

ON A CONJECTURE OF HARRIS

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ABSTRACT. For $d \geq 4$, the Noether-Lefschetz locus NL_d parametrizes smooth, degree d surfaces in \mathbb{P}^3 with Picard number at least 2. A conjecture of Harris states that there are only finitely many irreducible components of the Noether-Lefschetz locus of non-maximal codimension. Voisin showed that the conjecture is false for sufficiently large d , but is true for $d \leq 5$. She also showed that for $d = 6, 7$, there are finitely many *reduced*, irreducible components of NL_d of non-maximal codimension. In this article, we prove that for any $d \geq 6$, there are infinitely many *non-reduced* irreducible components of NL_d of non-maximal codimension.

1. INTRODUCTION

The underlying field is \mathbb{C} . By surface we will always mean a projective surface in \mathbb{P}^3 . A classical result in the theory of surfaces, stated by M. Noether and later proved by Lefschetz, says that for any $d \geq 4$, a very general, smooth, degree d surface in \mathbb{P}^3 is of Picard number 1 (by *Picard number* we mean the rank of the Néron-Severi group). Here, *very general* means that the points on the parametrizing space $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)))$ of degree d surfaces in \mathbb{P}^3 , corresponding to such surfaces, lie outside a countable union of proper, closed subsets of $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)))$. The *Noether-Lefschetz locus*, denoted NL_d , is then defined to be the locus of smooth, degree d surfaces in \mathbb{P}^3 with Picard number at least 2. The irreducible components of the Noether-Lefschetz locus have a natural (analytic) scheme structure, which we will now describe.

Denote by $U_d \subseteq \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)))$ the open subscheme parametrizing smooth projective hypersurfaces in \mathbb{P}^3 of degree d . Let

$$\pi : \mathcal{X} \rightarrow U_d$$

be the corresponding universal family. For a given $u \in U_d$, denote by $\mathcal{X}_u := \pi^{-1}(u)$. Denote by

$$\mathbb{H} := R^2\pi_*\mathbb{Z} \text{ and } \mathcal{H} := R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_{U_d}.$$

Using the Ehresmann's theorem, it is easy to check that \mathbb{H} is a local system, hence \mathcal{H} is a vector bundle. Fix a point $o \in \text{NL}_d$ and $U \subseteq U_d$ a simply connected open neighbourhood of o in U_d (under the analytic topology). It is easy to check that the restriction of \mathbb{H} to U is trivial and any class $\gamma_0 \in H^2(\mathcal{X}_o, \mathbb{Z})$ defines by flat transform, a section $\gamma \in \Gamma(U, \mathbb{H})$. Let $\bar{\gamma}$ be the image of γ in $\mathcal{H}/F^1\mathcal{H}$, where $F^1\mathcal{H} \subset \mathcal{H}$ is a vector subbundle such that for every $u \in U$, the fiber $(F^1\mathcal{H})_u \subset \mathcal{H}_u$ can be identified with $F^1H^2(\mathcal{X}_u, \mathbb{C}) \subset H^2(\mathcal{X}_u, \mathbb{C})$ (see [12, §10.2.1]). If γ_0 belongs to $H^{1,1}(\mathcal{X}_o, \mathbb{C})$ i.e., γ_0 is a Hodge class, then the *Hodge locus associated to γ_0* , denoted $\text{NL}(\gamma_0)$, is defined as

$$\text{NL}(\gamma_0) := \{u \in U \mid \bar{\gamma}(u) = 0\},$$

where $\bar{\gamma}(u)$ denotes the value at u of the section $\bar{\gamma}$. The Hodge locus is equipped with a natural scheme structure (see [13, §5.3.1]). The intersection of NL_d with U is the union of $\text{NL}(\gamma_0)$ as γ_0 ranges over the Hodge classes of \mathcal{X}_u for all $u \in U$, such that the Hodge class is not a multiple of $c_1(\mathcal{O}_{\mathcal{X}_u}(1))$ (see [13, §5.3.3]). We say that the closure $\overline{\text{NL}(\gamma_0)}$ of $\text{NL}(\gamma_0)$ (in the Zariski topology), is an *irreducible component* of NL_d , if the underlying topological space is irreducible and as an

