. For this goal, we use the following theorem. Let X be a continuous random variable having PDF f_X . Suppose that $g(x)$ is a strictly monotonic (increasing or decreasing), differentiable (and thus continuous) function of x . Then the random variable Y defined by $Y = g(X)$ has a PDF given by

$$f_Y(y) = \begin{cases} f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y = g(y) \text{ for some } x, \\ 0 & \text{if } y \neq g(y) \text{ for all } x, \end{cases} \quad (3.52)$$

where $g^{-1}(y)$ is defined to equal that value of x such that $g(x) = y$. Considering the functions in the theorem as

$$\sqrt{y} \equiv g(y), \quad y^2 \equiv g^{-1}(y), \quad f_X \equiv L_{\alpha/2}^{-\alpha/2}, \quad (3.53)$$

we have that the PDF of $\sqrt{\Lambda_{\alpha/2,1}}$ is

$$q(\ell) = 2\ell L_{\alpha/2}^{-\alpha/2}(\ell^2). \quad (3.54)$$

3.2.4 The symmetric time-fractional case

The stochastic process whose PDF is the solution of the symmetric time-fractional diffusion equation is actually the gBm proposed by Schneider [37, 38].

3.2.5 Some special cases of the space-time fractional diffusion

(i) Brownian motion

In the case $\beta = 1$ and $\alpha = 2$, the classical Brownian motion is recovered, because $H = 1/2$ and each one of the two Lévy variables result to be distributed according to a delta function.

(ii) Time-fractional diffusion: $0 < \beta < 1$

In the special case $\alpha = 2$, we obtain the grey Brownian motion whose PDF is the solution of the time-fractional diffusion equation.

(iii) Space-fractional diffusion: $1 < \alpha < 2$

In the special case $\beta = 1$ and then $1 < \alpha < 2$, the PDF of the resulting process is the solution of a space-fractional diffusion equation,

$$\mathcal{P}(x; t) = \frac{1}{t^{1/\alpha}} L_\alpha \left(\frac{x}{t^{1/\alpha}} \right). \quad (3.55)$$

(iv) Neutral diffusion: $0 < \alpha = \beta < 1$

In the special case $0 < \alpha = \beta < 1$, the PDF of the resulting process is the solution of the neutral fractional diffusion equation, and this PDF is a generalization of the Cauchy/Lorentz distribution:

$$\mathcal{P}(x; t) = \frac{1}{\pi} \frac{(x/t)^{\alpha-1} \sin[\alpha\pi/2]}{1 + 2(x/t)^\alpha \cos[\alpha\pi/2] + (x/t)^{2\alpha}}. \quad (3.56)$$

4 Stochastic processes with nonstationary increments

4.1 Definitions

The stochastic process under consideration is [25]

$$X(t) = \sqrt{t^\alpha \Lambda_\beta} X_H(t), \quad t \geq 0, \quad \alpha \geq -1, \quad (4.1)$$

with $X(0) = 0$. Here, $X_H(t)$, $0 < H < 1$, is the fBm whose displacement variance, or ensemble-average MSD (EAMSD), and correlation function are [1]

$$\langle X_H^2(t) \rangle = 2D_H t^{2H}, \quad (4.2)$$

$$\langle X_H(t) X_H(s) \rangle = D_H (t^{2H} + s^{2H} - |t - s|^{2H}), \quad (4.3)$$

and Λ_β is an independent positive random variable distributed according to the one-side M-Wright/Mainardi function, with $0 < \beta < 1$, and its mean is $\langle \Lambda_\beta \rangle = 1/\Gamma(1 + \beta)$. Note that when $\alpha = 0$, the stochastic process (4.1) reduces to the ggBm [29].

It is specially remarkable the fact that every parameter of the process, that is, H , β , and α , has a clear physical meaning. In particular, we show in the following that H controls the exponent of the power-law of the variance although it is affected by α . In the case of an underlying Gaussian diffusion, that is, $H = 1/2$, subdiffusion occurs when $\alpha < 0$ and superdiffusion when $\alpha > 0$. β is the only parameter which controls the ergodicity/nonergodicity transition. α determines the power-law of aging.

