GRIGORCHUK-GUPTA-SIDKI GROUPS AS A SOURCE FOR BEAUVILLE SURFACES

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Abstract. If G is a Grigorchuk-Gupta-Sidki group defined over a p-adic tree, where p is an odd prime, we study the existence of Beauville surfaces associated to the quotients of G by its level stabilizers $\mathrm{st}_G(n)$. We prove that if G is periodic then the quotients $G/\mathrm{st}_G(n)$ are Beauville groups for every $n \geq 2$ if $p \geq 5$ and $n \geq 3$ if $p = 3$. In this case, we further show that all but finitely many quotients of G are Beauville groups. On the other hand, if G is non-periodic, then none of the quotients $G/\mathrm{st}_G(n)$ are Beauville groups.

Contents

1. Introduction

Groups acting on regular rooted trees have been widely studied since the 1980's, when the first Grigorchuk group was defined by Rostilav Grigorchuk [13]. This group was designed to be a counterexample to the General Burnside Problem, so that it is a finitely generated, periodic and infinite group. More importantly, it was the first example of a group having intermediate word growth [14], answering the Milnor Problem [18]. Later on many different examples and generalizations came into the literature. The interest on this kind of groups resides in their *odd* properties, and because they have been useful to answer unsolved questions.

Some of the examples that came up were the Gupta-Sidki groups [15] and the second Grigorchuk group [13]. The Grigorchuk-Gupta-Sidki groups (GGS-groups for short) are a family of groups generalizing them. These groups act on the regular p-adic rooted tree where p is an odd prime. More concretely, each of them is generated by two automorphisms: a rooted automorphism a permuting the vertices hanging from the root according to the permutation $(1\,2\,\ldots\,p)$, and a recursively defined automorphism b which is defined according to a given

Date: June 25, 2020.

Both authors acknowledge financial support from the Spanish Government, grants MTM2014-53810-C2-2-P and MTM2017-86802-P, partly with FEDER funds, and from the Basque Government, grant IT974-16. This research is also supported by the Basque Government through the BERC 2018-2021 program and by the Spanish State Research Agency through BCAM Severo Ochoa excellence accreditation SEV-2017-0718 and through project RTI2018-093860-B-C21 funded by (AEI/FEDER, UE) and acronym "MathNEURO".

vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$. Vovkivsky [21] showed that G is always infinite and it is a periodic group if and only if $\Sigma_{i=1}^{p-1} e_i = 0$, so some of them are also counterexamples for the General Burnside Problem.

A group G acting faithfully on a regular rooted tree $\mathcal{T}$ is always residually finite, and a way to analyze the structure of a residually finite group is by looking at its finite quotients. For these groups there is a very natural family of normal subgroups of finite index, which are the level stabilizers $\mathrm{st}_G(n)$ for each $n \in \mathbb{N}$. The quotient $G/\mathrm{st}_G(n)$ can be naturally seen as a subgroup of the group of automorphisms of the subtree $\mathcal{T}_n$ consisting of the first n levels, since the kernel of the action of G on $\mathcal{T}_n$ is $\mathrm{st}_G(n)$. For the GGS-groups these quotients have been well studied. For instance, in [11] Fernández-Alcober and Zugadi-Reizabal have given the sizes of these quotients, and the profinite completion of each group has been compared to the completion with respect to the level stabilizers by Fernández-Alcober, Garrido and Uria-Albizuri in [10]. The main goal of this paper is to determine whether these quotients are Beauville groups or not.

A finite group G is called a *Beauville group* if it is a 2-generator group and there exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of G such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$, where

$$\Sigma(x_i, y_i) = \bigcup_{g \in G} \Big(\langle x_i \rangle^g \cup \langle y_i \rangle^g \cup \langle x_i y_i \rangle^g \Big),$$

for $i = 1, 2$. Then we say that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ form a *Beauville structure* for G.

Every Beauville group gives rise to a complex surface of general type which is known as a *Beauville surface*. Roughly speaking, a Beauville surface is a compact complex surface defined by taking a pair of complex curves C_1 and C_2 and letting a finite group G, which is called a Beauville group, act freely on their product to define the surface as the quotient $(C_1 \times C_2)/G$.

Beauville groups have been intensely studied in recent times; see surveys [5, 17]. For example, the abelian Beauville groups were classified by Catanese [4]: a finite abelian group G is a Beauville group if and only if $G \cong C_n \times C_n$ for $n > 1$ with $\gcd(n, 6) = 1$. After abelian groups, the most natural class of finite groups to consider are nilpotent groups. The study of nilpotent Beauville groups is reduced to that of Beauville p-groups.

If p is a prime, the state of the art on Beauville p-groups can be found in the survey papers [1], [3] and [6]. The smallest non-abelian Beauville p-groups were determined by Barker, Boston and Fairbairn in [1]. On the other hand, in [9], Fernández-Alcober and Gül extended Catanese's criterion in the case of p-groups from abelian groups to finite p-groups having a 'nice power structure', including in particular p-groups of class $< p$.

Also in [1], it was shown that there are non-abelian Beauville p-groups of order p^n for every $p \geq 5$ and every $n \geq 3$. The first explicit infinite family of Beauville 2-groups was constructed in [2]. Recently in [19], Stix and Vdovina constructed an infinite series of Beauville p-groups, for every prime p, by considering quotients of ordinary triangle groups. In particular this gives examples of non-abelian Beauville p-groups of arbitrarily large order. On the other hand, in [16], Gül showed that quotients by the terms of the lower p-central series in either the free group of rank 2 or in the free product of two cyclic groups of order p are Beauville groups. In [9], quotients of the Nottingham group over $\mathbb{F}_p$ have been studied in order to construct more infinite families of Beauville p-groups, for an odd prime p.

