

EQUIVARIANT MOTIVIC INTEGRATION AND PROOF OF THE INTEGRAL IDENTITY CONJECTURE FOR REGULAR FUNCTIONS

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Dedicated to Professor Gert-Martin Greuel on the occasion of his seventy-fifth birthday

ABSTRACT. We develop Denef-Loeser’s motivic integration to an equivariant version and use it to prove the full integral identity conjecture for regular functions. In comparison with Hartmann’s work, the equivariant Grothendieck ring defined in this article is more elementary and it yields the application to the conjecture.

1. INTRODUCTION

1.1. Introduced by Kontsevich in 1995, motivic integration has since become an important subject in algebraic geometry by virtue of its connection to many areas of mathematics, including mathematical physics, birational geometry, non-Archimedean geometry, tropical geometry, singularity theory, Hodge theory, model theory (see for instance, [DL98], [DL99]), [Ba99], [Se04], [LS03], [HK06], [NS07], [Ni09], [CL08], [KS08]). Let k be a field of characteristic zero, which plays the role of a base field in the setup of the classical motivic integration in [DL99]. Let X be an (algebraic) k -variety, i.e., a (not necessarily reduced and irreducible) k -scheme of finite type, and let $\mathcal{L}(X)$ be its arc space (see Section 2). On the family of semi-algebraic subsets of $\mathcal{L}(X)$ we can construct an additive measure μ which takes value in a completion of the Grothendieck ring of k -varieties \mathcal{M}_k . Let \mathbb{L} denote the class of \mathbb{A}_k^1 , which is invertible in \mathcal{M}_k . There is a subfamily \mathcal{F}_X consisting of stable semi-algebraic subsets of $\mathcal{L}(X)$ whose measure is exactly in \mathcal{M}_k ; the restriction of μ on \mathcal{F}_X will be denoted by $\tilde{\mu}$. Let A be a semi-algebraic subset of $\mathcal{L}(X)$, and let $\ell : A \rightarrow \mathbb{Z}$ be a simple function. The motivic integral of ℓ can be defined if all the fibers of ℓ are semi-algebraic (i.e., measurable) and if the completion of \mathcal{M}_k is “big enough” so that the sum $\sum_{n \in \mathbb{Z}} \mu(\ell^{-1}(n))\mathbb{L}^{-n}$ makes sense. If, furthermore, all the fibers of ℓ and A are in \mathcal{F}_X , then by a result in [DL99], $|\ell|$ is bounded and the sum

$$(1.1) \quad \sum_{n \in \mathbb{Z}} \tilde{\mu}(\ell^{-1}(n))\mathbb{L}^{-n}$$

is in \mathcal{M}_k ; we then denote this sum by $\int_A \mathbb{L}^{-\ell} d\tilde{\mu}$. A breakthrough feature of motivic integration is that it admits a change of variables formula with respect to a proper birational morphism

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onto X , which Kontsevich used to prove Batyrev's conjecture on the Betti numbers of birationally equivalent complex Calabi-Yau varieties (cf. [Ba99]).

1.2. We consider an algebraic group G and its good actions on k -schemes (see Section 2.1 for definition of good action). Let S be a k -variety endowed with the trivial G -action, and for abuse of notation let \mathbb{L} be the S -isomorphism class of $\mathbb{A}_S^1 := \mathbb{A}_k^1 \times_k S \rightarrow S$ endowed with the affine G -action (see Section 2.1). The context of our investigation will be the G -equivariant Grothendieck ring $K_0^G(\text{Var}_S)$ of S -varieties endowed with good G -action, and its localization $\mathcal{M}_S^G := K_0^G(\text{Var}_S)[\mathbb{L}^{-1}]$. The notion of piecewise trivial fibration for maps between constructible sets introduced in [DL99] can be easily generalized to that of G -equivariant piecewise trivial fibration. If $f : X \rightarrow Y$ is a morphism of k -varieties, and y is a point in Y with residue field $k(y)$, the fiber of f over y is defined to be the fiber product $X \times_Y k(y)$ and denoted by $f^{-1}(y)$ or X_y . For y in Y , we write G_y for the stabilizer subgroup of G with respect to y . By a categorical quotient we mean a morphism of varieties which satisfies certain conditions (see [MFK94, Definition 0.5]). Our first fundamental result is the following theorem.

Theorem 1 (Theorem 2.2). *Let X and Y be k -varieties endowed with a good G -action, and let $f : X \rightarrow Y$ be a G -equivariant morphism. Suppose that the categorical quotient $Y \rightarrow Y//G$ exists and is quasi-finite. In order that f is a G -equivariant piecewise trivial fibration, it is necessary and sufficient that there exists a k -variety F endowed with a good G -action, and for every y in Y , there exists a G_y -equivariant isomorphism of $k(y)$ -varieties $X_y \xrightarrow{\cong} F \times_k k(y)$.*

According to this theorem, if for every y in Y , there exists a G_y -equivariant isomorphism of $k(y)$ -varieties $X_y \cong F \times_k k(y)$, we get the identity $[X] = [Y] \cdot [F]$ in \mathcal{M}_k^G . In the special case where F is the affine variety \mathbb{A}_k^n , for some n in \mathbb{N} , the following theorem is much more applicable than the previous one.

Theorem 2 (Theorem 2.3). *Let X and Y be k -varieties endowed with good G -action. Let $f : X \rightarrow Y$ be a G -equivariant morphism. Suppose that the categorical quotient $Y \rightarrow Y//G$ exists and is quasi-finite. Suppose further that for every y in Y , there exists an isomorphism of $k(y)$ -varieties $X_y \cong \mathbb{A}_{k(y)}^n$, for a given n in \mathbb{N} . Then the identity*

$$[X] = [Y] \cdot \mathbb{L}^n$$

holds in $K_0^G(\text{Var}_k)$.

Remark that if G is a finite group and Y is endowed with a good G -action, then the categorical quotient $Y \rightarrow Y//G$ exists and is quasi-finite, which satisfies the hypothesis of the two above-mentioned theorems.

1.3. Let \hat{G} be a group scheme over k of the form $\hat{G} = \varprojlim_{i \in I} G_i$, where I is a partially ordered directed set and $\{G_i, G_j \rightarrow G_i \mid i \leq j \text{ in } I\}$ is a projective system of algebraic groups over k . We shall consider good \hat{G} -actions on k -schemes (\hat{G} acts on a scheme via some G_i) and enhance the integral (1.1) to the \hat{G} -equivariant version. We define $K_0^{\hat{G}}(\text{Var}_S) := \varinjlim K_0^{G_i}(\text{Var}_S)$ and $\mathcal{M}_S^{\hat{G}} := K_0^{\hat{G}}(\text{Var}_S)[\mathbb{L}^{-1}]$, and we have $\mathcal{M}_S^{\hat{G}} = \varinjlim \mathcal{M}_S^{G_i}$.

Assume that there exists a nice \hat{G} -action on $\mathcal{L}(X)$, i.e., for every i in I and n in \mathbb{N} , there is a good G_i -action on $\mathcal{L}_n(X)$ such that the morphism $\pi_n^m : \mathcal{L}_m(X) \rightarrow \mathcal{L}_n(X)$ is G_i -equivariant for all $i \in I$ and $m \geq n$. For each i in I , let $\mathcal{F}_X^{G_i}$ be the family of A in \mathcal{F}_X such that $\pi_m(A)$ is G_i -invariant for every $m \geq n$, assuming A is stable at level n . Define $\mathcal{F}_X^{\hat{G}} := \varinjlim \mathcal{F}_X^{G_i}$.

Using Theorem 2 (i.e., Theorem 2.3), we can construct a well defined \hat{G} -equivariant additive measure

$$(1.2) \quad \tilde{\mu}^{\hat{G}} : \mathcal{F}_X^{\hat{G}} \rightarrow \mathcal{M}_k^{\hat{G}}$$

and a natural \hat{G} -equivariant motivic integral

$$\int_A \mathbb{L}^{-\ell} d\tilde{\mu}^{\hat{G}} := \sum_{n \in \mathbb{N}} \tilde{\mu}^{\hat{G}}(\ell^{-1}(n)) \mathbb{L}^{-n},$$

which takes value in $\mathcal{M}_k^{\hat{G}}$, provided A and all the fibers of a natural-value simple function ℓ are in $\mathcal{F}_X^{\hat{G}}$. Furthermore, we perform in Theorem 3.1 that the \hat{G} -equivariant motivic integration admits a change of variables formula with respect to a proper birational morphism onto X (cf. Section 3.2).

1.4. In significant applications, one takes \hat{G} to be the profinite group scheme of roots of unity $\hat{\mu}$, i.e., the projective limit of the group schemes $\mu_n = \text{Spec}(k[T]/(T^n - 1))$ with transition morphisms $\mu_{mn} \rightarrow \mu_n$ given by $\lambda \mapsto \lambda^m$. For this case, and for the natural $\hat{\mu}$ -action on $\mathcal{L}(X)$ (see [DL98]), we write simply $\tilde{\mu}$ instead of $\tilde{\mu}^{\hat{\mu}}$, the $\hat{\mu}$ -equivariant measure on $\mathcal{F}_X^{\hat{\mu}}$, cf. (1.2).

Assume the k -variety X is smooth of pure dimension d . We consider a regular function f on X whose zero locus X_0 is nonempty. For every $n \geq 1$, the sets

$$\begin{aligned} \bar{\mathcal{X}}_n(f) &:= \{ \gamma \in \mathcal{L}(X) \mid f(\gamma) = t^n \pmod{t^{n+1}} \}, \\ \bar{\mathcal{X}}_{n,x}(f) &:= \{ \gamma \in \bar{\mathcal{X}}_n(f), \gamma(0) = x \}, \end{aligned}$$

with x a closed point in X_0 , are in $\mathcal{F}_X^{\mu_n}$ with stable level being n . Notice that the k -variety $\mathcal{X}_n(f) := \pi_n(\bar{\mathcal{X}}_n(f))$ admits the natural morphism to X_0 sending γ to $\gamma(0)$. In view of [DL98], the motivic zeta functions $\sum_{n \geq 1} [\mathcal{X}_n(f)] \mathbb{L}^{-nd} T^n$ in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$ and $\mathbb{L}^d \sum_{n \geq 1} \tilde{\mu}(\bar{\mathcal{X}}_{n,x}(f)) T^n$ in $\mathcal{M}_k^{\hat{\mu}}[[T]]$ are *rational series*. The *limit* of these series, \mathcal{S}_f and $\mathcal{S}_{f,x}$, are called the motivic nearby cycles of f and the motivic Milnor fiber of f at x , respectively (cf. Section 4.2).

We want to go further on the rationality of a formal power series with coefficients in $\mathcal{M}_k^{\hat{\mu}}$. The following theorem is the first attempt, it proves a special case of Conjecture 4.4. We do not need X to be smooth in our result.

Theorem 3 (Theorem 4.5). *Let X be a k -variety and f a regular function on X . Let A_α , α in \mathbb{N}^r , be a semi-algebraic family of semi-algebraic subsets of $\mathcal{L}(X)$ such that, for every covering of X by affine open subsets U , any semi-algebraic condition defining $A_\alpha \cap \mathcal{L}(U)$ contains only conditions of two first forms in (4.6). Assume that, for every α in \mathbb{N}^r , A_α is stable and disjoint with $\mathcal{L}(X_{\text{Sing}})$. For $n \geq 1$, we put*

$$A_{n,\alpha} := \{ \gamma \in A_\alpha \mid f(\gamma) = t^n \pmod{t^{n+1}} \},$$

which is in $\mathcal{F}_X^{\mu_n}$. Then the series $\sum_{(n,\alpha) \in \mathbb{N}^{r+1}} \tilde{\mu}(A_{n,\alpha}) T_0^n T_1^{\alpha_1} \cdots T_r^{\alpha_r}$ is rational.

A consequence of this theorem plays an important role in the last section of the article.

