Static and Dynamical, Fractional Uncertainty Principles

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Abstract

We study the process of dispersion of low-regularity solutions to the Schrödinger equation using fractional weights (observables). We give another proof of the uncertainty principle for fractional weights and use it to get a lower bound for the concentration of mass. We consider also the evolution when the initial datum is the Dirac comb in $\mathbb{R}$. In this case we find fluctuations that concentrate at rational times and that resemble a realization of a Lévy process. Furthermore, the evolution exhibits multifractality.

1 Introduction

This work grew out of the interest in understanding the process of dispersion of solutions to the Schrödinger equation with initial data with low regularity. By Schrödinger equation we mean the initial value problem:

$$\begin{cases} \partial_t u = \frac{i}{2}\hbar\Delta u \\ u(x,0) = f(x), \end{cases}$$

where $\hbar := 1/(2\pi)$.

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We measure the regularity using the space

$$\Sigma_\delta(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) \mid \|f\|_{\Sigma_\delta}^2 := \||x|^\delta f\|_2^2 + \|D^\delta f\|_2^2 < \infty\}, \tag{1}$$

where $D^\delta f := |\xi|^\delta \hat{f}(\xi)$ and

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\,dx$$

We will consider $0 < \delta \leqslant 1$, and refer to solutions with $u(x,0) \in \Sigma_\delta(\mathbb{R}^n)$, for $0 < \delta < 1$, as low-regularity solutions.

Similarly, we measure the dispersion of a solution u with the functional

$$h_\delta[f](t) := \int |x|^{2\delta} |u(x,t)|^2\,dx; \tag{2}$$

for simplicity, we may write $h_\delta(t)$. Nahas and Ponce studied this functional during their work on persistence properties of decay and regularity in the non-linear setting [21]. As a consequence of Lemma 2 in [21] we have

$$h_\delta[f](t) \leqslant C_\delta \|f\|_{\Sigma_\delta(\mathbb{R}^n)}^2 (1+t^2)^\delta, \tag{3}$$

where f is the initial datum, so the functional (2) makes sense for every time. Another proof of this persistence property is given in [1], where the motivation is to give sufficient conditions for uniqueness of linear and non-linear Schrödinger equations following the ideas in [9].

From another point of view, $h_\delta[f](t)$ is the evolution of the average value of a quantum observable and the corresponding quantity for a classical particle in free-motion is $h_\delta^{\mathrm{c}}[x_0, p_0](t) := |x_0 + p_0 t|^{2\delta}$, where x_0 and p_0 are the initial position and momentum, respectively. It is interesting to compare the quantum and classical behavior; for example, after computing h_1'' or by using the identity $e^{i\pi t|\xi|^2}(i\hbar\partial)e^{-i\pi t|\xi|^2} = i\hbar\partial + t\xi$ we can see that

$$h_1[f](t) = \langle (x_0 + p_0 t)^2 \rangle := \int f(x)\overline{(x - it\hbar\partial)^2 f}\;dx,$$

where $x_0 = x$ and $p_0 = -i\hbar\partial$ are the initial (in the Heisenberg picture) position and momentum operators, respectively. Does this simple and smooth behavior hold equally when $0 < \delta < 1$?

The classical Heisenberg's Uncertainty Principle asserts that

$$\left[\int |x|^2 |f(x)|^2\,dx \int |\xi|^2 |\hat{f}(\xi)|^2\,d\xi\right]^{\frac{1}{2}} \geqslant \frac{n}{4\pi}\|f\|_{L^2(\mathbb{R}^n)}^2. \tag{4}$$

Using translations in physical space and in phase space (i.e. Galilean transformations) it is always possible to assume that $\int x|f(x)|^2\,dx = \int \xi|\hat{f}(\xi)|^2\,d\xi = 0$, and (4) is then a measure of the concentration of $|f|$ and $|\hat{f}|$ around the origin. Finally, using translations in time and dilations we can also assume that $\langle x_0p_0 + p_0x_0\rangle = 0$ and that $a^2 := \langle x_0^2\rangle = \langle p_0^2\rangle$, so that in that case $h_1[f](t) = a^2(1+t^2)$. Hence, using (4) we conclude that if $\|f\|_{L^2(\mathbb{R}^n)} = 1$ then

$$h_1[f](t) \geqslant \frac{n}{4\pi}(1+t^2), \tag{5}$$

and the identity holds if and only if $f = cf_G(x) := c2^{n/4}e^{-\pi|x|^2}$, where $|c| = 1$. In fact, in that case the corresponding solution is explicitly given by $u_G = 2^{n/4}(1+it)^{-n/2}e^{-\pi|x|^2/(1+it)}$, so that

$$h_\delta[f_G](t) = h_\delta[f_G](0)(1+t^2)^\delta. \tag{6}$$

The above argument suggests that a lower bound of $h_\delta[f](t)$ might be proved by means of a generalization of the uncertainty principle (4) with weights $|x|^{2\delta}$ and $|\xi|^{2\delta}$, for $0 < \delta < 1$. As it is well known, the uncertainty principle has been already extended in several directions, see *e.g.* [7, 2, 11, 3, 19, 25], and the "fractional uncertainty principle" we are interested in was proved by Hirschman in [16]. One of the results in this paper is another proof of this fact.

