SCATTERING WITH CRITICALLY-SINGULAR AND δ-SHELL POTENTIALS

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Abstract. We consider a scattering problem for electric potentials that have a component which is critically singular in the sense of Lebesgue spaces, and a component given by a measure supported on a compact Lipschitz hypersurface. We study direct and inverse point-source scattering under the assumptions that the potentials are real-valued and compactly supported. To solve the direct scattering problem, we introduce two functional spaces —sort of Bourgain type spaces— that allow to refine the classical resolvent estimates of Agmon and Hörmander, and Kenig, Ruiz and Sogge. These spaces seem to be very useful to deal with the critically-singular and δ-shell components of the potentials at the same time. Furthermore, these spaces and their corresponding resolvent estimates turn out to have a strong connection with the estimates for the conjugated Laplacian used in the context of the inverse Calderón problem. In fact, we derive the classical estimates by Sylvester and Uhlmann, and the more recent ones by Haberman and Tataru after some embedding properties of these new spaces. Regarding the inverse scattering problem, we prove uniqueness for the potentials from point-source scattering data at fix energy. To address the question of uniqueness we combine some of the most advanced techniques in the construction of complex geometrical optics solutions.

1. Introduction

In this paper we study a point-source scattering problem for electric potentials that are a combination of *critically-singular potentials* and *δ-shell potentials.* More precisely, we are interested in real potentials of the form

$$V = V^0 + \alpha \,\mathrm{d}\sigma \tag{1}$$

where V^0 stands for the critically-singular component of the potential and $\alpha\,\mathrm{d}\sigma$ is its δ-shell component. Here $V^0 \in L^{d/2}(\mathbb{R}^d;\mathbb{R})$, σ denotes the surface measure of Γ, $\alpha \in L^\infty(\Gamma;\mathbb{R})$ and Γ is a compact hypersurface which is locally described by the graphs of Lipschitz functions. Additionally, we assume the support of V to be contained in the ball $B_0 = \{x \in \mathbb{R}^d : |x| < R_0\}$ with $R_0 \geq 1$. For this class of potentials, we study direct and inverse point-source scattering in dimension $d \geq 3$. However, we carry out part of our analysis in dimension $d \geq 2$, emphasizing when $d \geq 3$ is required.

1.1. **Direct scattering.** The direct scattering theory for potentials as V follows the general scheme of more regular potentials. First, we consider an incident wave u_{in}, which solves the equation $(\Delta + \lambda)u_{\mathrm{in}} = 0$ in $\mathbb{R}^d \setminus \{y\}$ with $|y| \geq R_0$. Then, the scattering solution u_{sc} solves

$$(\Delta + \lambda - V)u_{\mathrm{sc}} = V u_{\mathrm{in}} \text{ in } \mathbb{R}^d,$$

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and satisfies the ingoing or outgoing Sommerfeld radiation condition (SRC for short). There are at least two possible ways of showing the existence of the scattering solution u_{sc}. One based on a Neumann series argument, which consists of solving the problem

$$\begin{cases} (\Delta + \lambda) u_n = V u_{n-1} & \text{in } \mathbb{R}^d, \\ u_n \text{ satisfying SRC} \end{cases} \tag{2}$$

for each $n \in \mathbb{N}$ with $u_0 = u_{\mathrm{in}}$, and showing that $\sum_{n\in\mathbb{N}} u_n$ makes sense. In this case, the scattering solution is given by $u_{\mathrm{sc}} = \sum_{n\in\mathbb{N}} u_n$. The problem (2) can be solved using an appropriate inverse, denoted throughout the paper by $(\Delta+\lambda\pm i0)^{-1}$ —the sign $\pm$ accounts for the ingoing and outgoing radiation conditions. Thus,

$$u_n = (\Delta + \lambda \pm i0)^{-1}(V u_{n-1}) = [(\Delta + \lambda \pm i0)^{-1} \circ V]^n (u_0),$$

and consequently, in order for $\sum_{n\in\mathbb{N}} u_n$ to converge, we only have to see that the linear operator $(\Delta+\lambda\pm i0)^{-1}\circ V$ is bounded in certain Banach space and its norm is strictly less than 1 —in short, it is a contraction. Here and throughout the article, V denotes not only the potential but also the operator multiplication by V.

Another possible way to prove the existence of the scattering solution is via Fredholm theory, which consists in choosing u_{sc} as the solution of

$$\begin{cases} (I - (\Delta + \lambda \pm i0)^{-1} \circ V) u_{\mathrm{sc}} = (\Delta + \lambda \pm i0)^{-1}(V u_{\mathrm{in}}) & \text{in } \mathbb{R}^d, \\ u_{\mathrm{sc}} \text{ satisfying SRC}, \end{cases} \tag{3}$$

where I stands for the identity operator. In order to solve the problem (3) using the Fredholm alternative, one needs to ensure that $(\Delta + \lambda \pm i0)^{-1} \circ V$ is compact in the space where the solutions u_{sc} will belong to, and zero is the only solution to the homogeneous counterpart of the problem (3).

To apply any of these two schemes one needs appropriate estimates for the resolvent $(\Delta + \lambda \pm i0)^{-1}$ according to the character or behaviour of V. For example, the well-known resolvent estimate —due to Agmon [2]—

$$\lambda^{1/2} \|(\Delta + \lambda \pm i0)^{-1} f\|_{L^{2,-\delta}} \lesssim \|f\|_{L^{2,\delta}}, \tag{4}$$

with $\delta > 1/2$ and $\|f\|^2_{L^{2,\pm\delta}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |x|^2)^{\pm\delta} |f(x)|^2 \,\mathrm{d}x$, makes possible to prove that

$$\|(\Delta + \lambda \pm i0)^{-1} \circ V\|_{\mathcal{L}(L^{2,-\delta}(\mathbb{R}^d))} \lesssim \lambda^{-1/2}$$

with $V \in L^\infty(\mathbb{R}^d)$ and compactly supported. An improved version of Agmon's inequality is the following one —due to Agmon and Hörmander [1]—

$$\lambda^{1/2} \sup_{j\in\mathbb{N}_0} \big(2^{-j/2} \|(\Delta + \lambda \pm i0)^{-1} f\|_{L^2(D_j)}\big) \lesssim \sum_{j\in\mathbb{N}_0} 2^{j/2} \|f\|_{L^2(D_j)}, \tag{5}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $D_j = \{x \in \mathbb{R}^d : 2^{j-1} < |x| \leq 2^j\}$ for $j \in \mathbb{N}$ and $D_0 = \{x \in \mathbb{R}^d : |x| \leq 1\}$. It is very common to let the norm on the left-hand side be denoted by $\|\!| \cdot |\!\|_*$ and the one on the right by $\|\!| \cdot |\!\|$. Thus,

$$\|\!| f |\!\|_* = \sup_{j\in\mathbb{N}_0} \big(2^{-j/2} \|f\|_{L^2(D_j)}\big), \qquad \|\!| f |\!\| = \sum_{j\in\mathbb{N}_0} 2^{j/2} \|f\|_{L^2(D_j)}.$$

Another important and very celebrated resolvent estimate is the following —due to Kenig, Ruiz and Sogge [17]—

$$\|(\Delta + \lambda \pm i0)^{-1} f\|_{L^p} \lesssim \lambda^{\frac{d}{2}\left(\frac{1}{p'} - \frac{1}{p}\right) - 1} \|f\|_{L^{p'}}, \tag{6}$$

where $2/(d + 1) \leq 1/p' - 1/p \leq 2/d$, $1/p + 1/p' = 1$ and $d \geq 3$. The inequality (6) can be used to show that, for the range $2/(d + 1) \leq 1/p < 2/d$, the inequality

$$\|(\Delta + \lambda \pm i0)^{-1} \circ V\|_{\mathcal{L}(L^{2p'})} \lesssim \lambda^{\frac{d}{2p} - 1}$$

holds with $1/p' + 1/p = 1$ and $V \in L^q(\mathbb{R}^d)$ compactly supported, where $q > d/2$ and $d \geq 3$. The end-point case $V \in L^{d/2}(\mathbb{R}^d)$ does not follow directly from either the Neumann series argument —unless there is smallness for $\|V\|_{L^{d/2}}$— or the Fredholm alternative. The Neumann series argument fails in the end-point because we only have

$$\|(\Delta + \lambda \pm i0)^{-1} \circ V\|_{\mathcal{L}(L^{p_d})} \lesssim 1,$$

where p_d is the index of the $\dot{H}^1(\mathbb{R}^d)$ Hardy–Littlewood–Sobolev embedding, $1/p_d = 1/2 - 1/d$. The Fredholm theory does not seem to apply for the lack of the compactness, specially because $H^1(B_0)$ is not compactly embedded in $L^{p_d}(B_0)$. However, Lavine and Nachman managed to modify the procedure, with a formulation that reminds the operator used to prove the Birman–Schwinger principle, in order to reach the end-point. To make their argument work one needs to use the inequalities (4) and (6). We learnt it from [9]. Another improvement of Agmon's inequality is

$$\|(\Delta + \lambda \pm i0)^{-1} f\|_{H^{s,-\delta}_\lambda} \lesssim \lambda^{-(1/2-s)} \|f\|_{H^{-s,\delta}_\lambda}, \tag{7}$$

with $0 \leq s \leq 1/2$, $\delta > 1/2$ and

$$\|f\|_{H^{\pm s, \mp\delta}_\lambda} = \|(\lambda - \Delta)^{\pm s/2} f\|_{L^{2,\mp\delta}},$$

where $(\lambda - \Delta)^{\pm s/2}$ stands for the multiplier with symbol $(\lambda + |\xi|^2)^{\pm s/2}$. This inequality was proved in [5] to study scattering in the presence of a class of Gaussian random potentials called microlocally isotropic. The realizations of such potentials are compactly supported and belong to the potential Sobolev spaces $L^p_{-s}(\mathbb{R}^d)$ with $0 < s \leq 1/2$ and $d/s \leq p < \infty$, almost surely. Recall that the potential Sobolev space $L^p_s(\mathbb{R}^d)$ with $1 < p < \infty$ and $s \in \mathbb{R}$ is defined by $(I - \Delta)^{-s/2} L^p(\mathbb{R}^d)$ with $(I - \Delta)^{-s/2}$ the Bessel potential with symbol $(1 + |\xi|^2)^{-s/2}$. From (7), Caro, Helin and Lassas showed in [5] that, for compactly supported potentials in $L^p_{-s}(\mathbb{R}^d)$ with $0 < s \leq 1/2$ and $d/s \leq p < \infty$, one has

$$\|(\Delta + \lambda \pm i0)^{-1} \circ V\|_{\mathcal{L}(H^{s,-\delta}_\lambda)} \lesssim o(\lambda^{-(1/2-s)}).$$

The inequality (7) can be easily extended to the range $0 \leq s \leq 1$ and then used to prove —by the Neumann series argument— the existence of scattering solution for potentials as (1) with $V^0 \in L^\infty(\mathbb{R}^d; \mathbb{R})$ and $\|\alpha\|_{L^\infty(\Gamma)}$ small enough. Despite the fact that we do not know any reference dealing with this problem for every dimension $d \geq 2$, we believe that the truth challenge of the scattering theory arises when considering potentials that are the combination of critically-singular and δ-shell potentials. For such potentials, neither the inequality (7) —for the full range $0 \leq s \leq 1$— nor (6) with no adjustment seem to be enough to develop the scattering theory. On the other hand, because of the nature of the term $\alpha\, \mathrm{d}\sigma$, the Lavine–Nachman argument might not be easily adapted for potentials of the form in (1). In fact, in this article we develop an alternative path that we motivate in the next lines and explain right after.

The approach we propose is inspired by the most recent works studying the Calderón problem for dimension $d \geq 3$. This inverse problem consists in determining the electric conductivity of a medium from its corresponding Dirichlet-to-Neumann map. The key ingredient in the resolution of this problem is a type of solutions called complex geometrical optics (CGO for short), first constructed by Sylvester and Uhlmann [25]. Most of the progresses related to this problem have consisted in refining the construction of the CGO solutions, which boils down to inverting the conjugated Laplacian $\Delta + 2\tau\partial_{x_d} + \tau^2$ for at least $\tau \geq \tau_0 > 0$. In [12], Haberman and Tataru introduced a family of Bourgain spaces —denoted here by

$\dot{Y}^s_\tau$ with $s \in \mathbb{R}$— adapted to this differential operator[1], whose norms were of the form

$$\|f\|_{\dot{Y}^s_\tau} = \||q_\tau|^s \widehat{f}\|_{L^2}, \tag{8}$$

where $q_\tau(\xi) = -|\xi|^2 + i2\tau\xi_d + \tau^2$ stands for the symbol of the conjugated Laplacian. This family of spaces is very convenient for several reasons: the first one is because the inverse of the conjugated Laplacian is an isometry

$$\|(\Delta + 2\tau\partial_{x_d} + \tau^2)^{-1} f\|_{\dot{Y}^s_\tau} = \|f\|_{\dot{Y}^{s-1}_\tau}. \tag{9}$$

The regularity of V in (1) make the index $s = 1/2$ play a relevant role. The second reason is that, when functions are localized in space, the norm for $s = 1/2$ controls the L^2 norm of such functions with a gain of $\tau^{1/2}$. This fact was shown in [12]:

$$\tau^{1/2}\|\chi f\|_{L^2} \lesssim \|f\|_{\dot{Y}^{1/2}_\tau} \tag{10}$$

where $\chi \in \mathcal{S}(\mathbb{R}^d)$. Another reason that makes relevant this space is the following embedding —due to Haberman [11]—

$$\|f\|_{L^{p_d}} \lesssim \|f\|_{\dot{Y}^{1/2}_\tau}. \tag{11}$$

As a consequence of (11) and (9), one can derive the inequality

$$\|(\Delta + 2\tau\partial_{x_d} + \tau^2)^{-1} f\|_{L^{p_d}} \lesssim \|f\|_{L^{p'_d}} \tag{12}$$

for $d \geq 3$ with $1/p_d + 1/p'_d = 1$. The inequality (12) was proved by Kenig, Ruiz and Sogge [17] as a consequence of (6) for $1/p' - 1/p = 2/d$, however, this was written in the form of a Carleman estimate.

Our strategy in this article is to introduce two spaces X_λ and X^*_λ adapted to the resolvent operator $(\Delta + \lambda \pm i0)^{-1}$ for which analogues of (9), (10) and (11) hold. In fact, we will see the resolvent estimate

$$\|(\Delta + \lambda \pm i0)^{-1} f\|_{X^*_\lambda} \lesssim \|f\|_{X_\lambda} \tag{13}$$

and the embedding

$$\lambda^{1/4} |\!|\!| f |\!|\!|_* + \lambda^{\frac{d}{2}(\frac{1}{p} - \frac{1}{p_d})} \|f\|_{L^p} \lesssim \|f\|_{X^*_\lambda}, \tag{14}$$

where $p \in [q_d, p_d]$ with q_d so that $2/q_d = (d-1)/(d+1)$ —the index[2] in the extension form of the Tomas–Stein theorem. From the inequalities (13) and (14), one can prove that $(\Delta + \lambda \pm i0)^{-1} \circ V$ is a contraction on X^*_λ for $\lambda \geq \lambda_0 > 0$ under a smallness assumption on α. This would allow to construct the scattering solution $u_{\rm sc}$ by the Neumann series argument. In order to avoid assuming smallness for α, we adopt a strategy that combines the Neumann series argument and the Fredholm alternative. First, we use the Neumann series argument to construct the resolvent $(\Delta + \lambda \pm i0 - V^0)^{-1}$ and prove its boundedness from X_λ to X^*_λ. Then, we use the Fredhlom theory to solve the problem

$$\begin{cases} \big(I - (\Delta + \lambda \pm i0 - V^0)^{-1} \circ (\alpha \,\mathrm{d}\sigma)\big) u_{\rm sc} = (\Delta + \lambda \pm i0 - V^0)^{-1}(V u_{\rm in}) \quad \text{in } \mathbb{R}^d, \\ u_{\rm sc} \text{ satisfying SRC}. \end{cases}$$

Two ingredients are required to apply the Fredholm theory. The first one is the compactness from X^*_λ to X_λ of the operator multiplication by $\alpha \,\mathrm{d}\sigma$. The second ingredient is a unique continuation property for an equation with a potential as in

[1]Actually, the differential operator that Haberman and Tataru considered was $\Delta + 2\zeta \cdot \nabla$ with $\zeta = \Re\zeta + i\Im\zeta \in \mathbb{C}^d$ so that $\zeta \cdot \zeta = 0$, and consequently the family of Bourgain spaces they introduced, denoted in their work by $\dot{X}^s_\zeta$, had similar norms to $\dot{Y}^s_\tau$ but with $p_\zeta(\xi) = -|\xi|^2 + i2\zeta \cdot \xi$ instead of q_τ. Note that $\Delta + 2\zeta \cdot \nabla = e^{-i\Im\zeta \cdot x}(\Delta + 2\Re\zeta \cdot \nabla + |\Re\zeta|^2) \circ e^{i\Im\zeta \cdot x}$ and consequently, if $\Re\zeta = \tau T e_d$ with $T \in SO(d)$, then $q_\tau(\xi) = p_\zeta(T\xi - \Im\zeta)$.

[2]For some computations, it is useful to note that $1/q_d = 1/2 - 1/(d+1)$, that is $q_d = p_{d+1}$

(1). Here, we derive this unique continuation using a Carleman estimate that Caro and Rogers proved in [6] for the Bourgain spaces.

Intuitively, the elements of X_λ^* should be thought as functions with some integrability whose weak (up to first order) derivatives have also certain (but different) integrability properties. In fact,

$$X_\lambda^* \subset L^p_{d(1/p-1/p_d)}(\mathbb{R}^d) \cap (I-\Delta)^{-1/2} B_*$$

with $p \in [q_d, p_d]$, $L^p_{d(1/p-1/p_d)}(\mathbb{R}^d)$ the potential Sobolev space with differentiability index $d(1/p - 1/p_d)$ and integrability index p, and B_* the Banach space defined by the norm $\| \mathbf{.} \|_*$ —this inclusion follows by changing slightly the proofs of the lemmas 4.8 and 4.12 in the section 4. Contrarily, the elements of X_λ are actual distributions, an example of them are elements of

$$L^{p'}_{-d(1/p_d'-1/p')}(\mathbb{R}^d) + (I-\Delta)^{1/2} B$$

with p' and p_d' the dual exponents of p and p_d, and B the Banach space defined by the norm $\| \mathbf{.} \|$. Actually, the latter space is included in X_λ. Despite the nature of the spaces X_λ^* and X_λ, the inequality (13) is somehow equivalent to a combination of (5) and (6) (see the remarks 4.3 and 4.7 in section 4). However, the inequality (13) is better adapted than (5) and (6) to deal with potentials V as in (1), in this sense our new estimate is a refinement of the classical ones. The ideal situation would be to define the spaces X_λ^* and X_λ through the L^2 norms in the frequency side with the weights $\sqrt{m_\lambda}$ and $1/\sqrt{m_\lambda}$ respectively, where $m_\lambda(\xi) = |\lambda - |\xi|^2|$. However, it is not as straightforward as this since $1/\sqrt{m_\lambda}$ is not in $L^2_{\rm loc}(\mathbb{R}^d)$ —see how we overcome this issue in the definitions 2.1, 2.2 and 2.4 in the section 2.

Our approach provides a suitable framework to construct the scattering solution using a strategy that combines the Neumann series and the Fredholm alternative. Given $y \in \mathbb{R}^d$, consider $u^\pm_{\rm in}(x,y) = \Phi^\pm_\lambda(y-x)$ with

$$\Phi^\pm_\lambda = (\Delta + \lambda \pm i0)^{-1}\delta_0,$$

where δ_0 is the Dirac mass at 0.

Theorem 1. *Consider $d \geq 3$. There exists a positive $\lambda_0 = \lambda_0(d, V^0, R_0)$ so that, we can find $u^\pm_{\rm sc}(\mathbf{.}, y) \in X^*_\lambda$ solving the problem*

$$\begin{cases} (\Delta + \lambda - V) u^\pm_{\rm sc}(\mathbf{.}, y) = V u^\pm_{\rm in}(\mathbf{.}, y) & \text{in } \mathbb{R}^d, \\ \sup\limits_{|x|=R} \Big| \dfrac{x}{|x|} \cdot \nabla u^\pm_{\rm sc}(x,y) \mp i\lambda^{1/2} u^\pm_{\rm sc}(x,y) \Big| = o_y(R^{-\frac{d-1}{2}}) & R \geq R_0, \end{cases}$$

for every $\lambda \geq \lambda_0$ and $y \in \mathbb{R}^d \setminus B_0$. Moreover, $u^\pm_{\rm sc}(\mathbf{.}, y)$ is the only solution of the previous problem.

Remark 1.1. For dimension $d = 2$, we could have used the Neumann series argument and our estimates to state that there exist $\varepsilon = \varepsilon(d, \Gamma, R_0)$ and $\lambda = \lambda(d, V^0, R_0)$ so that, if $\|\alpha\|_{L^\infty(\Gamma)} \leq \varepsilon$, then there exists a unique scattering solution $u^\pm_{\rm sc}(\mathbf{.}, y) \in X^*_\lambda$ for every $\lambda \geq \lambda_0$. We have not combined the Neumann series argument and the Fredholm alternative in this situation because we have not found an appropriate unique continuation for a potential V as in (1) for $d = 2$.

Remark 1.2. When comparing this theorem with the results of Mantile, Posilicano and Sini [18], one immediately feels that the assumption on α could be weaken to belong to $L^p(\Gamma)$ with $p > d-1$. We believe that a modification of our results in the section 4.3 would be enough. We believe that this generalization is possible using our approach because the space Y^*_λ provides control on the square integrability of the first order derivatives, and consequently the restrictions of the functions in Y^*_λ to Γ have derivatives of order 1/2. This 1/2 derivatives on Γ could trade q-integrability on Γ with $2/q > 1 - 1/(d-1)$ using the compact Sobolev embedding (note that the compactness required for our argument would be loss for $2/q = 1 - 1/(d-1)$). This improved integrability of the restriction to Γ of elements in Y_λ^* would allow to weaken the integrability required for α, so it is enough to assume $\alpha \in L^p(\Gamma)$ to $p > d - 1$.

1.2. **Inverse scattering.** The inverse point-source problem we study in this paper consists in determining a potential V as in (1) from the knowledge of $u^\pm_{\rm sc}|_{\partial B_0 \times \partial B_0}$ for a fixed energy λ, where $u^\pm_{\rm sc}(\centerdot, y)$ is the scattering solution of the theorem 1 yielded by the incident wave $u^\pm_{\rm in}(\centerdot, y) = \Phi^\pm_\lambda(y - \centerdot)$.