Corollary 1 (Proposition 4.6). *Let X , f , A_α and $A_{n,\alpha}$ be as in Theorem 3. Let Δ be a cone in $\mathbb{R}_{\geq 0}^{r+1}$ and $\bar{\Delta}$ its closure. Let ℓ and ε be integral linear forms on \mathbb{Z}^{r+1} with $\ell(n,\alpha) > 0$ and $\varepsilon(n,\alpha) \geq 0$ for all (n,α) in $\bar{\Delta} \setminus \{0\}$. Then the series $\sum_{(n,\alpha) \in \Delta \cap \mathbb{N}^{r+1}} \tilde{\mu}(A_{n,\alpha}) \mathbb{L}^{-\varepsilon(n,\alpha)} T^{\ell(n,\alpha)}$ is rational, and its limit is independent of the forms ℓ and ε .*

In summary, we consider Theorems 1, 2, 3 and their corollaries to be the heart in our conceptual setting of equivariant motivic integration, which inherits Denef-Loeser's idea on the classical motivic integration for stable semi-algebraic subsets of an arc space. Unless taking the rationality result into account, we have developed the equivariant motivic integration with any group scheme being the limit of a projective system of finite algebraic groups.

Meanwhile, for formal schemes topologically of finite type, Hartmann [Ha15] has recently provided a preprint on equivariant motivic integration with respect to an abelian finite group. Hartmann's work may be regarded as the version with action of finite groups of Sebag and Loeser's motivic integration [Se04], [LS03]. It is important to remark that our equivariant motivic integration cannot be considered as a particular case of that of Hartmann, because the definition of the equivariant Grothendieck ring in [Ha15, Definition 4.1 (2)] is quite different to Denef-Loeser's definition that we use, and the key foundation in her work [Ha15, Lemma 4.7] is somewhat different to Theorems 1, 2 (i.e., Theorems 2.2, 2.3) in this article. Furthermore, a result as in [Ha15, Lemma 4.7] is not sufficient for our application in Section 5.3 where the fiber of our morphism is not an affine space.

1.5. We discuss a crucial application of the $\hat{\mu}$ -equivariant motivic integration to the full version of the integral identity conjecture for regular functions. It is well known that this conjecture is a building block in Kontsevich-Soibelman's theory of motivic Donaldson-Thomas invariants for noncommutative Calabi-Yau threefolds; it implies directly the existence of these invariants (cf. [KS08], [Le17, Section 1]). Let us state the version for regular functions of the conjecture (for the version for formal functions, see [KS08, Conjecture 4.4]).

Recall from [DL98] that any morphism of k -varieties $g : S \rightarrow S'$ induces a morphism of rings $g^* : \mathcal{M}_{S'}^{\hat{\mu}} \rightarrow \mathcal{M}_S^{\hat{\mu}}$ by fiber product, and induces a morphism of groups $g_! : \mathcal{M}_S^{\hat{\mu}} \rightarrow \mathcal{M}_{S'}^{\hat{\mu}}$ by composition. When S' is $\text{Spec } k$ we write \int_S for $g_!$.

Conjecture 1 (Kontsevich-Soibelman). *Let (x, y, z) be the standard coordinates of the vector space $k^d = k^{d_1} \times k^{d_2} \times k^{d_3}$. Let f be in $k[x, y, z]$ such that $f(0, 0, 0) = 0$ and $f(\lambda x, \lambda^{-1}y, z) = f(x, y, z)$ for all λ in $\mathbb{G}_{m,k}$. Then the integral identity $\int_{\mathbb{A}_k^{d_1}} i^* \mathcal{S}_f = \mathbb{L}^{d_1} \mathcal{S}_{\tilde{f},0}$ holds in $\mathcal{M}_k^{\hat{\mu}}$, where \tilde{f} is the restriction of f to $\mathbb{A}_k^{d_3}$, and i is the inclusion of $\mathbb{A}_k^{d_1}$ in $f^{-1}(0)$.*

The conjecture was first proved by Lê [Le12] for the case where f is either a function of Steenbrink type or the composition of a pair of regular functions with a polynomial in two variables. In [Le15, Theorem 1.2], in view of the formalism of Hrushovski-Kazhdan [HK06] and Hrushovski-Loeser [HL15], Lê showed that Conjecture 1 holds in $\mathcal{M}_{\text{loc}}^{\hat{\mu}}$, a ‘‘big’’ localization of $\mathcal{M}_k^{\hat{\mu}}$, as soon as the base field k is algebraically closed (see [Le17] for a short review about that work). Recently, Nicaise and Payne [NP17] have developed an effective method to compute motivic nearby cycles as motivic volume of semi-algebraic sets, the motivic Fubini theorem for the tropicalization map, on the foundation of [HK06] and tropical geometry. Their approach needs the condition that k contains all roots of unity, but it brings out a strong improvement when proving the conjecture with the context $\mathcal{M}_k^{\hat{\mu}}$.

Our proof of Conjecture 1 in this article is complete, no additional hypothesis is needed. The work is based on our equivariant motivic integration and it is devoted in Section 5 for performance of detailed arguments.

2. SOME RESULTS ON THE EQUIVARIANCE

2.1. Equivariant Grothendieck rings of varieties. Let k be a field of characteristic zero, and let S be a k -variety. By an S -variety we mean a k -variety X together with a morphism

$X \rightarrow S$. As usual (cf. [DL98], [DL99]), we denote by Var_S the category of S -varieties and $K_0(\text{Var}_S)$ its Grothendieck ring. By definition, $K_0(\text{Var}_S)$ is the quotient of the free abelian group generated by the S -isomorphism classes $[X \rightarrow S]$ in Var_S modulo the relations

$$[X \rightarrow S] = [X_{\text{red}} \hookrightarrow X \rightarrow S]$$

and

$$[X \rightarrow S] = [Y \rightarrow S] + [X \setminus Y \rightarrow S],$$

where X_{red} is the reduced subscheme of X , and Y is a Zariski closed subvariety of X . Together with fiber product over S , $K_0(\text{Var}_S)$ is a commutative ring with unity $1 = [\text{Id} : S \rightarrow S]$. Put $\mathbb{L} = [\mathbb{A}_k^1 \times_k S \rightarrow S]$ and write \mathcal{M}_S for the localization of $K_0(\text{Var}_S)$ which makes \mathbb{L} invertible.

Let X be a k -variety; let G be an algebraic group which acts on X . The action of G on X is said to be *good* if every G -orbit is contained in an affine open subset of X . Now we fix a good action of G on the k -variety S (we may choose the trivial action). By definition, the G -equivariant Grothendieck group $K_0^G(\text{Var}_S)$ of G -equivariant morphism of k -varieties $X \rightarrow S$, where X is endowed with a good G -action, is the quotient of the free abelian group generated by the G -equivariant isomorphism classes $[X \rightarrow S, \sigma]$, where X is a k -variety endowed with a good G -action σ and $X \rightarrow S$ is a G -equivariant morphism of k -varieties, modulo the relations

$$[X \rightarrow S, \sigma] = [Y \rightarrow S, \sigma|_Y] + [X \setminus Y \rightarrow S, \sigma|_{X \setminus Y}]$$

for Y being σ -invariant Zariski closed in X , and

$$(2.1) \quad [X \times_k \mathbb{A}_k^n \rightarrow S, \sigma] = [X \times_k \mathbb{A}_k^n \rightarrow S, \sigma']$$

if σ and σ' lift the same good G -action on X . As above, we have the commutative ring with unity structure on $K_0^G(\text{Var}_S)$ by fiber product, where G -action on the fiber product is through the diagonal G -action, and we may define the localization \mathcal{M}_S^G of the ring $K_0^G(\text{Var}_S)$ by inverting \mathbb{L} . Here, \mathbb{L} is the class of $\mathbb{A}_k^1 \times_k S \rightarrow S$ endowed with the trivial action of G .

Let \hat{G} be a group scheme over k of the form $\hat{G} = \varprojlim_{i \in I} G_i$, where I is a partially ordered set and $\{G_i, G_j \rightarrow G_i \mid i \leq j \text{ in } I\}$ is a projective system of algebraic groups over k . In particular, we may consider \hat{G} to be the profinite group scheme of roots of unity $\hat{\mu}$, the projective limit of the group schemes $\mu_n = \text{Spec}(k[T]/(T^n - 1))$ and transition morphisms $\mu_{mn} \rightarrow \mu_n$ sending λ to λ^m . We define $K_0^{\hat{G}}(\text{Var}_S) = \varinjlim_{i \in I} K_0^{G_i}(\text{Var}_S)$ and $\mathcal{M}_S^{\hat{G}} = K_0^{\hat{G}}(\text{Var}_S)[\mathbb{L}^{-1}]$, which implies the identity $\mathcal{M}_S^{\hat{G}} = \varinjlim_{i \in I} \mathcal{M}_S^{G_i}$.

2.2. Equivariant piecewise trivial fibrations. We start with Denef-Loeser's definition of piecewise trivial fibration in [DL99]. Let X, Y and F be k -varieties; let A and B be respectively constructible subsets of X and Y . Let $f : X \rightarrow Y$ be a morphism such that $f(A) \subseteq B$. The restriction map $f : A \rightarrow B$ is called a *piecewise trivial fibration with fiber F* if there is a stratification of B into finitely many locally closed subsets B_i in Y such that, for every i , $f^{-1}(B_i) := A \times_Y B_i$ is locally closed in X and isomorphic as a k -variety to $B_i \times_k F$, and that, over B , f corresponds to the projection $B_i \times_k F \rightarrow B_i$. More generally, given a constructible subset C of B , f is by definition a *piecewise trivial fibration over C* if $f|_{f^{-1}(C)} : f^{-1}(C) \rightarrow C$ is a piecewise trivial fibration.

Theorem 2.1 (Sebag [Se04], Théorème 4.2.3). *The map $f : A \rightarrow B$ is a piecewise trivial fibration with fiber F if and only if for every y in B , the fiber $f^{-1}(y)$ is isomorphic as a $k(y)$ -variety to $F \times_k k(y)$.*

Now we go to the equivariant framework. Let G be an algebraic group over k . Let X, Y and F be k -varieties endowed with good G -actions; let A and B be G -invariant constructible subsets of X and Y respectively. Consider a G -equivariant map of constructible sets $f : A \rightarrow B$ which is the restriction of a given G -equivariant morphism $f : X \rightarrow Y$ with $f(A) \subseteq B$. Then f is called a G -equivariant piecewise trivial fibration with fiber F if there exists a stratification of B into finitely many G -invariant locally closed subsets B_i such that, for every i , $f^{-1}(B_i)$ is a G -invariant constructible subset of A and $f^{-1}(B_i)$ is G -equivariant isomorphic to $B_i \times_k F$, with the action of G on $B_i \times_k F$ being the diagonal one, and moreover, over B_i , f corresponds to the projection $B_i \times_k F \rightarrow B_i$. In view of the definition of $K_0^G(\text{Var}_k)$, such a map f yields the identity

$$[X] = [Y] \cdot [F]$$

in $K_0^G(\text{Var}_k)$. In the following theorem we will give a criterion for which a morphism of k -varieties with G -action is a G -equivariant piecewise trivial fibration (we only consider the case $A = X$ and $B = Y$ in the previous definition). Let us fix the notation which concern. For a morphism of k -varieties $X \rightarrow Y$ and any immersion $S \rightarrow Y$, the notation X_S will stand for their fiber product. And, for each y in Y , the stabilizer subgroup G_y of G with respect to y is the subgroup of elements in G that fix y .