Note that if G is a GGS-group, then the quotients of G by its level stabilizers $\mathrm{st}_G(n)$ are finite p-groups generated by two elements of order p but whose exponent can be arbitrarily high. As a consequence, they do not fit into the family of finite p-groups having a 'nice power structure'. For this reason, they are natural candidates to search for Beauville p-groups of a very different type from the ones in [9]. It turns out that the property of being Beauville for these quotients depends on whether G is periodic or not.

The main results of this paper are as follows.

Theorem A. *Let G be a periodic GGS-group over the p-adic tree. Then the quotient $G/\operatorname{st}_G(n)$ is a Beauville group if $p \geq 5$ and $n \geq 2$, or $p = 3$ and $n \geq 3$.*

Then by using the fact that $G/\operatorname{st}_G(3)$ is a Beauville group for an odd prime p in Theorem A, we prove the following more general result.

Theorem B. *Let G be a periodic GGS-group over the p-adic tree for an odd prime p. Then all but finitely many quotients of G are Beauville groups.*

Theorem C. *Let G be a non-periodic GGS-group over the p-adic tree. Then the quotient $G/\operatorname{st}_G(n)$ is not a Beauville group for any $n \geq 1$.*

Theorem B shows that a periodic GGS-group is a source for the construction of an infinite series of Beauville p-groups. This gives yet another reason why GGS-groups constitute an important family in group theory.

Notation. If G is a group, then we denote by $\operatorname{Cl}_G(x)$ the conjugacy class of the element $x \in G$. Also, if p is a prime, then the exponent of a p-group G, denoted by $\exp G$, is the maximum of the orders of all elements of G.

2. Definitions and Preliminaries

In this section, we will establish a few properties of GGS-groups that will help us prove the main theorems of this paper. Before proceeding, we recall some facts about automorphisms of rooted trees.

If $d \geq 2$ is an integer and $X = \{1, ..., d\}$, the *d-adic tree* $\mathcal{T}$ is the rooted tree whose set of vertices is the free monoid X^*, where the root corresponds to the empty word $\emptyset$, and a word u is a descendant of v if $u = vx$ for some $x \in X$. The set L_n of all vertices of length n is called the *nth level of $\mathcal{T}$*, for every integer $n \geq 0$. If we consider only words of length $\leq n$, then we have a finite tree $\mathcal{T}_n$, which we refer to as *the tree $\mathcal{T}$ truncated at level n*.

An *automorphism of $\mathcal{T}$* is a bijection of the vertices that preserves incidence. The group of automorphisms of $\mathcal{T}$ is denoted by $\operatorname{Aut}\mathcal{T}$. The subgroup $\operatorname{st}(n)$ of $\operatorname{Aut}\mathcal{T}$ consisting of the automorphisms that fix pointwise L_n is called the *nth level stabilizer*. More generally, if $G \leq \operatorname{Aut}\mathcal{T}$, we define $\operatorname{st}_G(n) = \operatorname{st}(n) \cap G$.

If an automorphism g fixes a vertex u, then the restriction of g to the subtree hanging from u induces an automorphism g_u of $\mathcal{T}$. In particular, if $g \in \operatorname{st}(1)$ then g_i is defined for every $i = 1, ..., d$, and we have an isomorphism

$$\begin{aligned} \psi\colon \operatorname{st}(1) &\longrightarrow \operatorname{Aut}\mathcal{T} \times \overset{d}{\ldots} \times \operatorname{Aut}\mathcal{T} \\ g &\longmapsto (g_1, \ldots, g_d). \end{aligned}$$

An important automorphism of $\mathcal{T}$ is the automorphism that permutes the d subtrees hanging from the root rigidly according to the permutation $(12...d)$. This is called a *rooted automorphism* and will be denoted by the letter a. Since a has order d, it makes sense to write a^k for $k \in \mathbb{Z}/d\mathbb{Z}$. Now, given a non-zero vector $\mathbf{e} = (e_1, ..., e_{d-1}) \in (\mathbb{Z}/d\mathbb{Z})^{d-1}$, we can define recursively $b \in \operatorname{st}(1)$ via

$$\psi(b) = (a^{e_1}, \ldots, a^{e_{d-1}}, b).$$

Then the subgroup $G = \langle a, b\rangle$ of $\operatorname{Aut}\mathcal{T}$ is called the *GGS-group* corresponding to the *defining vector* $\mathbf{e}$.

If $d = 2$ then there is only one GGS-group, which is isomorphic to the infinite dihedral group D_∞. In case $d = 3$ there are only three essentially different defining vectors $\mathbf{e}$, and they are $(1, 0)$, $(1, 1)$ and $(1, 2)$. The corresponding groups are called the Fabrykowski-Gupta group, the Bartholdi-Grigorchuk group and the Gupta-Sidki group, respectively. If $d = 4$ then one example of GGS-group is the second Grigorchuk group, where the defining vector is $(1, 0, 1)$.

In the remainder of this paper, we will asume that $d = p$ for an odd prime p. Let $G = \langle a, b\rangle$ be a GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1})$. Observe that both a and b are of order p. For every integer i, we write $b_i = b^{a^i}$. Notice that $b_i = b_j$ if $i \equiv j \pmod p$. The images of the elements b_i under the map ψ can be described as:

$$\begin{aligned}\psi(b_0) &= (a^{e_1}, \ldots, a^{e_{p-1}}, b),\\ \psi(b_1) &= (b, a^{e_1}, \ldots, a^{e_{p-1}}),\\ &\vdots\\ \psi(b_{p-1}) &= (a^{e_2}, a^{e_3}, \ldots, b, a^{e_1}).\end{aligned} \tag{1}$$

Here we collect some basic results regarding GGS-groups.