Theorem 2. *Consider $d \geq 3$. Let $V_1 = V_1^0 + \alpha_1 \,\mathrm{d}\sigma_1$ and $V_2 = V_2^0 + \alpha_2 \,\mathrm{d}\sigma_2$ be two electric potentials as in (1), where σ_j is the surface measure of Γ_j. Let $\lambda_0 = \lambda_0(d, V_1^0, V_2^0, R_0)$ be so that the scattering solutions $u^\pm_{{\rm sc},1}(\centerdot, y)$ and $u^\pm_{{\rm sc},2}(\centerdot, y)$ of the theorem 1 with potentials V_1 and V_2 are available for every $\lambda \geq \lambda_0$. Then,*

$$u^\pm_{{\rm sc},1}|_{\partial B_0 \times \partial B_0} = u^\pm_{{\rm sc},2}|_{\partial B_0 \times \partial B_0} \text{ for a fixed } \lambda \geq \lambda_0 \implies V_1 = V_2.$$

Remark 1.3. The identity $u^\pm_{{\rm sc},1}|_{\partial B_0 \times \partial B_0} = u^\pm_{{\rm sc},2}|_{\partial B_0 \times \partial B_0}$ in this theorem is meant to be understood as

$$\text{either } u^+_{{\rm sc},1}|_{\partial B_0 \times \partial B_0} = u^+_{{\rm sc},2}|_{\partial B_0 \times \partial B_0} \text{ or } u^-_{{\rm sc},1}|_{\partial B_0 \times \partial B_0} = u^-_{{\rm sc},2}|_{\partial B_0 \times \partial B_0}.$$

Remark 1.4. *Note that $V_1 = V_2$ implies that $V_1^0 = V_2^0$, $\Gamma_1 = \Gamma_2$ and $\alpha_1 = \alpha_2$. Indeed, we can test $V_1 - V_2$ with a sequence of functions ϕ_n that concentrates around $\Gamma_1 \cup \Gamma_2$ so that $\int_{\mathbb{R}^d} (V_1^0 - V_2^0)\phi_n$ vanishes as the functions concentrates around this set of measure zero. This implies that $\Gamma_1 = \Gamma_2$ and consequently that $\alpha_1 = \alpha_2$. At this point obviously $V_1^0 = V_2^0$.*

To address the question of uniqueness for this fixed-energy inverse scattering problem, we adopt the approach that Hähner and Hohage followed in [13] to prove some stability estimates for a similar problem for the acoustic equation. We start by proving an *orthogonality* relation in the spirit of Alessandrini's identity for the Calderón problem, that is,

$$\langle (V_1 - V_2) v_1, v_2 \rangle = 0 \tag{15}$$

for all v_j solution of $(\Delta + \lambda - V_j) v_j = 0$ in B_0. Then, we construct CGO solutions —as Sylvester and Uhlmann did in [25]— in the form

$$v_j(x) = e^{\zeta_j \cdot x}(1 + w_j(x)),$$

where $\zeta_j \in \mathbb{C}^d$ so that $\zeta_j \cdot \zeta_j = -\lambda$ and $\zeta_1 + \zeta_2 = -i\kappa$ for an arbitrarily given $\kappa \in \mathbb{R}^d$ —which is possible in dimension $d \geq 3$—, and the correction term w_j vanishes in a specific sense when $|\zeta_j|$ grows. Because of the δ-shell components $\alpha_1 \,\mathrm{d}\sigma_1$ and $\alpha_2 \,\mathrm{d}\sigma_2$ of the potentials V_1 and V_2, we follow the ideas introduced by Haberman and Tataru in [12] in order to ensure the asymptotic behaviour of w_j when $|\zeta_j|$ grows. However, since no smallness is assumed for α_j, we also require at this stage the Carleman estimate proved by Caro and Rogers in [6]. The critically-singular components V_1^0 and V_2^0 can be treated thanks to the embedding (11) due to Haberman [11]. Finally, we plug in the CGO solutions to (15) and make $|\zeta_1|$ and $|\zeta_2|$ grow. Thus, we can conclude that the Fourier transform of $V_1 - V_2$ is identically zero, that is,

$$\mathcal{F}\big(V_1 - V_2\big)(\kappa) = 0, \qquad \forall \kappa \in \mathbb{R}^d.$$

The injectivity of the Fourier transform allows us to conclude that $V_1 = V_2$.

1.3. **Some previous results.** The spaces

$$L^p_{d(1/q_d-1/p_d)}(\mathbb{R}^d)\cap(I-\Delta)^{-1/2}B_* \quad \text{and} \quad L^{p'}_{-d(1/p'_d-1/q'_d)}(\mathbb{R}^d)+(I-\Delta)^{1/2}B$$

are the spaces chosen by Ionescu and Schlag in [16] to prove the limit absorption principle for a large class of perturbations. It turns out that their basic estimate —with an explicit control in λ— can be derived from (13) and the relation of these spaces with X^*_λ and X_λ. Another resolvent estimate that seems to follow from ours, after an adjustment in the norm of X^*_λ, is the one due to Ruiz and Vega —Theorem 1.2 in [23]. See also the work of Goldberg and Schlag [10].

Regarding previous results on inverse scattering with δ-shell potentials, see the work of Mantile, Posilicano and Sini [18] in dimension $d=3$—in their case $V^0\in L^2(\mathbb{R}^3)$ and $\alpha\in L^p(\Gamma)$ with $p>2$. The point source-scattering have been previously studied in [13] by Hähner and Hohage in acoustic media, and by Ola and Somersalo [21] for Maxwell equations.

The literature on inverse scattering is rather wide and we cite only a few works where the measurements are assumed to be modelled by the far-field pattern. Colton and Kirsch introduced in [7] the linear sampling method to determine the support of an imperfect conductor. Uniqueness and reconstruction for the inverse scatteing problem in an acoustic medium was proved by Nachman [19], Novikov [20], and Ramm [22]. The stability question was first addressed by Stefanov [24], and then improved by Hähner and Hohage [13].

1.4. **The outline of the paper and notation.** The section 2 is devoted to the study of the direct scattering from a point source. We first pose rigorously the point-source scattering problem. Then, we introduce the spaces X_λ and X^*_λ, and state rigorously the inequalities (13) and (14). Afterwards, we construct the resolvent $(\Delta+\lambda\pm i0-V^0)^{-1}$ by a Neumann series argument and then we use the Fredholm alternative to prove the existence of the scattering solution. The inverse problem is considered in the section 3. First, we prove a couple of lemmas that are required for the orthogonality identity (15). Then, we construct the CGO solutions and show the uniqueness of the potentials. In the section 4, we first state a couple of refined resolvent estimates in the spirit of (13). There, we also provide a rather simple proofs of (5) and (6). We find specially interesting the proof of (6), where we do not use Stein's interpolation theorem and reach the endpoint in the case $d=2$. The last part of the section 4 contains some connections of our refined resolvent estimates with the estimates that Sylvester and Uhlmann used to construct the CGO solutions, as well as, the inequalities (10) and (11) proved by Haberman and Tataru. The article finishes with an appendix where we address the most basic questions of the functional spaces X^*_λ and X_λ.

The section 4 may be read independently of the sections 2 and 3, only some notations and definitions from the previous sections would be required. However, the sections 2 and 3 are full of references and calls to the section 4. Thus, if readers choose to follow the order proposed by the authors, they would get a global picture of the direct and inverse problems from the sections 2 and 3 postponing the details for the section 4.

Notation. The index of the $\dot H^1(\mathbb{R}^d)$ Hardy–Littlewood–Sobolev embedding is

$$1/p_d=1/2-1/d.$$

The index in the extension form of the Tomas–Stein theorem is

$$2/q_d=(d-1)/(d+1).$$

The modulus of the symbol of $\Delta+\lambda$ is

$$m_\lambda(\xi)=|\lambda-|\xi|^2|.$$

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2. Scattering theory

In point-source scattering theory, the incident wave is typically chosen as certain fundamental solutions. More precisely, given $y \in \mathbb{R}^d$, the incident wave is given by $u_{\rm in}^\pm(x,y) = \Phi_\lambda^\pm(y-x)$ with

$$\Phi_\lambda^\pm = (\Delta + \lambda \pm i0)^{-1}\delta_0,$$

where δ_0 is the Dirac mass at 0. The previous identity is understood as

$$\langle \Phi_\lambda^\pm, f\rangle = \frac{1}{(2\pi)^{d/2}}\Big[\lim_{\epsilon\to 0}\int_{m_\lambda>\epsilon}\frac{\widehat{f}(\xi)}{\lambda-|\xi|^2}\,\mathrm{d}\xi \mp i\frac{\pi}{2\lambda^{1/2}}\int_{S_\lambda}\widehat{f}(\xi)\,\mathrm{d}S_\lambda(\xi)\Big]$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$, where $S_\lambda = \{\xi \in \mathbb{R}^d : |\xi| = \lambda^{1/2}\}$ and $\mathrm{d}S_\lambda$ stands for the volume form on S_λ. One can check that $\Phi_\lambda^\pm$ is the fundamental solution solving the problem

$$\begin{cases}\Delta\Phi_\lambda^\pm + \lambda\Phi_\lambda^\pm = \delta_0 & \text{in}\,\mathbb{R}^d,\\ \dfrac{x}{|x|}\cdot\nabla\Phi_\lambda^\pm(x) \mp i\lambda^{1/2}\Phi_\lambda^\pm(x) = \mathcal{O}(|x|^{-\frac{d+1}{2}}) & \text{for}\,|x|\geq 1.\end{cases}$$

The last condition corresponds to either the ingoing or the outgoing SRC. Our goal in this section is to construct the scattering solution $u_{\rm sc}^\pm(\centerdot, y)$ solving the problem

$$\begin{cases}(\Delta+\lambda-V)u_{\rm sc}^\pm(\centerdot,y) = Vu_{\rm in}^\pm(\centerdot,y) & \text{in}\,\mathbb{R}^d,\\ \sup\limits_{|x|=R}\Big|\dfrac{x}{|x|}\cdot\nabla u_{\rm sc}^\pm(x,y) \mp i\lambda^{1/2}u_{\rm sc}^\pm(x,y)\Big| = o_y(R^{-\frac{d-1}{2}}) & R\geq 1,\end{cases} \tag{16}$$

with $y \in \mathbb{R}^d \setminus B_0$ and V as in (1).

As we mention in the introduction, the scattering solution $u_{\rm sc}(\centerdot, y)$ will be constructed in a space X_λ^*. Ideally this space would be defined through the symbol $\sqrt{m_\lambda}$, with $m_\lambda(\xi) = |\lambda - |\xi|^2|$, however, this is not possible. If X_λ^* were defined by the symbol $\sqrt{m_\lambda}$, its pre-dual X_λ would have to be defined by the symbol $1/\sqrt{m_\lambda}$, which is not locally square-integrable around the *critical hypersurface* S_λ. For that reason, given $\lambda > 0$, the integer k_λ so that $2^{k_\lambda-1} < \lambda^{1/2} \leq 2^{k_\lambda}$ will play a special role. Thus, to avoid the *critical frequencies* around S_λ, we introduce the set

$$I = \{k_\lambda-2, k_\lambda-1, k_\lambda, k_\lambda+1\}$$

and use the Littlewood–Paley projectors P_k and $P_{\leq k}$. To define them, it is enough to consider $\phi \in \mathcal{S}(\mathbb{R}^d)$ supported in $\{\xi \in \mathbb{R}^d : |\xi| \leq 2\}$ and $\phi(\xi) = 1$ whenever $|\xi| \leq 1$, and the function $\psi(\xi) = \phi(\xi) - \phi(2\xi)$. Then,

$$\widehat{P_k f}(\xi) = \psi(\xi/2^k)\widehat{f}(\xi), \qquad \widehat{P_{\leq k}f}(\xi) = \phi(\xi/2^k)\widehat{f}(\xi).$$

In this paper, the projector $P_{\leq k_\lambda-3}$ will have a relevant importance, and will be denoted for simplicity by $P_{<I}$

The space X_λ^* will be introduced as the dual of X_λ which in turn is defined as the sum of two spaces Y_λ and Z_λ. These later spaces with their corresponding duals Y_λ^* and Z_λ^* come forth to refine the estimates (5) and (6), respectively.

Definition 2.1. *Let Y_λ be the set of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|m_\lambda^{-1/2}\widehat{P_{<I}f}\|_{L^2}^2 + \sum_{k\in I}\lambda^{-1/2}|\!|\!|P_k f|\!|\!|^2 + \sum_{k>k_\lambda+1}\|m_\lambda^{-1/2}\widehat{P_k f}\|_{L^2}^2 < \infty,$$

where $m_\lambda(\xi) = |\lambda - |\xi|^2|$. *For* $f \in Y_\lambda$, *define the norm*

$$\|f\|_{Y_\lambda}^2 = \|m_\lambda^{-1/2}\widehat{P_{<I}f}\|_{L^2}^2 + \sum_{k\in I}\lambda^{-1/2}|||P_kf|||^2 + \sum_{k>k_\lambda+1}\|m_\lambda^{-1/2}\widehat{P_kf}\|_{L^2}^2.$$

To introduce the space Z_λ, it is convenient to remember that q_d is $2/q_d = (d-1)/(d+1)$ for $d \geq 2$, while p_d is $1/p_d = 1/2 - 1/d$ if $d \geq 3$. In dimension $d = 2$, we write $p_2 = \infty$.

Definition 2.2. *Let* $Z_{\lambda,p'}$ *with* $p' \in [p'_d, q'_d]$ *be the set of* $g \in \mathcal{S}'(\mathbb{R}^d)$ *such that*

$$\|m_\lambda^{-1/2}\widehat{P_{<I}g}\|_{L^2}^2 + \sum_{k\in I}\lambda^{d(\frac{1}{p'}-\frac{1}{p'_d})}\|P_kg\|_{L^{p'}}^2 + \sum_{k>k_\lambda+1}\|m_\lambda^{-1/2}\widehat{P_kg}\|_{L^2}^2 < \infty,$$

where $m_\lambda(\xi) = |\lambda - |\xi|^2|$. *For* $g \in Z_{\lambda,p'}$, *define the norm*

$$\|g\|_{Z_{\lambda,p'}}^2 = \|m_\lambda^{-1/2}\widehat{P_{<I}g}\|_{L^2}^2 + \sum_{k\in I}\lambda^{d(\frac{1}{p'}-\frac{1}{p'_d})}\|P_kg\|_{L^{p'}}^2 + \sum_{k>k_\lambda+1}\|m_\lambda^{-1/2}\widehat{P_kg}\|_{L^2}^2.$$

Here q'_d *and* p'_d *are the dual exponents of* q_d *and* p_d *respectively, in particular,* $p'_2 = 1$. *For simplicity, we write* Z_λ *instead of* Z_{λ,q'_d}.

Remark 2.3. By Bernstein's inequality

$$\|g\|_{Z_\lambda} \lesssim \|g\|_{Z_{\lambda,p'}} \lesssim \|g\|_{Z_{\lambda,p'_d}},$$

and therefore,

$$Z_{\lambda,p'_d} \subset Z_{\lambda,p'} \subset Z_\lambda.$$

Now, we are in position to state the precise definitions of the spaces X_λ and X_λ^*.

Definition 2.4. *Let* X_λ *be the set of* $h \in \mathcal{S}'(\mathbb{R}^d)$ *such that* $h = f + g$ *with* $f \in Y_\lambda$ *and* $g \in Z_\lambda$. *For* $h \in X_\lambda$, *define the norm*

$$\|h\|_{X_\lambda} = \inf\{\|f\|_{Y_\lambda} + \|g\|_{Z_\lambda} : h = f + g\}.$$

Note that the infimum is taken over all representation $h = f + g$ *with* $f \in Y_\lambda$ *and* $g \in Z_\lambda$.

The Banach space $(X_\lambda^*, \|\cdot\|_{X_\lambda^*})$ *is defined as the dual space of* $(X_\lambda, \|\cdot\|_{X_\lambda})$.

To construct the solutions in this functional analytical framework, these spaces have to satisfy some basic properties that are stated below and proved in the appendix A.

Proposition 2.5. *The Schwartz class* $\mathcal{S}(\mathbb{R}^d)$ *is dense in* Y_λ *and* Z_λ *with their corresponding norms. In particular,* $\mathcal{S}(\mathbb{R}^d)$ *is also dense in* X_λ.

Proposition 2.6. *The pair* $(X_\lambda, \|\cdot\|_{X_\lambda})$ *is a Banach space. Its norm can be computed testing on duals elements as follows:*

$$\|f\|_{X_\lambda} = \sup_{u\in X_\lambda^*\setminus\{0\}}\frac{\langle f, u\rangle}{\|u\|_{X_\lambda^*}}. \tag{17}$$

Proposition 2.7. *The space* X_λ^* *is isomorphic to the space of* $u \in \mathcal{S}'(\mathbb{R}^d)$ *so that*

$$\|m_\lambda^{1/2}\widehat{P_{<I}u}\|_{L^2}^2 + \sum_{k\in I}\big[\lambda^{\frac{1}{2}}|||P_ku|||_*^2 + \lambda^{d(\frac{1}{q_d}-\frac{1}{p_d})}\|P_ku\|_{L^{q_d}}^2\big] + \sum_{k>k_\lambda+1}\|m_\lambda^{1/2}\widehat{P_ku}\|_{L^2}^2 < \infty,$$

endowed with the norm

$$\Big(\sum_{k\in I}\big[\lambda^{1/2}|||P_ku|||_*^2 + \lambda^{d(\frac{1}{q_d}-\frac{1}{p_d})}\|P_ku\|_{L^{q_d}}^2\big] + \|m_\lambda^{1/2}\widehat{P_{<I}u}\|_{L^2}^2 + \sum_{k>k_\lambda+1}\|m_\lambda^{1/2}\widehat{P_ku}\|_{L^2}^2\Big)^{1/2}. \tag{18}$$

Finally, $\mathcal{S}(\mathbb{R}^d)$ *is dense in* X_λ^*.

These spaces have been constructed to make the following theorems hold.

Theorem 2.8. *There exists a constant* $C > 0$ *only depending on* d *so that*

$$\|(\Delta + \lambda \pm i0)^{-1} f\|_{X_\lambda^*} \leq C \|f\|_{X_\lambda}$$

for all $f \in X_\lambda$.

Proof. A standard density argument together with the proposition 2.5 reduces the theorem to prove the inequality for every $f \in \mathcal{S}(\mathbb{R}^d)$. Now, by the proposition 2.7 and the lemmas 4.2 and 4.6 —in the section 4.1— we obtain that

$$\|(\Delta + \lambda \pm i0)^{-1} f\|_{X_\lambda^*} \lesssim \|f\|_{Y_\lambda} + \|f\|_{Z_\lambda}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. Since for $f \in Y_\lambda \cap Z_\lambda$ we have that $\|f\|_{X_\lambda} = (\|f\|_{Y_\lambda} + \|f\|_{Z_\lambda})/2$ we conclude that

$$\|(\Delta + \lambda \pm i0)^{-1} f\|_{X_\lambda^*} \lesssim \|f\|_{X_\lambda}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. □

Theorem 2.9. *Consider* $p \in [q_d, p_d]$. *There exists a constant* $C > 0$ *only depending on* d *and* p *so that*

$$\lambda^{1/4} \|\!|u|\!\|_* + \lambda^{\frac{d}{2}(\frac{1}{p} - \frac{1}{p_d})} \|u\|_{L^p} \leq C \|u\|_{X_\lambda^*}$$

for every $u \in \mathcal{S}(\mathbb{R}^d)$.

Proof. This theorem is a consequence of the lemmas 4.8 and 4.12 —in the section 4.2— and the proposition 2.7. □

Next, we use the previous embedding to estimate the norm of the operator multiplication by V^0.

Corollary 2.10. *There exists a constant* $C > 0$ *that only depends on* d *and* R_0 *so that*

$$\|V^0\|_{\mathcal{L}(X_\lambda^*; X_\lambda)} \leq C \big(\lambda^{-1/4} + \|\mathbf{1}_F V^0\|_{L^{d/2}} \big),$$

where $F = \{x \in \mathbb{R}^d : |V^0(x)| > \lambda^{1/4}\}$.

Proof. We use (17) in the proposition 2.6 to estimate $\|V^0\|_{\mathcal{L}(X_\lambda^*; X_\lambda)}$. Start by writing

$$\langle V^0 f, g \rangle = \int_{\mathbb{R}^d} V^0 f g \tag{19}$$

with f and g in $\mathcal{S}(\mathbb{R}^d)$. Since the support of V is contained in B_0, the support of V^0 is also contained in B_0. Then, f and g in (19) can be replaced by χf and χg with χ a smooth cut-off function supported in $2B_0$ and so that $\chi(x) = 1$ for all $x \in B_0$. Thus,

$$\int_{\mathbb{R}^d} V^0 f g = \int_{\mathbb{R}^d} \mathbf{1}_E V^0 \chi f \chi g + \int_{\mathbb{R}^d} \mathbf{1}_F V^0 f g\,,$$

where $E = \{x \in \mathbb{R}^d : |V^0(x)| \leq M\}$, $F = \mathbb{R}^d \setminus E$, and $\mathbf{1}_E$ and $\mathbf{1}_F$ stand for the characteristic functions of E and F. Using Hölder's inequality, we obtain

$$\begin{aligned} \Big| \int_{\mathbb{R}^d} V^0 \chi f \chi g \Big| &\leq M \|\chi f\|_{L^2} \|\chi g\|_{L^2} + \|\mathbf{1}_F V^0\|_{L^{d/2}} \|f\|_{L^{p_d}} \|g\|_{L^{p_d}} \\ &\lesssim M \|\!|f|\!\|_* \|\!|g|\!\|_* + \|\mathbf{1}_F V^0\|_{L^{d/2}} \|f\|_{L^{p_d}} \|g\|_{L^{p_d}} \\ &\lesssim M \lambda^{-1/2} \|f\|_{X_\lambda^*} \|g\|_{X_\lambda^*} + \|\mathbf{1}_F V^0\|_{L^{d/2}} \|f\|_{X_\lambda^*} \|g\|_{X_\lambda^*}. \end{aligned} \tag{20}$$

In the last inequality we have used the theorem 2.9. From the inequalities (20) together with the density of $\mathcal{S}(\mathbb{R}^d)$ in X_λ^* provided by the proposition 2.7, we conclude the statement of the corollary by choosing $M = \lambda^{1/4}$. □

As a direct consequence of the theorem 2.8 and the corollary 2.10 we can estimate $\|(\Delta+\lambda\pm i0)^{-1}\circ V^0\|_{\mathcal{L}(X_\lambda^*)}$.

