CARLEMAN TYPE INEQUALITIES FOR FRACTIONAL RELATIVISTIC OPERATORS

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ABSTRACT. In this paper, we derive Carleman estimates for the fractional relativistic operator. Firstly, we consider changing-sign solutions to the heat equation for such operators. We prove monotonicity inequalities and convexity of certain energy functionals to deduce Carleman estimates with linear exponential weight. Our approach is based on spectral methods and functional calculus. Secondly, we use pseudo-differential calculus in order to prove Carleman estimates with quadratic exponential weight, both in parabolic and elliptic contexts. The latter also holds in the case of the fractional Laplacian.

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References
1. Introduction and main results

In this paper we are interested in exponential (in space) decay estimates for solutions of the evolution equation

\begin{equation}
\begin{cases}
    u_t(t, x) + (-\Delta + m^2) u(t, x) = V(t, x) u(t, x), & x \in \mathbb{R}^N, \ t > 0, \\
    u(0, x) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\end{equation}

where \( s \in (0, 1) \) and \( m \geq 0 \). Here, the solution will be taken as \( u : (0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \) and the potential \( V : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \).

We will name the operator \((-\Delta + m^2)^s\) the fractional relativistic Schrödinger operator with mass \( m \) (or just fractional relativistic operator). For \( s = 1/2 \), \((-\Delta + m^2)^{1/2}\) is sometimes called the square root Klein–Gordon operator (see for instance [31]). To be more precise, the terminology relativistic Schrödinger operator concerns

\[ \sqrt{-\Delta + m^2} - m + V. \]

The above is motivated by the kinetic energy of a relativistic particle (that is, a particle travelling with speed close to the speed of light \( c \)) with mass \( m \), and \( V \) corresponds to the quantization of the potential energy. We refer the reader to [3] (where in particular the motivation and justification for the nomenclature of this operator is explained in the Introduction). The relativistic Schrödinger operator has been extensively studied ([36 Section 7.11], [21]) as well as the evolution Problem (1.1) which involves such an operator, see for instance [1]. The literature is very broad and we will not give an exhaustive account of the references. Related equations to (1.1) have been also considered, namely, the boson star equation was studied in [23].

1.1. Motivation and main results. One of our main motivations is the search of lower bounds for solutions of (1.1) for large \( |x| \) very much in the spirit of what is known for solutions of the heat equation. More concretely, consider the heat equation with a potential

\begin{equation}
\begin{cases}
    u_t(t, x) - \Delta u(t, x) = V(t, x) u(t, x), & x \in \mathbb{R}^N, \ t > 0, \\
    u(0, x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\end{equation}

It was proved in [13,14] that if the potential \( V \) is bounded and \( \|e^{\alpha(T)}|\cdot|^2 u_0\|_{L^2(\mathbb{R}^N)} + \|e^{\alpha(T)}|\cdot|^2 / u(T, \cdot)\|_{L^2(\mathbb{R}^N)} < \infty \) with \( \alpha(0) = 0 \) and \( \alpha(T) = 1/(4T) \), then \( u \equiv 0 \). The proof is obtained by contradiction after getting first some lower bounds that hold for all the solutions of (1.2). The ingredients to prove these lower bounds are of three types:

1. First, it is necessary to establish a monotonicity argument that gives the persistence of the Gaussian decay for positive times if the same is assumed at the initial time. In this first step it is proved that \( \alpha(t) \) can decrease with time, as it happens for example with the fundamental solution of (1.2).
2. The second kind of arguments are of convexity type. It is proved that if the solution has the same decay at two different times \( t_1 < t_2 \) (i.e. \( \alpha(t_1) = \alpha(t_2) \)), then for \( t_1 < t < t_2 \) the solution has a better gaussian decay, i.e. \( \alpha(t) > \alpha(t_1) \) for \( t_1 < t < t_2 \).
3. Finally, the last ingredient is to obtain some Carleman estimates that together with some localization procedure allow to prove the desired lower bounds.

This procedure has turned to be rather general. On the one hand it was proved in [12] that it also works for other local evolution equations of higher degree like the generalized Korteweg-De Vries equation which is a third order equation in the spatial variable. In that example the right decay is superlineal and it is given by \( e^{-\rho |x|^3/2} \) for some \( \rho > 0 \). On the other hand, the same procedure was also proved to be successful in the discrete case. That is to say, when one considers the discrete Laplacian [13], which can be understood as a non-local operator (due to that fact does not preserve the support of the function and this causes the absence of a “local” Leibniz rule). In that setting the decay rate is slightly superlineal and is given by \( e^{-\rho |x| \log(1+|x|)} \). In this article we explore up to what extent the three ingredients mentioned above hold for solutions of (1.1).
Let us describe the structure of the paper. In Section 2 we present several definitions of the operator, the fractional Leibniz rule and some positivity inequalities. Moreover, we construct a family of explicit eigenfunctions. In Section 3 we consider the initial value problem without potential. This is done by a fixed point argument. Finally in the last two sections we provide two families of solutions to Problem (1.1). We prove the quantity (1.4) is still finite in the case of a bounded potential. Then the following inequality holds

\[
\frac{1}{2} \int_0^1 t(1-t) \left\{ \int_{\mathbb{R}^N} \omega(t,x) u^2(t,x) dx + \int_{\mathbb{R}^N} \omega(t,x) u^2(t,x) dx \right\} dt + \frac{1}{2} \int_{\mathbb{R}^N} \omega(t,x) u^2(t,x) dx dt
\]

provided the same spatial decay is assumed for the initial data. Then, Section 4 is devoted to the study of solutions to Problem (1.1). We prove that the quantity (1.4) is still finite in the case of a bounded potential. This is done by a fixed point argument. Finally in the last two sections we provide two families of Carleman estimates. In Section 5 we show convexity estimates with linear exponential weight for solutions to Problem (1.1). This is stated in the theorem below, where the operator $H_m^{2s}$ is defined in Proposition 2.1.

**Theorem 1.1.** Let $N \geq 1$, $s \in (0,1/2]$, $m > 0$, $\lambda \in \mathbb{R}^N$ with $|\lambda| < m$, and $u$ be a solution to Problem (1.1). We prove that the quantity (1.4) is still finite in the case of a bounded potential. This is done by a fixed point argument. Finally in the last two sections we provide two families of Carleman estimates. In Section 5 we show convexity estimates with linear exponential weight for solutions to Problem (1.1). This is stated in the theorem below, where the operator $H_m^{2s}$ is defined in Proposition 2.1.

\[
(\lambda^2 + m^2)^s - A)^2 - \frac{1}{2} \int_0^1 t(1-t) \left\{ 2 \int_{\mathbb{R}^N} \omega(t,x) u^2(t,x) dx \right\} dt + \frac{1}{2} \int_{\mathbb{R}^N} \omega(t,x) u^2(t,x) dx dt
\]

for $A + m^{2s} > 0$ sufficiently small (that is, $|A|$ sufficiently large) satisfying

\[
(\lambda^2 + m^2)^s - A)^2 \geq C_2(N,s)m^{4s},
\]

where $C_1(N,s), C_2(N,s)$ are positive constants depending only on $N$ and $s$.

The latter is achieved by proving monotonicity estimates for the corresponding energy functionals for the solution $u : (0,T] \times \mathbb{R}^N \rightarrow \mathbb{R}$. Only linear exponential weights are admissible, and only for $m > 0$, a fact that is strongly related to the decay of the kernel associated to the fractional relativistic operator (in big contrast with the polynomial decay of the kernel of $(-\Delta)^s$). Our techniques are based on functional and spectral calculus and they hold for exponents $0 < s \leq 1/2$. Observe that this restriction in $s$ is due to the fact that we need the operator $H_m^{2s}$ to be negative to keep the corresponding energy term positive, and we are able to ensure this only by using the definition given in Proposition 2.1 which is valid for $0 < 2s < 1$. The case $s = 1/2$ corresponds to the local case. Theorem 1.1 is related to ingredients (1) and (2). We notice a major difference with respect to the classical diffusion process: the coefficient $\alpha(t)$ in the weight $e^{\lambda x}$ is always constant, thus the spatial decay does not change with time. Log-convexity still works, thus the decays at time 0 an 1 control the decay at intermediate times.

In Section 6 we prove Carleman inequalities with quadratic exponential weight both for parabolic and elliptic fractional operators, for all $m \geq 0$. This will be achieved via pseudo-differential calculus. The corresponding results are the following.

**Theorem 1.2** (Parabolic case). Let $f \in C^2_0((0,\infty) \times \mathbb{R})$, $m \geq 0$ and $1/2 < s \leq 1$. Let $\alpha, R > 0$ and $\varphi : [0,\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\varphi(t,x) = \alpha(x/R + \psi(t))^{2}$ where $\psi \in C^\infty([0,\infty))$ is such that $0 \leq \psi(t) \leq 3$. Assume that $f$ is supported in the set

\[
\left\{ 1 \leq \left| \frac{x}{R} + \psi(t) \right| \leq 4 \right\}.
\]
If moreover

\[ s \frac{2s-1}{R^{2s}} \geq c(\|\psi'\|_\infty, \|\psi''\|_\infty) \]

and \( m \leq 2 \frac{1}{2} \), the following inequality holds true

\[ c_1 s^2 \alpha \frac{R^2}{R^2} \|(-\Delta + m^2)^{\frac{2s-1}{2}} f\|_{L^2_{t,x}}^2 + c_2 s^2 \alpha^{\frac{2s-1}{2}} R^{4s} \|f\|_{L^2_{t,x}}^2 \leq \|e^{\varphi}(\partial_t + (-\Delta + m^2)^s)e^{-\varphi} f\|_{L^2_{t,x}}^2, \]

where \( c_1 \) and \( c_2 \) are positive constants, depending on \( s \) and \( m \).

The restriction \( s \in (1/2,1) \) is a necessary technical condition for (1.5) to make sense. Also, \( 2s - 1 > 0 \) is the order of the symbol of the commutator as it will be shown in Section 6.

**Theorem 1.3 (Elliptic case).** Let \( f \in C_0^2(\mathbb{R}) \), \( m \geq 0 \) and \( 1/2 \leq s \leq 1 \). Let \( \alpha, R > 0 \) and \( \varphi : \mathbb{R} \to \mathbb{R} \) defined by \( \varphi(x) = \alpha \left( \frac{x}{R} + 3 \right)^2 \). Assume that \( f \) is supported in the set \( \{ 1 \leq |x/R + 3| \leq 4 \} \). If \( \alpha^{4s-1} \geq CR^{4s} \) and \( m \leq 2 \frac{1}{2} \), the following inequality holds true

\[ c_1 s^2 \alpha \frac{R^2}{R^2} \|(-\Delta + m^2)^{\frac{2s-1}{2}} f\|_{L^2_{t,x}}^2 + c_2 s^2 \alpha^{\frac{2s-1}{2}} R^{4s} \|f\|_{L^2_{t,x}}^2 \leq \|e^{\varphi}(-\Delta + m^2)^s e^{-\varphi} f\|_{L^2_{t,x}}^2, \]

where \( c_1 \) and \( c_2 \) are positive constants, depending on \( s \) and \( m \).

Note that in the stationary case with \( m = 0 \) the condition \( \alpha^{4s-1} \geq CR^{4s} \) arising in Theorem 1.3 reminds the range of the parameters in the work by Rüland and Wang [42]. In fact, this kind of constraint is well known in the local setting, i.e. \( s = 1 \), due to an example of Meshkov of a non-trivial solution of

\[ \Delta u = Vu \]

in two dimensions, such that \( |u| \leq C e^{-\rho|x|^{4/3}} \) for some \( \rho > 0 \). In this example \( V \) is a bounded complex-valued potential.

Theorems 1.2 and 1.3 correspond to ingredient (3). Unfortunately, the above Carleman estimates are not sufficient to conclude any lower bounds. The problem comes from the non-local properties of the operator \((-\Delta + m^2)^s\). As we said before, in order to use the Carleman estimates some localization procedure is necessary. When this is done in the non-local setting the only way of closing the argument is to assume that the fundamental solution in the constant coefficient case (1.3) decays fast enough depending on (1.5). Hence, we should need that the fundamental solution decays at least as a superlinear exponential, which is not the case. For example the fundamental solution (see Section 3.1) for \( s = 1/2 \) is known to have an explicit form (see [2, 8], also [30], p. 185–186]):

\[ K^{1/2}_t(x) = (2\pi)^{N/2} \left( \frac{2}{\pi} \right)^{1/2} m^{N+1/2} t(|x|^2 + t^2)^{-N+1} K_{N+1}(m \sqrt{|x|^2 + t^2}), \]

and \( K^{1/2}_t(x) \sim e^{-m|x|} \) for large \( |x| \), see (2.5).

This rises the question of some other possible scenarios of non-local operators that exhibit super-linear exponential decay. There are examples of distribution densities of Lévy processes which show a “weakly super-linear” asymptotic behaviour. Let us explain this more precisely: let \( (Z_t)_{t \geq 0} \) be a real-valued Lévy process with characteristic exponent \( \psi \), i.e., \( E e^{i z Z_t} = e^{\psi(z)} \), \( t > 0 \). The function \( \psi : \mathbb{R} \to \mathbb{C} \) admits the Lévy–Khintchin representation

\[ \psi(z) = i a z - b z^2 + \int_\mathbb{R} (e^{i z u} - 1 - i z u 1_{\{|u| \leq 1\}}) \mu(du), \quad a \in \mathbb{R}, \quad b \geq 0 \]

and \( \mu(\cdot) \) is a Lévy measure, that is, \( \int_\mathbb{R} (1 + u^2) \mu(du) < \infty \) (the operator \((-\Delta + m^2)^s\) falls into this class). The function \( e^{t \psi} \) is integrable under some conditions of the process \( Z_t \) and hence the associated transition probability density \( p_t(x) \) has the integral representation as the inverse Fourier transform of the characteristic function, \( p_t(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{-i x + t \psi(z)} dz \). It is possible to investigate the latter oscillatory integral, under the
assumption that the characteristic exponent \( \psi \) admits an analytic extension to the complex plane. A usual assumption on the Lévy measure is that it is exponentially integrable

\[
\int_{|y| \geq 1} e^{Cy} \mu(dy) < \infty, \quad \text{for all} \ C \in \mathbb{R}.
\]

The latter assumption is satisfied, for instance, for a generalized tempered Lévy measure defined in terms of certain \( \psi \) with super-exponential decay, i.e., \( e^{C\psi}(u) \to 0, u \to \infty, \) for all \( C \in \mathbb{R}, \) see for example [30] 48]. Then we may find Lévy processes where the transition probability density satisfies a “weakly super-linear” asymptotic behavior, namely, for constants \( c_2 < c_1, \) there exists \( y = y(c_1, c_2) \) such that

\[
\exp \left( -c_1 x \log^{\beta} -s (x/t) \right) \leq p_t(x) \leq \exp \left( -c_2 x \log^{\beta} -s (x/t) \right), \quad x/t > y,
\]

where \( \beta > 1, \) if the Lévy measure \( \mu \) satisfies certain exponential estimates (see [30] (1.18)). Of course this decay is still far from the one of Meshkov example which is \( e^{-\rho|x|^{1/3}}. \) So it seems rather natural to look for non-local operators whose fundamental solutions have superlinear decay and that still have the analitycity properties mentioned above. We will explore this question in the future.

Let us recall that, as we already mentioned, in the particular case of the discrete Laplacian [19] the localization procedure still works while the operator is also non-local. In that scenario the fundamental solution has the right decay in the sense that it fits with the constraints imposed by the corresponding Carleman inequality (i.e. \( e^{-|x|\log(1+|x|/3)} \)). Hence it will be also very natural to try to extend the argument based on the pseudodifferential calculus that we use in this paper to the discrete setting. Again we leave this question for the future.

We finish this subsection with some further comments. The three steps to prove lower bounds above-mentioned are very much in the setting of unique continuation. In particular, for nonlocal models, in [10] 32 44 45 the authors use Carleman estimates. On the other hand, in [3] 16 17 the so-called Almgren monotonicity formulas (see [24]) are used. Lower bounds and Runge approximation results for fractional heat equation are proved in [31].

