ON THE ABSOLUTE DIVERGENCE OF FOURIER SERIES IN THE INFINITE DIMENSIONAL TORUS

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Abstract. In this note we present some simple counterexamples, based on quadratic forms in infinitely many variables, showing that the implication $f \in C^{(\infty}(\mathbb{T}^\omega) \Longrightarrow \sum_{\bar{p}\in\mathbb{Z}^\infty} |\widehat{f}(\bar{p})| < \infty$ is false. There are functions of the class $C^{(\infty}(\mathbb{T}^\omega)$ (depending on an infinite number of variables) whose Fourier series diverges absolutely. This fact establishes a significant difference to what happens in the finite dimensional case.

1. Introduction

In the following result it is established a sufficient condition of smoothness on a function defined on the n-dimensional torus $\mathbb{T}^n$ $(n \geq 1)$ for the absolute convergence of its Fourier series:

Theorem 1.1 ([13, p. 249])**.** *If $f \in C^{(k}(\mathbb{T}^n)$, $k > n/2$, then*

$$\sum_{m\in\mathbb{Z}^n} \left|\widehat{f}(m)\right| < \infty.$$

When $f \in C^{(\infty}(\mathbb{T}^n)$, more conclusive results are verified, for example (see [12, Th. 7.25, p. 202]):

Theorem 1.2. *If $f \in C^{(\infty}(\mathbb{T}^n)$, then*

$$\sum_{m\in\mathbb{Z}^n} (1+|m|)^N \left|\widehat{f}(m)\right| < \infty \quad \forall N = 0,1,\dots, \quad |m| = \left(\textstyle\sum_{i=1}^n m_i^2\right)^{1/2}.$$

This same result holds for *cylindrical* infinitely smooth functions defined on the infinite dimensional torus $\mathbb{T}^\omega$, which is the compact abelian group consisting of the complete direct sum of countably many copies of $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$. Let us remember that $f(x)$ is a cylindrical function on $\mathbb{T}^\omega$ if f depends only on a finite number of variables, i.e., if $\exists\, g_n\colon \Omega_n \to \mathbb{C}$, with $\Omega_n \subseteq \mathbb{T}^n$, such that $f = g_n \circ \pi_n$, being $\pi_n\colon \mathbb{T}^\omega \to \mathbb{T}^n$ the canonical projection and $n \geq 1$. The space of cylindrical functions of the class $C^{(\infty}$ in $\mathbb{T}^\omega$ (see Definition 2.3)

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is defined ([1, p. 73–75]) by

$$\mathcal{D}(\mathbb{T}^\omega) = \bigcup_{n=1}^{\infty} \{ g_n \circ \pi_n | g_n \in C^{(\infty}(\mathbb{T}^n) \}$$

so that, if $f \in \mathcal{D}(\mathbb{T}^\omega)$, then there exists $p \in \mathbb{N}$ and $g_p \in C^{(\infty}(\mathbb{T}^p)$ such that $f = g_p \circ \pi_p$.

The dual group of $\mathbb{T}^\omega$, denoted by $\mathbb{Z}^\infty$, is the direct sum of countably many copies of $\mathbb{Z}$, formed by the finitely nonzero sequences of integer numbers. Denote by dx the normalized Haar measure in $\mathbb{T}^\omega$. If $f \in L^1(\mathbb{T}^\omega)$, then the function $\widehat{f}$ defined on $\mathbb{Z}^\infty$ by

$$\widehat{f}(\bar{n}) = \int_{\mathbb{T}^\omega} f(x) e^{-2\pi i \bar{n}\cdot x}\, dx \qquad (\bar{n} \in \mathbb{Z}^\infty)$$

is the *Fourier transform* of f, the *Fourier series* of f being the formal series (observe that $\mathbb{Z}^\infty$ is a countable set)

$$\sum_{\bar{n} \in \mathbb{Z}^\infty} \widehat{f}(\bar{n}) e^{2\pi i \bar{n}\cdot x}.$$

By using the ideas in the proof of Theorem 1.2, the following result could be proved (see also [2, Proposition 1]):

Theorem 1.3. *If $\phi \in \mathcal{D}(\mathbb{T}^\omega)$, then*

$$\sum_{\bar{p} \in \mathbb{Z}^\infty} (1 + |\bar{p}|)^N \big|\widehat{\phi}(\bar{p})\big| < \infty \quad \forall N = 0, 1, \dots, \quad |\bar{p}| = \Big(\textstyle\sum_{i=1}^\infty p_i^2 \Big)^{1/2}.$$

In May 2016, in a private communication to the second author, Professor A. D. Bendikov conjectured that the implication $f \in C^{(\infty}(\mathbb{T}^\omega) \Rightarrow \sum_{\bar{p} \in \mathbb{Z}^\infty} |\widehat{f}(\bar{p})| < \infty$, which holds, as we have said, for functions depending only on a finite number of variables, is in general false. This fact establishes a significant difference to what happens in the finite dimensional case.

