A Bilinear Strategy for Calderón's Problem

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Abstract

Electrical Impedance Imaging would suffer a serious obstruction if two different conductivities yielded the same measurements of potential and current at the boundary. The Calderón's problem is to decide whether the conductivity is indeed uniquely determined by the data at the boundary. In $\mathbb{R}^d$, for $d \geqslant 5$, we show that uniqueness holds when the conductivity is in $W^{1+\frac{d-5}{2p}+,p}(\Omega)$, for $d \leqslant p < \infty$. This improves on recent results of Haberman, and of Ham, Kwon and Lee. The main novelty of the proof is an extension of Tao's Bilinear Theorem.

MSC(2010): 31B20 (Primary) 42B37 (Secondary)

Keywords: Inverse Problems, Calderón's Problem, Restriction Theory, Tao's Bilinear Theorem

1 Introduction

Electrical Impedance Imaging is a technique that exploits the differences in electrical conductivity inside a body to reconstruct its inner structure from measurements of potential and current at the boundary. At least since the 30', geophysicists have used this technique to identify different layers of earth underground [25]. In pioneering work, Calderón [10] posed the problem of deciding whether the conductivity is uniquely determined by the measurements at the boundary. Calderón went on to show uniqueness when the conductivity is infinitesimally close to one.

In a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary the electrical potential u solves the boundary value problem

$$\begin{aligned} &L_\gamma u := \operatorname{div}(\gamma \nabla u) = 0, \\ &u|_{\partial\Omega} = f, \end{aligned} \tag{1}$$

where γ is the conductivity and f is the potential at the boundary. We assume that $\gamma \in L^\infty(\Omega)$ and that $\gamma \geqslant c > 0$. If $f \in H^{1/2}(\partial\Omega)$ then a solution $u \in H^1(\Omega)$ exists. The electrical current at the boundary is $\gamma \partial_\nu u \,|_{\partial\Omega}$, where ν is the outward-pointing normal, and the operator $\Lambda_\gamma : u|_{\partial\Omega} \mapsto \gamma\partial_\nu u \,|_{\partial\Omega}$ is called the Dirichlet-to-Neumann map. We can define the map Λ_γ rigorously as an operator $\Lambda_\gamma : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ given by

$$\langle \Lambda_\gamma f, g\rangle := \int_\Omega \gamma \nabla u \cdot \nabla \overline{v}\,, \tag{2}$$

where u solves (1) and $v \in H^1(\Omega)$ is *any* extension of $g \in H^{1/2}(\partial\Omega)$. If we choose v such that $L_\gamma v = 0$, then we see that Λ_γ is symmetric. Uniqueness fails if two different conductivities γ_1 and γ_2 satisfy $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$; this were the case, for every $f_1, f_2 \in H^{\frac{1}{2}}(\partial\Omega)$ we would have

$$0 = \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_1, f_2\rangle = \int_\Omega (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla \overline{u}_2, \tag{3}$$

where $L_{\gamma_1} u_1 = 0$ and $L_{\gamma_2} u_2 = 0$ are extensions of f_1 and f_2 respectively. Most proofs of uniqueness show that the linear span of the functions $\{\nabla u_1 \cdot \nabla \overline{u}_2\}$ is dense, so γ_1 and γ_2 cannot be different.

Kohn and Vogelius [19] showed that for smooth conductivities γ_1 and γ_2, uniqueness holds at the boundary to all orders, so $\partial_\nu^N \gamma_1 = \partial_\nu^N \gamma_2$ at $\partial\Omega$ for every integer N. In particular, if the conductivities are analytic then $\gamma_1 = \gamma_2$ in Ω.

In [27], Sylvester and Uhlmann introduced the method that most proofs follow nowadays. If u_j solves (1) for γ_j, then the function $w_j := \gamma_j^{\frac{1}{2}} u_i$ solves the equation $(-\Delta + q_j) w_j = 0$, where $q_j := \gamma_j^{-\frac{1}{2}} \Delta \gamma_j^{\frac{1}{2}}$, and the relationship (3) is to be replaced by

$$\int_{\mathbb{R}^d} (q_1 - q_2) w_1 \overline{w}_2 = 0. \tag{4}$$

Then, Sylvester and Uhlmann proved that the linear span of the functions $\{w_1 \overline{w}_2\}$ is dense. The integral is evaluated over $\mathbb{R}^d$ because the functions γ_1 and γ_2 are extended to the whole space, and they are arranged so that $\gamma_1 = \gamma_2 = 1$ outside a ball containing Ω. Since $e^{\zeta \cdot x}$ is harmonic when $\zeta \in \mathbb{C}^d$ satisfies $\zeta \cdot \zeta = 0$, then they used the ansatz $w_j = e^{\zeta_j \cdot x}(1 + \psi_j)$, expecting that ψ_j is somehow negligible for $|\zeta_1|, |\zeta_2| \to \infty$. These highly oscillating solutions

w_j are called Complex Geometrical Optics (CGO) solutions. Sylvester and Uhlmann selected ζ_1 and ζ_2 such that $\zeta_1 + \overline{\zeta}_2 = i\xi$ for $\xi \in \mathbb{R}^d$; then, on the assumed smallness of ψ_j for $|\zeta_1|, |\zeta_2| \to \infty$, equation (4) becomes $\widehat{q}_1 - \widehat{q}_2 = 0$, and this implies that $\gamma_1 = \gamma_2$. The argument works well for conductivities in $C^2(\Omega)$.

In $\mathbb{R}^2$, Astala and Päivärinta [3] proved that uniqueness holds in $L^\infty(\Omega)$, the best possible result. In higher dimensions Brown [6] proved uniqueness for conductivities in $C^{\frac{3}{2}+}(\Omega)$, and this was improved to $W^{\frac{3}{2},2d+}(\Omega)$ by Brown and Torres [8]. By analogy with unique continuation, it is conjectured that the lowest possible regularity is $W^{1,d}(\Omega)$.

The function ψ in the CGO solution $w = e^{\zeta\cdot x}(1+\psi)$ satisfies the equation

$$\Delta_\zeta \psi := \Delta\psi + 2\zeta \cdot \nabla\psi = q(1+\psi). \tag{5}$$

Then, it is necessary to prove that a solution exists and is small. In [15], Haberman and Tataru introduced the following Bourgain-type spaces adapted to $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi$, the symbol of Δ_ζ:

$$\begin{aligned} \dot{X}^b_\zeta &:= \{u \mid \|u\|^2_{\dot{X}^b_\zeta} := \int_{\mathbb{R}^d} |p_\zeta(\xi)|^{2b} |\widehat{u}|^2 \, d\xi < \infty\}, \\ X^b_{\zeta,\sigma} &:= \{u \mid \|u\|^2_{X^b_{\zeta,\sigma}} := \int_{\mathbb{R}^d} (|p_\zeta(\xi)| + \sigma)^{2b} |\widehat{u}|^2 \, d\xi < \infty\}, \quad \text{for } \sigma > 0, \\ X^b_\zeta &:= X^b_{\zeta,|\zeta|}. \end{aligned} \tag{6}$$

It follows that $\|\Delta_\zeta^{-1}\|_{\dot{X}^{-\frac{1}{2}}_\zeta \to \dot{X}^{\frac{1}{2}}_\zeta} = 1$, and that the dual of $\dot{X}^b_\zeta$ is $\dot{X}^{-b}_\zeta$. If we define the multiplication operator $M_q : u \mapsto qu$, then the existence of ψ follows from $\|\Delta_\zeta^{-1} M_q\|_{\dot{X}^{\frac{1}{2}}_\zeta \to \dot{X}^{\frac{1}{2}}_\zeta} \leqslant \|M_q\|_{\dot{X}^{\frac{1}{2}}_\zeta \to \dot{X}^{-\frac{1}{2}}_\zeta} \leqslant c < 1$, and the smallness of ψ follows from the smallness of $\|q\|_{\dot{X}^{-\frac{1}{2}}_\zeta}$. Using these spaces Haberman and Tataru proved uniqueness for Lipschitz conductivities close to one.

Caro and Rogers [11] proved uniqueness for Lipschitz conductivities without further restriction. They used Carleman estimates, in the spirit of [9, 18, 12].

After an observation in [24], Haberman refined in [14] the method of Bourgain spaces, and proved uniqueness for conductivities in the conjectured spaces $W^{1,d}(\Omega)$, for $d = 3, 4$, and for conductivities in $W^{1+\frac{d-4}{2p},p}(\Omega)$, for $p \geqslant d$ and $d = 5, 6$. He argued as follows: for γ_1 and γ_2 he wanted to show that $\|M_{q_j}\|_{\dot{X}^{\frac{1}{2}}_{\zeta_j} \to \dot{X}^{-\frac{1}{2}}_{\zeta_j}}$ and $\|q_j\|_{\dot{X}^{-\frac{1}{2}}_{\zeta_j}}$ are small for some ζ_1 and ζ_2 that satisfy

$\zeta_1+\overline{\zeta}_2=i\xi$, so Haberman proved that there exist sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$ for which $\|M_{q_j}\|_{\dot X^{\frac12}_{\zeta_{j,k}}\to \dot X^{-\frac12}_{\zeta_{j,k}}}$ and $\|q_j\|_{\dot X^{-\frac12}_{\zeta_{j,k}}}$ tend to zero as $|\zeta_{1,k}|,|\zeta_{2,k}|\to\infty$. To find the sequences, he proved that the average of both norms goes to zero as $|\zeta_1|,|\zeta_2|\to\infty$.

Theorem 1 (Haberman, Theorem 5.3 in [14])**.** *Let us write $\zeta(U,\tau):=\tau(Ue_1-iUe_2)$ for $\tau\geqslant 1$ and $U\in O_d$ a rotation. If γ_1 and γ_2 are in $W^{1,d}(\mathbb{R}^d)$ for $d=3,4$, or in $W^{1+\frac{d-4}{2p},p}(\mathbb{R}^d)$ for $d\leqslant p<\infty$ and $d\geqslant 5$, then*

$$\frac{1}{M}\int_M^{2M}\int_{O_d}\|M_{q_j}\|^p_{\dot X^{\frac12}_{\zeta(U,\tau)}\to\dot X^{-\frac12}_{\zeta(U,\tau)}}\,dUd\tau \ \text{ and } \ \frac{1}{M}\int_M^{2M}\int_{O_d}\|q_j\|^2_{\dot X^{-\frac12}_{\zeta(U,\tau)}}\,dUd\tau\xrightarrow{M\to\infty}0.$$

The idea is that, when $|\zeta_j|$ is large, the set of bad pairs (ζ_1,ζ_2) for which $\|M_{q_j}\|_{\dot X^{\frac12}_{\zeta_{j,k}}\to\dot X^{-\frac12}_{\zeta_{j,k}}}$ or $\|q_j\|_{\dot X^{-\frac12}_{\zeta_{j,k}}}$ is large has a small measure, then it is possible to extract sequences such that these norms are small and such that $\zeta_{1,k}+\overline{\zeta}_{k,2}=i\xi$; see Theorem 7.3 in [14].

The estimates of Haberman are very good, and most of the argument works well just for $\gamma\in W^{1,d}(\mathbb{R}^d)$. In fact, the condition $\gamma\in W^{1,d}(\mathbb{R}^d)$ suffices to show that the average of $\|q_j\|_{\dot X^{-\frac12}_{\zeta(U,\tau)}}$ vanishes, and the bottle-neck to prove that the average of $\|M_{q_j}\|_{\dot X^{\frac12}_{\zeta(U,\tau)}\to\dot X^{-\frac12}_{\zeta(U,\tau)}}$ vanishes, for $\gamma\in W^{1,d}(\mathbb{R}^d)$, is to get a strong upper bound of $\|M_{\partial_i f}\|_{\dot X^{\frac12}_{\zeta(U,\tau)}\to\dot X^{-\frac12}_{\zeta(U,\tau)}}$, for $f\in L^d(\Omega)$. Stronger upper bounds were obtained by Ham, Kwon and Lee [16] using deep inequalities from restriction theory, and they got the following Theorem.

Theorem 2 (Ham, Kwon and Lee, Proposition 5.13 in [16])**.** *Let us write $\zeta(U,\tau):=\tau(Ue_1-iUe_2)$ for $\tau\geqslant 1$ and $U\in O_d$ a rotation. Suppose that f is a function supported in the unit ball. If $d=5,6$[1] and if*

$$f\in\begin{cases}W^{\frac{d-5}{2p}+,p}(\mathbb{R}^d) & \text{for } d+1\leqslant p<\infty\\ W^{\frac{d-5}{2p}+\frac{d+1-p}{2p(d-1)}+,p}(\mathbb{R}^d) & \text{for } d\leqslant p<d+1,\end{cases}$$

then

$$\frac{1}{M}\int_M^{2M}\int_{O_d}\|M_{\partial_i f}\|_{\dot X^{\frac12}_{\zeta(U,\tau)}\to\dot X^{-\frac12}_{\zeta(U,\tau)}}\,dUd\tau\xrightarrow{M\to\infty}0. \tag{7}$$

[1]For $d>6$ the exponents change, but for brevity I omit them and focus only on the cases $d=5,6$.

One of the main results of this paper, which we present in more detail in Section 2, is the following Theorem.

Theorem 3 (Vanishing of the Average)**.** *Let us write* $\zeta(U,\tau) := \tau(Ue_1 - iUe_2)$ *for* $\tau \geqslant 1$ *and* $U \in O_d$ *a rotation. Suppose that* f *is a function supported in the unit ball. If* $f \in W^{\frac{d-5}{2p}+,p}(\mathbb{R}^d)$ *for* $d \leqslant p < \infty$ *and* $d \geqslant 5$*, then*

$$\frac{1}{M}\int_M^{2M}\int_{O_d} \|M_{\partial_i f}\|_{\dot{X}^{\frac{1}{2}}_{\zeta(U,\tau)} \to \dot{X}^{-\frac{1}{2}}_{\zeta(U,\tau)}}\, dU d\tau \xrightarrow{M\to\infty} 0. \tag{8}$$

The main consequence of this Theorem is the following improvement on Calderón's problem.

Theorem 4. *For* $d = 5,6$ *suppose that* $\Omega \subset \mathbb{R}^d$ *is a bounded domain with Lipschitz boundary. If* γ_1 *and* γ_2 *are in* $W^{1+\frac{d-5}{2p}+,p}(\Omega) \cap L^\infty(\Omega)$ *for* $d \leqslant p < \infty$*, and if* $\gamma_1, \gamma_2 \geqslant c > 0$*, then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \textit{implies} \quad \gamma_1 = \gamma_2 \quad \textit{in } \Omega.$$

We write $\gamma \in W^{1+\frac{d-5}{2p}+,p}(\Omega) \cap L^\infty(\Omega)$ to emphasize that $\gamma \in L^\infty(\Omega)$, but it follows from Sobolev embedding for domains with Lipschitz boundary.

We summarize the results as: there is uniqueness as long as the conductivities belong to

$W^{1,d}(\Omega)$ for $d = 3,4$	Haberman [14]
$W^{1+\frac{d-4}{2p},p}(\Omega)$ for $d \leqslant p < \infty$ and for $d = 5,6$	Haberman [14]
$W^{1+\frac{d-5}{2p}+,p}(\Omega)$ for $d+1 \leqslant p < \infty$ and for $d = 5,6$	Ham *et al.* [16]
$W^{1+\frac{d-5}{2p}+\frac{d+1-p}{2p(d-1)}+,p}(\Omega)$ for $d \leqslant p < d+1$ and for $d = 5,6$	Ham *et al.* [16]
$W^{1+\frac{d-5}{2p}+,p}(\Omega)$ for $d \leqslant p < \infty$ and for $d = 5,6$	Theorem 4

Theorem 3 holds for $d \geqslant 5$, so the hypothesis $d = 5,6$ in Theorem 4 seems odd; in fact, we can state the following consequence of Theorem 3.

Theorem 5. *For* $d \geqslant 7$ *suppose that* $\Omega \subset \mathbb{R}^d$ *is a bounded domain with Lipschitz boundary. If* γ_1 *and* γ_2 *are in* $W^{1+\frac{d-5}{2p}+,p}(\Omega) \cap L^\infty(\Omega)$ *for* $d \leqslant p < \infty$*, if* $\partial_\nu\gamma_1 = \partial_\nu\gamma_2$ *at* $\partial\Omega$*, and if* $\gamma_1, \gamma_2 \geqslant c > 0$*, then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \textit{implies} \quad \gamma_1 = \gamma_2 \quad \textit{in } \Omega.$$

By the Trace Theorem the normal derivative $\partial_\nu \gamma$ is well-defined. The proofs of Theorem 4 and Theorem 5 have been already summarized in this Section, and we provide more details in Section 2. We refer the reader to the literature to reconstruct the whole argument, in particular to Haberman [14].

We have added an appendix with an example that shows that it is necessary to average.

> *Note Added in Proof:* By a recent result of the author, in Theorem 5 the condition $\partial_\nu \gamma_1 = \partial_\nu \gamma_2$ at $\partial\Omega$ is implied by the other hypotheses of the Theorem. Therefore, Theorem 4 can be extended to dimensions $d \geqslant 5$.

1.1 Restriction Theory

Ham, Kwon and Lee [16] applied deep estimates from restriction theory to improve on Harberman's results, and we will follow most of their arguments. We give here a brief introduction to restriction theory and the way it comes into Calderón's problem; a detailed exposition of restriction theory can be found in [23, part IV].

We control the norm $\|M_{\partial_i f}\|_{X_\zeta^{\frac{1}{2}} \to X_\zeta^{-\frac{1}{2}}}$ by duality, so we need an upper bound of

$$\langle (\partial_i f) u, v \rangle = \int_{\mathbb{R}^d} (\partial_i f) u \overline{v} \, dx \quad \text{for } u, v \in X_\zeta^{\frac{1}{2}}. \tag{9}$$

In general, it is very hard to control the contribution of frequencies around the null set of $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi$, which we call the characteristic set Σ_ζ. The characteristic set Σ_ζ is a $(d-2)$-sphere, so we have to understand functions u and v whose Fourier transform are supported around Σ_ζ. This is just the setting for which restriction theory has been developed; a few classical examples of applications are [13, 17, 4, 5].

In restriction theory the goal is to prove the best possible bounds $\|\widehat{f}|_S\|_{L^p(S)} \leqslant C\|f\|_{L^q(\mathbb{R}^n)}$, where S is a set, usually a manifold. One of the earliest and most important results is due to Tomas [30] and Stein (unpublished); for a proof see *e.g.* [26, Ch. 9].

Theorem 6. *(Tomas-Stein Inequality) Suppose that $S \subset \mathbb{R}^n$ is a compact manifold of dimension $n-1$ with non-vanishing curvature. If $f \in L^p(\mathbb{R}^n)$ for*

$1 \leqslant p \leqslant 2\,\frac{n+1}{n+3}$, *then*

$$\|\widehat{f}|_S\|_{L^2(S)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}. \tag{10}$$

The operator dual to restriction is called the extension operator, and it is the Fourier transform of a measure fdS supported on the set S. The function $(fdS)^\vee$ is the prototype of a function with frequencies highly concentrated around S. In the dual side, the Tomas-Stein inequality is

$$\|(fdS)^\vee\|_{L^{p'}(\mathbb{R}^n)} \leqslant C\|f\|_{L^2(S)}, \quad \text{for } 2\,\frac{n+1}{n-1} \leqslant p' \leqslant \infty. \tag{11}$$

Since the earliest days of restriction theory, a kind of stability of bilinear estimates was observed. For example, the bound $\|(fdS)^\vee\|_{L^4(\mathbb{R}^2)} \leqslant C\|f\|_2$ is false, but the bound $\|(f_1dS_1)^\vee(f_2dS_2)^\vee\|_{L^2(\mathbb{R}^2)} \leqslant C\|f_1\|_2\|f_2\|_2$ is true whenever the lines S_1 and S_2 are transversal; curvature is not required here. This stability of bilinear estimates was clarified and refined by Tao, Vargas and Vega [29], and they showed how to get linear bounds from bilinear bounds, an strategy that we will follow in Section 3.2

We need strong bilinear estimates to exploit the bilinear strategy. For example, we need inequalities like

$$\|(f_1dS_1)^\vee(f_2dS_2)^\vee\|_{L^{p'}(\mathbb{R}^n)} \leqslant C\|f_1\|_{L^2(S_1)}\|f_2\|_{L^2(S_2)}, \tag{12}$$

for some $p' < \frac{n+1}{n-1}$ when S_1 and S_2 are, for example, subsets of a surface with positive curvature; the case $p' = \frac{n+1}{n-1} = 1 + \frac{2}{n-1}$ follows by Hölder and Tomas-Stein inequalities. However, we have to impose a condition of separation on the surfaces S_1 and S_2 to get strong bilinear estimates. For example, if $\|(f_1dS_1)^\vee(f_2dS_2)^\vee\|_{L^2(\mathbb{R}^2)} \leqslant C\|f_1\|_2\|f_2\|_2$ were true in any case, then just setting $S_1 = S_2$ would provide a linear estimate, a false one in this case.

