On numerical regularity of the longest-edge bisection algorithm

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Abstract: The finite element method usually requires regular or strongly regular families of partitions in order to get guaranteed a priori or a posteriori error estimates. In this paper we examine the recently invented longest-edge bisection algorithm that produces face-to-face simplicial partitions of a given bounded polytopic domain in arbitrary space dimension. First, we prove that the regularity of the family of partitions generated by this algorithm is equivalent to its strong regularity. Second, we present $3d$ numerical examples, which demonstrate that the technique seems to produce regular (and therefore strongly regular) families of tetrahedral partitions. However, a mathematical proof of this statement is still an open problem.

Keywords: bisection algorithm, conforming finite element method, regular family of partitions, nested tetrahedral partitions, simplicial elements.

Mathematical Subject Classification: 65M50, 65N30, 65N50

1 Introduction

When solving problems of mathematical physics by the finite element method, we have to decompose the investigated domain into small subdomains – called *elements*. To achieve a high numerical accuracy, elements should be small. In addition, at those parts where the exact solution of a particular problem possesses some singularities, we have to use smaller elements than those at the parts, where the exact solution is smooth enough. In this paper we shall deal only with face-to-face simplicial partitions, i.e., conforming partitions into triangles, tetrahedra, 4-simplexes, etc.

The longest-edge bisection algorithm is one of the most popular refinement algorithm in the finite element method, since it is very simple and can be easily applied also in higher dimensions. Martin Stynes in his papers [26]–[29] (see also [1] and [24]) examined the regularity of a family triangulations obtained by a recursive use of the longest-edge bisection algorithm. This algorithm bisects simultaneously all triangles by medians to the longest edge of each triangle in a given triangulation (see Figure 1). In this way, a family of nested triangulations is obtained.

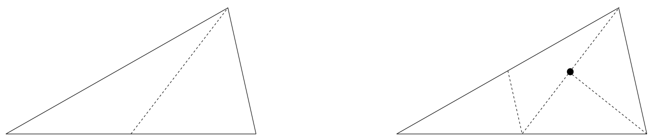

Figure 1: The classical longest-edge bisection algorithm may produce triangulations that are not face-to-face. Broken lines stand for new bisections.

M. Stynes proved that during the infinite bisection process only a finite number of similarity-distinct subtriangles are generated, which proves the regularity of the resulting family of triangulations, since the Zlámal's minimum angle condition (see [30]) is satisfied, i.e., all angles α of all triangles are not less than some fixed constant $\alpha_0 > 0$ when the mesh size h tends to zero,

$$\alpha \geq \alpha_0. \tag{1}$$

As usual, the mesh size h stands for the length of the longest edge in a given triangulation.

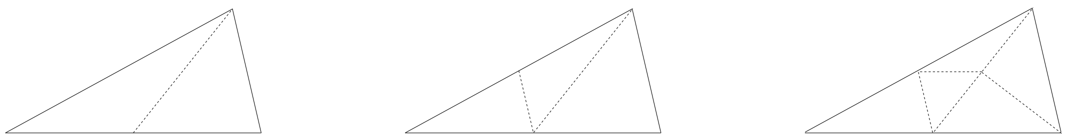

Figure 2: A modified longest-edge bisection algorithm that always produces face-to-face triangulations.

However, this may lead to the so-called hanging nodes and thus refined triangulations may not all be face-to-face, in general (see Figure 1). Therefore, in [12] a modified version of the bisection algorithm by medians was introduced, where only simplices sharing the longest edge of the whole triangulation are bisected at each step (see Figure 2). In this way, all produced triangulations are face-to-face and moreover, the corresponding family of triangulations satisfies the minimum angle condition (1) with computable and known α_0, see [12]. In [8] and [9] we modified the above bisection algorithm so that it may also generate locally refined simplicial face-to-face partitions. This goal was achieved by introducing a Lipschitz continuous mesh density function which tell us, which simplices have to be chosen for bisections. We proved that this generalized algorithm also produces nested face-to-face partitions.