In order to support the statement of Bendikov, we show in this note some counterexamples via quadratic forms depending on an infinite number of variables. The construction of such counterexamples is based on classical results of Toeplitz [14], Littlewood [10] and Bohnenblust and Hille [4][1].

The main result in this note is the following.

Theorem 1.4. *There exist functions of the class $C^{(\infty}(\mathbb{T}^\omega)$ (depending on an infinite number of variables) whose Fourier series diverges absolutely.*

[1]In the same private communication, Prof. Bendikov suggested that a counterexample could be constructed through an appropriate Jacobi Theta function in an infinite number of variables. The construction via quadratic forms that we show in the present note does not follow the path indicated by Bendikov.

Although we restrict ourselves to the case of the infinite dimensional torus, we point out that Bendikov and L. Saloff-Coste have studied, in [3], several scales of smooth functions in the more general setting of connected infinite-dimensional compact groups.

In Section 2 we introduce some definitions and show several basic results. We present in Section 3 a detailed account of the bilinear and quadratic forms in infinite number of variables used to construct our counterexamples. The proof of Theorem 1.4 and the counterexamples are contained in Section 4.

2. Premilinary definitions and results

We will begin by providing some basic principles.

Definition 2.1 ([5, p. 130])**.** The function $f\colon \mathbb{T}^\omega \to \mathbb{C}$ is *continuous at the point* $x^{(0)} = (x_1^0, x_2^0, \ldots)$ if $\forall \varepsilon > 0$ there is a positive integer m and a number $\delta > 0$ such that, in each point $(x_1, x_2, \ldots) \in \mathbb{T}^\omega$ verifying the m inequalities $\left|x_j - x_j^0\right| < \delta$ $(j = 1, 2, \ldots, m)$, it holds

$$\left|f(x_1, x_2, \ldots) - f(x_1^0, x_2^0, \ldots)\right| < \varepsilon.$$

Since $\mathbb{T}^\omega$ is compact, the vector space

$$C^{(0}(\mathbb{T}^\omega) = \{f\colon \mathbb{T}^\omega \to \mathbb{C} \mid f \text{ is continuous at all } x \in \mathbb{T}^\omega\}$$

is a Banach space with the norm $\|f\|_\infty = \max_{x\in\mathbb{T}^\omega} |f(x)|$.

Lemma 2.2. Let $\varphi(t) \in C^{(0}(\mathbb{T})$ and $\sum_{j=1}^\infty a_j$ be an absolutely convergent series of complex numbers. Then, the function $\Psi(x) = \sum_{j=1}^\infty a_j \varphi(x_j)$ is continuous on $\mathbb{T}^\omega$.

Proof. We can suppose that φ is not zero, otherwise the statement would be trivial. In which follows, let $x^{(0)} \in \mathbb{T}^\omega$ be fixed. Given $\varepsilon > 0$, since the series $\sum_{j=1}^\infty |a_j|$ converges, there exists $m_1 \in \mathbb{N}$ such that

$$\sum_{j=m_1+1}^{N} |a_j| < \frac{\varepsilon}{4\,\|\varphi\|_\infty}$$

for all $N > m_1$. On the other hand, for each $j = 1, \ldots, m_1$, the continuity of φ in $x_j^{(0)}$ ensures the existence of a number $\delta_j > 0$ such that, if $\left|x_j - x_j^{(0)}\right| < \delta_j$, then

$$\left|\varphi(x_j) - \varphi(x_j^{(0)})\right| < \frac{\varepsilon}{2m_1\,|a_j|}.$$

Let $\delta = \min_{1\le j\le m_1} \delta_j$. If x is a point of $\mathbb{T}^\omega$ verifying $\big|x_j - x_j^{(0)}\big| < \delta$ for $j = 1, \dots, m_1$, then we have, for all $N > m_1$,

$$\begin{aligned}\Big|\sum_{j=1}^{N} a_j\varphi(x_j) - \sum_{j=1}^{N} a_j\varphi(x_j^{(0)})\Big| &= \Big|\sum_{j=1}^{N} a_j\big(\varphi(x_j) - \varphi(x_j^{(0)})\big)\Big| \\ &\le \sum_{j=1}^{m_1} |a_j|\,\big|\varphi(x_j) - \varphi(x_j^{(0)})\big| + \sum_{j=m_1+1}^{N} |a_j|\,\big|\varphi(x_j) - \varphi(x_j^{(0)})\big| \\ &\le \sum_{j=1}^{m_1} |a_j| \cdot \frac{\varepsilon}{2m_1\,|a_j|} + 2\,\|\varphi\|_\infty \sum_{j=m_1+1}^{N} |a_j| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Moreover, we have that $\sum_{j=1}^\infty |a_j\varphi(x_j)| \le \|\varphi\|_\infty \sum_{j=1}^\infty |a_j|$ for all $x \in \mathbb{T}^\omega$. Therefore, the series defining the function $\Psi(x)$ is absolutely convergent, thus the function $\Psi(x)$ is defined for all $x \in \mathbb{T}^\omega$ and there exists $m_2 = m_2(\varepsilon)$ such that, if $N > m_2$, then

$$\Big|\Psi(x) - \sum_{j=1}^{N} a_j\varphi(x_j)\Big| < \varepsilon \quad \forall x \in \mathbb{T}^\omega.$$