Klainerman and Machedon conjectured that the inequality (12) holds true for every $p' > \frac{n+2}{n} = 1 + \frac{2}{n}$ when the surfaces S_1 and S_2 are separated subsets of a cone. Despite the intractability of the problem, Wolff proved the conjecture in [33]. Subsequently, Tao refined the method and proved (12) for $p' > \frac{n+2}{n}$ when the surfaces are subsets of a surface with positive curvature [28]. Vargas [32] and Lee [21] proved (12) for $p' > \frac{5}{3}$ when the surfaces are subsets of the hyperboloid in $\mathbb{R}^3$, dealing with unusual obstructions.

Since we are interested in the sphere Σ_ζ, we need to use the Bilinear Theorem for this case. To avoid antipodal points in the bilinear inequality, we restrict ourselves to the surface

$$S := \{(\xi', \xi_n) \mid \xi_n = 1 - \sqrt{1 - |\xi'|^2} \text{ and } |\xi'| < \frac{1}{\sqrt{2}} + \frac{1}{10}\} \tag{13}$$

Following [29], we define also surfaces of elliptic type.

Definition 7. (Surfaces of Elliptic Type) A surface S is of ε-elliptic type if:

- The surface is the graph of a C^∞ function $\Phi : B_1 \subset \mathbb{R}^{n-1} \to \mathbb{R}$.
- $\Phi(0) = 0$ and $\nabla\Phi(0) = 0$.
- The eigenvalues of $D^2\Phi(x)$ lie in $[1-\varepsilon, 1+\varepsilon]$ for every $x \in B_1$.

For every $\varepsilon > 0$ and for every point in a surface with positive curvature, we can find a sufficiently small neighborhood U so that U is of ε-elliptic type, up to a linear transformation.

In (9) we do not deal with measures supported in a sphere, but with two different functions with frequencies possibly highly concentrated around a sphere. In Section 4 we prove the following extension of Tao's Bilinear Theorem, which will allow us to handle more precisely this situation.

Theorem 8 (Bilinear Theorem)**.** *Suppose that $S_1, S_2 \subset \mathbb{R}^n$ are two open subsets of a surface of elliptic type or the hemisphere in* (13)*, and suppose that their diameter is $\lesssim 1$ and they lie at distance ~ 1 of each other. If f_μ and g_ν are functions with Fourier transforms supported in a μ-neighborhood of S_1 and a ν-neighborhood of S_2 respectively, for $\mu \leqslant \nu < \mu^{\frac{1}{2}} < 1$, then for every $\delta > 0$ it holds that*

$$\|f_\mu g_\nu\|_{p'} \leqslant C_\delta \mu^{\frac{n}{2p} - \delta} \nu^{\frac{1}{p} - \delta} \|f_\mu\|_2 \|g_\nu\|_2, \quad \text{for } 1 \leqslant p' \leqslant \frac{n}{n-1}. \tag{14}$$

For surfaces of ε-elliptic type, the constant C_δ may depend on ε and on the semi-norms $\|\partial^N \Phi\|_\infty$. The inequalities are best possible in μ and ν, up to δ-losses.

Tao's Bilinear Theorem holds under the same hypotheses of curvature and separation of S_1 and S_2, and it states that for any two functions $f \in L^2(S_1)$ and $g \in L^2(S_2)$ the following inequality holds for every $\delta > 0$:

$$\|(f dS_1)^\vee (g dS_2)^\vee\|_{L^{\frac{n+2}{n}}(B_R)} \leqslant C_\delta R^\delta \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)}. \tag{15}$$

In Theorem 8 the localization at a ball B_R is traded by delocalization in the surfaces. Inequality (15) leads to further inequalities by interpolation with the easier inequalities

$$\|fdS_1)^\vee (gdS_2)^\vee\|_{L^1(B_R)} \leqslant CR\|f\|_2\|g\|_2,$$
$$\|(fdS_1)^\vee (gdS_2)^\vee\|_{L^\infty} \leqslant C\|f\|_2\|g\|_2.$$

An averaging of the surfaces followed by an application of (15) leads to the constant $C_\delta(\mu\nu)^{\frac{n+2}{4p}-\delta}$ in (14) for $1 \leqslant p' \leqslant \frac{n+2}{n}$, which is actually the constant used in [16];[2] for details on the averaging see Lemma 2.4 in [20] or Theorem 12 below. It was surprising, at least to me, that this constant can be lowered to $C_\delta\mu^{\frac{n}{2p}-\delta}\nu^{\frac{1}{p}-\delta}$ when $\mu < \nu$, gaining so the regularity needed to get Theorem 4.

Another unexpected phenomenon appears: when μ is much smaller than ν, *i.e.* when $\mu^{\frac{1}{2}} \leqslant \nu$, bilinearity does not play any role; moreover, the curvature of the support of g_ν is of no importance, and the bounds that Tomas-Stein yield cannot be improved.

The reader can consult the symbols and notations we use at the end of the article.

Acknowledgments

I wish to thank Pedro Caro for his continuous support and stimulating conversations. This research is supported by the Basque Government through the BERC 2018-2021 program, and by the Spanish State Research Agency through BCAM Severo Ochoa excellence accreditation SEV-2017-0718 and through projects ERCEA Advanced Grant 2014 669689 - HADE and PGC2018-094528-B-I00. I thank the referee for his suggestions, which helped greatly to improve the paper.

2 Outline of the Proof of Theorem 4

The proof that Theorem 3 implies Theorem 4 is long, and many steps are already well described in the literature. The main source here is [14].

First, we extend carefully γ_1 and γ_2 to the whole space, and for this we need some results of uniqueness of the conductivity at the boundary. After

[2] Notice that the range of p' is larger than that in (14) when $n > 2$.

the early result of Kohn and Vogelius, it is worth mentioning the works of Alessandrini [1, 2] and Brown [7].

By the definition of $W^{s,p}(\Omega)$, we can extend γ_1 to a function in $W^{s,p}(\mathbb{R}^d)$. Since $\gamma_j \in W^{1+\frac{d-5}{2p}+,p}(\Omega)$, then by Brown's Theorem in [7] we have that $\gamma_1 = \gamma_2$ at $\partial\Omega$ if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. The mechanism that allows us to extend the conductivities is the following Theorem.

Theorem 9 (Marschall, Theorem 1 in [22])**.** *Let Ω be a domain with Lipschitz boundary. Suppose that $1 < p < \infty$ and that $k+\frac{1}{p} < s \leqslant k+1+\frac{1}{p}$, for $k \geqslant 0$ an integer. If $f \in W^{s,p}(\Omega)$ satisfies $f|_{\partial\Omega} = \cdots = \partial_\nu^k f|_{\partial\Omega} = 0$ then $f \in W_0^{s,p}(\Omega)$.*

We define the function

$$\eta := \begin{cases} \gamma_2 - \gamma_1 & \text{if } \Omega \\ 0 & \text{if } \Omega^c. \end{cases}$$

Since η is zero at $\partial\Omega$ and $\frac{1}{p} < 1 + \frac{d-5}{2p}+ \leqslant 1 + \frac{1}{p}$ for $d \leqslant 6$, then $\eta \in W^{1+\frac{d-5}{2p}+,p}(\mathbb{R}^d)$; this explains the condition $d \leqslant 6$ in Theorem 4. We can thus define the extension $\gamma_2 := \gamma_1 + \eta \in W^{1+\frac{d-5}{2p}+,p}(\mathbb{R}^d)$. Finally, we arrange the extensions so that $\gamma_1 = \gamma_2 = 1$ outside a ball containing Ω. For $d \geqslant 7$ we are in the case $1 + \frac{1}{p} < 1 + \frac{d-5}{2p}+ \leqslant 2 + \frac{1}{p}$, and we need additionally the condition $\partial_\nu\gamma_1 = \partial_\nu\gamma_2$ at $\partial\Omega$ to apply Theorem 9, which is the condition that we included in Theorem 5. For further details see [8, Cor. 3].

For all $w_1, w_2 \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ that solve $(-\Delta + q_j)w_j = 0$ with $q_j = \gamma_j^{-\frac{1}{2}}\Delta\gamma_j^{\frac{1}{2}}$, we want to show that the linear span of the functions $\{w_1\overline{w}_2\}$ is dense, which implies that $\gamma_1 = \gamma_2$; see [6, Prop. 8]. Notice that q_j is compactly supported.

For $\zeta_j \cdot \zeta_j = 0$, the functions $w_j = e^{\zeta_j\cdot x}(1 + \psi_j)$ are CGO solutions, and the functions $\psi_j \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ have to satisfy the equation

$$(-\Delta_{\zeta_j} + q_j)\psi_j = -q_j. \tag{16}$$

If we choose ζ_1 and ζ_2 such that $\zeta_1 + \overline{\zeta}_2 = i\xi$ and replace the CGO solutions in (4), then we get

$$\int_{\mathbb{R}^d} (q_1 - q_2)e^{i\xi\cdot x} = \int e^{i\xi\cdot x}\psi_1 q_1 - \overline{\int e^{-i\xi\cdot x}\psi_2 q_2} + \\ + \int e^{i\xi\cdot x}\overline{\psi}_2\Delta_{\zeta_1}\psi_1 - \overline{\int e^{-i\xi\cdot x}\overline{\psi}_1\Delta_{\zeta_2}\psi_2}\,. \tag{17}$$

We expect the functions ψ_j to be negligible, so if we ignore them, we could conclude that $\widehat{q}_1(\xi) = \widehat{q}_2(\xi)$ for every $\xi \in \mathbb{R}^d$, which implies $\gamma_1 = \gamma_2$.

The space $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ does not seem to be the best suited space to solve (16), so we use the spaces $\dot{X}^b_\zeta$ and X^b_ζ defined in (6). We need the following Lemma.

Lemma 10 (Haberman and Tataru, Lemma 2.2 in [15]). *Let φ be a Schwartz function. Then the following inequalities hold*

$$\|\varphi u\|_{\dot{X}_\zeta^{-\frac{1}{2}}} \lesssim \|u\|_{X_\zeta^{-\frac{1}{2}}} \tag{18}$$

$$\|\varphi u\|_{X_\zeta^{\frac{1}{2}}} \lesssim \|u\|_{\dot{X}_\zeta^{\frac{1}{2}}}, \tag{19}$$

where the implicit constants depend on φ.

By $\|u\|_{H^1} \lesssim \|u\|_{X_\zeta^{\frac{1}{2}}}$ and (19) we get the inclusion $\dot{X}_\zeta^{\frac{1}{2}} \subset H^1_{\mathrm{loc}}(\mathbb{R}^d)$, so we have that

$$(-\Delta_\zeta + q) : \dot{X}_\zeta^{\frac{1}{2}} \to \dot{X}_\zeta^{-\frac{1}{2}}.$$

The goal is to find a pair of sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$ that satisfy the following conditions:

- $\zeta_{1,k} + \overline{\zeta}_{2,k} = i\xi$ and $|\zeta_{j,k}| \to \infty$ as $k \to \infty$.
- There exist solutions $\psi_{j,k} \in \dot{X}^{\frac{1}{2}}_{\zeta_{j,k}}$ of the equation (16).
- $\|\psi_{j,k}\|_{\dot{X}^{\frac{1}{2}}_{\zeta_{j,k}}} \to 0$ as $k \to \infty$.

To solve (16) we write it as $(I - \Delta_\zeta^{-1} q)\psi = \Delta_\zeta^{-1} q$ and then we invert the operator $(I - \Delta_\zeta^{-1} M_q)$, where $M_q : u \mapsto qu$; for the latter it suffices to show that $\|M_q\|_{\dot{X}_\zeta^{\frac{1}{2}} \to \dot{X}_\zeta^{-\frac{1}{2}}} \leqslant c < 1$. We also have the upper bound

$$\|\psi\|_{\dot{X}_\zeta^{\frac{1}{2}}} \leqslant \|(I - \Delta_\zeta^{-1} M_q)^{-1}\|_{\dot{X}_\zeta^{\frac{1}{2}} \to \dot{X}_\zeta^{\frac{1}{2}}} \|q\|_{\dot{X}_\zeta^{-\frac{1}{2}}} \leqslant \frac{1}{1-c} \|q\|_{\dot{X}_\zeta^{-\frac{1}{2}}}.$$

Then, we can rewrite our goal as: to find a pair of sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$ that satisfy the following conditions:

- $\zeta_{1,k} + \overline{\zeta}_{2,k} = i\xi$ and $|\zeta_{j,k}| \to \infty$ as $k \to \infty$.

- $\|M_{q_j}\|_{X^{\frac12}_{\zeta_{j,k}}\to X^{-\frac12}_{\zeta_{j,k}}} \leqslant c < 1$ for sufficiently large k.
- $\|q_j\|_{X^{-\frac12}_{\zeta_{j,k}}} \to 0$ as $k\to\infty$.

For various technical reasons we have used X^b_ζ instead of $\dot X^b_\zeta$. To find the sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$, Haberman proved that the averages of $\|M_{q_j}\|_{X^{\frac12}_{\zeta}\to X^{-\frac12}_{\zeta}}$ and of $\|q_j\|_{X^{-\frac12}_{\zeta}}$ over $|\zeta|\sim M\geqslant 1$ tend to zero as $M\to\infty$; see in [14, Thm. 7.3] how to use Theorem 1 to select the sequences.

The selected sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$ allow us to show that each term at the right of (17) goes to zero. For example, for fixed $\xi\in\mathbb{R}^d$ we get

$$|\int e^{\pm i\xi\cdot x}\psi_{j,k}q_j| \leqslant \|\psi_j\|_{\dot X^{\frac12}_{\zeta_{j,k}}}\|e^{\pm i\xi\cdot x}q_j\|_{\dot X^{-\frac12}_{\zeta_{j,k}}} \leqslant C_\xi\|q_j\|^2_{X^{-\frac12}_{\zeta_{j,k}}}\to 0;$$

here we applied Lemma 10 with $e^{\pm i\xi\cdot x}\varphi$, where $\varphi=1$ in the support of q_j. The last two terms at the right of (17) are more difficult to control because they mix $\zeta_{1,k}$ and $\zeta_{2,k}$; see the proof of Theorem 7.3 in [14].

The condition $\gamma_j\in W^{1,d}(\mathbb{R}^d)$ suffices to prove that the average of $\|q_j\|_{X^{-\frac12}_{\zeta}}$ vanishes, so we will not turn our attention to it; see [14, Sec. 5]. To control $\|M_{q_j}\|_{X^{\frac12}_{\zeta}\to X^{-\frac12}_{\zeta}}$ we write

$$q=\frac12\Delta\log\gamma+\frac14|\nabla\log\gamma|^2=\frac12\mathrm{div}\,(\boldsymbol{f})+\frac14|\boldsymbol{f}|^2,$$

where the components of $\boldsymbol{f}=(f^1,\dots,f^n):=\nabla\log\gamma$ belong to $W^{s-1,p}(\mathbb{R}^d)$. We split M_q into two terms $M_{\partial_i f}$ and $M_{|f|^2}$. Haberman proved that the average of $\|M_{|f|^2}\|_{X^{\frac12}_{\zeta}\to X^{-\frac12}_{\zeta}}$ goes to zero if $f\in L^d(\mathbb{R}^d)$ —this is the term h in the proof of Theorem 5.3 in [14], so we are left with $\|M_{\partial_i f}\|_{X^{\frac12}_{\zeta}\to X^{-\frac12}_{\zeta}}$.

Whether or not the condition $f\in L^d(\mathbb{R}^d)$ suffices to prove that $\|M_{\partial_i f}\|_{X^{\frac12}_{\zeta}\to X^{-\frac12}_{\zeta}}$ is small on average is unknown, and in this paper we make progress on this problem. To prove Theorem 3 we need the following Theorem, which we prove in the next Section.

Theorem 11. *Suppose that f is supported in the unit ball. If $f\in W^{\frac{d-5}{2p}+,p}(\mathbb{R}^d)$ for $d\leqslant p<\infty$, then*

$$⨍_M\int_{O_d}\|M_{\partial_i f}\|_{X^{\frac12}_{\zeta(U,\tau)}\to X^{-\frac12}_{\zeta(U,\tau)}}\,dU\,d\tau \leqslant C\|f\|_{\frac{d-5}{2p}+,p}. \tag{20}$$

Proof of Theorem 3. Since f is compactly supported, then by Lemma 10 we get $\|M_{\partial_i f}\|_{\dot{X}_\zeta^{\frac12}\to\dot{X}_\zeta^{-\frac12}} \lesssim \|M_{\partial_i f}\|_{X_\zeta^{\frac12}\to X_\zeta^{-\frac12}}$. We estimate M_g by duality as

$$|\langle gu, v\rangle| \leqslant \|g\|_\infty \|u\|_2 \|v\|_2 \leqslant \frac{1}{|\zeta|}\|g\|_\infty \|u\|_{X_\zeta^{\frac12}} \|v\|_{X_\zeta^{\frac12}}.$$

For some $A \leqslant 1$ to be fixed later we define $g = P_{\leqslant A}\partial_i f$, where $P_{\leqslant A}$ is the projection to frequencies $\lesssim A$. By Young inequality for convolutions we get

$$\|M_g\|_{X_\zeta^{\frac12}\to X_\zeta^{-\frac12}} \leqslant \frac{1}{|\zeta|}\|g\|_\infty \lesssim \frac{A^2}{|\zeta|}\|f\|_d.$$

Using this and Theorem 11 we can control the average as

$$\begin{aligned}\fint_M \int_{O_d} \|M_{\partial_i f}\|_{X^{\frac12}_{\zeta(U,\tau)}\to X^{-\frac12}_{\zeta(U,\tau)}}\, dU d\tau &\lesssim \frac{A^2}{M}\|f\|_d + \fint_M \int_{O_d} \|M_{P_{>A}\partial_i f}\|_{X^{\frac12}_{\zeta(U,\tau)}\to X^{-\frac12}_{\zeta(U,\tau)}}\, dU d\tau \\ &\lesssim \frac{A^2}{M}\|f\|_d + \|P_{>A} f\|_{\frac{d-5}{2p}+,p}.\end{aligned}$$

If we choose $A = M^{\frac14}$ and let $M \to \infty$, then we get (8). $\square$

3 Proof of Theorem 11

In this Section we prove the upper bound (20). We estimate the norm of $\|M_{\partial_j f}\|_{X^{\frac12}_{\zeta(U,\tau)}\to X^{-\frac12}_{\zeta(U,\tau)}}$ by duality as

$$|\langle(\partial_j f)u, v\rangle| = |\int_{\mathbb{R}^d} (\partial_j f)u\bar{v}\, dx| \leqslant C(U,\tau,f)\|u\|_{X^{\frac12}_{\zeta(U,\tau)}} \|v\|_{X^{\frac12}_{\zeta(U,\tau)}}, \tag{21}$$

where $C(U,\tau,f)$ is a constant that can be related to $\|f\|_{W^{s,p}}$. In Section 3.1 we control the contribution of the frequencies of u and v away from the characteristic set Σ_ζ, which is a $(d-2)$-sphere, and then we use Tomas-Stein Theorem to control frequencies around Σ_ζ. In the next Section 3.2 we use the Bilinear Theorem 8 to get additional refinements on contributions around Σ_ζ, and we conclude that Section with Theorem 16, which contains the precise form of $C(U,\tau,f)$. We end the proof of Theorem 11 in Section 3.3, where we bound the average value of $\|M_{\partial_i f}\|_{X^{\frac12}_{\zeta(U,\tau)}\to X^{-\frac12}_{\zeta(U,\tau)}}$. The linear estimates are based on the work of Haberman [14], and the bilinear estimates on the work of Ham, Kwon and Lee [16].