Proposition 2.1. [11, Theorem 2.1] *If $G = \langle a, b\rangle$ is a GGS-group, then*

(i) $\operatorname{st}_G(1) = \langle b\rangle^G = \langle b_0, \ldots, b_{p-1}\rangle$, *and* $G = \langle a\rangle \ltimes \operatorname{st}_G(1)$.
(ii) $\operatorname{st}_G(2) \le G' \le \operatorname{st}_G(1)$.
(iii) $|G : G'| = p^2$ *and* $|G : \gamma_3(G)| = p^3$.

Remark 2.2. For all $k \ge 1$ we have $\psi(\operatorname{st}_G(k)) \subseteq \operatorname{st}_G(k-1) \times \overset{p}{\ldots} \times \operatorname{st}_G(k-1)$.

Remark 2.3. For every positive integer n, we can define an isomorphism ψ_n from the stabilizer of the first level in $\operatorname{Aut}\mathcal{T}_n$ to the direct product $\operatorname{Aut}\mathcal{T}_{n-1} \times \overset{p}{\ldots} \times \operatorname{Aut}\mathcal{T}_{n-1}$, in the same way as ψ is defined. Since $G_n = G/\operatorname{st}_G(n)$ can be seen as a subgroup of $\operatorname{Aut}\mathcal{T}_n$, we can consider the restriction of ψ_n to $\operatorname{st}_{G_n}(1)$. Then by the previous remark, we have

$$\psi_n(\operatorname{st}_{G_n}(k)) \subseteq \operatorname{st}_{G_{n-1}}(k-1) \times \overset{p}{\ldots} \times \operatorname{st}_{G_{n-1}}(k-1).$$

For simplicity, throughout this paper we are not going to use bar notation for the elements in the quotient groups $G/\operatorname{st}_G(n)$. However, the subscript n at the map ψ_n means that we are working in the quotient group $G/\operatorname{st}_G(n)$.

If $\mathbf{e} = (e_1, \ldots, e_{p-1})$ is the defining vector of a GGS-group, then we write $C(\mathbf{e}, 0)$ for the circulant matrix $C(e_1, ..., e_{p-1}, 0)$ over $\mathbb{F}_p$.

Proposition 2.4. [11, Theorem 2.4] *Let G be a GGS-group with defining vector $\mathbf{e}$, and put $C = C(\mathbf{e}, 0)$. Then*

(i) *The dimension of* $\operatorname{st}_{G_2}(1)$ *coincides with the rank t of C.*
(ii) *G_2 is a p-group of maximal class of order p^{t+1}.*

Lemma 2.5. [11, Lemma 2.7] *Let $C = C(e_1, \ldots, e_{p-1}, 0)$ be a circulant matrix over $\mathbb{F}_p$. Then*

(i) *The rank of C is $p - m$, where m is the multiplicity of 1 as a root of the polynomial $f(X) = e_1 + e_2X + \cdots + e_{p-1}X^{p-2}$.*
(ii) *The rank of C is strictly less than p if and only if $\Sigma_{i=1}^{p-1} e_i = 0$.*

Proposition 2.6. [11, Theorems 2.13, 2.14] *Let G be a GGS-group. Then*

(i) $\mathrm{st}_G(1)' \leq \mathrm{st}_G(2)$.
(ii) $|G : \mathrm{st}_G(1)'| = p^{p+1}$.

We say that the defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1})$ of a GGS-group is *symmetric* if $e_i = e_{p-i}$ for all $i = 1, \ldots, p-1$.

Proposition 2.7. [11, Lemma 3.4, Theorem 3.7] *Let G be a GGS-group with non-symmetric defining vector $\mathbf{e}$. Then*
(i) $\psi(st_G(1)') = G' \times \overset{p}{\ldots} \times G'$.
(ii) *If $G_n = G/\mathrm{st}_G(n)$ then for every $n \geq 2$*

$$\log_p |G_n| = tp^{n-2} + 1,$$

where t is the rank of $C(\mathbf{e}, 0)$.

3. Quotients of periodic GGS-groups

Let G be a periodic GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1})$, where p is an odd prime. In this section, we first determine whether the quotients of G by its level stabilizers $\mathrm{st}_G(n)$ are Beauville groups. We next show that all but finitely many quotients of G are Beauville groups.

We first deal with the quotient group $G_2 = G/\mathrm{st}_G(2)$.

Theorem 3.1. *Let G be a periodic GGS-group. Then the quotient $G/\mathrm{st}_G(2)$ is a Beauville group if and only if $p \geq 5$.*

Proof. By Proposition 2.4(ii), we know that G_2 is a group of maximal class of order p^{t+1}, where t is the rank of $C(\mathbf{e}, 0)$. Then Lemma 2.5(ii) implies that $|G_2| \leq p^p$, and hence G_2 is regular by Lemma 3.13 in [20]. Since G_2 is generated by two elements of order p, this, together with being regular, implies that $\exp G_2 = p$, by Corollary 2.11 in [7]. Then according to Corollary 2.10 in [9], G_2 is a Beauville group if and only if $p \geq 5$. □

We next deal with the quotient $G_3 = G/\mathrm{st}_G(3)$. To this purpose, first we need to calculate the orders of $a^{-1}b$ and ab^i for $1 \leq i \leq p-1$.

Lemma 3.2. *Let G be a periodic GGS-group. Then the orders of $a^{-1}b$ and ab^i for $1 \leq i \leq p-1$ are p^2 in both G and G_3.*

Proof. Let $1 \leq i \leq p-1$. Then we have

$$(ab^i)^p = a^p(b^i)^{a^{p-1}} \ldots (b^i)^a b^i = b^i_{p-1} \ldots b^i_1 b^i_0,$$

and hence

$$\psi((ab^i)^p) = \left(a^{i(e_2+\cdots+e_{p-1})} b^i a^{ie_1}, a^{i(e_3+\cdots+e_{p-1})} b^i a^{i(e_1+e_2)}, \ldots, a^{i(e_1+\cdots+e_{p-1})} b^i\right).$$

Then being $\Sigma_{i=1}^{p-1} e_i = 0$ implies that

$$\psi((ab^i)^p) = \left((b^i)^{a^{ie_1}}, (b^i)^{a^{i(e_1+e_2)}}, \ldots, b^i, b^i\right). \tag{2}$$

Since each component in equation (2) is of order p, the order of ab^i is p^2 for every $1 \leq i \leq p-1$. Observe that $(ab^i)^p \notin \mathrm{st}_G(3)$, and hence also in G_3 the order of ab^i is p^2.