Theorem 1 (Static, Fractional Uncertainty Principle)**.** *There exists a constant $a_\delta > 0$, for $0 < \delta < 1$, such that*

$$\||x|^\delta f\|_{L^2(\mathbb{R}^n)}\|D^\delta f\|_{L^2(\mathbb{R}^n)} \geqslant a_\delta\|f\|^2_{L^2(\mathbb{R}^n)}. \tag{7}$$

Equality is attained and the minimizer Q_δ is unique under the constraints: $Q_\delta > 0$, $\|Q_\delta\|_2 = 1$ and $\||x|^\delta Q_\delta\|_2 = \|D^\delta Q_\delta\|_2$. Furthermore, $Q_\delta(x) \simeq |x|^{-n-4\delta}$ for $|x| \gg 1$.

The decay result is direct consequence of the work of Kaleta and Kulczycki [20]. Observe that, interestingly, the minimizer of the fractional uncertainty principle does not decay exponentially.

As a consequence of the above theorem we easily obtain a lower bound for $h_\delta[f](t)$ as stated in our next theorem.

Theorem 2 (Dynamical, Fractional Uncertainty Principle). *If $f \in \Sigma_\delta(\mathbb{R}^n)$, for $0 < \delta < 1$, and $\|f\|_2 = 1$, then*

$$h_\delta[f](t) \geqslant \Big(\frac{a_\delta^2}{\||x|^\delta f\|_2\|D^\delta f\|_2}\Big)^2 \max\Big(\||x|^\delta f\|_2^2, \|D^\delta f\|_2^2|t|^{2\delta}\Big),$$

where a_δ is the constant in (12)*. Furthermore, for any $T \neq 0$*

$$h_\delta[f](0)h_\delta[f](T) \geqslant a_\delta^4|T|^{2\delta},$$

with equality if and only if

$$f(x) = ce^{-\pi i|x|^2/T}\lambda^{n/2}Q_\delta(\lambda x)$$

for some $\lambda > 0$ and $|c| = 1$.

One could wonder up to what extent the behavior exhibited by the gaussian in (6) is generic for $h_\delta[f](t)$. One of the main purposes of this paper is to start to explore the answer to this question. We first study the regularity of $h_\delta[f](t)$ and also give precise results about its Fourier transform. From the proofs of these results one can easily guess that the so called Talbot effect can generate plenty of fluctuations from the generic behavior $(1 + t^2)^\delta$; the reader is referred to [8] for more information on the Talbot effect.

Then, as a second step, we focus our attention in one space dimension and to the particular case when f is the Dirac comb

$$F_D(x) := \sum_{m\in\mathbb{Z}} \delta(x - m).$$

Even though F_D is not a proper function but a distribution, so that at first $h_\delta[F_D]$ does not make sense, we are able to extend, after renormalization, the functional h_δ to periodic functions and then to the Dirac comb. To approach the Dirac comb in $\mathbb{R}$ we use functions of the form for

$$f_{\varepsilon_1,\varepsilon_2}(x) := N_{\varepsilon_2}^{-1}\psi(\varepsilon_2 x)F_{\varepsilon_1}/\|F_{\varepsilon_1}\|_2, \qquad (8)$$

where ψ is a smooth function with $\psi(0) = 1$, N_{ε_2} is chosen so that $\|f_{\varepsilon_1,\varepsilon_2}\|_2 = 1$, and

$$F_{\varepsilon_1}(x) := \sum_{m\in\mathbb{Z}} \varepsilon_1^{-1}e^{-\pi((x-m)/\varepsilon_1)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi(\varepsilon_1 m)^2}e^{2\pi ixm}.$$

We will prove that in the limit $\varepsilon_2 \to 0$ (ε_1 fixed) the function $h_\delta[f_{\varepsilon_1,\varepsilon_2}]$ splits into a smooth background and a oscillating, periodic function that