Theorem 2.2. *Let X and Y be k -varieties endowed with good G -actions; let $f : X \rightarrow Y$ be a G -equivariant morphism. Suppose that the categorical quotient $Y \rightarrow Y//G$ exists and is quasi-finite. In order that f is a G -equivariant piecewise trivial fibration, it is necessary and sufficient that there exists a k -variety F endowed with a good G -action, and for every y in Y , there exists a G_y -equivariant isomorphism of $k(y)$ -varieties $X_y \xrightarrow{\cong} F \times_k k(y)$.*

Proof. Let S_0 be the set of all the generic points of $Y//G$ and let $\zeta \in S_0$. Since the categorical quotient $\phi_Y : Y \rightarrow Y//G$ is quasi-finite, the scheme Y_ζ is finite. Then $Y_\zeta = \bigsqcup_{\eta \in I} G \cdot \eta$, where I is a finite subset of Y . Furthermore, the orbits $G \cdot \eta$ are also finite, i.e., $G \cdot \eta = \{\eta, \eta_1, \dots, \eta_l\}$ for some natural number l . For each $1 \leq i \leq l$, we take an element g_i in G such that $\eta_i = g_i \cdot \eta$ and define the isomorphism $\Phi_{\eta_i} : X_{\eta_i} \cong F \times_k k(\eta_i)$ of $k(\eta_i)$ -schemes as the composition

$$X_{\eta_i} \xrightarrow{g_i^{-1}} X_\eta \cong F \times_k k(\eta) \xrightarrow{g_i} F \times_k k(\eta_i);$$

in other words, $\Phi_{\eta_i}(x) = g_i \Phi_\eta(g_i^{-1}x)$ where the middle isomorphism Φ_η is taken from the hypothesis. Combining all the isomorphisms Φ_{η_i} with ζ in S_0 and η in I we define the isomorphism

$$\Phi_{S_0} : X_{S_0} \cong F \times_k Y_{S_0},$$

by $\Phi_{S_0}(x) = \Phi_{\eta_i}(x)$ if $x \in X_{\eta_i}$. We claim that Φ_{S_0} is G -equivariant. In fact, take any $x \in X_{S_0}$ and $g \in G$. Assume that $\eta_1 = f(x), \eta_2 = f(gx)$ and $\eta_1 = g_1\eta, \eta_2 = g_2\eta$ as above. Then

$$g_2\eta = \eta_2 = f(gx) = gf(x) = g\eta_1 = gg_1\eta,$$

and therefore $g_2^{-1}gg_1 \in G_\eta$. This implies that

$$\Phi_{S_0}(gx) = \Phi_{\eta_2}(gx) = g_2\Phi_\eta(g_2^{-1}gx) = g_2\Phi_\eta(g_2^{-1}gg_1g_1^{-1}x) = g_2g_2^{-1}gg_1\Phi_\eta(g_1^{-1}x),$$

where the last equality follows from the G_η -equivariance of Φ_η . Hence

$$\Phi_{S_0}(gx) = gg_1\Phi_\eta(g_1^{-1}x) = g\Phi_{\eta_1}(x) = g\Phi_{S_0}(x),$$

whence the claim.

We now show that there exists a dense open subset U of $Y//G$ such that there is an isomorphism $X_U \cong U \times F$ through which f factors. Indeed, let \mathcal{T} be the set of open dense

subsets λ of $Y//G$. It is a directed partially ordered set with relation $\lambda \leq \lambda'$ if $\lambda' \subseteq \lambda$. To apply Grothendieck's descent theory, we define the initial objects

$$\alpha := Y//G \in \mathcal{T}, \quad S_\alpha := Y,$$

and S_α -schemes

$$A_\alpha := X \xrightarrow{f} Y, \quad B_\alpha := Y \times F \xrightarrow{\text{pr}_1} Y,$$

and consider the following projective systems with natural transition morphisms

$$S_\lambda := Y_\lambda, \quad A_\lambda := A_\alpha \times_{S_\alpha} S_\lambda = X_\lambda, \quad B_\lambda := B_\alpha \times_{S_\alpha} S_\lambda = Y_\lambda \times F.$$

Note that $\varprojlim S_\lambda = Y_{S_0}$, $A := \varprojlim A_\lambda = X_{S_0}$ and $B := \varprojlim B_\lambda = Y_{S_0} \times F$. It follows from [Gro66, Corollary 8.8.2.5] (see also, [Stack, Lemma 31.8.11 (Tag 081E)]) that there exist $U \geq \alpha$ and an S_U -isomorphism $\Phi_U: A_U \rightarrow B_U$ such that the diagram

$$\begin{array}{ccc} X_U & \xrightarrow{\Phi_U} & Y_U \times F \\ & \searrow f & \swarrow \text{pr}_1 \\ & Y_U & \end{array}$$

commutes. Let us now consider the two Y_U -schemes $G \times X_U$ and $Y_U \times F$, and the two Y_U -morphisms of these schemes $\varphi_U = \Phi_U \circ \rho_1$ and $\psi_U = \rho_2 \circ (\text{Id} \times \Phi_U)$

$$\begin{array}{ccc} & X_U & \\ \rho_1 \nearrow & & \searrow \Phi_U \\ G \times X_U & & Y_U \times F \\ \text{Id} \times \Phi_U \searrow & & \nearrow \rho_2 \\ & G \times Y_U \times F & \end{array}$$

where ρ_1 and ρ_2 are action morphisms of G on X_U and $Y_U \times F$, respectively. It follows from the G -equivariant isomorphism $X_{S_0} \cong Y_{S_0} \times F$ that $\varphi_{U, Y_{S_0}} = \psi_{U, Y_{S_0}}$, where $\varphi_{U, Y_{S_0}}$ and $\psi_{U, Y_{S_0}}$ are the base changes of φ_U and ψ_U by the inclusion $Y_{S_0} \rightarrow Y_U$, respectively. Applying [Stack, Lemma 31.10.1 (Tag 01ZM)], there exists $V \geq U$ such that $\varphi_{U, Y_V} = \psi_{U, Y_V}$. Note that Y_V is a G -invariant subset of Y_U , it then follows that $\varphi_{U, Y_V} = \varphi_V$ and $\psi_{U, Y_V} = \psi_V$, where $\varphi_V = \Phi_V \circ \rho_1$ and $\psi_V = \rho_2 \circ (\text{Id} \times \Phi_V)$. The equality $\varphi_V = \psi_V$ yields that the isomorphism

$$\Phi_V: X_V \rightarrow Y_V \times F$$

is G -equivariant. Iterating the above argument for the closed subset $Y \setminus Y_V$, we obtain a finite stratification $Y = \bigsqcup_{0 \leq i \leq n_0} Y_i$ into G -invariant locally closed subsets Y_i , with $Y_1 = Y_V$ and $\dim Y_i > \dim Y_{i+1}$, such that f is a G -equivariant trivial fibration with fiber F over each stratum Y_i . The theorem is proved. \square

Theorem 2.3. *Let X and Y be k -varieties endowed with good G -action. Let $f: X \rightarrow Y$ be a G -equivariant morphism. Suppose that the categorical quotient $Y \rightarrow Y//G$ exists and is quasi-finite. Suppose further that for every y in Y , there exists an isomorphism of $k(y)$ -varieties $X_y \cong \mathbb{A}_{k(y)}^n$, for a given n in \mathbb{N} . Then the identity*

$$[X] = [Y] \cdot \mathbb{L}^n$$

holds in $K_0^G(\text{Var}_k)$.

Proof. As in the proof of Theorem 2.2, it follows from [Gro63, Exposé V, Proposition 1.8] that, if G is finite, then the geometric quotient $\phi_Y: Y \rightarrow Y/G$ exists and it is a finite morphism. Let S_0 be the set of the generic points of $Y//G$. Then the scheme Y_{S_0} is G -invariant and it is a finite subset of Y . By assumption, for each η in Y_{S_0} , there exists an isomorphism $\Phi_\eta: X_\eta \rightarrow k(\eta) \times \mathbb{A}_k^n$. It yields an isomorphism $\Phi_{S_0}: X_{S_0} \rightarrow Y_{S_0} \times \mathbb{A}_k^n$ defined by $\Phi_{S_0}(x) = \Phi_\eta(x)$ if x is in X_η . Applying [Gro66, Corollary 8.8.2.5] (see the first part of the proof of Theorem 2.2) we obtain an open dense subset V in $Y//G$ and an isomorphism $\Phi_V: X_V \cong Y_V \times \mathbb{A}_k^n$ of Y_V -varieties, through which f factors, i.e $f = \pi_1 \circ \Phi_V$. We now endow the product variety $Y_V \times \mathbb{A}_k^n$ with a good action of G defined as $g \cdot z = \Phi_V(g \cdot \Phi_V^{-1}(z))$ for all $g \in G$ and $z \in Y_V \times \mathbb{A}_k^n$. Then the isomorphism Φ_V and the projection (on to the first component) π_1 becomes G -equivariant. It yields that, the action on $Y_V \times \mathbb{A}_k^n$ is a lifting of the action of G on Y_V . We deduce the identity

$$[X_V] = [Y_V \times \mathbb{A}_k^n],$$

which is equal to $[Y_V] \cdot \mathbb{L}^n$ in $K_0^G(\text{Var}_k)$, by the definition of the ring $K_0^G(\text{Var}_k)$ (in particular, see relation (2.1)). Note that V is open and dense in $Y//G$, so its complement $(Y//G) \setminus V$ is closed subset of $Y//G$. Repeating the above argument for the closed subset $(Y//G) \setminus V$ and so on, we get a stratification $Y//G = \bigsqcup_{0 \leq i \leq m_0} Z_i$ of G -invariant locally closed subsets Z_i , with $Z_1 = V$ and $\dim Z_i > \dim Z_{i+1}$, such that the identities

$$[X_{Z_i}] = [Y_{Z_i}] \cdot \mathbb{L}^n$$

hold in $K_0^G(\text{Var}_k)$. The theorem is now definitely proved. \square

Remark 2.4. (a) Theorems 2.2 and 2.3 can be applied to good actions of finite groups, because if G is a finite group and if Y is a variety endowed with a good G -action, then the categorical quotient $Y \rightarrow Y//G$ exists and is quasi-finite.

(b) The hypothesis in our Theorem 2.3 is easier to check than that from Hartmann's [Ha15, Lemma 4.7]. Moreover, Theorem 2.2 is stated in a more general form than that lemma.

3. ARC SPACES, EQUIVARIANT MOTIVIC MEASURE AND INTEGRATION

3.1. Arc spaces and semi-algebraic sets. Let X be a k -variety. For any n in \mathbb{N} , denote by $\mathcal{L}_n(X)$ the k -scheme of n -jets of X , which represents the functor from the category of k -algebras to the category of sets sending a k -algebra A to $\text{Mor}_{k\text{-schemes}}(\text{Spec}(A[t]/A(t^{n+1})), X)$. For $m \geq n$, the truncation $k[t]/(t^{m+1}) \rightarrow k[t]/(t^{n+1})$ induces a morphism of k -schemes

$$\pi_n^m: \mathcal{L}_m(X) \rightarrow \mathcal{L}_n(X)$$

and this is an affine morphism. If X is smooth of dimension d , the morphism π_n^m is a locally trivial fibration with fiber $\mathbb{A}_k^{(m-n)d}$. The n -jet schemes and the morphisms π_n^m form in a natural way a projective system of k -schemes, we call the projective limit

$$\mathcal{L}(X) := \varprojlim \mathcal{L}_n(X)$$

the *arc space* of X . Note that $\mathcal{L}(X)$ is a k -scheme but it is not of finite type. For any field extension $K \supseteq k$, the K -points of $\mathcal{L}(X)$ correspond one-to-one to the $K[[t]]$ -points of X . Denote by π_n the natural morphism

$$\mathcal{L}(X) \rightarrow \mathcal{L}_n(X).$$

Recall from [DL99, Section 2], for any algebraically closed field K containing k , that a subset of $K((t))^m \times \mathbb{Z}^r$ is *semi-algebraic* if it is a finite boolean combination of sets of the form

$$\{(x, \alpha) \in K((t))^m \times \mathbb{Z}^r \mid \text{ord}_t f_1(x) \geq \text{ord}_t f_2(x) + \ell_1(\alpha), \\ \text{ord}_t f_3(x) \equiv \ell_2(\alpha) \pmod{d}, \Phi(\overline{\text{ac}}(g_1(x)), \dots, \overline{\text{ac}}(g_n(x))) = 0\},$$

where f_i, g_j and Φ are polynomials over k , ℓ_1 and ℓ_2 are polynomials over \mathbb{Z} of degree at most 1, d is in \mathbb{N} , and $\overline{\text{ac}}(g_j(x))$ is the angular component of $g_j(x)$. One calls a collection of formulas defining a semi-algebraic set a *semi-algebraic condition*. A family $\{A_\alpha \mid \alpha \in \mathbb{N}^r\}$ of subsets A_α of $\mathcal{L}(X)$ is called a *semi-algebraic family of semi-algebraic subsets* if there exists a covering of X by affine Zariski open sets U such that $A_\alpha \cap \mathcal{L}(U)$ is defined by a semi-algebraic condition, that is,

$$A_\alpha \cap \mathcal{L}(U) = \{\gamma \in \mathcal{L}(U) \mid \theta(h_1(\tilde{\gamma}), \dots, h_m(\tilde{\gamma}); \alpha)\},$$

where h_i are regular functions on U , θ is a semi-algebraic condition, and $\tilde{\gamma}$ is the element in $\mathcal{L}(U)(k(\gamma))$ corresponding to a point γ in $\mathcal{L}(U)$ of residue field $k(\gamma)$ (cf. [DL99, Section 2.2]). In the case when $r = 0$, the unique element in the previous family is called a *semi-algebraic subset* of $\mathcal{L}(X)$.