### 1.2. Importance of analyticity: revisiting the fractional Landis conjecture

As it happens with the example of the Lévy process mentioned in the previous subsection, in our proof of the Carleman estimate with a quadratic weight we make a strong use of the analytic extension of the multiplier associated to the operator \(( -\Delta + m^2)^s.\) In this direction, let us recall a recent work by Rüland and Wang [42]. They proved that, for potentials with some a priori bounds, if a solution to Problem (1.6) below decays at a rate \( e^{-|x|^{1+}}, \) then this solution is trivial. On the other hand, for \( s \in (1/4, 1) \) and merely bounded non-differentiable potentials, if a solution decays at a rate \( e^{-|x|^{\alpha}} \) with \( \alpha > 4s/(4s - 1), \) then this solution must again be trivial. We remark that when \( s \to 1, 4s/(4s - 1) \to 4/3, \) which is the optimal exponent for the standard Laplacian, see [37]. In fact, in [42] Theorem 3 they provide a quantitative lower bound which leads to the mentioned unique continuation principle. Their result motivates us to point out how another result on unique continuation, of qualitative nature, can be obtained as a consequence of lack of analitycity:

Let \( N \geq 1 \) and \( s \in (0, 1). \) Let \( u: \mathbb{R}^N \to \mathbb{R} \) be a solution to the equation

\[
(-\Delta)^s u(x) = V(x) u(x), \quad x \in \mathbb{R}^N,
\]

with \( V \in L^\infty(\mathbb{R}^N). \) Assume that \( u \) decays exponentially fast at infinity, i.e.

\[
|u(x)| \leq e^{-c|x|^{1+}}.
\]

Then it follows that \( u \equiv 0. \) Indeed (for simplicity, we restrict ourselves to the one dimensional case), if we take the Fourier transform at both sides of equation (1.6)

\[
|\xi|^{2s} \hat{u}(\xi) = \hat{F}(\xi), \quad F := Vu,
\]

we notice that the right hand side is analytic in \( \mathbb{C} \) while the left hand side is not. This is justified as follows: observe that condition (1.7) implies that \( \hat{u}(\xi) \) is analytic (the exponential decay of \( u \) makes the Fourier transform of \( u \) to be well defined, as well as its derivatives). Moreover, since \( V \) is uniformly bounded then
the right hand side term \( F \) also decays exponentially fast at infinity and thus \( \hat{F} \) is analytic. Since \( |\xi|^{2s} \) is not analytic at the origin we conclude that \( \hat{u} \) has to be identically zero.

We note that condition \((L7)\) can be relaxed to an \( L^2 \) decay condition such as \( \|u(\cdot)e^{i|\cdot|^2} \|_{L^2(\mathbb{R}^N)} < \infty \), which is sufficient to ensure the analyticity of \( \hat{u} \). Observe also that the result can be seen as an optimal, qualitative Landis conjecture, valid for all \( s \in (0, \infty) \setminus N \).

Remark 1.4. Some comments are in order:

(i) In the case of the parabolic problem \( u_t(t,x) + (-\Delta + m^2)^s u(t,x) = V(t,x) u(t,x) \) \((m \geq 0)\) the argument above does not work anymore since the exponential decay of \( u \) assumed in \((L7)\) does not need to be inherited by \( u_t \).

(ii) The exponential decay is a sufficient condition that can not be improved with this technique. Less decay assumptions on \( u \) were considered for instance by Frank, Lenzmann and Silvestre [22], although their result concerns only radial solutions.

1.3. Remarks and notations. We want to emphasize that what corresponds in fact to a diffusion problem is the following equation:

\[
\begin{align*}
(1.8) & \quad \quad v_t(t,x) + ((-\Delta + m^2)^s - m^{2s}) v(t,x) = V(t,x) v(t,x), \quad x \in \mathbb{R}^N, \quad t > 0, \\
& \quad \quad \text{with data } v(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \quad \text{It is easy to check that mass is conserved when } V = 0, \text{i.e. } \int_{\mathbb{R}^N} v(t,x) dx = \int_{\mathbb{R}^N} u_0(x) dx \text{ for all } t > 0. \quad \text{Moreover,}
\end{align*}
\]

\[
(1.9) \quad u(t,x) := e^{-m^{2s}t} v(t,x)
\]

is a solution to Problem \((1.1)\) with the same initial data. Throughout the paper we will work with Problem \((1.1)\). Most of the results can be reformulated in terms of the solution to Problem \((1.8)\) via the transformation \((1.9)\) or by simply adapting the definition of the operator adding the term \( m^{2s}I \). For instance, the spatial behaviour for small times is the same for both \( u \) and \( v \).

We denote, for \( m \geq 0 \), the operator

\[
L_m := (-\Delta + m^2)
\]

(observe that \( L_0 = (-\Delta) \)). The main reason to work with \( L_m^s := (-\Delta + m^2)^s \) and not \( R_m^s := (-\Delta + m^2)^s - m^{2s} \) is that the composition law becomes simpler in the case of \( L_m^s \), namely

\[
L_m^s(L_m^s) = L_{2s}^m \quad \text{Unlike} \quad R_m^s(R_m^s) = R_{2s}^m - 2m^{2s}R_m^s.
\]

Along the paper we will use a fairly standard notation. We will just skip the variables \((x,t)\) of the functions in many of the instances e.g., we will sometimes use \( u \) instead of \( u(t,x) \). The complete expression will be used when relevant.

2. The fractional relativistic operator. Definitions and properties

There are various equivalent definitions for \( L_m^s \), we state two of them below. The proof of the equivalence is given in Appendix A. We will always consider real valued functions to avoid complex conjugates.

(1) Definition using the Fourier transform. For a function \( f \) in the Schwartz class \( S \) we can define its Fourier transform as

\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^N.
\]

The inversion formula is given, for \( f \in S \), by

\[
\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(\xi) e^{-i\xi \cdot x} d\xi, \quad x \in \mathbb{R}^N.
\]

Let \( 0 < s < 1, \ m \geq 0 \) and \( f \in S \). The operator \( L_m^s(f) \) is defined as a pseudo-differential operator

\[
(2.1) \quad L_m f(\xi) = (|\xi|^2 + m^2)^s \hat{f}(\xi) = (\xi \cdot \xi + m^2)^s \hat{f}(\xi), \quad \xi \in \mathbb{R}^N.
\]
(2) Definition via a singular integral. We first introduce some well known facts about the so called modified Bessel functions and Macdonald’s functions, that will be useful later. Let \( I_\nu(z) \) be the modified Bessel function of first kind given by the formula (see \[33\] Chapter 5, Section 5.7)

\[
I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |z| < \infty, \quad |\arg z| < \pi
\]

and let \( K_\nu \) be the Macdonald’s function of order \( \nu \) defined by (see also \[33\] Chapter 5, Section 5.7)

\[
K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu \pi}, \quad |\arg z| < \pi, \quad \nu \neq 0, \pm 1, \pm 2, \ldots
\]

and, for integral \( \nu = n \), \( K_n(z) = \lim_{\nu \to n} K_\nu(z), \quad n = 0, \pm 1, \pm 2, \ldots \) From (2.2) and (2.3) it is clear that

\[
K_\nu(z) \sim \frac{\Gamma(\nu)}{2} \left( \frac{z}{2} \right)^{-\nu}, \quad (\text{Re} \nu > 0) \quad \text{as} \quad z \to 0^+.
\]

Moreover, it is well known (see \[33\] Chapter 5, Section 5.11)) that

\[
K_\nu(z) = Ce^{-z}z^{-1/2} + \tilde{R}_\nu(z), \quad |\tilde{R}_\nu(z)| \leq C_\nu e^{-z}z^{-3/2}, \quad |\arg z| \leq \pi - \delta.
\]

We have the integral representation for the Macdonald’s functions, also called Sommerfeld integral (see for instance \[38\] p. 407 or \[33\] Chapter 5, (5.10.25)),

\[
K_\nu(z) = 2^{-\nu-1}z^\nu \int_0^\infty e^{-(t+\frac{1}{4}z^2)}t^{\nu-1}dt.
\]

Let \( s \in (0, 1) \), \( m > 0 \) and \( f \) with suitable decay at infinity, for instance \( f \in C^2_b(\mathbb{R}^N) \). Then \( L_m^s(f) \) has a pointwise representation as

\[
L_m^sf(x) = C_{N,s}m^{\frac{N+2s}{2}} \text{P. V.} \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x - y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x - y|)dy + m^{2s}f(x), \quad \forall x \in \mathbb{R}^N,
\]

where \( C_{N,s} \) is a normalization positive constant given by

\[
C_{N,s} = -\frac{2^{1+s-N/2}}{\pi^{N/2} \Gamma(-s)}.
\]

This definition appears already in \[17\].

2.1. Leibniz rule and pointwise estimates. Consider the operator

\[
H_m^s(f, g) := L_m^sf(g) - fL_m^sg - gL_m^sf.
\]

Indeed, \( H_m^s \) is the remainder arising in the fractional Leibniz rule associated to our operator \( L_m^s \). Moreover, \( 2H_m^s(f, g) \) is known in the literature as carré du champ operator, see definition and properties in \[3\] Subsection 1.4.2. Most of the properties we use are proved in \[3\] for a general Lévy process having an infinitesimal generator \( L \). All the necessary information is stated in the proposition below, whose proof is just a consequence of the symmetry of the kernel, so we omit the details.

**Proposition 2.1.** Let \( 0 < s < 1 \). For \( f, g \in C^2_b(\mathbb{R}^N) \) we have, for all \( x \in \mathbb{R}^N \),

\[
H_m^s(f, g)(x) = -C_{N,s}m^{-\frac{N+2s}{2}} \int_{\mathbb{R}^N} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x - y|)dy - m^{2s}f(x)g(x)
\]

where \( C_{N,s} \) is as in (2.8). In particular,

\[
H_m^s(f, f)(x) = -C_{N,s}m^{-\frac{N+2s}{2}} \int_{\mathbb{R}^N} \frac{(f(x) - f(y))^2}{|x - y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x - y|)dy - m^{2s}f(x)^2 \leq 0.
\]

Moreover,

\[
L_m^s(f^2)(x) = 2f(x)L_m^sf(x) + H_m^s(f, f)(x), \quad L_m^s(f^2)(x) \leq 2fL_m^s(f)(x).
\]
2.2. Construction of unbounded eigenfunctions. In this subsection we will construct a special family of eigenfunctions of the operator $L^s_m$. First we prove some integral formulas involving the Bessel functions.

**Lemma 2.2.** Let $N \geq 1$, $s \in (0, 1)$, $\lambda \in \mathbb{R}^N$ with $|\lambda| < 1$. Then

\[(2.10) \quad C_{N,s} \int_{\mathbb{R}^N} \frac{1 - e^{\lambda z}}{|z|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|z|) \, dz = (1 - |\lambda|^2)^s - 1,\]

where $C_{N,s}$ is given by (2.8).

Moreover, when $N - 2s < 1$ and $|\lambda| = 1$, the following also holds

\[(2.11) \quad C_{N,s} \int_{\mathbb{R}^N} \frac{1 - e^{\lambda z}}{|z|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|z|) \, dz = -1.\]

**Proof.** Observe that the integrals are well defined (this can be checked by using the asymptotics of the Bessel function (2.4) and (2.5)). The restriction for $N$ and $s$ for the second identity comes from the integrability of the integral near the origin.

The identities follow by using the integral representation of the Bessel function (2.6) and the identity

\[(2.12) \quad \gamma^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\gamma} - 1) \frac{dt}{t^{1+s}}, \quad \gamma > 0, \quad 0 < s < 1.\]

For the proof of (2.10) observe that

\[
C_{N,s} \int_{\mathbb{R}^N} \frac{1 - e^{\lambda z}}{|z|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|z|) \, dz = C_{N,s} 2^{-\frac{N+2s}{2}-1} \int_{\mathbb{R}^N} (1 - e^{\lambda z}) \int_0^\infty e^{-\frac{|t|^2}{4}} \frac{dt}{t^{\frac{N+2s}{2}+1}} \, dz
\]

Notice that the integral in $z$, after changing variables to $z = 2\sqrt{t}y$, equals

\[
\int_{\mathbb{R}^N} (1 - e^{2\sqrt{t}\lambda y}) e^{-|y|^2} (2\sqrt{t})^N dy = 2^N t^{\frac{N}{2}} \left( \int_{\mathbb{R}^N} e^{-|y|^2} dy - \int_{\mathbb{R}^N} e^{2\sqrt{t}\lambda y - |y|^2} dy \right)
\]

\[
= 2^N t^{\frac{N}{2}} \left( \sqrt{\pi}^N - e^{t|\lambda|^2} \int_{\mathbb{R}^N} e^{-(y-\sqrt{t}\lambda)^2} dy \right)
\]

\[
= 2^N t^{\frac{N}{2}} \sqrt{\pi}^N (1 - e^{t|\lambda|^2}).
\]

Then using the explicit form of the constant given in (2.8) we obtain that

\[
C_{N,s} \int_{\mathbb{R}^N} \frac{1 - e^{\lambda z}}{|z|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|z|) \, dz = C_{N,s} 2^{-\frac{N+2s}{2}-1+N} \sqrt{\pi}^N \int_0^\infty t^{\frac{N}{2}} \left( 1 - e^{t|\lambda|^2} \right) e^{-t} \frac{dt}{t^{\frac{N+2s}{2}+1}}
\]

\[
= C_{N,s} 2^{-\frac{N+2s}{2}-1} \sqrt{\pi}^N \int_0^\infty \left( e^{-t} - 1 + 1 - e^{-t(1-|\lambda|^2)} \right) \frac{dt}{t^{s+1}}
\]

\[
= C_{N,s} 2^{-\frac{N+2s}{2}-1} \sqrt{\pi}^N \Gamma(-s) \left( 1 - (1 - |\lambda|^2)^s \right) = (1 - |\lambda|^2)^s - 1.
\]

For (2.11) the proof is the same, except for the last integral is $\int_0^\infty (e^{-t} - 1) \frac{dt}{t^{s+1}}$, which equals $\Gamma(-s)$ according to (2.12).

\[\square\]

**Proposition 2.3.** Let $N \geq 1$, $s \in [0, 1]$, $\lambda \in \mathbb{R}^N$ with $|\lambda| < m$. Then

\[(2.13) \quad L^s_m e^{\lambda x} = (-|\lambda|^2 + m^2)^s e^{\lambda x}, \quad a.e. \ x \in \mathbb{R}^N.
\]

Moreover, when $N - 2s < 1$ and $|\lambda| = m$ then

\[
L^s_m e^{\lambda x} = 0, \quad a.e. \ x \in \mathbb{R}^N.
\]
Proof. First, observe that the cases \( s = 0 \) and \( s = 1 \) follow trivially. Let \( N \geq 1, s \in (0, 1) \) and \( \lambda \in \mathbb{R}^N \) as in the hypothesis. Then for \( x \in \mathbb{R}^N \)

\[
L^s_m(e^{\lambda \cdot \cdot})(x) = m^{2s}e^{\lambda \cdot \cdot} = C_{N,s}m^{2s}e^{\lambda \cdot \cdot}\int_{\mathbb{R}^N} \frac{1 - e^{-\frac{1}{m} \lambda \cdot z}}{|z|^{2s} + \nu} K_{N,s+2s}(|z|) \, dz.
\]

The integral is well defined, as proved in Lemma 2.2 with \( \lambda/m \) as the corresponding parameter. Now we use Lemma 2.2 with \( \lambda/m \in \mathbb{R}^N, |\lambda|/m \leq 1 \) and the result follows in each of the cases. \( \square \)

Remark 2.4. The identity (2.13) cannot be extended to \( |\lambda| > m \) because the function \( \mathbb{C} \ni z \to (z^2 + m^2)^s \) is not well defined in \( \{it : t > m\} \).

