EXTENSIONS OF THE JOHN–NIRENBERG THEOREM AND APPLICATIONS

JAVIER CANTO AND CARLOS PÉREZ

ABSTRACT. The John–Nirenberg theorem states that functions of bounded mean oscillation are exponentially integrable. In this article we give two extensions of this theorem. The first one relates the dyadic maximal function to the sharp maximal function of Fefferman–Stein, while the second one concerns local weighted mean oscillations, generalizing a result of Muckenhoupt and Wheeden. Applications to the context of generalized Poincaré type inequalities and to the context of the C_p class of weights are given. Extensions to the case of polynomial BMO type spaces are also given.

1. INTRODUCTION AND MAIN RESULTS

The classical John–Nirenberg (JN) theorem [JN] states that any function of bounded mean oscillation is locally exponentially integrable, see for example [GCRdF]. More precisely, we have the inequality

$$(1.1) \qquad |\{x \in Q : |f(x) - f_Q| > t\}| \le c_1 e^{-c_2 t / \|f\|_{\mathrm{BMO}}} |Q|.$$

The main purpose of this work is to provide two extensions of the classical JN theorem for BMO functions. The first extension constitutes an improvement of a result of Karagulyan [Kar], which is in turn a more precise version of the classical Fefferman–Stein inequalities relating the Hardy–Littlewood and the sharp maximal functions. The second extension is an improvement of some classical estimates by Muckenhoupt and Wheeden [MW] concerning weighted local mean oscillations. These estimates were already discussed in the recent work [OPRRR] in a more restrictive setting.

These two extensions, although different a priori, are obtained by a similar method, using some ideas from the recent work [PR].

1.1. **Improving Karagulyan's estimate.** The first extension is motivated by the work of Karagulyan [Kar], who already provided an extension of the JN theorem. We improve this interesting result by providing a different, more flexible, proof with many advantages. However, this first extension is also inspired by the recent work [PR], where a different approach to the main results from [FKS] concerning degenerate Poincaré–Sobolev inequalities is found with the bonus that a simpler proof of the classical JN theorem is implicit.

We obtain two different consequences of this improvement of the JN theorem. Firstly, we derive some degenerate Poincaré–Sobolev endpoint inequalities not available from the methods in [PR]. Secondly, this improvement will be applied within the context of the C_p class of weights (see Section 2) solving some questions left open in [Can].

To establish this result we recall the sharp maximal function introduced by C. Fefferman and E. Stein. It is defined by

$$(1.2) \qquad M^{\sharp} h(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |h(y) - h_R| \, dy,$$

where the supremum is taken over all cubes R that contain the point x, and $h_R = \frac{1}{|R|} \int_R h$ denotes the average of h over R. Karagulyan proved in [Kar] the following interesting exponential decay. If f is an L^1_{loc} function and B a ball in $\mathbb{R}^n$, then

$$|\{x \in B : \frac{|f(x) - f_B|}{M^{\sharp} f(x)} > \lambda\}| \lesssim e^{-c\,\lambda} |B|.$$

Our first main result, Theorem 1.1, improves this exponential decay in several ways. On one hand, we have the decay for the local maximal function and on the other hand, we obtain weighted estimates. Our method of proof is different from that of [Kar].

To state our first main result we set the following notation for $r > 1$ and for any cube Q

$$w_r(Q) = |Q| \left(\fint_Q w(x)^r dx \right)^{\frac{1}{r}} = |Q|^{\frac{1}{r'}} \left(\int_Q w(x)^r dx \right)^{\frac{1}{r}}.$$

For convenience, if w vanishes on a cube Q, we set $\frac{1}{w_r(Q)} \int_Q F(x)w(x)dx = 0$ for any function F. Also, we will denote by M_Q to the local dyadic maximal operator over Q, see Section 3 for the precise definition, and $w(Q) = w_1(Q) = \int_Q w dx$.

Theorem 1.1. *Let f be a locally integrable function and w a non-negative weight. Then for any cube Q, for any $1 \leq p < \infty$ and $1 < r < \infty$, the following estimate holds*

$$(1.3) \qquad \left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \right)^p w(x)dx \right)^{\frac{1}{p}} \leq c_n \, pr'.$$

Hence, if further $w \in A_\infty$ we have

$$(1.4) \qquad \left(\frac{1}{w(Q)} \int_Q \left(\frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \right)^p w(x)dx \right)^{\frac{1}{p}} \leq c_n \, p \, [w]_{A_\infty}.$$

In Section 7 we will extend this result to the context of polynomial type BMO.

Remark 1.2. We point out that we do not know how to extend this result into the context of spaces of homogeneous type since it is not clear if our method works. However, we believe that the result should hold in that context since it is well known that one of its consequences, namely the classical John–Nirenberg theorem (1.1), is true in spaces of homogeneous type. See the next corollary and the subsequent remarks.

Remark 1.3. The corresponding result replacing the L^p norm by the (larger) Lorentz norm $L^{p,q}$ with $1 \leq q < p$ cannot be proved even in the simplest situation $w = 1$ and without M.

Remark 1.4. The factor p in (1.3) (or (1.4)) is crucial since it yields the exponential type result as follows.

As a direct corollary of Theorem 1.1 and Lemma 2.3, we have the following exponential estimates.

Corollary 1.5. *Let f be a locally integrable function. Then we have*
  - *Improved John–Nirenberg estimate:*

$$\left\| \frac{M_Q(f - f_Q)}{M^\sharp f} \right\|_{\exp L(Q, \frac{w dx}{w(Q)})} \leq c_n \, [w]_{A_\infty}$$

  meaning that there exist dimensional constants $c_1, c_2 > 0$ such that

$$w \left(\left\{ x \in Q : \frac{M_Q(f - f_Q)}{M^\sharp f(x)} > t \right\} \right) \leq c_1 \, e^{-c_2 \, t/[w]_{A_\infty}} \, w(Q), \quad t > 0.$$

  - *For every cube and $\lambda, \gamma > 0$ we have the following good-λ type inequality*

$$w \big(\{ x \in Q : M_Q(f - f_Q) > \lambda, M^\sharp f(x) \leq \gamma\lambda \} \big) \leq c_1 \, e^{\frac{-c_2}{\gamma[w]_{A_\infty}}} \, w(Q).$$

We call this result improved John–Nirenberg estimate because if $w = 1$ and $f \in$ BMO, then $M^\sharp f(x) \leq \|f\|_{\text{BMO}}$ for a.e. x and, therefore,

$$|\{ x \in Q : M_Q(f - f_Q) > t \}| \leq c_1 \, e^{\frac{-c_2 \, t}{\|f\|_{\text{BMO}}}} \, |Q|, \quad t > 0.$$

This implies the JN theorem (1.1) by Lebesgue differentiation theorem, because $M_Q(f - f_Q) \geq f - f_Q$ a.e. in Q.

  - *Generalized Poincaré inequalities: sharp cuantitative A_∞ bounds.* As a first application of Theorem 1.1, we improve the main result in [FPW] (at least in the simplest situation of cubes) which at the same time provides a limiting result that could not be treated in Theorem 1.14 of [PR].