Consequently, taking $M = \max\{m_1, m_2\}$, we have

$$\begin{aligned}\big|\Psi(x) - \Psi(x^{(0)})\big| \le \Big|\Psi(x) - \sum_{j=1}^{M} a_j\varphi(x_j)\Big| + \Big|\sum_{j=1}^{M} a_j\varphi(x_j) - \sum_{j=1}^{M} a_j\varphi(x_j^{(0)})\Big| \\ + \Big|\Psi(x^{(0)}) - \sum_{j=1}^{M} a_j\varphi(x_j^{(0)})\Big| \\ < 3\varepsilon\end{aligned}$$

if $x \in \mathbb{T}^\omega$ verifies $\big|x_j - x_j^{(0)}\big| < \delta$ for $j = 1, \dots, M$, and therefore the function $\Psi(x)$ is continuous at the point $x^{(0)}$. □

Definition 2.3. Let $f(x)$ be a function defined on $\mathbb{T}^\omega$. For each multiindex $\alpha = (\alpha_1, \alpha_2, \dots)$ finitely nonzero, that is, such that $\alpha_j \neq 0$ for only finitely many j, it is defined the *partial differentiation operator* by

$$D^\alpha f = D_{j_1}^{\alpha_{j_1}} \cdots D_{j_m}^{\alpha_{j_m}} f = \frac{\partial^{\alpha_{j_1}}}{\partial x_{j_1}^{\alpha_{j_1}}} \cdots \frac{\partial^{\alpha_{j_m}}}{\partial x_{j_m}^{\alpha_{j_m}}} f \quad \text{if } \alpha_j = 0\ \forall j \notin \{j_1, \dots, j_m\}.$$

The *total order* of α is $|\alpha| = \alpha_{j_1} + \dots + \alpha_{j_m}$. When $|\alpha| = 0$, $D^\alpha f = f$.

For each k, the space $C^{(k}(\mathbb{T}^\omega)$ is defined as the class of the functions f with continuous everywhere *partial derivatives* up to the k-th order, i.e., such that $D^\alpha f \in C^{(0}(\mathbb{T}^\omega)$ for all multiindex α finitely nonzero such that

$|\alpha| \le k$. With the norm

$$\|f\|_{(k)} = \sup_{0\le|\alpha|\le k} \|D^\alpha f\|_\infty$$

where $\|D^\alpha f\|_\infty = \max_{x\in\mathbb{T}^\omega} |(D^\alpha f)(x)|$ for each fixed α, $C^{(k}(\mathbb{T}^\omega)$ is a Banach space ([7, 2.2.4]). The space of the infinitely differentiable functions is the intersection $C^{(\infty}(\mathbb{T}^\omega) = \bigcap_{k=0}^\infty C^{(k}(\mathbb{T}^\omega)$ and it is a Fréchet space ([7, 12.1]).

Double series. (See [6, p. 72–76]; also [11].) Consider a double series of complex numbers, as

$$\sum_{m,n=1}^\infty a_{mn}. \tag{2.1}$$

The *rectangular* partial (finite) sums of (2.1) are

$$s_{MN} := \sum_{m=1}^M \sum_{n=1}^N a_{mn}, \quad (M,N) \in \mathbb{N}^2.$$

It is said that the series (2.1) *converges to the sum* $s \in \mathbb{C}$ *in Pringsheim's sense* when $\forall\varepsilon > 0\ \exists\mu$ such that

$$|s_{MN} - s| < \varepsilon \quad \text{if } M, N \ge \mu.$$

A necessary and sufficient condition for the convergence of (2.1) in Pringsheim's sense is the following:

(2.2)

$$\forall\varepsilon > 0\ \exists\mu \text{ such that } |s_{PQ} - s_{MN}| < \varepsilon \text{ if } P > M \ge \mu \text{ and } Q > N \ge \mu.$$

When the series $\sum_{m,n} a_{mn}$ and $\sum_{m,n} b_{mn}$ converge in Pringsheim's sense, then the same happens to the series $\sum_{m,n}(a_{mn} + b_{mn})$, and

$$\sum_{m,n}(a_{mn} + b_{mn}) = \sum_{m,n} a_{mn} + \sum_{m,n} b_{mn}. \tag{2.3}$$

Hardy [8, p. 88] introduced the notion of regular convergence of a double series as follows: the series (2.1) is said *to converge regularly to the sum* $s \in \mathbb{C}$ if it converges to s in Pringsheim's sense and, in addition, each of its row and column series, $\sum_{n=1}^\infty a_{mn}$ for each $m = 1, 2, \ldots$, and $\sum_{m=1}^\infty a_{mn}$ for each $n = 1, 2, \ldots$, respectively, also converges as a single series.