3. The heat equation for the fractional relativistic operator

We devote this section to the linear heat equation

\[
\begin{cases}
  u_t(t, x) + (\Delta + m^2)^s u(t, x) = 0, & x \in \mathbb{R}^N, \ t > 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\] (3.1)

3.1. Fundamental solution. The fundamental solution \( K^\ast_t \) for the heat equation involving \( L^s_m \) is defined via the Fourier transform as

\[
\hat{K}^\ast_t(\xi) = e^{-t(|\xi|^2 + m^2)^s}.
\]

This will correspond to the probability density function of the associated stable relativistic process. Estimates for the fundamental solution are well-known and can be found in [43, Subsection 6.4] (see also [9, Theorem 1.2] and [10, Theorem 4.1]). For our purposes, we will emphasize that \( K^\ast_t \) has an exponential decay in \( |x| \) for small times:

\[
K^\ast_t(x) \sim e(t)|x|^{-N-2s} e^{-\tilde{c} m|x|}, \ x \in \mathbb{R}^N,
\]

with a positive constant \( \tilde{c} \) independent of time.

3.1.1. Integral representation. The following subordination formula is shown in [43, (7)], see also [27]

\[
K^\ast_t(x) = \int_0^\infty \Theta^\ast_t(\rho) e^{-m^2 \rho} \frac{e^{-|x|^2}}{(4\pi \rho)^{N/2}} \, d\rho.
\] (3.2)

Here, \( \Theta^\ast_t(\rho), \rho > 0 \), is the density function of the s-stable process whose Laplace transform is \( e^{-t \lambda^s} \). In the case \( s = 1/2 \), \( K^{1/2}_t \) has an explicit expression, as stated in the Introduction.

We compute the following weighted \( L^1 \) norm of \( K^\ast_t \).

Lemma 3.1. We have, for all \( |\lambda| \leq m \),

\[
\|e^{\lambda \cdot \cdot} K^\ast_t(\cdot)\|_{L^1(\mathbb{R}^N)} = e^{-(m^2 - |\lambda|^2)^s} t, \ t > 0.
\] (3.3)

Proof. In [43, p. 3], Ryznar defines the probability density function

\[
\Theta^\ast_t(\rho, m) = e^{-m^2 \rho + m^2 \xi^t \Theta^\ast_t(\rho)}, \ \rho > 0,
\]

so in particular this means that

\[
\int_0^\infty e^{-m^2 \rho \Theta^\ast_t(\rho)} \, d\rho = e^{-tm^2 s}.
\] (3.4)

Now, by (3.2) and Fubini,

\[
\int_{\mathbb{R}^N} e^{\lambda \cdot x} \int_0^\infty \Theta^\ast_t(\rho) e^{-m^2 \rho} \frac{e^{-|x|^2}}{(4\pi \rho)^{N/2}} \, d\rho \, dx = \int_0^\infty \Theta^\ast_t(\rho) e^{-m^2 \rho} (4\pi \rho)^{-N/2} \int_{\mathbb{R}^N} e^{\lambda \cdot x} e^{-|x|^2/4\rho} \, dx \, d\rho.
\] (3.5)

Observe that

\[
\int_{\mathbb{R}^N} e^{\lambda \cdot x} e^{-|x|^2/4\rho} \, dx = \prod_{i=1}^N \int_{\mathbb{R}} e^{\lambda_i x_i} e^{-x_i^2/4\rho} \, dx_i = e^{|\lambda|^2 \rho} \prod_{i=1}^N \int_{\mathbb{R}} e^{-\frac{(x_i - \lambda_i \sqrt{\rho})^2}{4\rho}} \, dx_i = e^{|\lambda|^2 \rho} 2^N (\pi \rho)^{N/2}.
\]
Therefore, (3.5) equals
\[ \int_0^\infty \Theta_t^s(\rho) e^{-\rho(m^2-|\lambda|^2)} \, d\rho = e^{-t(m^2-|\lambda|^2)^2}, \]
where the equality follows from (3.4).

3.2. Energy estimates. Let \( u \) be a solution to Problem (3.1). Thus \( u \) is obtained directly from the fundamental solution and the data
(3.6) \[ u(t, x) = u_0(x) * K_t^s(x), \quad x \in \mathbb{R}^N, \ t > 0. \]

**Proposition 3.2.** The solution \( u \) to the Problem (3.1) has the following properties:

- (Decay of total mass) We have
  (3.7) \[ \int_{\mathbb{R}^N} u(t, x) \, dx = e^{-m^2t} \int_{\mathbb{R}^N} u_0(x) \, dx, \quad t > 0. \]
- (Energy estimate) Assume \( u_0 \in L^2(\mathbb{R}^N) \). Then for all \( 0 < t < T \) we have
  (3.8) \[ 2 \int_0^t \int_{\mathbb{R}^N} |L_m^{s/2} u(\tau, x)|^2 \, d\tau \, dx + \int_{\mathbb{R}^N} u^2(t, x) \, dx = \int_{\mathbb{R}^N} u_0^2(x) \, dx. \]
- (Decay of weighted \( L^2 \) norm) Assume \( u_0 \in L^2(e^{\lambda \cdot x}, \mathbb{R}^N) \). Then for all \( 0 < t \) and \( |\lambda| \leq 2m \) we have
  (3.9) \[ \int_{\mathbb{R}^N} u^2(t, x) e^{\lambda \cdot x} \, dx \leq e^{-(m^2-|\lambda|^2/4)t} \int_{\mathbb{R}^N} u_0^2(x) e^{\lambda \cdot x} \, dx, \quad t \geq 0. \]

**Proof.** The first identity follows from (1.9) and mass conservation for Problem (1.8). On the other hand, (3.8) follows from the fact that, formally,
\[ \frac{d}{dt} \int_{\mathbb{R}^N} u^2(t, x) \, dx = -2 \int_{\mathbb{R}^N} u L_m^s u(t, x) \, dx = -2 \int_{\mathbb{R}^N} |L_m^{s/2} u(t, x)|^2 \, dx. \]
Notice that the identities in (3.7) and (3.8) are typical energy estimates for diffusion equations.

For the inequality (3.9) we proceed as follows. Let \( \lambda \in \mathbb{R}^N \) with \( |\lambda| \leq 2m \). Then, by using the representation (3.6) and (3.3) with \( \lambda/2 \), we derive that
\[ \int_{\mathbb{R}^N} e^{\lambda \cdot x} u^2(t, x) \, dx = \int_{\mathbb{R}^N} e^{\lambda \cdot x} ((K_t^s * u_0)(x))^2 \, dx \]
\[ = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} e^{\lambda/2 \cdot (x-y)} K_t^s(x-y) e^{\lambda/2 \cdot y} u_0(y) \, dy \right)^2 \, dx \]
\[ = \| e^{\lambda/2 \cdot (\cdot)} K_t^s(\cdot) * (e^{\lambda/2 \cdot (\cdot)} u_0(\cdot)) \|_{L^2(\mathbb{R}^N)}^2 \]
\[ \leq \| e^{\lambda/2 \cdot (\cdot)} K_t^s(\cdot) \|_{L^1(\mathbb{R}^N)}^2 \| (e^{\lambda/2 \cdot (\cdot)} u_0(\cdot)) \|_{L^2(\mathbb{R}^N)}^2 \]
\[ = e^{-(m^2-|\lambda|^2/4)t} \| (e^{\lambda/2 \cdot (\cdot)} u_0(\cdot)) \|_{L^2(\mathbb{R}^N)}^2. \]

3.3. Construction of separate variable solutions. With the tools we have so far, it is easy to construct explicit separate variables solutions \( \omega : (0, T) \times \mathbb{R}^N \to \mathbb{R} \) to the equation
(3.10) \[ \partial_t \omega(t, x) = -L_m^s \omega(t, x), \quad x \in \mathbb{R}^N, \ t > 0. \]
We are interested in spatial increasing solutions. By Proposition 2.3 we obtain that
\[ \omega_\lambda(t, x) = e^{-(|\lambda|^2+m^2)t} e^{\lambda \cdot x}, \quad x \in \mathbb{R}^N, \ t > 0, \]
verifies the equation (3.10) for every \( \lambda \in \mathbb{R}^N \) with \( |\lambda| < m \). Moreover
\[ L_m^s \omega(t, x) = (-|\lambda|^2 + m^2)^s \omega(t, x), \quad x \in \mathbb{R}^N, \ t > 0. \]
3.4. Log-convexity of the weighted functional for the linear heat equation. Let $u$ be the solution to Problem (3.1) and let $\mathcal{H}(t) := \int_{\mathbb{R}^N} e^{\lambda x} u^2(t, x) \, dx$, which is well defined for $|\lambda| \leq 2m$ according to (3.3). Moreover,

$$\mathcal{H}(t) = \int_{\mathbb{R}^N} |e^{\lambda/2} u(t, x)|^2 \, dx = \int_{\mathbb{R}^N} |\tilde{u}(\xi + \frac{i\lambda}{2})|^2 \, d\xi = \int_{\mathbb{R}^N} e^{-2t((\xi + \frac{i\lambda}{2})^2 + m^2)} |\tilde{u}_0(\xi + \frac{i\lambda}{2})|^2 \, d\xi.$$ 

**Theorem 3.3.** The functional $\mathcal{H}$ is logarithmically convex. In particular

$$\mathcal{H}(t) \leq \mathcal{H}(0)^{1-t} \mathcal{H}(1)^t, \quad t \in [0, 1].$$

**Proof.** Starting from (3.6), we use the Fourier representation of the solution $u$. Then, the functional $\mathcal{H}(t)$ can be written as follows

$$\mathcal{H}(t) = \|e^{\lambda/2} (K_t * u_0)(\cdot)\|_{L^2(\mathbb{R}^N)}^2 = \|e^{-t((\cdot + \frac{i\lambda}{2})^2 + m^2)} \tilde{u}_0(\cdot + \frac{i\lambda}{2})\|_{L^2(\mathbb{R}^N)}^2.$$

Since

$$(\log(\mathcal{H}))'' = \frac{\mathcal{H}(t) \mathcal{H}(t) - \dot{\mathcal{H}}(t)^2}{(\mathcal{H}(t))^2}$$

we only need to check that the numerator is positive. Indeed, by using $\frac{d}{dt} \hat{K}_t^s(\xi) = (\xi \cdot \xi + m^2)^s e^{-t(\xi \cdot \xi + m^2)^s}$, we have

$$(\dot{\mathcal{H}}(t))^2 = \|\left((\cdot + \frac{i\lambda}{2})^2 + m^2\right)^s e^{-t((\cdot + \frac{i\lambda}{2})^2 + m^2)} \hat{u}_0(\cdot + \frac{i\lambda}{2})\|_{L^2(\mathbb{R}^N)}^2$$

$$\leq \|\left((\cdot + \frac{i\lambda}{2})^2 + m^2\right)^{2s} e^{-t((\cdot + \frac{i\lambda}{2})^2 + m^2)} \hat{u}_0(\cdot + \frac{i\lambda}{2})\|_{L^2(\mathbb{R}^N)}^2 \|e^{-t((\cdot + \frac{i\lambda}{2})^2 + m^2)} \tilde{u}_0(\cdot + \frac{i\lambda}{2})\|_{L^2(\mathbb{R}^N)}^2$$

$$= \ddot{\mathcal{H}}(t) \mathcal{H}(t).$$

**Remark 3.4.** The logarithmic convexity is a strong tool that might lead to an uncertainty principle result for the corresponding equation, like in [13]. However, the method developed in [13] in order to prove uncertainty principles does not work here: the reason is that the decay of $u_0$ given by $\mathcal{H}(0) < \infty$ does not lead to better decay for $u(t, x)$, $t > 0$. This can be immediately seen from the definition $\mathcal{H}(t) := \int_{\mathbb{R}^N} e^{\lambda x} u^2(t, x) \, dx$, where the space decay is always the same ($\lambda$ is constant). In particular, if we look at the fundamental solution, it always has the same spatial decay $e^{-m|\tau|}$, thus it does not improve with time. This is in big contrast to what happens with self-similar processes (for instance, as in [13] dealing with the classical heat equation).

4. The heat equation with potential

We devote this section to the study of (any sign) solutions to Problem (1.1).

**Definition 4.1.** Let $u_0 \in L^\infty(\mathbb{R}^N)$ and $T > 0$ or $T = \infty$. A mild solution of Problem (1.1) is a function $u \in L^\infty([0, \infty) \times \mathbb{R}^N)$ which satisfies, for a.e. $(t, x) \in (0, T) \times \mathbb{R}^N$,

$$u(t, x) = (K_t^s * u_0)(x) + \int_0^t (K_{t-s}^s * (V(\tau, \cdot) u(\tau, \cdot)))(x) d\tau.$$

By classical theory (see [7] Théorème VII.10, also [39, 50]) there exists a mild solution to the initial value Problem (1.1). Moreover, this mild solution is in fact a strong solution $u \in C([0, T]: L^2(\mathbb{R}^N)) \cap C^1((0, T): L^2(\mathbb{R}^N))$. Since we will work with exponentially decaying solutions, we prove here that such solutions do exist. The procedure is similar as for the existence of the mild solution in $C([0, T]: L^2(\mathbb{R}^N))$, i.e. we will use a fixed point argument.

**Theorem 4.2.** Let $0 < s < 1$, $m > 0$ and $\lambda \in \mathbb{R}^N$ with $|\lambda| \leq 2m$. Let $u_0 \in L^\infty(\mathbb{R}^N)$, $T > 0$ and $V : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ such that $V(t, x) \in L^\infty([0, \infty] \times \mathbb{R}^N)$. Then there exists a solution $u \in L^\infty(0, T: L^2(e^{\lambda x} \, dx))$ to Problem (1.1).
Proof. Let \( X := L^\infty(0,T : L^2(e^{\lambda x} dx)) \) and let us define the map \( \mathcal{T} : X \to X \) by
\[
\mathcal{T}(u)(t,x) := (K_t^s * u_0)(x) + \int_0^t \left( K_{t-\tau}^s * (V(\tau,\cdot)u(\tau,\cdot))(x) \right) d\tau, \quad t \in [0,T].
\]
We prove that \( \mathcal{T} : B \to B \) is a contraction in every \( B = \{ u \in L^\infty(0,T : L^2(e^{\lambda x})) \text{ such that } \| u \| < R \} \).
Indeed, we have
\[
\| e^{\lambda x}(\mathcal{T}(u)(t,x) - \mathcal{T}(v)(t,x)) \|_{L^2(\mathbb{R}^N)} \leq \| e^{\lambda x} \int_0^t (K^s_{t-\tau} * (V(\tau,\cdot)(u(\tau,\cdot) - v(\tau,\cdot))))(x) d\tau \|_{L^2(\mathbb{R}^N)}
\]
\[
= \left( \int_{\mathbb{R}^N} \left( \int_0^t (e^{\lambda/2(\cdot)}K^s_{t-\tau}(\cdot)) * (e^{\lambda/2(\cdot)}V(\tau,\cdot)(u(\tau,\cdot) - v(\tau,\cdot)))(x) d\tau \right)^2 dx \right)^{1/2}
\]
\[
\leq \| V \|_{L^\infty} \int_0^t \| (e^{\lambda/2(\cdot)}K^s_{t-\tau}(\cdot)) \|_{L^1(\mathbb{R}^N)} \| e^{\lambda/2(\cdot)}(u(\tau,\cdot) - v(\tau,\cdot)) \|_{L^2(\mathbb{R}^N)} d\tau
\]
\[
\leq \| V \|_{L^\infty} \sup_{t \in [0,T]} \| (e^{\lambda/2(\cdot)}(u(t,\cdot) - v(t,\cdot))) \|_{L^2(\mathbb{R}^N)}^2 \int_0^t \| e^{\lambda/2(\cdot)}K^s_{t-\tau}(\cdot) \|_{L^1(\mathbb{R}^N)} d\tau
\]
\[
= \| V \|_{L^\infty} \| (u(t,\cdot) - v(t,\cdot)) \|_X \int_0^t e^{-(m^2-|\lambda|^2/4)(t-\tau)} d\tau < \| V \|_{L^\infty} T \| (u(t,\cdot) - v(t,\cdot)) \|_X
\]
where we used Minkowski’s integral inequality and (3.3). Thus \( \mathcal{T} \) is a contraction for sufficiently small \( T \). It follows that there is a fixed point for \( \mathcal{T} \).

\[ \square \]

4.1. Backward unique continuation for the heat equation with potential.

Theorem 4.3 (Backward unique continuation). Let \( N \geq 1, s \in (0,1) \) and \( m \geq 0 \). Let \( u \) be a solution to Problem (1.1) with initial data \( u_0 \in L^2(\mathbb{R}^N) \) and \( V \in L^\infty(0,T \times \mathbb{R}^N) \). Assume that \( u(T,\cdot) = 0 \) for some time \( T > 0 \). Then \( u \equiv 0 \).