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(analytic) *scheme* is not contained properly in any irreducible component of $\overline{\text{NL}(\gamma')}$ for some Hodge class γ' over \mathcal{X}_o , where γ' is not a multiple of $c_1(\mathcal{O}_{\mathcal{X}_o}(1))$. Two irreducible components of NL_d are isomorphic if they are isomorphic as analytic schemes (scheme structure as Hodge loci). It was shown by Ciliberto-Harris-Miranda [1] that any irreducible component L of NL_d satisfies the inequality:

$$d - 3 \leq \text{codim}(L, U_d) \leq \binom{d-1}{3}.$$

If $\text{codim}(L, U_d) = \binom{d-1}{3}$, then L is called a *general component*. Otherwise, L is called a *special component*. Harris conjectures the following on the special components of the Noether-Lefschetz locus (see [2, 11]):

Conjecture (Harris). Fix an integer $d \geq 5$. Then,

- (1) **Topological Harris conjecture:** Ignoring the natural analytic scheme structure (as Hodge locus) on the irreducible components of NL_d , there are finitely many topological, special components of NL_d .
- (2) **Analytic Harris conjecture:** The Noether-Lefschetz locus NL_d contains finitely many special, irreducible components (by irreducible component we mean as above).

For $d \leq 5$ the conjectures hold true (see [9, Theorem 0.2]). The conjectures have been shown to be false by Voisin [11] for sufficiently large d . For $d = 6, 7$, Voisin proved in [10] that NL_d has finitely many *reduced*, special components. But there are several questions that are still open: What is the largest d' such that the Harris conjectures hold true for all $d \leq d'$? For those d for which the Harris conjectures fail, what is the largest k such that there are finitely many special components of codimension at most k and infinitely many special components of codimension strictly greater than k ? In this article we give a complete answer to these questions for the analytic Harris conjecture. We show:

Theorem 1.1 (see Theorem 2.4). Let $d \geq 6$ and X be a general smooth degree d surface containing two coplanar lines, say L_1, L_2 . Denote by $\gamma_a := [L_1] + a[L_2] \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})$, for any $a \in \mathbb{Z}$. Then, for any $a, b \geq 0$ with $a \neq b$, we have $\text{NL}(\gamma_a) \neq \text{NL}(\gamma_b)$. Moreover,

$$\text{codim NL}(\gamma_0) = d - 3, \text{codim NL}(\gamma_1) = 2d - 7 \text{ and } \text{codim NL}(\gamma_a) = 2d - 6 \text{ for } a \geq 2.$$

In particular, we have following table which gives the number of irreducible components of NL_d with the given codimension:

codimension	number of irr. components
$< d - 3$	0
$d - 3$	1
$2d - 7$	1
$\leq 2d - 7$	2
$2d - 6$	∞

The theorem immediately disproves the analytic Harris conjecture for $d \geq 6$. Recall, Voisin in [11] uses the existence of infinitely many general components of NL_4 to produce infinitely many sufficiently high degree special components (after replacing the coordinates with general high degree polynomials). In particular, the topological space underlying the special components are distinct. Her counterexample relies on a numerical inequality that holds *only* for sufficiently large d (see [11, p. 686]). In contrast, we simply study the scheme structure of the Hodge locus corresponding to different linear combinations of coplanar lines. In particular, we show that the space parametrizing smooth, degree d surfaces containing 2 coplanar lines can be equipped with infinitely many (distinct) scheme structures *naturally* arising as the Hodge loci associated to different combinations of the two coplanar lines (see Theorem 2.4). As a result, the infinite

number of special components in this article have the same underlying topological space (but different analytic scheme structures), thus giving us an entirely different set of counterexamples from those in [11]. The topological Harris conjecture is still open for small values of d . In a recent preprint, Movasati in [4] uses computer calculations for explicit values of d to give a description of possible counterexamples to the topological Harris conjecture.

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2. PROOF OF MAIN THEOREM

In this section we prove Theorem 1.1. Fix an integer $d \geq 6$.

2.1. Cohomology computations of the invertible sheaves associated to lines. To prove Theorem 1.1 we need the following basic computation on the first and second cohomology group of the invertible sheaf associated to a line contained in a smooth, degree d surface in \mathbb{P}^3 .