Bisection-type algorithms are very favourite in finite element community, since they enable us to perform local mesh refinement quite easily. In [3, 13] local mesh refinements are generated by means of the famous red refinement. A given tetrahedron is first subdivided into eight tetrahedra by mid-lines (see Figure 3). Then each of its neighbouring tetrahedra has to be subdivided be means of the red-green refinement into four subtetrahedra or by means of the green refinement into two subtetrahedra to get face-to-face partitions. This process is quite complicated from a combinatorial and programming point of view, although the corresponding family of partitions is regular (see [13] for the proof). Thus, a practical realization of the longest-edge bisection algorithm is much simpler than those of algorithms based on red, red-green, and green refinement. However, for the dimension $d \geq 3$ it is not known whether the resulting family of partitions is regular (see Definition 2.1 below). Therefore in this work we investigate the regularity properties of the $3d$ meshes generated by the longest-edge bisection numerically.

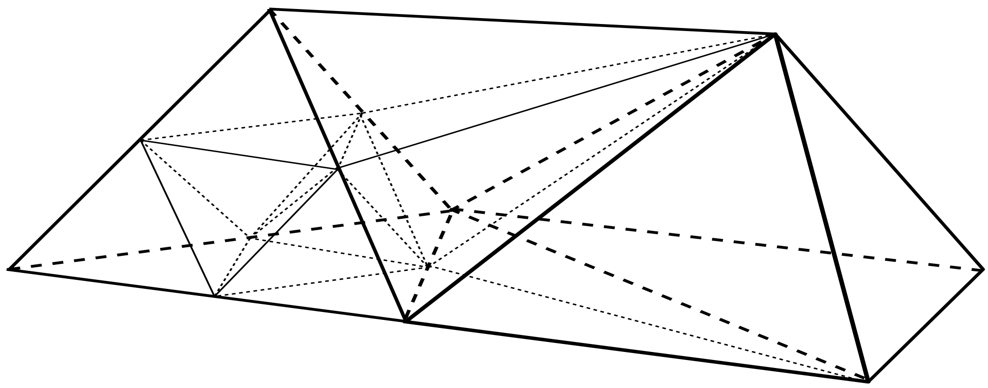

Figure 3: Red, red-green, and green refinements of several adjacent tetrahedra.

For arbitrary space dimension d, Kearfott [11] proved that the longest-edge algorithm produces d-simplices whose largest diameter tends to zero. In this algorithm each d-simplex is bisected by a $(d-1)$-dimensional hyperplane passing through the midpoint of its longest edge and through its vertices that do not belong to this edge.

In [2, 10, 14, 15, 16, 17, 18] several other bisection-like algorithms are investigated, some of them do not necessarily halve the longest edge. The longest-edge algorithm was originally designed for solving systems of nonlinear algebraic equations (see [25, 7]). Later, mainly due to effort of Rivara [19, 20, 21, 22, 23], bisection methods started to become popular also in practical use of the finite element method.

2 Theoretical background

The convex hull of $d+1$ points in $\mathbf{R}^d$ for $d \in \{1, 2, 3, \dots\}$, which are not contained in a hyperplane of dimension $d-1$, is called a *d-simplex* or just a *simplex*. The angles between its $(d-1)$-dimensional facets are called *dihedral*.

Next we define a *simplicial partition* $\mathcal{T}_h$ over a bounded polytope $\overline{\Omega} \subset \mathbf{R}^d$. We subdivide $\overline{\Omega}$ into a finite number of simplices so that their union is $\overline{\Omega}$, any two simplices have disjoint interiors and any facet of any simplex is a facet of another simplex from the partition or belongs to the boundary $\partial\Omega$ of $\overline{\Omega}$. The discretization parameter h is the length of the longest edge in a given simplicial partition. We see that simplicial partitions are face-to-face.

The set $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ is named a *family of simplicial partitions* if for any $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

Definition 2.1 *A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of simplicial partitions of a bounded polytope $\overline{\Omega}$ is called regular if there exists a constant $C > 0$ such that for all partitions $\mathcal{T}_h \in \mathcal{F}$ and for all simplices $S \in \mathcal{T}_h$ we have*

$$\operatorname{meas}_d S \geq C(\operatorname{diam} S)^d. \tag{2}$$

It can be easily checked that the regularity for $d = 2$ is equivalent to condition (1). Various equivalent definitions can be found in [4] and [5].