3.1 Linear Estimates

The characteristic set Σ_ζ of Δ_ζ, *i.e.* the null set of the symbol $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta\cdot\xi$, is a $(d-2)$-sphere in the hyperplane $\{\xi \mid \langle Ue_1, \xi\rangle = 0\}$ with center $\tau U e_2$ and radius $\tau \geqslant 1$. If $d(\xi, \Sigma_\zeta)$ denotes the distance from ξ to Σ_ζ, then

$$|p_\zeta(\xi)| \sim \begin{cases} \tau d(\xi, \Sigma_\zeta), & \text{for } d(\xi,\Sigma_\zeta) \leqslant \frac{1}{10}\tau, \\ \tau^2 + |\xi|^2, & \text{for } d(\xi,\Sigma_\zeta) > \frac{1}{10}\tau \end{cases}$$

We break up the frequencies accordingly into characteristics and non-characteristics, and define the corresponding projections as

$$\begin{aligned} (Q_l f)^\wedge(\xi) &:= \zeta(\tau^{-1} d(\xi, \Sigma_\zeta))\widehat{f}(\xi) \\ (Q_h f)^\wedge(\xi) &:= (1 - \zeta(\tau^{-1} d(\xi, \Sigma_\zeta)))\widehat{f}(\xi), \end{aligned}$$

where $\zeta \in C_c^\infty(\mathbb{R})$ is supported in $(-\frac{1}{10}, \frac{1}{10})$. It follows that

$$\|Q_h u\|_2 \leqslant \tau^{-1}\|u\|_{X^{\frac{1}{2}}_{\zeta(U,\tau)}} \tag{22}$$

$$\|\partial_j Q_h u\|_2 \leqslant \|u\|_{X^{\frac{1}{2}}_{\zeta(U,\tau)}}. \tag{23}$$

In Lemma 3.3 of [14] Haberman proved, using Tomas-Stein inequality, that

$$\|u\|_{\frac{2d}{d-2}} \lesssim \|u\|_{X^{\frac{1}{2}}_{\zeta(U,\tau)}}. \tag{24}$$

With the help of inequalities (22), (23) and (24), we can control in (21) all the terms involving non-characteristic frequencies. In fact,

$$\begin{aligned} \langle(\partial_j f)u, v\rangle = \langle(\partial_j f)Q_h u, Q_h v\rangle + \langle(\partial_j f)Q_h u, Q_l v\rangle + \\ + \langle(\partial_j f)Q_l u, Q_h v\rangle + \langle(\partial_j f)Q_l u, Q_l v\rangle. \end{aligned}$$

For the first term at the right, after integration by parts, we have

$$\begin{aligned} |\langle(\partial_j f)Q_h u, Q_h v\rangle| &\leqslant \|f\|_d(\|\partial_j Q_h u\|_2 \|Q_h v\|_{\frac{2d}{d-2}} + \\ &\qquad + \|Q_h u\|_{\frac{2d}{d-2}}\|\partial_j Q_h v\|_2) \\ &\lesssim \|f\|_d \|u\|_{X^{\frac{1}{2}}_{\zeta(U,\tau)}} \|v\|_{X^{\frac{1}{2}}_{\zeta(U,\tau)}}. \end{aligned} \tag{25}$$

For the mixed terms we have

$$\begin{aligned}|\langle(\partial_j f)Q_h u, Q_l v\rangle| &\leqslant \|f\|_d(\|\partial_j Q_h u\|_2\|Q_l v\|_{\frac{2d}{d-2}}+\\ &\qquad\qquad + \|Q_h u\|_2\|\partial_j Q_l v\|_{\frac{2d}{d-2}})\\ &\lesssim \|f\|_d\|u\|_{X^{\frac{1}{2}}_{\zeta(U,\tau)}}\|v\|_{X^{\frac{1}{2}}_{\zeta(U,\tau)}},\end{aligned} \tag{26}$$

where we used the localization of $Q_l v$ to frequencies $\leqslant 5\tau$, so that $\|\partial_j Q_l v\|_{\frac{2d}{d-2}} \lesssim \tau\|Q_l v\|_{\frac{2d}{d-2}}$; this follows from Young inequality. We are left then with the characteristic frequencies.

We assume that the support of the Fourier transform of u and v lie in a $\frac{1}{10}$-neighborhood of Σ_ζ. We define the transformation

$$u_{\tau U}(x) := \tau^{-d}u(\tau^{-1}Ux), \tag{27}$$

so that the frequencies of $u_{\tau U}$ are supported in a $\frac{1}{10}$-neighborhood of the S^{d-2} sphere with center e_2, radius 1 and that lies in the hyperplane normal to e_1. The Fourier transform of $u_{\tau U}$ is $\widehat{u}_{\tau U}(\xi) = \widehat{u}(\tau U\xi)$, and the $X^b_{\zeta(U,\tau)}$-norm scales as

$$\|u\|_{X^b_{\zeta(U,\tau)}} = \tau^{\frac{d}{2}+2b}\|u_{\tau U}\|_{X^b_{\zeta(1),1/\tau}}. \tag{28}$$

We change variables in the pairing (21) to get

$$\begin{aligned}\langle(\partial_j f)u, v\rangle &= \tau^{-d}\int(\partial_j f)(\tau^{-1}Ux)u(\tau^{-1}Ux)\bar{v}(\tau^{-1}Ux)\,dx\\ &= \tau^{2d+1}\int_{B_\tau}(\partial_j f_{\tau U})u_{\tau U}\bar{v}_{\tau U}\,dx\\ &= \tau^{2d+1}\langle(\partial_{Ue_j}f_{\tau U})u_{\tau U}, v_{\tau U}\rangle,\end{aligned} \tag{29}$$

where we used the identity

$$(\partial_j f)(\tau^{-1}Ux) = \int \xi_j\widehat{f}(\xi)e^{i(\tau^{-1}Ux)\cdot\xi}\,d\xi = \tau^{d+1}(\partial_{Ue_j}f_{\tau U})(x).$$

Therefore, we will assume that the characteristic sphere S^{d-2} is centered at e_2, has radius 1 and lies in the normal plane to e_1. We assume also that the function f is supported in $B_\tau(0)$.

We apply the Hardy-Littlewood decomposition to $f = \sum_{\tau^{-1} \leqslant \lambda} P_\lambda f$, and decompose u and v into dyadic projections u_μ and v_ν, where $\hat{u}_\mu := \zeta(\mu^{-1} d(\xi, \Sigma_\zeta))\hat{u}$ and $\zeta \in C_c^\infty(\mathbb{R})$ is supported in $(\frac{1}{2}, 2)$. Then, the pairing (21) gets into

$$\begin{aligned}\langle(\partial_w f)u, v\rangle &= \sum_{\tau^{-1} \leqslant \lambda, \mu, \nu \lesssim 1} \langle(\partial_w P_{\lambda, \sup(\mu,\nu)} f) u_\mu, v_\nu\rangle \\ &= \sum_{\substack{\tau^{-1} \leqslant \lambda \lesssim 1 \\ \tau^{-1} \leqslant \mu \leqslant \nu \lesssim 1}} \langle(\partial_w P_{\lambda,\nu} f) u_\mu, v_\nu\rangle + \sum_{\substack{\tau^{-1} \leqslant \lambda \lesssim 1 \\ \tau^{-1} \leqslant \mu > \nu \lesssim 1}} \cdots, \end{aligned} \tag{30}$$

where ∂_w is the derivative in some direction w, and $P_{\lambda,\sup(\mu,\nu)}$ is the projection to frequencies $|\xi| \sim \lambda$ and $|\xi_1| \lesssim \sup(\mu, \nu)$. By symmetry, we can assume that $\mu \leqslant \nu$.

We use Toma-Stein to control the low-frequency terms, $\lambda \lesssim \nu^{\frac{1}{2}}$, and the terms with very different characteristic regions, $\mu^{\frac{1}{2}} \leqslant \nu$.

Theorem 12 (Tomas-Stein Theorem). *If f_μ and g_ν are functions in $\mathbb{R}^n$, and their Fourier transform are supported in a μ- and ν-neighborhood of S^{n-1} respectively, where $\mu \leqslant \nu$, then*

$$\|f_\mu g_\nu\|_{p'} \lesssim \mu^{\frac{n+1}{2p}} \|f_\mu\|_2 \|g_\nu\|_2, \qquad \text{for } 1 \leqslant p' \leqslant \frac{n+1}{n}. \tag{31}$$

Proof. We use Hölder to get

$$\|f_\mu g_\nu\|_{p'} \leqslant \|f_\mu\|_{2p'/(2-p')} \|g_\nu\|_2. \tag{32}$$

Since $1 \leqslant p' \leqslant \frac{n+1}{n}$, then $2 \leqslant 2p'/(2-p') \leqslant 2\frac{n+1}{n-1}$, and the latter is the Tomas-Stein exponent. To bound the term $\|f_\mu\|_r$, for $r = \frac{2p'}{2-p'}$, we interpolate between $p' = 2$ and $p' = 2\frac{n+1}{n-1}$.

The point $p' = 2$ is immediate. For $p' = 2\frac{n+1}{n-1}$, we write $\widehat{f}_\mu$ as an average over spheres

$$f_\mu(x) = \int r^{n-1} \int_{S^{n-1}} \widehat{f}_\mu(r\theta) e(\langle rx, \theta\rangle)\, d\theta dr := \int r^{n-1} (f_\mu^r dS)^\vee (rx)\, dr$$

We apply Minkowski, Tomas-Stein and Cauchy-Schwarz to find $\|f_\mu\|_{2\frac{n+1}{n-1}} \leqslant C\mu^{\frac{1}{2}} \|f_\mu\|_2$; this leads to

$$\|f_\mu\|_r \lesssim \mu^{\frac{n+1}{2}(\frac{1}{2} - \frac{1}{r})} \|f_\mu\|_2, \qquad \text{for } 2 \leqslant r \leqslant 2\,\frac{n+1}{n-1}.$$

We replace it in (32) to get

$$\|f_\mu g_\nu\|_{p'} \lesssim \mu^{\frac{n+1}{2p}} \|f_\mu\|_2 \|g_\nu\|_2,$$

which is what we wanted. $\square$

By Hölder, we can bound each term in (30) as

$$|\langle(\partial_w P_{\lambda,\nu} f) u_\mu, v_\nu\rangle| \leqslant \lambda \|P_{\lambda,\nu} f\|_p \|u_\mu v_\nu\|_{p'}. \tag{33}$$

To bound the bilinear term, we begin by writing it as

$$\int |u_\mu v_\nu|^{p'} \, dx = \iint |u_\mu(x_1, \tilde{x}) v_\nu(x_1, \tilde{x})|^{p'} \, d\tilde{x} dx_1. \tag{34}$$

We fix x_1 as a parameter and define the function $u_\mu^{x_1}(\tilde{x}) = u_\mu(x_1, \tilde{x})$; its Fourier transform is the term in parentheses in

$$u_\mu(x_1, \tilde{x}) = \int \Big(\int \widehat{u}_\mu(\xi) e^{ix_1 \cdot \xi_1} \, d\xi_1 \Big) e^{i\tilde{x} \cdot \tilde{\xi}} \, d\tilde{\xi} = \int \widehat{u}_\mu^{x_1}(\tilde{\xi}) e^{i\tilde{x} \cdot \tilde{\xi}} \, d\tilde{\xi}.$$

The support of $\widehat{u}_\mu^{x_1}$ lies in a μ-neighborhood of the sphere $S^{d-2} \subset \mathbb{R}^{d-1}$. Hence, we can apply Theorem 12 with $n = d-1$ to the inner integral at the right of (34) to get

$$\int |u_\mu v_\nu|^{p'} \, dx \leqslant \mu^{p' \frac{d}{2p}} \int \|u_\mu(x_1, \cdot)\|_2^{p'} \|v_\nu(x_1, \cdot)\|_2^{p'} \, dx_1. \tag{35}$$

Since $\widehat{u}_\mu$ is supported in the μ-neighborhood of the hyperplane normal to e_1, then $u_\mu = u_\mu *_1 \phi_\mu$, where $\phi_\mu(t) := \mu\phi(\mu t)$ and $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a smooth function such that $\widehat{\phi}(\eta) = 1$ in a neighborhood of the origin. Hence, by Minkowski we have

$$\begin{aligned} \|u_\mu(x_1, \cdot)\|_2 &= \Big(\int \Big| \int u_\mu(x_1 - y_1, \tilde{x}) \phi_\mu(y_1) \, dy_1 \Big|^2 \, d\tilde{x} \Big)^{1/2} \\ &\leqslant \int \|u_\mu(x_1 - y_1, \cdot)\|_2 |\phi_\mu|(y_1) \, dy_1 \\ &= (\|u_\mu^{z_1}\|_{L^2_{\tilde{x}}} *_1 |\phi_\mu|)(x_1). \end{aligned}$$

This fact and the following Lemma allow us to bound the integral at the right of (35).

Lemma 13. *Let a and b be two functions in the real line, then*

$$\|(a * \phi_\mu)b\|_{p'} \leqslant C\mu^{\frac{1}{p}}\|a\|_2\|b\|_2, \qquad \textit{for } 1 \leqslant p' \leqslant 2. \tag{36}$$

The inequality is best possible in μ.

Proof. We use Hölder and Young inequalities to get

$$\|(a * \phi_\mu)b\|_{p'} \leqslant \|a * \phi_\mu\|_{2p'/(2-p')}\|b\|_2 \leqslant \|\phi_\mu\|_{p'}\|a\|_2\|b\|_2,$$

where $\|\phi_\mu\|_{p'} = \mu^{\frac{1}{p}}\|\phi_1\|_{p'}$. The example $a = b = \mathbb{1}_{(-\mu^{-1},\mu^{-1})}$ shows that the constant $\mu^{\frac{1}{p}}$ is best possible. $\square$

With the aid of Lemma 13 and $\|u_\mu\|_2 \lesssim \mu^{-\frac{1}{2}}\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}$, we continue (35) as

$$\|u_\mu v_\nu\|_{p'} \leqslant \mu^{\frac{d+2}{2p}}\|u_\mu\|_2\|v_\nu\|_2 \lesssim \mu^{\frac{d+2}{2p}-\frac{1}{2}}\nu^{-\frac{1}{2}}\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}. \tag{37}$$

Furthermore, when we are restricted to low frequencies $\lambda \lesssim \nu^{\frac{1}{2}}$, we can use this bound and (33) in the pairing (30) to get, for $p = d$,

$$\begin{aligned}
|\langle(\partial_w f)u, v\rangle| &\lesssim \Big(\sum_{\substack{\tau^{-1}\leqslant\lambda\lesssim\nu^{\frac{1}{2}}\\ \tau^{-1}\leqslant\mu\leqslant\nu\lesssim 1}} \lambda\mu^{\frac{1}{d}}\nu^{-\frac{1}{2}}\|P_\lambda f\|_d\Big)\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}} + \cdots \\
&\lesssim \Big(\sum_{\tau^{-1}\leqslant\lambda\lesssim\nu^{\frac{1}{2}}} \lambda\nu^{\frac{1}{d}-\frac{1}{2}}\|P_\lambda f\|_d\Big)\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}} \\
&\lesssim \Big(\sum_{\tau^{-1}\leqslant\lambda\lesssim 1} \lambda^{\frac{2}{d}}\|P_\lambda f\|_d\Big)\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}} \\
&\lesssim \Big(\sum_{\tau^{-1}\leqslant\lambda\lesssim 1} \|P_\lambda f\|_d^d\Big)^{\frac{1}{d}}\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}} \\
&\lesssim \|f\|_d\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}} + \cdots .
\end{aligned} \tag{38}$$

On the other hand, when the characteristic frequencies are very different, *i.e.* $\mu^{\frac{1}{2}} \leqslant \nu$, again by (37) and (33) in the pairing (30), we get

$$|\langle(\partial_w f)u, v\rangle| \leqslant \Big(\sum_{\substack{\nu^{\frac{1}{2}}\lesssim\lambda\lesssim 1\\ \tau^{-1}\leqslant\mu\leqslant\nu^2\lesssim 1}} \lambda\mu^{\frac{d+2}{2p}-\frac{1}{2}}\nu^{-\frac{1}{2}}\|P_{\lambda,\nu} f\|_p\Big)\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}} + \cdots . \tag{39}$$

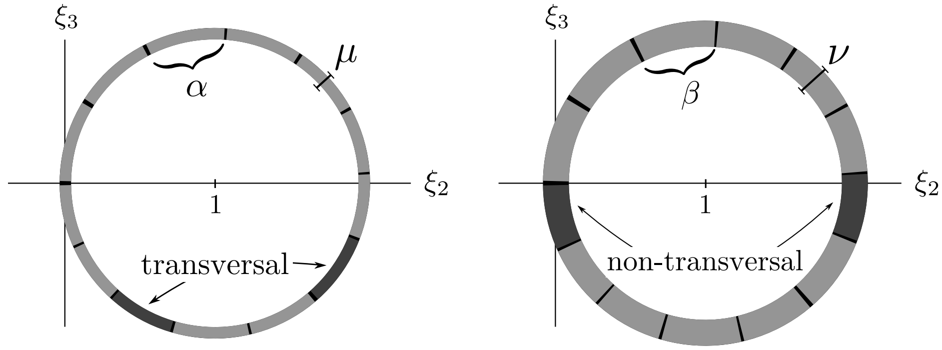

Figure 1: The decomposition of the μ- and the ν-neighborhoods of the sphere $S^{d-2} + e_2$ into caps α and β.

We are left thus with the case of high frequencies ($\lambda \gtrsim \nu^{\frac{1}{2}}$) and similar characteristic frequencies ($\mu \leqslant \nu \leqslant \mu^{\frac{1}{2}}$).

3.2 Bilinear Strategy

In this Section we assume that $\lambda \gtrsim \nu^{\frac{1}{2}}$ and that $\mu \leqslant \nu < \mu^{\frac{1}{2}}$, so that the bilinear inequality in Theorem 8 gives us a small window of stability on restriction estimates. To pass from bilinear to linear inequalities, we follow the strategy in [29].

Using smooth partitions of unity $\{\varphi_\alpha\}_\alpha$ and $\{\varphi_\beta\}_\beta$ we decompose, respectively, the μ- and the ν-neighborhoods of the sphere $S^{d-2} + e_2$ into caps α and β of radius $\rho_0 \ll 1$; see Figure 1. If the angle between the normal vectors to two caps α and β is $\gtrsim \rho_0$, then we call them transversal and denote it by $\alpha \sim \beta$; otherwise the caps are not transversal, $\alpha \not\sim \beta$. For the transversal caps we will use the Bilinear Theorem 8 for the sphere. We define the projections $\widehat{u}_{\mu,\alpha} := \varphi_\alpha \widehat{u}_\mu$ and $\widehat{v}_{\nu,\beta} := \varphi_\beta \widehat{v}_\nu$, and write so the bilinear term as

$$u_\mu \overline{v}_\nu = \sum_{\alpha,\beta} u_{\mu,\alpha} \bar{v}_{\nu,\beta} = \sum_{\alpha\sim\beta} u_{\mu,\alpha} \bar{v}_{\nu,\beta} + \sum_{\alpha\not\sim\beta} u_{\mu,\alpha} \bar{v}_{\nu,\beta}.$$

Since we cannot apply the Bilinear Theorem to non-transversal caps $\alpha \not\sim \beta$, we decompose them again into caps of radius $\rho_1 = \frac{1}{2}\rho_0$, and we still denote the smaller caps as α and β. If the angle between the normal vectors to two caps α and β is $\sim \rho_1$, then we call them transversal and denote it again by $\alpha \sim \beta$; otherwise the caps are not transversal, $\alpha \not\sim \beta$. For transversal

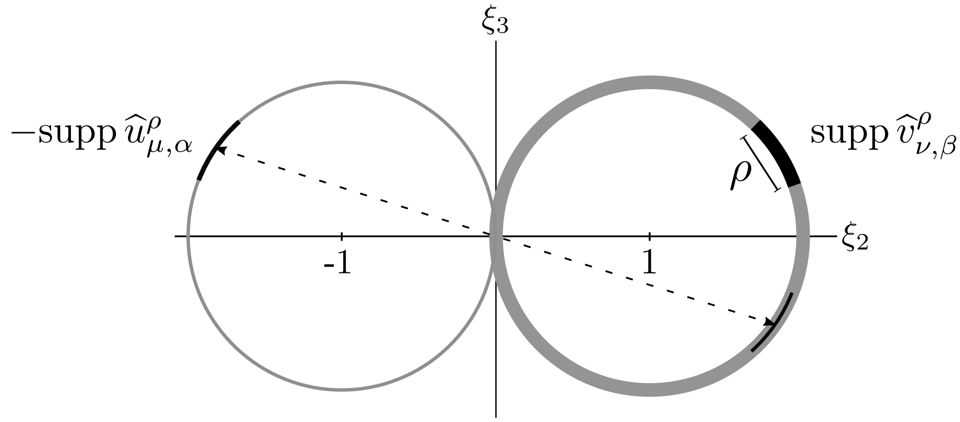

Figure 2: Two neighboring, transversal caps.

caps we will use a rescaled version of the Bilinear Theorem 8 for surfaces of elliptic type, after choosing ρ_0 sufficiently small. We continue the process of subdivision of non-transversal caps until the radius of the caps is $\rho \sim \nu^{\frac{1}{2}}$, and write

$$\langle(\partial_w f)u, v\rangle = \sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \lesssim 1 \\ \mu \leqslant \nu < \mu^{\frac{1}{2}}}} \Big[\sum_{\substack{\nu^{1/2} < \rho \lesssim 1 \\ \alpha \sim \beta}} \langle(\partial_w P_{\lambda,\nu} f)u^{\rho}_{\mu,\alpha}, v^{\rho}_{\nu,\beta}\rangle + \\ + \sum_{\alpha \nsim \beta} \langle(\partial_w P_{\lambda,\nu} f)u^{\rho^*}_{\mu,\alpha}, v^{\rho^*}_{\nu,\beta}\rangle \Big] + \cdots , \quad (40)$$

where the sum over non-transversal terms is at scale $\rho^* \sim \nu^{\frac{1}{2}}$. The superscript in $u^{\rho}_{\mu,\alpha}$ is to keep track of the radius of the caps α.