A double series absolutely convergent is also regularly convergent, but the regular convergence is sufficient to

$$\sum_{m,n=1}^\infty a_{mn} = \sum_{m=1}^\infty \sum_{n=1}^\infty a_{mn} = \sum_{n=1}^\infty \sum_{m=1}^\infty a_{mn}$$

hold ([11, Th. 1]).

3. Bilinear and quadratic forms in an infinite number of variables

Let us denote by $\mathcal{S} := \big\{(z_n)_{n=1}^{\infty} \,|\, z_n \in \mathbb{C},\ |z_n| \leq 1\ \forall n \in \mathbb{N}\big\}$ the infinite dimensional polydisc (the closed unit ball of $\ell_\infty(\mathbb{N})$). Analogously to $\mathbb{T}^\omega$, we will consider the space $\mathcal{S}$ with the topology of the cartesian product of infinitely many closed unit circles of the complex plane. Particularly, if $x \in \mathbb{T}^\omega$, then

$$z = e^{2\pi i x} := \big(e^{2\pi i x_1}, \dots, e^{2\pi i x_n}, \dots\big) \in \mathcal{S}.$$

We define a *bilinear form in* $\mathcal{S}$ (in principle only formally) by the expression

$$Q(x,y) := \sum_{m,n=1}^{\infty} a_{mn} x_m y_n \qquad (a_{mn} \in \mathbb{C},\ x, y \in \mathcal{S}). \tag{3.1}$$

The bilinear character and the very existence of the function $Q(x,y)$ depends on the convergence of the double series above.

Definition 3.1. The series (3.1) is *completely bounded in* $\mathcal{S}$ if there is a constant H such that

$$\left|\sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn} x_m y_n\right| \leq H \quad \forall x, y \in \mathcal{S},\ \forall M, N \in \mathbb{N}. \tag{3.2}$$

The following property is immediately deduced.

Lemma 3.2. Suppose that the series (3.1) is completely bounded in $\mathcal{S}$. Then, the series $\sum_{n=1}^{\infty} |a_{mn}|$ for each $m \in \mathbb{N}$, and $\sum_{m=1}^{\infty} |a_{mn}|$ for each $n \in \mathbb{N}$, are convergent.

Proof. For $M, N \in \mathbb{N}$, let $Q_{MN}(x,y)$ denote the rectangular partial sums (or sections) of $Q(x,y)$, i.e.,

$$Q_{MN}(x,y) := \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn} x_m y_n \quad (x, y \in \mathcal{S}).$$

The section $Q_{MN}(x,y)$ only depends on the M first components of x and on the N first components of y, and thus we can consider it as a bilinear form defined on $D^M \times D^N$, where D denotes the closed unit disc of the complex plane. Let us write

$$x^{(M)} := (x_1, \dots, x_M), \quad y^{(N)} := (y_1, \dots, y_N).$$

Therefore, by hypothesis we have

$$\big|Q_{MN}\big(x^{(M)}, y^{(N)}\big)\big| \leq H \quad \text{ if } \ \big\|x^{(M)}\big\|_\infty, \big\|y^{(N)}\big\|_\infty \leq 1.$$

Let, for example, $n_0 \in \mathbb{N}$ be fixed (we proceed in a similar way when we fix $m_0 \in \mathbb{N}$). Consider the points

$$\xi_{n_0} := \Big(\frac{\overline{a_{1n_0}}}{|a_{1n_0}|}, \dots, \frac{\overline{a_{mn_0}}}{|a_{mn_0}|}, \dots\Big) \quad \text{and} \quad \eta_{n_0} := (\delta_{1n_0}, \dots, \delta_{mn_0}, \dots)$$

(δ_{ij} is the Kronecker's symbol). Obviously, ξ_{n_0} and η_{n_0} belongs to $\mathcal{S}$, and, for each $M \in \mathbb{N}$ such that $M > n_0$, it holds

$$\sum_{m=1}^{M} |a_{mn_0}| = Q_{MM}\big(\xi_{n_0}^{(M)}, \eta_{n_0}^{(M)}\big) = Q_{MM}(\xi_{n_0}, \eta_{n_0}) = |Q_{MM}(\xi_{n_0}, \eta_{n_0})| \le H$$

with H independent of M. Consequently the series $\sum_{m=1}^{\infty} |a_{mn_0}|$ is convergent. $\square$

The theorem which follows is due to Littlewood ([10, p. 166–168]).

Theorem 3.3. *If the series* (3.1) *is completely bounded in $\mathcal{S}$ by a constant H, then it converges in Pringsheim's sense, uniformly in $\mathcal{S}^2$, to a bilinear form $Q(x,y)$ which verifies $|Q(x,y)| \le H$ $\forall x, y \in \mathcal{S}$* (and it is said, then, that the bilinear form $Q(x,y)$ is *completely bounded in $\mathcal{S}$*).