Lemma 2.1. Let X be a smooth, degree d surface containing two coplanar lines, say L, L' . Then,

$$H^0(\mathcal{O}_X(L')) = H^0(\mathcal{O}_X(L)) = \mathbb{C} = H^0(\mathcal{O}_X(L \cup L')) \text{ and } H^1(\mathcal{O}_X(L)) = 0 = H^1(\mathcal{O}_X(L')).$$

Proof. By the adjunction formula, $L^2 = -2 - (d - 4) = 2 - d$, which is less than zero for $d \geq 6$. Hence, $H^0(\mathcal{N}_{L|X}) = 0$. Using the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(L) \rightarrow \mathcal{N}_{L|X} \rightarrow 0 \quad (2.1)$$

we conclude that $\mathbb{C} = H^0(\mathcal{O}_X) = H^0(\mathcal{O}_X(L))$. Similarly, we can show that $\mathbb{C} = H^0(\mathcal{O}_X(L'))$. Next, we consider the short exact sequence:

$$0 \rightarrow \mathcal{O}_X(L) \rightarrow \mathcal{O}_X(L \cup L') \rightarrow \mathcal{O}_{L'} \otimes \mathcal{O}_X(L \cup L') \rightarrow 0 \quad (2.2)$$

obtained by tensoring with $\mathcal{O}_X(L \cup L')$ the short exact sequence:

$$0 \rightarrow \mathcal{O}_X(-L') \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{L'} \rightarrow 0.$$

Note that, $L' \cdot (L + L') = 1 + (2 - d) < 0$ for $d \geq 6$. This implies, $H^0(\mathcal{O}_{L'} \otimes \mathcal{O}_X(L \cup L')) = 0$. By (2.2), we then have $H^0(\mathcal{O}_X(L \cup L')) = H^0(\mathcal{O}_X(L)) = \mathbb{C}$. This proves the first part of the lemma. We now show that $H^1(\mathcal{O}_X(L)) = 0 = H^1(\mathcal{O}_X(L'))$.

Recall, the Castelnuovo-Mumford regularity of \mathcal{I}_L is one (see [6, Example 1.8.2]). This implies that $H^i(\mathcal{I}_L(j)) = 0$ for $i \geq 1$ and $j \geq 0$. Consider now the short exact sequence:

$$0 \rightarrow \mathcal{I}_X(d - 4) \rightarrow \mathcal{I}_L(d - 4) \rightarrow \mathcal{O}_X(-L)(d - 4) \rightarrow 0. \quad (2.3)$$

As $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-d)$, we have $H^2(\mathcal{I}_X(d - 4)) = 0$. Using the long exact sequence associated to (2.3), we conclude that $H^1(\mathcal{O}_X(-L)(d - 4)) = 0$. By Serre duality this implies

$$H^1(\mathcal{O}_X(L)) = H^1(\mathcal{O}_X(-L)(d - 4)) = 0.$$

Similarly, we can show that $H^1(\mathcal{O}_X(L')) = 0$. This proves the lemma. \square

2.2. Flag Hilbert schemes. Let X be a smooth, degree d surface in \mathbb{P}^3 containing two (distinct) coplanar lines, say L_1 and L_2 . Denote by P_0 (resp. P_1, Q_d) the Hilbert polynomial of L_1 (resp. $L_1 \cup L_2, X$). Denote by $\text{Hilb}_{P_0, P_1, Q_d}$ the flag Hilbert scheme parametrizing triples $(Z_1 \subset Z_2 \subset Z_3)$ with Z_1 (resp. Z_2, Z_3) having Hilbert polynomial P_0 (resp. P_1, Q_d). Denote by Hilb_{P_i} and Hilb_{Q_d} the Hilbert scheme associated to the Hilbert polynomial P_i and Q_d , respectively for $i = 0, 1$. See [8, §4.3, 4.5] for a detailed discussion on (flag) Hilbert schemes.

Proposition 2.2. The flag Hilbert scheme $\text{Hilb}_{P_0, P_1, Q_d}$ is reduced. In particular, the scheme-theoretic image under the natural projection map

$$\text{pr} : \text{Hilb}_{P_0, P_1, Q_d} \rightarrow \text{Hilb}_{Q_d}$$

is reduced.