On the other hand, since $(a^{-1}b)^{-1} = (ab^{-1})^b = (ab^{p-1})^b$, the order of $a^{-1}b$ is also p^2 in both G and G_3. □

The next lemma shows the relation between the conjugates of b by powers of a modulo $\mathrm{st}_G(2)$.

Lemma 3.3. *Let G be a GGS-group. If*

$$b^{a^i} \equiv b^{a^j} \pmod{\mathrm{st}_G(2)}$$

for $0 \le i, j \le p-1$, then $i = j$.

Proof. Suppose, on the contrary, that $i \neq j$. Then $G = \langle a^{j-i}, b\rangle$. The congruence $b^{a^i} \equiv b^{a^j} \pmod{\mathrm{st}_G(2)}$ implies that $[b, a^{j-i}] \in \mathrm{st}_G(2)$, and so $G/\mathrm{st}_G(2)$ is abelian. However, by Lemma 2.5(i), the rank of the circulant matrix $C(\mathbf{e}, 0)$ is at least 2. Thus, $G/\mathrm{st}_G(2)$ is a p-group of maximal class of order at least p^3, and hence it is not abelian. □

The following proposition is the key result for determining the existence of a Beauville structure for G_3.

Proposition 3.4. *Let G be a periodic GGS-group. If*

$$\langle (ab^i)^p \rangle = \langle (ab^j)^p \rangle^g$$

for $1 \le i, j \le p-1$ and $g \in G_3$, then $i = j$.

Proof. For simplicity we write $w_i = (ab^i)^p$ for every $1 \le i \le p-1$. Then the equality $\langle w_i \rangle = \langle w_j \rangle^g$ implies that

$$w_i^k = w_j^g \tag{3}$$

for some $1 \le k \le p-1$. By (2), we have

$$\psi_3(w_i) = \big((b^i)^{a^{ie_1}}, (b^i)^{a^{i(e_1+e_2)}}, \dots, b^i, b^i\big). \tag{4}$$

By Remark 2.3, each component in the equation (4) is an element of G_2.

Since $\mathrm{st}_{G_2}(1)$ is abelian, the components of $\psi_3(w_j^g)$ are of the form $(b^j)^{a^m}$, with $m \in \{0, \dots, p-1\}$. Also one of the components of $\psi_3(w_i^k)$ is b^{ik}. Then by equality (3), we have $(b^j)^{a^m} = b^{ik}$ in G_2 for some m. Thus, $b^{ik-j} = [b^j, a^m] \in G_2'$, and this is true only if $ik - j \equiv 0 \pmod p$. Therefore, $k \equiv i^{-1}j \pmod p$, and hence we have $w_i^{i^{-1}j} = w_j^g$. If we write $x_i = w_i^{i^{-1}}$ for every $1 \le i \le p-1$, then we have

$$x_i = x_j^g.$$

Note that

$$\psi_3(x_i) = \big(b^{a^{ie_1}}, b^{a^{i(e_1+e_2)}}, \dots, b, b\big)$$

for every $1 \le i \le p-1$.

Let $g = a^s h_s$ for some $0 \le s \le p-1$ and $h_s \in \mathrm{st}_{G_3}(1)$. Observe that

$$\psi_3(x_j^{a^s}) = \big(b^{a^{j(e_1+\cdots+e_{p-(s-1)})}}, \dots, b, b, \dots, b^{a^{j(e_1+\cdots+e_{p-s})}}\big),$$

where the first b appears at the $(s-1)$st component. Then the equality $x_i = x_j^g$ implies that

$$\psi_3(h_s) = \big(a^{ie_1 - j(e_1+\cdots+e_{p-(s-1)})}u_1, a^{i(e_1+e_2)-j(e_1+\cdots+e_{p-(s-2)})}u_2, \dots, a^{-j(e_1+\cdots+e_{p-s})}u_p\big),$$

where $u_\ell \in \mathrm{st}_{G_2}(1)$ for all $1 \le \ell \le p$. We next define recursively elements $h_{i-1} = h_i b_{i-1}^{-j}$ for $i = s, \dots, 1$. Now since G is periodic, we have $e_1 + \cdots + e_{p-1} = 0$ and consequently

$$\psi_3(h_0) = \big(a^{(i-j)e_1}v_1, a^{(i-j)(e_1+e_2)}v_2, \dots, v_{p-1}, v_p\big),$$

with $v_\ell \in \mathrm{st}_{G_2}(1)$ for all $1 \le \ell \le p$.

Notice that we have

$$\psi_3(h_0^a) \equiv \psi_3(h_0 b^{j-i}) \pmod{\operatorname{st}_{G_2}(1) \times \overset{p}{\cdots} \times \operatorname{st}_{G_2}(1)}.$$

Hence $\psi_3(b^{i-j}[h_0,a]) \in \big(\operatorname{st}_{G_2}(1) \times \overset{p}{\cdots} \times \operatorname{st}_{G_2}(1)\big) \cap \psi_3(\operatorname{st}_{G_3}(1)) = \psi_3(\operatorname{st}_{G_3}(2))$. Thus, $b^{i-j}[h_0,a] \in \operatorname{st}_{G_3}(2) \leq G_3'$, and hence $b^{i-j} \in G_3'$. This implies that $i = j$. □

We are now ready to prove that G_3 is a Beauville group. We deal separately with the cases $p \geq 5$ and $p = 3$.