Then the EAMSD of the process $X(t)$ stated in (4.1) results to be

$$\langle X^2 \rangle = \frac{2D_H}{\Gamma(1 + \beta)} t^{2H+\alpha}, \quad (4.4)$$

where the power-law of the growing in time is actually that of the fBm but affected by the parameter α with an increasing or decreasing effect according to its sign.

4.2 Probability density function

The PDF of the process results to be

$$\mathcal{P}(x; t) = \frac{1}{2t^{H+\alpha/2}} M_{\beta/2} \left(\frac{|x|}{t^{H+\alpha/2}} \right). \quad (4.5)$$

Special cases follow from special cases of the PDF and they are similar to the special cases of the ggBm. In particular, it is here we report the following:

- (a) Brownian diffusion $\langle x^2 \rangle \sim t$ is obtained when $\alpha = -2H$ and that means, in the case of the fBm, that the process $X_{\alpha/2}(t)$ is not the Wiener process but it shows a certain correlation.
- (b) Brownian diffusion $\langle x^2 \rangle \sim t$ with an exponential PDF is obtained in the case $\alpha = -2H$ and $\beta = 0$.

4.3 Ergodicity breaking

In order to investigate the ergodicity of the process, we study in the following the so-called Ergodicity Breaking (EB) parameter defined as

$$\text{EB} = \lim_{T \rightarrow \infty} \frac{\overline{\langle (\delta_X^2(t_a, T, \Delta))^2 \rangle}}{\langle \delta_X^2(t_a, T, \Delta) \rangle^2} - 1, \quad (4.6)$$

where

$$\overline{\delta_X^2(t_a, T, \Delta)} = \frac{\int_{t_a}^{t_a+T-\Delta} [X(\tau + \Delta) - X(\tau)]^2 d\tau}{T - \Delta} \quad (4.7)$$

is the temporal-average MSD (TAMSD) and T , Δ , and t_a stand for the measurement time, lag time, and aging time, respectively.

Using the definition of TAMSD, and formulae (4.2) and (4.3), we obtain the ensemble-average TAMSD (EATAMSD)

$$\langle \overline{\delta_X^2(t_a, T, \Delta)} \rangle = \frac{2D_H \int_{t_a}^{t_a+T-\Delta} f(\tau, \Delta) d\tau}{(T - \Delta)\Gamma(1 + \beta)}, \quad (4.8)$$

where

$$f(\tau, \Delta) = (\tau + \Delta)^{2H+\alpha} + \tau^{2H+\alpha} - \tau^{\alpha/2}(\tau + \Delta)^{\alpha/2}[\tau^{2H} + (\tau + \Delta)^{2H} - \Delta^{2H}].$$

We introduce the Taylor expansion of $f(\tau, \Delta)$ with $\Delta/\tau \ll 1$,

$$f(\tau, \Delta) \underset{\Delta \ll \tau}{=} \tau^\alpha \Delta^{2H} + O((\Delta/\tau) \wedge (\Delta/\tau)^{2-2H}), \quad (4.9)$$

where $a \wedge b = \min\{a, b\}$.

If $-1 < \alpha$, then the integration in (4.8) reads

$$\int_{t_a}^{t_a+T-\Delta} f(\tau, \Delta) d\tau \simeq \Delta^{2H} \frac{(t_a + T - \Delta)^{\alpha+1} - t_a^{\alpha+1}}{\alpha + 1},$$

and the EATAMSD can be approximated by [25]

$$\langle \overline{\delta_X^2(t_a, T, \Delta)} \rangle \underset{\Delta \ll t_a \ll T}{\simeq} \frac{2D_H \Delta^{2H} T^\alpha}{\Gamma(1 + \beta)(1 + \alpha)}. \quad (4.10)$$

This means that as long as the limit $T \rightarrow \infty$ holds, EATAMSD follows a power-law with exponent α , growing infinitely for positive α and tending to 0 for negative $\alpha \in (-1, 0)$. This provides aging effects in the trajectories generated by the proposed stochastic process and relates it with existing studies on diffusion in biological systems [39].