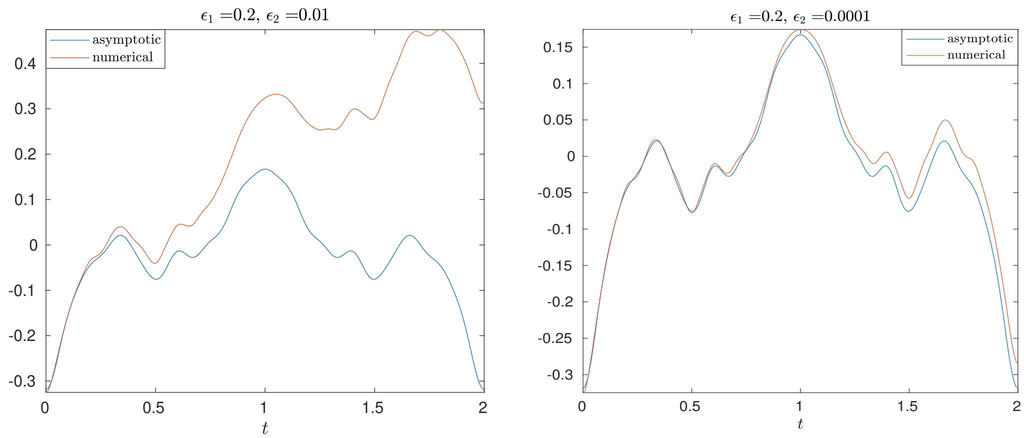

Figure 1: The red line is the plot of $h_\delta[f_{\varepsilon_1,\varepsilon_2}]$ using its definition in (2), and the blue line is $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$, to be defined in (71). In this plot we have removed from $h_\delta[f_{\varepsilon_1,\varepsilon_2}]$ a constant term C_{ε_2} and then multiplied by ε_2^{-1}; this will be clear when we reach (81). The choice of $\varepsilon_1 = 0.2$ is due to the high computational cost of taking a smaller value of ε_1 and then to diminish ε_2.

we call $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$. In Figure 1 we can see how $h_\delta[f_{\varepsilon_1,\varepsilon_2}]$ approaches, after renormalization, $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$.

The final step is to pass to the limit $\varepsilon_1 \to 0$. In this way we obtain a periodic, pure point distribution $h_{\mathrm{p},\delta}[F_D]$ with support at rational times, a fact which is very reminiscent of the Talbot effect. More concretely, we prove the following result; see Fig. 2.

Theorem 3.

$$\begin{aligned} h_{p,\delta}[F_D](2t) = -\frac{2b_{1,\delta}}{\|\psi\|_2^2}\zeta(2(1+\delta))\Big[\sum_{\substack{(p,q)=1 \\ q>0 \ odd}} \frac{1}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t) - \\ - \sum_{\substack{(p,q)=1 \\ q\equiv 2 \,(mod\, 4)}} \frac{2(2^{1+2\delta}-1)}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t) + \sum_{\substack{(p,q)=1 \\ q\equiv 0 \,(mod\, 4)}} \frac{2^{2(1+\delta)}}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t)\Big], \end{aligned} \tag{9}$$

where $\zeta(s)$ is the Riemann zeta function, and

$$b_{1,\delta} = \frac{1}{(2\pi)^{2\delta}} \frac{\Gamma(2\delta)}{|\Gamma(-\delta)|\Gamma(\delta)}.$$

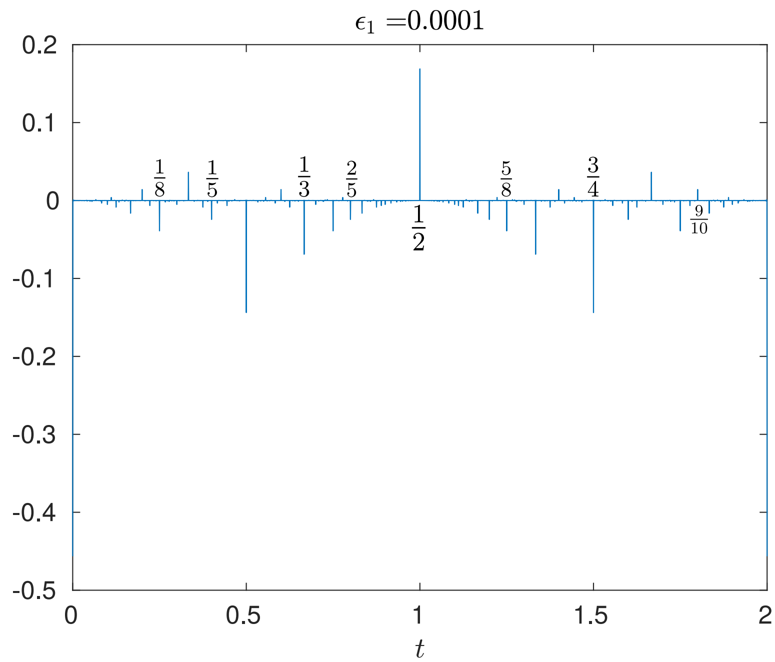

Figure 2: Plot of $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$, to be defined in (71). In Figure 1 the plot of $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$ lacks the rich structure suggested by (9) because ε_1 is still small there, however as ε_1 approaches zero the emergence of Dirac deltas is clearly visible.