Let A be a semi-algebraic subset of $\mathcal{L}(X)$; let r be a natural number. As introduced in [DL99], a function

$$\ell : A \times \mathbb{Z}^r \rightarrow \mathbb{Z} \cup \{+\infty\}$$

is called *simple* if the family of sets $\{x \in A \mid \ell(x, \alpha_1, \dots, \alpha_r) = \alpha_{r+1}\}$, with $(\alpha_1, \dots, \alpha_{r+1})$ in \mathbb{N}^{r+1} , is a semi-algebraic family of semi-algebraic subsets of $\mathcal{L}(X)$. For instance, if f is a regular function on X and A is a semi-algebraic subset of $\mathcal{L}(X)$, then $\text{ord}_t f$ is a simple function on A . A subset of \mathbb{Z}^r is called a *Presburger set* if it is defined by a finite boolean combination of sets of the form

$$\{\alpha \in \mathbb{Z}^r \mid \ell_1(\alpha) \geq 0, \ell_2(\alpha) \equiv 0 \pmod{d}\},$$

where ℓ_1 and ℓ_2 are polynomials over \mathbb{Z} of degree at most 1 and d is in $\mathbb{N}_{>0}$. In other words, a Presburger set is a subset of \mathbb{Z}^r (for some r) defined by a formula in the Presburger language. If ℓ is a \mathbb{Z} -valued function on \mathbb{Z}^r whose graph is a Presburger subset of \mathbb{Z}^{r+1} , then we call ℓ a *Presburger function*.

3.2. Equivariant motivic measure and integration. Let X be a k -variety of pure dimension d , and A be a semi-algebraic subset of the arc space $\mathcal{L}(X)$. The set A is said to be *weakly stable at level n* , for some n in \mathbb{N} , if A is a union of fibers of $\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$; the set A is said to be *weakly stable* if it is weakly stable at some level (see [DL99]). Further, A is said to be *stable at level n* if it is weakly stable at level n and, for every $m \geq n$, the map

$$(3.1) \quad \pi_{m+1}(\mathcal{L}(X)) \rightarrow \pi_m(\mathcal{L}(X))$$

is a piecewise trivial fibration over $\pi_m(A)$ with fiber \mathbb{A}_k^d ; and A is *stable* if it is stable at some level (see [DL99]). Let \mathcal{F}_X be the family of stable semi-algebraic subsets of $\mathcal{L}(X)$. Note that if A is a weakly stable semi-algebraic subset of $\mathcal{L}(X)$ and A is disjoint with $\mathcal{L}(X_{\text{Sing}})$, then A is in \mathcal{F}_X , where X_{Sing} is the locus of singular points of X . As noticed in [DL99], the family \mathcal{F}_X is closed for finite intersection and finite union operations. A direct corollary of the definition is that if A is in \mathcal{F}_X , then there exists a natural number n such that, for every $m \geq n$, the identity $[\pi_m(A)] = [\pi_n(A)]\mathbb{L}^{(m-n)d}$ holds in $K_0(\text{Var}_k)$, i.e., the identity

$$[\pi_m(A)]\mathbb{L}^{-md} = [\pi_n(A)]\mathbb{L}^{-nd}$$

holds in \mathcal{M}_k . One puts

$$\tilde{\mu}(A) := [\pi_n(A)]\mathbb{L}^{-(n+1)d},$$

for A in \mathcal{F}_X stable at level n , and obtains an additive measure $\tilde{\mu} : \mathcal{F}_X \rightarrow \mathcal{M}_k$.

Let $\hat{G} = \varprojlim G_i$ be the limit of a projective system of finite algebraic groups over a directed ordered set (I, \leq) . Assume \hat{G} acts nicely on $\mathcal{L}(X)$, that is, the given action of G_i on $\mathcal{L}_n(X)$ are good for every i in I and $m \geq n$, and the morphisms

$$\pi_n^m : \mathcal{L}_m(X) \rightarrow \mathcal{L}_n(X)$$

are G_i -equivariant. Let i be in I , and let A be a semi-algebraic subset of $\mathcal{L}(X)$ which is G_i -invariant stable at level n , i.e., A is stable at level n and $\pi_m(A)$ is invariant under the action of G_i for all $m \geq n$. Then the morphism (3.1) is G_i -equivariant and it is a piecewise trivial fibration with fiber \mathbb{A}_k^d for all $m \geq n$. By Theorem 2.3, the identities

$$[\pi_{m+1}(A)] = [\pi_m(A)]\mathbb{L}^d$$

holds in $K_0^{G_i}(\text{Var}_k)$, and hence $[\pi_m(A)]\mathbb{L}^{-md}$ is constant in $\mathcal{M}_k^{G_i}$ for every $m \geq n$. Putting $\tilde{\mu}^{G_i}(A) := [\pi_n(A)]\mathbb{L}^{-(n+1)d}$ we also get a G_i -equivariant additive measure $\tilde{\mu}^{G_i} : \mathcal{F}_X^{G_i} \rightarrow \mathcal{M}_k^{G_i}$, where $\mathcal{F}_X^{G_i}$ denotes the subfamily of \mathcal{F}_X consisting of G_i -equivariant stable semi-algebraic subsets of $\mathcal{L}(X)$. The family $\{\tilde{\mu}^{G_i}, i \in I\}$ then forms an inductive system and its limit defines a \hat{G} -equivariant additive measure

$$(3.2) \quad \tilde{\mu}^{\hat{G}} : \mathcal{F}_X^{\hat{G}} \rightarrow \mathcal{M}_k^{\hat{G}},$$

where $\mathcal{F}_X^{\hat{G}}$ is the limit of the system $\mathcal{F}_X^{G_i}$. When \hat{G} is $\hat{\mu}$, with $I = \mathbb{N}$, and it acts naturally on $\mathcal{L}(X)$ as $\lambda \cdot \gamma(t) := \gamma(\lambda t)$ for every λ in μ_n and γ in $\mathcal{L}_n(X)$, we shall write simply $\tilde{\mu}$ instead of $\tilde{\mu}^{\hat{\mu}}$.

Now, let A be in $\mathcal{F}_X^{\hat{G}}$ and let $\ell : A \rightarrow \mathbb{N}$ be a simple function such that all the fibers $\ell^{-1}(n)$ of ℓ are in $\mathcal{F}_X^{\hat{G}}$. By [DL99, Lemma 2.4], A is the disjoint union of finitely many subsets $\ell^{-1}(n)$. Then we may define \hat{G} -equivariant motivic integral of ℓ to be

$$(3.3) \quad \int_A \mathbb{L}^{-\ell} d\tilde{\mu}^{\hat{G}} := \sum_{n \in \mathbb{N}} \tilde{\mu}^{\hat{G}}(\ell^{-1}(n)) \mathbb{L}^{-n},$$

which takes value in $\mathcal{M}_k^{\hat{G}}$.

As in [DL99, Section 3.3], we can define the order $\text{ord}_t \mathcal{J}$ of a coherent sheaf of ideals \mathcal{J} on a k -variety Z of pure dimension d , which is a simple function. Denote by Ω_Z^1 the sheaf of differentials on Z , and by Ω_Z^d the d th exterior power of Ω_Z^1 . Let \mathcal{S} be a coherent sheaf on Z such that there exists a natural morphism of sheaves $\iota : \mathcal{S} \rightarrow \Omega_Z^d$. Assume that Z is smooth. Let $\mathcal{J}(\mathcal{S})$ be the sheaf of ideals on Z locally generated by functions $\iota(s)/dz$ with s a local section of \mathcal{S} and dz a local volume form on Z . Then we define $\text{ord}_t \mathcal{S} := \text{ord}_t \mathcal{J}(\mathcal{S})$.

Theorem 3.1. *Let X and Y be k -varieties of pure dimension d , with Y being smooth in addition. Let $h : Y \rightarrow X$ be a proper birational morphism. Let $\hat{G} = \varprojlim G_i$ act nicely on $\mathcal{L}(Y)$ and on $\mathcal{L}(X)$ such that all the morphisms $h_n : \mathcal{L}_n(Y) \rightarrow \mathcal{L}_n(X)$ are G_i -equivariant. Let A be in $\mathcal{F}_X^{\hat{G}}$ such that $A \cap \mathcal{L}(h(E)) = \emptyset$, where E is the exceptional locus of h . Let $\ell : A \rightarrow \mathbb{N}$ a simple function whose fibers are all in $\mathcal{F}_X^{\hat{G}}$. Then $h^{-1}(A)$ and the fibers of $\ell \circ h + \text{ord}_t h^*(\Omega_X^d)$ on $h^{-1}(A)$ are in $\mathcal{F}_Y^{\hat{G}}$, and furthermore, the identity*

$$\int_A \mathbb{L}^{-\ell} d\tilde{\mu}^{\hat{G}} = \int_{h^{-1}(A)} \mathbb{L}^{-\ell \circ h - \text{ord}_t h^*(\Omega_X^d)} d\tilde{\mu}^{\hat{G}}$$

holds in $\mathcal{M}_k^{\tilde{G}}$.

Proof. We assume that $A \in \mathcal{F}_X^{G_i}$ for some $i \in I$. The first statement that $h^{-1}(A)$ and the fibers of $\ell \circ h + \text{ord}_t h^*(\Omega_X^d)$ on $h^{-1}(A)$ are in $\mathcal{F}_Y^{G_i}$ is clear by [DL99, Lemma 3.3] any by the hypothesis that all h_n are G_i -equivariant. Now, for simplicity of notation, we shall write \tilde{A} for $h^{-1}(A)$, and write $\tilde{\ell}$ (resp. ν) for the simple function $\ell \circ h + \text{ord}_t h^*(\Omega_X^d)$ (resp. $\text{ord}_t h^*(\Omega_X^d)$). Since \tilde{A} and all the fibers of $\tilde{\ell}$ are stable, it follows from [DL99, Lemma 2.4] that the functions $\tilde{\ell}$ and ν are bounded. Choose a positive integer N such that all the fibers of ℓ are stable at level N and that $|\tilde{\ell}| \leq \frac{N}{2}$. Define

$$A_n := \pi_N(\ell^{-1}(n)), \quad \tilde{A}_n := h_N^{-1}(A_n), \quad \tilde{A}_{n,e} := \tilde{A}_n \cap \nu^{-1}(e), \quad A_{n,e} := h(\tilde{A}_{n,e}),$$

for every $e \geq 0$. Then we have that $\tilde{A}_n = \pi_N((\ell \circ h)^{-1}(n))$ and that, by [DL99, Lemma 3.4], the morphism $h|_{\tilde{A}_{n,e}} : \tilde{A}_{n,e} \rightarrow A_{n,e}$ is a piecewise trivial fibration with fiber \mathbb{A}_k^e . We then deduce from Theorem 2.3 that $[\tilde{A}_{n,e}] = [A_{n,e}]\mathbb{L}^e$ in $\mathcal{M}_k^{G_i}$. Therefore, we get

$$\begin{aligned} \int_A \mathbb{L}^{-\ell} d\tilde{\mu}^{G_i} &= \sum_n [A_n] \mathbb{L}^{-(N+1)d-n} = \sum_{n,e} [A_{n,e}] \mathbb{L}^{-(N+1)d-n} \\ &= \sum_{n,e} [\tilde{A}_{n,e}] \mathbb{L}^{-(N+1)d-(n+e)} = \sum_m \left(\sum_{n+e=m} [\tilde{A}_{n,e}] \right) \mathbb{L}^{-(N+1)d-m} \\ &= \sum_m \left[\pi_N \left(\tilde{\ell}^{-1}(m) \right) \right] \mathbb{L}^{-(N+1)d-m} = \int_{h^{-1}(A)} \mathbb{L}^{-\tilde{\ell}} d\tilde{\mu}^{G_i}, \end{aligned}$$

as desired. \square

4. RATIONALITY OF GENERALIZED MOTIVIC ZETA FUNCTIONS

4.1. Formal series and Hadamard product. Let \mathcal{M} be a commutative ring with unity which contains \mathbb{L} and \mathbb{L}^{-1} , and let $\mathcal{M}[[T]]$ be the set of formal power series in T with coefficients in \mathcal{M} , which is a ring and also a \mathcal{M} -module with respect to usual operations for series. Denote by $\mathcal{M}[[T]]_{\text{sr}}$ the submodule of $\mathcal{M}[[T]]$ generated by 1 and by finite products of terms $\frac{\mathbb{L}^a T^b}{(1-\mathbb{L}^a T^b)}$ for (a, b) in $\mathbb{Z} \times \mathbb{N}_{>0}$. An element of $\mathcal{M}[[T]]_{\text{sr}}$ is called a *rational series*. By [DL98], there exists a unique \mathcal{M} -linear morphism

$$\lim_{T \rightarrow \infty} : \mathcal{M}[[T]]_{\text{sr}} \rightarrow \mathcal{M}$$

such that

$$\lim_{T \rightarrow \infty} \frac{\mathbb{L}^a T^b}{(1 - \mathbb{L}^a T^b)} = -1$$

for any (a, b) in $\mathbb{Z} \times \mathbb{N}_{>0}$.