Observe that $Q(x,y)$ is a bilinear form if $\forall x, x', y, y' \in \mathcal{S}$ it is verified

$$Q(x, y + y') = Q(x,y) + Q(x,y') \quad \text{and} \quad Q(x + x', y) = Q(x,y) + Q(x',y). \tag{3.3}$$

When the bilinear form $Q(x,y)$ is completely bounded in $\mathcal{S}$ it is verified, in particular, that given $\varepsilon > 0$, there exists $\nu_1 = \nu_1(\varepsilon)$ such that $\big|Q(x,y) - Q_{\nu\nu}(x,y)\big| < \varepsilon$ $\forall (x,y) \in \mathcal{S}^2$ if $\nu \ge \nu_1$. From here, it follows easily:

Corollary 3.4. *A bilinear form $Q(x,y)$ completely bounded in $\mathcal{S}$ defines a continuous function in $\mathcal{S}^2$.*

Proof. Let $(x_0, y_0) \in \mathcal{S}^2$ be fixed and $\varepsilon > 0$. First, as we just said above, there exists $\nu_1(\varepsilon)$ such that

$$\big|Q(x,y) - Q_{\nu\nu}(x,y)\big| < \frac{\varepsilon}{3} \quad \forall (x,y) \in \mathcal{S}^2$$

if $\nu \ge \nu_1$. On the other hand, the bilinear form defined on $D^{\nu_1} \times D^{\nu_1}$ by

$$Q_{\nu_1\nu_1}\big(x^{(\nu_1)}, y^{(\nu_1)}\big) = \sum_{m=1}^{\nu_1} \sum_{n=1}^{\nu_1} a_{mn} x_m y_n$$

is continuous at the point $\big(x_0^{(\nu_1)}, y_0^{(\nu_1)}\big)$. Then, there exists $\delta > 0$ (depending on (x_0, y_0) and ε) such that, if $\max_{1 \le j \le \nu_1} \big\{|x_j - x_{0j}|, |y_j - y_{0j}|\big\} < \delta$, then

$$\big|Q_{\nu_1\nu_1}\big(x^{(\nu_1)}, y^{(\nu_1)}\big) - Q_{\nu_1\nu_1}\big(x_0^{(\nu_1)}, y_0^{(\nu_1)}\big)\big| < \frac{\varepsilon}{3}.$$

Thus, for every $(x,y) \in \mathcal{S}^2$ verifying $\max\{|x_j - x_{0j}|, |y_j - y_{0j}|\} < \delta$ for $j = 1, \dots, \nu_1$, we have

$$\begin{aligned} \big|Q(x,y) - Q(x_0,y_0)\big| &\le \big|Q(x,y) - Q_{\nu_1\nu_1}(x,y)\big| \\ &\quad + \big|Q_{\nu_1\nu_1}(x,y) - Q_{\nu_1\nu_1}(x_0,y_0)\big| + \big|Q_{\nu_1\nu_1}(x_0,y_0) - Q(x_0,y_0)\big| \\ &\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

(we have used that $Q_{\nu_1\nu_1}(x,y) = Q_{\nu_1\nu_1}\big(x^{(\nu_1)}, y^{(\nu_1)}\big)$ for all $(x,y) \in \mathcal{S}^2$), and the continuity of $Q(x,y)$ at the point (x_0,y_0) is proved. $\square$

Some more definitions and remarks. Let $Q(x,y) = \sum_{m,n=1}^{\infty} a_{mn} x_m y_n$ be a bilinear form completely bounded by a constant H in $\mathcal{S}$, and which defines, according to Corollary 3.4, a continuous function on $\mathcal{S}^2$. Let

$$C(x) := Q(x,x) = \sum_{m,n=1}^{\infty} a_{mn} x_m x_n$$

for $x \in \mathcal{S}$. The quadratic form $C(x)$ is also called *completely bounded in* $\mathcal{S}$, because it is verified

$$|C_M(x)| = \left|\sum_{m=1}^{M}\sum_{n=1}^{M} a_{mn} x_m x_n\right| \le H \quad \forall x \in \mathcal{S},\ \forall M \in \mathbb{N}.$$

When the bilinear form $Q(x,y)$ is completely bounded in $\mathcal{S}$, then its *partial derivatives* are well defined (see [9, p. 128]). Writing, for each $p \in \mathbb{N}$, $e_p = (\delta_{pn})_{n=1}^{\infty} \in \mathcal{S}$, and applying (3.3), we have

$$\begin{aligned} \frac{\partial Q}{\partial y_p}(x,y) &= \lim_{t\to 0} \frac{Q(x, y + te_p) - Q(x,y)}{t} = Q(x, e_p) = \sum_{m=1}^{\infty} a_{mp} x_m, \\ \frac{\partial Q}{\partial x_p}(x,y) &= \lim_{t\to 0} \frac{Q(x + te_p, y) - Q(x,y)}{t} = Q(e_p, y) = \sum_{n=1}^{\infty} a_{pn} x_n, \end{aligned}$$

and thus these partial derivatives are bounded linear forms. According to Lemma 3.2, the series $\sum_{n=1}^{\infty} |a_{pn}|$ y $\sum_{m=1}^{\infty} |a_{mp}|$ are convergent for all p, and from here it is possible to deduce that the functions $\frac{\partial Q}{\partial x_p}(x,y)$ and $\frac{\partial Q}{\partial y_p}(x,y)$ are continuous in $\mathcal{S}^2$ by applying a result (in $\mathcal{S}^2$) analogous to Lemma 2.2 (in $\mathbb{T}^\omega$).