Our final result is about the properties of $h_{\mathrm{p},\delta}[F_D]$. Let us consider its primitive, that is,

$$H_\delta(t) := \int_{[0,t]} h_{\mathrm{p},\delta}(2s)\, ds. \tag{10}$$

Quite surprisingly, we find out that H_δ can be seen as a "realization" of a pure jump α-Lévy process with $\alpha := 1/(1+\delta)$—see Fig. 3, which suggests strongly the presence of intermittency. To prove this we compute its Hölder exponent at each irrational time and show that it depends on its "irrationality" $\mu(t)$; the precise definition of $\mu(t)$ is given in Definition 29. We look also at the so called spectrum of singularities $d_{H_\delta}(\gamma) := \dim F_\gamma$, where

$$F_\gamma := \{t \in [0,1) \mid H_\delta \text{ has Hölder exponent } \gamma \text{ at } t\}. \tag{11}$$

Our main result in this direction is the following one.

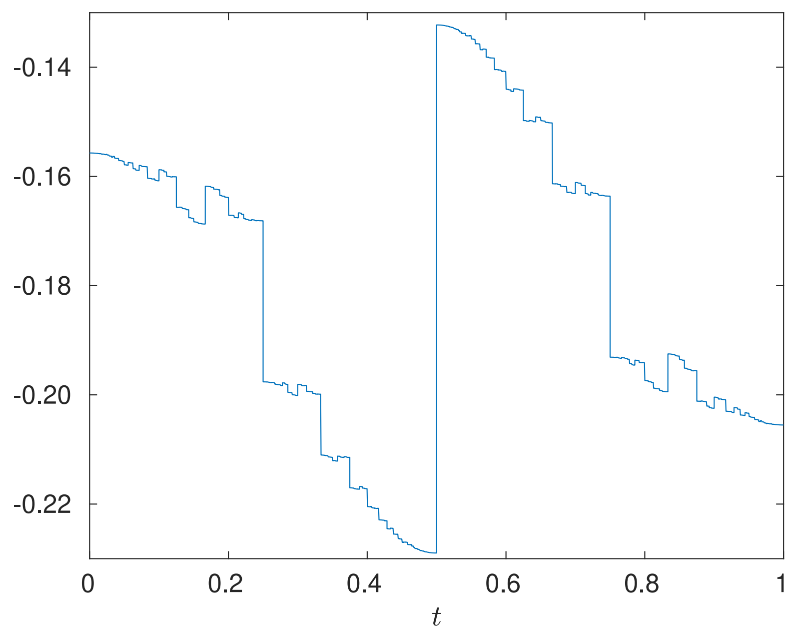

Figure 3: Plot of H_δ in (10). Even though H_δ has some symmetry, *e.g.* $H_\delta(1-t) = c_\delta - H_\delta(t-)$, the appearance of "unpredictable" large jumps resembles an α-Lévy process with small exponent α.

Theorem 4. *Let* $\alpha := 1/(1+\delta)$, *then*

$$d_{H_\delta}(\gamma) = \alpha\gamma, \quad \text{for } \gamma \in [0, 1/\alpha).$$

Jaffard proved in Thm. 1 of [18] that the spectrum of singularities of an α-Lévy process is almost surely

$$d_\alpha(\gamma) = \begin{cases} \alpha\gamma & \gamma \in [0, 1/\alpha] \\ -\infty & \gamma > 1/\alpha; \end{cases}$$

$d_\alpha(\gamma) = -\infty$ means that no point has Hölder exponent γ. This identity tightens our suggested relationship between H_δ and Lévy processes, and we suspect that $d_\alpha(\gamma) = d_{H_\delta}(\gamma)$ for every γ.

Structure of the paper:

- In section 2 we discuss the static, fractional uncertainty principle (Thm. 1) and prove some properties of the space $\Sigma_\delta(\mathbb{R}^n)$.

- In section 3 we discuss the dynamical, fractional uncertainty principle (Thm. 2); in sec. 3.1 we compute the Fourier transform of $h_\delta[f](t)$ and the main result there is Theorem 10; and in sec. 3.2 we exploit Theorem 10 to obtain regularity properties of $h_\delta[f](t)$.