More generally, we also consider the ring of formal power series $\mathcal{M}[[T_1, \dots, T_r]]$ in r variables (T_1, \dots, T_r) , and its subset $\mathcal{M}[[T_1, \dots, T_r]]_{\text{sr}}$ the polynomial ring with coefficients in \mathcal{M} and in variables $\frac{\mathbb{L}^a T_1^{b_1} \dots T_r^{b_r}}{1 - \mathbb{L}^a T_1^{b_1} \dots T_r^{b_r}}$ for (a, b_1, \dots, b_r) in $\mathbb{Z} \times (\mathbb{N}^r \setminus \{(0, \dots, 0)\})$. The set $\mathcal{M}[[T_1, \dots, T_r]]_{\text{sr}}$ is in fact a submodule of $\mathcal{M}[[T_1, \dots, T_r]]$, each element of $\mathcal{M}[[T_1, \dots, T_r]]_{\text{sr}}$ is called a *rational series*.

By definition, the Hadamard product of two formal power series $p(T) = \sum_{n \geq 1} p_n T^n$ and $q(T) = \sum_{n \geq 1} q_n T^n$ in $\mathcal{M}[[T]]$ is the series

$$(4.1) \quad p(T) * q(T) := \sum_{n \geq 1} p_n \cdot q_n T^n$$

in $\mathcal{M}[[T]]$. This product is commutative, associative, with unity $\sum_{n \geq 1} T^n$. It also preserves the rationality as seen in the following lemma.

Lemma 4.1 (Looijenga [Lo02], Lemma 7.6). *If $p(T)$ and $q(T)$ are rational series in $\mathcal{M}[[T]]$, so is $p(T) * q(T)$, and in this case,*

$$\lim_{T \rightarrow \infty} p(T) * q(T) = - \lim_{T \rightarrow \infty} p(T) \cdot \lim_{T \rightarrow \infty} q(T).$$

The Hadamard product may be also defined for two formal power series in several variables. Namely, for two formal power series $p = \sum p_{n_1, \dots, n_r} T_1^{n_1} \cdots T_r^{n_r}$ and $q = \sum q_{n_1, \dots, n_r} T_1^{n_1} \cdots T_r^{n_r}$ in $\mathcal{M}[[T_1, \dots, T_r]]$ (the sums run over \mathbb{N}^r), we define

$$(4.2) \quad p * q := \sum p_{n_1, \dots, n_r} \cdot q_{n_1, \dots, n_r} T_1^{n_1} \cdots T_r^{n_r},$$

which is an element of $\mathcal{M}[[T_1, \dots, T_r]]$. Similarly as above, the Hadamard product for formal power series in several variables is also rationality preserving, commutative, associative, and its unity is $\sum_{(n_1, \dots, n_r) \in \mathbb{N}^r} T_1^{n_1} \cdots T_r^{n_r}$.

4.2. Motivic zeta functions. Let X be a smooth k -variety of pure dimension d . Let $f : X \rightarrow \mathbb{A}_k^1$ be a regular function with the zero locus X_0 nonempty. For $n \geq 1$, we define

$$(4.3) \quad \mathcal{X}_n(f) := \{\gamma \in \mathcal{L}_n(X) \mid f(\gamma) = t^n \pmod{t^{n+1}}\}.$$

Then $\mathcal{X}_n(f)$ is naturally an X_0 -variety and invariant under the natural action σ of μ_n on $\mathcal{L}_n(X)$ given by $\lambda \cdot \gamma(t) := \gamma(\lambda t)$. For simplicity, we write $[\mathcal{X}_n(f)]$ for the class $[\mathcal{X}_n(f) \rightarrow X_0, \sigma]$ in the ring $\mathcal{M}_{X_0}^{\mu}$. The *motivic zeta function* of f is defined to be

$$(4.4) \quad Z_f(T) := \sum_{n \geq 1} [\mathcal{X}_n(f)] \mathbb{L}^{-nd} T^n,$$

which is a formal power series in $\mathcal{M}_{X_0}^{\mu}[[T]]$. If x is a closed point in X_0 , by setting

$$\mathcal{X}_{n,x}(f) = \{\gamma \in \mathcal{X}_n(f) \mid \gamma(0) = x\}$$

we obtain in the same way the *motivic zeta function of f at x*

$$(4.5) \quad Z_{f,x}(T) := \sum_{n \geq 1} [\mathcal{X}_{n,x}(f)] \mathbb{L}^{-nd} T^n,$$

which is a formal power series in $\mathcal{M}_k^{\mu}[[T]]$.

Remark 4.2. We can use the new terminology and notation in Section 3.2 as follows. We note that, for $n \geq 1$ and x as previous, the sets $\overline{\mathcal{X}}_n(f) := \{\gamma \in \mathcal{L}(X) \mid f(\gamma) = t^n \pmod{t^{n+1}}\}$ and $\overline{\mathcal{X}}_{n,x}(f) := \{\gamma \in \overline{\mathcal{X}}_n(f), \gamma(0) = x\}$ are in the family $\mathcal{F}_X^{\mu_n}$, they are stable at level n ; and furthermore, $\mathcal{X}_n(f) = \pi_n(\overline{\mathcal{X}}_n(f))$, $\mathcal{X}_{n,x}(f) = \pi_n(\overline{\mathcal{X}}_{n,x}(f))$, and with $\tilde{\mu}$ in Section 3.2,

$$Z_{f,x}(T) = \mathbb{L}^d \sum_{n \geq 1} \tilde{\mu}(\overline{\mathcal{X}}_{n,x}(f)) T^n.$$

As in Denef-Loeser [DL99], [DL02], to see the rationality of the series (4.4) and (4.5) we consider a log-resolution $h : Y \rightarrow X$ of X_0 . The exceptional divisors and irreducible components of the strict transform for h will be denoted by E_i , where i is in a finite set J . For every nonempty $I \subseteq J$, we put $E_I^\circ = (\bigcap_{i \in I} E_i) \setminus \bigcup_{j \notin I} E_j$, and consider an affine covering $\{U\}$ of Y such that on each piece $U \cap E_I^\circ \neq \emptyset$ the pullback of f has the form $u \prod_{i \in I} y_i^{N_i}$, with u a unit and y_i a local coordinate defining E_i . Denote by m_I the greatest common divisor of N_i , with i in I . Denef and Loeser [DL98] study the unramified Galois covering $\pi_I : \tilde{E}_I^\circ \rightarrow E_I^\circ$ with Galois group μ_{m_I} defined locally with respect to $\{U\}$ as follows

$$\{(z, y) \in \mathbb{A}_k^1 \times (U \cap E_I^\circ) \mid z^{m_I} = u(y)^{-1}\}.$$

The local pieces are glued over $\{U\}$ as in the proof of [DL98, Lemma 3.2.2] to get \tilde{E}_I° and π_I as mentioned, and the definition of the covering π_I is independent of the choice of $\{U\}$. Moreover, \tilde{E}_I° is endowed with a μ_{m_I} -action by multiplication of the z -coordinate with elements of μ_{m_I} , which gives rise to an element $[\tilde{E}_I^\circ] = [\tilde{E}_I^\circ \rightarrow E_I^\circ \rightarrow X_0]$ in $\mathcal{M}_{X_0}^\mu$ (cf. [DL02]). For every i in J , we denote by $\nu_i - 1$ the multiplicity of E_i in the canonical divisor of h .

Theorem 4.3 (Denef-Loeser [DL98]). *With the previous notation and hypothesis, we have*

$$Z_f(T) = \sum_{\emptyset \neq I \subseteq J} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

In other words, the motivic zeta function of f is a rational series.

An analogous formula can be also obtained for $Z_{f,x}(T)$ in (4.5), so it is a rational series. The following element of $\mathcal{M}_{X_0}^\mu$,

$$\mathcal{S}_f := - \lim_{T \rightarrow \infty} Z_f(T) = \sum_{\emptyset \neq I \subseteq J} (1 - \mathbb{L})^{|I|-1} [\tilde{E}_I^\circ],$$

is called the *motivic nearby cycles* of f . The element $\mathcal{S}_{f,x} := - \lim_{T \rightarrow \infty} Z_{f,x}(T)$ of \mathcal{M}_k^μ , which equals $(\{x\} \hookrightarrow X_0)^* \mathcal{S}_f$, is called the *motivic Milnor fiber* of f at x .

4.3. Generalizations. For simplicity of performance, we only consider generalizations of the motivic zeta functions in the case where the base variety is $\text{Spec } k$. Recall that a semi-algebraic condition θ is a finite boolean combination of the conditions of the forms

$$(4.6) \quad \begin{aligned} \text{ord}_t f_1(x) \geq \text{ord}_t f_2(x) + \ell_1(\alpha), \quad \text{ord}_t f_3(x) \equiv \ell_2(\alpha) \pmod{d}, \\ \Phi(\overline{\text{ac}}(g_1(x)), \dots, \overline{\text{ac}}(g_n(x))) = 0, \end{aligned}$$

where f_i, g_j, Φ are polynomials over k , ℓ_1 and ℓ_2 are polynomials over \mathbb{Z} of degree ≤ 1 , $x = (x_1, \dots, x_m)$ are free variables over $K((t))$ (with K being any algebraically closed field containing k), $\alpha = (\alpha_1, \dots, \alpha_r)$ are free variables over \mathbb{Z} , and d is in $\mathbb{N}_{>0}$. Suggested from [CL08, Section 14.5], we want to consider a so-called $k[t]$ -semi-algebraic condition. A $k[t]$ -*semi-algebraic* condition θ' is defined in the same way as the above θ but with f_i and g_j polynomials over $k[t]$ (instead of over k). Note that sometimes a semi-algebraic condition may be equivalent to a $k[t]$ -semi-algebraic condition, that is, they may define the same semi-algebraic subset of $\mathcal{L}(X)$. For instance, if f is a polynomial over k , the semi-algebraic condition

$$\text{ord}_t f(x) = n \wedge \overline{\text{ac}} f(x) = 1$$

and the $k[t]$ -semi-algebraic condition

$$(4.7) \quad \text{ord}_t(f(x) - t^n) \geq \text{ord}_t(t^n) + 1$$

are equivalent. Let us temporarily assume that $X = \mathbb{A}_k^d$. Let A be a semi-algebraic subset of $\mathcal{L}(X)$ which is defined by a $k[t]$ -semi-algebraic condition φ . For $n \geq 1$, let $\varphi[n]$ denote the $k[t]$ -semi-algebraic condition obtained from φ by replacing everywhere t by t^n . For instance, with a polynomial f over k , if φ is the $k[t]$ -semi-algebraic condition

$$\text{ord}_t(f(x) - t) \geq \text{ord}_t(t) + 1,$$

then $\varphi[n]$ is the condition (4.7). If A is a stable semi-algebraic subset of $\mathcal{L}(X)$ which is defined by a semi-algebraic condition θ , and if θ is equivalent to a $k[t]$ -semi-algebraic condition φ , then the subset $A[n]$ defined by $\varphi[n]$ is also a stable semi-algebraic subset of $\mathcal{L}(X)$. We can extend the definition to any k -variety X by using a covering by affine open subsets. In this case, the group μ_n acts naturally on $A[n]$ in such a way that $\lambda \cdot \gamma(t) = \gamma(\lambda t)$, so we can take the $\hat{\mu}$ -equivariant motivic measure $\tilde{\mu}(A[n])$ of $A[n]$, which is an element of $\mathcal{M}_k^{\hat{\mu}}$.