Corollary 3.5. (a) *If the bilinear form $Q(x,y)$ is completely bounded in $\mathcal{S}$, then the quadratic form $C(x) = Q(x,x)$ belongs to the the class $C^{(\infty}(\mathcal{S})$.*

(b) *If the quadratic form $C(x) = Q(x,x)$ is completely bounded in $\mathcal{S}$, then it belongs to the class $C^{(\infty}(\mathcal{S})$.*

Proof. (a) The quadratic form $C(x) = Q(x,x)$ is continuous in $\mathcal{S}$ according to Corollary 3.4. For each $p \in \mathbb{N}$ we have

$$\begin{aligned}\frac{\partial C}{\partial x_p}(x) &= \frac{\partial Q}{\partial x_p}(x,x) + \frac{\partial Q}{\partial y_p}(x,x)\\ &= \sum_{n=1}^{\infty} a_{pn}x_n + \sum_{m=1}^{\infty} a_{mp}x_m\\ &= \sum_{j=1}^{\infty}(a_{pj}+a_{jp})x_j\end{aligned}$$

due to the absolute convergence of each series. Then, by applying Lemma 2.2, the linear form $\frac{\partial C}{\partial x_p}(x)$ is continuous in $\mathcal{S}$, and its partial derivatives are constant functions.

(b) From the identity

$$Q(x,y) = Q\big(\tfrac{1}{2}(x+y), \tfrac{1}{2}(x+y)\big) - Q\big(\tfrac{1}{2}(x-y), \tfrac{1}{2}(x-y)\big)$$

it follows that the bilinear form $Q(x,y)$ is completely bounded in $\mathcal{S}$. Then apply part (a) and it is done. □

4. Functions in $C^{(\infty}(\mathbb{T}^\omega)$ whose Fourier series diverges absolutely

In this section, we prove Theorem 1.4. In 1913, Toeplitz [14, p. 427] introduced a quadratic form

$$C(z) = \sum_{m,n=1}^{\infty} a_{mn}z_m z_n \quad (z \in \mathcal{S}) \tag{4.1}$$

in infinitely many variables, symmetric (i.e., such that $a_{mn} = a_{nm}$), completely bounded in $\mathcal{S}$ in the above defined sense, and such that the series $\sum_{m,n=1}^{\infty} |a_{mn}|$ diverges. This quadratic form will be described below. We will simply replace in Toeplitz's form $z = e^{2\pi i x}$ (i.e., $z_j = e^{2\pi i x_j}$ for all j) with $x \in \mathbb{T}^\omega$, and consider the function

$$F(x) = C(e^{2\pi i x}) = \sum_{m,n=1}^{\infty} a_{mn}e^{2\pi i(x_m+x_n)}, \quad x = (x_j)_{j=1}^{\infty} \in \mathbb{T}^\omega. \tag{4.2}$$

From Corollary 3.5 (b) it follows that $F \in C^{(\infty}(\mathbb{T}^\omega)$. In particular, F is integrable.

Let us calculate now the Fourier coefficients of the function F. For this, we will use that $F(x) = \lim_{M\to\infty} F_M(x)$, where

$$F_M(x) = \sum_{m,n=1,\dots,M} a_{mn}e^{2\pi i(x_m+x_n)}.$$

Due to the fact that the quadratic form (4.1) is completely bounded in $\mathcal{S}$, i.e., $|C(z)| \le H$ for all $z \in \mathcal{S}$, we have that $|F_M(x)| \le H$ for all $M \in \mathbb{N}$ and $x \in \mathbb{T}^\omega$. This allows to apply Vitali's convergence theorem to write, for any $\bar{p} \in \mathbb{Z}^\infty$ fixed:

$$\begin{aligned}
\widehat{F}(\bar{p}) &= \int_{\mathbb{T}^\omega} \Big(\sum_{m,n=1}^{\infty} a_{mn} e^{2\pi i (x_m + x_n)} \Big) e^{-2\pi i \bar{p}\cdot x}\, dx \\
&= \int_{\mathbb{T}^\omega} \Big(\lim_{M\to\infty} \sum_{m,n=1,\dots,M} a_{mn} e^{2\pi i (x_m + x_n)} \Big) e^{-2\pi i \bar{p}\cdot x}\, dx \\
&= \int_{\mathbb{T}^\omega} \lim_{M\to\infty} \Big(\sum_{m,n=1,\dots,M} a_{mn} e^{2\pi i (x_m + x_n)} e^{-2\pi i \bar{p}\cdot x} \Big)\, dx \\
&= \lim_{M\to\infty} \sum_{m,n=1,\dots,M} a_{mn} \int_{\mathbb{T}^\omega} e^{2\pi i ((x_m + x_n) - \bar{p}\cdot x)}\, dx \\
&\overset{2}{=} \sum_{m,n=1}^{\infty} a_{mn} \int_{\mathbb{T}^\omega} e^{2\pi i ((x_m + x_n) - \bar{p}\cdot x)}\, dx \\
&= \begin{cases} a_{mn} + a_{nm} = 2a_{mn} & \text{if } \bar{p} = \bar{e}_m + \bar{e}_n,\ m \ne n \\ a_{mm} & \text{if } \bar{p} = 2\bar{e}_m, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where we denote by $\bar{e}_q$ the element $(\delta_{qj})_{j=1}^\infty$ belonging to $\mathbb{Z}^\infty$.

Thus, the above expression (4.2), which defines $F(x)$, is indeed its Fourier series, $\sum_{\bar{p}\in\mathbb{Z}^\infty} \widehat{F}(\bar{p}) e^{2\pi i \bar{p}\cdot x}$. Therefore we will have

$$\sum_{\bar{p}\in\mathbb{Z}^\infty} \big|\widehat{F}(\bar{p})\big| = \sum_{m,n=1}^{\infty} |a_{mn}|$$

and, since $\sum_{m,n=1}^\infty |a_{mn}| = \infty$, our function F is a counterexample showing that the implication $f \in C^{(\infty}(\mathbb{T}^\omega) \Longrightarrow \sum_{\bar{p}\in\mathbb{Z}^\infty} |\widehat{f}(\bar{p})| < \infty$ is false.

Let us proceed to describe the quadratic form $C(z)$. We first show an auxiliary lemma. Toeplitz [14, p. 423–426] gave it for real orthogonal matrices. In what follows, D denotes the closed unit disc of the complex plane.

[2] In fact, denoting by M_0 the *greatest nonzero index* of $\bar{p}$, i.e., the index such that $p_j = 0$ for all $j > M_0$, we have

$$\sum_{m,n=1}^{\infty} a_{mn} \int_{\mathbb{T}^\omega} e^{2\pi i ((x_m + x_n) - \bar{p}\cdot x)}\, dx = \sum_{m,n=1,\dots,M_0} a_{mn} \int_{\mathbb{T}^\omega} e^{2\pi i ((x_m + x_n) - \bar{p}\cdot x)}\, dx.$$

Lemma 4.1 (Littlewood, [10, p. 171]. See also [4, p. 609])**.** Let $A = (a_{mn})_{N\times N}$ be a unitary matrix, i.e., a matrix for which the following holds

$$\sum_{n=1}^{N} a_{rn}\overline{a_{sn}} = \delta_{rs} \quad \forall r, s = 1, \ldots, N,$$

and define $Q_{NN}(x) := N^{-1}\sum_{m,n=1}^{N} a_{mn}x_m x_n$ for $x \in D^N$. Then, it is verified

$$|Q_{NN}(x)| \leq 1 \quad \forall x \in D^N.$$

Toeplitz's quadratic form. Toeplitz begins by defining $C_1(z_1, \ldots, z_4)$ as the quadratic form in D^4 whose coefficients matrix is

$$C_1 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

The real symmetric matrix C_1 verifies $C_1^2 = 4I$, and so, by applying Lemma 4.1 it results

$$\left|C_1(z_1, \ldots, z_4)\right| \leq 4^{3/2} = 8$$

in D^4 (this maximum value is attained for $z_1 = \ldots = z_4 = 1$).

Next, he defines $C_2(z_1, \ldots, z_{4^2})$ as the quadratic form in D^{4^2} whose coefficients matrix is

$$C_2 = \begin{pmatrix} -C_1 & C_1 & C_1 & C_1 \\ C_1 & -C_1 & C_1 & C_1 \\ C_1 & C_1 & -C_1 & C_1 \\ C_1 & C_1 & C_1 & -C_1 \end{pmatrix}.$$

From Lemma 4.1 it results

$$\left|C_2(z_1, \ldots, z_{4^2})\right| \leq (4^2)^{3/2} = 8^2$$

in D^{4^2} (and the maximum modulus is attained for $z_1 = \ldots = z_{4^2} = 1$).