- In section 4 we define $h_\delta[f](t)$ for periodic initial data; and in sec. 4.1 we study the "dispersion" properties of the Dirac comb, and prove Theorems 3 and 4.

Finally, some questions that arise naturally for future work are:
1. What are the optimal constants in Theorems 1 and 2? Can $h_\delta[Q_\delta]$ be explicitly computed?
2. What is the result about the Dirac Comb in higher dimensions and in the non-linear setting?
3. Study different regimes for ε_1 and ε_2 in (8);
4. For other observables (weights) $W(x)$, can we estimate $\langle e^{-it\hbar\Delta/2} W e^{it\hbar\Delta/2}\rangle$ in terms of classical trajectories $W(x+tp)$?

Notations

- *Relations:* If $x \lesssim y$ then $x \leqslant Cy$, where $C > 0$ is a constant, and similarly for $x \gtrsim y$ and $x \simeq y$. If $x \ll 1$ then $x \leqslant c$, where $c > 0$ is a sufficiently small constant, and similarly for $x \gg 1$.

- *Miscellaneous:* $a+ := a + \varepsilon$, for $0 < \varepsilon \ll 1$. $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$. sgn is the sign function. The volume of the unit sphere is denoted by ω_m, and the standard measure on it as dS.

- If $A \subset \mathbb{R}^n$, then $|A|$ is its Lebesgue measure and $\mathbb{1}_A$ is the indicator function.

- The fractional derivative as $(D^\delta f)^\wedge(\xi) := |\xi|^\delta \hat{f}(\xi)$.

- Let $I \subset \mathbb{R}$ be an interval with center $c(I)$. The projection to frequencies $|\xi| \in I$ is the operator $(P_I f)^\wedge(\xi) := \zeta_I \hat{f}(\xi)$, where $\zeta_I(\xi) := \zeta((\xi - c(I))/|I|)$ and ζ is a fixed cutoff of $[-1,1]$.

- If X is a function space, then $X_{\text{loc}} := \{f \in \mathcal{S}' \mid \zeta f \in X \text{ for every } \zeta \in C_0^\infty\}$.

- Spaces: for $\Sigma_\delta(\mathbb{R}^n)$ see (1), and for $\Lambda^\alpha(\mathbb{R}^n)$ see (54). $H^s(\mathbb{R}^n)$ is the space of $f \in L^2$ with $D^s f \in L^2$.
- $h_f(t)$ is the Hölder exponent of a function f at $t \in \mathbb{R}$; see Def. 28; d_f is the spectrum of singularities; see (11).
- $\mu(t)$ is the irrationality measure of $t \in \mathbb{R}$; see Def. 29.

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2 Static, Fractional Uncertainty Principle

In this section we study the static, fractional uncertainty principle. We prove some general properties of $\Sigma_\delta(\mathbb{R}^n)$, which will play an important role in our investigation of h_δ.

The (static) uncertainty principle asserts that there exists $a_\delta > 0$ such that

$$\| |x|^\delta f\|_2 \|D^\delta f\|_2 \geqslant a_\delta \|f\|_2^2, \qquad \text{for } 0 < \delta \leqslant 1.$$

Actually, this is equivalent to the continuous embedding $\Sigma_\delta(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$. In fact, let us define

$$a_\delta^2 := \inf_{\|f\|_2=1} \| |x|^\delta f\|_2 \|D^\delta f\|_2, \tag{12}$$

We can exploit the symmetry $f_\lambda(x) := \lambda^{\frac{n}{2}} f(\lambda x)$ to force the condition $\| |x|^\delta f_\lambda\|_2 = \|D^\delta f_\lambda\|_2$ while preserving $\|f\|_2 = 1$, so that

$$2a_\delta^2 = \inf_{\substack{\|f\|_2=1 \\ \||x|^\delta f\|_2 = \|D^\delta f\|_2}} 2\| |x|^\delta f\|_2 \|D^\delta f\|_2 \geqslant \inf_{\|f\|_2=1} \|f\|_{\Sigma_\delta}^2$$

On the other hand, $2\| |x|^\delta f\|_2 \|D^\delta f\|_2 \leqslant \|f\|_{\Sigma_\delta}^2$ implies the reverse inequality $2a_\delta^2 \leqslant \inf_{\|f\|_2=1} \|f\|_{\Sigma_\delta}^2$, so

$$2a_\delta^2 = \inf_{\|f\|_2=1} \|f\|_{\Sigma_\delta}^2.$$

Lemma 5. *The class $\Sigma_\delta(\mathbb{R}^n)$ is a Hilbert space compactly embedded in $L^2(\mathbb{R}^n)$; in particular,*

$$\|f\|_2 \leqslant C(\||x|^\delta f\|_2^2 + \|D^\delta f\|_2^2)^{\frac{1}{2}}. \tag{13}$$