Corollary 2.11. *There exists a positive $\lambda_0=\lambda_0(d,V^0,R_0)$ so that*

$$\|(\Delta+\lambda\pm i0)^{-1}\circ V^0\|_{\mathcal{L}(X_\lambda^*)}<1$$

for all $\lambda\geq\lambda_0$.

Proof. Applying the theorem 2.8 and the corollary 2.10 and noting that $\|\mathbf{1}_F V^0\|_{L^{d/2}}$ tends to 0 as λ grows, we check that the statement holds. □

This corollary is the basic ingredient to perform the Neumann series argument sketched in the introduction. In fact, by the corollary 2.11 we have that the series

$$\sum_{n\in\mathbb{N}}[(\Delta+\lambda\pm i0)^{-1}\circ V^0]^{n-1}(u) \tag{21}$$

converges in X_λ^*, for every $u\in X_\lambda^*$. Thus, we can construct the resolvent

$$(\Delta+\lambda\pm i0-V^0)^{-1}$$

and prove its boundedness from X_λ to X_λ^*.

Proposition 2.12. *The operator defined by*

$$(\Delta+\lambda\pm i0-V^0)^{-1}f=\sum_{n\in\mathbb{N}}[(\Delta+\lambda\pm i0)^{-1}\circ V^0]^{n-1}((\Delta+\lambda\pm i0)^{-1}f),$$

for every $f\in X_\lambda$, is bounded from X_λ to X_λ^*. Moreover, $u^\pm=(\Delta+\lambda\pm i0-V^0)^{-1}f$ solves the equation

$$(\Delta+\lambda-V^0)u^\pm=f \text{ in } \mathbb{R}^d, \tag{22}$$

and, if f is compactly supported in B_0, then $u^\pm$ satisfies the Sommerfeld radiation condition

$$\sup_{|x|=R}\left|\frac{x}{|x|}\cdot\nabla u^\pm(x)\mp i\lambda^{1/2}u^\pm(x)\right|=o(R^{-\frac{d-1}{2}})$$

for all $R\geq R_0$.

Proof. The fact that $(\Delta+\lambda\pm i0-V^0)^{-1}$ is well-defined in X_λ follows from the convergence of the series (21), which is consequence of the corollary 2.11. The boundedness from X_λ to X_λ^* follows from the theorem 2.8 and the fact that the series (21) defines a bounded operator in X_λ^*. To check that $u^\pm$ solves (22) we just need to note that

$$\begin{aligned} u^\pm &= (\Delta+\lambda\pm i0)^{-1}f+\sum_{n\in\mathbb{N}}[(\Delta+\lambda\pm i0)^{-1}\circ V^0]^n((\Delta+\lambda\pm i0)^{-1}f)\\ &= (\Delta+\lambda\pm i0)^{-1}f+(\Delta+\lambda\pm i0)^{-1}(V^0u^\pm). \end{aligned} \tag{23}$$

Last identity holds by the corollary 2.11. Thus, testing the differential operator $\Delta+\lambda$ with $u^\pm$ and using the identity (23), we obtain that

$$(\Delta+\lambda-V^0)u^\pm=f \text{ in } \mathbb{R}^d.$$

To finish the proof of this proposition, we need to check that $u^\pm$ satisfies the corresponding radiation condition. Start by noting that

$$u^\pm=(\Delta+\lambda\pm i0)^{-1}\sum_{n\in\mathbb{N}}[V^0\circ(\Delta+\lambda\pm i0)^{-1}]^{n-1}(f).$$

To justify this identity, we use the boundedness of $(\Delta+\lambda\pm i0)^{-1}$ from X_λ to X_λ^* and that, for every $\lambda\geq\lambda_0$,

$$\|V^0\circ(\Delta+\lambda\pm i0)^{-1}\|_{\mathcal{L}(X_\lambda)}<1.$$

The contraction of $V^0 \circ (\Delta + \lambda \pm i0)^{-1}$ in X_λ is a consequence of the corollary 2.10 and the theorem 2.8. Note that $u^\pm = (\Delta + \lambda \pm i0)^{-1} g$, with

$$g = \sum_{n \in \mathbb{N}} [V^0 \circ (\Delta + \lambda \pm i0)^{-1}]^{n-1}(f) \in X_\lambda$$

and compactly supported in B_0. Since $g \in X_\lambda$, one can check that $u^\pm$ satisfies the equation $(\Delta + \lambda) u^\pm = g$. By Theorem 11.1.1 in [15], we have that the restriction of $u^\pm$ to $\mathbb{R}^d \setminus \operatorname{supp} g$ is smooth. On the other hand, since g is compactly supported and the function $y \mapsto \Phi^\pm_\lambda(x-y)$ is smooth in any open neighbourhood N_g of $\operatorname{supp} g$, for every $x \in \mathbb{R}^d \setminus \overline{N_g}$ then,

$$\langle u^\pm, \phi \rangle = \langle g, \int_{\mathbb{R}^d} \Phi^\pm_\lambda(x - \,\bullet\,) \phi(x) \,\mathrm{d}x \rangle = \int_{\mathbb{R}^d} \langle g, \Phi^\pm_\lambda(x - \,\bullet\,) \rangle \phi(x) \,\mathrm{d}x$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp} \phi \subset \mathbb{R}^d \setminus \operatorname{supp} g$. Then, the representation formula

$$u^\pm(x) = \langle g, \Phi^\pm_\lambda(x - \,\bullet\,) \rangle, \quad \forall x \in \mathbb{R}^d \setminus \operatorname{supp} g$$

holds. To check the radiation condition, we proceed as follows

$$\Big| \frac{x}{|x|} \cdot \nabla u^\pm(x) \mp i\lambda^{1/2} u^\pm(x) \Big| \leq \|g\|_{X_\lambda} \Big\| \chi \Big(\frac{x}{|x|} \cdot \nabla_x \mp i\lambda^{1/2} \Big) [\Phi^\pm_\lambda(x - \,\bullet\,)] \Big\|_{X^*_\lambda},$$

where χ is a smooth cut-off such that $\chi(y) = 1$ for all $y \in \operatorname{supp} g$, the subindex x in ∇_x indicates that the gradient is acting on the function $x \mapsto \Phi^\pm_\lambda(x-y)$. It remains to prove that

$$\sup_{|x|=R} \Big\| \chi \Big(\frac{x}{|x|} \cdot \nabla_x \mp i\lambda^{1/2} \Big) [\Phi^\pm_\lambda(x - \,\bullet\,)] \Big\|_{X^*_\lambda} = o(R^{-\frac{d-1}{2}}). \tag{24}$$

To do so, the first point we should notice is that

$$\begin{aligned} &\Big\| \chi \Big(\frac{x}{|x|} \cdot \nabla_x \mp i\lambda^{1/2} \Big) [\Phi^\pm_\lambda(x - \,\bullet\,)] \Big\|_{X^*_\lambda} \\ &\qquad \lesssim \sum_{|\alpha| \leq 1} \lambda^{\frac{1-|\alpha|}{2}} \Big\| \chi \Big(\frac{x}{|x|} \cdot \nabla_x \mp i\lambda^{1/2} \Big) [\partial^\alpha \Phi^\pm_\lambda(x - \,\bullet\,)] \Big\|_{L^2} \end{aligned} \tag{25}$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ denotes a multi-index and $|\alpha| = \alpha_1 + \dots + \alpha_d$. This inequality follows from the inequality

$$\|u\|_{X^*_\lambda} \lesssim \lambda^{1/4} |\!|\!| u |\!|\!|_* + \|(\lambda - \Delta)^{1/2} u\|_{L^2} \tag{26}$$

where $(\lambda - \Delta)^{1/2}$ denotes the multiplier with symbol $(\lambda + |\xi|^2)^{1/2}$. The inequality (26) is a consequence of a combination of three facts. The first one is the boundedness of P_k with respect to the norm $|\!|\!| \,\bullet\, |\!|\!|_*$. The second one is the inequality

$$\lambda^{d/2(1/q_d - 1/p_d)} \|P_k u\|_{L^{q_d}} \lesssim 2^k \|P_k u\|_{L^2}$$

for $k \in I$ —which follows from Bernstein's inequality and the equivalence $2^k \simeq 2^{k_\lambda} \simeq \lambda^{1/2}$ when $k \in I$. The third fact is that

$$m_\lambda(\xi)^{1/2} |\widehat{P_{<I} u}(\xi)| \simeq \lambda^{1/2} |\widehat{P_{<I} u}(\xi)|,$$

and

$$m_\lambda(\xi)^{1/2} |\widehat{P_k u}(\xi)| \simeq 2^k |\widehat{P_k u}(\xi)|$$

if $k > k_\lambda + 1$. Combining these three facts, one can derive the inequality (26). Finally, the condition (24) follows from the inequality (25) and the identity

$$\sup_{|x|=R} \Big| \frac{x}{|x|} \cdot \nabla_x \big(\partial_y^\alpha \Phi^\pm_\lambda(x-y) \big) \mp i\lambda^{1/2} \big(\partial_y^\alpha \Phi^\pm_\lambda(x-y) \big) \Big| = o_y(R^{-\frac{d-1}{2}}) \tag{27}$$

which holds uniformly for y in compact subsets. The identity (27) for $\alpha = 0$ is the standard radiation condition. The case $|\alpha| = 1$ is known but might not be so standard. It is consequence of a tedious computation, that is actually, the exactly same computation used to show that

$$\sup_{|x|=R}\Big|\frac{x}{|x|}\cdot\nabla_x\big(\nu_y\cdot\nabla_y\Phi^{\pm}_{\lambda}(x-y)\big)\mp i\lambda^{1/2}\big(\nu_y\cdot\nabla_y\Phi^{\pm}_{\lambda}(x-y)\big)\Big| = o_y(R^{-\frac{d-1}{2}}),$$

where ν denotes the unitary exterior vector normal to the boundary of a smooth bounded domain. The last identity is rather standard and is the basic ingredient to show that, if a solution of the homogeneous equation $(\Delta+\lambda)u=0$ in a exterior smooth bounded domain $\Omega=\mathbb{R}^d\setminus\overline{D}$ satisfies an integral representation in Ω, in terms of its values and those of $\Phi^{\pm}_{\lambda}(x-\centerdot)$ on $\partial\Omega$, then u has to satisfies the corresponding SRC. This shows that (24) holds and consequently the proof of this proposition is over. □

The next step will be to construct the scattering solution $u^{\pm}_{\rm sc}(\centerdot,y)$ as solution of the equation

$$\big(I-(\Delta+\lambda\pm i0-V^0)^{-1}\circ(\alpha\,\mathrm{d}\sigma)\big)u^{\pm}_{\rm sc}(\centerdot,y)=f^{\pm}(\centerdot,y)\ \text{in}\,\mathbb{R}^d \tag{28}$$

with $f^{\pm}(\centerdot,y)=(\Delta+\lambda\pm i0-V^0)^{-1}(Vu^{\pm}_{\rm in}(\centerdot,y))$. Assume for a moment that we have solved (28), then testing the operator $(\Delta+\lambda-V^0)$ with both sides of the identity (28), and applying the proposition 2.12, we would have that $u^{\pm}_{\rm sc}(\centerdot,y)$ solves the equation

$$(\Delta+\lambda-V)u^{\pm}_{\rm sc}(\centerdot,y)=Vu^{\pm}_{\rm in}(\centerdot,y)\ \text{in}\,\mathbb{R}^d.$$

Moreover, since

$$u^{\pm}_{\rm sc}(\centerdot,y)=(\Delta+\lambda\pm i0-V^0)^{-1}[(\alpha\,\mathrm{d}\sigma)u^{\pm}_{\rm sc}(\centerdot,y)+Vu^{\pm}_{\rm in}(\centerdot,y)] \tag{29}$$

we would also have, by the proposition 2.12 and the fact that $(\alpha\,\mathrm{d}\sigma)u^{\pm}_{\rm sc}(\centerdot,y)+Vu^{\pm}_{\rm in}(\centerdot,y)\in X_\lambda$ and is compactly supported, that $u^{\pm}_{\rm sc}(\centerdot,y)$ satisfies the Sommerfeld radiation condition:

$$\sup_{|x|=R}\Big|\frac{x}{|x|}\cdot\nabla u^{\pm}_{\rm sc}(x,y)\mp i\lambda^{1/2}u^{\pm}_{\rm sc}(x,y)\Big| = o_y(R^{-\frac{d-1}{2}}). \tag{30}$$

Thus, in order to prove the theorem 1 is enough to solve the equation (28).

To invert the operator $\big(I-(\Delta+\lambda\pm i0-V^0)^{-1}\circ(\alpha\,\mathrm{d}\sigma)\big)$ in X^*_λ we use the Fredholm alternative. The first point to be checked is the injectivity in X^*_λ of the operator

$$\big(I-(\Delta+\lambda\pm i0-V^0)^{-1}\circ(\alpha\,\mathrm{d}\sigma)\big). \tag{31}$$

The second point is to verify that $(\Delta+\lambda\pm i0-V^0)^{-1}\circ(\alpha\,\mathrm{d}\sigma)$ is compact in X^*_λ.

Start by proving the compactness. By the proposition 2.12 it is sufficient to show that the multiplication by $\alpha\,\mathrm{d}\sigma$ is compact from X^*_λ to X_λ. Note that multiplication by $\alpha\,\mathrm{d}\sigma$ is defined by

$$\langle f(\alpha\,\mathrm{d}\sigma),g\rangle=\langle\alpha\,\mathrm{d}\sigma,fg\rangle=\int_\Gamma\alpha fg\,\mathrm{d}\sigma.$$

Considering $\chi\in\mathcal{S}(\mathbb{R}^d)$ so that it does not vanish on Γ, we can write

$$\langle f(\alpha\,\mathrm{d}\sigma),g\rangle=\int_\Gamma\frac{\alpha}{\chi}(\chi f)g\,\mathrm{d}\sigma,$$

which means that the operator multiplication by $\alpha\,\mathrm{d}\sigma$ can be factorized as a composition of three operators, multiplication by χ, restriction to Γ —trace operator— and multiplication by $\alpha/\chi\,\mathrm{d}\sigma$. Multiplication by $\alpha/\chi\,\mathrm{d}\sigma$ is bounded from $L^2(\Gamma)$ to X_λ. This is a straightforward consequence of the Cauchy–Schwarz inequality, the theorem 4.15 and the definition 4.1 —in the sections 4.3 and 4.1, respectively— and the proposition 2.7. On the other hand, the trace on Γ is a bounded operator from

$\dot{B}^{1/2}_{2,1}(\mathbb{R}^d)$ to $L^2(\Gamma)$ —this is a Besov-space form of Theorem 14.1.1 in [15]. Recall that the semi-norm of the homogeneous Besov space $\dot{B}^{1/2}_{2,1}(\mathbb{R}^d)$ is given by

$$\|f\|_{\dot{B}^{1/2}_{2,1}} = \sum_{l\in\mathbb{Z}} 2^{l/2}\|P_l f\|_{L^2}.$$

Finally, multiplication by χ is a compact operator from X^*_λ to $\dot{B}^{1/2}_{2,1}(\mathbb{R}^d)$ at least when χ is defined by

$$\chi(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\delta/R_0 x\cdot\xi}\phi(\xi)\,\mathrm{d}\xi$$

with $\phi \in \mathcal{S}(\mathbb{R}^d)$ be a $[0,1]$-valued function supported in $\{\xi \in \mathbb{R}^d : |\xi| \le 1\}$ and it is not identically zero, and $\delta \in (0,1]$ chosen so that

$$\Big|\int_{\mathbb{R}^d} e^{ix\cdot\xi}\phi(\xi)\,\mathrm{d}\xi\Big| \ge \frac{1}{2}\int_{\mathbb{R}^d}\phi(\xi)\,\mathrm{d}\xi > 0$$

whenever $|x| \le \delta$. The compactness is a consequence of the lemma 4.17 and the definition 4.1 —in the sections 4.3 and 4.1, respectively— and the proposition 2.7. Therefore, the operator multiplication by $\alpha\,\mathrm{d}\sigma$ is a compact operator from X^*_λ to X_λ. This conclude the proof of the compactness of $(\Delta+\lambda\pm i0-V^0)^{-1}\circ(\alpha\,\mathrm{d}\sigma)$ in X^*_λ.

Continue by proving the injectivity. Let $v^\pm(\centerdot,y) \in X^*_\lambda$ be in the kernel of (31) and note that it satisfies that

$$v^\pm(\centerdot,y) = (\Delta+\lambda\pm i0-V^0)^{-1}[(\alpha\,\mathrm{d}\sigma)v^\pm(\centerdot,y)]. \tag{32}$$

Hence, by the proposition 2.12, $v^\pm(\centerdot,y)$ satisfies the Sommerfeld radiation condition (30). Furthermore, testing $(\Delta+\lambda-V^0)$ with $v^\pm(\centerdot,y)$, and using the identities (32) and the proposition 2.12, we obtain that $v^\pm(\centerdot,y)$ is solution of the equation

$$(\Delta+\lambda-V)v^\pm(\centerdot,y) = 0 \text{ in } \mathbb{R}^d.$$

A direct application of the lemma 2.13 below will show that $v^\pm(\centerdot,y)$ has to be identically zero. For that will need to show that $v^\pm$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, which is a consequence of the inclusion $X^*_\lambda \subset H^1_{\mathrm{loc}}(\mathbb{R}^d)$ proved in the lemma 2.14 below as well. Thus, we can use the Fredholm alternative to invert the operator (31), and construct $u^\pm_{\mathrm{sc}}(\centerdot,y)$ solving the equation (28). As we have already explained, this is the scattering solution we wanted, which ends the proof of the existence part of the theorem 1. The uniqueness part is again a direct application of the lemmas 2.13 and 2.14.

Lemma 2.13. *Consider $d \ge 3$. If $u^\pm \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a solution of*

$$(\Delta+\lambda-V)u^\pm = 0 \text{ in } \mathbb{R}^d$$

that satisfies the radiation condition

$$\sup_{|x|=R}\Big|\frac{x}{|x|}\cdot\nabla u^\pm(x) \mp i\lambda^{1/2}u^\pm(x)\Big| = o(R^{-\frac{d-1}{2}})$$

for all $R \ge R_0$, then $u^\pm$ has to be identically zero.

Proof. The restriction of $u^\pm$ to $\mathbb{R}^d \setminus \operatorname{supp} V$ is solution of $(\Delta+\lambda)u^\pm = 0$ in $\mathbb{R}^d \setminus \operatorname{supp} V$. By Theorem 11.1.1 in [15] this restriction is smooth, and we have that

$$\int_{\partial B}\big|\partial_\nu u^\pm \mp i\lambda^{1/2}u^\pm\big|^2\,\mathrm{d}S = \int_{\partial B}|\partial_\nu u^\pm|^2 + \lambda|u^\pm|^2 \mp 2\lambda^{1/2}\Im(\partial_\nu u^\pm\overline{u^\pm})\,\mathrm{d}S \tag{33}$$

where $\partial_\nu = \nu\cdot\nabla$ with ν the exterior unit normal vector to ∂B —the boundary of $B = \{x \in \mathbb{R}^d : |x| < R\}$— and $\Im$ denotes the imaginary part. Extending ν to be the exterior unit normal vector to $\partial(B\setminus\overline{B_0})$ and integrating by parts in $B\setminus\overline{B_0}$, we have that

$$2\int_{\partial B}\Im(\partial_\nu u^\pm\overline{u^\pm})\,\mathrm{d}S=-i\int_{\partial B}\partial_\nu u^\pm\overline{u^\pm}-\overline{\partial_\nu u^\pm}u^\pm\,\mathrm{d}S$$
$$=i\int_{\partial B_0}\partial_\nu u^\pm\overline{u^\pm}-\overline{\partial_\nu u^\pm}u^\pm\,\mathrm{d}S=-2\int_{\partial B_0}\Im(\partial_\nu u^\pm\overline{u^\pm})\,\mathrm{d}S.$$

Thus, taking limit, when R goes to infinity, in the identity (33) yields

$$\lim_{R\to\infty}\int_{\partial B}|\partial_\nu u^\pm|^2+\lambda|u^\pm|^2\,\mathrm{d}S=\mp2\lambda^{1/2}\int_{\partial B_0}\Im(\partial_\nu u^\pm\overline{u^\pm})\,\mathrm{d}S,$$

by the corresponding SRC. Since we assumed V^0 and α to be real-valued, we have integrating by parts now in B_0 that $\int_{\partial B_0}\Im(\partial_\nu u^\pm\overline{u^\pm})\,\mathrm{d}S=0$, which implies that $\lim_{R\to\infty}\int_{\partial B}|u^\pm|^2\,\mathrm{d}S=0$, and consequently, by Rellich's lemma, $\operatorname{supp}u^\pm\subset\overline{B_0}$ and $u^\pm\in H^1(\mathbb{R}^d)$.