Proof. Let \( \mathcal{H}(t) := \int_{\mathbb{R}^N} u^2(t,x) dx \). In case the potential is \( V = 0 \) we have
\[
\mathcal{H}(t) = \langle u, u_t \rangle + \langle u_t, u \rangle = 2Re\langle u, L_s^m u \rangle = 2\| L_{m/2}^s u \|_{L^2(\mathbb{R}^N)}^2
\]
and
\[
\mathcal{H}(t) = 2Re\langle u, u_{tt} \rangle + 2\langle u_t, u_t \rangle = 4\| L_m^s u \|_{L^2(\mathbb{R}^N)}^2.
\]
Then \( (\mathcal{H}(t))^2 \leq 4\langle u, u \rangle \cdot \langle L_m^s u, L_m^s u \rangle \) and thus \( \mathcal{H}(t) \) is logarithmically convex. Hence
\[
\mathcal{H}(t) \leq \mathcal{H}(0)^\theta \mathcal{H}(T)^{1-\theta}, \quad \theta \in [0,1].
\]
When the potential is non trivial, the functional \( \mathcal{H} \) is still logarithmically convex, since the operator \( L_m^s \) is symmetric. According to [13, Lemma 2, p. 6] there exists a constant \( N \) such that
\[
\mathcal{H}(t) \leq e^{N(\| V \|_{L^\infty} + \| V \|_{L^\infty})^2} \mathcal{H}(0)^t \mathcal{H}(1)^{1-t}, \quad t \in [0,1].
\]
Thus, up to scaling in time, if \( u(T) \equiv 0 \) then \( u \equiv 0 \). \( \square \)

5. Carleman inequality for the parabolic operator with linear exponential weight

This section is devoted to the study of convexity estimates for an exponential weighted norm of solutions \( u : (0,T] \times \mathbb{R}^N \to \mathbb{R} \) to the initial value problem
\[
\begin{cases}
u(t,x) + (-\Delta + m^2)^s u(t,x) = F(t,x) & x \in \mathbb{R}^N, \ t > 0, \\
u(0,x) = u_0(x), & x \in \mathbb{R}^N
\end{cases}
\]
for some \( T > 0 \). In particular, when \( F = Vu \) we are reduced to the Problem (1.1). We consider the functional
\[
\mathcal{H}(t) := \int_{\mathbb{R}^N} \omega(t,x)u^2(t,x) dx, \quad \omega(t,x) = e^{At + \lambda x}, \quad t \in [0,T],
\]
for $A \in \mathbb{R}$, $\lambda \in \mathbb{R}^N$. Note that $\mathcal{H}(t) < \infty$ for $|\lambda| < 2m$, by Proposition [3.2]

5.1. Persistence of the spatial decay: monotonicity of the energy functional. In what follows we prove that if the initial data decays at least exponentially fast in space, then the solution $u(t, x)$ will have a similar decay at every positive time $t > 0$. This will be a consequence of the monotonicity of the functional $\mathcal{H}(t)$.

**Proposition 5.1.** Let $0 < s \prec 1$, $m > 0$ and let $u$ be a solution to the initial value problem [5.1]. Assume that $F(t, x) \in L^2(0, T; L^2(e^{\lambda x} \, dx))$ and $u_0 \in L^2(e^{\lambda x} \, dx)$ for some $\lambda \in \mathbb{R}^N$ with $|\lambda| < m$. Let $\omega(t, x)$ be as in [5.2]. Then for all $t \in [0, T]$,

$$
\int_{\mathbb{R}^N} \omega(t, x) u^2(t, x) \, dx + \int_0^t \int_{\mathbb{R}^N} \omega(t, x)(-H^s_{m}(u, u)) \, dx \, dt
\leq e^{(A-(-|\lambda|^2+m^2)^s)t} \int_{\mathbb{R}^N} \omega(t, x) u_0^2(x) \, dx + e^{(A-(-|\lambda|^2+m^2)^s)t} \int_0^t \int_{\mathbb{R}^N} \omega(t, x) F^2(t, x) \, dx \, dt.
$$

**Proof.** Let $\mathcal{H}(t)$ be defined as in [5.2]. We have that

$$
\dot{\mathcal{H}}(t) = \int_{\mathbb{R}^N} (\omega_t - L^s_{m}(\omega)) u^2 \, dx + \int_{\mathbb{R}^N} \omega H^s_{m}(u, u) \, dx + 2 \int_{\mathbb{R}^N} \omega u F \, dx
\leq [A - (-|\lambda|^2 + m^2)^s] \int_{\mathbb{R}^N} \omega u^2 \, dx + \int_{\mathbb{R}^N} \omega H^s_{m}(u, u) \, dx + 2 \int_{\mathbb{R}^N} \omega u F \, dx.
$$

Let $a := [A - (-|\lambda|^2 + m^2)^s]$. For sufficiently negative $A$, the coefficient $a$ will be negative and thus we could ignore the term $a \int_{\mathbb{R}^N} \omega u^2 \, dx$. However, we keep this term to avoid imposing conditions on the parameter $A$ in this proposition. Thus we have

$$
\dot{\mathcal{H}}(t) - a \mathcal{H}(t) \leq \int_{\mathbb{R}^N} \omega H^s_{m}(u, u) \, dx + \int_{\mathbb{R}^N} \omega F^2 \, dx,
$$

that can be rewritten as

$$
\frac{d}{dt} (e^{-at} \mathcal{H}(t)) \leq e^{-at} \int_{\mathbb{R}^N} \omega H^s_{m}(u, u) \, dx + e^{-at} \int_{\mathbb{R}^N} \omega F^2 \, dx.
$$

We integrate from $t_1$ to $t_2$ in time and therefore

$$
\mathcal{H}(t_2) + e^{at_2} \int_{t_1}^{t_2} e^{-ar} \int_{\mathbb{R}^N} \omega (-H^s_{m}(u, u)) \, dx \, dt \leq e^{a(t_2 - t_1)} \mathcal{H}(t_1) + e^{at_2} \int_{t_1}^{t_2} e^{-ar} \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt.
$$

This implies that, for all $0 \leq t_1 < t_2 \leq 1$,

$$
\mathcal{H}(t_2) + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \omega (-H^s_{m}(u, u)) \, dx \, dt \leq e^{at_2} \mathcal{H}(t_1) + e^{at_2} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt.
$$

The conclusion follows just by taking $t_1 = 0$ and $t_2 = t$.

**Remark 5.2.** In particular, if we take $t_2 = 1$ in (5.3) and renaming $t_1$ into $t$, we obtain that, for all $t \in (0, 1)$,

$$
\mathcal{H}(1) + \int_{t}^{1} \int_{\mathbb{R}^N} \omega (-H^s_{m}(u, u)) \, dx \, dt \leq e^a \mathcal{H}(t) + e^a \int_{t}^{1} \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt.
$$

5.2. Convexity arguments. In this subsection we will consider a similar weight as in Subsection [3.4], but with a correction in time needed in order to absorb the effects of the potential. We will prove a convexity result related to it, that will be the key point to get a Carleman inequality for $\partial_t + L^s_{m}$ in Subsection [5.3].

Note that the proof carried out in Subsection [5.4] is not valid anymore due to the presence of the potential.

Let $\omega(t, x)$ be defined as in [5.2], where $A$ is a constant to be chosen later. Let

$$
D(t) := \int_{\mathbb{R}^N} \omega_t u^2 \, dx - 2 \int_{\mathbb{R}^N} \omega u L^s_{m} u \, dx = \int_{\mathbb{R}^N} (\omega_t - L^s_{m} \omega) u^2 \, dx + \int_{\mathbb{R}^N} \omega H^s_{m}(u, u) \, dx
$$

where the second equality follows easily from [2.9]. Actually, (5.5) is a formal definition, for any $\omega$. Recall also the definition of $\mathcal{H}(t)$ in [5.2]. We will prove the following.
Proposition 5.3. Let $N \geq 1$, $s \in (0, 1/2]$, $m > 0$, $\lambda \in \mathbb{R}^N$ with $|\lambda| < m$, and $u$ be a solution to Problem (5.1) such that $u \in L^2(e^{\lambda x}dx)$ and $u^2 \in \text{Dom}(D^s_m)$. Let $\omega(t, x)$ be as in (5.2) and $F \in L^2(0,T; L^2(e^{\lambda x}dx))$. For $A + m^{2s} \leq 0$ sufficiently small (that is, $|A|$ sufficiently large) satisfying

\begin{equation}
\frac{1}{4}((-|\lambda|^2 + m^2)^s - A)^2 \geq C_2(N, s)m^{4s},
\end{equation}

where $C_2(N, s)$ is a positive constant depending only on $N$ and $s$, we obtain the following lower bound

\begin{equation}
\dot{D}(t) \geq \frac{3}{4}((-|\lambda|^2 + m^2)^s - A)^2 \mathcal{H}(t) - C_1(N, s) \int_{\mathbb{R}^N} \omega F^2 \, dx + 2 \int_{\mathbb{R}^N} \omega (u_t)^2 \, dx
\end{equation}

where $C_1(N, s)$ is a positive constant depending only on $N$ and $s$.

Proof. First of all, direct calculations and Proposition 2.3 give

\begin{equation}
\omega_t = A \omega, \quad \omega_{tt} = A^2 \omega, \quad L^s_m \omega = (-|\lambda|^2 + m^2)^s \omega, \quad L^s_m (\omega_t) = (-|\lambda|^2 + m^2)^s A \omega,
\end{equation}

\begin{equation}
L^{2s} \omega = (-|\lambda|^2 + m^2)^{2s} \omega,
\end{equation}

and

\begin{equation}
H^s_m (\omega, \omega) = ((-4|\lambda|^2 + m^2)^s - 2(-|\lambda|^2 + m^2)^s) \omega^2.
\end{equation}

Let $\mathcal{H}(t)$ be defined as in (5.2). Then

\begin{equation}
\dot{\mathcal{H}}(t) = \int_{\mathbb{R}^N} \omega_t u^2 \, dx + 2 \int_{\mathbb{R}^N} \omega uu_t \, dx = \int_{\mathbb{R}^N} (\omega_t - L^s_m \omega)u^2 \, dx + \int_{\mathbb{R}^N} \omega H^s_m (u, u) \, dx + 2 \int_{\mathbb{R}^N} \omega F^2 \, dx.
\end{equation}

Thus

\begin{equation}
\dot{\mathcal{H}}(t) = D(t) + 2 \int_{\mathbb{R}^N} \omega F^2 \, dx.
\end{equation}

We focus on $D(t)$. We have, by using the relation (2.9) and after tedious computations,

\begin{equation}
D(t) = \int_{\mathbb{R}^N} (\omega_t - 2L^s_m (\omega_t) + L^{2s}_m \omega)u^2 \, dx + 2 \int_{\mathbb{R}^N} \omega_t H^s_m (u, u) \, dx + 2 \int_{\mathbb{R}^N} (\omega_t - L^s_m \omega \cdot u F \, dx
\end{equation}

\begin{equation}
+ 2 \int_{\mathbb{R}^N} \omega (L^s_m u - F)^2 \, dx - 2 \int_{\mathbb{R}^N} \omega F^2 \, dx - \int_{\mathbb{R}^N} \omega H^{2s}_m (u, u) \, dx - 2 \int_{\mathbb{R}^N} H^s_m (\omega, u) F \, dx.
\end{equation}

In view of the expression of $\omega$ in (5.2), we have that, by (5.8) and (5.9),

\begin{equation}
\int_{\mathbb{R}^N} (\omega_t - 2L^s_m (\omega_t) + L^{2s}_m \omega)u^2 \, dx = ((-|\lambda|^2 + m^2)^s - A)^2 \mathcal{H}(t)
\end{equation}

and

\begin{equation}
2 \int_{\mathbb{R}^N} \omega_t H^s_m (u, u) \, dx = 2A \int_{\mathbb{R}^N} \omega H^s_m (u, u) \, dx.
\end{equation}

Concerning the term $2 \int_{\mathbb{R}^N} (\omega_t - L^s_m \omega) \cdot u F \, dx$, by (5.8) and by applying arithmetic-geometric inequality (AM-GM inequality), we get

\begin{equation}
2 \left| \int_{\mathbb{R}^N} (\omega_t - L^s_m \omega)u F \, dx \right| = 2 \left| \int_{\mathbb{R}^N} (A - (-|\lambda|^2 + m^2)^s) \omega u F \, dx \right|
\end{equation}

\begin{equation}
\leq \frac{1}{4}((-|\lambda|^2 + m^2)^s - A)^2 \mathcal{H}(t) + 4 \int_{\mathbb{R}^N} \omega F^2 \, dx.
\end{equation}

Let us see now how to estimate the term $-2 \int H^s_m (\omega, u) F$ in (5.11). By definition we have that

\begin{equation}
-2 \int_{\mathbb{R}^N} H^s_m (\omega, u) F \, dx
\end{equation}
\[ = 2C_{N,s}m^{N+2s}e^{At} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(e^{\lambda x} - e^{\lambda y})}{|x - y|^{N+2s}} K_{N+2s}(m|x - y|) dy F \, dx + 2m^{2s} \int_{\mathbb{R}^N} F u \omega \, dx \]

where the integrals \( I_{|x-y|<1/m} \) and \( I_{|x-y|>1/m} \) are determined by the splitting \( \int_{\mathbb{R}^N} \int_{|x-y|<1/m} \) and \( \int_{\mathbb{R}^N} \int_{|x-y|>1/m} \), respectively. The integral close to the origin is bounded as follows (we use the asymptotics of Macdonald’s function in (2.4) that involve constants depending on \( s \) that do not blow up)

\[ I_{|x-y|<1/m} \approx c C_{N,s} \Gamma \left( \frac{N + 2s}{2} \right) 2^{N+2s-1} m^{N+2s} \]

Let us estimate the quantity

\[ \frac{1}{2} \left( \int_{|z|<1/m} \frac{(1 - e^{-\lambda z})^2}{|z|^{N+2s}} \, dz \right)^{1/2} C_{N,s} \Gamma \left( \frac{N + 2s}{2} \right) 2^{N+2s-1} m^{N+2s} \]

Using again the asymptotics for the Macdonald’s function, and applying Cauchy-Schwartz in the integral in the variable \( y \), we get

\[ |I_{|x-y|<1/m}| \leq c \left( \int_{|z|<1/m} \frac{(1 - e^{-\lambda z})^2}{|z|^{N+2s}} \, dz \right)^{1/2} C_{N,s} \Gamma \left( \frac{N + 2s}{2} \right) 2^{N+2s-1} m^{N+2s} \]

We will prove that for \( |\lambda| < m \), the positive constant (5.14) is bounded from above independently of \( \lambda \) and \( m \). Indeed, for \( |\lambda| < m \), using the Mean Value Theorem we obtain

\[ \int_{|z|<1/m} \frac{(1 - e^{-\lambda z})^2}{|z|^{N+2s}} \, dz \leq \int_{|z|<1/m} \frac{e^{2|m|\lambda |z|^2}}{|z|^{N+2s}} \, dz = e^{2\frac{|\lambda|^2}{m}} \frac{1}{2m^{2-2s}} \omega_N \leq e^{2\frac{1}{2} \omega_N m^{2s}}. \]