Furthermore, there exists a function Q_δ with $\|Q_\delta\|_2 = 1$ such that

$$\inf_{\|f\|_2=1} \|f\|_{\Sigma_\delta} = \|Q_\delta\|_{\Sigma_\delta} \tag{14}$$

Proof. We choose a sequence of functions $\{f_n\}_n$ with $\|f_n\|_2 = 1$ that minimizes $\|g\|_{\Sigma_\delta}$, that is, $\|f_n\|_{\Sigma_\delta} \to \inf_{\|g\|_2=1} \|g\|_{\Sigma_\delta}$.

By the Fréchet-Kolmogorov theorem, the sequence $\{f_n\}$ will be relatively compact in $L^2(\mathbb{R}^n)$ if the following two conditions holds uniformly in n:

(1) $\displaystyle\int_{|x|>R} |f_n|^2 \, dx < \varepsilon,$ for every $\varepsilon > 0$ and $R \gg 1$

(2) $\|f_n(\cdot - h) - f_n\|_2 < \varepsilon,$ for every $\varepsilon > 0$ and $|h| \ll 1.$

The condition (1) follows from

$$\int_{|x|>R} |f_n|^2 \, dx \leqslant R^{-2\delta} \int |x|^{2\delta} |f_n|^2 \, dx \lesssim R^{-2\delta}.$$

The condition (2) follows from

$$\begin{aligned} \|f_n(\cdot - h) - f_n\|_2^2 &= \int |\hat{f}_n|^2 |e^{-2\pi i \xi \cdot h} - 1|^2 \, d\xi \\ &\leqslant \int_{|\xi| \leqslant |h|^{-\frac{1}{2}}} + \int_{|\xi| > |h|^{-\frac{1}{2}}} |\hat{f}_n|^2 |e^{-2\pi i \xi \cdot h} - 1|^2 \, d\xi \\ &\lesssim |h| + |h|^\delta. \end{aligned}$$

Hence, we can choose a sub-sequence $\{f_{n_k}\}_k$ that converges in $L^2(\mathbb{R}^n)$ to some function $Q_\delta \in L^2(\mathbb{R}^n)$ with $\|Q_\delta\|_2 = 1$.

If $\inf_{\|g\|_2=1} \|g\|_{\Sigma_\delta} = 0$ then $\||x|^\delta f_{n_k}\|_2 \to 0$ and, passing to a sub-sequence if necessary, we see that $f_{n_k} \to 0$ a.e., which contradicts $\|Q_\delta\|_2 = 1$. Thus, $\inf_{\|g\|_2=1} \|g\|_{\Sigma_\delta} > 0$ and $\Sigma_\delta(\mathbb{R}^n)$ is continuously embedded in $L^2(\mathbb{R}^n)$, which is (13). Incidentally, the proof shows that the ball $\{\|g\|_{\Sigma_\delta} \leqslant 1\}$ is relatively compact in $L^2(\mathbb{R}^n)$, so the embedding is compact.

We prove now that $Q_\delta \in \Sigma_\delta(\mathbb{R}^n)$. Since $\Sigma_\delta(\mathbb{R}^n)$ is a Hilbert space, we can pass to a sub-sequence, say $\{f_{n_k}\}_k$, that converges weakly to some $f^* \in \Sigma_\delta(\mathbb{R}^n)$. By (13) every $h \in L^2(\mathbb{R}^n)$ defines a continuous linear map $g \mapsto \int gh$ in $\Sigma_\delta(\mathbb{R}^n)$, then $\int Q_\delta h = \int f^* h$ and $Q_\delta = f^* \in \Sigma_\delta(\mathbb{R}^n)$. □

The minimizer is the ground state of a differential equation.

Lemma 6. *If* $\|Q_\delta\|_{\Sigma_\delta} = \inf_{\|u\|_2=1}\|u\|_{\Sigma_\delta}$ *and* $\|Q_\delta\|_2 = 1$*, then*

$$D^{2\delta}Q_\delta + |x|^{2\delta}Q_\delta = 2a_\delta^2 Q_\delta. \tag{15}$$

Proof. We take $v \in \Sigma_\delta(\mathbb{R}^n)$, with $\|v\|_2 = 1$, orthogonal to Q_δ in $L^2(\mathbb{R}^n)$. Let us define $w(\theta) := \cos\theta\, Q_\delta + \sin\theta\, v$ so that $f(\theta) := \|w(\theta)\|_{\Sigma_\delta}^2$ has a minimum at $\theta = 0$. Since the derivative is

$$f'(\theta) = \sin(2\theta)\big(\|v\|_{\Sigma_\delta}^2 - 2a_\delta^2\big) + 2\cos(2\theta)(Q_\delta, v)_{\Sigma_\delta},$$

then $(Q_\delta, v)_{\Sigma_\delta} = 0$; considering $\tilde{v} = v/\|v\|_2$, we can remove the condition $\|v\|_2 = 1$.