Let w be an A_∞ weight and let a be a functional over cubes of $\mathbb{R}^n$. We will assume that a satisfies the $D_r(w)$ condition for some $r > 1$ as introduced in [FPW]. More precisely, for every cube Q and every collection Λ of pairwise disjoint subcubes of Q, the following inequality holds:

$$(1.5) \qquad \sum_{P \in \Lambda} w(P)\, a(P)^r \leq \|a\|^r w(Q) a(Q)^r,$$

for some constant $\|a\| > 0$ that plays the role of the "norm" of a.

These kind of functionals were studied in relation with self improvement properties of generalized Poincaré inequalities in [FPW], further studied in [MP] and more recently improved in [PR]. We establish now an new endpoint result in the spirit of Theorem 1.14 in [PR] which was missing since Theorem 1.1 was not available.

Theorem 1.6. *Let $w \in A_\infty$ and a a functional satisfying* (1.5). *Let f be a locally integrable function such that for every cube Q,*

$$(1.6) \qquad \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq a(Q).$$

Then, for every cube Q,

$$\|f - f_Q\|_{L^{r,\infty}\left(Q, \frac{w}{w(Q)}\right)} \leq c_n\, r\, [w]_{A_\infty} \|a\|\, a(Q).$$

Remark 1.7. The method in [FPW], based on the good-λ method of Burkholder–Gundy [BG], yields an exponential bound in $[w]_{A_\infty}$. We still use here the good-λ method but we use instead Corollary 1.5.

• *The C_p condition.* As a second application of Theorem 1.1 we provide an improvement of a theorem of K. Yabuta [Yab] concerning a classical inequality of Fefferman–Stein relating the usual (uncentered, cubic) Hardy–Littlewood maximal function M defined by

$$Mh(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |h(y)| dy,$$

where the supremum is taken over all cubes R containing x, and the sharp maximal function $M^\sharp$ defined in (1.2) and introduced in [FS].

This result of Yabuta considers the C_p class of weights which is a larger class of weights than the A_∞ class of Muckenhoupt (see Section 2).

Theorem 1.8 (Quantitative norm inequality)**.** *Let $1 < p < q < \infty$ and $w \in C_q$. Then for any $f \in L_c^\infty(\mathbb{R}^n)$ we have*

$$\|Mf\|_{L^p(w)} \leq c_n \frac{pq}{q-p} \max\{1, [w]_{C_q} \log^+[w]_{C_q}\} \|M^\sharp f\|_{L^p(w)},$$

where the constant c_n only depends on n.

Remark 1.9. We remark that, as a consequence of Corollary 1.5, we can also obtain the following weighted inequality for A_∞ weights by standard arguments:

$$\|Mf\|_{L^p(w)} \leq c[w]_{A_\infty} \|M^\sharp f\|_{L^p(w)}, \quad 0 < p < \infty.$$

This inequality is not new, see for example [L06].

Very recently, A. Lerner [L19] proved a characterization of weights satisfying a weak Fefferman–Stein inequality

$$\|f\|_{L^{p,\infty}(w)} \leq C\|M^\sharp f\|_{L^p(w)}.$$

The weights satisfying this inequality are of a different class of weights, called SC_p (strong C_p). This class is contained in C_p and contains $C_{p+\varepsilon}$ for every $\varepsilon > 0$.

Theorem 1.8 has a straight application to the wide class of operators described in [CLPR]. Indeed, we say that an operator T satisfies the (D) property if there are some constants $\delta \in (0, 1)$ and $c > 0$ such that for all f,

$$(D) \qquad M_\delta^\sharp(Tf)(x) \leq cMf(x), \quad a.e.\ x.$$

Here M denotes the standard Hardy–Littlewood maximal operator and $M_\delta^\sharp f = M^\sharp(f^\delta)^{\frac{1}{\delta}}$. This property is modeled by a result in [AP] where (D) was proved for any Calderón–Zygmund operator. It also holds for some square function operators and some pseudo-differential operators. The

multilinear Calderón–Zygmund operators version was obtained in [LOPTT]. There is a more exhaustive list in [CLPR].

Corollary 1.10. *Let $1 < p < q < \infty$ and T be an operator that satisfies the property* (D) *with constant C_T for some $\frac{p}{q} < \delta < 1$. Then for $w \in C_q$ we have*

$$\|Tf\|_{L^p(w)} \le c_n \, C_T \, \left(\frac{pq}{\delta q - p} \, \max\{1, [w]_{C_q} \log^+[w]_{C_q}\} \right)^{\frac{1}{\delta}} \|Mf\|_{L^p(w)}.$$

1.2. **Weighted mean oscillation.** The second extension of the JN theorem we consider in this paper is motivated by the following classical result of Muckenhoupt and Wheeden in [MW].

Following the language in [MW], a function f is said to be bounded mean oscillation with weight w if there exists some $C > 0$ such that for every cube Q,

$$\int_Q |f(x) - f_Q|dx \le C \, w(Q).$$

This class of functions is interesting because it is connected to the theory of weighted Hardy spaces [GC] and to the context of extrapolation [HMS] (see more details and a new proof in [CPR]).

Theorem (Muckenhoupt–Wheeden). *Let $1 \le p < \infty$ and $w \in A_p$. Then f is of bounded mean oscillation with weight w if and only if for every $1 \le r < \infty$ satisfying $1 \le r \le p'$, there exists a constant such that, for all cubes Q,*

$$(1.7) \qquad \int_Q |f(x) - f_Q|^r w(x)^{1-r} dx \le c \, w(Q).$$

As was shown in [MW], the range $1 \le r \le p'$ is optimal, since for any given $p > 1$ there exist f, w for which $w \in A_q$ for all $q > p$ but (1.7) fails for $r = p'$.

In [OPRRR] the authors obtained a mixed-type A_p–A_∞ quantitative estimate of inequality (1.7). Here we are going to improve Theorem 1.7 from that paper, using a simplified and more transparent argument that avoids completely the use of sparse domination.

The main idea is closely related to the proof of Theorem 1.1 above.

For a weight w and $p > 1, r \ge 1$, we define the following bumped A_p constant

$$[w]_{A_p^r} = \sup_Q \left(\fint_Q w^r \right)^{\frac{1}{r}} \left(\fint_Q w^{1-p'} \right)^{p-1}.$$

Note that $[w]_{A_p} \le [w]_{A_p^r}$ for $r \ge 1$.