The support of the inverse Fourier transform of $u^{\rho}_{\mu,\alpha}\overline{v}^{\,\rho}_{\nu,\beta}$ has some special properties, and they determine when the pairing $\langle(\partial_w P_{\lambda,\nu} f)u^{\rho}_{\mu,\alpha}, v^{\rho}_{\nu,\beta}\rangle$ either vanishes or not. Recall that the support of the convolution $\breve{u}^{\rho}_{\mu,\alpha} * \overline{\widehat{v}}^{\,\rho}_{\nu,\beta}$ lies in the Minkowski sum of the sets $-\alpha \supset \mathrm{supp}\,\breve{u}^{\rho}_{\mu,\alpha}$ and $\beta \supset \mathrm{supp}\,\widehat{v}^{\rho}_{\nu,\beta}$; see Figure 2. The reader will find easier to evaluate the Minkowski sum of $-\alpha + e_2$ and $\beta - e_2$.

When the caps α and β have radius ρ_0 and are transversal, then we have that

$$-\alpha + \beta \subset \{(\xi_1, \tilde{\xi}) \mid \frac{\rho_0}{2} \leqslant |\tilde{\xi}| \leqslant 2 - \frac{\rho_0^2}{2}, \ \ |\xi_1| \leqslant 2\nu\};$$

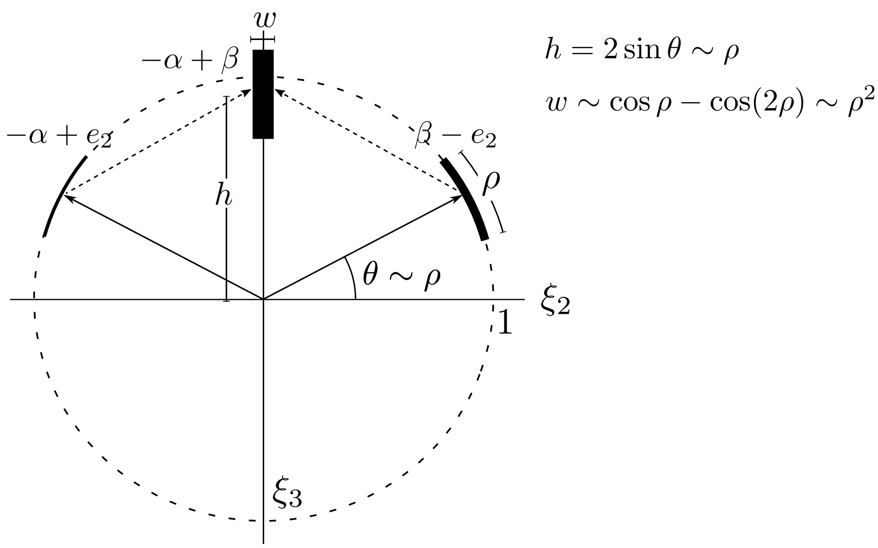

Figure 3: The Minkowski sum of two neighboring, transversal caps at scale ρ.

Hence, all the terms $\langle(\partial_w P_{\lambda,\nu} f) u^{\rho_0}_{\mu,\alpha}, v^{\rho_0}_{\nu,\beta}\rangle$ vanish for $\lambda \leqslant c\rho_0$; Figure 3 may help.

When the caps have radius $\rho < \rho_0$ we have to distinguish between neighboring and antipodal caps. Two caps are neighboring if there exists a ball of radius $2\rho_0$ that contains both of them, and two caps are antipodal if they lie in different and opposite balls of radius $2\rho_0$. We refer to neighboring and antipodal, transversal caps as $\alpha \sim_n \beta$ and $\alpha \sim_a \beta$ respectively.

If two caps of radius $\nu^{\frac{1}{2}} \leqslant \rho < \rho_0$ are neighboring and transversal, then for the Minkwoski sum we get

$$-\alpha + \beta \subset \{(\xi_1, \tilde{\xi}) \mid |\tilde{\xi}| \sim \rho,\ \ |\xi_1| \leqslant 2\nu\};$$

see Figure 3. Hence, only the terms $\langle(\partial_w P_{\lambda,\nu} f) u^{\rho}_{\mu,\alpha}, v^{\rho}_{\nu,\beta}\rangle$ for which $\lambda \sim \rho$ survive. When the caps are non-transversal, the Minkowski sum lies in $\{|\tilde{\xi}| \leqslant c\nu^{\frac{1}{2}}\}$, but we already considered the low frequency terms $\lambda \lesssim \nu^{\frac{1}{2}}$ in the previous Section, so $\langle(\partial_w P_{\lambda,\nu} f) u^{\rho*}_{\mu,\alpha}, v^{\rho*}_{\nu,\beta}\rangle$ always vanishes.

If two caps of radius $\nu^{\frac{1}{2}} \leqslant \rho < \rho_0$ are antipodal and transversal, then for the Minkwoski sum we get

$$-\alpha + \beta \subset S_{\nu,\rho} := \{(\xi_1, \tilde{\xi}) \mid 2 - |\tilde{\xi}| \sim \rho^2,\ \ |\xi_1| \leqslant 2\nu\}; \tag{41}$$

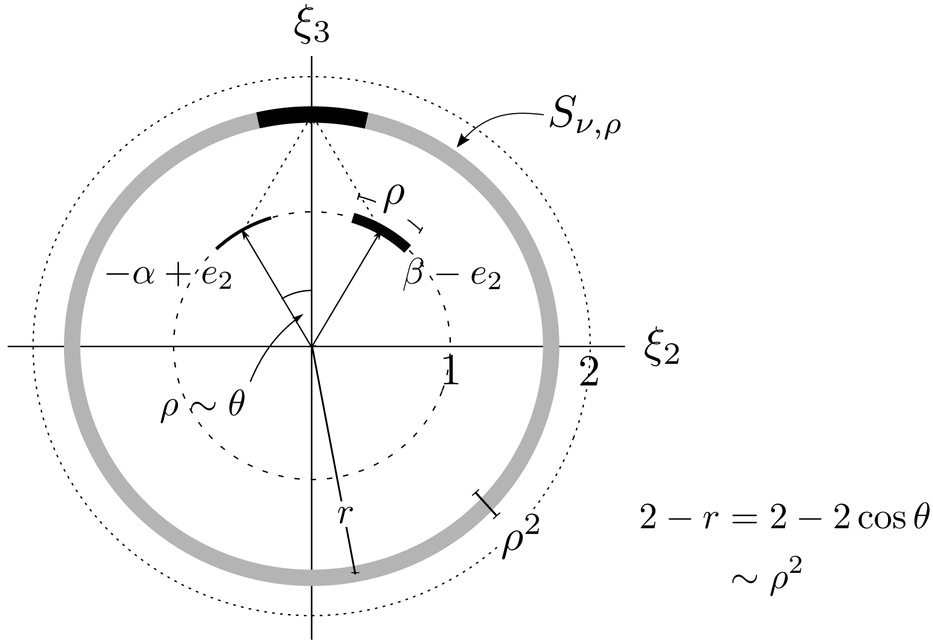

Figure 4: The Minkowski sum of two antipodal, transversal caps at scale ρ.

see Figure 4. Only the terms $\langle (\partial_w P_{\lambda,\nu} f) u^{\rho}_{\mu,\alpha}, v^{\rho}_{\nu,\beta} \rangle$ for which $\lambda \sim 1$ survive, but now we need more detailed information about $-\alpha + \beta$. We can see that $-\alpha + \beta$ forms a cap of radius $\sim \rho$ lying in the ρ^2-neighborhood of the sphere with radius $2 - \rho^2$ centered at zero, which we called $S_{\nu,\rho}$. Fixing ρ, the collection of all the the caps $\{-\alpha + \beta\}$, where $\alpha \sim_a \beta$, is an almost disjoint covering of $S_{\nu,\rho}$. In fact, let x be a point in $S_{\nu,\rho}$, c_α be the center of α and c_β be the center of β; if x and $-c_\alpha + e_2$ make an angle $\gtrsim \rho$, since $\alpha \sim_a \beta$ then the sum $-\alpha + \beta$ necessarily lies away from x. Hence, only the caps $-\alpha + \beta$ for which α and β make an angle $\lesssim \rho$ with x can cover it. For future reference let us write it down as a Lemma.

Lemma 14. *For fixed μ, ν and $\nu^{\frac{1}{2}} < \rho < \rho_0$, let α and β denote caps at scale ρ, then*

$$\sum_{\alpha \sim_a \beta} \mathbb{1}_{-\alpha+\beta} \leqslant C_d \mathbb{1}_{S_{\nu,\rho}}, \tag{42}$$

where $S_{\nu,\rho}$ is defined in (41), and C_d does not depend either on μ, on ν or on ρ.

A similar statement holds for non-transversal caps at scale $\nu^{\frac{1}{2}}$, but the caps $-\alpha+\beta$ lie now in a ν-neighborhood of $2S^{d-2}$.

We will follow the argument of the previous Section to bound the terms $\langle(\partial_w P_{\lambda,\nu} f)u^{\rho}_{\mu,\alpha}, v^{\rho}_{\nu,\beta}\rangle$. However, the Bilinear Theorem 8 is only stated for transversal caps at scale $\sim 1 \sim \rho_0$. To remedy this situation, we use parabolic rescaling.

Theorem 15. *Let $f_{\mu,\alpha}$ and $g_{\nu,\beta}$ be two functions with Fourier transform supported in a μ- and ν-neighborhood of S^{n-1}, where $\mu \leqslant \nu \leqslant \mu^{\frac{1}{2}}$. If the caps α and β are transversal at scale $\rho \leqslant \rho_0$, then for $1 \leqslant p' \leqslant \frac{n+1}{n}$ it holds that*

$$\begin{aligned} \|f_{\mu,\alpha}g_{\nu,\beta}\|_{p'} &\leqslant C_\varepsilon \rho^{-\frac{1}{p}}\mu^{\frac{n}{2p}-\varepsilon}\nu^{\frac{1}{p}-\varepsilon}\|f_{\mu,\alpha}\|_2\|g_{\nu,\beta}\|_2 && \text{for } \rho > \nu\mu^{-\frac{1}{2}}, \\ \|f_{\mu,\alpha}g_{\nu,\beta}\|_{p'} &\leqslant C\mu^{\frac{n+1}{2}}\|f_{\mu,\alpha}\|_2\|g_{\nu,\beta}\|_2 && \text{for } \nu^{\frac{1}{2}} \leqslant \rho \leqslant \nu\mu^{-\frac{1}{2}}. \end{aligned} \tag{43}$$

Proof. The case $\rho_0 \sim 1 \gtrsim \nu\mu^{-\frac{1}{2}}$ is Theorem 8 for the sphere, so we assume that $\rho < \rho_0$. By conjugation, rotation and modulation of $f_{\mu,\alpha}$ and of $g_{\nu,\beta}$ we assume further that both caps lie in the surface given by the graph of

$$\varphi(\eta') = 1 - \sqrt{1-|\eta'|^2} = \frac{1}{2}|\eta'|^2 + O(|\eta'|^4),$$

where $\eta = (\eta', \eta_n) \in \mathbb{R}^n$; we also assume that the centers of the caps are symmetrically placed along the axis η_1. Since the caps are at distance $\sim \rho$ of each other, after applying the scaling $\xi \mapsto (\rho^{-1}\bar{\eta}, \rho^{-2}\eta_d)$ the support of the new functions $\widehat{F}(\eta) := \widehat{f}_{\mu,\alpha}(\rho\eta', \rho^2\eta_n)$ and $\widehat{G} := \widehat{g}_{\nu,\beta}(\rho\eta', \rho^2\eta_n)$ lie at distance ~ 1 of each other, and the surface transforms accordingly to the graph of

$$\varphi_\rho(\eta') := \rho^{-2}\varphi(\rho\eta') = \rho^{-2} - \sqrt{\rho^{-4} - |\rho^{-1}\bar{\eta}|^2} = \frac{1}{2}|\eta'|^2 + O(\rho_0^2|\eta'|^4).$$

If $\rho < \rho_0$ is sufficiently small, then the semi-norms $\|\partial^N\varphi_\rho\|_\infty$ are uniformly bounded, and the Bilinear Theorem holds uniformly. The rescaled functions F and G are

$$\begin{aligned} F(x) &= \rho^{-n-1} f_{\mu,\alpha}(\rho^{-1}x', \rho^{-2}x_n) \\ G(x) &= \rho^{-n-1} g_{\nu,\beta}(\rho^{-1}x', \rho^{-2}x_n). \end{aligned}$$

Since the Fourier transforms of F and G are supported now in sets of width $\rho^{-2}\mu$ and $\rho^{-2}\nu$ respectively, then we should apply the Bilinear Theorem 8 whenever $\rho^{-2}\nu < (\rho^{-2}\mu)^{\frac{1}{2}}$, and Tomas-Stein otherwise.

If $\rho > \nu\mu^{-\frac{1}{2}}$, then we apply the Bilinear Theorem to F and G to find

$$\begin{aligned}\|f_{\mu,\alpha}g_{\nu,\beta}\|_{p'} &= \rho^{2(n+1)-\frac{n+1}{p'}}\|FG\|_{p'} \\ &\leqslant C_\varepsilon \rho^{2(n+1)-\frac{n+1}{p'}-\frac{n+2}{p}}\mu^{\frac{n}{2p}-\varepsilon}\nu^{\frac{1}{p}-\varepsilon}\|F\|_2\|G\|_2 \\ &= C_\varepsilon \rho^{-\frac{1}{p}}\mu^{\frac{n}{2p}-\varepsilon}\nu^{\frac{1}{p}-\varepsilon}\|f_{\mu,\alpha}\|_2\|g_{\nu,\beta}\|_2;\end{aligned}$$

if we use Tomas-Stein instead, then we get the result for $\rho \leqslant \nu\mu^{-\frac{1}{2}}$ $\square$

If we define the quantity

$$K^\rho_{\mu,\nu}(p') := \sup_{\substack{\|f_{\mu,\alpha}\|_2=1 \\ \|g_{\nu,\beta}\|_2=1}} \|f_{\mu,\alpha}g_{\nu,\beta}\|_{p'}, \tag{44}$$

where the supremum runs over functions $f_{\mu,\alpha}$ and $g_{\nu,\beta}$ with Fourier transform supported in caps at scale ρ, then we can restate Theorem 15 as

$$K^\rho_{\mu,\nu}(p') \leqslant \begin{cases} C_\varepsilon \rho^{-\frac{1}{p}}\mu^{\frac{n}{2p}-\varepsilon}\nu^{\frac{1}{p}-\varepsilon} & \text{for } \rho > \nu\mu^{-\frac{1}{2}} \\ C\mu^{\frac{n+1}{2p}} & \text{for } \nu^{\frac{1}{2}} \leqslant \rho \leqslant \nu\mu^{-\frac{1}{2}} \end{cases}$$

By Lemma 13 and Theorem 15 for $n = d-1$, we get

$$\begin{aligned}\sum_{\alpha\sim\beta}\|u^\rho_{\mu,\alpha}v^\rho_{\nu,\beta}\|_{p'} &\lesssim \mu^{\frac{1}{p}}K^\rho_{\mu,\nu}\sum_{\alpha\sim\beta}\|u_{\mu,\alpha}\|_2\|v_{\nu,\beta}\|_2 \\ &\lesssim \mu^{\frac{1}{p}}K^\rho_{\mu,\nu}\|u_\mu\|_2\|v_\nu\|_2 \\ &\lesssim \mu^{\frac{1}{p}-\frac{1}{2}}\nu^{-\frac{1}{2}}K^\rho_{\mu,\nu}\|u\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac{1}{2}}_{\zeta(1),1/\tau}}.\end{aligned} \tag{45}$$

Now let us consider only transversal, neighboring caps at scale ρ. By the decomposition (40) we get

$$
\begin{aligned}
|\langle(\partial_w f)u,v\rangle| &\leqslant \sum_{\substack{\nu^{\frac12}\lesssim\lambda\\ \mu\leqslant\nu<\mu^{\frac12}}}\ \sum_{\substack{\nu^{1/2}\leqslant\rho\sim\lambda\\ \alpha\sim_n\beta}} |\langle(\partial_w P_{\lambda,\nu} f)u^{\rho}_{\mu,\alpha}, v^{\rho}_{\nu,\beta}\rangle| + \cdots\\
&\lesssim \sum_{\substack{\nu^{\frac12}\lesssim\lambda\\ \mu\leqslant\nu<\mu^{\frac12}}} \lambda\mu^{\frac1p-\frac12}\nu^{-\frac12}\|P_{\lambda,\nu}f\|_p \sum_{\nu^{1/2}\leqslant\rho\sim\lambda} K^{\rho}_{\mu,\nu}\|u\|_{X^{\frac12}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac12}_{\zeta(1),1/\tau}}\\
&\lesssim_\varepsilon \Bigg(\sum_{\substack{\nu^{\frac12}\lesssim\lambda\leqslant\nu\mu^{-\frac12}\\ \mu\leqslant\nu<\mu^{\frac12}}} \lambda\mu^{\frac{d+2}{2p}-\frac12}\nu^{-\frac12}\|P_{\lambda,\nu}f\|_p +\\
&\qquad \sum_{\substack{\nu\mu^{-\frac12}\leqslant\lambda\lesssim 1\\ \mu\leqslant\nu<\mu^{\frac12}}} \lambda^{1-\frac1p}\mu^{\frac{d+1}{2p}-\frac12}\nu^{\frac1p-\frac12-\varepsilon}\|P_{\lambda,\nu}f\|_p\Bigg)\|u\|_{X^{\frac12}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac12}_{\zeta(1),1/\tau}}.
\end{aligned}
\tag{46}
$$

The operator $P_{\lambda,\nu}$ is the projection to frequencies $|\xi|\sim\lambda$ and $|\xi_1|\lesssim\nu$.

When the caps α and β are antipodal, we have to refine the projection $P_{\lambda,\nu}$, so we project also to the cap $-\alpha+\beta$ and denote this projection as $P_{\lambda,\nu,\alpha,\beta}$. We argue as above to get

$$
\begin{aligned}
|\langle(\partial_w f)u,v\rangle| &\leqslant \sum_{\substack{\lambda\sim1\\ \mu\leqslant\nu<\mu^{\frac12}}} \Bigg(\sum_{\substack{\nu^{1/2}<\rho\\ \alpha\sim_a\beta}} |\langle(\partial_w P_{\lambda,\nu} f)u^{\rho}_{\mu,\alpha}, v^{\rho}_{\nu,\beta}\rangle| + |\langle(\partial_w P_{\lambda,\nu} f)u^{\rho^*}_{\mu,\alpha}, v^{\rho^*}_{\nu,\beta}\rangle|\Bigg)\\
&\lesssim \sum_{\substack{\lambda\sim1\\ \mu\leqslant\nu<\mu^{\frac12}}} \lambda\mu^{\frac1p-\frac12}\nu^{-\frac12}\sum_{\nu^{1/2}\leqslant\rho} K^{\rho}_{\mu,\nu}\sup_{\alpha\sim_a\beta}\|P_{\lambda,\nu,\alpha,\beta}f\|_p\|u\|_{X^{\frac12}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac12}_{\zeta(1),1/\tau}} + \cdots.
\end{aligned}
\tag{47}
$$

We have already bounded all the contributions, and we can say that for some functional $A(f)$ we got an upper bound

$$|\langle(\partial_w f)u,v\rangle| \leqslant (\|f\|_d + A(f))\|u\|_{X^{\frac12}_{\zeta(1),1/\tau}}\|v\|_{X^{\frac12}_{\zeta(1),1/\tau}}$$

If we return to the original variables, and replace u and v by $u_{\tau U}$ and $v_{\tau U}$, and w by Ue_j, then by (27), (28) and (29) we get

$$\begin{aligned}\langle(\partial_j f)u,v\rangle &= \tau^{2d+1}\langle(\partial_w f_{\tau U})u_{\tau U},v_{\tau U}\rangle\\ &\lesssim \tau^{2d+1}(\|f_{\tau U}\|_d + A(f_{\tau U}))\|u_{\tau U}\|_{X^{1/2}_{\zeta(1),1/\tau}}\|v_{\tau U}\|_{X^{1/2}_{\zeta(1),1/\tau}}\\ &= (\|f\|_d + \tau^{d-1}A(f_{\tau U}))\|u\|_{X^{1/2}_{\zeta(\tau,U)}}\|v\|_{X^{1/2}_{\zeta(\tau,U)}}.\end{aligned}$$

If $m_{\lambda,\nu,\alpha,\beta}$ is the multiplier of $P_{\lambda,\nu,\alpha,\beta}$, then

$$\begin{aligned}(P_{\lambda,\nu,\alpha,\beta}f_{\tau U})(x) &= (m_{\lambda,\nu,\alpha,\beta}(\cdot)\widehat{f}(\tau U\cdot))\check{}(x)\\ &= \tau^{-d}(P^U_{\tau\lambda,\tau\nu,\alpha,\beta}f)(\tau^{-1}Ux),\end{aligned}$$

where the multiplier of P^U is $m(U^{-1}\xi)$. Hence,

$$\|P_{\lambda,\nu,\alpha,\beta}f_{\tau U}\|_p = \tau^{-\frac{d}{p'}}\|P^U_{\tau\lambda,\tau\nu,\alpha,\beta}f\|_p$$

We collect all the estimates (25), (26), (38), (39), (46) and (47) to conclude this Section with the following Theorem.