Conjecture 4.4. *Let X be a k -variety, and let $A = \{A_\alpha \mid \alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r\}$ be a semi-algebraic family of semi-algebraic subsets of $\mathcal{L}(X)$ where there exists a covering of X by affine open subsets U such that a semi-algebraic condition defining each $A_\alpha \cap \mathcal{L}(U)$ is equivalent to a $k[t]$ -semi-algebraic condition. Assume A_α is weakly stable (hence stable) and disjoint with $\mathcal{L}(X_{\text{Sing}})$, for every α in \mathbb{N}^r (hence $A_\alpha[n]$ is in $\mathcal{F}_X^{\mu_n}$ for every α in \mathbb{N}^r , $n \geq 1$). Then the formal power series*

$$Z_A(T_0, T_1, \dots, T_r) := \sum_{(n, \alpha) \in \mathbb{N}^{r+1}} \tilde{\mu}(A_\alpha[n]) T_0^n T_1^{\alpha_1} \dots T_r^{\alpha_r}$$

is a rational series, i.e., an element of $\mathcal{M}_k^{\hat{\mu}}[[T_0, T_1, \dots, T_r]]_{\text{sr}}$.

In what follows we are going to prove Conjecture 4.4 in a special case.

Theorem 4.5. *Let X be a k -variety and f a regular function on X . Let A_α , α in \mathbb{N}^r , be a semi-algebraic family of semi-algebraic subsets of $\mathcal{L}(X)$ such that, for every covering of X by affine open subsets U , any semi-algebraic condition defining $A_\alpha \cap \mathcal{L}(U)$ contains only conditions of two first forms in (4.6). Assume that, for every α in \mathbb{N}^r , A_α is weakly stable (hence stable) and disjoint with $\mathcal{L}(X_{\text{Sing}})$. For $n \geq 1$, we put*

$$A_{n, \alpha} := \{\gamma \in A_\alpha \mid f(\gamma) = t^n \pmod{t^{n+1}}\},$$

which is in $\mathcal{F}_X^{\mu_n}$ for every α in \mathbb{N}^r , $n \geq 1$. Then the formal power series

$$Z(T_0, T_1, \dots, T_r) := \sum_{(n, \alpha) \in \mathbb{N}^{r+1}} \tilde{\mu}(A_{n, \alpha}) T_0^n T_1^{\alpha_1} \dots T_r^{\alpha_r}$$

is an element of $\mathcal{M}_k^{\hat{\mu}}[[T_0, T_1, \dots, T_r]]_{\text{sr}}$.

Proof. With the same reason as in the proof of [DL99, Theorem 5.1'], we may assume that X is smooth and affine of dimension d . Let θ be a semi-algebraic condition which defines A_α . Let f_i , $1 \leq i \leq m$, be all the polynomials in $k[x_1, \dots, x_e]$ (for some e) occurring in θ (we may assume that X is a closed subvariety of \mathbb{A}_k^e). By the hypothesis, θ contains only conditions of first two forms in (4.6), so it is of the form

$$\theta = \theta'(\text{ord}_t f_1, \dots, \text{ord}_t f_m, \alpha),$$

where θ' defines a Presburger subset of \mathbb{Z}^{m+r} (i.e., θ' is a formula in the Presburger language). For every $\beta = (\beta_1, \dots, \beta_m)$ in \mathbb{N}^m and n in $\mathbb{N}_{>0}$, we put

$$D_\beta := \{\gamma \in \mathcal{L}(X) \mid \text{ord}_t f_i(z_1(\gamma), \dots, z_e(\gamma)) = \beta_i, 1 \leq i \leq m\}$$

and

$$D_{n,\beta} := \{\gamma \in D_\beta \mid f(\gamma) = t^n \pmod{t^{n+1}}\},$$

where z_j , $1 \leq j \leq e$, are regular functions on X . We observe that $D_{n,\beta}$ is invariant under the μ_n -action $\lambda \cdot \gamma(t) = \gamma(\lambda t)$. Then we get the decomposition

$$A_{n,\alpha} = \bigsqcup_{\beta \in \mathbb{N}^m, \theta'(\beta, \alpha)} D_{n,\beta},$$

hence the identity

$$(4.8) \quad \tilde{\mu}(A_{n,\alpha}) = \sum_{\beta \in \mathbb{N}^m, \theta'(\beta, \alpha)} \tilde{\mu}(D_{n,\beta})$$

in \mathcal{M}_k^μ . Since $D_{n,\beta}$ is stable of level $N := n + \sum_{i=1}^m \beta_i$, we have $\tilde{\mu}(D_{n,\beta}) = [\pi_N(D_{n,\beta})] \mathbb{L}^{-Nd}$. Define

$$(4.9) \quad g := f \prod_{i=1}^m f_i$$

and consider a log-resolution $h : Y \rightarrow X$ of the zero locus $X_0(g)$ of g . We use the notation about h as in Section 4.2. In particular, we consider an affine covering $\{U\}$ of Y such that, on $U \cap E_I^\circ \neq \emptyset$, with $h(E_I^\circ)$ contained in $X_0(g)$, we have

$$(4.10) \quad f \circ h = u \prod_{j \in I} y_j^{N_j(f)}, \quad f_i \circ h = u_i \prod_{j \in I} y_j^{N_j(f_i)}, \quad 1 \leq i \leq m,$$

where u and u_i do not vanish on U , and for each j , y_j is a local coordinate defining E_j . We now use the idea in the proof of [DL02, Lemma 2.5] and slightly modify it. Consider the solutions $(k_j)_{j \in I} \in \mathbb{N}_{\geq 1}^I$ of the system of diophantine equations

$$(4.11) \quad \sum_{j \in I} k_j N_j(f) = n, \quad \sum_{j \in I} k_j N_j(f_i) = \beta_i, \quad 1 \leq i \leq m.$$

When emphasising the free coefficient vector $(n, \beta)^t$ in this system (4.11) we write $(4.11)_{n,\beta}$ for it. For such a solution $(k_j)_{j \in I}$ in $\mathbb{N}_{\geq 1}^I$ of (4.11), we put

$$U_{(k_j)} := \{\gamma \in \mathcal{L}_N(U) \mid \overline{\text{ac}}f(h_{n*}(\gamma)) = 1, \text{ord}_t y_j(\gamma) = k_j, j \in I\}.$$

Similarly as in the proof of [DL02, Lemma 2.5] we obtain the identity

$$[U_{(k_j)}] = [V_I] \mathbb{L}^{Nd - \sum_{j \in I} k_j},$$

which in fact lies in \mathcal{M}_k^μ , where

$$V_I := \left\{ ((c_j)_{j \in I}, y) \in \mathbb{G}_{m,k}^I \times_k (E_I^\circ \cap U) \mid u(y) \prod_{j \in I} c_j^{N_j(f)} = 1 \right\}$$

whose class in \mathcal{M}_k^μ is nothing but $(\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ \cap U]$. Using Lemma 3.4 of [DL99] (or more particularly, Lemma 2.2 of [DL02]), the set $U_{(k_j)}$ modified with N replaced by a sufficiently large natural number l and augmented by the condition $\text{ord}_t \det \text{Jac}_h(x) = a$ is a piecewise trivial fibration with fiber \mathbb{A}_k^a onto a subset of $\pi_l(D_{n,\beta})$. Note that, on $U \cap E_I^\circ \neq \emptyset$, with $h(E_I^\circ) \subseteq X_0(g)$, we have $\text{ord}_t \det \text{Jac}_h = v \prod_{j \in I} y_j^{\nu_j - 1}$, with v a unit on U . As in the proof of

[DL02, Theorem 2.4], we may glue $U_{(k_j)}$ and use the properties that $\tilde{\mu}$ is additive and that π_N^l is a locally trivial fibration as X and Y are smooth. We thus get the identity

$$[\pi_N(D_{n,\beta})] = \mathbb{L}^{Nd} \sum_{\substack{\emptyset \neq I \subseteq J \\ h(E_I^\circ) \subseteq X_0(g)}} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ] \sum_{\substack{(k_j)_{j \in I} \in \mathbb{N}_{\geq 1}^I \\ (4.11)_{n,\beta}}} \mathbb{L}^{-\sum_{j \in I} k_j \nu_j},$$

and hence, by (4.8),

$$\tilde{\mu}(A_{n,\alpha}) = \sum_{\substack{\emptyset \neq I \subseteq J \\ h(E_I^\circ) \subseteq X_0(g)}} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ] \sum_{\substack{\beta \in \mathbb{N}^m \\ \theta'(\beta, \alpha)}} \sum_{\substack{(k_j)_{j \in I} \in \mathbb{N}_{\geq 1}^I \\ (4.11)_{n,\beta}}} \mathbb{L}^{-\sum_{j \in I} k_j \nu_j}$$

in the ring \mathcal{M}_k^μ . It follows that

$$(4.12) \quad Z(T_0, T_1, \dots, T_r) = \sum_{\emptyset \neq I \subseteq J, h(E_I^\circ) \subseteq X_0(g)} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ] S_I(T_0, T_1, \dots, T_r),$$

where, for every $I \subseteq J$ nonempty,

$$S_I(T_0, T_1, \dots, T_r) := \sum_{(n,\alpha) \in \mathbb{N}^{r+1}} \sum_{\substack{\beta \in \mathbb{N}^m \\ \theta'(\beta, \alpha)}} \sum_{\substack{(k_j)_{j \in I} \in \mathbb{N}_{>0}^I \\ (4.11)_{n,\beta}}} \mathbb{L}^{-\sum_{j \in I} k_j \nu_j} T_0^n T_1^{\alpha_1} \dots T_r^{\alpha_r}.$$

Now we fix a nonempty subset I of J and consider the following $(r+2)$ -variable formal power series

$$S(T_0, T_1, \dots, T_r, T_{r+1}) := \sum_{(n,\alpha) \in \mathbb{N}^{r+1}} \sum_{\substack{\beta \in \mathbb{N}^m \\ \theta'(\beta, \alpha)}} \sum_{\substack{(k_j)_{j \in I} \in \mathbb{N}_{>0}^I \\ (4.11)_{n,\beta}}} T_0^n T_1^{\alpha_1} \dots T_r^{\alpha_r} T_{r+1}^{\sum_{j \in I} k_j \nu_j}.$$

Denote by P the following Presburger subset of \mathbb{N}^{m+r} ,

$$P := \left\{ (n, \beta, \alpha, \alpha_{r+1}) \in \mathbb{N}^{m+r+2} \mid \theta'(\beta, \alpha), (4.11)_{n,\beta}, \sum_{j \in I} k_j \nu_j = \alpha_{r+1}, k_j \geq 1 \forall j \in I \right\}.$$