For any $v \in \Sigma_\delta(\mathbb{R}^n)$ the function $Pv = v - (Q_\delta, v)_2 Q_\delta$ is orthogonal to Q_δ in $L^2(\mathbb{R}^n)$, so we have $(Q_\delta, Pv)_{\Sigma_\delta} = 0$ or

$$(Q_\delta, v)_{\Sigma_\delta} = 2a_\delta^2(Q_\delta, v)_2,$$

which is (15). □

By the Perron–Frobenius theorem—see Ch. XIII.12 of [22]—the lowest eigenvalue of the operator $D^{2\delta} + |x|^{2\delta}$ is unique and can be chosen positive. To apply this method we need to know that the heat kernel $e^{-tD^{2\delta}}$ is positive; see *e.g.* [6] or Lemma A.1 in [13]. Uniqueness implies that $\hat{Q}_\delta = Q_\delta$ and that Q_δ is radial.

In Corollary 3 of [20], Kaleta and Kulczycki proved that the lowest eigenvalue satisfies $Q_\delta(x) \simeq 1/|x|^{n+4\delta}$ $(0 < \delta < 1)$, for $|x| \gg 1$.

We summarize the discussion so far in the following theorem, which was stated in the introduction.

Theorem 1. *There exists a constant* $a_\delta > 0$*, for* $0 < \delta < 1$*, such that*

$$\||x|^\delta f\|_{L^2(\mathbb{R}^n)}\|D^\delta f\|_{L^2(\mathbb{R}^n)} \geqslant a_\delta \|f\|_{L^2(\mathbb{R}^n)}^2. \tag{16}$$

Equality is attained and the minimizer Q_δ *is unique under the constraints:* $Q_\delta > 0$*,* $\|Q_\delta\|_2 = 1$ *and* $\||x|^\delta Q_\delta\|_2 = \|D^\delta Q_\delta\|_2$*. Furthermore,* $Q_\delta(x) \simeq |x|^{-n-4\delta}$ *for* $|x| \gg 1$*.*

We prove now a few additional properties of $\Sigma_\delta(\mathbb{R}^n)$.

Lemma 7. *The space* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $\Sigma_\delta(\mathbb{R}^n)$*.*

Proof. We choose a symmetric function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $\zeta \geqslant 0$; we might replace ζ by $\zeta * \zeta$ to assume also that $\hat{\zeta} \geqslant 0$. By dilation and multiplication by a constant, we assume that $\zeta(0) = 1$ and $\int \zeta = 1$; we define $\zeta_\lambda(x) := \zeta(x/\lambda)$.

We prove first that functions with compact support are dense in $\Sigma_\delta(\mathbb{R}^n)$. We fix $\varepsilon > 0$ and choose $R \gg_\varepsilon 1$ such that $\||x|^\delta(1-\zeta_R)f\|_2 < \varepsilon$, so we only have to prove that $\||\xi|^\delta(\hat{f} - (\zeta_R f)^\wedge)\|_2 \lesssim \varepsilon$ for $R \gg 1$.

We choose $\lambda \gg_\varepsilon 1$ such that $\||\xi|^\delta \mathbb{1}_{|\xi|>\lambda}\hat{f}\|_{L^2} < \varepsilon$. Since $(\zeta_R f)^\wedge \to \hat{f}$ in L^2, then for $R \gg_{\varepsilon,\lambda} 1$ we have that $\||\xi|^\delta \mathbb{1}_{|\xi|<2\lambda}[\hat{f} - (\zeta_R u)^\wedge]\|_2 < \varepsilon$. By Jensen's inequality $|(\zeta_R f)^\wedge|^2 \leqslant \hat{\zeta}_R * |\hat{f}|^2$, so

$$\begin{aligned}\int_{|\xi|>2\lambda} |\xi|^{2\delta}|(\zeta_R f)^\wedge|^2\, d\xi &\leqslant \int (|\xi|^{2\delta}\mathbb{1}_{|\xi|>2\lambda}) * \hat{\zeta}_R\, |\hat{f}|^2\, d\xi \\ &\lesssim \frac{1}{R\lambda}\int_{|\xi|<\lambda} |\hat{f}|^2\, d\xi + \int_{|\xi|>\lambda} |\xi|^{2\delta}|\hat{f}|^2\, d\xi \\ &\lesssim \frac{1}{R\lambda}\int |\hat{f}|^2\, d\xi + \varepsilon^2,\end{aligned}$$