Theorem 16. *For $d \leqslant p \leqslant \infty$, the norm of the operator $M_{\partial_j f} : u \in X^{\frac{1}{2}}_{\zeta(U,\tau)} \mapsto (\partial_j f)u \in X^{-\frac{1}{2}}_{\zeta(U,\tau)}$ has the upper bound*

$$\|M_{\partial_j f}\|_{X^{1/2}_{\zeta(\tau,U)}\mapsto X^{-1/2}_{\zeta(\tau,U)}} \lesssim_\varepsilon \|f\|_d + \tau^{\frac{d}{p}-1}A(\tau,U,f), \tag{48}$$

where

$$\begin{aligned}A(\tau,U,f) := &\sum_{\substack{\nu^{\frac{1}{2}}\lesssim\lambda\lesssim 1\\ \tau^{-1}\leqslant\mu\leqslant\nu}} Q(\lambda,\mu,\nu)\|P^U_{\tau\lambda,\tau\nu}f\|_p +\\ &+ \sum_{\substack{\lambda\sim 1\\ \mu\leqslant\nu<\mu^{\frac{1}{2}}}} \lambda\mu^{\frac{1}{p}-\frac{1}{2}}\nu^{-\frac{1}{2}} \sum_{\nu^{1/2}\leqslant\rho} K^\rho_{\mu,\nu}\sup_{\alpha\sim_a\beta}\|P^U_{\tau\lambda,\tau\nu,\alpha,\beta}f\|_p.\end{aligned} \tag{49}$$

The constant $K^\rho_{\mu,\nu}$ is defined in (44), and

$$Q(\lambda,\mu,\nu) := \begin{cases}\lambda^{1-\frac{1}{p}}\mu^{\frac{d+1}{2p}-\frac{1}{2}}\nu^{\frac{1}{p}-\frac{1}{2}-\varepsilon} & \text{for } \lambda > \nu\mu^{-\frac{1}{2}} \text{ and } \nu \leqslant \mu^{\frac{1}{2}}\\ \lambda\mu^{\frac{d+2}{2p}-\frac{1}{2}}\nu^{-\frac{1}{2}} & \text{otherwise.}\end{cases}$$

3.3 End of the Proof

In this Section we average the norm $\|M_{\partial_j f}\|_{X^{1/2}_{\zeta(\tau,U)}\mapsto X^{-1/2}_{\zeta(\tau,U)}}$ over τ and U. We follow the ideas in Haberman [14], and in Ham, Kwon and Lee [16].

By Theorem 16 we have

$$\begin{aligned}
&\fint_M \int_{O_d} \|M_{\partial_i f}\|_{X^{1/2}_{\zeta(\tau,U)}\mapsto X^{-1/2}_{\zeta(\tau,U)}}\, dU d\tau \lesssim_\varepsilon \|f\|_d + \\
&\qquad + M^{\frac{d}{p}-1} \sum_{\substack{\nu^{\frac12}\lesssim\lambda\lesssim 1\\ M^{-1}\leqslant\mu\leqslant\nu}} Q(\lambda,\mu,\nu) \fint_M \int_{O_d} \|P^U_{\tau\lambda,\tau\nu} f\|_p\, dU d\tau + \\
&+ M^{\frac{d}{p}-1} \sum_{\substack{\lambda\sim 1\\ \mu\leqslant\nu<\mu^{\frac12}}} \lambda \mu^{\frac1p-\frac12}\nu^{-\frac12} \sum_{\nu^{1/2}\leqslant\rho} K^\rho_{\mu,\nu} \fint_M \int_{O_d} \sup_{\alpha,\beta} \|P^U_{\tau\lambda,\tau\nu,\alpha,\beta} f\|_p\, dU d\tau. \qquad (50)
\end{aligned}$$

The first average at the right has been already bounded by Haberman.

Lemma 17. *(Haberman, Lemma 5.1 in [14]) Let $P^U_{\tau\lambda,\tau\nu}$ be the projection to frequencies $|\xi|\sim\tau\lambda$ and to frequencies $|\langle Ue_1,\xi\rangle|\leqslant 2\tau\nu$. If $f\in L^p(\mathbb{R}^d)$, then*

$$\Big(\int_{O_d} \|P^U_{\tau\lambda,\tau\nu} f\|_p^p\, dU\Big)^{\frac1p} \leqslant C\Big(\frac{\nu}{\lambda}\Big)^{\frac1p}\|f\|_p \qquad \text{for } 2\leqslant p\leqslant\infty. \qquad (51)$$

The second average at the right of (50) has been already bounded by Ham, Kwon and Lee.

Lemma 18. *(Ham, Kwon and Lee, Lemma 4.3 in [16]) For fixed $\tau^{-\frac12}\leqslant\nu^{\frac12}<\rho<\rho_0$ and $\tau^{-1}\leqslant\lambda\leqslant 1$, let α and β denote all the transversal, antipodal caps at scale ρ, or all the non-transversal, antipodal caps at scale $\sim\nu^{\frac12}$, as described in Section 3.2. If $P^U_{\tau\lambda,\tau\nu,\alpha,\beta}$ is the projection to frequencies $|\xi|\sim\tau\lambda$, $|\langle Ue_1,\xi\rangle|\leqslant 2\tau\nu$ and $\{\xi \mid \xi\in\tau U(-\alpha+\beta)\}$, then*

$$\Big(\fint_M \int_{O_d} \sup_{\alpha\sim_a\beta} \|P^U_{\tau\lambda,\tau\nu,\alpha,\beta} f\|_p^p\, dU d\tau\Big)^{\frac1p} \leqslant C\Big(\frac{\nu}{\lambda}\Big)^{\frac1p}\rho^{\frac2p}\|f\|_p \qquad \text{for } 2\leqslant p\leqslant\infty. \qquad (52)$$

Sketch of the proof. The proof is by interpolation. Since $-\alpha+\beta$ forms a cap of dimensions $\rho\times\cdots\times\rho\times\rho^2$, then for the point $p=\infty$ we get

$$\sup_{\alpha,\beta,U,\tau} \|P^U_{\tau\lambda,\tau\nu,\alpha,\beta} f\|_\infty \lesssim \|f\|_\infty.$$

Let us denote by $m^U_{\tau\lambda,\tau\nu,\alpha,\beta}$ the multiplier of $P^U_{\tau\lambda,\tau\nu,\alpha,\beta}$. For $p=2$ we get

$$\fint_M \int_{O_d} \sum_{\alpha,\beta} \|m^U_{\tau\lambda,\tau\nu,\alpha,\beta}\hat{f}\|_2^2 \, dU d\tau = \\ \int |\hat{f}(\xi)|^2 \fint_M \int_{S^{d-1}} \sum_{\alpha,\beta} |m_{\tau\lambda,\tau\nu,\alpha,\beta}|^2(|\xi|\omega)\, d\omega d\tau d\xi.$$

By Lemma 14 we have that $\sum_{\alpha,\beta} |m_{\tau\lambda,\tau\nu,\alpha,\beta}|^2 \lesssim \mathbb{1}_{\tau S_{\nu,\rho}}$, where $S_{\nu,\rho}$ is defined in (41). The set $\tau S_{\nu,\rho}$ is a $(d-2)$-sphere of radius $\tau(2-\rho^2)$, width $2\tau\nu$ in the direction e_1, and width $\tau\rho^2$ in $\{\xi_1 = 0\}$. For fixed ξ we get

$$\fint_M \int_{S^{d-1}} \sum_{\alpha\sim_a\beta} |m_{\tau\lambda,\tau\nu,\alpha,\beta}|^2(|\xi|\omega)\, d\omega d\tau \lesssim \mathbb{1}_{\{|\xi|\sim M\}} \frac{\nu}{\lambda}\rho^2 |\xi| M^{-1},$$

which leads to

$$\begin{aligned} \fint_M \int_{O_d} \sum_{\alpha,\beta} \|m^U_{\tau\lambda,\tau\nu,\alpha,\beta}\hat{f}\|_2^2 \, dU d\tau &\lesssim \frac{\nu}{\lambda}\rho^2 \int_{\{|\xi|\sim M\}} |\hat{f}|^2 d\xi \\ &\leqslant \frac{\nu}{\lambda}\rho^2 \|f\|_2^2, \end{aligned}$$

and then (52) follows. □

We use Lemma 17, Lemma 18 and Hölder in (50) to get

$$\begin{aligned} \fint_M \int_{O_d} \|M_{\partial_j f}\|_{X^{1/2}_{\zeta(\tau,U)} \mapsto X^{-1/2}_{\zeta(\tau,U)}} \, dU d\tau \lesssim_\varepsilon \|f\|_d + & \\ + M^{\frac{d}{p}-1} \sum_{\substack{\nu^{\frac12} \lesssim \lambda \lesssim 1 \\ M^{-1} \leqslant \mu \leqslant \nu}} Q(\lambda,\mu,\nu)\nu^{\frac1p}\lambda^{-\frac1p} \|P_{M\lambda} f\|_p + & \\ + M^{\frac{d}{p}-1} \sum_{\substack{\lambda\sim 1 \\ \mu\leqslant\nu<\mu^{\frac12}}} \lambda^{1-\frac1p}\mu^{\frac1p-\frac12}\nu^{\frac1p-\frac12} \sum_{\nu^{1/2}\leqslant\rho} K^\rho_{\mu,\nu}\rho^{\frac2p}\|P_{M\lambda}f\|_p & \\ \sim_\varepsilon \|f\|_d + A_1 + A_2. & \end{aligned} \tag{53}$$

To bound A_1 we use the definition of $Q(\lambda,\mu,\nu)$ in Theorem 16:

$$Q(\lambda,\mu,\nu) := \begin{cases} \lambda^{1-\frac1p}\mu^{\frac{d+1}{2p}-\frac12}\nu^{\frac1p-\frac12-\varepsilon} & \text{for } \lambda > \nu\mu^{-\frac12} \text{ and } \nu \leqslant \mu^{\frac12} \\ \lambda\mu^{\frac{d+2}{2p}-\frac12}\nu^{-\frac12} & \text{otherwise.} \end{cases}$$

We sum first in ν, then in μ and finally in λ to get

$$\begin{aligned}
A_1 &= \Big(\sum_{\substack{\nu^{\frac12}\leqslant\lambda\leqslant 1\\ \mu^{\frac12}\leqslant\nu}} + \sum_{\substack{\nu^{\frac12}\leqslant\lambda\leqslant\nu\mu^{-\frac12}\\ \mu\leqslant\nu\leqslant\mu^{\frac12}}} + \sum_{\substack{\nu\mu^{-\frac12}\leqslant\lambda\leqslant 1\\ \mu\leqslant\nu\leqslant\mu^{\frac12}}} \Big) Q(\lambda,\mu,\nu)\nu^{\frac1p}\lambda^{-\frac1p}M^{\frac dp-1}\|P_{M\lambda}f\|_p\\
&= \Big(\sum_{\mu^{\frac14}\leqslant\lambda} \lambda^{1-\frac1p}\mu^{\frac{d+2}{2p}-\frac12} \sum_{\mu^{\frac12}\leqslant\nu\leqslant\lambda^2} \nu^{\frac1p-\frac12}+\\
&\quad + \sum_{\mu^{\frac12}\leqslant\lambda} \lambda^{1-\frac1p}\mu^{\frac{d+2}{2p}-\frac12} \sum_{\lambda\mu^{\frac12}\leqslant\nu\leqslant\min(\lambda^2,\mu^{\frac12})} \nu^{\frac1p-\frac12}+\\
&\quad + \sum_{\mu^{\frac12}\leqslant\lambda} \lambda^{1-\frac2p}\mu^{\frac{d+1}{2p}-\frac12} \sum_{\mu\leqslant\nu\leqslant\lambda\mu^{\frac12}} \nu^{\frac2p-\frac12-}\Big) M^{\frac dp-1}\|P_{M\lambda}f\|_p\\
&\lesssim \Big(\sum_{M^{-\frac14}\leqslant\lambda} \lambda^{1-\frac1p}\Big[\sum_{\mu\leqslant\lambda^4}\mu^{\frac{d+3}{2p}-\frac34}\Big] + \sum_{M^{-\frac12}\leqslant\lambda}\lambda^{\frac12}\Big[\sum_{\mu\leqslant\lambda^2}\mu^{\frac{d+3}{2p}-\frac34}\Big]+\\
&\quad + M^{1-\frac{d+5}{2p}+}\sum_{\lambda}\lambda^{1-\frac2p}\Big) M^{\frac dp-1}\|P_{M\lambda}f\|_p.
\end{aligned}$$

During the summation we used the condition $p\geqslant d\geqslant 5$. At the end we get

$$A_1 \leqslant CM^{\frac{d-5}{2p}+} \sum_{M^{-\frac12}\lesssim\lambda\lesssim 1} \lambda^{\frac12}\|P_{M\lambda}f\|_p \leqslant C_\varepsilon \|f\|_{W^{\frac{d-5}{2p}+,p}}.$$

We bound now A_2, recalling that:

$$K^\rho_{\mu,\nu}(p') \leqslant \begin{cases} C_\varepsilon \rho^{-\frac1p}\mu^{\frac{d-1}{2p}}\nu^{\frac1p-\varepsilon} & \text{for } \rho > \nu\mu^{-\frac12}\\ C\mu^{\frac{d}{2p}} & \text{for } \nu^{\frac12}\leqslant\rho\leqslant\nu\mu^{-\frac12}. \end{cases}$$

We sum first in ρ, then in ν, in μ and finally in λ to get

$$\begin{aligned}
A_2 &\lesssim_\varepsilon M^{\frac dp-1} \sum_{\substack{\lambda\sim 1\\ \mu\leqslant\nu\leqslant\mu^{\frac12}}} \lambda^{1-\frac1p}\mu^{\frac{d+1}{2p}-\frac12}\nu^{\frac1p-\frac12}\Big(\mu^{\frac{1}{2p}}\sum_{\nu^{\frac12}\leqslant\rho\leqslant\nu\mu^{-\frac12}}\rho^{\frac2p} + \nu^{\frac1p-}\sum_{\nu\mu^{-\frac12}\leqslant\rho\leqslant 1}\rho^{\frac1p}\Big)\|P_{M\lambda}f\|_p\\
&\lesssim_\varepsilon \sum_{\lambda\sim 1}\lambda^{1-\frac1p}\mu^{\frac{d+1}{2p}-\frac12}\sum_{\mu\leqslant\nu\leqslant\mu^{\frac12}}\nu^{\frac2p-\frac12-}\|P_{M\lambda}f\|_p\\
&\lesssim_\varepsilon M^{\frac{d-5}{2p}+}\sum_{\lambda\sim 1}\|P_{M\lambda}f\|_p\\
&\lesssim_\varepsilon \|f\|_{W^{\frac{d-5}{2p}+,p}}.
\end{aligned}$$

The statement of Theorem 11 follows.

We conclude this Section with some informal remarks. Haberman's method could settle the conjectured regularity $\gamma \in L^d(\Omega)$ for Calderón's problem only if we are able to prove the inequality

$$\int_1^2 \int_{O_d} \sup_{\|u_\mu\|_2=\|v_\nu\|_2=1} |\langle (|D| f_{\tau U}) u_\mu, v_\nu \rangle| \, dU d\tau \leqslant C(\mu\nu)^{\frac{1}{2}} \|f\|_d, \tag{54}$$

where $f_{\tau U}(x) := \tau^{-d} f(\tau^{-1} U x)$, and $(|D|f)^\wedge(\xi) := |\xi| \hat{f}(\xi)$. I do not know of any example that shows that (54) is false; see the Appendix for a discussion on the non-averaged estimate. It is intriguing that Haberman's method only works for $d = 3, 4$.

The strategy that we have followed, as in previous works, is to apply Hölder inequality to the inner term and to bound the product $\|u_\mu v_\nu\|_{\frac{d}{d-1}}$, for which the following bounds were morally obtained:

$$\|u_\mu v_\nu\|_{\frac{d}{d-1}} \text{ ``} \lesssim \text{''} \begin{cases} \mu^{\frac{1}{2}+\frac{1}{d}} & \text{by Tomas-Stein; see (37),} \\ \rho^{-\frac{1}{d}} \mu^{\frac{1}{2}+\frac{1}{2d}} \nu^{\frac{1}{d}} & \text{by Theorem 8; see (45),} \\ \rho^{-\frac{1}{d}} \mu^{\frac{1}{4}+\frac{5}{4d}} \nu^{\frac{1}{4}+\frac{1}{4d}} & \text{by Tao's Theorem.} \end{cases}$$

Recall that we set $\|u_\mu\|_2 = \|v_\nu\|_2 = 1$ in (54), and that ρ is the parameter of transversality.

Tomas-Stein Theorem gives us the term $\mu^{\frac{1}{2}}$ and an additional term $\mu^{\frac{1}{d}}$, which does not suffice to offset the term $\nu^{\frac{1}{2}}$ unless $\mu \ll \nu$, but the averaging in Lemma 17 gives us an additional term $\nu^{\frac{1}{d}}$, with which the total gain is $(\mu\nu)^{\frac{1}{d}} \leqslant \nu^{\frac{2}{d}}$; hence, Tomas-Stein suffices to get (54) for $d = 3, 4$. The Bilinear Theorem gives us the term $\mu^{\frac{1}{2}}$ and an additional gain of $\mu^{\frac{1}{2d}} \nu^{\frac{1}{d}}$, plus $\nu^{\frac{1}{d}}$ after averaging to get $\mu^{\frac{1}{2d}} \nu^{\frac{2}{d}}$, which improves on Tomas-Stein as long as $\nu < \mu^{\frac{1}{2}}$, but not as much as to get $\nu^{\frac{1}{2}}$ for $d > 5$. In high dimensions we only get (54) when $\mu \ll \nu$, and the Bilinear Theorem falls short of getting (54) unless μ is very small.

The term $\rho^{-\frac{1}{d}}$ is almost irrelevant when we control non-antipodal caps, in which case $\rho \sim \lambda$ can be absorbed into the term λ; see (46) or the term A_1 in (53). On the other hand, the term $\rho^{-\frac{1}{d}}$ is especially troublesome when we control antipodal caps, in which case $\lambda \sim 1$; see the term A_2 in (53). We would get again Haberman's result, were not by the efficient averaging in Lemma 18, due to the smallness of the supports of $(u^\rho_{\mu,\alpha} \overline{v}^{\,\rho}_{\nu,\beta})^\vee$.