Taking the composition of the inclusion of P in \mathbb{N}^{m+r+2} with the projection $\mathbb{N}^{m+r+2} \rightarrow \mathbb{N}^{r+2}$ sending $(n, \beta, \alpha, \alpha_{r+1})$ to $(n, \alpha, \alpha_{r+1})$ we get a map $\rho : P \rightarrow \mathbb{N}^{r+2}$. Because for any α_{r+1} in \mathbb{N} fixed the diophantine equation $\sum_{j \in I} k_j \nu_j = \alpha_{r+1}$ has only finitely many positive solutions, every fiber of ρ is a finite set. By [DL99, Lemma 5.2], the series $S(T_0, T_1, \dots, T_r, T_{r+1})$ belongs to the subring of $\mathbb{Z}[[T_0, T_1, \dots, T_r, T_{r+1}]]$ generated by $\mathbb{Z}[[T_0, T_1, \dots, T_r, T_{r+1}]]$ and the series $(1 - T_0^{c_0} T_1^{c_1} \dots T_r^{c_r} T_{r+1}^{c_{r+1}})^{-1}$, where $(c_0, c_1, \dots, c_r, c_{r+1})$ are in $\mathbb{N}^{r+2} \setminus \{(0, \dots, 0)\}$. It follows that the formal power series $S_I(T_0, T_1, \dots, T_r) = S(T_0, T_1, \dots, T_r, \mathbb{L}^{-1})$ is a rational series, from which $Z(T_0, T_1, \dots, T_r)$ is an element of $\mathcal{M}_k^\mu[[T_0, T_1, \dots, T_r]]_{\text{sr}}$. \square

Proposition 4.6. *Let X , f , A_α and $A_{n,\alpha}$ be as in Theorem 4.5. Let Δ be a rational polyhedral convex cone in $\mathbb{R}_{\geq 0}^{r+1}$ and $\bar{\Delta}$ its closure. Let ℓ and ε be integral linear forms on \mathbb{Z}^{r+1} with $\ell(n, \alpha) > 0$ and $\varepsilon(n, \alpha) \geq 0$ for all (n, α) in $\bar{\Delta} \setminus \{0\}$. Then the formal power series*

$$Z(T) := \sum_{(n,\alpha) \in \Delta \cap \mathbb{N}^{r+1}} \tilde{\mu}(A_{n,\alpha}) \mathbb{L}^{-\varepsilon(n,\alpha)} T^{\ell(n,\alpha)}$$

is an element of $\mathcal{M}_k^\mu[[T]]_{\text{sr}}$, and the limit $\lim_{T \rightarrow \infty} Z(T)$ is independent of such an ℓ and ε .

Proof. The first statement is direct corollary of Theorem 4.5. We now prove the second one, that $\lim_{T \rightarrow \infty} Z(T)$ is independent of the linear form ℓ . As in the proof of Theorem 4.5, we may assume that X is smooth of dimension d and a closed subvariety of \mathbb{A}_k^e for some $e \geq d$. Using notation and arguments in the proof of Theorem 4.5 we get

$$Z(T) = \sum_{\emptyset \neq I \subseteq J, h(E_I^\circ) \subseteq X_0(g)} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ] S_{I,\varepsilon,\ell}(T),$$

where g is as in (4.9) and

$$S_{I,\varepsilon,\ell}(T) := \sum_{(n,\alpha) \in \Delta \cap \mathbb{N}^{r+1}} \sum_{\substack{\beta \in \mathbb{N}^m \\ \theta'(\beta,\alpha)}} \sum_{\substack{(k_j)_{j \in I} \in \mathbb{N}_{>0}^I \\ (4.11)_{n,\beta}}} \mathbb{L}^{-\sum_{j \in I} k_j \nu_j} \mathbb{L}^{-\varepsilon(n,\alpha)} T^{\ell(n,\alpha)}.$$

Let us consider the Presburger set $Q := \{(\beta, \alpha) \in \mathbb{N}^{m+r} \mid \theta'(\beta, \alpha)\}$. We can assume that there is no congruence relations in the description of Q , because if necessary we replace (β, α) by $w(\beta, \alpha) + \delta$ for some w in $\mathbb{N}_{>0}$ and δ in \mathbb{N}^{m+r} . Since for $Q = Q_1 \cup Q_2$, the sum taking over Q satisfies $\sum_Q = \sum_{Q_1} + \sum_{Q_2} - \sum_{Q_1 \cap Q_2}$, we may thus assume that Q is the set of integral points in a rational polyhedral convex cone \bar{Q} in $\mathbb{R}_{\geq 0}^{m+r}$ (cf. proof of Lemma 5.2 of [DL99]). We now assume that Q is defined by a system of linear equations and inequations $p_u(\beta, \alpha) \geq 0$ and $q_v(\beta, \alpha) = 0$ for linear forms p_u and q_v with integer coefficients. We add new variables a_u in \mathbb{N} and consider the system $p_u(\beta, \alpha) = a_u$, $q_v(\beta, \alpha) = 0$. Then there exist rational linear forms l_s such that

$$\alpha_s = l_s(\beta', (a_u)_u, (b_w)_w), \quad 1 \leq s \leq r,$$

where $\beta' = (\beta_i)_{i \in R}$ consists of part of components of β (i.e., $R \subseteq \{1, \dots, m\}$), which only equals β when the maximum number of linearly independent equations in variables α of the system is greater than or equal to r , and b_w are new variables in \mathbb{N} , which only appear when the maximum number of linearly independent equations in variables α of the system is strictly smaller than r . Since when replacing ε and ℓ by $e\varepsilon$ and $e\ell$, respectively, for any e in $\mathbb{N}_{>0}$, the dependence of $\lim_{T \rightarrow \infty} S_{I,\varepsilon,\ell}(T)$ on ε and ℓ does not change, we may assume further that l_s are integral linear forms. Now we put $\delta = ((k_j)_{j \in I}, (a_u)_u, (b_w)_w)$ and

$$\omega(\delta) := \ell \left(\sum_{j \in I} k_j N_j(f), \left(l_s \left(\left(\sum_{j \in I} k_j N_j(f_i) \right)_{i \in R}, (a_u)_u, (b_w)_w \right) \right)_{1 \leq s \leq r} \right)$$

and

$$\omega'(\delta) := \sum_{j \in I} k_j \nu_j + \varepsilon \left(\sum_{j \in I} k_j N_j(f), \left(l_s \left(\left(\sum_{j \in I} k_j N_j(f_i) \right)_{i \in R}, (a_u)_u, (b_w)_w \right) \right)_{1 \leq s \leq r} \right).$$

By the hypothesis on ℓ , ε and l_s , the forms $\omega(\delta)$ and $\omega'(\delta)$ are integral linear forms which are positive for all δ in $\mathbb{R}_{\geq 0}^N \setminus \{0\}$, where N is the number of components of the vector δ . On the other hand, there exists a rational polyhedral convex cone C in $\mathbb{R}_{\geq 0}^N$ such that (n, α) is in $\Delta \cap \mathbb{N}^{r+1}$ if and only if δ is in $C \cap \mathbb{N}^N$. Then we have

$$S_{I,\varepsilon,\ell}(T) = \sum_{\delta \in C \cap \mathbb{N}^N} \mathbb{L}^{-\omega'(\delta)} T^{\omega(\delta)}.$$

By Guibert's result [Gu02, Lemme 2.1.5] (see also [GLM06, Section 2.9]), $\lim_{T \rightarrow \infty} S_{I,\varepsilon,\ell}(T)$ is independent of ω and ω' , hence it is independent of ℓ and ε . This proves the second statement of the proposition. \square

5. PROOF OF THE INTEGRAL IDENTITY CONJECTURE

5.1. Decomposition of the integral identity's LHS. Let us consider the polynomial f in Conjecture 1, which induces a regular function, also denoted by f , on \mathbb{A}_k^d , with the zero locus X_0 containing 0 in \mathbb{A}_k^d . By the homogeneity of f on the (x, y) -variables we have

$$f(x, 0, z) = f(0, 0, z) = \tilde{f}(z),$$

where \tilde{f} is the restriction of f to $\mathbb{A}_k^{d_3}$ (we consider $\mathbb{A}_k^{d_3}$ as $\{0\} \times \{0\} \times_k \mathbb{A}_k^{d_3}$). Then $\mathbb{A}_k^{d_1}$ is regarded as a k -subvariety of X_0 , and the inclusion is denoted by i . In what follows, for simplicity of notation, we shall sometimes write γ instead of (x, y, z) , for x, y and z in $\mathcal{L}_n(\mathbb{A}_k^{d_1})$, $\mathcal{L}_n(\mathbb{A}_k^{d_2})$ and $\mathcal{L}_n(\mathbb{A}_k^{d_3})$, respectively. Consider the rational series

$$\int_{\mathbb{A}_k^{d_1}} i^* Z_f(T) = \sum_{n \geq 1} \int_{\mathbb{A}_k^{d_1}} i^* [\mathcal{X}_n(f)] \mathbb{L}^{-nd} T^n,$$

with coefficients in $\mathcal{M}_k^{\hat{\mu}}$. The minus limit of this rational series is nothing but the left hand side of the integral identity, namely,

$$(5.1) \quad \int_{\mathbb{A}_k^{d_1}} i^* \mathcal{S}_f = - \lim_{T \rightarrow \infty} \int_{\mathbb{A}_k^{d_1}} i^* Z_f(T).$$

Clearly, the identity

$$i^* [\mathcal{X}_n(f)] = \left[\left\{ \gamma \in \mathcal{X}_n(f) \mid \gamma(0) \in \mathbb{A}_k^{d_1} \times_k \{0\} \times_k \{0\} \right\} \rightarrow \mathbb{A}_k^{d_1}, \gamma \mapsto \gamma(0) \right]$$

holds in $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{A}_k^{d_1}})$. It is convenient to use the following order of n -jets $x(t)$, and similarly, that of $y(t)$:

$$\text{ord}_t x(t) = \min_{1 \leq j \leq d_1} \text{ord}_t x_j(t), \quad \text{ord}_t y(t) = \min_{1 \leq j \leq d_2} \text{ord}_t y_j(t).$$

In the rest of the present article, instead of writing $\gamma(0) \in \mathbb{A}_k^{d_1} \times_k \{0\} \times_k \{0\}$ we shall write for short $\gamma(0) \in \mathbb{A}_k^{d_1}$ for γ in $\mathcal{L}(\mathbb{A}_k^d)$. We observe that the sets

$$U_n := \left\{ \gamma \in \mathcal{X}_n(f) \mid \gamma(0) \in \mathbb{A}_k^{d_1}, \text{ord}_t x(t) + \text{ord}_t y(t) > n \right\}$$

and

$$W_n := \left\{ \gamma \in \mathcal{X}_n(f) \mid \gamma(0) \in \mathbb{A}_k^{d_1}, \text{ord}_t x(t) + \text{ord}_t y(t) \leq n \right\}$$

are closed and open k -subvarieties of $\mathcal{X}_n(f) \times_{X_0} \mathbb{A}_k^{d_1}$, respectively, and they are invariant under the natural μ_n -action $\lambda \cdot \gamma(t) = \gamma(\lambda t)$. Hence we get the following identity in $\mathcal{M}_k^{\hat{\mu}}$:

$$(5.2) \quad \int_{\mathbb{A}_k^{d_1}} i^* [\mathcal{X}_n(f)] \mathbb{L}^{-nd} = [U_n] \mathbb{L}^{-nd} + [W_n] \mathbb{L}^{-nd}.$$

By Theorem 4.5, the series

$$U(T) := \sum_{n \geq 1} [U_n] \mathbb{L}^{-nd} T^n \quad \text{and} \quad W(T) := \sum_{n \geq 1} [W_n] \mathbb{L}^{-nd} T^n$$

are rational series. Denoting $\mathbb{U} := - \lim_{T \rightarrow \infty} U(T)$ and $\mathbb{W} := - \lim_{T \rightarrow \infty} W(T)$ we obtain

$$(5.3) \quad \int_{\mathbb{A}_k^{d_1}} i^* \mathcal{S}_f = \mathbb{U} + \mathbb{W}.$$

The theorem hence follows directly from the following computations of \mathbb{U} and \mathbb{W} .