Theorem 3.5. *Let G be a periodic GGS-group over the p-adic tree. If $p \geq 5$ then the quotient $G/\operatorname{st}_G(3)$ is a Beauville group.*

Proof. Note that $\{a^{-2}, ab\}$ and $\{ab^2, b\}$ are both systems of generators of $G_3 = G/\operatorname{st}_G(3)$. We claim that they yield a Beauville structure for G_3. If $X = \{a^{-2}, ab, a^{-1}b\}$ and $Y = \{ab^2, b, ab^3\}$, we have to see that

$$\langle x \rangle \cap \langle y^g \rangle = 1 \tag{5}$$

for all $x \in X$, $y \in Y$, and $g \in G_3$. Observe that $\langle x\Phi(G_3)\rangle$ and $\langle y\Phi(G_3)\rangle$ have trivial intersection for every $x \in X$ and $y \in Y$, since a and b are linearly independent modulo $\Phi(G_3)$. Thus, x and y^g lie in different maximal subgroups of G_3 in every case.

Assume first that $x = a^{-2}$, which is an element of order p. If (5) does not hold, then $\langle x \rangle \subseteq \langle y^g \rangle$, and consequently $\langle x\Phi(G_3)\rangle = \langle y\Phi(G_3)\rangle$, which is a contradiction. The same argument holds if $y = b$ since it is also of order p.

The remaining elements in X and Y are of order p^2, by Lemma 3.2. Thus, in order to prove our claim, we need to show that

$$\langle x^p \rangle \neq \langle y^p \rangle^g \tag{6}$$

for all $g \in G_3$, and for those $x \in X$ and $y \in Y$. If $x = ab$ and $y = ab^2$ or ab^3, then we apply Proposition 3.4 to conclude that (6) holds.

It remains to deal with $x = a^{-1}b$ and $y = ab^2$ or $y = ab^3$. Since $(a^{-1}b)^{-1} = (ab^{p-1})^b$, if (6) does not hold, then $\langle (ab^{p-1})^p \rangle^b = \langle y^p \rangle^g$, that is,

$$\langle (ab^{p-1})^p \rangle = \langle y^p \rangle^{gb^{-1}},$$

for some $g \in G_3$. This contradicts with Proposition 3.4, and hence (6) holds. This completes the proof. □

Now we assume that $p = 3$.

Since proportional nonzero vectors define the same GGS group, if G is periodic and $p = 3$, then the defining vector of G can only be $\mathbf{e} = (1, -1)$. So G is the Gupta-Sidki 3-group. In this case, observe that the rank of $C(\mathbf{e}, 0)$ is 2. Then Proposition 2.4(ii), together with Proposition 2.1(iii), implies that $\operatorname{st}_G(2) = \gamma_3(G)$, which is of index 3^3 in G. By Proposition 2.7(ii), the order of G_3 is 3^7. Since the smallest Beauville 3-group is of order 3^5 [1], we can ask whether G_3 is Beauville or not.

Lemma 3.6. *Let $G = \langle a, b \rangle$ be the Gupta-Sidki 3-group. Then $Z(G_3)$ is a subgroup of $\operatorname{st}_{G_3}(1)'$ of order 3. More precisely, if $Z(G_3) = \langle z \rangle$ then*

$$\psi_3(z) = ([a,b],[a,b],[a,b]). \tag{7}$$

Proof. First of all, we observe that $Z(G_3) \leq \mathrm{st}_{G_3}(1)$. Otherwise, if there were $z \in Z(G_3)$ such that $z \notin \mathrm{st}_{G_3}(1)$, then $G_3 = \langle z, b\rangle$ would be an abelian group, which is not true.

Since for every $n \geq 1$, $\mathrm{st}_G(n)$ is a subdirect product of 3^n copies of G, this, together with

$$\psi_3(\mathrm{st}_{G_3}(1)) \subseteq G_2 \times G_2 \times G_2,$$

implies that

$$\psi_3(Z(G_3)) \subseteq Z(G_2) \times Z(G_2) \times Z(G_2) = G_2' \times G_2' \times G_2'. \tag{8}$$

On the other hand, by Proposition 2.7 (i), we have

$$\psi_3(\mathrm{st}_{G_3}(1)') = G_2' \times G_2' \times G_2'. \tag{9}$$

Thus, by (8) and (9), we have $Z(G_3) \leq \mathrm{st}_{G_3}(1)'$. Let $z \in Z(G_3)$. Since $z = z^a$, this yields that $\psi_3(z) = (c, c, c)$ for some $c \in G_2' = \langle [a, b]\rangle$. Hence $|Z(G_3)| = 3$ and $Z(G_3) = \langle z\rangle$, where $\psi_3(z) = ([a, b], [a, b], [a, b])$. □

Lemma 3.7. *Let $G = \langle a, b\rangle$ be the Gupta-Sidki 3-group. Then*

$$Z(G_3) \cap \{[b, g] \mid g \in G_3\} = 1.$$

Proof. Suppose that $1 \neq [b, g] \in Z(G_3)$. Then since $Z(G_3) \leq \mathrm{st}_{G_3}(1)'$, b and g commute modulo $\mathrm{st}_{G_3}(1)'$. We will first show that $g \in \mathrm{st}_{G_3}(1)$.

Since $|G_3/\mathrm{st}_{G_3}(1)' : \gamma_2(G_3/\mathrm{st}_{G_3}(1)')| = 3^2$ and $\mathrm{st}_{G_3}(1)/\mathrm{st}_{G_3}(1)'$ is an abelian maximal subgroup of $G_3/\mathrm{st}_{G_3}(1)'$, the quotient group $G_3/\mathrm{st}_{G_3}(1)'$ is a p-group of maximal class with an abelian maximal subgroup. Then being $b \in \mathrm{st}_{G_3}(1)$ yields that $g \in \mathrm{st}_{G_3}(1)$.