Even though it is sensible to try to place the exponent d in f when using Hölder to get $\|f\|_d$, this is not the only possibility. The projection $P^U_{\tau\lambda,\tau\nu}$ is roughly equal to the operator $f \mapsto \tau^d\lambda^{d-1}\nu f * \mathbb{1}_{T^U_{\tau\lambda,\tau\nu}}$, where $T^U_{\tau\lambda,\tau\nu}$ is a tube of length $(\tau\nu)^{-1}$ and radius $(\tau\lambda)^{-1}$ pointing in the direction Ue_d. It is worth noting that Lemma 17 for $p = 2$ is reminiscent of the smoothing estimate $\||D|^{\frac{1}{2}}Xf\|_2 \lesssim \|f\|_2$, where X is the X-ray transform $f \mapsto Xf(y,\omega) := \int_{l(y,\omega)} f$; the last integral is over the line $l(y,\omega) \subset \mathbb{R}^d$ with direction $\omega \in S^{d-1}$ passing through the point $y \in \mathbb{R}^{d-1} \cong \mathbb{R}^d/\omega\mathbb{R}$. The strongest conjectured smoothing effect for the X-ray transform is, up to ε-losses, $\|Xf\|_{L^d(\omega\mapsto L^\infty_y)} \lesssim \|f\|_d$, which is the famous Kakeya conjecture; for more information see [23, Ch. 22]. To exploit this conjecture we begin with

$$|\langle(|D|f_{\tau U})u_\mu, v_\nu\rangle| \lesssim \|P_\nu f_{\tau U}\|_\infty \|u_\mu v_\nu\|_1 \leqslant \|P_\nu f_{\tau U}\|_\infty \|u_\mu\|_2 \|v_\nu\|_2.$$

Since the projection satisfies

$$|P_\nu f_{\tau U}(x', x_d)| \leqslant \nu \int |f_{\tau U}(x', x_d)|\, dx_d = \nu X|f_\tau|(Ux', Ue_d),$$

we get by replacing it in (54) that

$$\begin{aligned}\int_{O_d} \sup_{\|u_\mu\|_2 = \|v_\nu\|_2 = 1} |\langle(|D|f_{\tau U})u_\mu, v_\nu\rangle|\, dU &\leqslant \nu \int_{O_d} \|X|f_\tau|(\cdot, Ue_d)\|_{L^\infty}\, dU \\ &\lesssim \nu \|f_\tau\|_d;\end{aligned}$$

the last inequality follows from Hölder and Kakeya conjecture. Hence, the Kakeya conjecture would allow us to control the terms $\mu \sim \nu$, but it would not imply yet the full inequality (54). Since Haberman got this inequality for $d = 3, 4$ without resorting to the full smoothing of Kakeya conjecture, then either (54) is false in general or we are overlooking a better method to bound the left side of (54), a method which would allow us to cover seamlessly every dimension $d \geqslant 3$

4 The Bilinear Theorem

In this Section we prove the Bilinear Theorem 8 for two open subsets of the paraboloid. The paraboloid is technically simpler, so the exposition runs more smoothly. After concluding the proof, we explain how we should modify the proof to get Theorem 8. The proof follows closely the ideas presented by Tao in [28], and we include here the argument for the sake of completeness.

Theorem 8'. *Suppose that S_1 and S_2 are two open subsets of the paraboloid in $\mathbb{R}^n$ with diameter $\lesssim 1$ and at distance ~ 1 of each other. If f_μ and g_ν are functions with Fourier transforms supported in a μ-neighborhood of S_1 and a ν-neighborhood of S_2 respectively, for $\mu \leqslant \nu < \mu^{\frac{1}{2}} < 1$, then for every $\varepsilon > 0$ it holds that*

$$\|f_\mu g_\nu\|_{p'} \leqslant C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f_\mu\|_2 \|g_\nu\|_2, \quad \text{for } 1 \leqslant p' \leqslant \frac{n}{n-1}. \tag{55}$$

The inequalities are best possible, up to ε-losses, in μ and ν.

We can restate the Theorem in terms of the quantity

$$K_{\mu,\nu}(p') := \sup_{\|f_\mu\|_2 = \|g_\nu\|_2 = 1} \|f_\mu g_\nu\|_{p'}.$$

We will bound $K_{\mu,\nu}(p')$ by an argument of induction in scales. In the examples below we will show that the upper bound $K_{\mu,\nu}(p') \leqslant C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon}$ is the best possible, up to ε-losses.

When $\mu^{\frac{1}{2}} \leqslant \nu$, the separation between supports does not yield any improvement on Theorem 12, at least in the range $1 \leqslant p' \leqslant \frac{n+1}{n}$.

Example 19 (Case $\mu^{\frac{1}{2}} \leqslant \nu$)**.** Let $N_\mu(S_1)$ and $N_\nu(S_2)$ be neighborhoods of two open subsets of the paraboloid with diameter ~ 1 and at distance ~ 1 of each other; see Figure 5. In $N_\mu(S_1)$ let C_1 be a cap of radius $\mu^{\frac{1}{2}}$ and width μ. In $N_\nu(S_2)$ let $C_2 := C_1 + a \subset N_\nu(S_2)$ for some vector a; this is possible owing to the hypothesis $\mu^{\frac{1}{2}} \leqslant \nu$. After replacing for $\widehat{u}_\mu = \mathbb{1}_{C_1}$ and for $\widehat{v}_\nu = \mathbb{1}_{C_2}$ in the bilinear inequality, we get $K_{\mu,\nu}(p') \geqslant c\mu^{\frac{n+1}{2p}}$.

Theorem 8 holds in $\mathbb{R}^2$ without ε-losses. The proof is by averaging over translations of the parabola; see for example Lemma 2.4 in [20].

Example 20 (Case $\mathbb{R}^2$ and $\mu \leqslant \nu \leqslant \mu^{\frac{1}{2}}$)**.** Let $N_\mu(S_1)$ and $N_\nu(S_2)$ be separated in the parabola as in Theorem 8'. In $N_\mu(S_1)$ let C_1 be a cap of radius ν and width μ. In $N_\nu(S_2)$ let $C_2 := C_1 + a \subset N_\nu(S_2)$ for some vector a. After replacing for $\widehat{u}_\mu = \mathbb{1}_{C_1}$ and for $\widehat{v}_\nu = \mathbb{1}_{C_2}$ in the bilinear inequality, we get $K_{\mu,\nu}(p') \geqslant c\mu^{\frac{1}{p}}\nu^{\frac{1}{p}}$.

In higher dimensions we consider as example a modification of the squashed caps in Section 2.7 of [29].

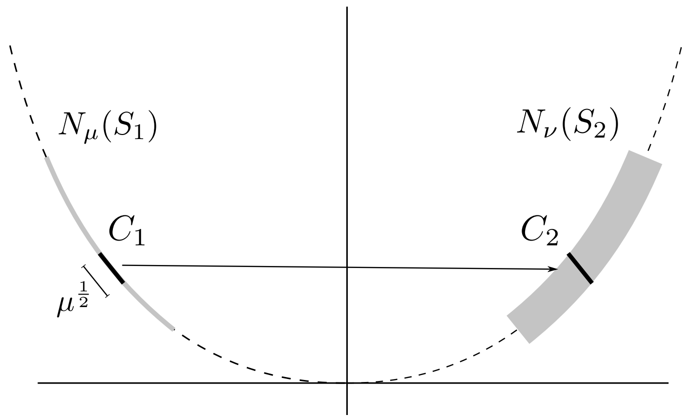

Figure 5: The case $\mu^{\frac{1}{2}} \leqslant \nu$ in Example 19.

Example 21 (Case $n \geqslant 3$ and $\mu \leqslant \nu \leqslant \mu^{\frac{1}{2}}$)**.** Let $N_\mu(S_1)$ and $N_\nu(S_2)$ be separated in the paraboloid as in Theorem 8'. Let $L_\mu \subset \mathbb{R}^{n-1}$ be a $\mu^{\frac{1}{2}}$-neighborhood of the plane $\{x_1 = \cdots = x_{n-2} = 0\}$. In L_μ choose a box $\widetilde{C}_1$ of dimensions $\nu \times \mu^{\frac{1}{2}} \times \cdots \times \mu^{\frac{1}{2}}$, so that its lift to the paraboloid lies in S_1, and thicken it in $N_\mu(S_1)$ creating so a cap C_1 of dimensions $\nu \times \mu^{\frac{1}{2}} \times \cdots \times \mu^{\frac{1}{2}} \times \mu$; see Figure 6. Now, let $C_2 := C_1 + a \subset N_\nu(S_2)$ for some vector a. After replacing for $\widehat{u}_\mu = \mathbb{1}_{C_1}$ and for $\widehat{v}_\nu = \mathbb{1}_{C_2}$ in the bilinear inequality, we get $K_{\mu,\nu}(p') \geqslant c\mu^{\frac{n}{2p}}\nu^{\frac{1}{p}}$.

The rest of this Section is devoted to the proof of the inequality (55) in Theorem 8'. We do first some reductions.

By Galilean and rotational symmetry, we can assume that

$$\begin{aligned} S_1 &= \{(\xi', \frac{1}{2}|\xi'|^2) \mid |\xi' - c_1 e_1| \leqslant c_2\} \\ S_2 &= \{(\xi', \frac{1}{2}|\xi'|^2) \mid |\xi' + c_1 e_1| \leqslant c_2\}; \end{aligned}$$

the constant C_ε in (55) depends on c_1 and c_2.

It suffices to prove the local inequality

$$\|f_\mu g_\nu\|_{L^{p'}(B_{\mu^{-1}})} \leqslant C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon}\nu^{\frac{1}{p}-\varepsilon}\|f_\mu\|_2\|g_\nu\|_2. \tag{56}$$

In fact, cover $\mathbb{R}^n$ with balls $B_{\mu^{-1}}$ and choose a bump function $\zeta_{B_\mu^{-1}} \sim 1$ in

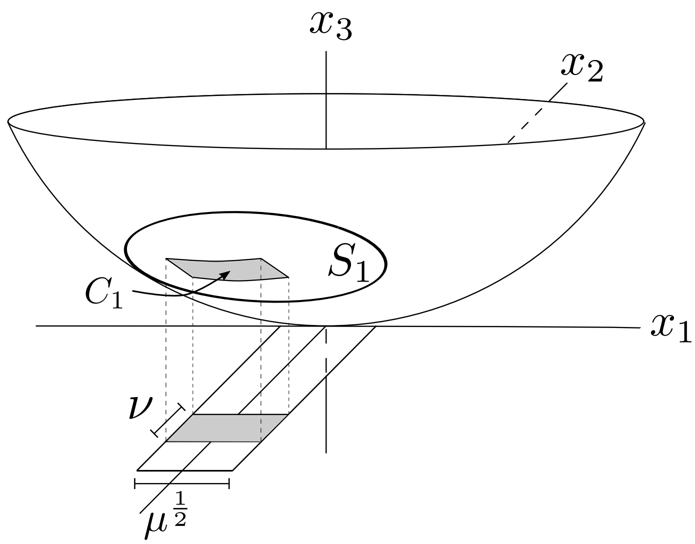

Figure 6: The construction of the cap C_1 in Example 21.

$B_{\mu^{-1}}$ so that $\operatorname{supp}\widehat{\zeta}_{B_\mu^{-1}} \subset B_\mu(0)$. Then,

$$\begin{aligned}\|f_\mu g_\nu\|_{p'} &\leqslant \sum_{B_{\mu^{-1}}} \|f_\mu g_\nu\|_{L^{p'}(B_{\mu^{-1}})}\\ &\lesssim \sum_{B_{\mu^{-1}}} \|(\widehat{f}_\mu * \widehat{\zeta}_{B_\mu^{-1}})^\vee (\widehat{g}_\nu * \widehat{\zeta}_{B_\mu^{-1}})^\vee\|_{L^{p'}(B_{\mu^{-1}})}\end{aligned}$$

The width of the supports of $\widehat{f}_\mu * \widehat{\zeta}_{B_\mu^{-1}}$ and $\widehat{g}_\nu * \widehat{\zeta}_{B_\mu^{-1}}$ are essentially μ and ν respectively. Hence, we can apply the local bilinear inequality (56) to get

$$\begin{aligned}\|f_\mu g_\nu\|_{p'} &\leqslant C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \sum_{B_{\mu^{-1}}} \|f_\mu \zeta_{B_{\mu^{-1}}}\|_2 \|g_\nu \zeta_{B_{\mu^{-1}}}\|_2\\ &\leqslant C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \Big(\sum_{B_{\mu^{-1}}} \|f_\mu \zeta_{B_{\mu^{-1}}}\|_2^2\Big)^{\frac{1}{2}} \Big(\sum_{B_{\mu^{-1}}} \|g_\nu \zeta_{B_{\mu^{-1}}}\|_2^2\Big)^{\frac{1}{2}}\\ &\leqslant C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f_\mu\|_2 \|g_\nu\|_2,\end{aligned}$$

which is what we wanted to prove.

At scale μ^{-1} the function f_μ looks like $(f dS)^\vee$ for some function f in the paraboloid, so it suffices to prove the following Theorem.

Theorem 22. *Suppose that S_1 and S_2 are two open subsets of the paraboloid in $\mathbb{R}^n$ with diameter ~ 1 and at distance ~ 1 of each other. If fdS is a measure supported in S_1 and g_ν a function with Fourier transform supported in a ν-neighborhood of S_2, then for $1 < R^{\frac{1}{2}} \leqslant \nu^{-1} \leqslant R$ and for every $\varepsilon > 0$ it holds*

$$\|(fdS)^\vee g_\nu\|_{L^{p'}(B_R)} \leqslant C_\varepsilon R^{\frac{1}{2}(1-\frac{n}{p})+\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f\|_{L^2(S)} \|g_\nu\|_2, \tag{57}$$

where $1 \leqslant p' \leqslant \frac{n}{n-1}$.

In fact, after a change of variables $\xi \mapsto (\xi', \frac{1}{2}|\xi'|^2 + t)$ we can write f_μ as

$$\begin{aligned} f_\mu(x) &= \int_{-\mu}^{\mu} \Big(\int \widehat{f}_\mu(\xi', \frac{1}{2}|\xi|^2 + t) e(\langle x', \xi' \rangle + x_n \frac{1}{2}|\xi'|^2)\, d\xi \Big) e(x_n t)\, dt \\ &= \int_{-\mu}^{\mu} (\widehat{f}_{\mu,t} dS)^\vee e(x_n t)\, dt, \end{aligned} \tag{58}$$

where $\widehat{f}_{\mu,t}$ is a parabolic slice of $\widehat{f}_\mu$. To bound the local bilinear inequality (56) we use Minkowski to get

$$\|f_\mu g_\nu\|_{L^{p'}(B_{\mu^{-1}})} \leqslant \int_{-\mu}^{\mu} \|(\widehat{f}_{\mu,t} dS)^\vee g_\nu\|_{L^{p'}(B_{\mu^{-1}})}\, dt.$$

Then, writing $\mu^{-1} = R$, we can use Theorem 22 and Cauchy-Schwarz inequality to get

$$\begin{aligned} \|f_\mu g_\nu\|_{p'} &\leqslant C_\varepsilon \mu^{\frac{1}{2}(\frac{n}{p}-1)-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \int_{-\mu}^{\mu} \|f_{\mu,t}\|_2\, dt\ \|g_\nu\|_2 \\ &\leqslant C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f_\mu\|_2 \|g_\nu\|_2. \end{aligned}$$

Therefore, we must prove now Theorem 22.

The point $p' = 1$ of Theorem 22 can be proven readily. By Cauchy-Schwarz and by the trace inequality $\|(fdS)^\vee\|_2 \leqslant CR^{\frac{1}{2}} \|f\|_{L^2(S)}$ we get

$$\|(fdS)^\vee g_\nu\|_{L^1(B_R)} \leqslant CR^{\frac{1}{2}} \|f\|_2 \|g_\nu\|_2.$$

Hence, it suffices to prove the inequality (57) at the point $p' = \frac{n}{n-1}$.

We begin the proof in the next Section with the wave packet decomposition. This decomposition is nowadays a classical change of basis, so we only outline it.

4.1 Wave Packet Decomposition

Let f be a function in $\mathbb{R}^{n-1}$, and decompose the space into caps α of radius $R^{-\frac{1}{2}}$ and center $c_\alpha \in \mathbb{R}^{n-1}$. Choose a smooth partition of unity $\{\zeta_\alpha\}$ adapted to the caps α so that $\sum_\alpha \zeta_\alpha^2 = 1$. Use Fourier series adapted to each α to expand $f\zeta_\alpha$ into frequencies ω, and develop f as

$$f(\xi) = |\alpha|^{-\frac{1}{2}} \sum_{\alpha,\omega} a(\alpha,\omega)\zeta_\alpha(\xi)e(\langle \omega, \xi - c_\alpha\rangle),$$

where $\omega = R^{\frac{1}{2}}\mathbb{Z}^{n-1}$. The coefficients a satisfy the next properties:

$$a(\alpha,\omega) = \frac{1}{|\alpha|^{\frac{1}{2}}} \int f\zeta_\alpha e(-\langle \omega, \xi - c_\alpha\rangle)\, d\xi, \tag{59}$$

$$\sum_{\alpha,\omega} |a(\alpha,\omega)|^2 = \|f\|_2^2. \tag{60}$$

By the linearity of the extension operator, we can write $(fdS)^\vee$ as

$$(fdS)^\vee(x) = \sum_{\alpha,\omega} a(\alpha,\omega)\phi_{T(\alpha,\omega)},$$

where ϕ_T is a function essentially supported in a tube T of dimensions $R^{\frac{1}{2}} \times \cdots \times R^{\frac{1}{2}} \times R$; the angle and position of T are determined by α and ω respectively. Furthermore,

$$|\phi_T(x)| \leqslant C_M R^{-\frac{n-1}{2}} \frac{1}{\langle R^{-\frac{1}{2}}(x' + \omega + x_n c_\alpha)\rangle^M}, \quad \text{for } |x_n| \leqslant R;$$

so ϕ_T is concentrated in a tube T of direction $(-c_\alpha, 1)$ whose main axis passes through $(-\omega, 0)$. We deduce also that for $\delta > 0$, for $x \notin R^\delta T$, and for $|x_n| \leqslant R$ it holds

$$|\phi_T(x)| \leqslant C_\delta R^{-100n}, \tag{61}$$

where possibly $C_\delta \to \infty$ as $\delta \to 0$.

The function g_ν can be written similarly. We decompose $N_\nu(S_2)$ into rectangles β of dimensions $\nu \times R^{-\frac{1}{2}} \times \cdots \times R^{-\frac{1}{2}}$ and center $c_\beta \in \mathbb{R}^n$, where c_β is now a point in S_2. Arguing as before we have

$$\widehat{g}_\nu(\xi) = |\beta|^{-\frac{1}{2}} \sum_{\beta,\omega} b(\alpha,\omega)\zeta_\beta(\xi)e(\langle \omega, \xi - c_\beta\rangle),$$

where ω belongs to some rotation of the grid $\nu^{-1}\mathbb{Z} \times R^{\frac{1}{2}}\mathbb{Z}^{n-1}$. Again, we get

$$b(\beta, \omega) = \frac{1}{|\beta|^{\frac{1}{2}}} \int \widehat{g}_\nu \zeta_\beta e(-\langle \omega, \xi - c_\beta \rangle)\, d\xi \tag{62}$$

$$\sum_{\beta,\omega} |b(\beta, \omega)|^2 = \|g_\nu\|_2^2. \tag{63}$$

By the linearity of the Fourier transform, we can write g_ν as

$$g_\nu = \sum_{\beta,\omega} b(\beta, \omega) \phi_{T(\beta,\omega)},$$

where T are now tubes of dimensions $\nu^{-1} \times R^{\frac{1}{2}} \times \cdots \times R^{\frac{1}{2}}$. Again, we get

$$\begin{aligned} |\phi_T(x)| &\leqslant C_M \nu R^{-\frac{n-1}{2}} \frac{1}{\langle R^{-\frac{1}{2}} |x' + \omega' + x_n c'_\beta| + \nu |x_n + \omega_n| \rangle^M}, \\ |\phi_T(x)| &\leqslant C_\delta \nu R^{-100n}, \quad \text{for } x \notin R^\delta T \text{ and for } \delta > 0. \end{aligned} \tag{64}$$

We replace the wave packet decomposition into the bilinear inequality (57), so we must prove that for $\|a\|_2 = 1$ and $\|b\|_2 = 1$ we have

$$\| \sum_{T_1,T_2} a_{T_1} b_{T_2} \phi_{T_1} \phi_{T_2} \|_{L^{\frac{n}{n-1}}(B_R)} \leqslant C_\varepsilon R^\varepsilon \nu^{\frac{1}{n} - \varepsilon}.$$

Since $|\phi_{T_1}|$ and $|\phi_{T_2}|$ decay strongly outside the tubes, then we can ignore all the tubes that do not intersect the ball $10B_R$, so the number of tubes in each group is $\lesssim R^{Cn}$; recall that $\nu^{-1} \geqslant R^{\frac{1}{2}}$.