Theorem 1.11. *Let $p, r > 1$, w such that $[w]_{A_p^r} < \infty$ and let f locally integrable such that*

$$\|f\|_{\mathrm{BMO}_{w,r}} := \sup_Q \frac{1}{w_r(Q)} \int_Q |f(x) - f_Q|dx < \infty.$$

Then we have the estimate

$$(1.8) \qquad \left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f(x) - f_Q|}{w(x)} \right)^{p'} w(x)dx \right)^{\frac{1}{p'}} \le c_n p' [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \|f\|_{\mathrm{BMO}_{w,r}}.$$

Corollary 1.12. *Let $f \in \mathrm{BMO}_{w,1}$, namely $\sup_Q \frac{1}{w(Q)} \int_Q |f - f_Q| < \infty$.*

(1) *If $w \in A_1$ we have that for every $q > 1$,*

$$\left(\frac{1}{w(Q)} \int_Q \left| \frac{f(x) - f_Q}{w(x)} \right|^q w(x)dx \right)^{\frac{1}{q}} \le c_n q \, [w]_{A_1}^{\frac{1}{q'}} [w]_{A_\infty}^{\frac{1}{q}} \|f\|_{\mathrm{BMO}_{w,1}},$$

and hence for any cube Q

$$(1.9) \qquad \left\| \frac{f - f_Q}{w} \right\|_{\exp L\left(Q, \frac{w(x)dx}{w(Q)}\right)} \le c_n \, [w]_{A_1} \|f\|_{\mathrm{BMO}_{w,1}}.$$

(2) *If $w \in A_p$ with $1 < p < \infty$ then,*

$$\left(\frac{1}{w(Q)} \int_Q \left| \frac{f(x) - f_Q}{w(x)} \right|^{p'} w(x)dx \right)^{\frac{1}{p'}} \le c_n \, p' \, [w]_{A_p}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{p'}} \|f\|_{\mathrm{BMO}_{w,1}}.$$

Remark 1.13. It follows from (1.9) that for any $t > 0$ then

$$w(\{x \in Q : |f(x) - f_Q| > t\, w(x)\}) \leq 2e^{-\frac{t}{c_n[w]_{A_1}\|f\|_{\mathrm{BMO}_{w,1}}}}\, w(Q).$$

We extend this result to the context of polynomial type BMO in Section 7. It is not clear how to obtain these new polynomial BMO type estimates from the method in [OPRRR].

2. Some preliminaries and notation

A weight is a non-negative, locally integrable function, which will be denoted by w. Although usually weights are assumed to be positive a.e., we will allow them to vanish on a set of positive measure. Abusing notation, we may identify the weight with the measure it defines, and use the symbol w for both. For $1 \leq p < \infty$, we say that $w \in A_p$ if w is positive a.e. and if $[w]_{A_p} < \infty$, where

$$[w]_{A_1} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)},$$

$$[w]_{A_p} = \sup_Q \left(\fint_Q w\right) \left(\fint_Q w^{1-p'}\right)^{p-1}, \quad 1 < p < \infty.$$

The A_∞ class is defined as the union of all other A_p classes, that is,

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p.$$

The A_∞ constant is defined as

$$[w]_{A_\infty} = \sup_Q \frac{\int_Q M(w\chi_Q)}{w(Q)}.$$

We state the sharp reverse Hölder inequality for A_∞ weights obtained in [HP] (see also [HPR]).

Theorem 2.1 (Sharp RHI, [HP]). *Let $w \in A_\infty$ and $\delta = \frac{1}{c_n[w]_{A_\infty}}$. Then for every cube Q, we have*

$$\left(\fint_Q w^{1+\delta}\right)^{\frac{1}{1+\delta}} \leq 2 \fint_Q w.$$

For $1 < p < \infty$, the C_p class of weights is defined as the weights w for which there exist C, ε such that for every cube Q and $E \subset Q$,

$$w(E) \leq C \left(\frac{|E|}{|Q|}\right)^{\varepsilon} \int_{\mathbb{R}^n} (M\chi_Q)^p w.$$

Weights of C_p class may vanish on a set of positive measure, for example $w = \chi_{(0,\infty)} \in C_p$ for all $1 < p < \infty$, see [Muc]. The C_p constant was defined in [Can] as

$$[w]_{C_p} = \sup_Q \frac{\int_Q M(w\chi_Q)}{\int_{\mathbb{R}^n} (M\chi_Q)^p w}.$$

These weights also satisfy a sharp RHI as found in [Can]:

Theorem 2.2 (Sharp RHI, [Can]). *Let $1 < p < \infty$, $w \in C_p$ and $\delta = \frac{1}{c_{n,p}([w]_{C_p}+1)}$. Then for every cube Q, we have*

$$\left(\fint_Q w^{1+\delta}\right)^{\frac{1}{1+\delta}} \leq \frac{2}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w.$$

Finally, we state the following key lemma concerning exponential integrability.

Lemma 2.3. *Suppose that (X, μ) is a probability space and f a non-negative function such that for every $1 < p < \infty$ we have the L^p bound*

$$\left(\int_X f(x)^p d\mu(x)\right)^{\frac{1}{p}} \leq \gamma\, p,$$

for some constant γ independent from p. Then $f \in \exp(L)(X, \mu)$, meaning

$$\mu(\{x \in X : f(x) > t\}) \leq e^{-\frac{t}{4\gamma}}, \quad t > 0.$$

Sketch of the proof. It suffices to note that

$$\mu(\{x \in X : f(x) > t\}) = \mu(\{x \in X : \frac{f(x) - t}{4\gamma} > 0\}) \le e^{-\frac{t}{4\gamma}} \int_X \exp\left(\frac{f(x)}{4\gamma}\right) d\mu(x),$$

and use the Taylor expansion of the exponential together with the hypothesis. $\qquad\square$

3. KARAGULYAN'S RESULT REVISITED

In this section we give a new inequality that can be seen as an extension of the JN theorem that requires minimal hypothesis on the function. This new inequality concerns the dyadic maximal operator and the sharp maximal operator. Given a cube Q we define M_Q, the *localized dyadic maximal operator*, acting on a function h by

$$M_Q h(x) = \sup_{\substack{R \in \mathcal{D}(Q) \\ x \in R}} \frac{1}{|R|} \int_R |h|.$$

Here $\mathcal{D}(Q)$ denotes the collection of all dyadic descendants of Q. We don't need that Q belongs to any particular dyadic family, even though the supremum is taken over the dyadic collection generated by Q.

Proof of Theorem 1.1. The main idea for the proof comes from [PR]. Fix a cube Q and consider the following function on Q,

$$F(x) = \frac{|f(x) - f_Q|}{\operatorname{osc}(f, Q)}$$

where we use the notation

$$\operatorname{osc}(f, Q) = \fint_Q |f - f_Q|.$$

Observing that $\fint_Q F(x)\, dx = 1$ we can consider the standard Calderón–Zygmund decomposition of F in Q at height $\lambda > 1$ that will be chosen later (see [Whe, Lemma 14.55] for a thorough description of the decomposition.) The choice of λ will only depend on p and r'. Then, there is a family of dyadic pairwise disjoint subcubes $\{Q_j\}$ with respect to Q, which satisfy the following properties:

- $\operatorname{osc}(f, Q)\lambda < \fint_{Q_j} |f - f_Q| \le 2^n \lambda \operatorname{osc}(f, Q)$,
- $\sum_j |Q_j| \le \frac{|Q|}{\lambda}$,
- For $x \notin \cup_j Q_j$, $\quad M_Q(f - f_Q)(x) \le \lambda \operatorname{osc}(f, Q)$.