It remains to prove that $u^\pm$ also vanishes in B_0, we do it using a Carleman estimate that Caro and Rogers proved in [6]. This estimate holds for a modified family of Bourgain-type spaces whose norms were

$$\|u\|_{Y^s_{\tau,M}}=\|(M\tau^2+M^{-1}|q_\tau|^2)^{s/2}\widehat{u}\|_{L^2}\tag{34}$$

with $M,\tau\in[1,\infty)$, $s\in\mathbb{R}$ and $q_\tau(\xi)=-|\xi|^2+i2\tau\xi_d+\tau^2$. The estimate, stated in Theorem 2.1 from [6], reads as follows. Set $\varphi(x)=\tau x_d+Mx_d^2/2$ and $R\geq1$. There exists an absolute constant $C>0$, such that, if $M>CR^2$, then

$$\|u\|_{Y^{1/2}_{\tau,M}}\leq CR\|e^{\varphi}\Delta(e^{-\varphi}u)\|_{Y^{-1/2}_{\tau,M}}\tag{35}$$

for all $u\in\mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}u\subset\{x\in\mathbb{R}^d:|x|<R\}$ and $\tau>8MR$. This inequality can be perturbed to consider the operator $\Delta+\lambda-V$ tested in any function in $H^1(\mathbb{R}^d)$ with support in $\overline{B_0}$. Indeed, start by estimating $(\lambda-V)u$ in $Y^{-1/2}_{\tau,M}$, by duality, with $u\in\mathcal{S}(\mathbb{R}^d)$ supported in B_0:

$$\langle(\lambda-V)u,v\rangle=\lambda\int_{\mathbb{R}^d}uv+\int_{\mathbb{R}^d}V^0uv+\int_\Gamma\alpha uv\,\mathrm{d}\sigma.\tag{36}$$

The first term on the right-hand side of (36) can be easily bounded by the Cauchy–Schwarz inequality

$$\Big|\lambda\int_{\mathbb{R}^d}uv\Big|\leq\lambda\|u\|_{L^2}\|v\|_{L^2}\leq\lambda M^{-1/2}\tau^{-1}\|u\|_{Y^{1/2}_{\tau,M}}\|v\|_{Y^{1/2}_{\tau,M}}.$$

To estimate the second term on the right-hand side of (36), we do as in the corollary 2.10

$$\int_{\mathbb{R}^d}V^0uv=\int_{\mathbb{R}^d}\mathbf{1}_EV^0uv+\int_{\mathbb{R}^d}\mathbf{1}_FV^0uv\,,$$

where $E=\{x\in\mathbb{R}^d:|V^0(x)|\leq N\}$, $F=\mathbb{R}^d\setminus E$ and N to be chosen. Thus, we have by the Cauchy–Schwarz and Hölder's inequalities

$$\Big|\int_{\mathbb{R}^d}V^0uv\Big|\leq N\|u\|_{L^2}\|v\|_{L^2}+\|\mathbf{1}_FV^0\|_{L^{d/2}}\|u\|_{L^{p_d}}\|v\|_{L^{p_d}}$$
$$\leq NM^{-1/2}\tau^{-1}\|u\|_{Y^{1/2}_{\tau,M}}\|v\|_{Y^{1/2}_{\tau,M}}+\|\mathbf{1}_FV^0\|_{L^{d/2}}\|u\|_{\dot{Y}^{1/2}_\tau}\|v\|_{\dot{Y}^{1/2}_\tau}.$$

In the last inequality we have used Haberman's embedding —see the corollary 4.23 in the section 4.4. By the definition of the norm of the space $Y^{1/2}_{\tau,M}$ we have that

$$\Big|\int_{\mathbb{R}^d}V^0uv\Big|\leq(NM^{-1/2}\tau^{-1}+M^{1/2}\|\mathbf{1}_FV^0\|_{L^{d/2}})\|u\|_{Y^{1/2}_{\tau,M}}\|v\|_{Y^{1/2}_{\tau,M}}.$$

Finally, we estimate the third term on the right-hand side of the identity (36). To do so, we use the Besov-space form of Theorem 14.1.1 in [15] and see that

$$\begin{aligned}\Big|\int_\Gamma \alpha uv\,\mathrm{d}\sigma\Big| \leq \|\alpha\|_{L^\infty(\Gamma)}\Big(&\sum_{k\leq l_\tau+1} 2^{k/2}\|P_k u\|_{L^2}\sum_{l\leq l_\tau+1} 2^{k/2}\|P_l v\|_{L^2}\\ &+\sum_{k\leq l_\tau+1} 2^{k/2}\|P_k u\|_{L^2}\sum_{l>l_\tau+1} 2^{l/2}\|P_l v\|_{L^2}\\ &+\sum_{k>l_\tau+1} 2^{k/2}\|P_k u\|_{L^2}\sum_{l\in\mathbb{Z}} 2^{l/2}\|P_l v\|_{L^2}\Big),\end{aligned}$$

where $l_\tau \in \mathbb{Z}$ satisfies that $2^{l_\tau-1} < \tau \leq 2^{l_\tau}$. If $k > l_\tau + 1$, we have that $2^{k/2}|\widehat{P_k u}(\xi)| \simeq 2^{-k/2}|q_\tau(\xi)|^{1/2}|\widehat{P_k u}(\xi)|$ for all $\xi \in \mathbb{R}^d$. Hence, for the high frequencies we have

$$\begin{aligned}\sum_{k>l_\tau+1} 2^{k/2}\|P_k u\|_{L^2} &\simeq \sum_{k>l_\tau+1} 2^{-k/2}\||q_\tau|^{1/2}\widehat{P_k u}\|_{L^2}\\ &\lesssim \tau^{-1/2}\|u\|_{\dot Y^{1/2}_\tau} \leq \tau^{-1/2}M^{1/4}\|u\|_{Y^{1/2}_{\tau,M}}.\end{aligned}$$

On the other hand, for the low frequencies we have that

$$\sum_{k\leq l_\tau+1} 2^{k/2}\|P_k u\|_{L^2} \lesssim \tau^{1/2}\|u\|_{L^2} \leq M^{-1/4}\|u\|_{Y^{1/2}_{\tau,M}}.$$

Combining the previous inequalities for the high and low frequencies we obtain that there exists an absolute constant $C' > 0$ such that

$$\Big|\int_\Gamma \alpha uv\,\mathrm{d}\sigma\Big| \leq C'\|\alpha\|_{L^\infty(\Gamma)}(M^{-1/2}+\tau^{-1/2}+\tau^{-1}M^{1/2})\|u\|_{Y^{1/2}_{\tau,M}}\|v\|_{Y^{1/2}_{\tau,M}}.$$

We now choose M so that $CR_0C'\|\alpha\|_{L^\infty(\Gamma)}M^{-1/2} \leq 1/4$, then we choose N such that $CR_0M^{1/2}\|\mathbf{1}_F V^0\|_{L^{d/2}} \leq 1/4$, and finally, we consider τ to have

$$CR_0[(\lambda+N)M^{-1/2}\tau^{-1} + C'\|\alpha\|_{L^\infty(\Gamma)}(\tau^{-1/2}+\tau^{-1}M^{1/2})] < 1/4.$$

Therefore, we can conclude that there exists a $\tau_0 = \tau_0(R_0, \|\alpha\|_{L^\infty(\Gamma)}, V^0, \lambda)$ such that

$$\|u\|_{Y^{1/2}_{\tau,M}} \leq 4CR_0\|e^{\varphi}\big(\Delta+\lambda-V\big)(e^{-\varphi}u)\|_{Y^{-1/2}_{\tau,M}} \tag{37}$$

for all $u \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp} u \subset B_0$ and $\tau > \tau_0$. One can check that $Y^{1/2}_{\tau,M}$ and $H^1(\mathbb{R}^d)$ are equal as sets, and that, for every $u \in H^1(\mathbb{R}^d)$ with $\operatorname{supp} u \subset \overline{B_0}$, we have $e^{\varphi}\big(\Delta+\lambda-V\big)(e^{-\varphi}u) \in Y^{-1/2}_{\tau,M}$. Thus , by a density argument

$$\|e^{\varphi}u\|_{Y^{1/2}_{\tau,M}} \leq 4CR_0\|e^{\varphi}\big(\Delta+\lambda-V\big)u\|_{Y^{-1/2}_{\tau,M}} \tag{38}$$

for all $u \in H^1(\mathbb{R}^d)$ with $\operatorname{supp} u \subset \overline{B_0}$ and $\tau > \tau_0$. Since $u^\pm$ is supported in $\overline{B_0}$, belongs to $H^1(\mathbb{R}^d)$ and solves $(\Delta+\lambda-V)u^\pm = 0$ in $\mathbb{R}^d$, we have that $u^\pm$ is identically zero by applying the inequality (38). □

Lemma 2.14. *Every $u \in X^*_\lambda$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$.*

Proof. Consider $u \in X^*_\lambda$ and set $u_I = \sum_{k\in I} P_k u$ and $u_{\mathbb{Z}\setminus I} = u - u_I$. Let us show that u_I belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $u_{\mathbb{Z}\setminus I}$ is in $H^1(\mathbb{R}^d)$. Let K be a compact subset of $\mathbb{R}^d$ and $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$ denote a multi-index such that $|\alpha| = \alpha_1+\cdots+\alpha_d \leq 1$, we have that

$$\|\partial^\alpha u_I\|_{L^2(K)} \leq \sum_{k\in I}\|\partial^\alpha P_k u\|_{L^2(K)}.$$

By Hölder's inequality, Bernstein's inequality for ∂^α, and the fact that I only contains four elements, we have

$$\|\partial^\alpha u_I\|_{L^2(K)} \lesssim \sum_{k\in I} \|\partial^\alpha P_k u\|_{L^{q_d}} \lesssim \sum_{k\in I} 2^{|\alpha|k} \|P_k u\|_{L^{q_d}} \lesssim \Big(\sum_{k\in I} 2^{2k} \|P_k u\|^2_{L^{q_d}}\Big)^{\frac{1}{2}}.$$

Thus,

$$\|\partial^\alpha u_I\|_{L^2(K)} \lesssim \lambda^{\frac{1}{2}-\frac{d}{2}(\frac{1}{q_d}-\frac{1}{p_d})} \Big(\sum_{k\in I} \lambda^{d(\frac{1}{q_d}-\frac{1}{p_d})} \|P_k u\|^2_{L^{q_d}}\Big)^{\frac{1}{2}},$$

which shows that u_I belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. Next we prove that $u_{\mathbb{Z}\setminus I}$ belongs to $H^1(\mathbb{R}^d)$. Let $(\lambda-\Delta)^{1/2}$ denote the multiplier with symbol $(\lambda+|\xi|^2)^{1/2}$. By Plancherel's identity and the finite overlapping of the supports of $\{P_k u : k > k_\lambda+1\}$, we have

$$\begin{aligned}\|(\lambda-\Delta)^{1/2} u_{\mathbb{Z}\setminus I}\|^2_{L^2} &= \int_{\mathbb{R}^d} (\lambda+|\xi|^2)|\widehat{u_{\mathbb{Z}\setminus I}}(\xi)|^2 \,\mathrm{d}\xi \\ &\simeq \int_{\mathbb{R}^d} (\lambda+|\xi|^2)|\widehat{P_{<I}u}(\xi)|^2 \,\mathrm{d}\xi + \sum_{k>k_\lambda+1} \int_{\mathbb{R}^d} (\lambda+|\xi|^2)|\widehat{P_k u}(\xi)|^2 \,\mathrm{d}\xi \\ &\simeq \lambda \int_{\mathbb{R}^d} |\widehat{P_{<I}u}(\xi)|^2 \,\mathrm{d}\xi + \sum_{k>k_\lambda+1} 2^{2k} \int_{\mathbb{R}^d} |\widehat{P_k u}(\xi)|^2 \,\mathrm{d}\xi.\end{aligned}$$

Note that $\lambda^{1/2}|\widehat{P_{<I}u}(\xi)| \simeq m_\lambda(\xi)^{1/2}|\widehat{P_{<I}u}(\xi)|$ for all $\xi \in \mathbb{R}^d$. While, if $k > k_\lambda+1$, we have that $2^k|\widehat{P_k u}(\xi)| \simeq m_\lambda(\xi)^{1/2}|\widehat{P_k u}(\xi)|$. Hence,

$$\|(\lambda-\Delta)^{1/2} u_{\mathbb{Z}\setminus I}\|^2_{L^2} \simeq \|m_\lambda^{1/2}\widehat{P_{<I}u}\|^2_{L^2} + \sum_{k>k_\lambda+1} \|m_\lambda^{1/2}\widehat{P_k u}\|^2_{L^2},$$

which proves that $u_{\mathbb{Z}\setminus I}$ belongs to $H^1(\mathbb{R}^d)$. This ends the proof of this lemma. □

We finish this section by stating an inequality which will be essential to address the inverse scattering problem.

Lemma 2.15. *Consider $d \geq 3$ and $\varphi(x) = \tau x_d + M x_d^2/2$. There exist positive constants $M = M(R_0, \|\alpha\|_{L^\infty(\Gamma)})$, $C = C(R_0)$ and $\tau_0 = \tau_0(R_0, \|\alpha\|_{L^\infty(\Gamma)}, V^0, \lambda)$ such that*

$$\|u\|_{Y^{1/2}_{\tau,M}} \leq C\|e^{\varphi}\big(\Delta+\lambda-T^*V\big)(e^{-\varphi}u)\|_{Y^{-1/2}_{\tau,M}}$$

for all $u \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp} u \subset B_0$ and $\tau > \tau_0$. Here T^*V denotes the following potential

$$\langle T^*V, \phi\rangle = \int_{\mathbb{R}^d} T^*V^0\,\phi + \int_{T^t\Gamma} T^*\alpha\,\phi \,\mathrm{d}T^*\sigma,$$

where $T \in SO(d)$, $T^*V^0(x) = V^0(Tx)$, $T^*\alpha(x) = \alpha(Tx)$, $T^t\Gamma = \{T^t x : x \in \Gamma\}$ and $T^*\sigma(E) = \sigma(TE)$ with $TE = \{Tx : x \in E\}$.

Recall that the norms of the spaces $Y^{1/2}_{\tau,M}$ and $Y^{-1/2}_{\tau,M}$ were given in (34).

Proof. We start from the inequality (35) —due to Caro and Rogers [6]— and perturb it to include the potential T^*V. This procedure is exactly the same as the one used in proof of the lemma 2.13 to derive the inequality (37) and we will not repeat it. □

3. Inverse scattering

In this section we adapt to our framework the approach we learnt from [13] by Hähner and Hohage. The first step is to obtain the orthogonality identity (15). In order to prove it, we need two lemmas regarding the single layer potential $\mathcal{S}^\pm$ whose kernel is given by the total wave

$$u^\pm_{\rm to}|_{\partial B_0\times\partial B_0} = u^\pm_{\rm in}|_{\partial B_0\times\partial B_0} + u^\pm_{\rm sc}|_{\partial B_0\times\partial B_0}.$$

For f continuous on ∂B_0, we define the *single layer potential* as

$$\mathcal{S}^\pm f(x) = \int_{\partial B_0} u^\pm_{\rm to}(x, \cdot) f \,\mathrm{d}S,$$

for $x \in \partial B_0$ where $\mathrm{d}S$ denotes the volume form on ∂B_0.

Lemma 3.1. *The scattering solution of the theorem 1 satisfies the following reciprocity relation*

$$u^\pm_{\rm sc}(x, y) = u^\pm_{\rm sc}(y, x), \quad \forall x, y \in \mathbb{R}^d \setminus \operatorname{supp} V.$$

In particular, the single layer potential $\mathcal{S}^\pm$ is symmetric, that is,

$$\int_{\partial B_0} \mathcal{S}^\pm f g \,\mathrm{d}S = \int_{\partial B_0} f \mathcal{S}^\pm g \,\mathrm{d}S$$

for all f and g continuous on ∂B_0.

Proof. Given $x, y \in \mathbb{R}^d \setminus \operatorname{supp} V$, there exists a bounded domain D containing $\operatorname{supp} V$ so that $x, y \in \mathbb{R}^d \setminus \overline{D}$ and its boundary is locally described by the graphs of twice continuously differentiable functions. The restrictions of $u^\pm_{\rm sc}(\cdot, x)$ and $u^\pm_{\rm sc}(\cdot, y)$ to $\mathbb{R}^d \setminus \operatorname{supp} V$ are solutions of the equation $(\Delta + \lambda)u = 0$ in $\mathbb{R}^d \setminus \operatorname{supp} V$. By Theorem 11.1.1 in [15] these restrictions are smooth. Thus, integrating by parts in a $B \setminus \overline{D}$, with $B = \{x \in \mathbb{R}^d : |x| < R\}$, and making R go to infinity we have that

$$\int_{\partial D} \left[\partial_\nu u^\pm_{\rm sc}(\cdot, y) u^\pm_{\rm sc}(\cdot, x) - u^\pm_{\rm sc}(\cdot, y)\partial_\nu u^\pm_{\rm sc}(\cdot, x)\right] \mathrm{d}S = 0 \tag{39}$$

where $\mathrm{d}S$ denotes the volume form on ∂D and ν stands for the unit exterior normal vector on ∂D. In order to make the integration on ∂B vanish when R goes to infinity, we just need to use the corresponding SRC. On the other hand, since the restrictions of $u^\pm_{\rm in}(\cdot, x)$ and $u^\pm_{\rm in}(\cdot, y)$ to D are solutions of the equation $(\Delta + \lambda)u = 0$ in D, we have, integrating by parts in D, that

$$\int_{\partial D} \left[\partial_\nu u^\pm_{\rm in}(\cdot, y) u^\pm_{\rm in}(\cdot, x) - u^\pm_{\rm in}(\cdot, y)\partial_\nu u^\pm_{\rm in}(\cdot, x)\right] \mathrm{d}S = 0. \tag{40}$$

Finally, it is well-known that smooth solutions of $(\Delta + \lambda)u = 0$ in $\mathbb{R}^d \setminus \operatorname{supp} V$ can be represented by a boundary integral expression. In particular, since $u^\pm_{\rm in}(\cdot, z) = \Phi^\pm_\lambda(z - \cdot)$ the functions $u^\pm_{\rm sc}(\cdot, y)$ and $u^\pm_{\rm sc}(\cdot, x)$ can be represented respectively by

$$u^\pm_{\rm sc}(z, y) = \int_{\partial D} \left[\partial_\nu u^\pm_{\rm sc}(\cdot, y) u^\pm_{\rm in}(\cdot, z) - u^\pm_{\rm sc}(\cdot, y)\partial_\nu u^\pm_{\rm in}(\cdot, z)\right] \mathrm{d}S, \quad \forall z \in \mathbb{R}^d \setminus \overline{D}, \tag{41}$$

and

$$u^\pm_{\rm sc}(z, x) = \int_{\partial D} \left[\partial_\nu u^\pm_{\rm sc}(\cdot, x) u^\pm_{\rm in}(\cdot, z) - u^\pm_{\rm sc}(\cdot, x)\partial_\nu u^\pm_{\rm in}(\cdot, z)\right] \mathrm{d}S, \quad \forall z \in \mathbb{R}^d \setminus \overline{D}. \tag{42}$$

Note that evaluating (41) at $z = x$ and (42) at $z = y$, we can compute $u^\pm_{\rm sc}(x, y) - u^\pm_{\rm sc}(y, x)$. Now, using the identities (39) and (40) we have

$$u^\pm_{\rm sc}(x, y) - u^\pm_{\rm sc}(y, x) = \int_{\partial D} \left[\partial_\nu u^\pm_{\rm to}(\cdot, y) u^\pm_{\rm to}(\cdot, x) - u^\pm_{\rm to}(\cdot, y)\partial_\nu u^\pm_{\rm to}(\cdot, x)\right] \mathrm{d}S.$$

Integrating by parts the right-had side of last identity in D and using that $u^\pm_{\mathrm{to}}(\centerdot, y)$ and $u^\pm_{\mathrm{to}}(\centerdot, x)$ are solutions of $(\Delta + \lambda - V)u = 0$ in D, we get that

$$u^\pm_{\mathrm{sc}}(x, y) - u^\pm_{\mathrm{sc}}(y, x) = \langle V u^\pm_{\mathrm{to}}(\centerdot, y), u^\pm_{\mathrm{to}}(\centerdot, x)\rangle - \langle V u^\pm_{\mathrm{to}}(\centerdot, x), u^\pm_{\mathrm{to}}(\centerdot, y)\rangle = 0.$$

This finishes the proof of the first part of this lemma. The second part is a direct consequence of first one since

$$u^\pm_{\mathrm{to}}|_{\partial B_0\times\partial B_0} = u^\pm_{\mathrm{in}}|_{\partial B_0\times\partial B_0} + u^\pm_{\mathrm{sc}}|_{\partial B_0\times\partial B_0}$$

is the kernel of the single layer potential and $u^\pm_{\mathrm{in}}(x, y) = \Phi^\pm_\lambda(y-x)$ with $\Phi^\pm_\lambda$ radially symmetric. □

Lemma 3.2. *Consider $d \geq 3$. Let f be continuous on ∂B_0. Then, the function*

$$u^\pm(x) = \int_{\partial B_0} u^\pm_{\mathrm{to}}(x, \centerdot) f \,\mathrm{d}S$$

is the unique solution in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ of the problem

$$\begin{cases} (\Delta + \lambda - V)u^\pm = 0 & \text{in}\, \mathbb{R}^d \setminus \partial B_0, \\ \partial_\nu u^\pm|_{\mathbb{R}^d\setminus\overline{B_0}} - \partial_\nu u^\pm|_{B_0} = f & \text{on}\, \partial B_0, \\ \sup_{|x|=R} \left| \frac{x}{|x|}\cdot\nabla u^\pm(x) \mp i\lambda^{1/2}u^\pm(x)\right| = o(R^{-\frac{d-1}{2}}) & \text{for}\, R \geq R_0. \end{cases} \tag{43}$$

Here ν is the unit exterior normal vector on ∂B_0.

Proof. Start by showing that the problem (43) has a unique solution in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. Note that it is enough to show that if $u^\pm$ is a solution for $f = 0$, then $u^\pm = 0$. The restrictions of $u^\pm$ to $B_0 \setminus \operatorname{supp} V$ and $\mathbb{R}^d \setminus \overline{B_0}$ are solutions of the equation $(\Delta + \lambda)u^\pm = 0$ in $B_0 \setminus \operatorname{supp} V$ and $\mathbb{R}^d \setminus \overline{B_0}$ respectively. By Theorem 11.1.1 in [15] these restrictions are smooth and can be extended by continuity up to the boundary of B_0. Since $u^\pm \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, the extensions from both sides of the boundary must coincide. The facts that $f = 0$ and $u^\pm$ is continuous across ∂B_0 make $u^\pm$ be a solution of $(\Delta + \lambda - V)u^\pm = 0$ in $\mathbb{R}^d$. Since it satisfies the SRC and belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, it has to vanish everywhere by the lemma 2.13.

Now we show that $u^\pm$ is solution of (43). The function $u^\pm$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ because $u^\pm_{\mathrm{sc}}(\centerdot, y)$ is in X^*_λ (recall the lemma 2.14) and the function

$$v^\pm(x) = \int_{\partial B_0} u^\pm_{\mathrm{in}}(x, \centerdot) f \,\mathrm{d}S$$

is continuous in $\mathbb{R}^d$ and smooth in $\mathbb{R}^d \setminus \partial B_0$ —recall that $u^\pm_{\mathrm{in}}(\centerdot, y = \Phi^\pm_\lambda(y - \centerdot))$. Moreover, $u^\pm$ solves the problem (43) because $u^\pm_{\mathrm{sc}}(\centerdot, y)$ is smooth in $\mathbb{R}^d \setminus \operatorname{supp} V$ for $y \in \mathbb{R}^d \setminus B_0$ (Theorem 11.1.1 in [15]) and solves the problem (16), and $v^\pm$ solves the problem

$$\begin{cases} (\Delta + \lambda)v^\pm = 0 & \text{in}\, \mathbb{R}^d \setminus \partial B_0, \\ \partial_\nu v^\pm|_{\mathbb{R}^d\setminus\overline{B_0}} - \partial_\nu v^\pm|_{B_0} = f & \text{on}\, \partial B_0, \\ \sup_{|x|=R} \left| \frac{x}{|x|}\cdot\nabla v^\pm(x) \mp i\lambda^{1/2}v^\pm(x)\right| = o(R^{-\frac{d-1}{2}}) & \text{for}\, R \geq R_0, \end{cases}$$

—again the fact that $v^\pm$ solves this problem is classical (see for example [8]). □

Proposition 3.3. *Consider $d \geq 3$. Let V_1 and V_2 be two electric potentials as in the theorem 2. Let $u^\pm_{\mathrm{sc},1}(\centerdot, y)$ and $u^\pm_{\mathrm{sc},2}(\centerdot, y)$ with $y \in \mathbb{R}^d \setminus B_0$ be the scattering solutions of the theorem 1 associated to V_1 and V_2. If*

$$u^\pm_{\mathrm{sc},1}|_{\partial B_0\times\partial B_0} = u^\pm_{\mathrm{sc},2}|_{\partial B_0\times\partial B_0},$$

then

$$\langle (V_1 - V_2)v_1, v_2\rangle = 0$$

for all v_1 *and* v_2 *in* $H^1(B_0)$ *such that* $(\Delta + \lambda - V_j)v_j = 0$ *in* B_0.