If $g \in \mathrm{st}_{G_3}(1)'$ then by (9), $\psi_3(g) \in Z(G_2 \times G_2 \times G_2)$. This implies that $\psi_3([b, g]) = (1, 1, 1)$, which is a contradiction. Hence $g \in \mathrm{st}_{G_3}(1) \smallsetminus \mathrm{st}_{G_3}(1)'$.

Write $g = b_0^{i_0} b_1^{i_1} b_2^{i_2} c$ for some $c \in \mathrm{st}_{G_3}(1)'$. Then

$$\begin{aligned}\psi_3([b, g]) &= \big([a, a^{i_0} b^{i_1} a^{2i_2}], [a^2, a^{2i_0+i_1} b^{i_2}], [b, b^{i_0} a^{2i_1+i_2}]\big)\\ &= \big([a, a^{i_0} b^{i_1} a^{2i_2}],\ b_2^{-i_2} b_0^{i_2},\ b_0^{-1} b_{2i_1+i_2}\big).\end{aligned}$$

Since $[b, g] \in Z(G_3)$, it follows that $\psi_3([b, g]) = \big([a, b], [a, b], [a, b]\big)^{\pm 1}$. Note that in G_2, we have $b_0 b_1 b_2 = 1$, and thus $[a, b] = b_1^{-1} b_0 = b_2 b_0^{-1}$. Then by the second and third components, we get $i_1 = 0$. Then the first component will be 1, which is a contradiction. □

Lemma 3.8. *Let $G = \langle a, b\rangle$ be the Gupta-Sidki 3-group. Then the element $v \in \mathrm{st}_{G_3}(1)'$ such that $\psi_3(v) = ([a, b], 1, 1)$ is not in the set $\{[a, g] \mid g \in G_3\}$.*

Proof. Since $\psi_3(\mathrm{st}_{G_3}(1)') = G_2' \times G_2' \times G_2'$, such an element v exists in $\mathrm{st}_{G_3}(1)'$. Suppose that $v = [a, g]$ for some $g \in G_3$. If we write $g = a^i h$ for some $h \in \mathrm{st}_{G_3}(1)$ then $v = [a, h]$. Write $\psi_3(h) = (h_1, h_2, h_3)$. Then

$$\psi_3((h^{-1})^a h) = (h_3^{-1} h_1, h_1^{-1} h_2, h_2^{-1} h_3) = ([a, b], 1, 1).$$

This implies that $h_1 = h_2 = h_3$ in G_2. Then $[a, b] = h_3^{-1} h_1 = 1$ in G_2, which is a contradiction. Thus, $v \in \mathrm{st}_{G_3}(1)' \smallsetminus \{[a, g] \mid g \in G_3\}$. □

In order to deal with the prime 3, we also need the following lemma.

Lemma 3.9. [9, Lemma 3.8] *Let G be a finite p-group and let $x \in G \smallsetminus \Phi(G)$ be an element of order p. If $t \in \Phi(G) \smallsetminus \{[x, g] \mid g \in G\}$ then*

$$\Big(\bigcup_{g \in G} \langle x\rangle^g\Big) \bigcap \Big(\bigcup_{g \in G} \langle xt\rangle^g\Big) = 1.$$

Theorem 3.10. *Let G be the Gupta-Sidki 3-group. Then the quotient $G/\operatorname{st}_G(3)$ is a Beauville group.*

Proof. Let $1 \neq u \in Z(G_3)$ and let $v \in \operatorname{st}_{G_3}(1)'$ be such that $\psi_3(v) = ([a,b],1,1)$. We claim that $\{a,b\}$ and $\{av, b^2u\}$ form a Beauville structure for G_3. Let $X = \{a,b,ab\}$ and $Y = \{av, b^2u, avb^2u\}$.

If $x = a$, which is of order 3, and $y = b^2u$ or avb^2u then $\langle x\rangle \cap \langle y\rangle^g = 1$ for all $g \in G_3$, as in the proof of Theorem 3.5. When $x = b$ and $y = av$ or avb^2u, the same argument applies. If we are in one of the following cases: $x = a$ and $y = av$, or $x = b$ and $y = b^2u = (bu^2)^2$, then the condition $\langle x\rangle \cap \langle y\rangle^g = 1$ follows from Lemma 3.9.

It remains to check the case $x = ab$ and $y \in Y$. If $y = b^2u$, which is of order 3, then we have $\langle x\rangle \cap \langle y\rangle^g = 1$, as in the previous paragraph. Now assume that $y = av$. Since $(av)^3 = v^{a^2}v^av$, we have

$$\psi_3((av)^3) = ([a,b],[a,b],[a,b]).$$

By Lemma 3.6, $(av)^3 \in Z(G_3)$. On the other hand,

$$\psi_3((ab)^3) = (b^a, b, b),$$

and hence $(ab)^3 \notin Z(G_3)$. Thus, the condition $\langle x\rangle \cap \langle y\rangle^g = 1$ follows.