Proof. Consider the natural projection map:

$$\text{pr}_0 : \text{Hilb}_{P_0, P_1, Q_d} \rightarrow \text{Hilb}_{P_0}.$$

Clearly, this map is surjective as for every line L there exists infinitely many lines L' lying on the same plane as L such that $L \cup L'$ is contained in a smooth, degree d surface in \mathbb{P}^3 . Note that for any point $t \in \text{Hilb}_{P_0}$, we have

$$\dim T_t \text{Hilb}_{P_0} = h^0(\mathcal{N}_{L_t|\mathbb{P}^3}) = h^0(\mathcal{O}_{L_t}(1)) + h^0(\mathcal{O}_{L_t}(1)) = 4 = \dim \text{Hilb}_{P_0},$$

where L_t is the line corresponding to the point t . Hence, Hilb_{P_0} is smooth. A standard exercise in commutative algebra tells us that given a morphism of schemes, the domain is reduced if the scheme-theoretic image and every fiber is reduced. Therefore, it is sufficient to check that the every fiber to the morphism pr_0 is reduced.

Note that, the morphism pr_0 factors through Hilb_{P_0, P_1} . Denote by

$$\text{pr}_1 : \text{Hilb}_{P_0, P_1} \rightarrow \text{Hilb}_{P_0} \text{ and } \text{pr}_2 : \text{Hilb}_{P_0, P_1, Q_d} \rightarrow \text{Hilb}_{P_0, P_1}$$

the natural projections. Since every conic can be embedded in a degree d surface in \mathbb{P}^3 , the scheme-theoretic image of $\text{pr}_0^{-1}(t)$ under the morphism pr_2 coincides with the fiber $\text{pr}_1^{-1}(t)$. The dimension of $\text{pr}_1^{-1}(t)$ equals $\dim \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1))) + 1$, where the first term is the dimension of the space of lines contained in the same plane as L_t (after fixing the plane) and the second term is the dimension of the space of planes in \mathbb{P}^3 containing L_t . For any line L' contained in the same plane as L_t , we have

$$h^0(\mathcal{O}_{L_t+L'}(1)) = h^0(\mathcal{O}_{L_t}(1)) + 1 \text{ and } h^0(\mathcal{O}_{L_t+L'}(2)) = 2 + 1 = 5$$

where the last equality follows from the fact that the Castelnuovo-Mumford regularity of $\mathcal{O}_{L_t+L'}$ is one, which implies $h^0(\mathcal{O}_{L_t+L'}(2))$ equals $P_{L_t+L'}(2)$ for the Hilbert polynomial $P_{L_t+L'}(n)$ of $L_t + L'$. Similarly, $h^0(\mathcal{O}_{L_t}(2)) = 2 + 1 = 3$. Since $\mathcal{N}_{L_t+L'|\mathbb{P}^3} \cong \mathcal{O}_{L_t+L'}(1) \oplus \mathcal{O}_{L_t+L'}(2)$, [5, Ex. II.8.4] implies that the restriction morphism

$$\rho : H^0(\mathcal{N}_{L_t+L'|\mathbb{P}^3}) \rightarrow H^0(\mathcal{N}_{L_t+L'|\mathbb{P}^3} \otimes_{\mathcal{O}_{\mathbb{P}^3}} \mathcal{O}_{L_t})$$

is surjective. Hence,

$$\dim \ker(\rho) = h^0(\mathcal{O}_{L_t+L'}(1)) + h^0(\mathcal{O}_{L_t+L'}(2)) - h^0(\mathcal{O}_{L_t}(1)) - h^0(\mathcal{O}_{L_t}(2)) = 1 + 5 - 3 = 3.$$

Using [8, Remarks 4.5.4], we have $\dim T_{(L_t \subset L_t+L')} \text{pr}_1^{-1}(t) = \ker \rho$, which by our computation equals $\dim \text{pr}_1^{-1}(t)$. Hence, $\text{pr}_1^{-1}(t)$ is reduced.

The fiber over the point corresponding to the pair $(L_t \subset L_t \cup L')$ for the composed morphism

$$\text{pr}_0^{-1}(t) \hookrightarrow \text{Hilb}_{P_0, P_1, Q_d} \xrightarrow{\text{pr}_2} \text{Hilb}_{P_0, P_1}$$

is isomorphic to $\mathbb{P}(H^0(\mathcal{I}_{L_t \cup L'}(d)))$, which is reduced. Since $\text{pr}_2(\text{pr}_0^{-1}(t)) = \text{pr}_1^{-1}(t)$ is reduced, this implies that $\text{pr}_0^{-1}(t)$ is reduced. Hence, $\text{Hilb}_{P_0, P_1, Q_d}$ is reduced. The second part of the

lemma is direct (scheme-theoretic image of a reduced scheme is reduced). This proves the proposition. \square

2.3. Proof of Theorem 1.1. Let X be a general, smooth, degree d surface in \mathbb{P}^3 containing 2 distinct coplanar lines, say L_1, L_2 . We use the notations as in §2.2. Denote by $o \in U_d$ the point corresponding to X . Let $U \subset U_d$ be a simply connected neighbourhood of o in U_d as before.