Proof. By Theorem 11.1.1 in [15], we know that the restriction of v_j to $B_0 \setminus \operatorname{supp} V_j$ is smooth. We extend v_j up to ∂B_0 by continuity. Let $w_j^\pm$ be the solution of $(\Delta + \lambda)w_j^\pm = 0$ in $\mathbb{R}^d \setminus \overline{B_0}$ satisfying the corresponding SRC and the Dirichlet boundary condition $w_j^\pm|_{\partial B_0} = v_j|_{\partial B_0}$. The solution $w_j^\pm$ is continuous in $\mathbb{R}^d \setminus B_0$ and smooth in $\mathbb{R}^d \setminus \overline{B_0}$ (see Theorem 3.11 in [8]). Then, the function

$$u_j^\pm = \mathbf{1}_{\overline{B_0}} v_j + \mathbf{1}_{\mathbb{R}^d \setminus \overline{B_0}} w_j^\pm \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$$

—$\mathbf{1}_{\overline{B_0}}$ and $\mathbf{1}_{\mathbb{R}^d \setminus \overline{B_0}}$ stand for the characteristic functions of $\overline{B_0}$ and its complement— and, by the lemma 3.2, satisfies that

$$u_j^\pm(x) = \int_{\partial B_0} u^\pm_{\mathrm{to},j}(x, \cdot)\big(\partial_\nu w_j^\pm - \partial_\nu v_j\big)\,\mathrm{d}S,$$

where ν is the unit exterior normal vector on ∂B_0. In particular,

$$v_j|_{\partial B_0} = \mathcal{S}_j^\pm\big(\partial_\nu w_j^\pm - \partial_\nu v_j\big). \tag{44}$$

Note that, integrating by parts in B_0 we have that

$$\langle (V_1 - V_2)v_1, v_2\rangle = \int_{\partial B_0} \big(v_2 \partial_\nu v_1 - v_1 \partial_\nu v_2\big)\mathrm{d}S;$$

while integrating by parts in $B \setminus \overline{B_0}$, where $B = \{x \in \mathbb{R}^d : |x| < R\}$, and making R go to infinity we have that

$$\int_{\partial B_0} \big(v_2 \partial_\nu w_1^\pm - v_1 \partial_\nu w_2^\pm\big)\mathrm{d}S = 0 \tag{45}$$

by the SRC. Then, by the identity (45) first and then by (44), we have that

$$\begin{aligned}
\langle (V_1 - V_2)v_1, v_2\rangle &= -\int_{\partial B_0} \big[v_2\big(\partial_\nu w_1^\pm - \partial_\nu v_1\big) - v_1\big(\partial_\nu w_2^\pm - \partial_\nu v_2\big)\big]\mathrm{d}S \\
&= -\int_{\partial B_0} \big[\mathcal{S}_2^\pm\big(\partial_\nu w_2^\pm - \partial_\nu v_2\big)\big(\partial_\nu w_1^\pm - \partial_\nu v_1\big) \\
&\qquad - \mathcal{S}_1^\pm\big(\partial_\nu w_1^\pm - \partial_\nu v_1\big)\big(\partial_\nu w_2^\pm - \partial_\nu v_2\big)\big]\mathrm{d}S.
\end{aligned}$$

By the symmetry of $\mathcal{S}_j^\pm$ stated in the lemma 3.1, we have

$$\langle (V_1 - V_2)v_1, v_2\rangle = \int_{\partial B_0} \big[\mathcal{S}_1^\pm - \mathcal{S}_2^\pm\big]\big(\partial_\nu w_2^\pm - \partial_\nu v_2\big)\big(\partial_\nu w_1^\pm - \partial_\nu v_1\big)\mathrm{d}S.$$

Thus, if $u^\pm_{\mathrm{sc},1}|_{\partial B_0 \times \partial B_0} = u^\pm_{\mathrm{sc},2}|_{\partial B_0 \times \partial B_0}$ the kernel of the operator $\mathcal{S}_1^\pm - \mathcal{S}_2^\pm$ is zero, and consequently, $\langle (V_1 - V_2)v_1, v_2\rangle = 0$. □

As we mentioned in the introduction, we will test the identity of the proposition 3.3 with a family of CGO solutions of the form

$$v_j(x) = e^{\zeta_j \cdot x}(1 + w_j(x)), \tag{46}$$

where $\zeta_j \in \mathbb{C}^d$ so that $\zeta_j \cdot \zeta_j = -\lambda$ and $\zeta_1 + \zeta_2 = -i\kappa$ for an arbitrarily given $\kappa \in \mathbb{R}^d$, and the correction term w_j vanishing in a specific sense when $|\zeta_j|$ grows. In order to state the existence of this type of solutions, we will need to introduce some spaces in the spirit of Haberman and Tataru in [12], and Caro and Rogers in [6]. First we introduce the non-homogeneous Bourgain space X^s_ζ with $s \in \mathbb{R}$, which consists of $u \in \mathcal{S}'(\mathbb{R}^d)$ so that $\widehat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\|u\|^2_{X^s_\zeta} = \int_{\mathbb{R}^d} (|\zeta| + |p_\zeta(\xi)|)^{2s} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty,$$

endowed with the norm $\|\,\cdot\,\|_{X^s_\zeta}$. Here $p_\zeta(\xi) = -|\xi|^2 + i2\zeta\cdot\xi$ for $\xi \in \mathbb{R}^d$. Then, for $s \geq 0$, we introduce the space

$$X^s_\zeta(B_0) = \{u|_{B_0} : u \in X^s_\zeta\}$$

endowed with the norm

$$\|u\|_{X^s_\zeta(B_0)} = \inf\{\|v\|_{X^s_\zeta} : v|_{B_0} = u\}.$$

For us, the only relevant indices will be $s = 1/2$ and $s = -1/2$. In addition to these spaces, there is another family of spaces that will be useful for us. This is given, for $s \in \mathbb{R}$, by the set

$$X^s_{\zeta,c}(B_0) = \{u \in X^s_\zeta : \operatorname{supp} u \subset \overline{B_0}\},$$

endowed with the same norm $\|\,\cdot\,\|_{X^s_\zeta}$. As it was stated in [6], $X^{-s}_{\zeta,c}(B_0)$ can be identified with dual space of $X^s_\zeta(B_0)$ for $s \geq 0$.

Proposition 3.4. *Consider $d \geq 3$ and τ_0 as in the lemma 2.15. For every $\zeta = \Re\zeta + i\Im\zeta \in \mathbb{C}^d$ such that $|\Re\zeta| = \tau$, $|\Im\zeta| = (\tau^2+\lambda)^{1/2}$ and $\Re\zeta\cdot\Im\zeta = 0$ with $\tau \geq \tau_0$, we have that there exists $w_\zeta \in X^{1/2}_\zeta(B_0)$ so that $v_\zeta = e^{\zeta\cdot x}(1 + w_\zeta)$ is solution of the equation $(\Delta + \lambda - V)v_\zeta = 0$ in B_0 and*

$$\|w_\zeta\|_{X^{1/2}_\zeta(B_0)} \lesssim \|V\|_{X^{-1/2}_\zeta}.$$

Proof. The lemma 2.15 is the analogue of Lemma 2.1 in [6]. Then, considering $T \in SO(d)$ so that $\Re\zeta = \tau T e_d$ and arguing as in Lemma 2.2, Lemma 2.3 and Proposition 2.4 in [6] we can derive the following inequality:

$$\|u\|_{X^{1/2}_\zeta} \lesssim \|(\Delta + 2\zeta\cdot\nabla - V)u\|_{X^{-1/2}_\zeta}$$

for all $u \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp} u \subset B_0$ and $\tau > \tau_0$. The implicit constant in the previous inequality depends on R_0 and $\|\alpha\|_{L^\infty(\Gamma)}$, while τ_0 is as in the lemma 2.15. Here we are implicitly using that $V \in X^{-1/2}_\zeta$. This fact is derived from the lemma 3.5 proved at the end of this section.

This inequality is the analogue of the one stated in Proposition 2.4 from [6], and it represents the key ingredient to perform the method of *a priori estimates* which yields the existence of w_ζ and its corresponding bound by the norm of V. For the details, see the pages 11 and 12, and Proposition 2.5 in [6]. □

The proposition 3.4 yields directly pairs of solutions as in (46), however, we also need that these pairs satisfy $\zeta_1 + \zeta_2 = -i\kappa$ for an arbitrarily given $\kappa \in \mathbb{R}^d$. Thus, let $\kappa \in \mathbb{R}^d$ be given and choose $\eta, \theta \in \mathbb{R}^d$ so that $|\eta| = |\theta| = 1$ and $\eta\cdot\theta = \eta\cdot\kappa = \theta\cdot\kappa = 0$. Then, for τ such that $\tau^2 \geq |\kappa|^2/4 - \lambda$ we set

$$\begin{aligned} \zeta_1 &= \tau\eta + i\Big[-\frac{\kappa}{2} + \Big(\tau^2 + \lambda - \frac{|\kappa|^2}{4}\Big)^{1/2}\theta\Big], \\ \zeta_2 &= -\tau\eta + i\Big[-\frac{\kappa}{2} - \Big(\tau^2 + \lambda - \frac{|\kappa|^2}{4}\Big)^{1/2}\theta\Big]. \end{aligned} \tag{47}$$

Since ζ_1 and ζ_2 satisfy the conditions of the proposition 3.4, there exists solutions v_1 and v_2 as in (46) solving the equation $(\Delta + \lambda - V_j)v_j = 0$ in B_0 with $w_j \in X^{1/2}_{\zeta_j}(B_0)$ such that

$$\|w_j\|_{X^{1/2}_{\zeta_j}(B_0)} \lesssim \|V_j\|_{X^{-1/2}_{\zeta_j}} \tag{48}$$

for $j = 1, 2$ and $\tau^2 \geq \max\{\tau_0^2, |\kappa|^2/4 - \lambda\}$. Considering any extension of w_j in $X^{1/2}_{\zeta_j}(B_0)$ to $X^{1/2}_{\zeta_j}$, one can check that this extension is in $H^1(\mathbb{R}^d)$ and consequently,

w_j belongs to $H^1(B_0)$ and so does v_j. Therefore, the solutions v_1 and v_2 can be plugged in to the identity of the proposition 3.3, and obtain that

$$\langle V_1-V_2, e^{-i\kappa\cdot x}\rangle = -\langle V_1-V_2, e^{-i\kappa\cdot x}w_1\rangle-\langle V_1-V_2, e^{-i\kappa\cdot x}w_2\rangle-\langle (V_1-V_2)w_1, e^{-i\kappa\cdot x}w_2\rangle.$$

The first two terms on the right-hand side can be bounded as follows

$$|\langle V_1 - V_2, e^{-i\kappa\cdot x}w_j\rangle| \le \|V_1 - V_2\|_{X^{-1/2}_{\zeta_j}} \|e^{-i\kappa\cdot x}w_j\|_{X^{1/2}_{\zeta_j}(B_0)},$$

since $\mathrm{supp}(V_1 - V_2) \subset B_0$ and because of the duality between $X^{-1/2}_{\zeta_j,c}(B_0)$ and $X^{1/2}_{\zeta_j}(B_0)$. One can check that

$$\|e^{-i\kappa\cdot x}w_j\|_{X^{1/2}_{\zeta_j}(B_0)} \lesssim (1 + |\kappa|)\|w_j\|_{X^{1/2}_{\zeta_j}(B_0)} \tag{49}$$

and consequently, by (49) and (48), one obtains

$$|\langle V_1 - V_2, e^{-i\kappa\cdot x}w_j\rangle| \lesssim (1 + |\kappa|)\|V_1 - V_2\|_{X^{-1/2}_{\zeta_j}}\|V_j\|_{X^{-1/2}_{\zeta_j}}. \tag{50}$$

On the other hand, the third term can be bounded, by duality, as follows

$$|\langle (V_1 - V_2)w_1, e^{-i\kappa\cdot x}w_2\rangle| \le \|(V_1 - V_2)w_1\|_{X^{-1/2}_{\zeta_2}} \|e^{-i\kappa\cdot x}w_2\|_{X^{1/2}_{\zeta_2}(B_0)}$$

since again $(V_1 - V_2)w_1$ is supported in B_0. We will show that the operator multiplication by $V_1 - V_2$ is a bounded from $X^{1/2}_{\zeta_1}(B_0)$ to $X^{-1/2}_{\zeta_2}$. For the time being, let us assume that such boundedness holds. Then, we have by (49) and (48) that

$$\begin{aligned} |\langle (V_1 - V_2)w_1, e^{-i\kappa\cdot x}w_2\rangle| &\lesssim (1 + |\kappa|)\|w_1\|_{X^{1/2}_{\zeta_1}(B_0)}\|w_2\|_{X^{1/2}_{\zeta_2}(B_0)} \\ &\lesssim (1 + |\kappa|)\|V_1\|_{X^{-1/2}_{\zeta_1}}\|V_2\|_{X^{-1/2}_{\zeta_2}}. \end{aligned} \tag{51}$$

Gathering the inequalities (50) and (51), one obtains the following bound

$$\big|\langle V_1 - V_2, e^{-i\kappa\cdot x}\rangle\big| \lesssim (1 + |\kappa|) \sum_{j,k=1}^{2} \|V_j\|_{X^{-1/2}_{\zeta_k}} \sum_{l,m=1}^{2} \|V_l\|_{X^{-1/2}_{\zeta_m}}. \tag{52}$$

Before going on, prove the boundedness of the operator multiplication by V_1-V_2 from $X^{1/2}_{\zeta_1}(B_0)$ to $X^{-1/2}_{\zeta_2}$. To do so, let V denote a potential of the form (1), consider $w \in X^{1/2}_{\zeta_1}(B_0)$, and show that there exists a positive $C = C(d, \Gamma, \alpha, V^0)$ such that

$$\|Vw\|_{X^{-1/2}_{\zeta_2}} \le C\|w\|_{X^{1/2}_{\zeta_1}(B_0)}. \tag{53}$$

We will prove this boundedness by duality. Let $u \in X^{1/2}_{\zeta_1}$ denote an arbitrary extension of $w \in X^{1/2}_{\zeta_1}(B_0)$ and note that

$$\langle Vw, \phi\rangle = \int_{\mathbb{R}^d} V^0 u\phi + \int_\Gamma \alpha u\phi \,\mathrm{d}\sigma.$$

The first of these terms on the right-hand side can be easily bounded using Hölder's inequality and Haberman's embedding (see[11])

$$\Big|\int_{\mathbb{R}^d} V^0 u\phi\Big| \le \|V^0\|_{L^{d/2}}\|u\|_{L^{p_d}}\|\phi\|_{L^{p_d}} \lesssim \|V^0\|_{L^{d/2}}\|u\|_{X^{1/2}_{\zeta_1}}\|\phi\|_{X^{1/2}_{\zeta_2}}.$$

The second term, can be rewritten as follows

$$\int_\Gamma \alpha u\phi \,\mathrm{d}\sigma = \int_{T^t\Gamma} \alpha_{\zeta_1+\zeta_2} u_{\zeta_1}\phi_{\zeta_2} \,\mathrm{d}T^*\sigma$$

with

$$\alpha_{\zeta_1+\zeta_2}(y) = e^{-i\Im(\zeta_1+\zeta_2)\cdot Ty}\alpha(Ty),$$
$$u_{\zeta_1}(y) = e^{i\Im\zeta_1\cdot Ty}u(Ty),$$
$$\phi_{\zeta_2}(y) = e^{i\Im\zeta_2\cdot Ty}\phi(Ty),$$

where $T \in SO(d)$ is so that $\Re\zeta_1 = \tau Te_d = -\Re\zeta_2$, $T^t\Gamma = \{T^tx : x \in \Gamma\}$ and $T^*\sigma(E) = \sigma(TE)$ with $TE = \{Tx : x \in E\}$. Thus, the Cauchy–Schwarz inequality, the theorem 4.15 and the lemma 4.18 —in the sections 4.3 and 4.4— imply that

$$\Big| \int_\Gamma \alpha u \phi \,\mathrm{d}\sigma \Big| \lesssim \|\alpha_\zeta\|_{L^\infty(T^t\Gamma)} \|u_\zeta\|_{\dot{Y}^{1/2}_\tau} \|\phi_\zeta\|_{\dot{Y}^{1/2}_\tau}.$$

Since $\widehat{u_{\zeta_1}}(\xi) = \widehat{u}(T\xi - \Im\zeta_1)$, $\widehat{\phi_{\zeta_2}}(\xi) = \widehat{\phi}(T\xi - \Im\zeta_2)$ and $|q_\tau(\xi)| = |p_{\zeta_j}(T\xi - \Im\zeta_j)|$ with $j = 1, 2$, we have that

$$\Big| \int_\Gamma \alpha u \phi \,\mathrm{d}\sigma \Big| \lesssim \|\alpha\|_{L^\infty(\Gamma)} \||p_{\zeta_1}|^{1/2}\widehat{u}\|_{L^2} \||p_{\zeta_2}|^{1/2}\widehat{\phi}\|_{L^2} \le \|\alpha\|_{L^\infty(\Gamma)} \|u\|_{X^{1/2}_{\zeta_1}} \|\phi\|_{X^{1/2}_{\zeta_2}}.$$

Gathering the inequalities for V^0 and $\alpha \,\mathrm{d}\sigma$, we obtain that

$$\langle Vw, \phi\rangle \lesssim (\|V^0\|_{L^{d/2}} + \|\alpha\|_{L^\infty(\Gamma)})\|u\|_{X^{1/2}_{\zeta_1}} \|\phi\|_{X^{1/2}_{\zeta_2}},$$

and, consequently that

$$\|Vw\|_{X^{-1/2}_{\zeta_2}} \lesssim (\|V^0\|_{L^{d/2}} + \|\alpha\|_{L^\infty(\Gamma)})\|u\|_{X^{1/2}_{\zeta_1}}$$

where u is an arbitrary extension of w. Taking the infimum, between the norm of all the possible extensions of w, we get the inequality (53).

We now go back to the inequality (52). Our aim is to show that its right-hand side tends to zero in some sense as τ in (47) goes to infinity. Due to the δ-shell parts of V_1 and V_2, this decay will be possible in average as Haberman and Tataru showed for the Calderón problem in [12].

Lemma 3.5. *Let V be a potential of the form of* (1) *and $T \in SO(d)$. If $\zeta = \zeta(\tau, T) \in \mathbb{C}^d$ is so that $\zeta \cdot \zeta = -\lambda$ with $\Re\zeta = \tau Te_1$, then for every $s < 1/2$, we have that*

$$\frac{1}{M}\int_M^{2M}\int_{SO(d)} \|V\|^2_{X^{-1/2}_\zeta} \,\mathrm{d}\mu(T)\,\mathrm{d}\tau \lesssim \begin{cases} M^{-\frac{1}{2}}\|V^0\|^2_{L^{d/2}} + M^{-s}\|\alpha\|^2_{L^\infty(\Gamma)} & d = 3 \\ M^{-1}\|V^0\|^2_{L^{d/2}} + M^{-s}\|\alpha\|^2_{L^\infty(\Gamma)} & d \ge 4 \end{cases}$$

where the implicit constant depends on R_0, Γ and d. The measure μ denotes the Haar measure on $SO(d)$.