We take into account that \( C_{N,s} \) is given by the formula (2.5), thus the constant (5.14) above is bounded by

\[ c e^{2 \frac{1}{4} \omega_N C_{N,s} \Gamma \left( \frac{N + 2s}{2} \right) 2^{N+2s-1} m^{N+2s}} = -c e^{2 \frac{1}{2} \omega_N \frac{4s}{\pi^{N/2}} \Gamma \left( \frac{N+2s}{2} \right)} m^{2s} = C_1(N,s) m^{2s}. \]
Now we apply AM-GM in the variable \(x\) to obtain

\[
|I_{|x-y|<1/m}| \leq \frac{m^{2s}}{2} \int_{\mathbb{R}^N} e^{\lambda x} H_m^s(u,u) \, dx + C_1(N,s) \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx.
\]

For the integral away from the origin we have

\[
I_{|x-y|>1/m} = C_{N,s} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} \frac{e^{\lambda x} u(t,x) - e^{\lambda y} u(t,x) - e^{\lambda y} u(t,y) + e^{\lambda y} u(t,y)}{|x-y|^{\frac{N+2s}{2}}} K_{N+2s} (m|x-y|) \, dy \, dx
\]

For \(E_1\), we get

\[
E_1 = C_{N,s} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} (e^{\lambda x} - e^{\lambda y}) u(t,x) \frac{K_{N+2s} (m|x-y|)}{|x-y|^{\frac{N+2s}{2}}} \, dy \, dx
\]

Observe that

\[
\int_{|z|>1/m} \frac{1-e^{\lambda z}}{z^{\frac{N+2s}{2}}} K_{N+2s} (m|z|) \, dz = m^{-\frac{N+2s}{2}} - m \int_{|y|>1} \frac{1-e^{\lambda y}}{y^{\frac{N+2s}{2}}} K_{N+2s} (|y|) \, dy.
\]

Thus, applying AM-GM inequality, we get

\[
|E_1| \leq C_{N,s} m^{2s} \int_{\mathbb{R}^N} e^{\lambda x} |u| \, |F| \, dx \leq C_{N,s} m^{2s} \int_{\mathbb{R}^N} e^{\lambda x} u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx.
\]

For \(E_2\) we have

\[
|E_2| \leq C_{N,s} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} \frac{(e^{\lambda x} - e^{\lambda y}) u(t,y)}{|x-y|^{\frac{N+2s}{2}}} K_{N+2s} (m|x-y|) \, dy \, dx
\]

\[
= C_{N,s} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} e^{\frac{\lambda y}{2}} u(t,y) \left| e^{\frac{\lambda x}{2}} - e^{\frac{\lambda y}{2}} \right| \frac{K_{N+2s} (m|x-y|)}{|x-y|^{\frac{N+2s}{2}}} \, dy \, dx
\]

\[
\leq C_{N,s} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} e^{\frac{\lambda x}{2}} F^2 \, dx \cdot \left( \int_{\mathbb{R}^N} \int_{|x-y|>1/m} \left| e^{\frac{\lambda y}{2}} u(t,y) \right| \cdot k(x-y) \, dy \right) \, dx
\]

\[
= C_{N,s} \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx \cdot \|e^{\frac{\lambda x}{2}} u \cdot k\|_{L^1(\mathbb{R}^N)}
\]

\[
\leq C_{N,s} \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx \cdot \|e^{\frac{\lambda x}{2}} u \|_{L^1(\mathbb{R}^N)}.
\]
where \( k(z) = \chi_{\{|z| > 1/m\}}(\|z\|) \cdot \frac{\lambda z}{|z|^{N+2s}} K_{N+2s}(m|z|) \). Observe that if \( |\lambda| < m \) then

\[
\|k(x)\|_{L^1(\mathbb{R}^N)} \leq 2\sqrt{\frac{\pi}{2}} m^{-1/2} \int_{\{x|>1/m\}} \frac{1}{|x|^{N+2s}} e^{-m|x|} \, dx = \sqrt{\frac{\pi}{2}} \frac{2^{N-2s}}{m^{N-2s}} \Gamma \left( \frac{N-2s-1}{2} \right).
\]

We use AM-GM to get

\[
|E_2| \leq C_{N,s} \left( \sqrt{\frac{\pi}{2}} \frac{2^{N-2s}}{m^{N-2s}} \Gamma \left( \frac{N-2s-1}{2} \right) \right)^2 m^{4s} \int_{\mathbb{R}^N} e^{\lambda^+ x} u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} e^{\lambda^+ x} F^2 \, dx.
\]

Thus using (5.15), (5.16) and (5.17) we conclude that

\[
-2 \int H^s_m(\omega, u) F \, dx \leq -m^{2s} \int_{\mathbb{R}^N} e^{At+\lambda^+ x} H^s_m(u, u) \, dx + 8C_1(N, s) \int_{\mathbb{R}^N} e^{At+\lambda^+ x} F^2 \, dx
\]

\[
\quad + C_2(N, s)m^{4s} \int_{\mathbb{R}^N} e^{At+\lambda^+ x} u^2 \, dx + \int_{\mathbb{R}^N} e^{At+\lambda^+ x} F^2 \, dx
\]

\[
\quad + C_3(N, s)m^{4s} \int_{\mathbb{R}^N} e^{At+\lambda^+ x} u^2 \, dx + \int_{\mathbb{R}^N} e^{At+\lambda^+ x} F^2 \, dx
\]

\[
\quad + 4m^{4s} \int_{\mathbb{R}^N} e^{At+\lambda^+ x} u^2 \, dx + \int_{\mathbb{R}^N} e^{At+\lambda^+ x} F^2 \, dx,
\]

where \( C_1(N, s) \) is as above, and \( C_2(N, s) \) and \( C_3(N, s) \) are positive and depend only on \( N \) and \( s \). We rename the constants (i.e., below \( C_1(N, s) \) and \( C_2(N, s) \) mean different constants, but still positive and depending only on \( N \) and \( s \)) and we have

\[
-2 \int H^s_m(\omega, u) F \, dx \leq -m^{2s} \int_{\mathbb{R}^N} e^{At+\lambda^+ x} H^s_m(u, u) \, dx + C_1(N, s) \int_{\mathbb{R}^N} e^{At+\lambda^+ x} F^2 \, dx + C_2(N, s)m^{4s} \int_{\mathbb{R}^N} e^{At+\lambda^+ x} u^2 \, dx.
\]

Summing up, from (5.11), (5.12), (5.13) and (5.18) we have

\[
\hat{D}(t) \geq \frac{3}{4} \left( (-|\lambda|^2 + m^2)^s - A \right)^2 \mathcal{H}(t) - C_1(N, s) \int_{\mathbb{R}^N} \omega F^2 \, dx + 2 \int_{\mathbb{R}^N} \omega(u_t)^2 \, dx
\]

\[
\quad + (A + m^{2s}) \int_{\mathbb{R}^N} H^s_m(u, u) \omega \, dx - \int_{\mathbb{R}^N} \omega H^s_m(u, u) \, dx - C_2(N, s)m^{4s} \int_{\mathbb{R}^N} \omega u^2 \, dx.
\]

Thus if we take \( A + m^{2s} < 0 \) sufficiently small (that is \( |A| \) sufficiently large) such that \( A \) satisfies (5.6), we get the conclusion.

Notice that since we want our energy terms to be positive, the presence of \( H^s_m \) imposes the restriction \( s \in (0, 1/2] \).

5.3. Carleman inequality. Once we have Proposition 5.3 at our disposal, we will be able to prove Theorem 1.3 in the Introduction. We state the result here again, with a slightly different reformulation.

Theorem 5.4. Let \( N \geq 1 \), \( s \in (0, 1/2] \), \( m > 0 \), \( \lambda \in \mathbb{R}^N \) with \( |\lambda| < m \), and \( u \) be a solution to Problem (5.1) such that \( u \in L^2(\mathbb{R}^N) \) and \( u^2 \in \text{Dom}(L^2_m) \). Let \( \omega(t, x) \) be as in (5.2) and \( F \in L^2(0,T;L^2(\mathbb{R}^N) \). Then the following Carleman inequality holds

\[
\frac{1}{2} \int_0^1 \mathcal{H}(t) \, dt + \frac{1}{2} (-|\lambda|^2 + m^2)^s - A)^2 \int_0^1 t(1-t) \mathcal{H}(t) \, dt
\]

\[
+ \frac{1}{2} \int_0^1 t(1-t) \left\{ 2 \int_{\mathbb{R}^N} \omega(u_t)^2 \, dx - \int_{\mathbb{R}^N} \omega H^s_m(u, u) \, dx + (A + m^{2s}) \int_{\mathbb{R}^N} H^s_m(u, u) \, dx \right\} \, dt
\]

\[
\leq \frac{1}{2} \mathcal{H}(0) + \frac{1}{2} \mathcal{H}(1) + C_1(N, s) \int_0^1 \int_{\mathbb{R}^N} \omega((\partial_t + L^s_m)(u))^2 \, dx \, dt,
\]
for $A + m^{2s} < 0$ sufficiently small (that is, $|A|$ sufficiently large) satisfying
\[((-|\lambda|^2 + m^2)^s - A)^2 \geq C_2(N,s)m^{4s},\]
where $C_1(N,s), C_2(N,s)$ are positive constants depending only on $N$ and $s$.

**Proof.** We will consider the following tent function
\[
\eta(\tau) = \begin{cases} 
\frac{\tau}{t}, & 0 \leq \tau \leq t, \\
\frac{1}{1-t} - \frac{\tau}{t}, & t \leq \tau \leq 1.
\end{cases}
\]

Then $\dot{\eta}$ is a decreasing step function
\[
\dot{\eta}(\tau) + \frac{1}{1-t} = \begin{cases} 
\frac{1}{1-t}, & 0 \leq \tau \leq t, \\
0, & t \leq \tau \leq 1.
\end{cases}
\]

Thus by denoting $\delta$ the distributional derivative of the Heaviside function with values 0 and 1 we obtain that $\dot{\eta} = \frac{1}{1-t}\delta_t$ in the distributional sense. Let $\overline{\mathcal{H}}(t) = -(1 - t)\mathcal{H}(0) - t\mathcal{H}(1) + \mathcal{H}(t)$, so that $\overline{\mathcal{H}}(0) = 0$ and $\overline{\mathcal{H}}(1) = 0$. Then, integrating by parts we obtain
\[
\int_0^1 \overline{\mathcal{H}}(\tau)\dot{\eta}(\tau) d\tau = -\int_0^1 \dot{\eta}(\tau)\overline{\mathcal{H}}(\tau) d\tau
\]
and thus we infer that
\[
\mathcal{H}(t) = (1 - t)\mathcal{H}(0) + t\mathcal{H}(1) + (1 - t)\int_0^1 \dot{\mathcal{H}}(\tau)\eta(\tau) d\tau.
\]

Then, taking into account (5.10), we have
\[
\mathcal{H}(t) = (1 - t)\mathcal{H}(0) + t\mathcal{H}(1) + (1 - t)\int_0^1 \eta(\tau)D(\tau) d\tau + 2t(1 - t)\int_0^1 \dot{\eta}(\tau) \int_{\mathbb{R}^N} \omega u F dxd\tau.
\]

Integrating by parts,
\[
\mathcal{H}(t) = (1 - t)\mathcal{H}(0) + t\mathcal{H}(1) - (1 - t)\int_0^1 \eta(\tau)\dot{D}(\tau) d\tau + 2t(1 - t)\int_0^1 \dot{\eta}(\tau) \int_{\mathbb{R}^N} \omega u F dxd\tau.
\]

We integrate in $t$ between 0 and 1. Notice that
\[
\int_0^1 t(1 - t)\eta(\tau) dt = \frac{1}{2}(1 - \tau) \quad \text{and} \quad \int_0^1 t(1 - t)\dot{\eta}(\tau) dt = \frac{1 - 2\tau}{2}.
\]

Then
\[
\int_0^1 \mathcal{H}(t) dt = \frac{1}{2}\mathcal{H}(0) + \frac{1}{2}\mathcal{H}(1) - \frac{1}{2}\int_0^1 (1 - \tau)\dot{D}(\tau) d\tau + \int_0^1 (1 - 2\tau) \int_{\mathbb{R}^N} \omega u F dxd\tau.
\]

By renaming the integrals in $\tau$, this is equivalent to
\[
\int_0^1 \mathcal{H}(t) dt + \frac{1}{2}\int_0^1 t(1 - t)\dot{D}(t) dt = \frac{1}{2}\mathcal{H}(0) + \frac{1}{2}\mathcal{H}(1) + \int_0^1 (1 - 2t) \int_{\mathbb{R}^N} \omega u F dxd\tau.
\]

Now we use the estimate (5.7) of Proposition 5.3 (under the assumptions on $A$), so that
\[
\int_0^1 \mathcal{H}(t) dt + \frac{1}{2}\int_0^1 t(1 - t)\frac{3}{4}(A - (\lambda^2 + m^{2s})^s)^2 \mathcal{H}(t) dt - C_1(N,s) \int_{\mathbb{R}^N} t(1 - t) \int_{\mathbb{R}^N} \omega F^2 dxd\tau
\]
\[+ \frac{1}{2} \int_{\mathbb{R}^N} t(1 - t) \int_{\mathbb{R}^N} \omega (u^2) dxd\tau - \int_{\mathbb{R}^N} \omega H_m^s(u,u) dxd\tau + (A + m^{2s}) \int_{\mathbb{R}^N} H_m^s(u,u) \omega dxd\tau
\]
\[
\leq \frac{1}{2}\mathcal{H}(0) + \frac{1}{2}\mathcal{H}(1) + \int_0^1 (1 - 2t) \int_{\mathbb{R}^N} \omega u F dxd\tau,
\]
or, equivalently,
\[
\int_0^1 \mathcal{H}(t) dt + \frac{1}{2}\int_0^1 t(1 - t)\frac{3}{4}(\lambda^2 + m^2)^s - A)^2 \mathcal{H}(t) dt
\]
\[ + \frac{1}{2} \int_0^1 t(1-t) \left\{ 2 \int_{\mathbb{R}^N} \omega(u_t)^2 - \int_{\mathbb{R}^N} \omega H^2_{m}(u, u) + (A + m^2) \int_{\mathbb{R}^N} H^s_{m}(u, u) \omega dx \right\} \]
\[ \leq \frac{1}{2} \mathcal{H}(0) + \frac{1}{2} \mathcal{H}(1) + C_1(N, s) \int_0^1 t(1-t) \int_{\mathbb{R}^N} \omega F^2 dxdt + \int_0^1 (1-2t) \int_{\mathbb{R}^N} \omega uF dx dt. \]

Applying the AM-GM inequality \((1-2t)\omega uF \leq \frac{1}{2}(1-2t)^2 \omega F^2 + \omega u^2 \) yields
\[ \frac{1}{2} \int_0^1 \mathcal{H}(t) dt + \frac{3}{8}((-|\lambda|^2 + m^2)^s - A)^2 \int_0^1 t(1-t) \mathcal{H}(t) dt \]
\[ + \frac{1}{2} \left( \int_0^1 t(1-t) \left\{ 2 \int_{\mathbb{R}^N} \omega(u_t)^2 - \int_{\mathbb{R}^N} \omega H^2_{m}(u, u) + (A + m^2) \int_{\mathbb{R}^N} H^s_{m}(u, u) \omega dx \right\} dt \right) \]
\[ \leq \frac{1}{2} \mathcal{H}(0) + \frac{1}{2} \mathcal{H}(1) + \int_0^1 \left( \frac{1}{2}(1-2t)^2 + C_1(N, s)t(1-t) \right) \int_{\mathbb{R}^N} \omega F^2 dx dt. \]

Finally, by considering the maximum of the weight functions in \( t \) we obtain the conclusion. \( \square \)

**Corollary 5.5.** Due to the monotonicity of \( \mathcal{H}(t) \) as a function of \( t \) (in particular, by (5.19)), we have that
\[ \mathcal{H}(1) \leq \int_0^1 \mathcal{H}(t) dt + e^{A-(-|\lambda|^2+m^2)^s} \int_0^1 \int_{\mathbb{R}^N} \omega F^2 dx dt. \]

So the term \( \frac{1}{2} \mathcal{H}(1) \) can be hidden in the left hand side into the term \( \frac{1}{2} \int \mathcal{H}(t) dt \). Hence, the resulting terms turn out to be still positive, and we derive, from the Carleman inequality (5.19), that
\[ \frac{1}{2}((-|\lambda|^2 + m^2)^s - A)^2 \int_0^1 \mathcal{H}(t)t(1-t)dt + \text{positive energy terms} \]
\[ \leq \frac{1}{2} \mathcal{H}(0) + \left( C_1(N, s) + e^{A-(-|\lambda|^2+m^2)^s} \right) \int_0^1 \int_{\mathbb{R}^N} \omega ((\partial_t + L_m)(u))^2 dx dt, \]

where \( A \) satisfies (5.6).