Finally, we have to take $x = ab$ and $y = avb^2u$. Since $v \in \operatorname{st}_{G_3}(1)'$ and $\operatorname{st}_{G_3}(1)$ is of nilpotency class 2, $v \in Z(\operatorname{st}_{G_3}(1))$. So, $y = ab^2vu$, and $y^3 = (ab^2v)^3$. Observe that

$$(ab^2v)^3 = b_2^2b_1^2b_0^2v^{a^2}v^av.$$

By taking into account that $b_0b_1b_2 = 1$ in G_2, we get

$$\psi_3(y^3) = (b^{-1}, (b^{-1})^a, (b^{-1})^a).$$

If $\langle (ab)^3\rangle = \langle y^3\rangle^g$ for some $g \in G_3$, then $(ab)^3 = (y^{3i})^g$ for $i = 1$ or 2. Since $\operatorname{st}_{G_2}(1)$ is abelian, the components of $\psi_3((y^{3i})^g)$ are of the form $(b^{-i})^{a^m}$, with $m \in \{0,1,2\}$. Also one of the components of $\psi_3((ab)^3)$ is b. Then by equality $(ab)^3 = (y^{3i})^g$, we have $(b^{-i})^{a^m} = b$ in G_2 for some m. Thus, $[a^m, b^i] = b^{1+i} \in G_2'$, and this is true only if $1 + i \equiv 0 \pmod 3$. Thus

$$(ab)^3 = (y^{-3})^g.$$

Note that

$$\psi_3((ab)^3) = (b_1, b_0, b_0) \quad \text{and} \quad \psi_3(y^{-3}) = (b_0, b_1, b_1). \tag{10}$$

We write $g = a^ih$ for some $h \in \operatorname{st}_{G_3}(1)$ and $0 \leq i \leq 2$. Then the equality $(ab)^3 = (y^{-3})^g$ and (10) imply that $\psi_3(h)$ has to be congruent to one of the following modulo $\operatorname{st}_{G_2}(1) \times \operatorname{st}_{G_2}(1) \times \operatorname{st}_{G_2}(1)$: (a, a^2, a^2), $(1,1,a^2)$ or $(1,a^2,1)$. However, since G is periodic, the product of the powers of a in the components of $\psi_3(h)$ has to be 1. Thus, we conclude that there is no such h in G_3, and therefore $\langle (ab)^3\rangle \neq \langle y^3\rangle^g$ for any $g \in G_3$. This completes the proof. □

The following result, which gives a sufficient condition to lift a Beauville structure from a quotient group, is Lemma 4.2 in [12].

Lemma 3.11. *Let G be a finite group and let $\{x_1, y_1\}$ and $\{x_2, y_2\}$ be two sets of generators of G. Assume that, for a given $N \trianglelefteq G$, the following hold:*

(i) *$\{x_1N, y_1N\}$ and $\{x_2N, y_2N\}$ form a Beauville structure for G/N.*
(ii) *$o(g) = o(gN)$ for every $g \in \{x_1, y_1, x_1y_1\}$.*

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ form a Beauville structure for G.

We are now ready to give the proof of Theorem A.

Theorem 3.12. *Let G be a periodic GGS-group over the p-adic tree. Then the quotient $G/\operatorname{st}_G(n)$ is a Beauville group if $p \geq 5$ and $n \geq 2$, or $p = 3$ and $n \geq 3$.*

Proof. By Theorem 3.1, $G/\operatorname{st}_G(2)$ is a Beauville group if and only if $p \geq 5$. Now we assume that $n \geq 3$. If $p \geq 5$ then by Theorem 3.5, $G/\operatorname{st}_G(3)$ is a Beauville group with Beauville triples $X = \{a^{-2}, ab, a^{-1}b\}$ and $Y = \{ab^2, b, a^3b\}$. Since $o(ab\operatorname{st}_G(n)) = o(a^{-1}b\operatorname{st}_G(n)) = p^2$ for any $n \geq 3$, we can apply Lemma 3.11, and hence $G/\operatorname{st}_G(n)$ is a Beauville group for every $n \geq 3$. Similarly, if $p = 3$ then the Beauville structure of $G/\operatorname{st}_G(3)$ given in Theorem 3.10 is inherited by $G/\operatorname{st}_G(n)$ for any $n \geq 3$. □

We close this section with the proof of Theorem B.

Theorem 3.13. *Let G be a periodic GGS-group over the p-adic tree for an odd prime p. Then all but finitely many quotients of G are Beauville groups.*

Proof. We first note that by Theorem 4 in [21], the group G is just infinite, that is, all proper quotients of G are finite.

Let N be a non-trivial normal subgroup of G. We first assume that $N \leq \operatorname{st}_G(3)$. Then by using the same idea in the proof of Theorem 3.12, we can lift the Beauville structure of $G/\operatorname{st}_G(3)$ given in Theorems 3.5 and 3.10 to the quotient G/N. Hence G/N is a Beauville group if $1 \neq N \leq \operatorname{st}_G(3)$.

We now assume that $N \not\leq \operatorname{st}_G(3)$. Then by Proposition 2.4 in [10] and by the proof of Theorem 2.7 in [10], we have $\operatorname{st}_G(6) \leq N$ or $\operatorname{st}_G(7) \leq N$ according as the defining vector $\mathbf{e}$ is non-symmetric or symmetric. Consequently, there are only finitely many normal subgroup N of G such that $N \not\leq \operatorname{st}_G(3)$. This completes the proof. □

4. Quotients of non-periodic GGS-groups

Let G be a non-periodic GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1})$. In this section, we will prove that the quotients of G by its level stabilizers $\operatorname{st}_G(n)$ are not Beauville groups.

Since G is non-periodic, we have $\Sigma_{i=1}^{p-1} e_i = \alpha \neq 0$. Thus, by Lemma 2.5, the rank of the circulant matrix $C(\mathbf{e}, 0)$ is p, and according to Proposition 2.4, $G/\operatorname{st}_G(2)$ is a p-group of maximal class of order p^{p+1}. Since

$$G/\operatorname{st}_G(2) \lesssim \operatorname{Aut}\mathcal{T}/\operatorname{st}(2) \cong C_p \wr C_p,$$

this implies that $G/\operatorname{st}_G(2) \cong C_p \wr C_p$, and thus it is of exponent p^2.