Proposition 2.3. The (Zariski) closure of $\text{NL}([L_1]) \cap \text{NL}([L_2])$ in U_d is isomorphic to the scheme-theoretic image of the morphism pr as in Proposition 2.2, intersected with U_d .

Proof. Denote by $W := \text{Im}(\text{pr}) \cap U_d$, where pr is as in Proposition 2.2. Note that, W parametrizes smooth, degree d surfaces in \mathbb{P}^3 containing two coplanar lines. Hence, $\overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}$ contains W . We now prove the reverse inclusion i.e., $\overline{\text{NL}([L_1]) \cap \text{NL}([L_2])} \subset W$. Denote by

$$\pi' : \mathcal{X}' \rightarrow \overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}$$

the restriction of π to $\overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}$. By Lefschetz (1, 1)-theorem, there exist invertible sheaves \mathcal{L}_1 and \mathcal{L}_2 over \mathcal{X}' such that $\mathcal{L}_1|_X \cong \mathcal{O}_X(L_1)$ and $\mathcal{L}_2|_X \cong \mathcal{O}_X(L_2)$. Using Lemma 2.1 and the upper semi-continuity of cohomology (see [5, Theorem III.12.8]), there exists an open neighbourhood $V \subset \overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}$ of o such that for all $v \in V$, we have

$$h^0(\mathcal{L}_{1,v}) = 1 = h^0(\mathcal{L}_{2,v}) \text{ and } h^1(\mathcal{L}_{1,v}) = 0 = h^1(\mathcal{L}_{2,v}), \text{ where } \mathcal{L}_{1,v} := \mathcal{L}_1|_{\mathcal{X}_v} \text{ and } \mathcal{L}_{2,v} := \mathcal{L}_2|_{\mathcal{X}_v}.$$

By [5, Theorem III.12.11], for every $v \in V$, the natural morphisms

$$\pi'_* \mathcal{L}_1 \otimes k(v) \rightarrow H^0(\mathcal{L}_{1,v}) \text{ and } \pi'_* \mathcal{L}_2 \otimes k(v) \rightarrow H^0(\mathcal{L}_{2,v})$$

are isomorphisms. Hence, after contracting V if necessary, there exist sections $s_1 \in \Gamma(V, \pi'_* \mathcal{L}_1)$ and $s_2 \in \Gamma(V, \pi'_* \mathcal{L}_2)$ such that its image $s_{1,v}$ and $s_{2,v}$ in $H^0(\mathcal{L}_{1,v})$ and $H^0(\mathcal{L}_{2,v})$, respectively are non-zero for all $v \in V$. The sections s_1 and s_2 give rise to the short exact sequence:

$$0 \rightarrow \mathcal{L}_i^\vee|_V \xrightarrow{\cdot s_i} \mathcal{O}_{\mathcal{X}_V} \rightarrow \mathcal{O}_{Z(s_i)} \rightarrow 0,$$

where $Z(s_i)$ is the zero locus of the section s_i in $\mathcal{X}_V := \pi^{-1}(V)$, for $i = 1, 2$. Since $s_{i,v}$ is non-zero, the natural morphism

$$\mathcal{L}_{i,v}^\vee \xrightarrow{\cdot s_{i,v}} \mathcal{O}_{\mathcal{X}_v}$$

is injective for $i = 1, 2$. By the local criterion of flatness (see [7, p. 150, (20.E)]), we conclude that $Z(s_i)$ is flat over V for $i = 1, 2$. Denote by $\mathcal{L}_1^{\text{eff}} := Z(s_1)$ and $\mathcal{L}_2^{\text{eff}} := Z(s_2)$. It is easy to check that the effective divisor $\mathcal{L}_1^{\text{eff}} + \mathcal{L}_2^{\text{eff}}$ of \mathcal{X}_V is also flat over V . By the universal property, of the flag Hilbert schemes (see [8, Theorem 4.5.1]), the triple