**Remark 5.6.** Observe that, from (5.20), we could also immediately deduce the following convexity inequality:
\[ \| \sqrt{t(1-t)\omega t^{1/2}} u \|_{L^2(\mathbb{R}^N \times [0, 1])} \lesssim \sup_{t \in [0, 1]} \| \omega^{1/2} F \|_{L^2(\mathbb{R}^N)} + \mathcal{H}(0) + \mathcal{H}(1). \]

The Carleman inequality in (5.21) reminds the one contained in [13, Lemma 4]. Such an inequality is used therein to obtain a convexity inequality (see [13, Theorem 3]).

**6. CARLEMAN INEQUALITY FOR ELLIPTIC AND PARABOLIC OPERATORS WITH QUADRATIC EXPONENTIAL WEIGHT**

In this section we will obtain Carleman estimates for the operator \( L^s_{m} \) with the approach of the pseudo-differential operators. The basic references in our case are [19 Chapter 0], [20 Chapter 2], [26 Chapter 6], and [31, 32, 33, 34, 35], where the latter ones deal with the so-called Weyl or semiclassical pseudo-differential calculus.

We will first introduce some definitions. A smooth function \( p(x, \xi) \) defined on \( \mathbb{R}^N \times \mathbb{R}^N \) belongs to the class \( S^m_{\rho, \delta} \) if
\[ |\partial_x^\alpha \partial_{\xi}^\beta p(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{n-\rho|\beta|+\delta|\alpha|} \]
for all multi-indices \( \alpha, \beta \), where \( n \in \mathbb{R} \) and \( 0 \leq \rho, \delta \leq 1 \). If \( \rho = 1 \) and \( \delta = 0 \), we then say that \( p \) is a multiplier or a symbol of order \( n \), and we write \( p \in S^n \). Given \( p \in S^n_{\rho, \delta} \), the pseudo-differential operator \( P(x, D) \) with symbol \( p(x, \xi) \) is defined by
\[ P(\cdot, D)f(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} p(x, \xi) \hat{f}(\xi) e^{ix \cdot \xi} d\xi. \]
We say that the pseudo-differential operator $P(x, D)$ (or just $P(D)$) with symbol $p(x, \xi) \in S^n$ has principal symbol $a_n(x, \xi)$ if
\[
p(x, \xi) - a_n(x, \xi) \in S^{n-1}.
\]
We will sometimes use the notation $\text{Op}(p(x, \xi))$ to denote the pseudo-differential operator whose symbol is $p(x, \xi)$ and the notation symbol$(P)$ to denote the symbol associated to the operator $P$.

Coming back to the definitions of the fractional relativistic operator in Section 2, we will focus on the one given by the Fourier transform. In other words, calling $L^s_m =: P_m(D)$, we will consider the pseudo-differential operator
\[
P_m(D) = (-\Delta + m^2)^s := \left( \sum_{j=1}^{N} (D_{x_j})^2 + m^2 \right)^s,
\]
with $D_{x_j} = \frac{1}{i} \partial_{x_j}$, so that $D = (D_{x_1}, \ldots, D_{x_N})$. Observe that $D^2 = -\Delta$ and that $P_0(D) = (-\Delta)^s$. The pseudo-differential operator $P_m$ has symbol $p(x, \xi) = (|\xi|^2 + m^2)^s$. Fractional powers of the Laplacian have been treated as pseudo-differential operators extensively by Grubb, see for instance [25] [26].

Let $\varphi \in C^\infty(\mathbb{R}^N)$. We consider the conjugate operator $e^\varphi P_m(D)e^{-\varphi}$, we will represent it as the sum of its symmetric part $S$ and antisymmetric part $A$. Indeed, functional calculus allows us to prove the following formula (see Lemma A.2)
\[
Q(x, D) := e^\varphi P_m(D)e^{-\varphi} = (-\Delta + m^2 - (\nabla \varphi)^2 + \nabla \circ \nabla \varphi + \nabla \varphi \cdot \nabla)^s
\]
\[
= \left( D^2 + m^2 - (\nabla \varphi)^2 + i \sum (D_{x_j} \circ \partial_{x_j} \varphi + \partial_{x_j} \varphi \cdot D_{x_j}) \right)^s =: S(x, D) + A(x, D),
\]
with symbol
\[
q(x, \xi) = (\xi^2 + m^2 - (\varphi'_{x})^2 + 2i\xi\varphi'_{x})^s =: a(x, \xi) + ib(x, \xi),
\]
where
\[
a(x, \xi) = |\xi^2 + m^2 - (\varphi'_{x})^2 + 2i\xi\varphi'_{x}|^s \cos(s\theta(x, \xi)),
b(x, \xi) = |\xi^2 + m^2 - (\varphi'_{x})^2 + 2i\xi\varphi'_{x}|^s \sin(s\theta(x, \xi)),
\]
and
\[
\theta(x, \xi) := \text{arctan} \frac{2\xi\varphi'_{x}}{\xi^2 + m^2 - (\varphi'_{x})^2}.
\]
Observe that $q(x, \xi) \in S^{2s}$ and $a$ and $b$ are positive. The operators $S$ and $A$ above have symbols $a$ and $ib$
\[
S(x, D) := \text{Op}(a(x, \xi)), \quad A(x, D) := \text{Op}(ib(x, \xi)).
\]
We compute the Poisson bracket
\[
\{a, b\} = a_{\xi}b_{x} - a_{x}b_{\xi} = 4s^2\varphi''_{x}\left[ (\xi^2 + m^2 - (\varphi'_{x})^2)^2 + 4\xi^2(\varphi'_{x})^2 \right]^{s-1} \cdot (\xi^2 + (\varphi'_{x})^2).
\]
It is known that, if $a \in S^n, b \in S^{n'}$, then
\[
\text{symbol}([S, A]) - \{a, b\}(\xi) \in S^{n+n'-1}.
\]
We will be mostly interested in knowing to which class the Poisson bracket $\{a, b\}$ belongs.

6.1. **Towards a Carleman estimate using pseudo-differential calculus.** For the sake of the reading, we will study the case of dimension $N = 1$, although the reasoning remains completely valid for higher dimensions. Let us consider now a weight $\varphi := \varphi(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R})$ to be determined. In order to get a Carleman estimate for the operator $P_m(D)$, we observe that
\[
\left\| e^\varphi P_m(D)(e^{-\varphi}f) \right\|_{L^2}^2 = \left\| (S + A)f, (S + A)f \right\|_{L^2} = \left\| Sf \right\|_{L^2}^2 + \left\| Af \right\|_{L^2}^2 + \left\langle [S, A]f, f \right\rangle.
\]
On the other hand, we have
\[
e^\varphi(\partial_t + P_m(D))e^{-\varphi} = -\varphi_t + \partial_t + e^\varphi P(D)e^{-\varphi} = \left( -\varphi_t + S(x, D) \right) + (\partial_t + A(x, D)).
\]
The commutator corresponds to
\[ [\tilde{S}, \tilde{A}] := [-\varphi_t + S, \partial_t + A] = [-\varphi_t, \partial_t] + [S, \partial_t] + [-\varphi_t, A] + [S, A]. \]

We use pseudo-differential calculus also to define the operators with \( \partial_t \) since it is more convenient to work in the symbol side for the conjugated operator \( e^{\varphi}(\partial_t + P_m(D))e^{-\varphi} \). Let \((x,t,\xi,\tau)\) be the variables involved in the symbol. Thus
\[ \tilde{S} := -\varphi_t + S(x, D) = \text{Op}(-\varphi_t(t,x) + a(x,t,\xi,\tau)) \]
and
\[ \tilde{A} := \partial_t + A(x, D) = \text{Op}(i\tau + b(x, t, \xi, \tau)). \]

Therefore, we can write
\[
(6.2) \quad ||e^{\varphi}(\partial_t + P_m(D))(e^{-\varphi} f)||_{L^2}^2 = (\langle \tilde{S} + \tilde{A} \rangle f, (\tilde{S} + \tilde{A}) f) = ||\tilde{S} f||_{L^2}^2 + ||\tilde{A} f||_{L^2}^2 + \langle [\tilde{S}, \tilde{A}] f, f \rangle.
\]

We denote by \( \bar{a} \) and \( \bar{b} \) the real and imaginary part of the symbols of \( \tilde{S} \) and \( \tilde{A} \).
\[
\bar{a}(x,t,\xi,\tau) = -\partial_t \varphi(t,x) + a(x,t,\xi,\tau), \quad \bar{b}(x,t,\xi,\tau) = \tau + b(x,t,\xi,\tau),
\]
where we write
\[
a(x,t,\xi,\tau) = |\xi^2 + m^2 - (\partial_x \varphi)^2 + 2i\xi\partial_x \varphi|^s \cos(s\theta(x,t,\xi,\tau))
\]
\[
b(x,t,\xi,\tau) = |\xi^2 + m^2 - (\partial_x \varphi)^2 + 2i\xi\partial_x \varphi|^s \sin(s\theta(x,t,\xi,\tau)),
\]
and
\[
\theta := \theta(x,t,\xi,\tau) := \arctan \frac{2\xi\partial_x \varphi}{\xi^2 + m^2 - (\partial_x \varphi)^2}.
\]

The Poisson bracket \( \{\bar{a}, \bar{b}\} \) is as follows
\[
\{\bar{a}, \bar{b}\} = \bar{a}_x \bar{b}_x - \bar{a}_x \bar{b}_x - \bar{a}_t \bar{b}_t = a_x b_x - (-\varphi_t + a_t) \cdot 1
\]
\[
= \{a, b\} + \partial_x^2 \varphi b_x + \partial^2 \varphi - a_t.
\]

Here, recall that \( \{a, b\} \) was computed in (6.1). Some more computations deliver
\[
b_x = 2s[(\xi^2 + m^2 - (\partial_x \varphi)^2)^2 + 4\xi^2(\partial_x \varphi)^2]^{\frac{s}{2} - 1} [\xi(\xi^2 + m^2 - (\partial_x \varphi)^2) \sin(s\theta) + \partial_x \varphi(m^2 - (\partial_x \varphi)^2) \cos(s\theta)]
\]
and
\[
a_t = 2s\partial_t^2 \varphi \partial_x \varphi [(\xi^2 + m^2 - (\partial_x \varphi)^2)^2 + 4\xi^2(\partial_x \varphi)^2]^{\frac{s}{2} - 1} [-\xi(\xi^2 + m^2 - (\partial_x \varphi)^2) \sin(s\theta) + \xi \cos(s\theta)].
\]

In order to get the Carleman inequality we will prove that \( \{a, b\} \geq 0 \). Observe first that we need \( \partial_x^2 \varphi \geq 0 \) in order to have \( \{a, b\} \geq 0 \), so this requires convexity in the chosen function \( \varphi \). The strategy to prove the Carleman inequality will be as follows:

**Step I.** We will introduce a Carleman parameter so that we take \( \varphi \to \alpha \varphi \). Then we have to prove that for \( \alpha \) large enough, the terms \( \partial_t^2 \varphi b_x, \partial_t^2 \varphi \) and \( a_t \) can be hidden into the dominant term \( \{a, b\} \) (of course this is not needed when one of these terms is positive). Moreover, we will consider the scales \( \varphi = \varphi(x/R) \) and we will see that the dominant term will correspond to the coefficient \( \alpha^{4s-1}/R^{4s} \). In the local case \( s = 1 \) this equals \( \alpha^3/R^4 \) (the so-called Meshkov exponent, see [37]).

**Step II.** The next step will be to apply the Gårding Inequality for \( \{a, b\} \) to get the Carleman estimate.
6.2. Step I: Choice of the weight function and positivity of the symbol. Let \( \alpha > 0 \), \( R > 0 \) and \( \psi : [0, \infty) \to \mathbb{R} \) be a smooth function. We will choose

\[
(6.4) \quad \varphi(t, x) = \alpha \left( \frac{x}{R} + \psi(t) \right)^2.
\]

Then
\[
\begin{align*}
\partial_x \varphi(t, x) &= 2 \frac{\alpha}{R} \left( \frac{x}{R} + \psi(t) \right), \\
\partial_t \varphi(t, x) &= 2 \alpha \left( \frac{x}{R} + \psi(t) \right) \psi'(t), \\
\partial_{xx} \varphi(t, x) &= 2 \frac{\alpha}{R^2}, \\
\partial_{tt} \varphi(t, x) &= 2 \alpha (\psi'(t))^2 + 2 \alpha \left( \frac{x}{R} + \psi(t) \right) \psi''(t),
\end{align*}
\]
and
\[
\partial_{xt} \varphi(t, x) = \partial_{tx} \varphi(t, x) = 2 \frac{\alpha}{R} \psi'(t).
\]

**Proposition 6.1.** Let \( \varphi \) be the function defined in (6.4) and let \( \bar{a} + \bar{b} \) be the symbol of the conjugate operator \( e^\varphi (\partial_t + P_m(D)) e^{-\varphi} \), where \( (t, x) \in K := \{ 1 \leq | \frac{x}{R} + \psi(t) | \leq 4 \} \cap \{|x| \leq R \} \). Let \( 1/2 < s < 1 \). Then if
\[
s \frac{\alpha^{2s-1}}{R^{2s}} \geq c \| \psi' \|_\infty + \| \psi'' \|_\infty^{1/2} \quad \text{and} \quad m \leq 2 \frac{\alpha}{R}
\]
it holds that
\[
\{ \bar{a}, \bar{b} \} \geq 2s^2 \partial_x^2 \varphi \left[ \left( \xi^2 + m^2 - (\partial_x \varphi)^2 \right)^2 + 4 \xi^2 (\partial_x \varphi)^2 \right] s^{-1} \cdot \left( \xi^2 + (\partial_x \varphi)^2 \right).
\]
In particular,
\[
\{ \bar{a}, \bar{b} \} \geq c s^2 \frac{\alpha}{R^2} \left( \xi^2 + 4 \frac{\alpha^2}{R^2} \right)^{2s-1} \quad \text{for all} \ \xi \in \mathbb{R}.
\]

**Proof.** Let us compute each of the terms in \( \{ \bar{a}, \bar{b} \} \) in (6.3). With the choice of \( \varphi \), by (6.1) we have

\[
I := \{a, b\} = \frac{8s \alpha^2}{R^2} \frac{\xi^2 + 4 \frac{\alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2}{\left[ \left( \xi^2 + m^2 - \frac{4 \alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2 \right)^2 + 16 \xi^2 \frac{\alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2 \right]^{1-s}}.
\]

The next term in the Poisson bracket is
\[
\partial_{t,x}^2 \varphi \beta_{x\xi} = 2 \frac{\alpha}{R} \psi'(t) 2s \xi \left[ \xi \left( \xi^2 + m^2 - (\partial_x \varphi)^2 \right) \sin(s \theta) + \partial_x \varphi (m^2 - (\partial_x \varphi)^2) \cos(s \theta) \right] = II + III
\]
where
\[
II := \frac{2 \alpha}{R} \psi'(t) 2s \xi \left[ \xi \left( \xi^2 + m^2 - (\partial_x \varphi)^2 \right) \sin(s \theta) \right] \left[ \left( \xi^2 + m^2 - (\partial_x \varphi)^2 \right)^2 + 4 \xi^2 (\partial_x \varphi)^2 \right]^{1-s/2}
\]
\[
\begin{align*}
&\frac{2 \alpha}{R} \psi'(t) 2s \xi |\xi|^{2s-2} \left( 1 + m^2/\xi^2 + 4/\xi^2 \frac{4 \alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2 \right) \sin(s \theta) \\
&\left[ \left( 1 + m^2/\xi^2 - 4/\xi^2 \frac{4 \alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2 \right)^2 + 4/\xi^2 \frac{4 \alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2 \right]^{1-s/2}
\end{align*}
\]
and
\[
III := \frac{2 \alpha}{R} \psi'(t) 2s \xi \left[ \left( \xi^2 + m^2 - (\partial_x \varphi)^2 \right) \cos(s \theta) \right] \left[ \left( \xi^2 + m^2 - (\partial_x \varphi)^2 \right)^2 + 4 \xi^2 (\partial_x \varphi)^2 \right]^{1-s/2}.
\]

Next, we have the term
\[
IV := \partial_x^2 \varphi = 2 \alpha (\psi')^2 + 2 \alpha \left( \frac{x}{R} + \psi(t) \right) \psi''(t) =: IV_a + IV_b.
\]

The first term \( IV_a \) is positive, and for the second \( |IV_b| \leq 8 \alpha |\psi''(t)| \). The last term is
\[
V := a_t = 2s \partial_{t,x} \varphi \partial_{x,t} \varphi \left[ \left( \xi^2 + m^2 - (\varphi_x)^2 \right)^2 + 4 \xi^2 (\partial_x \varphi)^2 \right]^{\frac{1}{2}-1}
\]
\[
\times \left[ - \left( \xi^2 + m^2 - (\varphi_x)^2 \right) \sin(s \theta) + \left( \xi^2 - m^2 - (\partial_x \varphi)^2 \right) \cos(s \theta) \right].
\]

In what follows we will need the observation that the function
\[
h(\xi) = \left( \xi^2 + m^2 - (\partial_x \varphi)^2 \right)^2 + 4 \xi^2 (\partial_x \varphi)^2 = \left( \xi^2 + (\partial_x \varphi)^2 \right)^2 + 2m^2 \xi^2 + m^4 - 2m^2 (\partial_x \varphi)^2
\]
is increasing in $|\xi|$. Thus
\[
\min_{\xi \in \mathbb{R}^N} h(\xi) = h(0) = (m^2 - (\partial_x \varphi)^2)^2 \geq 0.
\]

Now we shall find sufficient conditions on the parameters $\alpha$ and $R$ in order to hide the terms $II$, $III$, $IV$ and $V$ into $I$.