When $\alpha = 0$ and $\Lambda_\beta = 1$, the result obtained in [1] for the fBm is recovered

$$\langle \overline{\delta_{X_H}^2(t_a, T, \Delta)} \rangle = 2D_H \Delta^{2H}.$$

In the special case of $\alpha = -1$, the EATAMSD also tends to zero but with the following law [25]:

$$\langle \overline{\delta_X^2(t_a, T, \Delta)} \rangle \underset{\Delta \ll t_a \ll T}{\simeq} \frac{2D_H \Delta^{2H}}{\Gamma(1 + \beta)} \frac{\ln T}{T}. \quad (4.11)$$

Since the limit in (4.6) does not depend on the value of t_a , to simplify notation, hereinafter we consider $\overline{\delta_X^2(T, \Delta)} \equiv \overline{\delta_X^2(0, T, \Delta)}$.

With reference to (4.6), we now consider

$$\langle (\overline{\delta_X^2(T, \Delta)})^2 \rangle = \frac{\int_0^{T-\Delta} d\tau_1 \int_0^{T-\Delta} d\tau_2 G(\tau_1, \tau_2)}{(T - \Delta)^2},$$

where

$$G(\tau_1, \tau_2) = \langle [X(\tau_1 + \Delta) - X(\tau_1)]^2 [X(\tau_2 + \Delta) - X(\tau_2)]^2 \rangle. \quad (4.12)$$

Using the following formula for Gaussian process with zero mean [11]:

$$\begin{aligned} \langle x(t_1) x(t_2) x(t_3) x(t_4) \rangle &= \langle x(t_1) x(t_2) \rangle \langle x(t_3) x(t_4) \rangle \\ &\quad + \langle x(t_1) x(t_3) \rangle \langle x(t_2) x(t_4) \rangle \\ &\quad + \langle x(t_1) x(t_4) \rangle \langle x(t_2) x(t_3) \rangle, \end{aligned}$$

it results that

$$G(\tau_1, \tau_2) \underset{\Delta \ll \tau_1, \tau_2}{\approx} G_0,$$

where

$$\begin{aligned} G_0 &= 4D_H^2 \Delta^{4H} \langle \Lambda_\beta^2 \rangle [\tau_1 \tau_2 (\tau_1 + \Delta) (\tau_2 + \Delta)]^{\alpha/2} \\ &\quad + 2D_H^2 \langle \Lambda_\beta^2 \rangle \{ |\tau_1 + \Delta - \tau_2|^{2H} \tau_2^{\alpha/2} (\tau_1 + \Delta)^{\alpha/2} \\ &\quad - |\tau_1 - \tau_2|^{2H} [(\tau_1 \tau_2)^{\alpha/2} + (\tau_1 + \Delta)^{\alpha/2} (\tau_2 + \Delta)^{\alpha/2}] \\ &\quad + |\tau_2 + \Delta - \tau_1|^{2H} \tau_1^{\alpha/2} (\tau_2 + \Delta)^{\alpha/2} \}^2. \end{aligned} \quad (4.13)$$

After the approximation,

$$(\tau_i + \Delta)^{\alpha/2} \approx \tau_i^{\alpha/2} \left(1 + \frac{\alpha}{2} \frac{\Delta}{\tau_i} \right), \quad i = \overline{1, 2},$$

we have

$$\begin{aligned} G(\tau_1, \tau_2) &= \langle \Lambda_\beta^2 \rangle \tau_1^\alpha \tau_2^\alpha [4D_H^2 \Delta^{4H} + 2D_H^2 \{ |\tau_1 + \Delta - \tau_2|^{2H} \\ &\quad + |\tau_2 + \Delta - \tau_1|^{2H} - 2|\tau_1 - \tau_2|^{2H} \}^2], \end{aligned}$$