The first two properties follow from the stopping time and the maximality. To prove the third one, note that $\fint_R |f - f_Q| \le \lambda \operatorname{osc}(f, Q)$ for all dyadic R that contains x.

Now, by maximality of the cubes, for $x \in Q_j$ we can localize the maximal function in the following way

$$(3.1) \qquad M_Q(f - f_Q)(x) = M_{Q_j}(f - f_Q)(x) \le M_{Q_j}(f - f_{Q_j})(x) + |f_Q - f_{Q_j}|.$$

Moreover, for $x \in Q_j$, we have

$$(3.2) \qquad \frac{|f_Q - f_{Q_j}|}{M^\sharp f(x)} = \frac{|\fint_{Q_j}(f - f_Q)|}{M^\sharp f(x)} \le \frac{\fint_{Q_j} |f - f_Q|}{\operatorname{osc}(f, Q)} \le 2^n \lambda,$$

by the Calderón–Zygmund decomposition. Thus, we have found the following pointwise bound, for a.e. $x \in Q$,

$$\frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} = \frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \chi_{Q \setminus \cup_j Q_j}(x) + \sum_j \frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \chi_{Q_j}(x)$$

$$\le \lambda \chi_{Q \setminus \cup_j Q_j}(x) + \sum_j \left(\frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} + \frac{|f_Q - f_{Q_j}|}{M^\sharp f(x)} \right) \chi_{Q_j}(x)$$

$$\le \lambda \chi_{Q \setminus \cup_j Q_j}(x) + \sum_j \left(\frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} + 2^n \lambda \right) \chi_{Q_j}(x)$$

$$\le 2^n \lambda + \sum_j \frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} \chi_{Q_j}(x).$$

We have used (3.1) and (3.2) in the first and second inequalities respectively.

Now we compute the norm. Using the triangle inequality, Jensen's inequality and the fact that the Q_j are pairwise disjoint, we get

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \right)^p w(x) dx \right)^{\frac{1}{p}}$$

$$\leq 2^n \lambda + \left(\sum_j \frac{w_r(Q_j)}{w_r(Q)} \frac{1}{w_r(Q_j)} \int_{Q_j} \left(\frac{M_{Q_j}(f - f_{Q_j}(x)}{M^\sharp f(x)} \right)^p w(x) dx \right)^{\frac{1}{p}}$$

$$\leq 2^n \lambda + X \left(\frac{1}{w_r(Q)} \sum_j w_r(Q_j) \right)^{\frac{1}{p}}$$

$$\leq 2^n \lambda + \frac{X}{\lambda^{\frac{1}{pr'}}}.$$

We have used that, by Hölder's and one of the main properties of the family Q_j,

$$\sum_j w_r(Q_j) \leq w_r(Q) \left(\frac{1}{\lambda} \right)^{\frac{1}{r'}},$$

and we have set

$$X = \sup_{R \in \mathcal{D}} \left(\frac{1}{w_r(R)} \int_R \left(\frac{M_R(f - f_R)(x)}{M^\sharp f(x)} \right)^p w(x) dx \right)^{\frac{1}{p}}.$$

Now take the supremum over all dyadic cubes Q and obtain, for arbitrary $\lambda > 1$,

$$X \leq 2^n \lambda + \frac{X}{\lambda^{\frac{1}{pr'}}}.$$

This in turn implies, if we assume $X < \infty$, that

$$X \leq 2^n \lambda \frac{\lambda^{\frac{1}{pr'}}}{\lambda^{\frac{1}{pr'}} - 1}.$$

Minimizing over $\lambda > 1$, we see $X \leq c_n \, pr'$. This finishes the proof if we assume that $X < \infty$.

In order to remove the hypothesis $X < \infty$, it is enough to work with

$$X_{\varepsilon, K} := \sup_{Q \in \mathcal{D}} \left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f_K - (f_K)_Q)}{M^\sharp(f_K) + \varepsilon} \right)^p w \right)^{\frac{1}{p}} \leq 2 \frac{K}{\varepsilon} < \infty$$

for a suitable truncation f_K of f at height K. For example, one can take

$$f_K(x) = \begin{cases} -K, & f(x) < -K, \\ f(x), & -K \leq f(x) \leq K, \\ K, & K < f(x). \end{cases}$$

Making the same computations as above with some trivial changes, we can obtain the bounds for $X_{\varepsilon, K}$ independently of ε and K. Finally, monotone convergence finishes the argument. This ends the proof of (1.3).

To prove (1.4) we assume $w \in A_\infty$. Let us choose $r = 1 + \delta$ with δ as in Theorem 2.1; thus, $w_r(Q) \leq 2w(Q)$ and $r' \lesssim [w]_{A_\infty}$. From this observation, estimate (1.4) in Theorem 1.1 follows. $\square$

Remark 3.1. Since throughout the proof the only cubes that appear are dyadic descendants of Q, we actually obtain the estimate

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f - f_Q)(x)}{M_Q^\sharp f(x)} \right)^p w(x) dx \right)^{\frac{1}{p}} \leq c_n \, pr',$$

where $M_Q^\sharp$ is the sharp operator taking the supremum over dyadic descendants of Q. Since $M_Q^\sharp \leq M^\sharp$, this last estimate is stronger.

4. Weighted local mean oscillation

In this section we provide the proof of Theorem 1.11. Recall that for $p, r > 1$ we use the following notation $w_r(Q) = |Q| \left(\fint_Q w^r \right)^{\frac{1}{r}}$ and $[w]_{A_p^r} = \sup_Q \left(\fint_Q w^r \right)^{\frac{1}{r}} \left(\fint_Q w^{1-p'} \right)^{p-1}$.

Proof of Theorem 1.11. For a fixed a cube Q, we have to prove

$$(4.1) \qquad \left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f(x) - f_Q|}{w(x)} \right)^{p'} w(x) dx \right)^{\frac{1}{p'}} \leq c_n p' [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \|f\|_{\mathrm{BMO}_{w,r}}.$$

We recall that $\|f\|_{\mathrm{BMO}_{w,r}} := \sup_Q \frac{1}{w_r(Q)} \int_Q |f - f_Q|$. We may suppose by homogeneity that $\|f\|_{\mathrm{BMO}_{w,r}} = 1$. Hence, if we let $L > 1$, to be chosen later, we can choose, as in the proof of Theorem 1.1, a family of maximal subcubes $\{Q_j\}$ in Q such that

$$(4.2) \qquad \frac{1}{w_r(Q_j)} \int_{Q_j} |f - f_Q| > L.$$

Observe that if the family is empty we can see that $|f(x) - f_Q| \leq Lw(x)$ a. e. $x \in Q$ and the result is trivial. Also since $\|f\|_{\mathrm{BMO}_{w,r}} = 1$, we have that Q is not one of the selected cubes. One can check that, if Q'_j denotes the ancestor of Q_j, the following properties hold:

- $\frac{1}{w_r(Q'_j)} \int_{Q'_j} |f - f_Q| \, dx \leq L$,
- $|f_{Q_j} - f_Q| \leq 2^n L \left(\fint_{Q'_j} w^r \right)^{\frac{1}{r}}$,
- $\sum_j w_r(Q_j) \leq \frac{w_r(Q)}{L}$ by (4.2) and since since $\|f\|_{BMO_{w,r}} = 1$,
- $|f(x) - f_Q| \leq Lw(x)$ for a.e. $x \notin \cup_j Q_j$.