(1) We prove that for suitable $\alpha$ and $R$ it holds that $\frac{1}{10}I \geq II$. To this aim it is sufficient to show, for all $\xi \in \mathbb{R}$,
\[
\frac{s}{5R} \left[ (\xi^2 + m^2 - (\partial_x \varphi)^2)^2 + 4\xi^2(\partial_x \varphi)^2 \right]^{s/2} \geq \|\psi\|_\infty \cdot |\xi|(\xi^2 + m^2 + (\partial_x \varphi)^2).
\]

We will consider two cases. First, for $|\xi| \leq 2\frac{\alpha}{R}$, it is sufficient to have
\[
\frac{s}{5R} \left[ (m^2 - (\partial_x \varphi)^2)^2 \right]^{s/2} \cdot (\partial_x \varphi)^2 \geq \|\psi\|_\infty \cdot \frac{\alpha R}{m^2 + (\frac{\alpha}{R})^2 + (\partial_x \varphi)^2}
\]
that, with the choice of $\varphi$, is equivalent to
\[
\frac{s}{5R} \left[ (m^2 - 4\frac{\alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2)^2 \right]^{s/2} \cdot 4\frac{\alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2 \geq \|\psi\|_\infty \cdot \frac{\alpha}{R^2} \left( m^2 + 4\frac{\alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2 \right).
\]

Since $\varphi$ is supported in $\{1 \leq |x/R + \psi(t)| \leq 4\}$ thus it is sufficient to impose
\[
\frac{s}{5R} \left[ (m^2 - 4\frac{\alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2)^2 \right]^{s/2} \cdot 4\frac{\alpha^2}{R^2} \geq \|\psi\|_\infty \cdot \frac{\alpha}{R^2} \left( m^2 + 4\frac{\alpha^2}{R^2} (\frac{x}{R} + \psi(t))^2 \right).
\]

Moreover, if $2\frac{\alpha}{R} \geq m$ then it is sufficient to take
\[
\frac{s}{5R} \left( 3\frac{\alpha^2}{R^2} \right)^s \cdot 4\frac{\alpha^2}{R^2} \geq \|\psi\|_\infty \cdot \frac{\alpha}{R^2} \frac{6\alpha^2}{R^2},
\]
equivalently
\[
\frac{s}{5R} \frac{\alpha^{2s-1}}{R^2} \geq c\|\psi\|_\infty.
\]

Secondly, for $|\xi|$ away from 0, i.e., for $|\xi| > 2\frac{\alpha}{R}$ (and in its turn $2\frac{\alpha}{R} \geq m$ by assumption) we impose
\[
\frac{s}{5R} |\xi|^{2s-1} \left[ (1 + \frac{m^2}{\xi^2} - (\partial_x \varphi)^2)^2 + 4\frac{(\partial_x \varphi)^2}{\xi^2} \right]^{s/2} \cdot \left( 1 + \frac{\partial_x \varphi}{\xi} \right) \geq \|\psi(t)\|_\infty \cdot \left( 1 + \frac{m^2}{\xi^2} + \frac{(\partial_x \varphi)^2}{\xi^2} \right).
\]

Observe that the function
\[
h(y) = (1 + m^2y - (\partial_x \varphi)^2)^2 + 4(\partial_x \varphi)^2 y
\]
has derivative $h'(y) = 2[ (m^2 - (\partial_x \varphi)^2)^2 y + m^2 + (\partial_x \varphi)^2 ] \geq 0$ for all $y \geq 0$, thus $h(y)$ is increasing. In particular $h(1/\xi^2) \geq h(0) = 1$ for all $|\xi| \geq \xi_0 > 0$. Thus if $m \leq 2\alpha/R$, it is sufficient to have, for $|\xi| \geq 2\alpha/R$,
\[
\frac{s}{5R} |\xi|^{2s-1} \geq \|\psi\|_\infty \cdot \left( 1 + \frac{m^2}{\xi^2} + \frac{\partial_x \varphi}{\xi^2} \right).
\]

Thus if $|\xi| \geq 2\alpha/R$ it is sufficient to have $\frac{s}{5R} (\alpha/R)^{2s-1} \geq 18\|\psi\|_\infty$, which is condition (6.6).

(2) We prove that for suitable $\alpha$ and $R$ it holds that $\frac{1}{10}I \geq III$. Indeed, this is equivalent to proving that
\[
\frac{4}{10} \frac{s^2}{R^2} \left[ (\xi^2 + m^2 - (\partial_x \varphi)^2)^2 + 4\xi^2(\partial_x \varphi)^2 \right]^{s/2} \cdot (\xi^2 + (\partial_x \varphi)^2)^2 \geq \frac{2\alpha}{R} \|\psi\|_\infty \cdot 2s \frac{2\alpha}{R} |x/R + \psi(t)||m^2 - (\partial_x \varphi)^2|
\]
or equivalently
\[
\frac{1}{10} s \left[ (\xi^2 + m^2 - (\partial_x \varphi)^2)^2 + 4\xi^2(\partial_x \varphi)^2 \right]^{s/2} \cdot (\xi^2 + (\partial_x \varphi)^2)^2 \geq \alpha \|\psi\|_\infty \cdot |x/R + \psi(t)||m^2 - (\partial_x \varphi)^2|.
\]
Similarly as above, taking into account that the expression into square brackets attains its minimum at $\xi = 0$, then a sufficient condition is also (6.6).
(3) We prove that for suitable $\alpha$ and $R$ it holds that $\frac{1}{10}I \geq IV_b$. This is equivalent to
\[4s^2 \partial_{xx} \varphi \frac{\xi^2 + (\partial_x \varphi)^2}{|\xi|^{4-4s} \left[ (1 + m^2/\xi^2 - (\partial_x \varphi)^2/\xi^2)^2 + 4/\xi^2 (\partial_x \varphi)^2 \right]}^{1-s} \geq 8\alpha \|\psi''\|_\infty.\]

If $|\xi| \geq 2\frac{\alpha}{R}$ then a sufficient condition is
\[4s^2 \partial_{xx} \varphi |\xi|^{4s-2} \frac{1}{\left[ (1 + m^2(R/\alpha)^2 - (\partial_x \varphi)^2(R/\alpha)^2)^2 + 4(R/\alpha)^2(\partial_x \varphi)^2 \right]^{1-s}} \geq 8\alpha \|\psi''\|_\infty,\]

thus
\[4s^2 \partial_{xx} \varphi |\alpha/R|^{4s-2} \frac{1}{\left[ (1 + m^2(R/\alpha)^2 - (\partial_x \varphi)^2(R/\alpha)^2)^2 + 4(R/\alpha)^2(\partial_x \varphi)^2 \right]^{1-s}} \geq 8\alpha \|\psi''\|_\infty.\]

With the assumption $m \leq 2\alpha/R$ then just take
\[\frac{s^2}{R^{2s}} \alpha^{2s-1} \geq c \|\psi''\|_\infty^{1/2}.\]

If $|\xi| \leq 2\frac{\alpha}{R}$ then a sufficient condition is
\[4s^2 \partial_{xx} \varphi \frac{(\partial_x \varphi)^2}{\left[ (\alpha/R)^2 + m^2 - (\partial_x \varphi)^2)^2 + 4(\alpha/R)^2(\partial_x \varphi)^2 \right]^{1-s}} \geq 8\alpha \|\psi''\|_\infty.\]

So again under the assumption $m \leq 2\alpha/R$ just impose (6.7).

(4) We prove that for suitable $\alpha$ and $R$ it holds that $\frac{1}{10}I \geq V$. Indeed, we have to prove that, under suitable conditions on $\alpha$ and $R$, it holds
\[\frac{4}{10} s^2 \partial_{xx}^2 \varphi \left[ (\xi^2 + m^2 - (\partial_x \varphi)^2)^2 + 4\xi^2(\partial_x \varphi)^2 \right]^{s/2} \cdot (\xi^2 + (\partial_x \varphi)^2)^{\frac{s}{2}} \geq 2s \partial_{x,t} \varphi \partial_{x} \varphi \times 2(\xi^2 + m^2 + (\varphi_x)^2).\]

Thus a sufficient condition is to impose that
\[\frac{4}{10} s^2 \partial_{xx}^2 \varphi \left[ (m^2 - (\partial_x \varphi)^2)^2 \right]^{s/2} \cdot (\partial_x \varphi)^2 \geq 2s \partial_{x,t} \varphi \partial_{x} \varphi \cdot 2(\alpha^2/R^2 + m^2 + (\varphi_x)^2)\]

thus if $m < \alpha/R$ we arrive again at (6.7). For $|\xi| \geq 2\alpha/R$ then take
\[\frac{4}{10} s^2 \partial_{xx}^2 \varphi |\xi|^{2s} \left[ (1 + m^2/\xi^2 - (\partial_x \varphi)^2/\xi^2)^2 + 4/\xi^2 (\partial_x \varphi)^2 \right]^{s/2} \geq 2s \partial_{x,t} \varphi \partial_{x} \varphi \cdot 2(1 + m^2/\xi^2 + (\varphi_x)^2/\xi^2)\]

so it is sufficient to have
\[\frac{4}{10} s^2 \partial_{xx}^2 \varphi |\xi|^{2s} \geq 4\partial_{x,t} \varphi \partial_{x} \varphi \left(1 + m^2/\xi^2 + (\varphi_x)^2/\xi^2 \right),\]

then we just impose (6.6). \(\square\)

Remark 6.2. Observe that Proposition 6.1 remains valid in the case in which $m = 0$, i.e., for the operator involving $P_0(D) = (-\Delta)^s$. We also remark that the restriction $1/2 < s < 1$ is a technical one for (1.5) to make sense. Notice as well that $2s - 1 > 0$ is the order of the symbol of the commutator.
6.3. Step II: A Carleman inequality via Gårding inequality. Now we prove the Carleman inequality with the quadratic function in the weight.

For pedagogical reasons, we will first show a Carleman inequality corresponding to $P_0(D) = -\Delta$ and $s = 1$. We will make use of the Gårding inequality due to C. Fefferman and Phong [18] as stated in [49, Theorem 0.7.C] and [46, p. 321].

**Theorem 6.3** (Sharp Gårding inequality). Suppose $p(x, \xi) \in S^0_{1,0}$, and assume that $p(x, \xi) \geq 0$ everywhere. Then there is a constant $c > 0$, such that

$$\text{Re}(P(x, D)f, f) \geq -c\|f\|^2_{L^2((x-1/2)^2)}.$$  

Observe that in the case $s = 1$, the Poisson bracket $\{a, b\}$ computed in (6.1) coincides with the formula in [29, (3.5)]

$$\{a, b\} = 4\varphi''(\xi^2 + (\varphi')^2).$$

Thus

$$[S, A] = 4\varphi''(-\Delta + (\varphi')^2).$$

When $\varphi(x) = \alpha(\frac{x}{R} + 1)^2$ then we have, in particular,

$$\varphi'(x) = \frac{2\alpha}{R}\left(\frac{x}{R} + 1\right), \quad \varphi''(x) = \frac{2\alpha}{R^2}, \quad [S, A] = \frac{8\alpha}{R^2}\left(-\Delta + 4\frac{\alpha^2}{R^2}\left(\frac{x}{R} + 1\right)^2\right),$$

and therefore

$$\langle [S, A]f, f \rangle = \frac{8\alpha}{R^2} \int |\nabla f|^2 dx + \frac{32\alpha^3}{R^4} \int \left(\frac{x}{R} + 1\right)^2 f^2 dx.$$  

Motivated by the fact that we can understand the function $\varphi(t, x) := \alpha(\frac{x}{R} + \psi(t))^2$ with $\psi(t)$ independent of $t$ as $\psi(t) = 1$, we write

$$\varphi(x) := \alpha\left(\frac{x}{R} + 1\right)^2 = \alpha\left(\frac{x^2}{R^2} + \frac{2x}{R} + 1\right).$$

Then

$$\varphi' = \frac{2\alpha}{R} + \frac{2\alpha x}{R^2}, \quad \varphi'' = \frac{2\alpha}{R^2}.$$  

Let us assume $x \in K$ for some $K$ compact, such that

$$\text{supp} \varphi \subseteq \left\{1 \leq \left|\frac{x^2}{R^2} + \frac{2x}{R} + 1\right| \leq 4\right\}.$$  

Then, in view of (6.8)

$$\{a, b\} = \frac{8\alpha}{R^2}\left(\xi^2 + \left(\frac{2\alpha}{R} + \frac{2\alpha x}{R^2}\right)^2\right) = \frac{8\alpha}{R^2}\left(\xi^2 + 4\frac{\alpha^2}{R^2}\left(1 + \frac{x}{R}\right)^2\right) \geq \frac{8\alpha}{R^2}\xi^2 + 32\frac{\alpha^3}{R^4}.$$  

Thus

$$\{a, b\} - \frac{8\alpha}{R^2}\xi^2 - 32\frac{\alpha^3}{R^4} \geq 0.$$  

Applying the Gårding inequality in Theorem 6.3 with $n = 2$, we have

$$\langle \text{Op}(\{a, b\})f, f \rangle - \frac{8\alpha}{R^2}\langle (-\Delta) f, f \rangle_{L^2} \geq \left(32\frac{\alpha^3}{R^4} - c\right)\|f\|^2_{L^2},$$

for certain $c$. So, if $\frac{\alpha^3}{R^4}$ is large enough, then $32\frac{\alpha^3}{R^4} - c \geq 0$. Equivalently, it is enough to take $\alpha^3 > CR^4$, for certain $C$ large enough (recovering, in this way, the results in [11]).

Now we will turn into the case $s \in (1/2, 1)$ and this time we will use certain improvement of Fefferman-Phong estimate. In particular, J.-M. Bony proved in [5, Théorème 3.2] the following result.