Lemma 2.1 *Let $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ be a regular family of simplicial partitions of a bounded polytope $\overline{\Omega}$ and let S be an arbitrary d-simplex from an arbitrary partition $\mathcal{T}_h \in \mathcal{F}$. Denote by a the length of the shortest edge of S and by e the length of its longest edge. Then*

$$a \geq \kappa e,$$

where $\kappa > 0$ is a fixed constant independent of h.

P r o o f : Let $A_1, A_2, \dots, A_{d+1}$ be vertices of S, let F_i be the facet of S opposite to the vertex A_i, and let v_i be the altitude from the vertex A_i to the facet F_i. Without loss of generality we may assume that the shortest edge is A_1A_2. Then clearly $a \geq v_1$ and from (2) we obtain

$$Ce^d = C(\text{diam } S)^d \leq \text{meas}_d S = \frac{1}{d} v_1 \text{meas}_{d-1} F_1 \leq \frac{a}{d} \text{meas}_{d-1} F_1 \leq \frac{a}{d} e^{d-1},$$

where the last inequality is due to the fact that the $(d-1)$-simplex F_1 is contained in a $(d-1)$-dimensional hypercube whose edge has length e. Dividing the left-hand and right-hand sides of the above inequality by e^{d-1}, the lemma follows. ■

Definition 2.2 *A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of simplicial partitions of a bounded polytope $\overline{\Omega}$ is called strongly regular if there exists a constant $C > 0$ such that for all partitions $\mathcal{T}_h \in \mathcal{F}$ and for all simplices $S \in \mathcal{T}_h$ we have*

$$\operatorname{meas}_d S \geq C h^d. \tag{3}$$

One step of *the longest-edge bisection algorithm* in any space dimension is defined as follows:

I) Choose the longest edge ℓ in a given face-to-face simplicial partition.

II) Bisect each simplex S sharing ℓ by a hyperplane passing through the midpoint of ℓ and the vertices of S that do not belong to ℓ.

From the second point II) we can easily find by induction that all partitions generated by the longest-edge bisection algorithm remain face-to-face. In fact, this algorithm performs the so-called green refinements.

Theorem 2.1 *Let d be an arbitrary space dimension and let $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ be a family of simplicial partitions of $\overline{\Omega}$ generated by the longest-edge bisection algorithm. Then $\mathcal{F}$ is regular if and only if it is strongly regular.*

P r o o f : Every strongly regular family is clearly regular.

To prove the converse implication assume that all edges from the initial partition were already halved at least once, which occurs after a finite number of steps. It is enough to analyse only partitions generated after these initial refinement steps. Denote such partitions by $\mathcal{T}_h$, where h is the length of the longest edge. Let S be any d-simplex (with the shortest edge of the length a) from some partition $\mathcal{T}_h$. Since all edges from the initial partition were already halved, there exists exactly one mother d-simplex S' from some previous partition such that the bisection of S' in the next step yielded S and let the diameter of S' be h', i.e. $S' \in \mathcal{T}_{h'}$. Denoting by e the longest edge of S, we find that

$$2e \geq h' \geq h. \tag{4}$$

From (2), Lemma 2.1, and (4) we get

$$\text{meas}_d S \geq C(\text{diam } S)^d \geq Ca^d \geq C\kappa^d e^d \geq C\kappa^d 2^{-d} h^d.$$

Therefore, (3) is valid. ■

Corollary 2.1 *In a view of the above theorem, it is enough to prove only one of the two regularity results.*

Definition 2.3 *A d-simplex S is said to be similar to a d-simplex S', if there exists a nonsingular $d \times d$-matrix A, a vector $b \in \mathbf{R}^d$, and a positive constant c such that*

1) $A^\top A = c^2 I$, where I is the identity matrix,

2) $F(S) = S'$, where $F(x) = Ax + b$ for $x \in S \subset \mathbf{R}^d$.

The constant c is the scaling parameter and b is the translation vector. From 1) we observe that A/c is the orthogonal matrix. If $\det A > 0$ then S and S' have the same shape and they are *directly similar*. If $\det A < 0$ then they are *indirectly similar*. The linear mapping F thus consists of translation, rotation, scaling and for $\det A < 0$ also of mirroring. Therefore, S and S' have the same dihedral angles.