Proof. Start by estimating the critically singular part of V. If $d \ge 4$,

$$\|V^0\|_{X^{-1/2}_{\zeta(\tau,T)}} \le \tau^{-1/2}\|V^0\|_{L^2} \lesssim \tau^{-1/2}\|V^0\|_{L^{d/2}}$$

since $\operatorname{supp} V^0 \subset B_0$ and $d/2 \ge 2$ for $d \ge 4$. In the case $d = 3$, by the dual inequality to Haberman's embedding (see [11]),

$$\|V^0\|_{X^{-1/2}_{\zeta(\tau,T)}} \lesssim \tau^{-d(1/p'_d - 1/q'_d)}\|V^0\|_{L^{q'_d}} \lesssim \tau^{-1/4}\|V^0\|_{L^{d/2}}$$

since $\operatorname{supp} V^0 \subset B_0$ and $d/2 \ge q'_d$ for $d = 3$. This proves the part of the estimate corresponding to the critically singular component of V^0. We focus now on the δ-shell component. By Lemma 5.2 in [11] we have that

$$\frac{1}{M}\int_M^{2M}\int_{SO(d)} \|\alpha \,\mathrm{d}\sigma\|^2_{X^{-1/2}_{\zeta(\tau,T)}} \,\mathrm{d}\mu(T)\,\mathrm{d}\tau \lesssim M^{-1}\|L(\alpha \,\mathrm{d}\sigma)\|^2_{\dot{H}^{-1/2}} + \|H(\alpha \,\mathrm{d}\sigma)\|^2_{\dot{H}^{-1}}$$

where $\mathcal{F}H(\alpha\,\mathrm{d}\sigma) = \mathbf{1}_{|\xi|\geq 2M}\mathcal{F}(\alpha\,\mathrm{d}\sigma)$ and $L(\alpha\,\mathrm{d}\sigma) = \alpha\,\mathrm{d}\sigma - H(\alpha\,\mathrm{d}\sigma)$. Thus, for every $\varepsilon \in (0, 1/2)$, we have

$$\frac{1}{M}\int_M^{2M}\int_{SO(d)} \|\alpha\,\mathrm{d}\sigma\|_{X^{-1/2}_{\zeta(\tau,T)}}\,\mathrm{d}\mu(T)\,\mathrm{d}\tau \lesssim M^{-1+2\varepsilon}\|\alpha\,\mathrm{d}\sigma\|^2_{\dot H^{-1/2-\varepsilon}}.$$

Since $\operatorname{supp}(\alpha\,\mathrm{d}\sigma) \subset B_0$, we have

$$\|\alpha\,\mathrm{d}\sigma\|_{\dot H^{-1/2-\varepsilon}} \lesssim \|\alpha\,\mathrm{d}\sigma\|_{H^{-1/2-\varepsilon}}.$$

Using the dual of the usual trace theorem for Sobolev spaces we have that the right-hand side of last inequality is bounded by $\|\alpha\|_{L^2(\Gamma)}$. Since Γ is compact, we have that $\|\alpha\|_{L^2(\Gamma)} \lesssim \|\alpha\|_{L^\infty(\Gamma)}$, which proves the part of the inequality corresponding to the δ-shell component of the potential. □

We will apply this lemma to show that $\langle V_1 - V_2, e^{-i\kappa\cdot x}\rangle = 0$. For that, consider ζ_1 and ζ_2 as in (47) with $\kappa/|\kappa| = T_\kappa e_d$, $\eta = T_\kappa S e_1$ and $\theta = T_\kappa S e_2$, where $S \in SO(d)$ such that $Se_d = e_d$. Identifying the set $\{S \in SO(d) : Se_d = e_d\}$ with $SO(d-1)$, we have that

$$\begin{aligned}\left|\langle V_1 - V_2, e^{-i\kappa\cdot x}\rangle\right| &= \frac{1}{M}\int_M^{2M}\int_{SO(d-1)} \left|\langle V_1 - V_2, e^{-i\kappa\cdot x}\rangle\right| \mathrm{d}\mu(S)\,\mathrm{d}\tau \\ &\lesssim (1+|\kappa|)\sum_{j,k,l,m=1}^{2}\frac{1}{M}\int_M^{2M}\int_{SO(d-1)} \|V_j\|_{X^{-1/2}_{\zeta_k}}\|V_l\|_{X^{-1/2}_{\zeta_m}}\,\mathrm{d}\mu(S)\,\mathrm{d}\tau,\end{aligned}$$

where $\zeta_j = \zeta_j(\tau, S)$. Applying Cauchy–Schwarz in the integration with respect to S and τ, and the lemma 3.5, we obtain that

$$\langle V_1 - V_2, e^{-i\kappa\cdot x}\rangle = 0, \qquad \forall \kappa \in \mathbb{R}^d$$

after making M tend to infinity. By the injectivity of the Fourier transform, we know that $V_1 = V_2$. This proves the theorem 2.

4. Well-suited estimates for the resolvent

In this section we prove the lemmas that we used in the section 2 to derive the resolvent estimates in the spaces X^*_λ and X_λ. Additionally, we derive as a consequence the classical resolvent estimates (5) and (6), together with some inequalities for the conjugated Laplacian —including (10) and (11).

4.1. The refined estimates.

We start by stating a modification of (5) that turns out to be better adapted for our goal. To do so, we need to introduce the spaces Y^*_λ, and to call the definition of Y_λ given in the section 2.

Definition 4.1. *Let Y^*_λ be the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ so that*

$$\|m_\lambda^{1/2}\widehat{P_{<I}u}\|^2_{L^2} + \sum_{k\in I}\lambda^{1/2}\|\!|P_k u|\!\|^2_* + \sum_{k>k_\lambda+1}\|m_\lambda^{1/2}\widehat{P_k u}\|^2_{L^2} < \infty,$$

where $m_\lambda(\xi) = |\lambda - |\xi|^2|$. For $u \in Y^*_\lambda$, define the norm

$$\|u\|^2_{Y^*_\lambda} = \|m_\lambda^{1/2}\widehat{P_{<I}u}\|^2_{L^2} + \sum_{k\in I}\lambda^{1/2}\|\!|P_k u|\!\|^2_* + \sum_{k>k_\lambda+1}\|m_\lambda^{1/2}\widehat{P_k u}\|^2_{L^2}.$$

We now state the inequality, and prove it later.

Lemma 4.2. *There exists a constant $C > 0$ only depending on d so that*

$$\|(\Delta + \lambda \pm i0)^{-1}f\|_{Y^*_\lambda} \leq C\|f\|_{Y_\lambda}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Remark 4.3. The resolvent estimate in the lemma 4.2 is equivalent to (5), the fact that the latter inequality implies the former one is straightforward. The converse implication is proved in the Corollary 4.14 in the section 4.2.

We continue with our next refined estimate, which consists of a well-suited version of (6). Again, we start by introducing the space Z^*_λ, and calling the definition of Z_λ in the section 4.2.

Definition 4.4. *Let $Z^*_{\lambda,p}$ with $p \in [q_d, p_d]$ be the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ so that*

$$\|m_\lambda^{1/2} \widehat{P_{<I} u}\|^2_{L^2} + \sum_{k\in I} \lambda^{d(\frac{1}{p}-\frac{1}{p_d})} \|P_k u\|^2_{L^p} + \sum_{k>k_\lambda+1} \|m_\lambda^{1/2} \widehat{P_k u}\|^2_{L^2} < \infty,$$

where $m_\lambda(\xi) = |\lambda - |\xi|^2|$. For $u \in Z^*_{\lambda,p}$ we define the norm

$$\|u\|^2_{Z^*_{\lambda,p}} = \|m_\lambda^{1/2} \widehat{P_{<I} u}\|^2_{L^2} + \sum_{k\in I} \lambda^{d(\frac{1}{p}-\frac{1}{p_d})} \|P_k u\|^2_{L^p} + \sum_{k>k_\lambda+1} \|m_\lambda^{1/2} \widehat{P_k u}\|^2_{L^2}.$$

For simplicity, we write Z^*_λ instead of Z^*_{λ,q_d}.

Remark 4.5. By Bernstein's inequality

$$\|u\|_{Z^*_{\lambda,p_d}} \lesssim \|u\|_{Z^*_{\lambda,p}} \lesssim \|u\|_{Z^*_\lambda},$$

and therefore

$$Z^*_\lambda \subset Z^*_{\lambda,p} \subset Z^*_{\lambda,p_d}.$$

Lemma 4.6. *There exists a constant $C > 0$ only depending on d so that*

$$\|(\Delta + \lambda \pm i0)^{-1} f\|_{Z^*_\lambda} \leq C \|f\|_{Z_\lambda}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Remark 4.7. The resolvent estimate in the lemma 4.6 is equivalent to (6), the fact that the latter inequality implies the former one is straightforward. The converse implication is proved in the Corollary 4.10 in the section 4.2.

Proof of the lemma 4.6. The inequality to be proved follows from (6) for $d \geq 3$. The case $d = 2$ was not considered in [17]. We include here an argument that does not require Stein's interpolation theorem, which was the approach followed in [17], and works for dimension $d = 2$.

Start by providing an explicit formula of $(\Delta + \lambda \pm i0)^{-1}$:

$$\langle (\Delta + \lambda \pm i0)^{-1} f, \overline{g} \rangle = \lim_{\epsilon \to 0} \int_{m_\lambda > \epsilon} \frac{\widehat{f}(\xi)\overline{\widehat{g}(\xi)}}{\lambda - |\xi|^2} \, \mathrm{d}\xi \mp i \frac{\pi}{2\lambda^{1/2}} \int_{S_\lambda} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \, \mathrm{d}S_\lambda(\xi), \tag{54}$$

where $\mathrm{d}S_\lambda$ stands for the volume form on S_λ. If $k > k_\lambda + 1$,

$$\langle P_k (\Delta + \lambda \pm i0)^{-1} f, \overline{g} \rangle = \int_{\mathbb{R}^d} \frac{\widehat{P_k f}(\xi)\overline{\widehat{g}(\xi)}}{\lambda - |\xi|^2} \, \mathrm{d}\xi$$

and consequently, $\|m_\lambda^{1/2} \mathcal{F} P_k (\Delta + \lambda \pm i0)^{-1} f\|_{L^2} = \|m_\lambda^{-1/2} \widehat{P_k f}\|_{L^2}$, where $\mathcal{F}$ denotes the Fourier transform. The same holds with the projector $P_{<I}$. For the critical frequencies $k \in I$, the identity (54) does not become simpler. Start by the second term. Re-scaling the integral to bring S_λ back to $\mathbb{S}^{d-1}$, and then applying Cauchy–Schwarz we get

$$\Big| i \frac{\pi}{2\lambda^{1/2}} \int_{S_\lambda} \widehat{P_k f}(\xi)\overline{\widehat{g}(\xi)} \, \mathrm{d}S_\lambda(\xi) \Big| \lesssim \lambda^{d/2-1} \|\widehat{(P_k f)_\lambda}\|_{L^2(\mathbb{S}^{d-1})} \|\widehat{g_\lambda}\|_{L^2(\mathbb{S}^{d-1})}, \tag{55}$$

where $(P_k f)_\lambda(x) = \lambda^{-d/2} P_k f(x/\lambda^{1/2})$ and $g_\lambda(x) = \lambda^{-d/2} g(x/\lambda^{1/2})$. The restriction version of the Tomas–Stein theorem, together with an appropriate scale change, yields

$$\Big| i\frac{\pi}{2\lambda^{1/2}} \int_{S_\lambda} \widehat{P_k f}(\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}S_\lambda(\xi) \Big| \lesssim \lambda^{-d/2-1}\lambda^{d/q_d'} \|P_k f\|_{L^{q_d'}} \|g\|_{L^{q_d'}}.$$

Since $\lambda^{-d/2-1}\lambda^{d/q_d'} = \lambda^{d(1/q_d'-1/p_d')} = \lambda^{-d/2(1/q_d-1/p_d)}\lambda^{d/2(1/q_d'-1/p_d')}$, the inequality for the second term of the right-hand side of (54) follows by duality. To prove the inequality for the first term, we introduce

$$\mathcal{P}_\lambda f(x) = \frac{1}{(2\pi)^{d/2}} \lim_{\epsilon\to 0} \int_{m_\lambda>\epsilon} \frac{e^{ix\cdot\xi}}{\lambda - |\xi|^2} \widehat{f}(\xi)\,\mathrm{d}\xi$$

since to finish the proof of this lemma is enough to show that

$$\lambda^{\frac{d}{2}(\frac{1}{q_d}-\frac{1}{p_d})} \|P_k \mathcal{P}_\lambda f\|_{L^{q_d}} \lesssim \lambda^{\frac{d}{2}(\frac{1}{q_d'}-\frac{1}{p_d'})} \|P_k f\|_{L^{q_d'}}. \tag{56}$$

We analyse $\mathcal{P}_\lambda$ by distinguishing the frequencies inside a neighbourhood of S_λ of width $2\delta\lambda^{1/2}$, from those outside. Consider $\delta \in (0,1)$ —to be chosen— and set

$$\mathcal{P}^1_\lambda f(x) = \frac{1}{(2\pi)^{d/2}} \lim_{\epsilon\to 0} \int_{m_\lambda>\epsilon} \frac{e^{ix\cdot\xi}}{\lambda - |\xi|^2} \phi\Big(\frac{\lambda - |\xi|^2}{\lambda\delta}\Big) \widehat{f}(\xi)\,\mathrm{d}\xi$$

and $\mathcal{P}^2_\lambda f = \mathcal{P}_\lambda f - \mathcal{P}^1_\lambda f$, with $\phi \in C^\infty_0(\mathbb{R};[0,1])$ so that $\phi(t) = 1$ for all $t \in [-1,1]$ and $\phi(t) = 0$ whenever $|t| \geq 2$. By Bernstein's inequality, Plancherel's identity and the fact that

$$\operatorname{supp}\Big[1 - \phi\Big(\frac{\lambda - |\,\cdot\,|^2}{\lambda\delta}\Big)\Big] \subset \{\xi \in \mathbb{R}^d : \lambda\delta \leq |\lambda - |\xi|^2|\},$$

we have that, for $p \geq 2$,

$$\begin{aligned} \|P_k \mathcal{P}^2_\lambda f\|_{L^p} &\lesssim \lambda^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})} \|\mathcal{P}^2_\lambda P_k f\|_{L^2} \lesssim \lambda^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})} \lambda^{-1} \|\widehat{P_k f}\|_{L^2} \\ &\lesssim \lambda^{\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})} \lambda^{-1} \|P_k f\|_{L^{p'}}, \end{aligned}$$

where p' is the dual exponent of p. In the previous chain of inequalities, we used that $\lambda^{1/2} \simeq 2^k$ since $k \in I$. Therefore, we have that, for $p \geq 2$,

$$\|P_k \mathcal{P}^2_\lambda f\|_{L^p} \lesssim \lambda^{\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})} \lambda^{-1} \|P_k f\|_{L^{p'}} \tag{57}$$

with p' its dual exponent. In particular, considering $p = q_d$, we have for $\mathcal{P}^2_\lambda$ the corresponding inequality to (56) since $\lambda^{d/2(1/q_d'-1/q_d)-1} = \lambda^{d/2(1/q_d'-1/d-1/q_d-1/d)} = \lambda^{d/2(1/q_d'-1/p_d'-1/q_d+1/p_d)}$. It remains to prove (56) for $\mathcal{P}^1_\lambda$. Start by rescaling so that $P_k \mathcal{P}^1_\lambda f(x) = \lambda^{d/2-1} \mathcal{P}^1_1 (P_k f)_\lambda(\lambda^{1/2}x)$, then it is enough to prove

$$\|\mathcal{P}^1_1 g\|_{L^{q_d}} \lesssim \|g\|_{L^{q_d'}}. \tag{58}$$

Covering the 2δ-width neighbourhood of $\mathbb{S}^{d-1}$ with balls centred at points on $\mathbb{S}^{d-1}$ and radius $2\delta^{1/2}$, we can reduce the study to understand an operator of the form

$$\mathcal{Q}g(x) = \frac{1}{(2\pi)^{d/2}} \lim_{\epsilon\to 0} \int_{m_1>\epsilon} \frac{e^{ix\cdot\xi}}{1 - |\xi|^2} \phi\Big(\frac{1 - |\xi|^2}{\delta}\Big) \psi\Big(\frac{e_d - \xi}{\delta^{1/2}}\Big) \widehat{g}(\xi)\,\mathrm{d}\xi \tag{59}$$

with $\psi \in C^\infty_0(\mathbb{R}^d;[0,1])$ so that $\psi(0) \neq 0$ and $\psi(\eta) = 0$ whenever $|\eta| \geq 2$. The reduction to understand $\mathcal{Q}$ instead of $\mathcal{P}^1_1$ comes from the fact that, we can construct a partition of unity subordinated to the covering made of such balls so that, the latter can be written as a sum of operators $\{\mathcal{Q}_l : l = 1 \dots N_\delta\}$ with these looking as the former after a rotation. Thus, if (58) holds for operators as $\mathcal{Q}$, then (58) is also valid for $\mathcal{P}_1^1$. Indeed,

$$\|\mathcal{P}_1^1 g\|_{L^{q_d}} \leq \sum_{l=1}^{N_\delta} \|\mathcal{Q}_l g\|_{L^{q_d}} \lesssim \|g\|_{L^{q_d'}}$$

where N_δ is the number of ball needed to cover the 2δ-width neighbourhood of $\mathbb{S}^{d-1}$. Therefore, in order to finish the proof we just need to prove that (58) holds for $\mathcal{Q}$. Observe that $\mathcal{Q}g = K * g$ with

$$K(x) = \frac{1}{(2\pi)^d} \lim_{\epsilon \to 0} \int_{m_1 > \epsilon} \frac{e^{ix\cdot\xi}}{1-|\xi|^2} \phi\Big(\frac{1-|\xi|^2}{\delta}\Big) \psi\Big(\frac{e_d - \xi}{\delta^{1/2}}\Big)\,\mathrm{d}\xi.$$

Write $1-|\xi|^2 = (\Phi(\xi')+\xi_d)(\Phi(\xi')-\xi_d)$ with $\xi = (\xi', \xi_d)$ and $\Phi(\xi') = (1-|\xi'|^2)^{1/2}$, and take $\delta < 1/8$ so that

$$\Phi(\xi') + \xi_d \geq 1 \qquad \forall \xi \in \operatorname{supp} \psi\Big(\frac{e_d - \cdot}{\delta^{1/2}}\Big).$$

Changing variables, in the integrand defining the kernel K, according to $\eta' = \xi'$ and $\eta_d = \Phi(\xi') - \xi_d$ we have that

$$K(x) = \frac{1}{(2\pi)^{d-1/2}} \int_{\mathbb{R}^{d-1}} e^{i(x'\cdot\eta' + x_d \Phi(\eta'))} a(\eta', x_d)\,\mathrm{d}\eta' \tag{60}$$

with

$$a(\eta', x_d) = \frac{1}{(2\pi)^{1/2}} \mathrm{p.v.} \int_{\mathbb{R}} \frac{e^{-ix_d\eta_d}}{\eta_d} \Psi(\eta)\,\mathrm{d}\eta_d, \tag{61}$$

where

$$\Psi(\eta) = \frac{1}{\Phi(\xi') + \xi_d} \phi\Big(\frac{1-|\xi|^2}{\delta}\Big) \psi\Big(\frac{e_d - \xi}{\delta^{1/2}}\Big).$$

The term $a(\eta', x_d)$ reminds the well-known identity

$$\frac{1}{(2\pi)^{1/2}} \mathrm{p.v.} \int_{\mathbb{R}} \frac{e^{-ist}}{t} \varphi(t)\,\mathrm{d}t = -i\frac{1}{2} \int_{\mathbb{R}} \operatorname{sign}(s-t) \widehat{\varphi}(t)\,\mathrm{d}t,$$

and consequently,

$$a(\eta', x_d) = -i\frac{1}{2} \int_{\mathbb{R}} \operatorname{sign}(x_d - y_d) \mathcal{F}_d \Psi(\eta', y_d)\,\mathrm{d}y_d,$$

where $\mathcal{F}_d$ stands for the 1-dimensional Fourier transform applied to the last variable. Let us now get back to estimate $\mathcal{Q}g$. Note that, on the one hand

$$\begin{aligned} \Big\| \int_{\mathbb{R}^{d-1}} K(\cdot - y', x_d - y_d) g(y', y_d)\,\mathrm{d}y' \Big\|_{L^\infty(\mathbb{R}^{d-1})} \\ \leq \|K(\cdot, x_d - y_d)\|_{L^\infty(\mathbb{R}^{d-1})} \|g(\cdot, y_d)\|_{L^1(\mathbb{R}^{d-1})}. \end{aligned} \tag{62}$$

On the other hand, Plancherel's identity applied to the first $d-1$ variables yields

$$\begin{aligned} \Big\| \int_{\mathbb{R}^{d-1}} K(\cdot - y', x_d - y_d) g(y', y_d)\,\mathrm{d}y' \Big\|_{L^2(\mathbb{R}^{d-1})} \\ \lesssim \|\mathcal{F}' K(\cdot, x_d - y_d)\|_{L^\infty(\mathbb{R}^{d-1})} \|g(\cdot, y_d)\|_{L^2(\mathbb{R}^{d-1})}, \end{aligned} \tag{63}$$

where $\mathcal{F}'$ stands for the $(d-1)$-dimensional Fourier transform applied to the first $d-1$ variables. Furthermore, from the expression (60) one sees that

$$\|\mathcal{F}' K(\cdot, x_d - y_d)\|_{L^\infty(\mathbb{R}^{d-1})} \simeq \|a(\cdot, x_d - y_d))\|_{L^\infty(\mathbb{R}^{d-1})} \lesssim \int_{\mathbb{R}} \|\mathcal{F}_d \Psi(\cdot, y_d)\|_{L^\infty(\mathbb{R}^{d-1})}\,\mathrm{d}y_d.$$

Interpolating the inequalities (62) and (63), we get

$$\Big\| \int_{\mathbb{R}^{d-1}} K(\cdot - y', x_d - y_d) g(y', y_d) \,\mathrm{d}y' \Big\|_{L^q(\mathbb{R}^{d-1})} \\ \leq \|K(\cdot, x_d - y_d)\|_{L^\infty(\mathbb{R}^{d-1})}^{\frac{1}{q'} - \frac{1}{q}} \|g(\cdot, y_d)\|_{L^{q'}(\mathbb{R}^{d-1})}. \tag{64}$$

As a consequence of the stationary phase theorem (which exploits the curvature of $\mathbb{S}^{d-1}$),

$$\|K(\cdot, x_d - y_d)\|_{L^\infty(\mathbb{R}^{d-1})} \lesssim (1 + |x_d - y_d|)^{-\frac{d-1}{2}}.$$

Thus,

$$\|\mathcal{Q}g\|_{L^q} \lesssim \Big\| \int_{\mathbb{R}} |x_d - y_d|^{-\frac{d-1}{2}\left(\frac{1}{q'} - \frac{1}{q}\right)} \|g(\cdot, y_d)\|_{L^{q'}(\mathbb{R}^{d-1})} \,\mathrm{d}y_d \Big\|_{L^q(\mathbb{R}_{x_d})}.$$

Considering $q = q_d$, we can apply the Hardy–Littlewood–Sobolev inequality and conclude that

$$\|\mathcal{Q}g\|_{L^{q_d}} \lesssim \|g\|_{L^{q_d'}}$$

holds, which was the last ingredient to finish the proof of the lemma 4.6. □

Proof of the lemma 4.2. The wanted inequality follows from (5), however, we give here a simple proof for completeness. The argument follows the general scheme of the proof of the lemma 4.6 but simpler, since no interpolation is required, neither the curvature of S_λ is exploited.