Now, for all the terms that satisfy $|a_{T_1}|$ or $|b_{T_2}| \lesssim R^{-C_n}$ the contribution to the bilinear inequality is negligible, so we can ignore all these terms and do pigeonholing in $|a_{T_1}|$ and $|b_{T_2}|$; here, we introduce logarithmic losses. Hence, for two collections of tubes $\mathbb{T}_1$ and $\mathbb{T}_2$ that intersect the ball $10B_R$ we must prove that

$$\| \sum_{T_1 \in \mathbb{T}_1, T_2 \in \mathbb{T}_2} \phi_{T_1} \phi_{T_2} \|_{L^{\frac{n}{n-1}}(B_R)} \leqslant C_\varepsilon R^\varepsilon \nu^{\frac{1}{n} - \varepsilon} |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}. \tag{65}$$

The proof of this inequality begins with an induction on scales in the next Section.

4.2 Induction on Scales

We want to control the quantity

$$K_\nu(R) := \sup_{\|f\|_2=\|g_\nu\|_2=1} \|(f dS)^\vee g_\nu\|_{L^{p'}(B_R)}.$$

Rough estimates show that $K_\nu(R)$ is finite, thus well defined, and we want to prove that $K_\nu(R) \leqslant C_\varepsilon R^\varepsilon \nu^{\frac{1}{n}-\varepsilon}$.

The induction on scales consists in controlling $K_\nu(R)$ in terms of $K_\nu(R^{1-\delta})$ for some $\delta > 0$, which we keep fixed in what follows, so we lower scales and stop at scale $\sim \nu^{-1}$, when Tao's Bilinear Theorem provides the best possible upper bound, up to ε-losses. From now on, we write R' instead of $R^{1-\delta}$.

We begin the induction by breaking up the ball B_R into balls $B_{R'}$. Now, we define a relationship between balls and tubes, so that a tube is related to a ball if the contribution of ϕ_T to the bilinear term is large in that ball. We need first decompose B_R into balls q of radius $R^{\frac{1}{2}}$, and now we introduce the following set of definitions for a dyadic number μ_2:

$$\mathbb{T}_2(q) := \{T_2 \in \mathbb{T}_2 \mid R^\delta T_2 \cap q \neq \varnothing\} \tag{66}$$

$$q(\mu_2) := \{q \subset B_R \mid \mu_2 \leqslant |\mathbb{T}_2(q)| < 2\mu_2\} \tag{67}$$

$$\lambda(T_1, \mu_2, B_{R'}) := |\{q \in q(\mu_2) \mid q \subset B_{R'} \text{ and } R^\delta T_1 \cap q \neq \varnothing\}|. \tag{68}$$

Definition 23 (Relation between tubes and balls)**.** For every number μ_2 and every tube $T_1 \in \mathbb{T}_1$ choose a ball $B^*_{R'}(\mu_2, T_1)$, if it exists, such that

$$\lambda(T_1, \mu_2, B^*_{R'}) = \max_{B_{R'}} \lambda(T_1, \mu_2, B_{R'}) > 0.$$

We say that a tube $T_1 \in \mathbb{T}_1$ is related to a ball $B_{R'} \subset B_R$, or $T_1 \sim B_{R'}$, if $B_{R'} \subset 10B^*_{R'}(\mu_2, T_1)$ for some μ_2. The negation of $T_1 \sim B_{R'}$ is $T_1 \nsim B_{R'}$. Symmetrically, we can define a relation between tubes $T_2 \in \mathbb{T}_2$ and balls $B_{R'}$.

Every tube in $\mathbb{T}_j$ intersects a number $\lesssim R^\delta$ of balls $B_{R'} \subset B_R$, but each tube is related only to $\lesssim \log R$ balls. The latter follows from the fact that μ_2 is dyadic and that $1 \leqslant \mu_2 \lesssim R^{\frac{n-1}{2}+C\delta}$.

Now, we bound the bilinear term as

$$
\begin{aligned}
\| \sum_{\substack{T_1 \in \mathbb{T}_1 \\ T_2 \in \mathbb{T}_2}} \phi_{T_1}\phi_{T_2} \|_{L^{p'}(B_R)} &\leqslant \sum_{B_{R'} \subset B_R} \| \sum_{T_1, T_2} \phi_{T_1}\phi_{T_2} \|_{L^{p'}(B_{R'})} \\
&\leqslant \sum_{B_{R'} \subset B_R} \Big(\| \sum_{T_1 \sim B_{R'}, T_2 \sim B_{R'}} \phi_{T_1}\phi_{T_2} \|_{L^{p'}(B_{R'})} + \\
&+ \| \sum_{T_1 \nsim B_{R'}, T_2} \phi_{T_1}\phi_{T_2} \|_{L^{p'}(B_{R'})} + \| \sum_{T_1 \sim B_{R'}, T_2 \nsim B_{R'}} \phi_{T_1}\phi_{T_2} \|_{L^{p'}(B_{R'})} \Big). \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} \qquad (69)
\end{aligned}
$$

For the first term I at the right we use the inductive hypothesis, Cauchy-Schwarz, and the bound $|\{B_{R'} \mid T_j \sim B_{R'}\}| \lesssim \log R$ to get

$$
\begin{aligned}
\sum_{B_{R'} \subset B_R} \| \sum_{\substack{T_1 \sim B_{R'} \\ T_2 \sim B_{R'}}} \phi_{T_1}\phi_{T_2} \|_{L^{p'}(B_{R'})} &\leqslant K(R') \sum_{B_{R'} \subset B_R} |\{T_1 \sim B_{R'}\}|^{\frac{1}{2}} |\{T_2 \sim B_{R'}\}|^{\frac{1}{2}} \\
&\leqslant K(R') \Big(\sum_{B_{R'}, T_1} \mathbb{1}_{\{T_1 \sim B_{R'}\}} \Big)^{\frac{1}{2}} \Big(\sum_{B_{R'}, T_2} \mathbb{1}_{\{T_2 \sim B_{R'}\}} \Big)^{\frac{1}{2}} \\
&\leqslant C(\log R) K(R') |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}. \qquad (70)
\end{aligned}
$$

We have bounded so the main contribution with an acceptable logarithmic loss.

We turn now to II in (69); the term III can be similarly controlled, so we will not describe it. We bound the $L^{\frac{n}{n-1}}$-norm by interpolation between the points $p' = 1$ and $p' = 2$. For $p' = 1$ we use Cauchy-Schwarz and the trace inequality to get

$$
\| \sum_{T_1 \nsim B_{R'}, T_2} \phi_{T_1}\phi_{T_2} \|_{L^1(B_{R'})} \lesssim R^{\frac{1}{2}} |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}; \qquad (71)
$$

recall that $\sum_{T_1 \nsim B_{R'}} \phi_{T_1} = (f dS)^\vee$ for some function f in S, and $\sum_{T_2} \phi_{T_2} = g_\nu$ for some function g_ν, so we only applied the Trace Theorem to $(f dS)^\vee$, and used (60) and (63). We are left with the point $p' = 2$.

If we are to prove (65) by interpolation, we must get the upper bound

$$
\| \sum_{T_1 \nsim B_{R'}, T_2} \phi_{T_1}\phi_{T_2} \|_{L^2(B_{R'})} \lesssim_\delta R^{\frac{1}{2}(1-\frac{n}{2})+C\delta} \nu^{\frac{1}{2}} |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}.
$$

This inequality is in general false, if we do not put some constrains over the tubes. The simple example $f = 1$ and $g_\nu = 1$ in $N_\nu(S_2)$ is enough, and worst examples can be given. Hence, we have to exploit the special structure of the tubes $T_1 \not\sim B_{R'}$.

We use the decomposition of B_R into cubes q of radius $R^{\frac{1}{2}}$ and the definition (67) to write the L^2-norm as

$$\| \sum_{T_1 \not\sim B_{R'}, T_2} \phi_{T_1} \phi_{T_2} \|^2_{L^2(B_{R'})} = \sum_{\mu_2} \sum_{q \in q(\mu_2)} \| \sum_{T_1 \not\sim B_{R'}, T_2} \phi_{T_1} \phi_{T_2} \|^2_{L^2(q)}.$$

By pigeonholing, it suffices to control the norm for a fixed μ_2. We introduce now the definitions

$$\lambda(T_1, \mu_2) := |\{q \in q(\mu_2) \mid R^\delta T_1 \cap q \neq \varnothing\}| \tag{72}$$

$$\mathbb{T}_1[\mu_2, \lambda_1] := \{T_1 \in \mathbb{T}_1 \mid \lambda_1 \leqslant \lambda(T_1, \mu_2) < 2\lambda_1\}. \tag{73}$$

Since $1 \leqslant \lambda_1 \lesssim R^{\frac{1}{2}+C\delta}$, by pigeonholing again it suffices to prove

$$\sum_{q \in q(\mu_2)} \| \sum_{\substack{T_1 \not\sim B_{R'}, T_1 \in \mathbb{T}_1[\mu_2, \lambda_1] \\ T_2}} \phi_{T_1} \phi_{T_2} \|^2_{L^2(q)} \lesssim_\delta R^{1-\frac{n}{2}+C\delta} \nu |\mathbb{T}_1| |\mathbb{T}_2|. \tag{74}$$

The case $\lambda(T_1, \mu_2) = 0$ is handled with (61). In the next Section we use the special nature of the L^2-norm to decouple the frequencies.

4.3 Decoupling at Scale $R^{\frac{1}{2}}$

We need first a L^2 upper bound of the bilinear operator. Recall that the extension operator is defined as

$$(f dS)^\vee(x) = \int_{\mathbb{R}^{n-1}} f(\xi) e(\langle x', \xi' \rangle + x_n \varphi(\xi'))\, d\xi',$$

where $\varphi(\xi') = \frac{1}{2}|\xi'|^2$ and $\xi = (\xi', \xi_n)$. For an open subset S_1 of the paraboloid, we denote by $\pi(S_1)$ its projection to $\mathbb{R}^{n-1}$.

We need also the Radon transform of a function, and we define it as

$$Rf(\xi', \theta) := \int_{\mathbb{R}^{n-1}} f(\xi' + \eta) \delta\big(\langle \eta, \theta \rangle\big)\, d\eta;$$

the Radon transform $Rf(\xi', \theta)$ is the integral over the hyperplane with normal θ that passes through ξ'.

Lemma 24. *Let S_1 and S_2 be two open subsets of the paraboloid with radius ~ 1 and at distance ~ 1 of each other. Suppose that fdS and gdS are measures with support in S_1 and S_2 respectively. Then, it holds that*

$$\|(fdS)^\vee(gdS)^\vee\|_2^2 \leqslant C\|f\|_1 \sup_{\substack{\xi'\in\pi(S_1)\\ \xi''\in\pi(S_2)}} R|f|\big(\xi', \frac{\xi'-\xi''}{|\xi'-\xi''|}\big)\|g\|_1\|g\|_\infty \tag{75}$$

Proof. We compute the square of the extension operator as

$$\begin{aligned}|(fdS)^\vee(x)|^2 &= \int_{\mathbb{R}^{2(n-1)}} f(\xi_1'+\xi_2')\overline{f}\,(\xi_2') \\ &\qquad e(\langle x',\xi_1'\rangle + x_n(\varphi(\xi_1'+\xi_2')-\varphi(\xi_2')))\,d\xi_1'd\xi_2' \\ &= \int\Big(\int f(\xi_1'+\xi_2')\overline{f}\,(\xi_2')\delta(\varphi(\xi_1'+\xi_2')-\varphi(\xi_2')-t)\,d\xi_2'\Big)e(\langle x,\xi_1\rangle)\,d\xi_1 \\ &:= \check{F}(x),\end{aligned}$$

where F is the function in parentheses. Thus, we get

$$\|(fdS)^\vee(gdS)^\vee\|_2^2 = \int (F*G)^\vee(x)\,dx = (F*G)^\vee(0).$$

We develop the convolution and change variables, so that

$$\begin{aligned}&\|(fdS)^\vee(gdS)^\vee\|_2^2 = \int f(\xi_2')\overline{g}\,(\xi_2'') \\ &\int \overline{f}\,(\xi_2'+\xi_1')g(\xi_2''+\xi_1')\delta(\varphi(\xi_2')-\varphi(\xi_1'+\xi_2')+\xi_{1,n})\delta(\varphi(\xi_1'+\xi_2'')-\varphi(\xi_2'')-\xi_{1,n})\,d\xi_1 \\ &\qquad\qquad d\xi_2'd\xi_2''.\end{aligned}$$

We can use Fubini to put inside the integral with respect to $\xi_{1,n}$, so that after the change of variables $\xi_{1,n}\mapsto \xi_{1,n}+\varphi(\xi_1'+\xi_2'')-\varphi(\xi_2'')$ we get

$$\begin{aligned}I &:= \int \delta(\varphi(\xi_2')-\varphi(\xi_1'+\xi_2')+\xi_{1,n})\delta(\varphi(\xi_1'+\xi_2'')-\varphi(\xi_2'')-\xi_{1,n})\,d\xi_{1,n} \\ &= \delta(\langle\xi_1',\xi_2'-\xi_2''\rangle).\end{aligned} \tag{76}$$

Then, the L^2 norm gets into

$$\begin{aligned}\|(fdS)^\vee(gdS)^\vee\|_2^2 &\leqslant \int |f|(\xi_2')|g|(\xi_2'')\int |f|(\xi_2'+\xi_1')|g|(\xi_2''+\xi_1')\delta(\langle\xi_1',\xi_2'-\xi_2''\rangle)\,d\xi_1'd\xi_2'd\xi_2'' \\ &\leqslant \|f\|_1\|g\|_1\|g\|_\infty \sup_{\xi_2',\xi_2''}\int |f|(\xi_2'+\xi_1')\delta(\langle\xi_1',\xi_2'-\xi_2''\rangle)\,d\xi_1'.\end{aligned}$$

Finally, by the identity $\delta(at) = a^{-1}\delta(t)$, and the condition of separation between S_1 and S_2, we get

$$\int |f|(\xi_2' + \xi_1')\delta(\langle \xi_1', \xi_2' - \xi_2''\rangle)\, d\xi_1' \leqslant CR|f|\big(\xi_2', \frac{\xi_2' - \xi_2''}{|\xi_2' - \xi_2''|}\big),$$

which concludes the proof. □

We use now Lemma 24 to bound each term at the left side of the inequality (74). To simplify, let us define $\mathbb{T}_1' := \{T_1 \nsim B_{R'}\} \cap \mathbb{T}_1[\mu_2, \lambda_1]$. By (59) and (62) we can neglect the contribution from tubes such that $R^\delta T \cap q = \varnothing$. We define so the functions

$$\begin{aligned} f_q(\xi) &:= |\alpha|^{-\frac{1}{2}} \sum_{T_1 \in \mathbb{T}_1'(q)} \zeta_\alpha(\xi) e(\langle \omega, \xi - c_\alpha\rangle) \\ \widehat{g}_{\nu,q}(\xi) &:= |\beta|^{-\frac{1}{2}} \sum_{T_2 \in \mathbb{T}_2(q)} \zeta_\beta(\xi) e(\langle \omega, \xi - c_\beta\rangle). \end{aligned}$$

We write $g_{\nu,q}$ as an average over paraboloids as in (58), and by Minkowski and Cauchy-Schwarz we get

$$\begin{aligned} \| \sum_{T_1 \in \mathbb{T}_1'(q), T_2 \in \mathbb{T}_2(q)} \phi_{T_1}\phi_{T_2}\|_{L^2(q)}^2 &\leqslant \|(f_q dS)^\vee g_{\nu,q}\|_2^2 \\ &\leqslant \|(f_q dS)^\vee \int (\widehat{g}_{\nu,q}^t dS)^\vee e(x_n t)\, dt\|_2^2 \\ &\leqslant \nu \int \|(f_q dS)^\vee (\widehat{g}_{\nu,q}^t dS)^\vee\|_2^2\, dt \end{aligned}$$

We apply Lemma 24 to the integrand, using the inequalities

$$\|f_q\|_1 \leqslant R^{-\frac{n-1}{4}}|\mathbb{T}_1'(q)|$$
$$\|\widehat{g}_{\nu,q}^t\|_1 \leqslant \nu^{-\frac{1}{2}} R^{-\frac{n-1}{4}}|\mathbb{T}_2(q)|, \qquad \|\widehat{g}_{\nu,q}^t\|_\infty \leqslant \nu^{-\frac{1}{2}} R^{\frac{n-1}{4}+C\delta},$$

to get

$$\| \sum_{\substack{T_1 \in \mathbb{T}_1'(q) \\ T_2 \in \mathbb{T}_2(q)}} \phi_{T_1}\phi_{T_2}\|_{L^2(q)}^2 \leqslant C\nu R^{-\frac{n-1}{4}+C\delta}|\mathbb{T}_1'(q)||\mathbb{T}_2(q)| \sup_{\substack{\xi' \in \pi(S_1) \\ \xi'' \in \pi(S_2)}} R|f_q|\big(\xi', \frac{\xi' - \xi''}{|\xi' - \xi''|}\big). \tag{77}$$

Let $\mathbb{T}_1'(q)(\xi', \xi' - \xi'')$ denote the collection of tubes in $\mathbb{T}_1'(q)$ such that the corresponding cap α intersects the hyperplane with normal $(\xi' - \xi'')/|\xi' - \xi''|$ that passes through ξ'. Then,

$$\sup_{\substack{\xi' \in \pi(S_1) \\ \xi'' \in \pi(S_2)}} R|f_q|\big(\xi', \frac{\xi' - \xi''}{|\xi' - \xi''|}\big) \leqslant R^{-\frac{n-1}{4}+\frac{1}{2}} \sup_{\substack{\xi' \in \pi(S_1) \\ \xi'' \in \pi(S_2)}} |\mathbb{T}_1'(q)(\xi', \xi' - \xi'')|$$
$$:= R^{-\frac{n-1}{4}+\frac{1}{2}} \nu(q, \mu_2, \lambda_1);$$

in the last definition we use the same notation as Tao in [28]. We replace in (77) to find

$$\| \sum_{T_1 \in \mathbb{T}_1'(q), T_2 \in \mathbb{T}_2(q)} \phi_{T_1} \phi_{T_2} \|^2_{L^2(q)} \leqslant C\nu R^{1-\frac{n}{2}+C\delta} \nu(q, \mu_2, \lambda_1) |\mathbb{T}_1'(q)| |\mathbb{T}_2(q)|,$$

where $\mathbb{T}_1' := \{T_1 \nsim B_{R'}\} \cap \mathbb{T}_1[\mu_2, \lambda_1]$. Summing over all the cubes $q \in q(\mu_2)$ we get

$$\sum_{q \in q(\mu_2)} \| \sum_{T_1 \in \mathbb{T}_1'(q), T_2 \in \mathbb{T}_2(q)} \phi_{T_1} \phi_{T_2} \|^2_{L^2(q)} \leqslant C\nu R^{1-\frac{n}{2}+C\delta} \sum_{q \in q(\mu_2)} \nu(q, \mu_2, \lambda_1) |\mathbb{T}_1'(q)| |\mathbb{T}_2(q)|. \tag{78}$$

The term at the right does not involve oscillations, so we achieved a decoupling of the oscillating tubes at the left. To conclude the proof of (74), we must get an upper bound of $\nu(q, \mu_2, \lambda_1)$, which we do in the next Section.