To check whether two simplices arising during a real bisection process on computer are similar we shall apply the following simple lemma. Denote by $\|\cdot\|$ the standard Euclidean matrix norm.

Proposition 2.1 *Let S and S' be two d-simplices. Let A be the $d \times d$ matrix, $C = (C_{ij}) = A^\top A$, and $b \in \mathbf{R}^d$ be such that*

$$\|I - C/C_{11}\| < \varepsilon \quad \forall \varepsilon > 0 \tag{5}$$

and

$$F(S) = S',$$

where $F(x) = Ax + b$. Then S is similar to S'.

P r o o f : Formula (5) implies that $I = C/C_{11} = A^\top A/C_{11}$ and $C_{11} = \cdots = C_{dd}$. Since the matrix $A^\top A$ is symmetric and positive definite, C_{11} is positive. Setting $c = \sqrt{C_{11}}$, we find that the condition 1) from the previous Definition 2.3 is valid. Thus S is similar to S', because condition 2) follows trivially from the assumptions of the proposition. ■

Since there are $(d+1)!$ linear mappings F such that 2) of Definition 2.3 holds, we have to find at least one associated matrix A such that (5) is true. For instance, in the case $d = 3$ we have to calculate at most 24 matrices and verify whether inequality (5) holds. This is, of course, impossible in computer arithmetic. Therefore, in double precision arithmetic we have chosen $\varepsilon = 10^{-10}$ and assume that two simplices are similar if the norm appearing in (5) is smaller than 10^{-10}.

Remark 2.1 *Since the criterion (5) may be sensitive to rounding errors, we propose another criterion, based on the following inequality*

$$\|I\|C\| - C\|I\|\| < \varepsilon \quad \forall \varepsilon > 0. \tag{6}$$

3 Bisection of tetrahedral partitions

In this section we examine numerically the regularity properties of simplicial partitions generated by the longest-edge bisection algorithm for $d = 3$. Obviously, it is enough to study only particular tetrahedra from some initial face-to-face tetrahedral partition.

Tetrahedra have much more angles than triangles, which obviously complicates the analysis and numerical implementation in $3d$. Moreover, it is often even not possible to formulate or observe the properties of bisected tetrahedra, which could be natural analogues of performance of bisections in $2d$.

To illustrate the latter fact, we give several observations. Set $A = (-1, 0, 0)$, $B = (1, 0, 0)$, $C = (-2, 2, -1)$, and $D = (2, 2, 1)$. Then there are the following (dihedral) angles at edges: $25.21°$ at $|CD| = 4.47$, $28.56°$ at $|BD| = |AC| = 2.45$, $53.13°$ at $|AB| = 2$, $133.09°$ at $|BC| = |AD| = 3.75$.

Observation 1: The largest (dihedral) angle is not opposite to the longest edge CD.

Observation 2: The tertahedron is obtuse. However, after the first bisection of the longest edge CD the largest (dihedral) angle increases to $150.79°$. This does not happen for obtuse triangles in $2d$.

Observation 3: If we bisect the edge AD opposite the biggest (dihedral) angle $133.09°$, then one of the new angles is bigger, namely $140.77°$.

In some examples in below, for a given fixed tetrahedron we shall denote by s_n the number of non-similar subtetrahedra that have been generated during the first n steps of the longest-edge bisection algorithm (i.e., not during the n–th refinement step only).

The sequence $\{s_n\}_{n=0}^{\infty}$, where $s_0 = 1$, is obviously nondecreasing. In many cases we get $s_n < s_{n+1}$ for all n, but sometimes the sequence remains constant after several initial steps, i.e. there may exist an integer $n_0 > 0$ such that

$$s_n = s_{n+1} \quad \forall n \geq n_0. \tag{7}$$

This property will be now illustrated in several numerical examples. Proposition 2.1 was applied to check whether two or more subtetrahedra have the same shape.

Example 3.1 *First, consider the path tetrahedron with vertices* $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 1)$, *and* $(1, 1, 1)$. *Then* $s_0 = 1, s_1 = 2, s_2 = 3$ *(see Figure 4) and the whole sequence* s_n *is*

$$1, 2, 3, 3, 3, 3, 3, \ldots$$

In this case $n_0 = 2$ and there appear only 3 different shapes of subtetrahedra for the longest-edge bisection algorithm.