$$(\mathcal{L}_1^{\text{eff}} \subset \mathcal{L}_1^{\text{eff}} + \mathcal{L}_2^{\text{eff}} \subset \mathcal{X}_V)$$

induces a morphism from V to $\text{Hilb}_{P_0, P_1, Q_d}$ such that the composition

$$V \rightarrow \text{Hilb}_{P_0, P_1, Q_d} \xrightarrow{\text{pr}} \text{Hilb}_{Q_d}$$

is the natural inclusion. This implies, a dense open subscheme of $\overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}$ lies in the scheme-theoretic image of pr . Since the morphism pr is proper, we conclude that $\overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}$ lies in the scheme-theoretic image of pr . So, we have the reverse inclusion. Hence, $\overline{\text{NL}([L_1]) \cap \text{NL}([L_2])} = \text{Im}(\text{pr})$. This proves the proposition. \square

Theorem 2.4. For any $a \in \mathbb{Z}$, denote by $\gamma_a := [L_1] + a[L_2] \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})$. Then, for any $a, b \geq 0$ with $a \neq b$, we have $\text{NL}(\gamma_a) \neq \text{NL}(\gamma_b)$. Moreover,

$$\text{codim NL}(\gamma_0) = d - 3, \text{ codim NL}(\gamma_1) = 2d - 7 \text{ and } \text{codim NL}(\gamma_a) = 2d - 6 \text{ for } a \geq 2.$$

Proof. Denote by $\overline{\text{NL}(\gamma_a)}$ the closure of $\text{NL}(\gamma_a)$ in U_d under the Zariski topology. By [9, Theorem 0.2] $\overline{\text{NL}(\gamma_0)}$ (resp. $\overline{\text{NL}(\gamma_1)}$) parametrizes smooth, degree d surfaces containing a line (resp. a conic) and is of codimension $d - 3$ (resp. $2d - 7$) in U_d . Furthermore, both $\text{NL}(\gamma_0)$ and $\text{NL}(\gamma_1)$ are reduced. We first note that for $a \geq 2$, we have $\text{NL}(\gamma_a) \neq \text{NL}(\gamma_0)$, $\text{NL}(\gamma_a) \neq \text{NL}(\gamma_1)$ and $\text{codim } \text{NL}(\gamma_a) = 2d - 6$. Indeed, $\overline{\text{NL}(\gamma_a)}$ contains the space W parametrizing smooth, degree d surfaces containing 2 coplanar lines. It is easy to compute that $\text{codim } W = 2d - 6$. Hence, $\text{codim } \text{NL}(\gamma_a) \leq 2d - 6$. By [9, Theorem 0.2], either $\text{NL}(\gamma_a) = \text{NL}(\gamma_0)$ or $\overline{\text{NL}(\gamma_a)} = \overline{\text{NL}(\gamma_1)}$ or $\text{codim } \text{NL}(\gamma_a) = 2d - 6$. If $\text{NL}(\gamma_a) = \text{NL}(\gamma_0)$ or $\text{NL}(\gamma_a) = \text{NL}(\gamma_1)$, then $\overline{\text{NL}(\gamma_a)}$ parametrizes smooth, degree d surfaces such that both $[L_1]$ and $[L_2]$ remains a Hodge class, in particular

$$\text{NL}(\gamma_a) = \text{NL}([L_1]) \cap \text{NL}([L_2]).$$

But, Proposition 2.3 then implies that $\overline{\text{NL}(\gamma_a)} = \text{Im}(\text{pr}) = W$, which is of codimension $2d - 6$. This gives us a contradiction. Hence, for $a \geq 2$, $\text{NL}(\gamma_a) \neq \text{NL}(\gamma_0)$, $\text{NL}(\gamma_a) \neq \text{NL}(\gamma_1)$ and $\text{codim } \overline{\text{NL}(\gamma_a)} = 2d - 6$.