Note that the $p+1$ maximal subgroups of G are $\langle a, G' \rangle$, $\langle b, G' \rangle$ and $M_i = \langle ab^i, G' \rangle$ for all $1 \leq i \leq p-1$. We write $M_{n,i}$ instead of $M_i/\operatorname{st}_G(n)$, for every $1 \leq i \leq p-1$ and for $n \geq 2$. Then $M_{n,i} = \langle ab^i, G_n' \rangle$ is a maximal subgroup of $G_n = G/\operatorname{st}_G(n)$.

The following proposition gives the relation between the pth powers of elements in $M_{n,i} \smallsetminus G_n'$ for all $1 \leq i \leq p-1$.

Proposition 4.1. *Let $G_n = G/\operatorname{st}_G(n)$ for $n \geq 2$. Then the following hold:*

(i) *All elements in $M_{n,i} \smallsetminus G_n'$ are of order p^n for every $1 \leq i \leq p-1$.*

(ii) *If g is an element in $M_{n,i} \smallsetminus G_n'$ such that $g = (ab^i)^k w$ for some $w \in G_n'$ and for some $1 \leq k \leq p-1$, then*

$$g^{p^{n-1}} = (ab^i)^{kp^{n-1}}.$$

(iii) *Cyclic subgroups generated by p^{n-1}st powers of elements in $\bigcup_{i=1}^{p-1} M_{n,i} \smallsetminus G_n'$ coincide.*

Proof. We will first show the result for $n = 2$. Since $G_2 \cong C_p \wr C_p$, all elements in $M_{2,i} \smallsetminus G_2'$ are of order p^2 for $1 \leq i \leq p-1$, and hence (i) holds. We know that for any element $u \in M_{2,i} \smallsetminus G_2'$, we have $\mathrm{Cl}_{G_2}(u) = uG_2'$ and $u^p \in Z(G_2)$. Thus, if $g = (ab^i)^k w$ for some $w \in G_2'$ then g and $(ab^i)^k$ are conjugate in G_2. Since $(ab^i)^{kp} \in Z(G_2)$, this implies that in G_2

$$g^p = (ab^i)^{kp},$$

and so (ii) holds. It remains to show that (iii) holds. Since all elements in $\bigcup_{i=1}^{p-1} M_{2,i} \smallsetminus G_2'$ are of order p^2, cyclic subgroups generated by their pth powers are equal to $Z(G_2)$.

In order to prove the proposition we will use induction on n. Before proceeding to the induction step, consider the element ab^i of G for $1 \leq i \leq p-1$. We know that $(ab^i)^p = b^i_{p-1} b^i_{p-2} \dots b^i_1 b^i_0$. Then the condition $\Sigma_{i=1}^{p-1} e_i = \alpha$ implies that

$$\psi((ab^i)^p) = \big(a^{i\alpha}(b^i)^{a^{ie_1}}, a^{i\alpha}(b^i)^{a^{i(e_1+e_2)}}, \dots, b^i a^{i\alpha}, a^{i\alpha} b^i\big).$$

Then

$$\psi((ab^i)^{kp}) = \big((a^\alpha b)^{ki} u_1, (a^\alpha b)^{ki} u_2, \dots, (a^\alpha b)^{ki} u_p\big),$$

for $u_\ell \in G'$. Let $g = (ab^i)^k w$ for some $w \in G'$. By the previous paragraph we know that

$$g^p \equiv (ab^i)^{kp} \pmod{\mathrm{st}_G(2)}.$$

Write $g^p = (ab^i)^{kp} t$ for some $t \in \mathrm{st}_G(2)$. By Proposition 2.6, we have $\mathrm{st}_G(2) = [\mathrm{st}_G(1), \mathrm{st}_G(1)]$, and hence $\psi(t) \in G' \times \overset{p}{\dots} \times G'$. Thus

$$\psi(g^p) = \big((a^\alpha b)^{ki} w_1, (a^\alpha b)^{ki} w_2, \dots (a^\alpha b)^{ki} w_p\big),$$

where $w_\ell \in G'$.

Now assume that the proposition holds for $n \geq 2$. Then

$$\psi_{n+1}((ab^i)^{kp^n}) = \big(((a^\alpha b)^{ki} u_1)^{p^{n-1}}, ((a^\alpha b)^{ki} u_2)^{p^{n-1}}, \dots ((a^\alpha b)^{ki} u_p)^{p^{n-1}}\big),$$

$$\psi_{n+1}(g^{p^n}) = \big(((a^\alpha b)^{ki} w_1)^{p^{n-1}}, ((a^\alpha b)^{ki} w_2)^{p^{n-1}}, \dots ((a^\alpha b)^{ki} w_p)^{p^{n-1}}\big),$$

where each component is an element of G_n. By the induction hypothesis, all components are equal in G_n, and of order p. This completes the proof. □

Theorem 4.2. *Let G be a non-periodic GGS-group over the p-adic tree. Then the quotient $G/\mathrm{st}_G(n)$ is not a Beauville group for any $n \geq 1$.*

Proof. Let $G_n = G/\mathrm{st}_G(n)$. Clearly, G_1 is not a Beauville group since it is cyclic of order p. Thus, we assume that $n \geq 2$.

We argue by way of contradiction. Suppose $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are two systems of generators of G_n such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$. Since no two of the elements x_1, y_1 and $x_1 y_1$ can lie in the same maximal subgroup of G_n, it follows from Proposition 4.1(i) that one of these elements, say x_1, is of order p^n. Similarly, we may assume that the order of x_2 is also p^n. Then by Proposition 4.1(iii), we conclude that $\langle x_1^{p^{n-1}} \rangle = \langle x_2^{p^{n-1}} \rangle$, which is a contradiction. □

Acknowledgments

We would like to thank G. A. Fernández-Alcober for proposing us this research problem and for helpful comments.

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