Inductively, from the quadratic form in 4^α variables ($\alpha \geq 1$) with matrix C_α, one can construct the quadratic form in $4^{\alpha+1}$ variables and matrix

$$C_{\alpha+1} = \begin{pmatrix} -C_\alpha & C_\alpha & C_\alpha & C_\alpha \\ C_\alpha & -C_\alpha & C_\alpha & C_\alpha \\ C_\alpha & C_\alpha & -C_\alpha & C_\alpha \\ C_\alpha & C_\alpha & C_\alpha & -C_\alpha \end{pmatrix}.$$

According to Lemma 4.1 we have that, for all $\alpha \in \mathbb{N}$, it holds

$$\left|C_\alpha(z_1, \ldots, z_{4^\alpha})\right| \leq (4^\alpha)^{3/2} = 8^\alpha$$

in D^{4^α}. Finally, for $x \in \mathcal{S}$, Toeplitz defines

$$C(x) = \frac{\mu_1}{8} C_1(x_1, \dots, x_4) + \frac{\mu_2}{8^2} C_2(x_{4+1}, \dots, x_{4+4^2}) + \frac{\mu_3}{8^3} C_3(x_{4^2+4+1}, \dots, x_{4^2+4+4^3}) + \cdots \tag{4.3}$$

where $(\mu_\alpha)_{\alpha=1}^\infty$ is a sequence of positive numbers determined below, and he shows the following Lemma (see [14, p. 426–427]):

Lemma 4.2. If $\mu_\alpha > 0$ are chosen so that the series $\sum \mu_\alpha$ is convergent, then the quadratic form (4.3) is completely bounded in $\mathcal{S}$.

Moreover, the sum of the moduli of all coefficients of the form $C(x)$ is $\sum 2^\alpha \mu_\alpha$. It is easy to choose μ_α so that $\sum \mu_\alpha < \infty$ and $\sum 2^\alpha \mu_\alpha = \infty$ (for example, $\mu_\alpha = \frac{1}{\alpha^2}$, $\mu_\alpha = 2^{-\alpha}$, etc.). Thus, the function

$$F(x) = C(e^{2\pi i x}) \quad (x \in \mathbb{T}^\omega)$$

constructed with these μ_α is our first announced counterexample.

Littlewood's quadratic forms. From [10, p. 171–173] and [4, p. 609–612] we can get a variety of counterexamples that generalize the preceding, based on quadratic forms in $\mathcal{S}$ for which not all the coefficients are real.

For example, let $N > 2$ be a fixed integer, and consider the infinite collection of matrices

$$M_1 = \left(e^{2\pi i \frac{rs}{N}}\right)_{N\times N}, \quad r, s = 1, \dots, N,$$
$$M_\mu = \left(e^{2\pi i \frac{rs}{N}} \cdot M_{\mu-1}\right)_{N^\mu \times N^\mu}, \quad r, s = 1, \dots, N, \quad \text{if } \mu > 1.$$

All entries in M_μ are N-th roots of unity, and M_μ is an unitary matrix, for all $\mu \in \mathbb{N}$. Let us denote by $M_\mu\big(x_1^{(\mu)}, \dots x_{N^\mu}^{(\mu)}\big)$ the quadratic form associated with the matrix M_μ and the variables of a generic point $x \in \mathcal{S}$ on which it acts, and then define the quadratic form in infinitely many variables

$$\begin{aligned} M(x) &= N^{-3/2} M_1(x_1, \dots, x_N) + \frac{1}{4} N^{-3} M_2(x_{N+1}, \dots, x_{N+N^2}) \\ &\qquad + \frac{1}{9} N^{-9/2} M_3(x_{N+N^2+1}, \dots, x_{N+N^2+N^3}) + \cdots \\ &= \sum_{\mu=1}^\infty \frac{N^{-3\mu/2}}{\mu^2} M_\mu\big(x_1^{(\mu)}, \dots x_{N^\mu}^{(\mu)}\big). \end{aligned}$$

According to Lemma 4.1 we have

$$\Big| M_\mu\big(x_1^{(\mu)}, \dots x_{N^\mu}^{(\mu)}\big) \Big| \le N^{3\mu/2},$$

from where we get now

$$|M(x)| \le \sum_{\mu=1}^\infty \frac{1}{\mu^2} < \infty.$$

Thus, $M(x)$ is completely bounded and, applying Corollary 3.5 (b), it belongs to the class $C^{(\infty}$ in $\mathcal{S}$. But, if we denote $M(x) = \sum_{m,n=1}^{\infty} a_{mn} x_m x_n$, since all the moduli of the nonzero coefficients are equal to 1, we have

$$\sum_{m,n=1}^{\infty} |a_{mn}| = N^{-3/2} \cdot N^2 + \frac{1}{4} N^{-3} \cdot N^4 + \frac{1}{9} N^{-9/2} \cdot N^6 + \ldots = \sum_{j=1}^{\infty} \frac{N^{j/2}}{j^2} = \infty$$

and so, the Fourier series of the function $G(x) = M(e^{2\pi i x})$, $x \in \mathbb{T}^{\omega}$, diverges absolutely.

Bohnenblust and Hille ([4, p. 608–614]) generalized for m-ic forms ($m > 2$) the results of Littlewood. This would provide new counterexamples, this time based on m-ic forms ($m > 2$) in infinitely many variables.

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