We now show that for $a \geq 2$ and $u \in \text{NL}(\gamma_a)$ general, we have $\text{codim } T_u \text{NL}(\gamma_a) \leq 2d - 7$. Indeed, using [3, (4.a.4)], we have for all $a \geq 2$,

$$H^0(K_X(-L_1 - L_2)) \subset H^{2,0}(-\gamma_a) := \{\psi \in H^{2,0}(X, \mathbb{C}) \mid (t \cup \psi) \cup \gamma_a = 0 \text{ for all } t \in H^1(\mathcal{T}_X)\}.$$

Recall from [3, p. 211] that

$$\text{codim } T_o \text{NL}(\gamma_a) = \dim H^{2,0}(X, \mathbb{C}) - \dim H^{2,0}(-\gamma_a)$$

which by our calculations is bounded above by $h^0(K_X) - h^0(K_X(-L_1 - L_2))$. Consider now the short exact sequence:

$$0 \rightarrow K_X(-L_1 - L_2) \rightarrow K_X \rightarrow K_X \otimes \mathcal{O}_{L_1 \cup L_2} \rightarrow 0. \quad (2.4)$$

Using [6, Example 1.8.2], the Castelnuovo-Mumford regularity of $\mathcal{O}_{L_1 \cup L_2}$ is one. Hence,

$$H^1(\mathcal{I}_{L_1 \cup L_2}(d - 4)) = 0 \text{ for } d \geq 6.$$

Using the exact sequence

$$0 \rightarrow \mathcal{I}_X(d - 4) \rightarrow \mathcal{I}_{L_1 \cup L_2}(d - 4) \rightarrow \mathcal{O}_X(-L_1 - L_2)(d - 4) \rightarrow 0,$$

we conclude that $H^1(\mathcal{O}_X(-L_1 - L_2)(d - 4)) = 0$ for $d \geq 6$. In particular,

$$H^1(K_X(-L_1 - L_2)) = H^1(\mathcal{O}_X(-L_1 - L_2)(d - 4)) = 0.$$

By the short exact sequence (2.4), we conclude that

$$h^0(K_X) - h^0(K_X(-L_1 - L_2)) = h^0(K_X \otimes \mathcal{O}_{L_1 \cup L_2}) = h^0(\mathcal{O}_{L_1 \cup L_2}(d - 4)) = P_{L_1 \cup L_2}(d - 4),$$

where $P_{L_1 \cup L_2}(t)$ is the Hilbert polynomial of $L_1 \cup L_2$ and the last equality follows from the fact that the Castelnuovo-Mumford regularity of $\mathcal{O}_{L_1 \cup L_2}$ is one. Now, the Hilbert polynomial of $L_1 \cup L_2$ is $2(d - 4) + 1 = 2d - 7$. Therefore, $\text{codim } T_o \text{NL}(\gamma_a) \leq 2d - 7$. Since $\text{NL}([L_1]) \cap \text{NL}([L_2]) \subset \text{NL}(\gamma_a)$ and both spaces are of the same dimension, we have $\text{NL}([L_1]) \cap \text{NL}([L_2]) = \text{NL}(\gamma_a)_{\text{red}}$. Hence, for a general $u \in \text{NL}(\gamma_a)$, L_1 (resp. L_2) deforms to a line $L_{1,u}$ (resp. $L_{2,u}$) in \mathcal{X}_u (use the construction of $\mathcal{L}_1^{\text{eff}}$ and $\mathcal{L}_2^{\text{eff}}$ from the proof of Proposition 2.3). Then, γ_a deforms to $\gamma_{a,u} := [L_{1,u}] + a[L_{2,u}]$. Hence, $\text{NL}(\gamma_a) = \text{NL}(\gamma_{a,u})$. Similarly, as before, we get

$$\text{codim } T_u \text{NL}(\gamma_a) = \text{codim } T_u \text{NL}(\gamma_{a,u}) \leq 2d - 7.$$

This proves our claim.

If for $a, a' \geq 2$ with $a \neq a'$, we have $\text{NL}(\gamma_a) = \text{NL}(\gamma_{a'})$, then clearly

$$\overline{\text{NL}(\gamma_a)} = \overline{\text{NL}(\gamma_{a'})} \subset \overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}.$$

Since the three spaces have the same dimension and $\overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}$ is reduced (use Proposition 2.2 combined with Proposition 2.3), they coincide. But, $\overline{\text{NL}([L_1]) \cap \text{NL}([L_2])}$ is reduced

and $\text{NL}(\gamma_a), \text{NL}(\gamma_{a'})$ are generically non-reduced, as observed earlier. This gives a contradiction, hence $\text{NL}(\gamma_a) \neq \text{NL}(\gamma_{a'})$. \square

Remark 2.5. After Theorem 2.4, the remaining parts of Theorem 1.1 follows directly from [9, Theorem 0.2].

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