Using the disjointness, we have for *a.e.* $x \in Q$,

$$f(x) - f_Q = (f(x) - f_Q)\chi_{(\cup_j Q_j)^c}(x) + \sum_j (f_{Q_j} - f_Q)\chi_{Q_j}(x) + \sum_j (f(x) - f_{Q_j})\chi_{Q_j}(x).$$

Since $p' > 1$ we can use the triangular inequality to get

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f - f_Q|}{w} \right)^{p'} w \right)^{\frac{1}{p'}} \leq \left(\frac{1}{w_r(Q)} \int_{(\cup_j Q_j)^c} \left(\frac{|f - f_Q|}{w} \right)^{p'} w \right)^{\frac{1}{p'}}$$

$$+ \left(\frac{1}{w_r(Q)} \sum_j \int_{Q_j} \left(\frac{|f_{Q_j} - f_Q|}{w} \right)^{p'} w \right)^{\frac{1}{p'}}$$

$$+ \left(\frac{1}{w_r(Q)} \sum_j \int_{Q_j} \left(\frac{|f - f_{Q_j}|}{w} \right)^{p'} w \right)^{\frac{1}{p'}}$$

$$= A_1 + A_2 + B.$$

Now, since $w(Q) \leq w_r(Q)$ the first term is $A_1 \leq L$. To bound B we denote

$$X = \sup_R \left(\frac{1}{w_r(R)} \int_R \left(\frac{|f - f_R|}{w} \right)^{p'} w \right)^{\frac{1}{p'}},$$

and use that $\sum_j w_r(Q_j) \leq \frac{w_r(Q)}{L}$, the third property of the family of the cubes $\{Q_j\}$, to obtain:

$$B \leq X \left(\frac{1}{w_r(Q)} \sum_j w_r(Q_j) \right)^{\frac{1}{p'}} \leq X \left(\frac{1}{L} \right)^{\frac{1}{p'}}.$$

The argument for bounding A_2 is more delicate. We start the computations:

$$A_2 = \left(\frac{1}{w_r(Q)} \sum_j \int_{Q_j} |f_{Q_j} - f_Q|^{p'} w^{p'-1} \right)^{\frac{1}{p'}}$$

$$\leq 2^n L \left(\frac{1}{w_r(Q)} \sum_j \left(\frac{1}{|Q'_j|} \int_{Q'_j} w^r \right)^{\frac{p'}{r}} \int_{Q_j} w^{p'-1} \right)^{\frac{1}{p'}}$$

$$\leq 2^n \ L \ \left(\frac{1}{w_r(Q)} \sum_j w_r(Q'_j) \left(\frac{1}{|Q'_j|} \int_{Q'_j} w^r \right)^{\frac{p'-1}{r}} \left(\frac{1}{|Q'_j|} \int_{Q_j} w^{p'-1} \right)^{(p'-1)(p-1)} \right)^{\frac{1}{p'}}$$

$$\leq 2^n \ L \ [w]_{A_p^r}^{\frac{1}{p}} \left(\frac{1}{w_r(Q)} \sum_j |Q'_j| \left(\frac{1}{|Q'_j|} \int_{Q'_j} w^r \right)^{\frac{1}{r}} \right)^{\frac{1}{p'}}.$$

In order to bound the term in the sum, we recall the following result by Kolmogorov. If (X, μ) is a probability space, then for $\varepsilon < 1$

$$\|g\|_{L^\varepsilon(X)} \leq \left(\frac{1}{1-\varepsilon} \right)^{\frac{1}{\varepsilon}} \|g\|_{L^{1,\infty}(X)}.$$

We have

$$\sum_j |Q'_j| \left(\frac{1}{|Q'_j|} \int_{Q'_j} w^r \right)^{\frac{1}{r}} \leq 2^n \sum_j |Q_j| \inf_{z \in Q_j} M_Q(w^r \chi_Q)(z)^{\frac{1}{r}}$$

$$\leq 2^n |Q| \fint_Q M_Q(w^r \chi_Q)^{\frac{1}{r}}$$

$$\leq 2^n \frac{1}{1 - \frac{1}{r}} \|M_Q(w^r \chi_Q)\|_{L^{1,\infty}(Q, \frac{dx}{|Q|})}^{\frac{1}{r}} |Q|$$

$$\leq 2^n r' |Q| \left(\fint_Q w^r \right)^{\frac{1}{r}}$$

$$= 2^n \ r' \ w_r(Q),$$

where, as before, M_Q is the local dyadic maximal operator over Q whose weak-type $(1,1)$ bound is one. Thus, we have the bound

$$A_2 \leq 2^n [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} L.$$

Combining the bounds for A_1, A_2 and B, we have for every cube Q and $L > 1$

(4.3) $$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f - f_Q|}{w} \right)^{p'} w \right)^{\frac{1}{p'}} \leq L + 2^n [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} L + X \left(\frac{1}{L} \right)^{\frac{1}{p'}},$$

and thus, for each L,

$$X \leq 2^{n+1} [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} L + X \left(\frac{1}{L} \right)^{\frac{1}{p'}}.$$

Hence, if we assume $X < \infty$,

$$X \leq c_n \, p' [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}}.$$

This finishes the proof in the case that $X < \infty$. In order to remove the hypothesis $X < \infty$, it is enough to replace first for each cube Q

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f - f_Q|}{w} \right)^{p'} w \right)^{\frac{1}{p'}}$$

by

$$\left(\frac{1}{w_r(Q)} \int_Q \min\left\{ \frac{|f - f_Q|}{w}, m \right\}^{p'} w \right)^{\frac{1}{p'}}.$$

The argument done before works exactly to get the following variant of (4.3): For every $L > 1$ and $m \geq 1$,

$$\left(\frac{1}{w_r(Q)}\int_Q \min\left\{\frac{|f - f_Q|}{w}, m\right\}^{p'} w\right)^{\frac{1}{p'}} \leq L + 2^n [w]_{A_p^r}^{\frac{1}{p}}(r')^{\frac{1}{p'}} L + X_m \left(\frac{1}{L}\right)^{\frac{1}{p'}},$$

where now, instead of X we have X_m defined by

$$X_m := \sup_{Q \in \mathcal{D}} \left(\frac{1}{w_r(Q)}\int_Q \min\left\{\frac{|f - f_Q|}{w}, m\right\}^{p'} w\right)^{\frac{1}{p'}}, \qquad m \geq 1.$$