**Theorem 6.4** ([5] Théorème 3.2). If $p(x, \xi)$ is a nonnegative smooth function defined on $\mathbb{R}^N \times \mathbb{R}^N$ such that

$$|\partial_\xi^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha, \beta}, \quad \text{for } |\alpha| + |\beta| \geq 4,$$

then there exists $c$ such that, for all $f \in S(\mathbb{R}^N)$,

$$\text{Re}(P(x, D)f, f) \geq -c\|f\|^2_{L^2}.$$
Actually, since we need a precise control on the parameters $\alpha$ and $R$, we will use a quantitative version of the latter, see [35 Corollary 1.3.2 (ii)], see also [34 Chapter 2.5.3] (the result is stated therein in the semiclassical setting with a parameter $h \in (0, 1]$; we will take the instance $h = 1$). The linear dependence is not explicitly stated in [35 Corollary 1.3.2 (ii)], but this can be verified after a careful tracking of the proof. Up to our knowledge, the most precise result in that direction (which states the linear dependence) is contained in [6 Theorem 4.1].

**Theorem 6.5 ([35] Corollary 1.3.2 (ii)).** If $p(x, \xi)$ is a nonnegative smooth function defined on $\mathbb{R}^N \times \mathbb{R}^N$ such that

$$|\partial_\alpha^\beta p(x, \xi)| \leq C_{\alpha, \beta}, \quad \text{for } 4 \leq |\alpha| + |\beta| \leq 2N + 5,$$

then there exists a constant $c$ that depends linearly only on $N$ and on $\max_{4 \leq |\alpha| + |\beta| \leq 2N + 5} C_{\alpha, \beta}$ such that, for all $f \in \mathcal{S}(\mathbb{R}^N)$

$$\text{Re}(P(x, D)f, f) \geq -c\|f\|_{L^2}^2.$$

We will focus on dimension $N = 1$ for the sake of the reading, but the results below could be stated in higher dimensions. First we are going to show that, in the case $m = 0$, the symbol $\{\tilde{a}, \tilde{b}\}$ satisfies the condition (6.9), for $1/2 < s < 1$.

**Lemma 6.6.** Let $m \geq 0$. Let $\varphi$ be the function defined in (6.4) and let $\tilde{a} + i\tilde{b}$ be the symbol of the conjugate operator $e^\varphi(\partial_t + P_m(D))e^{-t\varphi}$, where $(t, x) \in K := \{1 \leq |t/R + \psi(t)| \leq 4\} \cap \{|x| \leq R\}$. Let $1/2 < s < 1$. Then if

$$s\alpha^{2s-1}/R^{2s} \geq c(\|\psi\|_{\infty} + \|\psi''\|_{1/2}),$$

we have that the symbol $\{\tilde{a}, \tilde{b}\}$ in (6.3) satisfies (6.10) if $1/2 < s < 1$.

**Proof.** Let us first study the case $m = 0$ in detail. In view of the proof of Proposition 6.1, it suffices to study $\{a, b\}$. Observe that $\{a, b\}$ in (6.5) reads as

$$\{a, b\} = \frac{8\alpha s^2}{R^2} \left|\xi^2 + \frac{4\alpha^2}{R^2} \left(\frac{x}{R} + \psi(t)\right)^2\right|^{2s-1}.$$

Therefore we have, for $|\delta| + |\beta| \geq 4$ (actually, the estimate below is performed for $|\delta| + |\beta| = 4$, but more derivatives deliver better decay),

$$|\partial_\delta^\beta \partial_\xi^\alpha \{a, b\}| \leq C \left(\xi^2 + \frac{4\alpha^2}{R^2} \left(\frac{x}{R} + \psi(t)\right)^2\right)^{2s-3} \leq C \left(\xi^2 + \frac{4\alpha^2}{R^2}\right)^{2s-3} \leq C \left(\frac{4\alpha^2}{R^2}\right)^{2s-3} =: \tilde{C},$$

where we used that $2s - 3 < 0$ and the support $K$ in the hypothesis. Here, the constant $\tilde{C}$ depends on $s, \alpha, R, \|\psi\|_{\infty}, \|\psi^{(1)}\|_{\infty}, \|\psi^{(2)}\|_{\infty}, \|\psi(3)\|_{\infty}$ and $\|\psi^{(4)}\|_{\infty}$.

For the case $m > 0$, the proof follows as above, but taking into account that the expression of the derivatives is more involved. Indeed, to get the estimates, it is important to notice that, for $0 \leq |\xi| \leq 64\alpha^2/R^2$, we have

$$C_m \geq \left(\xi^2 + m^2 - \frac{4\alpha^2}{R^2} \left(\frac{x}{R} + \psi(t)\right)^2\right)^2 + 16|\xi|^2 \frac{\alpha^2}{R^2} \left(\frac{x}{R} + \psi(t)\right)^2 \geq c_m > 0,$$

since the terms in the sum cannot both cancel at the same time. On the other hand, for $|\xi|$ large, the upper bound follows analogously as in the case $m = 0$ since the dominant term carries the exponent $2s - 3$. \hfill \Box

We are in position to show our Carleman inequality in Theorem 6.7 (it is actually Theorem 1.2 we state it here again for the sake of reading).

**Theorem 6.7.** Let $f \in C^0_0((0, \infty) \times \mathbb{R}), \ m \geq 0$ and $1/2 < s \leq 1$. Let $\alpha, R > 0$ and $\varphi : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\varphi(t, x) = \alpha \left(\frac{x}{R} + \psi(t)\right)^2$ where $\psi \in C^\infty((0, \infty))$ is such that $0 \leq \psi(t) \leq 3$. Assume that $f$ is supported in the set

$$\left\{1 \leq \left|\frac{x}{R} + \psi(t)\right| \leq 4\right\}.$$
If \( s \frac{2s-1}{R^2} \geq c(\|\psi\|_\infty, \|\psi''\|_\infty) \) and \( m \leq 2 \frac{a}{R} \), the following inequality holds true
\[
c_1 s^2 \frac{\alpha}{R^2} \|(-\Delta + m^2)^{\frac{2s-1}{2s}} f\|^2_{L^2} + c_2 s^2 \frac{\alpha^{4s-1}}{R^{4s}} ||f||^2_{L^2} \leq ||e^\varphi (\partial_t + P_m(D))e^{-\varphi} f||^2_{L^2},
\]
where \( c_1 \) and \( c_2 \) are positive constants, depending on \( s \) and \( m \).

**Proof.** We use the positivity of the Poisson bracket proved in Proposition 6.1 and the Gårding inequality in Theorem 6.5. Thus, for \( \alpha \) and \( R \) as in the hypothesis we have that
\[
(6.11) \quad \{\tilde{a}, b\} \geq c s^2 \frac{\alpha}{R^2}(\xi^2 + 4 \frac{\alpha^2}{R^2})^{2s-1} \quad \text{for} \quad \xi \in \mathbb{R}.
\]
As a consequence of the above, we get
\[
\{\tilde{a}, b\} - cs^2 \frac{\alpha}{R^2}(\xi + m^2)^{2s-1} - cs^2 \frac{\alpha}{R^2} \frac{2s-1}{R^2} \geq 0 \quad \text{for} \quad \xi \in \mathbb{R}
\]
(here \( c \) is a suitable constant different from the one in the previous step). Applying the Gårding inequality in Theorem 6.5 we obtain
\[
\langle \text{Op}(\{\tilde{a}, b\}), f, f \rangle - cs^2 \frac{\alpha}{R^2}(\xi + m^2)^{2s-1}f, f \rangle_{L^2} - cs^2 \frac{\alpha^{4s-1}}{R^{4s}} ||f||^2_{L^2} \geq -C(\alpha, R)||f||^2_{L^2},
\]
for certain \( C(\alpha, R) \) depending linearly on \( (\frac{4s^2}{R^2})^{2s-3} \). Thus we obtain
\[
\langle [S, \tilde{A}]f, f \rangle - cs^2 \frac{\alpha}{R^2}(\xi + m^2)^{2s-1}f, f \rangle_{L^2} \geq \left( cs^2 \frac{\alpha^{4s-1}}{R^{4s}} - C(\alpha, R) \right) ||f||^2_{L^2}.
\]
So, if \( s \frac{4s-1}{R^2} \) is large enough, then \( cs^2 \frac{\alpha^{4s-1}}{R^{4s}} - C(\alpha, R) \geq 0 \). Equivalently, it is enough to take \( \alpha^{4s-1} \geq CR^{4s} \), for certain \( C \) large enough. Finally, in view of (6.2) we conclude the proof. \( \square \)

**Remark 6.8.** Observe that the condition \( s \frac{2s-1}{R^2} \geq c(\|\psi\|_\infty, \|\psi''\|_\infty) \) is more restrictive than \( \alpha^{4s-1} \geq CR^{4s} \), but this is because we have to apply Proposition 6.1 to get the positivity of the symbol associated to the parabolic problem.

**Proof of Theorem 6.3.** The Carleman estimate can be proved for the elliptic problem similarly as in the proof of Theorem 6.7. In this case, the symbol \( \{a, b\} \) is already positive and Proposition 6.1 is not needed anymore. In particular, we only need the condition \( \alpha^{4s-1} \geq CR^{4s} \). Moreover, the estimate in (6.11) (that is the same for \( \{a, b\} \)) simplifies when \( s = 1/2 \) and the Carleman estimate can be also accomplished for that value of the parameter. \( \square \)

**Appendix A.**

Apart from the definitions for \( L^s_m \) given in Section 2 we introduce the definition using the subordination formula. Motivated by the formula (2.12), we define the operator \( L^s_m(f) \) as follows. Let \( 0 < s < 1 \), \( m \geq 0 \) and \( f \in \mathcal{S} \). The operator \( L^s_m(f) \) is obtained as a weighted integral of the associated heat semigroup, by means of the Spectral Theorem
\[
(A.1) \quad L^s_m f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t(\Delta - m^2)} f(x) - f(x)) \frac{dt}{t^{1+s}}.
\]

We notice that the fractional power could be also defined using functional calculus as in Kato [28, p. 286], Paży [39] p. 69] or Yosida [50, p. 260].

The following lemma is the analogous to [47, Lemma 2.1] for the fractional relativistic operator.

**Lemma A.1.** For \( f \in \mathcal{S}(\mathbb{R}^N) \) and \( 0 < s < 1 \), the definitions given in (2.11), (A.1) and (2.7) are equivalent.
Proof. We will first prove that (2.1) and (A.1) are equivalent. Observe that, by the inverse Fourier transform,
\[ e^{t(\Delta - m^2)} f(x) - f(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} (e^{-t(|\xi|^2 + m^2)} - 1) \hat{f}(\xi) e^{ix\cdot\xi} d\xi. \]

With this and the change of variables \( w = t(|\xi|^2 + m^2) \) we obtain
\[ \int_0^\infty \left| e^{t(\Delta - m^2)} f(x) - f(x) \right| \frac{dt}{t^{1+s}} \leq C_N \int_0^\infty \int_{\mathbb{R}^N} |e^{-t(|\xi|^2 + m^2)} - 1| |\hat{f}(\xi)| d\xi \frac{dt}{t^{1+s}} \]
\[ = C_N \int_{\mathbb{R}^N} \int_0^\infty |e^{-w} - 1| \frac{dw}{w^{1+s}} (|\xi|^2 + m^2)^s |\hat{f}(\xi)| d\xi \]
\[ = C_{s,N} \int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s |\hat{f}(\xi)| d\xi < \infty, \]

since we are considering \( f \in S(\mathbb{R}^N) \). Therefore, by Fubini’s Theorem,
\[ \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t(\Delta - m^2)} f(x) - f(x) \right) \frac{dt}{t^{1+s}} = \frac{1}{\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^N} e^{-t(|\xi|^2 + m^2)} - 1 \frac{dt}{t^{1+s}} \hat{f}(\xi) e^{ix\cdot\xi} d\xi \]
\[ = \frac{1}{\Gamma(-s)} \int_{\mathbb{R}^N} \int_0^\infty (e^{-w} - 1) \frac{dw}{w^{1+s}} (|\xi|^2 + m^2)^s \hat{f}(\xi) e^{ix\cdot\xi} d\xi \]
\[ = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s |\hat{f}(\xi)| d\xi = F^{-1}(\chi \cdot m^2 \hat{f}(\cdot))(x). \]

We will check the equivalence between (A.1) and (2.7). Let us denote \( W_{t,m}(x) := e^{-tm^2} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^N} \). By Fubini’s Theorem,
\[ \int_0^\infty \left( e^{t(\Delta - m^2)} f(x) - f(x) \right) \frac{dt}{t^{1+s}} = \int_0^\infty \int_{\mathbb{R}^N} W_{t,m}(x-y)(f(y) - f(x)) dy \frac{dt}{t^{1+s}} \]
\[ + f(x) \int_0^\infty \left( \int_{\mathbb{R}^N} W_{t,m}(x-y) dy \right) (1 - e^{-tm^2}) \frac{dt}{t^{1+s}}. \]

The integral in the second summand boils down to
\[ \int_0^\infty \left( \int_{\mathbb{R}^N} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^N} dy \right) (e^{-tm^2} - 1) \frac{dt}{t^{1+s}} = \Gamma(-s)m^{2s}, \]

On the other hand, the integral in the first summand reads as
\[ \int_{\mathbb{R}^N} (f(y) - f(x)) \int_0^\infty e^{-tm^2} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^N} \frac{dt}{t^{1+s}} dy = \Gamma(-s) C_{N,s} m^{N+2s} \int_{\mathbb{R}^N} \frac{\hat{f}(x) - f(y)}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|) dy \]

where we used the integral representation (2.6) of the Macdonald’s function \( K_{\nu} \), after a change of variable. The applications of Fubini’s Theorem can be justified following an analogous argument as in [47, Lemma 2.1], by using the asymptotics (2.4) and (2.5). □

In the following lemma (proved in [15]), it is shown that for sufficiently good functions, the conjugated of the fractional power \( L^s_m \) is the fractional power of the local conjugated operator. We provide the formal proof of a more general result to keep our paper self-contained.

Lemma A.2. Let \( L \) be a lineal second order partial differential operator that we assume to be nonnegative, densely defined and self-adjoint on a Hilbert space \( L^2 \). Then for any \( s \in (-1,1) \) we have
\[ e^{\varphi L^s} e^{-\varphi f} = (e^{\varphi L} e^{-\varphi})^s [f], \]
for any \( \varphi \) and \( f \) such that \( e^{-\varphi} f \) is in the domain of \( L \) and \( L^s \).
Proof. The proof will be at a formal level. Let us start with the negative powers. We use the definition motivated by the identity for the gamma function

\[ \gamma^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} e^{-t\gamma} \frac{dt}{t^{1-s}}, \quad \gamma > 0, \quad 0 < s < 1. \]

Let \( v = e^{-tL}[e^{-\varphi}f] \) be the solution to

\[ v_t - Lv = 0, \quad v(0, x) = e^{-\varphi(x)}f(x). \]

Now, define \( w = e^{\varphi}v \). Then, \( w \) is the solution to

\[ w_t = e^{\varphi}v = e^{\varphi}Lv = e^{\varphi}L[e^{-\varphi}w] = L^*[w], \quad w(0, x) = e^{\varphi}v(0, x) = f, \]

where \( L^* (\cdot) := e^{\varphi}L[e^{-\varphi}] \). Let us write \( w = e^{-tL^*}f \). Hence, in view of (A.2),

\[
\begin{align*}
\gamma^{-\varphi} & = e^{\varphi} \frac{1}{\Gamma(s)} \int_{0}^{\infty} v(x, t) \frac{dt}{t^{1-s}} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} w(x, t) \frac{dt}{t^{1-s}} = e^{\varphi} \frac{1}{\Gamma(s)} \int_{0}^{\infty} e^{-tL^*} f \frac{dt}{t^{1-s}} \\
& = (L^*)^{-\varphi} f = (e^{\varphi}L[e^{-\varphi}])^{-\varphi}[f].
\end{align*}
\]

The case of positive powers works analogously, just by taking into account the identity (2.12). We leave details to the reader. \( \square \)

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