The estimate for the non-critical frequencies is straightforward, and works exactly as in the lemma 4.6. To study the critical frequencies, we start analysing the second term on the right-hand side of (54), and obtain again the inequality (55). Applying the trace theorem —dual version of Theorem 7.1.26 in [14] (see also Theorem 14.1.1 in [15]), together with a change of scale, we have

$$\Big| i \frac{\pi}{2\lambda^{1/2}} \int_{S_\lambda} \widehat{P_k f}(\xi) \overline{\widehat{g}(\xi)} \,\mathrm{d}S_\lambda(\xi) \Big| \lesssim \lambda^{-d/2-1} \lambda^{(d+1)/2} \|\!|\!| P_k f |\!|\!\| \|\!|\!| g |\!|\!\|.$$

Since $\lambda^{-d/2-1}\lambda^{(d+1)/2} = \lambda^{-1/4}\lambda^{-1/4}$, the inequality for the second term of the right-hand side of (54) follows by duality. To prove the inequality for the first term, we split again $\mathcal{P}_\lambda = \mathcal{P}_\lambda^1 + \mathcal{P}_\lambda^2$. Note that using (57) with $p = 2d/(d-1)$, we obtain

$$\|P_k \mathcal{P}_\lambda^2 f\|_{L^2(D_j)} \lesssim 2^{\frac{j}{2}} \|P_k \mathcal{P}_\lambda^2 f\|_{L^p} \lesssim 2^{\frac{j}{2}} \lambda^{-\frac{1}{2}} \|P_k f\|_{L^{p'}} \leq 2^{\frac{j}{2}} \lambda^{-\frac{1}{2}} \sum_{l \in \mathbb{N}_0} \|P_k f\|_{L^{p'}(D_l)}.$$

Hence, by Hölder's inequality

$$\lambda^{\frac{1}{4}} \|\!|\!| P_k \mathcal{P}_\lambda^2 f |\!|\!\|_* \lesssim \lambda^{-\frac{1}{4}} \|\!|\!| P_k f |\!|\!\|. \tag{65}$$

We next prove (65) for $\mathcal{P}_\lambda^1$. After rescaling $P_k \mathcal{P}_\lambda^1 f(x) = \lambda^{d/2-1} \mathcal{P}_1^1 (P_k f)_\lambda (\lambda^{1/2} x)$, it is enough to prove

$$\sup_{j \in \mathbb{N}_0} \left((\lambda^{1/2} 2^j)^{-1/2} \|\mathcal{P}_1^1 g\|_{L^2(\lambda^{1/2} D_j)} \right) \lesssim \sum_{j \in \mathbb{N}_0} (\lambda^{1/2} 2^j)^{1/2} \|g\|_{L^2(\lambda^{1/2} D_j)}. \tag{66}$$

As in the lemma 4.6, the analysis can be reduced to study the operator $\mathcal{Q}$ in (59). In fact, we only need to check that (66) holds for $\mathcal{Q}$. Indeed, applying (63) and

(60) we have

$$
\begin{aligned}
&\|\mathcal{Q}g\|_{L^2(\lambda^{1/2}D_j)} \\
&\quad\leq \Big\| \int_{\mathbb{R}} \| \int_{\mathbb{R}^{d-1}} K(\bullet - y', x_d - y_d) g(y', y_d)\,\mathrm{d}y' \|_{L^2(\mathbb{R}^{d-1})}\,\mathrm{d}y_d \Big\|_{L^2(|x_d|<\lambda^{1/2}2^j)} \\
&\quad\lesssim \Big\| \int_{\mathbb{R}} \|g(\bullet, y_d)\|_{L^2(\mathbb{R}^{d-1})}\,\mathrm{d}y_d \Big\|_{L^2(|x_d|<\lambda^{1/2}2^j)} \\
&\quad\lesssim (\lambda^{1/2}2^j)^{1/2} \sum_{l\in\mathbb{N}_0} \int_{\mathbb{R}} \|[\mathbf{1}_{\lambda^{1/2}D_l} g](\bullet, y_d)\|_{L^2(\mathbb{R}^{d-1})}\,\mathrm{d}y_d \\
&\quad\lesssim (\lambda^{1/2}2^j)^{1/2} \sum_{l\in\mathbb{N}_0} (\lambda^{1/2}2^l)^{1/2} \|g\|_{L^2(\lambda^{1/2}D_l)},
\end{aligned}
$$

where $\mathbf{1}_{\lambda^{1/2}D_l}$ holds for the characteristic function of the set $\lambda^{1/2}D_l$. Therefore, (66) holds for $\mathcal{Q}$ and the lemma 4.2 is proved. □

4.2. **The classical resolvent estimates.** The classical resolvent estimates (5) and (6) follows from the lemmas 4.2 and 4.6 respectively, together with appropriate embeddings.

Lemma 4.8. *For $p, q \in [q_d, p_d]$ with $q \leq p$, there exists a constant $C > 0$ depending on d, p and q such that*

$$\lambda^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p_d})}\|f\|_{L^p} \leq C\|f\|_{Z^*_{\lambda,q}}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. By the Littlewood–Paley theorem, Bernstein's inequalities and Plancherel identity, we have that

$$
\begin{aligned}
\|f\|_{L^p}^2 &\leq \|P_{<I}f\|_{L^p}^2 + \sum_{k\geq k_\lambda - 2} \|P_k f\|_{L^p}^2 \\
&\lesssim 2^{2k_\lambda d(\frac{1}{2}-\frac{1}{p})}\|\widehat{P_{<I}f}\|_{L^2}^2 + \sum_{k\in I} 2^{2kd(\frac{1}{q}-\frac{1}{p})}\|P_k f\|_{L^q}^2 + \sum_{k>k_\lambda+1} 2^{2kd(\frac{1}{2}-\frac{1}{p})}\|\widehat{P_k f}\|_{L^2}^2.
\end{aligned}
$$

We have that $|\widehat{P_{<I}f}(\xi)| \simeq \lambda^{-1/2} m_\lambda(\xi)^{1/2} |\widehat{P_{<I}f}(\xi)|$ for all $\xi \in \mathbb{R}^d$. Hence,

$$\|\widehat{P_{<I}f}\|_{L^2} \simeq \lambda^{-1/2}\|m_\lambda^{1/2}\widehat{P_{<I}f}\|_{L^2}. \tag{67}$$

If $k > k_\lambda + 1$, we have that $|\widehat{P_k f}(\xi)| \simeq 2^{-k} m_\lambda(\xi)^{1/2} |\widehat{P_k f}(\xi)|$ for all $\xi \in \mathbb{R}^d$. Hence,

$$\|\widehat{P_k f}\|_{L^2} \simeq 2^{-k}\|m_\lambda^{1/2}\widehat{P_k f}\|_{L^2}. \tag{68}$$

Therefore,

$$
\begin{aligned}
\|f\|_{L^p}^2 \lesssim& \sum_{k\in I} 2^{2kd(\frac{1}{q}-\frac{1}{p})}\|P_k f\|_{L^q}^2 + \lambda^{-1} 2^{2k_\lambda d(\frac{1}{2}-\frac{1}{p})}\|m_\lambda^{1/2}\widehat{P_{<I}f}\|_{L^2}^2 \\
&+ 2^{-2k_\lambda d(\frac{1}{p}-\frac{1}{p_d})} \sum_{k>k_\lambda+1} \|m_\lambda^{1/2}\widehat{P_k f}\|_{L^2}^2.
\end{aligned}
$$

Since the critical scale 2^{k_λ} is of the order of $\lambda^{1/2}$, we have

$$
\begin{aligned}
\|f\|_{L^p}^2 \lesssim& \sum_{k\in I} \lambda^{d(\frac{1}{q}-\frac{1}{p})}\|P_k f\|_{L^q}^2 \\
&+ \lambda^{-d(\frac{1}{p}-\frac{1}{p_d})}\Big(\|m_\lambda^{1/2}\widehat{P_{<I}f}\|_{L^2}^2 + \sum_{k>k_\lambda+1} \|m_\lambda^{1/2}\widehat{P_k f}\|_{L^2}^2\Big).
\end{aligned}
$$

Finally, multiplying both sides by $\lambda^{d(1/p-1/p_d)}$ and taking square root, we obtain the embedding we were looking for. □

Lemma 4.9. *For $p', q' \in [p'_d, q'_d]$ with $p' \leq q'$, there exists a constant $C > 0$ depending on d, p' and q' such that*

$$\|f\|_{Z_{\lambda,q'}} \leq C\lambda^{\frac{d}{2}(\frac{1}{p'} - \frac{1}{p'_d})}\|f\|_{L^{p'}},$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. It follows from the lemma 4.8 by a standard duality argument, since the Banach space $(Z_{\lambda,q'}, \| \centerdot \|_{Z_{\lambda,q'}})$ is reflexive, its dual can be identified with $(Z^*_{\lambda,q}, \| \centerdot \|_{Z^*_{\lambda,q}})$ and $\mathcal{S}(\mathbb{R}^d)$ is dense in the latter space —see the lemma A.2 in the appendix A. □

Corollary 4.10. *For $p \in [q_d, p_d]$ with $d \geq 2$, there exists a constant $C > 0$ depending on d and p such that*

$$\|(\Delta + \lambda \pm i0)^{-1} f\|_{L^p} \lesssim \lambda^{\frac{d}{2}\left(\frac{1}{p'} - \frac{1}{p}\right) - 1}\|f\|_{L^{p'}}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. This is an immediate consequence of the lemmas 4.8, 4.9 and 4.6. □

Remark 4.11. The corollary 4.10 was stated in [17] for $d \geq 3$. This corollary also holds for $d = 2$ including the endpoint $p = p_2$.

Lemma 4.12. *There exists a constant $C > 0$ depending on d such that*

$$\lambda^{1/4}|||f|||_* \leq C\|f\|_{Y^*_\lambda},$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. By the triangle inequality and extending the domain of integration from D_j to $\mathbb{R}^d$, we have that

$$\begin{aligned}\|f\|_{L^2(D_j)} &\leq \|P_{<I} f\|_{L^2(D_j)} + \sum_{k \geq k_\lambda - 2} \|P_k f\|_{L^2(D_j)} \\ &\lesssim \|P_{<I} f\|_{L^2} + \sum_{k \in I} \|P_k f\|_{L^2(D_j)} + \sum_{k > k_\lambda + 1} \|P_k f\|_{L^2}.\end{aligned}$$

Multiplying by $2^{-j/2}$ with $j \in \mathbb{N}_0$ and using the equivalences (67) and (68):

$$\begin{aligned}2^{-j/2}\|f\|_{L^2(D_j)} \lesssim &\sum_{k \in I} 2^{-j/2}\|P_k f\|_{L^2(D_j)} + \lambda^{-1/2}\|m_\lambda^{1/2}\widehat{P_{<I} f}\|_{L^2} \\ &+ \sum_{k > k_\lambda + 1} 2^{-k}\|m_\lambda^{1/2}\widehat{P_k f}\|_{L^2}.\end{aligned}$$

Since there are only four critical frequencies, on has

$$\sum_{k \in I} 2^{-j/2}\|P_k f\|_{L^2(D_j)} \simeq \Big(\sum_{k \in I} 2^{-j}\|P_k f\|^2_{L^2(D_j)}\Big)^{1/2}.$$

Using the Cauchy–Schwarz inequality, we can proceed as follows:

$$\sum_{k > k_\lambda + 1} 2^{-k}\|m_\lambda^{1/2}\widehat{P_k f}\|_{L^2} \lesssim 2^{-k_\lambda}\Big(\sum_{k > k_\lambda + 1} \|m_\lambda^{1/2}\widehat{P_k f}\|^2_{L^2}\Big)^{1/2}.$$

The fact that the critical scale 2^{k_λ} is of the order of $\lambda^{1/2}$ implies that, after taking square, we obtain

$$\begin{aligned}\lambda^{1/2}2^{-j}\|f\|^2_{L^2(D_j)} \lesssim &\lambda^{-1/2}\|m_\lambda^{1/2}\widehat{P_{<I} f}\|^2_{L^2} \\ &+ \sum_{k \in I} \lambda^{1/2}2^{-j}\|P_k f\|^2_{L^2(D_j)} + \lambda^{-1/2}\sum_{k > k_\lambda + 1} \|m_\lambda^{1/2}\widehat{P_k f}\|^2_{L^2}.\end{aligned}$$

Taking the corresponding supremum of $j \in \mathbb{N}_0$, we obtain the embedding stated in the lemma. □

Lemma 4.13. *There exists a constant $C > 0$ depending on d such that*

$$\|f\|_{Y_\lambda} \leq C\lambda^{-1/4} |||f|||$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. This lemma is consequence of the lemma 4.12 together with a duality argument that uses that $(Y_\lambda^*, \|\,\boldsymbol{\cdot}\,\|_{Y_\lambda^*})$ is the dual of $(Y_\lambda, \|\,\boldsymbol{\cdot}\,\|_{Y_\lambda})$, and the density of $\mathcal{S}(\mathbb{R}^d)$ in the former space. This duality argument is based on the Hahn–Banach theorem (see the corollary 1.4 in [4]). □

Corollary 4.14. *There exists a constant $C > 0$ depending on d such that*

$$\lambda^{1/2} |||(\Delta + \lambda \pm i0)^{-1} f|||_* \leq C |||f|||$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. This is an immediate consequence of the lemmas 4.12, 4.13 and 4.2. □

4.3. **A trace theorem.** In this section we prove a trace theorem for the space Y_λ^*. This is an essential piece to construct the scattering solution for critically-singular and δ-shell potentials, specially to deal with the δ-shell component.

Theorem 4.15. *Let Γ be a compact hypersurface locally described by the graphs of Lipschitz functions. There exists a constant $C > 0$ only depending on d and Γ such that*

$$\|f\|_{L^2(\Gamma)} \leq C\|f\|_{Y_\lambda^*}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $\lambda > 1$.

Proof. We first introduce a localization function denoted by χ, which is not compactly supported. To do so, let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be a $[0, 1]$-valued function so that its support is contained in $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$ and it is not identically zero.

Then, there exists $\delta \in (0, 1]$ such that

$$\Big| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \phi(\xi)\,\mathrm{d}\xi \Big| \geq \frac{1}{2} \int_{\mathbb{R}^d} \phi(\xi)\,\mathrm{d}\xi > 0$$

whenever $|x| \leq \delta$. Let $\chi \in \mathcal{S}(\mathbb{R}^d)$ be defined by

$$\chi(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\delta/R x\cdot\xi} \phi(\xi)\,\mathrm{d}\xi, \tag{69}$$

with $R \geq 1$ so that $\Gamma \subset B = \{x \in \mathbb{R}^d : |x| < R\}$. Note that $|\chi(x)| \gtrsim 1$ whenever $|x| \leq R$ and $\operatorname{supp} \widehat{\chi} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. Since Γ is contained in B,

$$\|f\|_{L^2(\Gamma)} \lesssim \|\chi f\|_{L^2(\Gamma)} \lesssim \sum_{l\in\mathbb{Z}} 2^{l/2} \|P_l(\chi f)\|_{L^2}.$$

In the last inequality we have used the trace theorem —a Besov-space form of Theorem 14.1.1 in [15]. We now show that

$$\sum_{l\in\mathbb{Z}} 2^{l/2} \|P_l(\chi f)\|_{L^2} \lesssim \|f\|_{Y_\lambda^*}. \tag{70}$$

Start considering the low frequencies $l \leq k_\lambda + 4$. The continuity of P_l in $L^2(\mathbb{R}^d)$ and the fact that the sum of low frequencies is at most of the order of $2^{k_\lambda/2}$ imply that

$$\sum_{l\leq k_\lambda+4} 2^{l/2} \|P_l(\chi f)\|_{L^2} \lesssim 2^{k_\lambda/2} \|\chi f\|_{L^2} \simeq \lambda^{1/4} \|\chi f\|_{L^2}. \tag{71}$$

On the other hand,

$$\|\chi f\|_{L^2} = \Big(\sum_{j\in\mathbb{N}_0} \|\chi f\|^2_{L^2(D_j)} \Big)^{1/2} \leq \Big(\sum_{j\in\mathbb{N}_0} 2^j \|\chi\|^2_{L^\infty(D_j)} \Big)^{1/2} \|\!|f|\!\|_*. \tag{72}$$

Since $\sup_{x\in\mathbb{R}^d}(1+|x|)^{-N}|\chi(x)| < \infty$ for any $N \in \mathbb{N}$, the series on the right-hand side of the inequality in (72) converges. Thus, (71), (72) and the lemma 4.12 shows that

$$\sum_{l\leq k_\lambda+4} 2^{l/2}\|P_l(\chi f)\|_{L^2} \lesssim \lambda^{1/4}\|\!|f|\!\|_* \lesssim \|f\|_{Y^*_\lambda}. \tag{73}$$

Finally, we consider the high frequencies $l > k_\lambda + 4$. By the triangle inequality,

$$\sum_{l>k_\lambda+4} 2^{l/2}\|P_l(\chi f)\|_{L^2} \leq \sum_{l>k_\lambda+4}\sum_{k\in\mathbb{Z}} 2^{l/2}\|P_l(\chi P_k f)\|_{L^2}. \tag{74}$$

Since the support of $\widehat{\chi}$ is contained $\{\xi\in\mathbb{R}^d : |\xi|\leq 1\}$, we have that

$$\operatorname{supp}\widehat{\chi P_k f} \subset \left\{ \begin{array}{ll} \{\xi\in\mathbb{R}^d : |\xi|\leq 2^3\} & \text{if}\, k<2, \\ \{\xi\in\mathbb{R}^d : 2^{k-2}\leq|\xi|\leq 2^{k+2}\} & \text{if}\, k\geq 2. \end{array}\right. \tag{75}$$

Thus, $P_l(\chi P_k f) = 0$ whenever $k<2$ and $l>4$, or whenever $k\geq 2$ and $|l-k|>3$. Consequently, the sum on the right-hand side of (74) only has the terms $l > k_\lambda + 4$ and $k\geq 2$ —if $k<2$ the non-zero terms satisfy $l\leq 4$, but there are no l satisfying $k_\lambda + 4 < l \leq 4$ with $\lambda > 1$. Therefore,

$$\begin{aligned}\sum_{l>k_\lambda+4} 2^{l/2}\|P_l(\chi f)\|_{L^2} &\leq \sum_{l>k_\lambda+4}\sum_{|k-l|\leq 3} 2^{l/2}\|P_l(\chi P_k f)\|_{L^2} \\ &= \sum_{k>k_\lambda+1}\sum_{|l-k|\leq 3} 2^{l/2}\|P_l(\chi P_k f)\|_{L^2} \lesssim \sum_{k>k_\lambda+1} 2^{k/2}\|P_k f\|_{L^2}.\end{aligned}$$

In the last inequality we used the continuity in $L^2(\mathbb{R}^d)$ of the operators P_l and multiplication by χ, and the fact that $\sum_{|l-k|<3} 2^{l/2} \simeq 2^{k/2}$. Then, by Plancherel's identity, (68) and Cauchy–Schwarz applied to the sum, we obtain

$$\sum_{l>k_\lambda+4} 2^{l/2}\|P_l(\chi f)\|_{L^2} \lesssim \sum_{k>k_\lambda+1} 2^{-k/2}\|m_\lambda^{1/2}\widehat{P_k f}\|_{L^2} \lesssim 2^{-k_\lambda/2}\|f\|_{Y^*_\lambda}.$$

This inequality, together with (73), shows that (70) holds, and consequently the theorem is proved. □

Remark 4.16. The novelty of this trace theorem bases on showing that the operator multiplication by χ, defined as in (69), is bounded from Y^*_λ to $\dot{B}^{1/2}_{2,1}(\mathbb{R}^d)$ with a norm independent of λ. Our next step will be to show that such an operator is in fact compact.

Lemma 4.17. *Let χ be as in* (69) *and $\lambda > 1$. Multiplication by χ defines a compact operator from Y^*_λ to $\dot{B}^{1/2}_{2,1}(\mathbb{R}^d)$.*

Proof. In order to prove the compactness of the operator multiplication by χ, we will consider a bounded sequence $\{u_n : n\in\mathbb{N}\}$ in Y^*_λ and show that there exist a subsequence $\{u_{n(m)} : m\in\mathbb{N}\}$ and $u\in Y^*_\lambda$ so that

$$\lim_{m\to\infty}\|\chi u_{n(m)} - \chi u\|_{\dot{B}^{1/2}_{2,1}} = 0.$$

We will show in the appendix A that Y^*_λ is the dual space of Y_λ (see the lemma A.1). Thus, given a bounded sequence $\{u_n : n\in\mathbb{N}\}$ in Y^*_λ, we know by the Banach–Alaoglu–Bourbaki theorem that there exist a subsequence $\{u_{n(m)} : m\in\mathbb{N}\}$ and

$u \in Y_\lambda^*$ so that

$$\lim_{m\to\infty} \langle u_{n(m)} - u, f\rangle = 0 \tag{76}$$

for all $f \in Y_\lambda$. Here $\langle \cdot, \cdot \rangle$ stands for the duality pairing between Y_λ^* and Y_λ. For convenience, let v_m denote the difference $u_{n(m)} - u$. We will show that

$$\lim_{m\to\infty} \sum_{l\in\mathbb{Z}} 2^{l/2} \|P_l(\chi v_m)\|_{L^2} = 0.$$

To do so, we will use the dominate convergence theorem (DCT for short), which could be applied after we shown that, for every $l \in \mathbb{Z}$, $\|P_l(\chi v_m)\|_{L^2}$ tends to 0 as m goes to infinity, and

$$2^{l/2} \|P_l(\chi v_m)\|_{L^2} \lesssim \lambda^{-1/4} 2^{l/2} \mathbf{1}_{l\le k_\lambda+4} + 2^{-l/2} \mathbf{1}_{l>k_\lambda+4}, \tag{77}$$

where the implicit constant does not depend on m and, $\mathbf{1}_{l\le k_\lambda+4}$ and $\mathbf{1}_{l>k_\lambda+4}$ stand for the characteristic functions of the set $\{l \in \mathbb{Z} : l \le k_\lambda+4\}$ and $\{l \in \mathbb{Z} : l > k_\lambda+4\}$. Note that we can apply the DCT because the sequence on the right-hand side of (77) belongs to $l^1(\mathbb{Z})$.

Let us first check that (77) holds. Start by analysing the case $l \le k_\lambda + 4$. The boundedness of P_l in $L^2(\mathbb{R}^d)$, the inequality (72) and the lemma 4.12 implies that

$$2^{l/2} \|P_l(\chi v_m)\|_{L^2} \lesssim \lambda^{-1/4} 2^{l/2} \|v_m\|_{Y_\lambda^*}. \tag{78}$$

This inequality is only useful if $l \le k_\lambda + 4$. Continue now with the case $l > k_\lambda + 4$. Using (75) for $l > k_\lambda+4$, the boundedness of P_l and multiplication by χ in $L^2(\mathbb{R}^d)$, Plancherel's identity and (68), we have that

$$\begin{aligned} 2^{l/2} \|P_l(\chi v_m)\|_{L^2} &\le 2^{l/2} \sum_{|k-l|\le 3} \|P_l(\chi P_k v_m)\|_{L^2} \lesssim 2^{l/2} \sum_{|k-l|\le 3} \|P_k v_m\|_{L^2} \\ &\simeq 2^{l/2} \sum_{|k-l|\le 3} 2^{-k} \|m_\lambda^{1/2} \widehat{P_k v_m}\|_{L^2} \lesssim 2^{-l/2} \|v_m\|_{Y_\lambda^*}. \end{aligned} \tag{79}$$

The inequalities (78) and (79), together with the fact that $\{v_m : m \in \mathbb{Z}\}$ is bounded in Y_λ^*, yields (77).