Then, $X_m \leq 2^{n+1} [w]_{A_p^r}^{\frac{1}{p}}(r')^{\frac{1}{p'}} L + X_m \left(\frac{1}{L}\right)^{\frac{1}{p'}}$, $L > 1$, $m \geq 1$. Therefore, since $X_m \leq m$

$$X_m \leq c_n\, p'[w]_{A_p^r}^{\frac{1}{p}}(r')^{\frac{1}{p'}} \qquad m \geq 1.$$

Hence for each cube Q

$$\left(\frac{1}{w_r(Q)}\int_Q \min\left\{\frac{|f - f_Q|}{w}, m\right\}^{p'} w\right)^{\frac{1}{p'}} \leq c_n\, p'[w]_{A_p^r}^{\frac{1}{p}}(r')^{\frac{1}{p'}} \qquad m \geq 1.$$

Finally, let $m \to \infty$ to finish the proof. $\qquad\square$

Proof of Corollary 1.12. Part (1) follows from part (2) since $[w]_{A_1} \geq [w]_{A_p}$, $p > 1$. In order to prove part (2), choose $r = 1 + \delta$ with δ as in Theorem 2.1. This way $[w]_{A_p^r} \leq 2[w]_{A_p}$, $r' \lesssim [w]_{A_\infty}$ and $w_r(Q) \leq 2w(Q)$. The result follows from Theorem 1.11. $\qquad\square$

5. Application to C_p weights

In this section we will prove Theorem 1.8, namely we give a quantitative weighted norm inequality between the Hardy–Littlewood maximal operator and the Fefferman–Stein maximal function, for weights in class C_p. We are going to use the improved John–Nirenberg Theorem 1.1 to give a quantitative version of Theorem II in [Yab].

Before proving Theorem 1.8, we need to obtain a non-dyadic unweighted version of Corollary 1.5.

Theorem 5.1. *Let Q be an arbitrary cube and f a locally integrable function, non constant on Q. Then for any $\lambda > 0$ we have*

$$\left|\left\{x \in Q : \frac{M\big((f - f_Q)\chi_Q\big)(x)}{M^\sharp f(x)} > \lambda\right\}\right| \leq Ce^{-c\lambda}|Q|,$$

where $C, c > 0$ are dimensional constants. Here M denotes the standard Hardy–Littlewood maximal operator.

We will use a variant of a result from [Con], which will allow us to obtain the general case from the dyadic setting.

Lemma 5.2. *Let $Q \subset \mathbb{R}^n$ be a cube. Then there exist $n + 1$ dyadic systems $\{\mathcal{A}_j\}_{j=0}^n$ and $n + 1$ cubes, $Q_j \in \mathcal{A}_j$ such that the following two conditions are satisfied:*
  (1) *$Mf(x) \leq c_n \sum_{j=0}^n M_j f(x)$ a.e. for any function f, where M_j is the dyadic maximal function with respect to the dyadic system $\mathcal{A}_j$, $j = 0, ..., n$.*
  (2) *$Q \subset \cap_{j=0}^n Q_j$ and $|Q| \simeq |Q_j|$ for all j.*

Proof of Theorem 5.1. Fix the cube Q and the function f, and choose $Q_0, ..., Q_n$ and $\mathcal{A}_0, ..., \mathcal{A}_n$ as in Lemma 5.2. We have

$$\left|\left\{x \in Q : \frac{M\big((f - f_Q)\chi_Q\big)}{M^\sharp f} > \lambda\right\}\right| \leq \sum_{j=0}^n \left|\left\{x \in Q : \frac{M_j\big((f - f_Q)\chi_Q\big)}{M^\sharp f} > \frac{\lambda}{n+1}\right\}\right|$$

$$\leq \sum_{j=0}^n \left|\left\{x \in Q_j : \frac{M_j\big((f - f_Q)\chi_{Q_j}\big)}{M^\sharp f} > \frac{\lambda}{n+1}\right\}\right|$$

$$= \sum_{j=0}^n \left|\left\{x \in Q_j : \frac{M_{Q_j}(f - f_Q)}{M^\sharp f} > \frac{\lambda}{n+1}\right\}\right|.$$

Now, since Q and Q_j have comparable size, we have for $x \in Q_j$,

$$\frac{|f_Q - f_{Q_j}|}{M^\sharp f(x)} \leq \frac{|Q_j|}{|Q|} \frac{\fint_{Q_j} |f - f_{Q_j}|}{\fint_{Q_j} |f - f_{Q_j}|} \leq c_n.$$

So, for $\frac{\lambda}{n+1} \geq c_n$ we get for each j,

$$\left| \left\{ x \in Q_j : \frac{M_{Q_j}(f - f_Q)}{M^\sharp f} > \frac{\lambda}{n+1} \right\} \right| \leq \left| \left\{ x \in Q_j : \frac{M_{Q_j}(f - f_{Q_j})}{M^\sharp f} + c_n > \frac{\lambda}{n+1} \right\} \right|$$

$$\leq \left| \left\{ x \in Q_j : \frac{M_{Q_j}(f - f_{Q_j})}{M^\sharp f} > \frac{\lambda}{n+1} - c_n \right\} \right|$$

$$\leq Ce^{-c(\lambda - c_n)}|Q_j| \leq Ce^{-c(\frac{\lambda}{n+1} - c_n)}|Q|.$$

This finishes the proof for $\frac{\lambda}{n+1} > c_n$. The other case follows since in that case $e^{-\lambda}$ is bounded from bellow. $\qquad \square$

We now give the key estimate, which is a good-λ estimate between M and $M^\sharp$ with exponential decay. We will use a Whitney decomposition of open sets by cubes (see [Gr], Appendix J.)

Proposition 5.3. *Let f be a bounded function with compact support and let $\lambda > 0$. Consider the open set $\Omega_\lambda = \{Mf(x) > \lambda\}$ and let $\{Q_j\}_j$ be a Whitney decomposition of Ω_λ. Then for any Q_j and $\gamma > 0$,*

$$|\{x \in Q : Mf(x) > 4^n \lambda, M^\sharp f(x) \leq \gamma\lambda\}| \leq Ce^{-\frac{c}{\gamma}}|Q|.$$

Proof. Let $\overline{Q}$ be the multiple of Q such that $\overline{Q} \cap (\Omega_\lambda)^c \neq \emptyset$, as in the Whitney decomposition. We prove that if $x \in Q$ satisfies $Mf(x) > 4^n \lambda$ and $M^\sharp f(x) \leq \gamma\lambda$ then

$$(5.1) \qquad\qquad \frac{M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x)}{M^\sharp f(x)} > \frac{1}{\gamma}.$$

Then we can directly apply Theorem 5.1 and we will be done.