5.2. Computation of \mathbb{U} . In this paragraph we are going to prove the following proposition.

Proposition 5.1. *The identity $\mathbb{U} = \mathbb{L}^{d_1} \mathcal{S}_{\tilde{f},0}$ holds in $\mathcal{M}_k^{\hat{\mu}}$.*

Proof. Note that f is of the form

$$f(x, y, z) = \sum_{|\alpha|=|\beta|>0} c_{\alpha,\beta,\delta} x^\alpha y^\beta z^\delta + \tilde{f}(z),$$

where $|\alpha| = \alpha_1 + \dots + \alpha_{d_1}$ and $|\beta| = \beta_1 + \dots + \beta_{d_2}$. It implies that

$$\begin{aligned} U_n &= \left\{ \gamma \in \mathcal{L}_n(\mathbb{A}_k^d) \mid \gamma(0) \in \mathbb{A}_k^{d_1}, \text{ord}_t x(t) + \text{ord}_t y(t) > n, f(\gamma) = t^n \pmod{t^{n+1}} \right\} \\ &= \left\{ \gamma \in \mathcal{L}_n(\mathbb{A}_k^d) \mid \gamma(0) \in \mathbb{A}_k^{d_1}, \text{ord}_t x(t) + \text{ord}_t y(t) > n, \tilde{f}(z) = t^n \pmod{t^{n+1}} \right\} \\ &= U'_n \times \mathcal{X}_{n,0}(\tilde{f}), \end{aligned}$$

where

$$U'_n := \left\{ (x, y) \in \mathcal{L}_n(\mathbb{A}_k^{d_1} \times \mathbb{A}_k^{d_2}) \mid \text{ord}_t y(t) > 0, \text{ord}_t x(t) + \text{ord}_t y(t) > n \right\}.$$

Denote by I the index set $\{1, 2, \dots, n, \infty\}$, and for each m in I , put

$$\begin{aligned} U'_{n,m} &:= \left\{ (x, y) \in U'_n \mid \text{ord}_t y(t) = m \right\} \\ &= \left\{ x \in \mathcal{L}_n(\mathbb{A}_k^{d_1}) \mid \text{ord}_t x(t) > n - m \right\} \times \left\{ y \in \mathcal{L}_n(\mathbb{A}_k^{d_2}) \mid \text{ord}_t y(t) = m \right\} \end{aligned}$$

Then we have

$$U'_n = \bigsqcup_{m \in I} U'_{n,m},$$

and, for every $1 \leq m \leq n$,

$$[U'_{n,m}] = \mathbb{L}^{md_1} \cdot (\mathbb{L}^{d_2} - 1) \mathbb{L}^{(n-m)d_2},$$

while for $m = \infty$,

$$U'_{n,\infty} = \mathbb{L}^{(n+1)d_1}.$$

It yields the identity

$$[U'_n] = \sum_{m \in I} [U'_{n,m}] = \sum_{m=1}^n \mathbb{L}^{md_1} \cdot (\mathbb{L}^{d_2} - 1) \mathbb{L}^{(n-m)d_2} + \mathbb{L}^{(n+1)d_1}$$

in $\mathcal{M}_k^{\hat{\mu}}$, which implies that

$$(5.4) \quad \sum_{n \geq 1} [U'_n] \mathbb{L}^{-n(d_1+d_2)} T^n = (\mathbb{L}^{d_2} - 1) \sum_{1 \leq m \leq n} \mathbb{L}^{-(nd_1+(d_2-d_1)m)} T^n + \mathbb{L}^{d_1} \sum_{n \geq 1} \mathbb{L}^{-nd_2} T^n.$$

Applying [GLM06, Lemma 2.10] we get

$$\lim_{T \rightarrow \infty} \sum_{1 \leq m \leq n} \mathbb{L}^{-(nd_1+(d_2-d_1)m)} T^n = 0,$$

hence

$$- \lim_{T \rightarrow \infty} \sum_{n \geq 1} [U'_n] \mathbb{L}^{-n(d_1+d_2)} T^n = \mathbb{L}^{d_1}.$$

We then conclude that

$$\begin{aligned} \mathbb{U} &= - \lim_{T \rightarrow \infty} \left(\left(\sum_{n \geq 1} [U'_n] \mathbb{L}^{-n(d_1+d_2)} T^n \right) * Z_{\tilde{f},0}(T) \right) \\ &= \left(- \lim_{T \rightarrow \infty} \sum_{n \geq 1} [U'_n] \mathbb{L}^{-n(d_1+d_2)} T^n \right) \cdot \left(- \lim_{T \rightarrow \infty} Z_{\tilde{f},0}(T) \right) \\ &= \mathbb{L}^{d_1} \mathcal{S}_{\tilde{f},0}. \end{aligned}$$

Here, the symbol $*$ stands for the Hadamard product of two series in $\mathcal{M}_k^{\hat{\mu}}[[T]]$ defined in (4.1), and the second equality follows from Lemma 4.1. \square

5.3. Computation of \mathbb{W} . We are now in position to prove the last proposition of the article and finish the proof of the integral identity conjecture for regular functions.

Proposition 5.2. *The identity $\mathbb{W} = 0$ holds in $\mathcal{M}_k^{\hat{\mu}}$.*

Proof. For $n \geq m \geq 1$, we define

$$(5.5) \quad W_{n,m} := \{ \gamma \in W_n \mid \text{ord}_t x + \text{ord}_t y = m \},$$

which is invariant under the natural $\hat{\mu}$ -action $\lambda \cdot \gamma(t) = \gamma(\lambda t)$ (we recall that $\gamma = (x, y, z)$). It follows that

$$(5.6) \quad W_n = \coprod_{1 \leq m \leq n} W_{n,m},$$

therefore,

$$(5.7) \quad W(T) = \sum_{1 \leq m \leq n} [W_{n,m}] \mathbb{L}^{-nd} T^m,$$

a formal power series in $\mathcal{M}_k^{\hat{\mu}}[[T]]$.

Let us consider the action of $\mathbb{G}_{m,k}$ on the affine k -variety $X := \mathbb{A}_k^{d_1} \times_k \mathbb{A}_k^{d_2} \times_k \mathbb{A}_k^{d_3}$ given by

$$\lambda \cdot (x, y, z) := (\lambda x, \lambda^{-1} y, z),$$

for λ in $\mathbb{G}_{m,k}$ and (x, y, z) in X . Since the group $\mathbb{G}_{m,k}$ is reductive and X is affine, a categorical quotient $\phi : X \rightarrow Y$ exists and the space Y has a structure of a k -variety (cf. [MFK94, Chapter 1, Section 2]). We consider the induced morphism

$$\phi_n : \mathcal{L}_n(X) \rightarrow \mathcal{L}_n(Y)$$

and its restriction to $W_{n,m}$, also denoted by ϕ_n ,

$$\phi_n : W_{n,m} \rightarrow V_{n,m}.$$

Here, $V_{n,m}$ denotes the image of $W_{n,m}$ in $\mathcal{L}_n(Y)$ under ϕ_n . Since $W_{n,m}$ is invariant under the action of μ_n , so is $V_{n,m}$, and furthermore the morphism ϕ_n is μ_n -equivariant. We are going to show that ϕ_n is in fact a μ_n -equivariant piecewise trivial fibration with fiber

$$F := \{ \tau \in \mathcal{L}_n(\mathbb{A}_k^1) \mid \text{ord}_t \tau < m \},$$

where the action of μ_n on F is induced from the natural action of μ_n on $\mathcal{L}_n(\mathbb{A}_k^1)$. Indeed, for every field extension $K \supseteq k$, let us take an arbitrary K -arc ψ in $V_{n,m}$ and consider the fiber $W_{n,m,\psi}$ of ϕ_n over ψ . Note that, for all $1 \leq i \leq d_1$, $1 \leq j \leq d_2$ and $1 \leq l \leq d_3$, the morphisms f_{ij} and z_l from X to \mathbb{A}_k^1 defined respectively by $x_i y_j$ and z_l are $\mathbb{G}_{m,k}$ -equivariant with respect to the trivial action of $\mathbb{G}_{m,k}$ on \mathbb{A}_k^1 . By the universality of ϕ , these morphisms are constant on every fiber of ϕ . Hence, the induced morphisms $(f_{ij})_n, (z_l)_n : \mathcal{L}_n(X) \rightarrow \mathcal{L}_n(\mathbb{A}_k^1)$ are constant

on $W_{n,m,\psi}$. It implies that, for any two elements $\gamma = (x, y, z)$ and $\gamma' = (x', y', z')$ in $W_{n,m,\psi}$, the following identities hold in $K[[t]]/(t^{n+1})$:

$$\begin{aligned} x_i y_j &= x'_i y'_j, & \text{for } 1 \leq i \leq d_1, 1 \leq j \leq d_2, \\ z_i &= z'_i, & \text{for } 1 \leq i \leq d_3. \end{aligned}$$

Fix an element $\gamma^\circ = (x^\circ, y^\circ, z^\circ)$ in $W_{n,m,\psi}$. We may assume that $\text{ord}_t x^\circ = \text{ord}_t x_1^\circ$, and under this assumption, may prove that $\text{ord}_t x = \text{ord}_t x_1$ for all $\gamma = (x, y, z)$ in $W_{n,m,\psi}$. Let us define a morphism

$$\chi_\psi: W_{n,m,\psi} \rightarrow F \times_k K$$

which sends a K -arc $\gamma = (x, y, z)$ to its first component x_1 . It is easy to see that χ_ψ is a $(\mu_n)_\psi$ -equivariant morphism. Let us now prove that χ_ψ is isomorphic. For an arbitrary v in $F \times_k K$, put $\tau = v(x_1^\circ)^{-1} \in K((t))$ and

$$(x, y, z) = (\tau x^\circ \pmod{t^{n+1}}, \tau^{-1} y^\circ \pmod{t^{n+1}}, z^\circ).$$

Then (x, y, z) is in $W_{n,m,\psi}$ and $\chi_\psi(x, y, z) = v$, which proves that χ_ψ is a surjection. The injectivity of χ_ψ follows from the fact that $(f_{ij})_n$ and $(z_l)_n$ are constant on $W_{n,m,\psi}$. Therefore, the morphism χ_ψ is a $(\mu_n)_\psi$ -equivariant isomorphism. By Theorem 2.2, ϕ_n is μ_n -equivariant piecewise trivial fibration with fiber F , and we have

$$[W_{n,m}] = [V_{n,m}] \cdot [F] = [V_{n,m}] \cdot (\mathbb{L}^{n+1} - \mathbb{L}^{n-m+1})$$

in \mathcal{M}_k^μ . We consider the induced morphism of ϕ at the level of arc spaces

$$\phi_\infty: \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$$

and define a semi-algebraic family A_m of semi-algebraic subsets of $\mathcal{L}(Y)$ as the image of the family $\{\gamma \in \mathcal{L}(X) \mid \text{ord}_t x + \text{ord}_t y = m\}$ under ϕ_∞ . On the other hand, by the hypothesis, the regular function f is $\mathbb{G}_{m,k}$ -equivariant, it thus induces a regular function $g: Y \rightarrow \mathbb{A}_k^1$ satisfying $f = g \circ \phi$, by the universal property of the quotient ϕ . In view of Theorem 4.5, we define

$$A_{n,m} := \{\gamma \in A_m \mid g(\gamma) = t^n \pmod{t^{n+1}}\}.$$

Then $A_{n,m}$ is in $\mathcal{F}_Y^{\mu_n}$ and stable at level n , and

$$\tilde{\mu}(A_{n,m}) = [\pi_n(A_{n,m})] \mathbb{L}^{-(n+1)(d-1)} = [V_{n,m}] \mathbb{L}^{-(n+1)(d-1)}$$

(note that $\dim_k Y = d-1$). Therefore, we have the decomposition of $W(T)$ into the difference of two series as follows

$$\begin{aligned} W(T) &= \sum_{1 \leq m \leq n} [W_{n,m}] \mathbb{L}^{-nd} T^n \\ &= \mathbb{L}^d \sum_{1 \leq m \leq n} [V_{n,m}] \mathbb{L}^{-(n+1)(d-1)} T^n - \mathbb{L}^d \sum_{1 \leq m \leq n} [V_{n,m}] \mathbb{L}^{-(n+1)(d-1)} \mathbb{L}^{-m} T^n \\ &= \mathbb{L}^d \sum_{1 \leq m \leq n} \tilde{\mu}(A_{n,m}) T^n - \mathbb{L}^d \sum_{1 \leq m \leq n} \tilde{\mu}(A_{n,m}) \mathbb{L}^{-m} T^n. \end{aligned}$$

This implies that $\mathbb{W} = 0$ in \mathcal{M}_k^μ , because the two series in the previous difference decomposition have the same limit, according to Proposition 4.6. \square

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