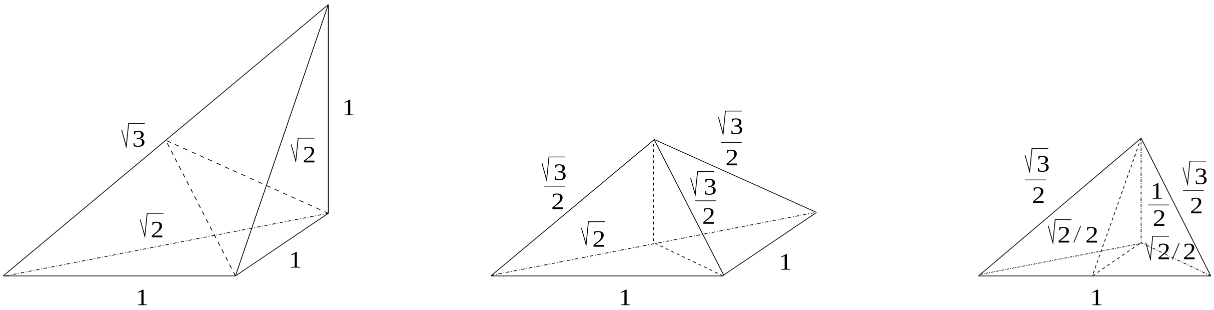

Figure 4: The first three steps of the longest-edge algorithm applied to the path tetrahedron from Example 3.1.

Example 3.2 *Now consider the Sommerville space-filler tetrahedron with vertices $(-1,0,0)$, $(1,0,0)$, $(0,-1,1)$, and $(0,1,1)$. After the first bisection step we get two congruent tetrahedra whose shape is sketched in the right part of Figure 4. Further bisections thus produce subtetrahedra that are sketched in Figure 4 up to scaling. Hence, the sequence s_n reads as*

$$1,\ 2,\ 3,\ 4,\ 4,\ 4,\ 4, \ldots$$

and $n_0 = 3$.

Remark 3.1 *Since subtetrahedra in previous examples have similar shapes, the construction of the stiffness matrix can be performed much faster. The element matrix can be computed only once for each equivalence class of tetrahedra.*

Example 3.3 *Consider the cube corner tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Its three longest edges have length $\sqrt{2}$. Figure 5 illustrates a few bisection steps. For the sake of clarity, we depiced only new shapes of subtetrahedra to reduce the number of lines in Figure 5. We see that the bisection process is not uniquely determined. After four steps, there are two longest edges. In this special case, the process bifurcates yielding different shapes of resuling tetrahedra in each bifurcation branch (see Figure 5). This makes the theoretical analysis difficult. Nevertheless, in Figure 6 we observe that the aspect ratio $\mathrm{vol}\, S/(\mathrm{diam}\, S)^3$ seems to be numerically bounded from below by a positive constant for randomly chosen bifurcation branches.*

Example 3.4 *For the regular tetrahedron with vertices $(0,0,0)$, $(1,1,0)$, $(0,1,1)$, and $(1,0,1)$ similar nonuniqueness appear as in Example 3.3. In Figure 7 we see the behaviour of the aspect ratio $\mathrm{vol}\, S/(\mathrm{diam}\, S)^3$ for the bisections of the regular tetrahedron.*

Example 3.5 *Now we make similar numerical experiments with two highly irregular tetrahedra – the needle tetrahedron having vertices $(0, \frac{1}{10\sqrt{3}}, 0)$, $(\frac{1}{20}, -\frac{1}{20\sqrt{3}}, 0)$, $(-\frac{1}{20}, -\frac{1}{20\sqrt{3}}, 0)$, $(0,0,1)$ (its base is an equilateral triangle in xy-plane with the edge length equal to 0.1), and the wedge element with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,0.1)$. In both cases, the aspect ratio $\mathrm{vol}\, S/(\mathrm{diam}\, S)^3$ seems to be numerically bounded from below by a positive constant.*