Because of the Whitney decomposition, $Mf(x) > 4^n \lambda$ implies $M(f\chi_{\overline{Q}})(x) > 4^n \lambda$. Also as a consequence of the Whitney decomposition, $|f|_{\overline{Q}} \leq \lambda$, so

$$4^n \lambda \leq M(f\chi_{\overline{Q}})(x) \leq M((f - f_{\overline{Q}})\chi_{\overline{Q}}) + |f|_{\overline{Q}} \leq M((f - f_{\overline{Q}})\chi_{\overline{Q}}) + \lambda,$$

which implies $M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x) > \lambda$. This proves (5.1). Therefore we have

$$|\{x \in Q : Mf(x) > 4^n \lambda, M^\sharp f(x) \leq \gamma\lambda\}|$$

$$\leq |\{x \in Q : M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x) > 4^n \lambda, M^\sharp f(x) \leq \gamma\lambda\}|$$

$$\leq \left| \left\{ x \in \overline{Q} : \frac{M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x)}{M^\sharp f(x)} > \frac{1}{\gamma} \right\} \right|$$

$$\leq ce^{-\frac{1}{c\gamma}}|\overline{Q}|$$

This ends the proof, since Q and $\overline{Q}$ have comparable size. $\qquad \square$

Now we prove Theorem 1.8. The proof follows mainly the one in [Yab], but we use the good-λ inequality from Proposition 5.3. We also keep an eye for the dependence on the constant of the weight, which is in fact our main objective.

Proof of Theorem 1.8. We may assume, arguing as in [Yab], that both norms are finite. Define $\Omega_k = \{x \in \mathbb{R}^n : Mf(x) > 2^k\}$ for $k \in \mathbb{Z}$. Let $\{Q_j\}_j$ be a Whitney decomposition of Ω_λ. In particular, $\Omega_k = \bigcup_j Q_j^k$ and the cubes $\{Q_j^k\}$ are pairwise disjoint. By Proposition 5.3 we have

$$|\{x \in Q_j^k : Mf(x) > 4^n 2^k, M^\sharp f(x) \leq \gamma\lambda\}| \leq Ce^{-\frac{c}{\gamma}}|Q_j^k|,$$

which in turn yields, using Theorem 2.2,

$$w(\{x \in Q_j^k : Mf(x) > 4^n 2^k, M^\sharp f(x) \leq \gamma\lambda\}) \leq Ce^{-c\frac{\varepsilon}{\gamma}} \int_{\mathbb{R}^n} (M\chi_{Q_j^k})^q w,$$

where $\varepsilon = \frac{c_n}{\max(1,[w]_{C_q})}$. These computations, together with the standard argument that uses the good-λ technique yield

$$\int Mf^p w \leq 2^p \sum_{k \in \mathbb{Z}} 2^{kp} w(\Omega_k)$$

$$\leq (c_n)^p \sum_{k \in \mathbb{Z}} 2^{kp} w(M^\sharp f > \gamma 2^k) + c_n e^{-\frac{c\varepsilon}{\gamma}} \sum_{k,j} 2^{kp} \int_{\mathbb{R}^n} (M\chi_{Q_j^k})^q w$$

$$\leq \left(\frac{c_n}{\gamma}\right)^p \int M^\sharp f^p w + c_n e^{-\frac{c\varepsilon}{\gamma}} \int (M_{p,q} Mf)^p w,$$

where $M_{p,q}$ is the Marcinkiewicz operator as in [Can], [Saw]. We now use Lemma 5.8 from [Can] and obtain

$$\int (M_{p,q} Mf)^p w \leq 2^{c_n \frac{pq}{q-p}} \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \int Mf^p w.$$

So, if we choose

$$\frac{1}{\gamma} = c_n \frac{pq}{q-p} \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} = c_n \frac{pq}{q-p} \max(1, [w]_{C_q} \log^+[w]_{C_q}),$$

we can absorb the last term to the left side and we obtain

$$\|Mf\|_{L^p(w)} \leq c_n \frac{pq}{q-p} \max(1, [w]_{C_q} \log^+[w]_{C_q}) \|M^\sharp f\|_{L^p(w)}.$$

This finishes the proof. $\qquad \square$

Next we provide a proof of Theorem 1.10. Recall that we say that an operator T satisfies the (D) property if there are some constants $\delta \in (0,1)$ and $c > 0$ such that for all f,

$$(D) \qquad\qquad M_\delta^\sharp (Tf)(x) \leq cMf(x), \quad a.e.\ x.$$

Here M denotes the standard Hardy–Littlewood maximal operator and $M_\delta^\sharp f = M^\sharp(f^\delta)^{\frac{1}{\delta}}$

Proof of Theorem 1.10. Since $\frac{p}{\delta} < q$, we can make the following computations:

$$\|Tf\|_{L^p(w)} \leq \|M_\delta(Tf)\|_{L^p(w)} = \left\|M(Tf^\delta)\right\|_{L^{\frac{p}{\delta}}(w)}^{\frac{1}{\delta}}$$

$$\leq c_n \left(\frac{pq}{\delta q - p} \max(1, [w]_{C_q} \log^+[w]_{C_q})\right)^{\frac{1}{\delta}} \left\|M^\sharp(Tf^\delta)\right\|_{L^{\frac{p}{\delta}}(w)}^{\frac{1}{\delta}}$$

$$= c_n \left(\frac{pq}{\delta q - p} \max(1, [w]_{C_q} \log^+[w]_{C_q})\right)^{\frac{1}{\delta}} \left\|M_\delta^\sharp(Tf)\right\|_{L^p(w)}$$

$$\leq c_n\, C_T \left(\frac{pq}{\delta q - p} \max(1, [w]_{C_q} \log^+[w]_{C_q})\right)^{\frac{1}{\delta}} \|Mf\|_{L^p(w)}. \qquad \square$$

6. GENERALIZED POINCARÉ INEQUALITIES: PROOF OF THEOREM 1.6

In this section we provide a proof of Theorem 1.6. Let $w \in A_\infty$ and let a be a functional satisfying the $D_r(w)$ condition for some $r > 1$. More precisely, for every cube Q and every collection Λ of pairwise disjoint subcubes of Q, the following inequality has to hold:

$$\sum_{P \in \Lambda} w(P)\, a(P)^r \leq \|a\|^r w(Q) a(Q)^r.$$

This kind of functionals were studied in relation with self improvement properties of Poincaré and Poincaré–Sobolev inequalities in [PR]. Here we present a result similar to the ones obtained there.

Proof of Theorem 1.6. Fix a cube Q. We have to prove that for every $t > 0$,

$$(6.1) \qquad t^r w(\{x \in Q : |f(x) - f_Q| > t\}) \leq (c_n \|a\| [w]_{A_\infty})^r a(Q)^r w(Q),$$

with c_n independent from everything but the dimension.