and finally

$$\langle (\overline{\delta_X^2(T, \Delta)})^2 \rangle = \frac{\langle \Lambda_\beta^2 \rangle 4D_H^2}{(T - \Delta)^2} \times \left\{ \int_0^{T-\Delta} \int_0^{T-\Delta} d\tau_1 d\tau_2 \tau_1^\alpha \tau_2^\alpha \Delta^{4H} + \frac{I}{2} \right\}, \quad (4.14)$$

where

$$I = \int_0^{T-\Delta} \int_0^{T-\Delta} d\tau_1 d\tau_2 \tau_1^\alpha \tau_2^\alpha \{ |\tau_1 - \tau_2 + \Delta|^{2H} + |\tau_2 - \tau_1 + \Delta|^{2H} - 2|\tau_1 - \tau_2|^{2H} \}^2.$$

According to the value of H , the integral I follows two different power-laws of Δ and T [25]:

$$I \sim \begin{cases} \Delta^{4H+1} T^{2\alpha+1}, & 0 < H < \frac{3}{4}, \\ \Delta^4 T^{4H+2\alpha-2}, & \frac{3}{4} < H < 1. \end{cases} \quad (4.15)$$

The EB parameter can now be written as

$$\text{EB} = \lim_{T \rightarrow \infty} \frac{\langle \Lambda_\beta^2 \rangle}{\langle \Lambda_\beta \rangle^2} \left[1 + \frac{(1+\alpha)^2 I}{2\Delta^{4H} T^{2\alpha+2}} \right] - 1. \quad (4.16)$$

Taking into account (4.15), it holds

$$\text{EB} = \frac{\langle \Lambda_\beta^2 \rangle}{\langle \Lambda_\beta \rangle^2} - 1 = \frac{\beta \Gamma^2(\beta)}{\Gamma(2\beta)} - 1. \quad (4.17)$$

The second equality in (4.17) is obtained from the Mellin transform of M_β , that is, $\int_0^\infty \lambda^{s-1} M_\beta(\lambda) d\lambda = \Gamma[1+(s-1)]/\Gamma[1+\beta(s-1)]$, $s > 0$ [21]. We have that $\text{EB} \neq 0$, $\forall \beta \in (0, 1)$ and only in the limit $\beta \rightarrow 1$ the process becomes ergodic.

5 Conclusions and final remarks

In the present chapter, we have presented a modeling approach for fractional diffusion based on randomly-scaled Gaussian processes. When the Gaussian process is the fBm and the scales are distributed according to the M-Wright/Mainardi function, the ggBm is recovered.

The main features of this approach is that the distribution of the scales leads to non-Gaussian densities of particle displacement and then for proper choice of scale fluctuations fractional diffusion is modeled. Moreover, the fluctuations of the scale leads also to the ergodicity breaking. Hence, we observe that since the distribution of the scale fluctuations causes non-Gaussianity and ergodicity breaking we claim that fractional diffusion is a specific ergodicity breaking due to the population of scales that characterizes the random/complex medium where diffusion occurs.

With reference to definition (2.1), it is here reported that the formalism holds for any choice of $\ell_\beta = \ell_\beta(t)$, with $\langle \ell_\beta^2 \rangle \sim t^\alpha$, and of any Gaussian processes such that

$$\langle X^2 \rangle = \langle \ell_\beta^2 \rangle \langle X_G^2 \rangle \sim t^\alpha \langle X_G^2 \rangle \sim t^\gamma, \quad (5.1)$$

where α is equal to 0 or not if the process has stationary or nonstationary increments, respectively, and γ corresponds to the desired power law for the particle variance. In particular, when $\ell_\beta(t) = \sqrt{t^\alpha \Lambda_\beta}$ the process defined in (4.1) is recovered. The choice of

the fBm as Gaussian process, that is, $X_G(t) = X_H(t)$, is only the most simple choice for meeting this constraint.

The present approach has been used to model anomalous diffusion in biological systems [26]. In particular, the proposed stochastic process interpolates between the fBm and the CTRW, generating nonergodicity and nontrivial p-variation, as it is observed in experimental data [15]. Recently, it has been compared against the so-called approach *diffusion diffusivity* in [43], and again in view of the application in biological systems.

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