4.4 A Kakeya-type Estimate

In this Section we aim to prove the inequality

$$\nu(q_0, \mu_2, \lambda_1) \lesssim R^{C\delta} \frac{|\mathbb{T}_2|}{\mu_2 \lambda_1}, \tag{79}$$

for some fixed $q_0 \in q(\mu_2)$, μ_2 and λ_1. For any $\xi' \in \pi(S_1)$ and $\xi'' \in \pi(S_2)$ we consider then the following bilinear expression

$$B := \int_{\substack{q \in q(\mu_2) \\ B_R \setminus 10 B_{R'}}} \sum_{T_1 \in \mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')} \mathbb{1}_{2R^\delta T_1} \sum_{T_2 \in \mathbb{T}_2} \mathbb{1}_{2R^\delta T_2}.$$

By the definition of $q(\mu_2)$ we get

$$B \gtrsim \mu_2 \sum_{T_1 \in \mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')} \int_{\substack{q \in q(\mu_2) \\ B_R \setminus 10 B_{R'}}} \mathbb{1}_{2R^\delta T_1}.$$

Since for $T_1 \in \{T_1 \not\sim B_{R'}\} \cap \mathbb{T}_1[\mu_2, \lambda_1]$ it holds that $|\{q \in q(\mu_2) \mid R^\delta T_1 \cap q \neq \varnothing\}| \sim \lambda_1$, we see that

$$|\{q \in q(\mu_2) \mid q \subset B_R \setminus 10 B_{R'} \text{ and } R^\delta T_1 \cap q \neq \varnothing\}| \gtrsim R^{-\delta} \lambda_1.$$

Then,

$$B \gtrsim R^{\frac{n}{2} - C\delta} \lambda_1 \mu_2 |\mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')| \tag{80}$$

To get an upper bound of B, we re-order the summations so that

$$B \leqslant \sum_{T_2 \in \mathbb{T}_2} \int_{B_R \setminus 10 B_{R'}} \mathbb{1}_{2R^\delta T_2} \sum_{T_1 \in \mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')} \mathbb{1}_{2R^\delta T_1}.$$

Since all the tubes intersect $q_0 \subset B_{R'}$, we see that

$$\sum_{T_1 \in \mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')} \mathbb{1}_{2R^\delta T_1}(x) \lesssim R^{C\delta} \quad \text{for } x \in B_R \setminus 10 B_{R'}.$$

The tubes in $\mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')$ have directions $(-c_\alpha, 1)$, where c_α lies at distance $< R^{-\frac{1}{2}}$ from a hyperplane with normal direction $\xi' - \xi''$ that passes through ξ'. Then, the main axis of all the tubes in $\mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')$ make an angle $< R^{-\frac{1}{2}}$ with a hyperplane with normal direction $(\xi' - \xi'', \langle \xi', \xi' - \xi'' \rangle)$ that passes through q_0. It amounts to saying that the support of $\sum_{T_1 \in \mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')} \mathbb{1}_{2R^\delta T_1}$ lies inside the $R^{\frac{1}{2}+\delta}$-neighborhood of a hyperplane that passes through q_0. Furthermore, every tube from $\mathbb{T}_2$ intersects the hyperplane transversally, making an angle $> c$ uniformly. Then,

$$B \lesssim R^{\frac{n}{2} + C\delta} |\mathbb{T}_2|. \tag{81}$$

We use (80) and (81) to conclude that

$$|\mathbb{T}_1'(q_0)(\xi', \xi' - \xi'')| \lesssim R^{C\delta} \frac{|\mathbb{T}_2|}{\lambda_1 \mu_2},$$

which is what we wanted to prove.

4.5 End of the Proof

In this Section we reap the fruits of all the bounds we have obtained. We plug (79) into (78) to get

$$\begin{aligned}\sum_{q\in q(\mu_2)}\|\sum_{\substack{T_1\in\mathbb{T}_1'(q)\\T_2\in\mathbb{T}_2(q)}}\phi_{T_1}\phi_{T_2}\|_{L^2(q)}^2 &\leqslant \nu R^{1-\frac{n}{2}+C\delta}|\mathbb{T}_2|\sum_{q\in q(\mu_2)}\lambda_1^{-1}|\mathbb{T}_1[\mu_2,\lambda_1](q)|\\ &\lesssim \nu R^{1-\frac{n}{2}+C\delta}|\mathbb{T}_2|\sum_{T_1\in\mathbb{T}_1[\lambda_1,\mu_2]}\lambda_1^{-1}\sum_{q\in q(\mu_2)}\mathbb{1}_{\{T_1\cap R^\delta q\neq\varnothing\}}\\ &\lesssim \nu R^{1-\frac{n}{2}+C\delta}|\mathbb{T}_1||\mathbb{T}_2|,\end{aligned}$$

This concludes the proof of (74).

We interpolate the bilinear norm between the points $p'=1$ in (71) and $p'=2$ in (74) to get

$$\|\sum_{T_1\nsim B_{R'},T_2}\phi_{T_1}\phi_{T_2}\|_{L^{\frac{n}{n-1}}(B_{R'})}\leqslant C_\delta(\log R)^C R^{C\delta}\nu^{\frac{1}{n}}|\mathbb{T}_1|^{\frac12}|\mathbb{T}_2|^{\frac12}.$$

This bound joins the inequalities (69) and (70) to yield

$$\|\sum_{\substack{T_1\in\mathbb{T}_1\\T_2\in\mathbb{T}_2}}\phi_{T_1}\phi_{T_2}\|_{L^{\frac{n}{n-1}}(B_R)}\leqslant C_\delta(\log R)^C(K_\nu(R')+R^{C\delta}\nu^{\frac1n})|\mathbb{T}_1|^{\frac12}|\mathbb{T}_2|^{\frac12};$$

in other words,

$$K_\nu(R)\leqslant C_\delta(\log R)^C(K_\nu(R^{1-\delta})+R^{C\delta}\nu^{\frac1n}).$$

When we iterate, we get at the N-th step

$$K_\nu(R)\leqslant C_\delta^N(\log R)^{NC}(K_\nu(R^{(1-\delta)^N})+NR^{C\delta}\nu^{\frac1n}).$$

We stop when $R^{(1-\delta)^N}\leqslant\nu^{-1}<R^{(1-\delta)^{N-1}}$; the number of steps is

$$N\leqslant-\frac{1}{\log(1-\delta)}+1\leqslant 2\delta^{-1}.$$

If $r\leqslant\nu^{-1}$, then we can average over translations of the paraboloid and apply Tao's Bilinear Theorem to get $K_\nu(r)\leqslant C_\varepsilon r^{1-\frac{n+2}{2p}+\varepsilon}\nu^{\frac12}$. We have thus that

$$K_\nu(R)\leqslant C_\delta R^{C\delta}(\nu^{-1+\frac{n+2}{2n}+\frac12}+\nu^{\frac1n})\leqslant C_\delta R^{C\delta}\nu^{\frac1n}.$$

This concludes the proof of Theorem 22, which implies Theorem 8'.

4.5.1 Additional Remarks

We indicate here the changes we need to do for surfaces of elliptic type or the hemisphere. The argument is sufficiently robust to admit perturbations.

For surfaces of ε-elliptic type, the semi-norms $\|\partial^N\Phi\|_\infty$ enter in the constants C_δ of (61) and (64). Since the eigenvalues of $D^2\Phi$ are close to one, then the tubes have approximately the same length.

The delta function in (76) gets into

$$\delta(\Phi(\xi_2') - \Phi(\xi_1' + \xi_2') + \Phi(\xi_1' + \xi_2'') - \Phi(\xi_2'')) = \delta(\langle A(\xi_2' - \xi_2''), \xi_1'\rangle)$$

for some matrix A with eigenvalues in $[1-\varepsilon, 1+\varepsilon]$. Then $|\langle A(\xi_2' - \xi_2''), \xi_1'\rangle - \langle \xi_2' - \xi_2'', \xi_1'\rangle| \leqslant C\varepsilon$, and instead of an integral over the hyperplane H with normal direction $\xi_2' - \xi_2''$ that passes through ξ_2', we integrate over a $(n-2)$-surface $\tilde{H}$ that lies in a ε-neighborhood of H and passes through ξ_2'.

A tube associated with a cap with center c_α has velocity $(-\nabla\Phi(c_\alpha), 1)$. If $\tilde{P} \subset \mathbb{R}^n$ is a $(n-1)$-cone with center in a cube q generated by all the lines with directions $(-\nabla\Phi(\eta), 1)$ for $\eta \in \tilde{H}$, then we must verify that all the tubes coming from the separated set S_2 are transversal to $\tilde{P}$. In fact, notice that for any point $\xi_2' + \xi_1' \in \tilde{H}$, a vector v tangent to $\tilde{H}$ satisfies the equation

$$\langle \nabla\Phi(\xi_1' + \xi_2'') - \nabla\Phi(\xi_1' + \xi_2'), v\rangle = 0;$$

hence, $\langle A(\xi_2'' - \xi_2'), v\rangle = 0$ for some matrix A close to I. Then, the vectors normal to $\tilde{P}$ have the form $(A(\xi_2'' - \xi_2'), \langle \nabla\Phi(\xi_2' + \xi_1'), A(\xi_2'' - \xi_2')\rangle)$. If we take the inner product of these vectors with $(-\nabla\Phi(\eta_2), 1)$ for $\eta_2 \in \pi(S_2)$, then we get

$$\langle A(\xi_2'' - \xi_2'), \nabla\Phi(\xi_2' + \xi_1') - \nabla\Phi(\eta_2)\rangle = \langle A(\xi_2'' - \xi_2'), A'(\xi_2' + \xi_1' - \eta_2)\rangle;$$

hence, the inner product is basically equal to $\langle \eta_1 - \eta_2, \eta_1' - \eta_2'\rangle$ for all the pairs $\eta_1, \eta_1' \in \pi(S_1)$ and $\eta_2, \eta_2' \in \pi(S_2)$, and $|\langle \eta_1 - \eta_2, \eta_1' - \eta_2'\rangle| \geqslant c > 0$, then $\tilde{P}$ is uniformly transversal to all the tubes coming from S_2. The estimates hold uniformly in $\varepsilon \ll 1$.

The case of the hemi-sphere is similar. The term (76) is almost as simple as for the paraboloid. By symmetry, we can assume that $\xi_2' = -ae_1$ and $\xi_2'' = ae_1$ for some $0 < a \leqslant \frac{1}{\sqrt{2}} + \frac{1}{10}$. Then, the $(n-2)$-surface $\tilde{H}$ is again a hyperplane H with normal direction e_1 that passes through ξ_2'. The cone $\tilde{P}$ is a translation of a portion of the quadratic cone $\{\xi \mid \xi_1^2 = a^2|\xi|^2\}$. It is intuitively clear that the portion of the cone generated by direction from S_1 is uniformly transversal to tubes from S_2.

Appendix: Non-averaged Upper Bounds

We may wonder whether it is really necessary to average, or we just have not pushed as much as possible the estimates for $\|M_{\partial_i f}\|_{X_\zeta^{\frac{1}{2}} \to X_\zeta^{-\frac{1}{2}}}$. We show that averaging is indeed necessary.

Theorem 25. *If $f \in W^{\frac{d-2}{2p},p}(\mathbb{R}^n)$, for $d \leqslant p < \infty$, is a function with support in B_1, then*

$$\|M_{\partial_i f}\|_{X_\zeta^{\frac{1}{2}} \to X_\zeta^{-\frac{1}{2}}} \leqslant C\|f\|_{\frac{d-2}{2p},p}, \tag{82}$$

where C does not depend on ζ. The inequality is best possible, in the sense that it is not possible to lower the regularity of f.

Proof. It is not necessary to use bilinear theory to get (82), the computations of Haberman in Section 4 of [14] are enough. To see that the result is best possible, we fix $\zeta = \tau(e_1 - ie_2)$ and consider the $\tau^{-\frac{1}{2}}$-neighborhood of a 2-plane of side-length 1 lying in the plane (x_1, x_2), and denote this set by F. We define $f(x) := e^{2\pi i(2\tau)x_2}\varphi_F(x)$, where φ_F is a smooth cut-off function of F —see Figure 7(a), and we have that $\|f\|_{s,p} \sim \tau^{s-\frac{d-2}{2p}}$. To estimate the operator norm of $\partial_2 f$ we consider the box B of dimensions $1 \times 1 \times \tau^{\frac{1}{2}} \times \cdots \times \tau^{\frac{1}{2}}$ centered at zero, and take $\hat{u} = \varphi_B$ and $\hat{v} = \varphi_B(\cdot - 2\tau e_2)$, for which $\|u\|_{X_\zeta^{\frac{1}{2}}} = \|v\|_{X_\zeta^{\frac{1}{2}}} = \tau^{\frac{d}{4}}$; see Figure 7(b). The duality pairing gives $|\langle \partial_2 f u, v\rangle| \gtrsim \tau^{\frac{d}{2}}$. If K is the best constant in (82), then we get

$$\tau^{\frac{d}{2}} \lesssim K\tau^{s-\frac{d-2}{2p}}\tau^{\frac{d}{2}};$$

if K is to be uniformly bounded in ζ, then necessarily $s \geqslant \frac{d-2}{2p}$. □

If we did not need bilinearity to get the sharp upper bound (82), then what would it happen if we tried to use bilinearity? We can answer this question

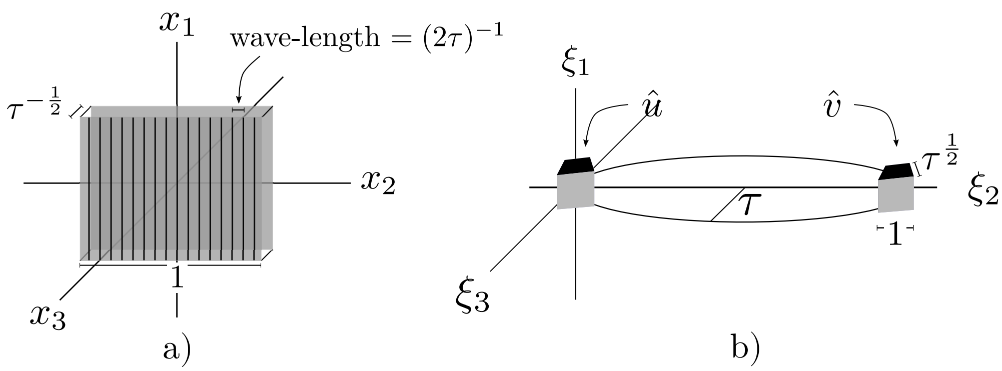

Figure 7: (a) Representation of the function f, and (b) of the Fourier transforms of u and v.

with the aid of Theorem 16. The contribution A_1' from non-antipodal caps is

$$
\begin{aligned}
A_1' &:= \sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \lesssim 1 \\ \tau^{-1} \leqslant \mu \leqslant \nu}} Q(\lambda,\mu,\nu)\tau^{\frac{d}{p}-1}\|P_{\tau\lambda}f\|_p \\
&= \Big(\sum_{\substack{\nu^{\frac{1}{2}} \leqslant \lambda \leqslant 1 \\ \mu^{\frac{1}{2}} \leqslant \nu}} + \sum_{\substack{\nu^{\frac{1}{2}} \leqslant \lambda \leqslant \nu\mu^{-\frac{1}{2}} \\ \mu \leqslant \nu \leqslant \mu^{\frac{1}{2}}}} + \sum_{\substack{\nu\mu^{-\frac{1}{2}} \leqslant \lambda \leqslant 1 \\ \mu \leqslant \nu \leqslant \mu^{\frac{1}{2}}}} \Big) Q(\lambda,\mu,\nu)\tau^{\frac{d}{p}-1}\|P_{\tau\lambda}f\|_p \\
&\lesssim \Big(\sum_{\tau^{-\frac{1}{4}} \leqslant \lambda} \lambda \Big[\sum_{\mu \leqslant \lambda^4} \mu^{\frac{d+2}{2p}-\frac{3}{4}} \Big] + \sum_{\tau^{-\frac{1}{2}} \leqslant \lambda} \lambda^{\frac{1}{2}} \Big[\sum_{\mu \leqslant \lambda^2} \mu^{\frac{d+2}{2p}-\frac{3}{4}} \Big] + \\
&\qquad\qquad + \sum_{\tau^{-\frac{1}{2}} \leqslant \lambda} \lambda^{1-\frac{1}{p}} \Big[\sum_{\mu \leqslant \lambda^2} \mu^{\frac{d+3}{2p}-1} \Big] \Big) \tau^{\frac{d}{p}-1-}\|P_{\tau\lambda}f\|_p \\
&\lesssim \|f\|_{W^{\frac{d-3}{2p}+,p}}.
\end{aligned}
$$

This bound is actually better than (82). Now let us see what happens with the antipodal caps

$$
\begin{aligned}
A_2' &:= \sum_{\substack{\lambda\sim 1\\ \mu\leqslant\nu<\mu^{\frac12}}} \lambda\mu^{\frac1p-\frac12}\nu^{-\frac12}\sum_{\nu^{1/2}\leqslant\rho} K^{\rho}_{\mu,\nu}\tau^{\frac{d}{p}-1}\|P_{\tau\lambda}f\|_p\\
&\leqslant \sum_{\substack{\lambda\sim 1\\ \mu\leqslant\nu\leqslant\mu^{\frac12}}} \lambda\mu^{\frac{d+1}{2p}-\frac12}\nu^{-\frac12}\Big(\mu^{\frac{1}{2p}}\sum_{\nu^{\frac12}\leqslant\rho\leqslant\nu\mu^{-\frac12}} 1+\nu^{\frac1p-}\sum_{\nu\mu^{-\frac12}\leqslant\rho\leqslant 1}\rho^{-\frac1p}\Big)\tau^{\frac{d}{p}-1}\|P_{\tau\lambda}f\|_p\\
&\lesssim \sum_{\lambda\sim 1}\lambda\mu^{\frac{d+2}{2p}-\frac12}\sum_{\mu\leqslant\nu\leqslant\mu^{\frac12}}\nu^{-\frac12-}\tau^{\frac{d}{p}-1}\|P_{\tau\lambda}f\|_p\\
&\lesssim \|f\|_{W^{\frac{d-2}{2p}+,p}}.
\end{aligned}
$$

This bound is, up to ε-losses, equal to (82); in fact, the example in Theorem 25 corresponds to antipodal caps. The reader may want to compare these computations with those for A_1 and A_2 in (53) and thereafter. As we cannot get a linear estimate stronger than Tomas-Stein by using Tao's Bilinear Theorem, so we cannot expect that bilinearity necessarily will improve on (82).

Notations

- Relations: $A\lesssim_\epsilon B$ if $A\leqslant C_\epsilon B$; $A\sim B$ if $A\lesssim B\lesssim A$; $A\ll 1$ if $A\leqslant c$, where c is chosen sufficiently small.

- Miscellaneous: $e(z):=e^{2\pi iz}$. $\langle x\rangle=(1+|x|^2)^{\frac12}$. $B_r(x)$ a ball of radius r with center at x. $⨍_M\, d\tau := \frac1M\int_M^{2M} d\tau$. $a+ := a+\varepsilon$ for $\varepsilon\ll 1$. If E is a set, then $\mathbb{1}_E$ is the characteristic function of the set, and $|E|$ is its measure, where the measure can be deduced from the context. If T is a tube with main axis l, then AT is a dilation of T by a factor $A>0$ and same main axis l.

- Multipliers: $m(D)f=(m\widehat{f})^\vee$, where m stands for *multiplier*; $Pf=m(D)f$, where m is a smooth cut-off for a set of frequencies where we want to project to.

- The operator $\Delta_\zeta:=\Delta+\zeta\cdot\nabla$ has symbol $p_\zeta(\xi):=-|\xi|^2+2i\zeta\cdot\xi$ and characteristic $\Sigma_\zeta:=\{\xi\mid p_\zeta(\xi)=0\}$.

- $\zeta(U,\tau) := \tau(Ue_1 - iUe_2)$, where $\{e_i\}$ is the canonical basis, $\tau \geqslant 1$ and $U \in O_d$ is a rotation.

- $\|u\|^2_{\dot{X}^b_\zeta} := \int |p_\zeta(\xi)|^{2b} |\widehat{u}(\xi)|^2 \, d\xi$.

- $\|u\|^2_{X^b_{\zeta,\sigma}} := \int (|p_\zeta(\xi)| + \sigma)^{2b} |\widehat{u}(\xi)|^2 \, d\xi$ for $\sigma > 0$; $\|u\|_{X^b_\zeta} = \|u\|_{X^b_{\zeta,|\zeta|}}$.

- Sobolev-Slobodeckij spaces: For $1 < p < \infty$, $W^{s,p}(\mathbb{R}^d)$ is the space of distributions f such that

$$\|f\|_{s,p} := \sum_{|\alpha| \leqslant s} \|D^\alpha f\|_p < \infty \quad \text{for } s \text{ integer}.$$

$$\|f\|_{s,p} := \|P_{\leqslant 1} f\|_p + \Big(\sum_{k>0} 2^{skp} \|P_k f\|_p^p \Big)^{\frac{1}{p}} < \infty \quad \text{for } 0 < s \neq \text{integer}.$$

For a domain $\Omega \subset \mathbb{R}^d$, we define $W^{s,p}(\Omega) := \{f|_\Omega \mid f \in W^{s,p}(\mathbb{R}^d)\}$. The space $W^{s,p}_0(\Omega)$ is the completion in $W^{s,p}(\mathbb{R}^d)$ of test functions $D(\Omega) := \{\varphi \in C^\infty(\Omega) \mid \operatorname{supp} \varphi \Subset \Omega\}$. For further details, see *e.g.* [31, 22].

- $(f dS)^\vee(x) := \int_{\mathbb{R}^{n-1}} f(\xi) e(\langle x', \xi \rangle + x_n \varphi(\xi)) \, d\xi$, where S is the graph of φ and $(x', x_n) \in \mathbb{R}^n$.

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