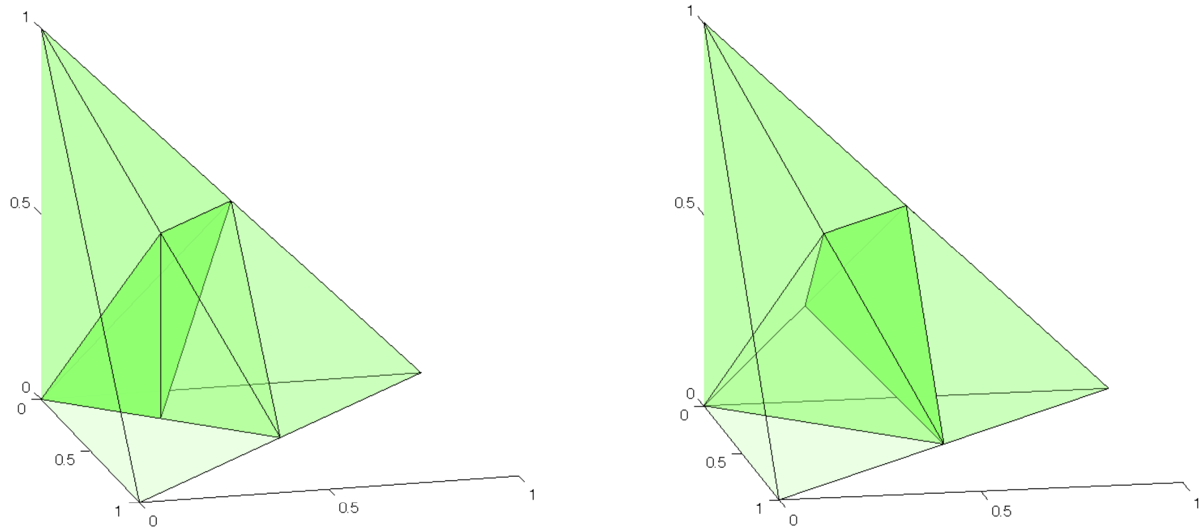

Figure 5: The bisection process of the cube corner tetrahedron does not produce uniquely determined shapes of subtetrahedra – see dark green subtetrahedra. For the sake of clarity, only different shapes are illustrated.

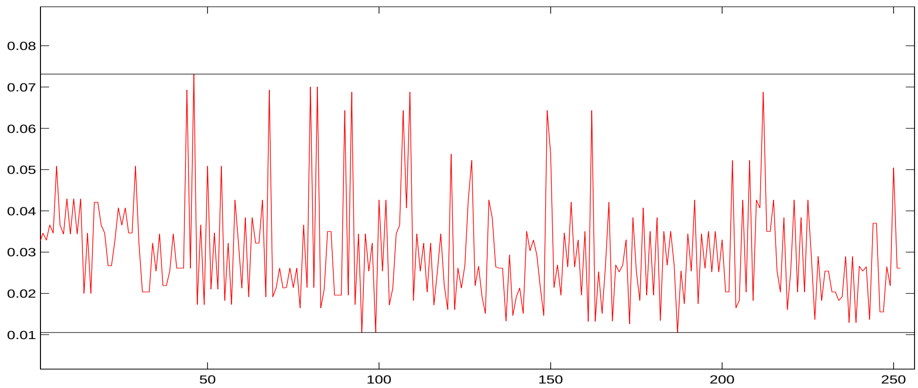

Figure 6: During the bisection process of the cube corner tetrahedron, the aspect ratio $\mathrm{vol}\, S/(\mathrm{diam}\, S)^3$ seems to be numerically bounded from below by a positive constant.

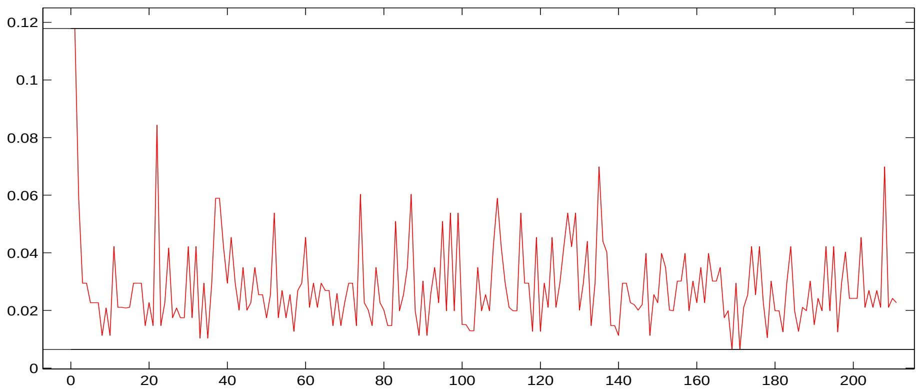

Figure 7: During the bisection process of the regular tetrahedron, the aspect ratio $\mathrm{vol}\, S/(\mathrm{diam}\, S)^3$ seems to be numerically bounded from below by a positive constant.