M_Q will denote the dyadic maximal operator localized in Q. Since $(f - f_Q) \leq M_Q(f - f_Q)$, we can just estimate the bigger set $\Omega_t = \{x \in Q : M_Q(f - f_Q)(x) > t\}$. Let Q_j be the maximal cubes that form Ω_t. They can be found by the Calderón–Zygmund decomposition. Let $q = 2^n + 1$ as in [PR], and let us make the same computations that they do. We arrive to $w(\Omega_{qt}) \leq \sum_j w(E_{Q_j})$,

where $E_{Q_j} = \{x \in Q_j : M_Q(f - f_{Q_j})(x) > t\} = \{x \in Q_j : M_{Q_j}(f - f_{Q_j})(x) > t\}$, by the maximality of the cubes Q_j. Now we will use the good-λ from Corollary 1.5. We use the version with the dyadic sharp maximal function in Remark 3.1. Let $\gamma > 0$ to be chosen later. Then

$$E_{Q_j} \subseteq \{M_{Q_j}(f - f_{Q_j}) > t, M_d^\sharp f \leq \gamma t\} \cup \{M_d^\sharp f > \gamma t\} = A_j \cup B_j$$

and therefore $w(E_{Q_j}) \leq w(A_j) + w(B_j)$.

For A_j-s, let $s > 1$ be the exponent for the Reverse Hölder inequality for $w \in A_\infty$ as in Theorem 2.1. Then, using Corollary 1.5, we have

$$\sum_j w(A_j) \leq c_1 e^{-\frac{c_2}{s\gamma}} \sum_j w(Q_j) = c_1 e^{-\frac{c_2}{s\gamma}} w(\Omega_t).$$

Remember that $c_1, c_2 > 0$ are dimensional constants. On the other hand, for B_j we can argue as follows. We have

$$\bigcup_j B_j \subseteq \{x \in Q : M_d^\sharp f(x) > \gamma t\} = \bigcup_i R_i,$$

where R_i are the maximal dyadic subcubes of Q such that

$$\gamma t < \frac{1}{|R_i|} \int_{R_i} |f - f_{R_i}|.$$

Now, using the starting point (1.6), we clearly have $\gamma t \leq a(R_i)$. Therefore, using that a satisfies the $D_r(w)$ condition, we have

$$\sum_j w(B_j) \leq w\big(\{x \in Q : M_d^\sharp f(x) > \gamma t\}\big) = \sum_i w(R_i)$$

$$\leq \left(\frac{1}{\gamma t}\right)^r \sum_i w(R_i) a(R_i)^r \leq \|a\|^r \left(\frac{1}{\gamma t}\right)^r w(Q) a(Q)^r.$$

Now, if we put everything together, we get

$$(qt)^r w(\Omega_{qt}) \leq c_1 (tq)^r e^{-\frac{c_2}{\gamma s}} w(\Omega_t) + \left(q\frac{\|a\|}{\gamma}\right)^r w(Q) a(Q)^r.$$

Since we have qt on the left and t on the right, we define the function

$$\varphi(N) = \sup_{0 < t \leq N} t^r w(\Omega_t).$$

This function is increasing, so we have

$$\varphi(N) \leq \varphi(Nq) \leq c_1 q^r e^{-\frac{c_2}{\gamma s}} \varphi(N) + \left(q\frac{\|a\|}{\gamma}\right)^r w(Q) a(Q)^r.$$

The parameter γ is free, and we make the choice so that $c_1 q^r e^{-\frac{c_2}{s\gamma}} = \frac{1}{2}$, namely $\gamma = \frac{c_n}{r[w]_{A_\infty}}$. This yields the result, since $\|f - f_Q\|_{L^{r,\infty}(Q,w)} \leq \sup_N \varphi(N)$. $\qquad\square$

7. FURTHER EXTENSIONS: POLYNOMIAL APPROXIMATION

In this section we generalize Theorems 1.1 and 1.11 to the context of polynomials. More precisely, we show that the average f_Q can be replaced with an appropriate polynomial $P_Q f$. It is not clear how to prove these results using the method in [OPRRR].

Let $\mathcal{P}_k(Q)$ denote the space of polynomials of degree at most k restricted to the cube Q. The degree k will be frozen from now on, so we omit the subscript k if there is no room for confusion. We denote by P_Q the orthogonal projection onto $\mathcal{P}_k(Q)$. The following optimality property holds: when $p \in [1, \infty)$:

$$\inf_{\pi \in \mathcal{P}_k} \left(\fint_Q |f - \pi|^p\right)^{1/p} \approx \left(\fint_Q |f - P_Q f|^p\right)^{1/p}.$$

Before we state the main results of this section, we introduce the sharp maximal function in this polynomial context, which has the expected form

$$M_k^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f - P_Q f|$$

The case $k = 0$ corresponds to the usual sharp maximal function. We first state the maximal polynomial theorem:

Theorem 7.1. *Let $f \in L^1_{loc}$, Q a cube, $1 < r < \infty$ and $1 \le p < \infty$. Then*

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f - P_Q f)(x)}{M^\sharp_k f(x)} \right)^p w(x) dx \right)^{\frac{1}{p}} \le c_n \, r' \, \gamma \, p.$$

We now state the polynomial version of Theorem 1.11. We introduce the weighted polynomial BMO norm, that is, for a certain weight w we define

$$\|f\|_{\mathrm{BMO}^r_k(w)} := \sup_Q \frac{1}{w_r(Q)} \int_Q |f - P_Q f|.$$

Theorem 7.2. *Let $1 < p < \infty$ and $r > 1$. Let w a weight and f a function satisfying $[w]_{A^r_p} < \infty$ and $\|f\|_{\mathrm{BMO}^r_k(w)} < \infty$. Then*

$$\left(\frac{1}{w_r(Q)} \int_Q \left| \frac{f(x) - P_Q f(x)}{w(x)} \right|^{p'} w(x) dx \right)^{\frac{1}{p'}} \le c_n \, \gamma \, p' \left([w]_{A^r_p} \right)^{\frac{1}{p}} \left(r' \right)^{\frac{1}{p'}} \|f\|_{\mathrm{BMO}^r_k(w)}.$$

Remark 7.3. One can also obtain A_∞ results analogous to Corollaries 1.5 and 1.12.

Since the proofs of these theorems are very similar to the zero degree case but making only the appropriate changes, we are only going to give the proof of Theorem 7.1, and just a sketch of the proof of Theorem 7.2.

Proof of Theorem 7.1. The idea is now to make the Calderón–Zygmund decomposition of the function

$$F(x) = \frac{|f(x) - P_Q f(x)|}{\mathrm{osc}_k(f, Q)},$$

where now

$$\mathrm{osc}_k(f, Q) = \fint_Q |f - P_Q f|.$$

The rest of the proof goes as in Theorem 1.1 using the special known properties of P_Q. $\square$

Sketch of the proof of Theorem 7.2. In this case, we make the mixed-type Calderón–Zygmund decomposition at height L of the function

$$|f - P_Q f|.$$

That is, we select the maximal cubes $\{Q_j\}$ that satisfy

$$\frac{1}{w_r(Q_j)} \int_{Q_j} |f - P_Q f| > L.$$

The proof follows as in the proof of Theorem 1.11. $\square$

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(Javier Canto) BCAM –Basque Center for Applied Mathematics, Bilbao, Spain
Email address: `jcanto@bcamath.org`

(Carlos Pérez) Department of Mathematics, University of the Basque Country, IKERBASQUE (Basque Foundation for Science) and BCAM –Basque Center for Applied Mathematics, Bilbao, Spain
Email address: `cperez@bcamath.org`