It remains to prove that

$$\lim_{m\to\infty} \|P_l(\chi v_m)\|_{L^2} = 0.$$

We will show this using the DCT again. Start by checking the point-wise convergence:

$$P_l(\chi v_m)(x) = \frac{2^{ld}}{(2\pi)^{d/2}} \langle v_m, \chi \widehat{\psi}(2^l(x - \cdot))\rangle$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between Y_λ^* and Y_λ, and ψ denotes the base function used to construct the Littlewood–Paley projectors. Since $\chi\widehat{\psi}(2^l(x-\cdot))$ belongs to Y_λ for all $x \in \mathbb{R}^d$, the convergence (76) implies that

$$\lim_{m\to\infty} P_l(\chi v_m)(x) = 0$$

for all $x \in \mathbb{R}^d$. Continue with the domination:

$$|P_l(\chi v_m)(x)| \lesssim 2^{ld} \|v_m\|_{Y_\lambda^*} \|\chi\widehat{\psi}(2^l(x - \cdot))\|_{Y_\lambda}.$$

Note that, since $\{v_m : m \in \mathbb{Z}\}$ is bounded in Y_λ^*, it is enough to see that the function $x \mapsto \|\chi\widehat{\psi}(2^l(x-\cdot))\|_{Y_\lambda}$ belongs to $L^2(\mathbb{R}^d)$. We finish the proof of this lemma showing that this is the case. By the lemma 4.13, and then using Cauchy–Schwarz, we know that

$$\begin{aligned}\|\chi\widehat{\psi}(2^l(x-\bullet))\|_{Y_\lambda} &\lesssim \lambda^{-1/4}\sum_{j\in\mathbb{N}_0} 2^{j/2}\|\chi\widehat{\psi}(2^l(x-\bullet))\|_{L^2(D_j)}\\ &\lesssim \|(\sum_{j\in\mathbb{N}_0} 2^{cj}\mathbf{1}_{D_j})^{1/2}\chi\widehat{\psi}(2^l(x-\bullet))\|_{L^2}\end{aligned}$$

where c is a constant so that $c>1$. Consequently,

$$\begin{aligned}\int_{\mathbb{R}^d}\|\chi\widehat{\psi}(2^l(x-\bullet))\|_{Y_\lambda}^2\,\mathrm{d}x &\lesssim \int_{\mathbb{R}^d}\|(\sum_{j\in\mathbb{N}_0} 2^{cj}\mathbf{1}_{D_j})^{1/2}\chi\widehat{\psi}(2^l(x-\bullet))\|_{L^2}^2\,\mathrm{d}x\\ &= \|(\sum_{j\in\mathbb{N}_0} 2^{cj}|\chi|^2\mathbf{1}_{D_j})*|\widehat{\psi}(2^l\bullet)|^2\|_{L^1}\\ &\le \|\sum_{j\in\mathbb{N}_0} 2^{cj}|\chi|^2\mathbf{1}_{D_j}\|_{L^1}\|\widehat{\psi}(2^l\bullet)\|_{L^2}^2.\end{aligned}$$

Since χ and ψ are in $\mathcal{S}(\mathbb{R}^d)$, the right-hand side of the previous chains of inequalities is bounded, which concludes the proof of this lemma. □

4.4. **Other estimates.** In this section we state and derive several consequences from the embeddings and inequalities proved in the sections 4.1 and 4.2. In particular, (10), (11) and (12), beside a slightly different version of the Sylvester–Uhlmann inequality. At this point it might be convenient recall that the definition of the norm of $\dot{Y}^s_\tau$ was given in (8).

Lemma 4.18. *Whenever $d\ge 3$, there exists a constant $C>0$ depending on d so that*

$$\|f\|_{Y^*_\lambda}\le C\|f\|_{\dot{Y}^{1/2}_\tau}$$

for all $f\in\mathcal{S}(\mathbb{R}^d)$ and $\lambda=\tau^2$.

Proof. The fact that $m_\lambda(\xi)\le|q_\tau(\xi)|$ for all $\xi\in\mathbb{R}^d$ implies that

$$\|f\|_{Y^*_\lambda}^2\le \||q_\tau|^{1/2}\widehat{P_{<I}f}\|_{L^2}^2+\sum_{k\in I}\lambda^{1/2}\|\!|P_kf|\!\|_*^2+\sum_{k>k_\lambda+1}\||q_\tau|^{1/2}\widehat{P_kf}\|_{L^2}^2.$$

Thus, if we prove that for $k\in I$ we have

$$\tau^{1/2}\|\!|P_kf|\!\|_*\lesssim \||q_\tau|^{1/2}\widehat{P_kf}\|_{L^2}, \tag{80}$$

then the result follows since there exists a constant $c>0$ so that

$$|\widehat{P_{<I}f}(\xi)|^2+\sum_{k\ge k_\lambda-2}|\widehat{P_kf}(\xi)|^2\le c|\widehat{f}(\xi)|^2,\qquad \forall\xi\in\mathbb{R}^d.$$

Last inequality is a known property of the Littlewood–Paley projectors. To finish the proof of this lemma, we show that (80) holds. Let g be defined by

$$\widehat{g}(\xi)=|q_\tau(\xi)|^{1/2}\widehat{f}(\xi),\ \forall\xi\in\mathbb{R}^d. \tag{81}$$

Thus, using the inversion formula of the Fourier transform and changing variables to $\rho=|\xi'|$ and $\theta=\xi'/|\xi'|$ with $\xi=(\xi',\xi_d)\in\mathbb{R}^{d-1}\times\mathbb{R}$, we have that

$$P_kf(x)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}}e^{ix_d\xi_d}\int_0^\infty\int_{\mathbb{S}^{d-2}}e^{i\rho x'\cdot\theta}\frac{\widehat{P_kg}(\rho\theta,\xi_d)}{|q_\tau(\rho\theta,\xi_d)|^{1/2}}\,\mathrm{d}S(\theta)\rho^{d-2}\,\mathrm{d}\rho\,\mathrm{d}\xi_d, \tag{82}$$

where $\mathrm{d}S$ denotes the volume form on $\mathbb{S}^{d-2}$. Note that, extending the integration from D_j to $\{x\in\mathbb{R}^d:|x'|\le 2^j\}$, applying Plancherel's identity in the variable x_d

and then Minkowski's inequality, we have

$$
\begin{aligned}
&2^{-j/2}\|P_k f\|_{L^2(D_j)}\\
&\lesssim 2^{-j/2}\Big(\int_{B'_j}\|\int_0^\infty\int_{\mathbb{S}^{d-2}} e^{i\rho x'\cdot\theta}\frac{\widehat{P_k g}(\rho\theta,\centerdot)}{|q_\tau(\rho\theta,\centerdot)|^{1/2}}\,\mathrm{d}S(\theta)\rho^{d-2}\,\mathrm{d}\rho\|^2_{L^2(\mathbb{R})}\,\mathrm{d}x'\Big)^{\frac12} \qquad (83)\\
&\lesssim 2^{-j/2}\Big\|\int_0^\infty\Big(\int_{B'_j}\Big|\int_{\mathbb{S}^{d-2}} e^{i\rho x'\cdot\theta}\frac{\widehat{P_k g}(\rho\theta,\centerdot)}{|q_\tau(\rho\theta,\centerdot)|^{1/2}}\,\mathrm{d}S(\theta)\Big|^2\,\mathrm{d}x'\Big)^{\frac12}\rho^{d-2}\,\mathrm{d}\rho\Big\|_{L^2(\mathbb{R})},
\end{aligned}
$$

where $B'_j=\{x'\in\mathbb{R}^{d-1}:|x'|<2^j\}$. Next we change variables $\rho x'=y'$ so that

$$
\begin{aligned}
&2^{-j/2}\Big(\int_{B'_j}\Big|\int_{\mathbb{S}^{d-2}} e^{i\rho x'\cdot\theta}\frac{\widehat{P_k g}(\rho\theta,\centerdot)}{|q_\tau(\rho\theta,\centerdot)|^{1/2}}\,\mathrm{d}S(\theta)\Big|^2\,\mathrm{d}x'\Big)^{\frac12}\\
&\qquad=\rho^{1-d/2}(\rho 2^j)^{-1/2}\Big(\int_{\rho B'_j}\Big|\int_{\mathbb{S}^{d-2}} e^{iy'\cdot\theta}\frac{\widehat{P_k g}(\rho\theta,\centerdot)}{|q_\tau(\rho\theta,\centerdot)|^{1/2}}\,\mathrm{d}S(\theta)\Big|^2\,\mathrm{d}y'\Big)^{\frac12}.
\end{aligned}
$$

Applying the extension version of the trace theorem —Theorem 7.1.26 in [14] valid here for $d\geq 3$— we have that the right-hand side of the previous identity can be bounded so that the inequality (83) becomes

$$
2^{-\frac{j}{2}}\|P_k f\|_{L^2(D_j)}\lesssim\Big\|\int_0^\infty\rho^{1-\frac{d}{2}}\Big(\int_{\mathbb{S}^{d-2}}\frac{|\widehat{P_k g}(\rho\theta,\centerdot)|^2}{|q_\tau(\rho\theta,\centerdot)|}\,\mathrm{d}S(\theta)\Big)^{\frac12}\rho^{d-2}\,\mathrm{d}\rho\Big\|_{L^2(\mathbb{R})}. \qquad (84)
$$

Note that $|q_\tau(\rho\theta,\xi_d)|^2=|\tau^2-\rho^2-\xi_d^2|^2+|2\tau\xi_d|^2$, which does not depend on θ. Hence, by the Cauchy–Schwarz inequality applied to the integration in ρ, we have that the right-hand side of the previous inequality is bounded by a constant multiple of

$$
\sup_{|\xi_d|\leq 2^{k+1}}\Big(\int_0^{2^{k+1}}|q_\tau(\rho\theta,\xi_d)|^{-1}\,\mathrm{d}\rho\Big)^{\frac12}\Big\|\int_0^\infty\int_{\mathbb{S}^{d-2}}|\widehat{P_k g}(\rho\theta,\centerdot)|^2\,\mathrm{d}S(\theta)\rho^{d-2}\,\mathrm{d}\rho\Big\|_{L^2(\mathbb{R})}.
$$

Since $k\in I$, one can check that this term is bounded by $\tau^{-1/2}\|\widehat{P_k g}\|_{L^2}$. Thus, the inequality (84) becomes

$$
2^{-j/2}\|P_k f\|_{L^2(D_j)}\lesssim\tau^{-1/2}\||q_\tau|^{1/2}\widehat{P_k f}\|_{L^2}.
$$

Taking supremum in $j\in\mathbb{N}_0$ we obtain (80). □

Corollary 4.19 (Haberman–Tataru)**.** *Whenever $d\geq 3$, there exists a constant $C>0$ depending on d so that, if $\chi\in\mathcal{S}(\mathbb{R}^d)$, then*

$$
\tau^{1/2}\|\chi f\|_{L^2}\leq C\|f\|_{\dot{Y}^{1/2}_\tau}
$$

for all $f\in\mathcal{S}(\mathbb{R}^d)$.

Proof. This is a consequence of the inequality (72) and the lemmas 4.12 and 4.18. □

Lemma 4.20. *Whenever $d\geq 3$, there exists a constant $C>0$ depending on d so that*

$$
\|f\|_{\dot{Y}^{-1/2}_\tau}\leq C\|f\|_{Y_\lambda}
$$

for all $f\in\mathcal{S}(\mathbb{R}^d)$ and $\lambda=\tau^2$.

Proof. It follows from the lemma 4.18 by a standard duality argument. □

Corollary 4.21 (Sylvester–Uhlmann)**.** *Whenever $d\geq 3$, there exists a constant $C>0$ depending on d so that*

$$
\|(\Delta+2\tau\partial_{x_d}+\tau^2)^{-1}f\|_*\leq\frac{C}{\tau}\|f\|
$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. It follows from the identity (9) and the lemmas 4.18, 4.20, 4.12 and 4.13. □

Lemma 4.22. *Whenever* $d \geq 3$, *there exists a constant* $C > 0$ *depending on* d *so that*

$$\|f\|_{Z^*_{\lambda,p_d}} \leq C\|f\|_{\dot{Y}^{1/2}_\tau}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ *and* $\lambda = \tau^2$.

Proof. By the same argument as in the proof of the lemma 4.18, it is enough to show that for $k \in I$ we have

$$\|P_k f\|_{L^{p_d}} \lesssim \||q_\tau|^{1/2}\widehat{P_k f}\|_{L^2}. \tag{85}$$

Let g be as in (81), and write $P_k f$ as in (82). Applying Bernstein's and Plancherel's identities in the variable x_d and after this Minkowski's inequality, we have

$$\begin{aligned} &2^{-k(\frac{1}{2}-\frac{1}{p_d})}\|P_k f\|_{L^{p_d}(\mathbb{R}^d)} \\ &\lesssim \Big(\int_{\mathbb{R}^{d-1}} \Big\| \int_0^\infty \int_{\mathbb{S}^{d-2}} e^{i\rho x'\cdot\theta} \frac{\widehat{P_k g}(\rho\theta,\,\cdot\,)}{|q_\tau(\rho\theta,\,\cdot\,)|^{\frac{1}{2}}}\, \mathrm{d}S(\theta)\rho^{d-2}\,\mathrm{d}\rho\Big\|^{p_d}_{L^2(\mathbb{R})}\,\mathrm{d}x'\Big)^{\frac{1}{p_d}} \\ &\lesssim \Big\| \int_0^\infty \Big(\int_{\mathbb{R}^{d-1}} \Big| \int_{\mathbb{S}^{d-2}} e^{i\rho x'\cdot\theta} \frac{\widehat{P_k g}(\rho\theta,\,\cdot\,)}{|q_\tau(\rho\theta,\,\cdot\,)|^{1/2}}\, \mathrm{d}S(\theta)\Big|^{p_d}\,\mathrm{d}x'\Big)^{\frac{1}{p_d}} \rho^{d-2}\,\mathrm{d}\rho\Big\|_{L^2(\mathbb{R})}. \end{aligned} \tag{86}$$

As in the proof of the lemma 4.18, we change variables $\rho x' = y'$ so that

$$\begin{aligned} &\Big(\int_{\mathbb{R}^{d-1}} \Big| \int_{\mathbb{S}^{d-2}} e^{i\rho x'\cdot\theta} \frac{\widehat{P_k g}(\rho\theta,\,\cdot\,)}{|q_\tau(\rho\theta,\,\cdot\,)|^{1/2}}\, \mathrm{d}S(\theta)\Big|^{p_d}\,\mathrm{d}x'\Big)^{\frac{1}{p_d}} \\ &\qquad = \rho^{(1-d)/p_d}\Big(\int_{\mathbb{R}^{d-1}} \Big| \int_{\mathbb{S}^{d-2}} e^{iy'\cdot\theta} \frac{\widehat{P_k g}(\rho\theta,\,\cdot\,)}{|q_\tau(\rho\theta,\,\cdot\,)|^{1/2}}\, \mathrm{d}S(\theta)\Big|^{p_d}\,\mathrm{d}y'\Big)^{\frac{1}{q_d}}. \end{aligned}$$

Applying the extension version of the Tomas–Stein theorem —valid here for $d \geq 3$ since $p_d = q_{d-1}$— we have that the right-hand side of the previous identity can be bounded so that the inequality (86) becomes

$$\|P_k f\|_{L^{p_d}(\mathbb{R}^d)} \lesssim 2^{k(\frac{1}{2}-\frac{1}{p_d})}\Big\| \int_0^\infty \rho^{\frac{1-d}{p_d}} \Big(\int_{\mathbb{S}^{d-2}} \frac{|\widehat{P_k g}(\rho\theta,\,\cdot\,)|^2}{|q_\tau(\rho\theta,\,\cdot\,)|}\, \mathrm{d}S(\theta)\Big)^{\frac{1}{2}} \rho^{d-2}\,\mathrm{d}\rho\Big\|_{L^2(\mathbb{R})}.$$

As we noted in the proof of the lemma 4.18, $|q_\tau(\rho\theta,\xi_d)|$ does not depend on θ, and consequently, applying the Cauchy–Schwarz inequality to the integration in ρ, we have that the norm on the right-hand side of the previous inequality is bounded by a constant multiple of

$$\sup_{|\xi_d|\leq 2^{k+1}} \Big(\int_0^{2^{k+1}} \frac{\rho^{2/p_d}}{|q_\tau(\rho\theta,\xi_d)|}\,\mathrm{d}\rho\Big)^{\frac{1}{2}} \Big\| \int_0^\infty \int_{\mathbb{S}^{d-2}} |\widehat{P_k g}(\rho\theta,\,\cdot\,)|^2\, \mathrm{d}S(\theta)\rho^{d-2}\,\mathrm{d}\rho\Big\|_{L^2(\mathbb{R})}.$$

Since $k \in I$, one can check that the first term of the previous product is bounded by $\tau^{1/p_d - 1/2}$. Thus, we end up with the inequality

$$\|P_k f\|_{L^{p_d}} \lesssim 2^{k(1/2-1/p_d)}\tau^{1/p_d-1/2}\||q_\tau|^{1/2}\widehat{P_k f}\|_{L^2}.$$

Since $k \in I$ and $2^k \simeq \tau$, we get the inequality (85). □

Corollary 4.23 (Haberman)**.** *Assume* $d \geq 3$. *There exists a constant* $C > 0$ *depending on* d *so that*

$$\|f\|_{L^{p_d}} \leq C\|f\|_{\dot{Y}^{1/2}_\tau}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. It is a consequence of the lemmas 4.8 and 4.22. □

Lemma 4.24. *Whenever $d \geq 3$, there exists a constant $C > 0$ depending on d so that*

$$\|f\|_{\dot{Y}_\tau^{-1/2}} \leq C\|f\|_{Z_{\lambda,p'_d}}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda = \tau^2$.

Proof. It follows from the lemma 4.22 by a standard duality argument. □

Corollary 4.25 (Kenig–Ruiz–Sogge)**.** *Assume $d \geq 3$. There exists a constant $C > 0$ depending on d so that*

$$\|(\Delta + 2\tau\partial_{x_d} + \tau^2)^{-1} f\|_{L^{p_d}} \leq C\|f\|_{L^{p'_d}}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. It follows from the identity (9), the corollary 4.23 and the lemmas 4.24 and 4.9. □

Appendix A. The functional analytical framework

Here we prove the propositions 2.5, 2.6 and 2.7 which describe some basic properties of the functional spaces X_λ and X_λ^*. As we see, these propositions will be immediately derived from some properties related to the spaces Y_λ, Y_λ^*, Z_λ and Z_λ^*.

Lemma A.1. *The pair $(Y_\lambda, \|\cdot\|_{Y_\lambda})$ is a Banach space and its dual is isomorphic to $(Y_\lambda^*, \|\cdot\|_{Y_\lambda^*})$. The Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in Y_λ and Y_λ^* with their corresponding norms.*

Lemma A.2. *The pair $(Z_{\lambda,p'}, \|\cdot\|_{Z_{\lambda,p'}})$ is a reflexive Banach space and its dual is isomorphic to $(Z_{\lambda,p}^*, \|\cdot\|_{Z_{\lambda,p}^*})$ with p and p' duals. The Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $Z_{\lambda,p'}$ and $Z_{\lambda,p}^*$ with their corresponding norms.*

Note that $2^k \simeq \lambda^{1/2}$ when $k \in I$, $m_\lambda(\xi)^{1/2}|\widehat{P_{<I}f}(\xi)| \simeq \lambda^{1/2}|\widehat{P_{<I}f}(\xi)|$, and $m_\lambda(\xi)^{1/2}|\widehat{P_k f}(\xi)| \simeq 2^k|\widehat{P_k f}(\xi)|$ when $k > k_\lambda + 1$. Thus, the norms of the spaces Y_λ, Y_λ^*, $Z_{\lambda,p'}$ and $Z_{\lambda,p}^*$ can be re-written similarly to the norms of non-homogeneous Besov spaces with different weights and norms on the critical scales $k \in I$. This remark is the key to justify that these spaces are Banach and $\mathcal{S}(\mathbb{R}^d)$ is dense with respect to the corresponding topologies. The duality also works because of the same principle —since the norms in the corresponding pieces are taken to be dual. To be more precise, note that $|||\cdot|||_*$ is the dual norm of $|||\cdot|||$ and not the other way around, while $\|\cdot\|_{L^{q'_d}}$ and $\|\cdot\|_{L^{q_d}}$ are dual of each other. Hence, $Z_{\lambda,p'}$ is reflexive and Y_λ is not.

Now, we show how to derive the propositions 2.5, 2.6 and 2.7 stated in the section 2. Start by the first of these three propositions. The density of $\mathcal{S}(\mathbb{R}^d)$ in Y_λ and Z_λ is explicitly stated in the lemmas A.1 and A.2, in particular, the density also holds for $X_\lambda = Y_\lambda + Z_\lambda$ with its corresponding norm. This proves the proposition 2.5. Now, we turn to the proposition 2.6. Since $(Y_\lambda, \|\cdot\|_{Y_\lambda})$ and $(Z_\lambda, \|\cdot\|_{Z_\lambda})$ are Banach spaces and Y_λ and Z_λ are subspaces of $\mathcal{S}'(\mathbb{R}^d)$, we have by Theorem 1.3 in [3] that $(X_\lambda, \|\cdot\|_{X_\lambda})$ is a Banach space. The identity (17) is a standard property of Banach spaces (see Corollary 1.4 in [4]). This concludes the proof of the proposition 2.6. Finally, let us focus on the last of these three propositions. It is a well-known fact —since $\mathcal{S}(\mathbb{R}^d)$ is dense in Y_λ and Z_λ— that $(X_\lambda^*, \|\cdot\|_{X_\lambda^*})$ is isomorphic to the space $Y_\lambda^* \cap Z_\lambda^*$ endowed with the norm $\max\{\|\cdot\|_{Y_\lambda^*}, \|\cdot\|_{Z_\lambda^*}\}$ (see 2 in the section Exercises and Further Results for Chapter 3 of the book [3]). Note that this later space is actually isomorphic to the space described in the proposition 2.7 endowed with the norm (18). To finish the proof of this proposition, it is enough to check the density of $\mathcal{S}(\mathbb{R}^d)$ in X_λ^* with its corresponding norm. Note that this holds because $\mathcal{S}(\mathbb{R}^d)$ is dense in Y_λ^* and Z_λ^*, and the norm $\|\cdot\|_{X_\lambda^*}$ is equivalent to $\max\{\|\cdot\|_{Y_\lambda^*}, \|\cdot\|_{Z_\lambda^*}\}$.

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