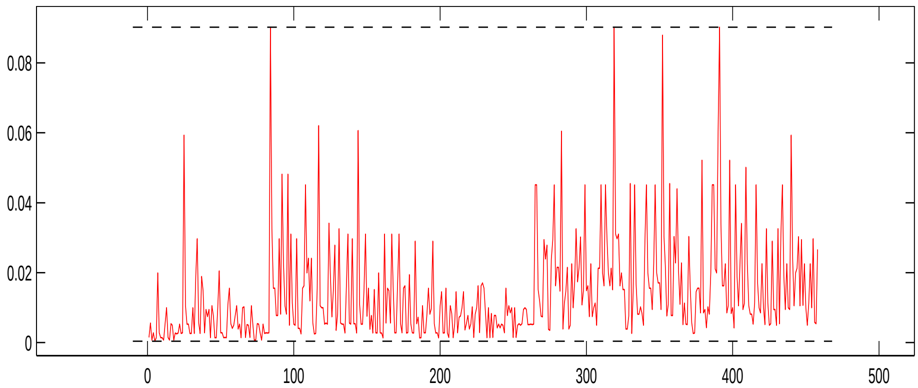

Figure 8: The behaviour of the aspect ratio for the above defined wedge tetrahedron.

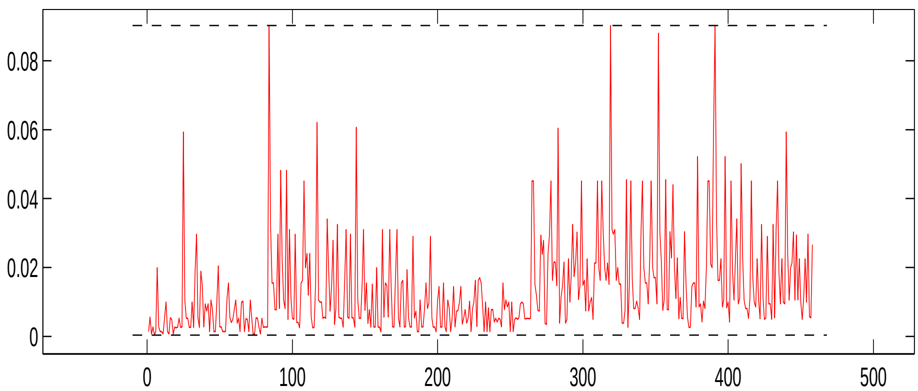

Figure 9: The behaviour of the aspect ratio for the above defined needle tetrahedron.

Remark 3.2 *In many cases, we can end up with several edges in the mesh having the same length. Depending in which order these edges are bisected, different number of shapes will be generated. For example, the C++ code gives* 8 *shapes for the cube corner tetrahedron, compared to* 7 *in the MATLAB code. The order of bisections in C++ code is different. This makes the analysis in 3d much harder that in 2d.*

4 Conclusions

Our numerical examples of Section 3 indicate that the longest-edge bisection algorithm seems to produce the regular (and, therefore, by Theorem 2.1, strongly regular) families of partitions. However, a mathematical proof of this statement is an open problem.

If the family of tetrahedral partitions is proved to be regular, then an analogue of the minimal angle condition (1) is valid. For $d = 3$ it says that all dihedral angles and also all angles between edges are greater than some fixed positive angle α_0. The generalization of the Zlámal minimum angle condition to an arbitrary space dimension is given in [6].

Regular families of tetrahedral partitions enable us to achieve the optimal interpolation order, to obtain the convergence of the finite element method, to derive efficient a posteriori error estimates, etc. Subtetrahedra produced by the longest-edge bisection algorithm thus do not degenerate to a plane when $h \to 0$.

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