where we exploited the rapid decay of $\hat{\zeta}_R$; if $R \gg_{\varepsilon,\lambda} 1$, then $\||\xi|^\delta \mathbb{1}_{|\xi|>2\lambda}(\zeta_R f)^\wedge\|_2 \leqslant C\varepsilon$. Hence,

$$\begin{aligned}\||\xi|^\delta(\hat{f} - (\zeta_R f)^\wedge)\|_2 \leqslant \||\xi|^\delta \mathbb{1}_{|\xi|<2\lambda}[\hat{f} - (\zeta_R f)^\wedge]\|_2 + \\ + \||\xi|^\delta \mathbb{1}_{|\xi|>2\lambda}\hat{f}\|_2 + \||\xi|^\delta \mathbb{1}_{|\xi|>2\lambda}(\zeta_R f)^\wedge\|_2 \leqslant C\varepsilon,\end{aligned}$$

which shows that functions with compact support are dense in $\Sigma_\delta(\mathbb{R}^n)$.

A similar, though simpler argument shows that a function $f \in \Sigma_\delta$ with compact support can be approximated by functions $\zeta_\rho * f \in C_0^\infty(\mathbb{R}^n)$. □

We can give a description of the dual space $\Sigma_\delta^*(\mathbb{R}^n)$.

Lemma 8 (Dual space $\Sigma_\delta^*(\mathbb{R}^n)$)**.** *The dual space of $\Sigma_\delta(\mathbb{R}^n)$ can be represented as the space of distributions*

$$\Sigma_\delta^* = \{v \mid v = v_1 + v_2, \text{ such that } |x|^{-\delta}v_1 \in L^2(\mathbb{R}^n) \text{ and } D^{-\delta}v_2 \in L^2(\mathbb{R}^n)\} \tag{17}$$

with norm

$$\|v\|_{\Sigma_\delta^*}^2 := \inf_{v_1+v_2=v}(\||x|^{-\delta}v_1\|_2^2 + \|D^{-\delta}v_2\|_2^2), \tag{18}$$

and duality pairing $\langle v, f\rangle := \int_{\mathbb{R}^n} fv\, dx$.

Proof. We define the space

$$X_\delta := \{(f_1, f_2) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \mid \||x|^\delta f_1\|_2^2 + \|D^\delta f_2\|_2^2 < \infty\},$$

so that $\Sigma_\delta(\mathbb{R}^n)$ is the subspace $\{f_1 = f_2\}$. The dual space of X_δ is $X_{-\delta}$ under the pairing

$$\langle (v_1, v_2), (f_1, f_2) \rangle := \int_{\mathbb{R}^n} f_1 v_1 + f_2 v_2 \, dx, \text{ for } v \in X_{-\delta} \text{ and } f \in X_\delta, \tag{19}$$

so by the Hahn-Banach Theorem we can extend a functional $w \in \Sigma_\delta^*$ to a functional $v \in X_{-\delta}$ with norm $\|v\|_{X_{-\delta}} = \|w\|_{\Sigma_\delta^*}(\mathbb{R}^n)$, which proves (17). The identity (18) holds because the norm of a functional does not decrease after extension. ☐

The next lemma contains some embedding properties.

Lemma 9. *If $f \in \Sigma_\delta(\mathbb{R}^n)$, then $f, \hat{f} \in H^\delta(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, where p satisfies:*

$$\begin{aligned} \frac{1}{2} - \frac{\delta}{n} \leqslant \frac{1}{p} < \frac{1}{2} + \frac{\delta}{n} \qquad & \text{if } n \geqslant 2, \text{ or } n = 1 \text{ and } \delta < \frac{1}{2}, \\ 0 < \frac{1}{p} < 1 \qquad & \text{if } n = 1 \text{ and } \delta = \frac{1}{2}, \\ 0 \leqslant \frac{1}{p} \leqslant 1 \qquad & \text{if } n = 1 \text{ and } \delta > \frac{1}{2}. \end{aligned} \tag{20}$$

Proof. The inequalities at the left follow from the Sobolev Embedding Theorem, and those at the right follow from Hölder inequality. ☐

We cannot improve the strict inequalities in (20), and we can use the examples $f(x) := \zeta(|x|)|x|^{-\frac{n}{2}-\delta}(\log|x|)^{-\frac{1}{2}-\varepsilon}$, for $0 < \varepsilon < \delta/n$, where $\zeta \in C^\infty(\mathbb{R})$ vanishes around zero. When $n = 1$ and $\delta = 1/2$, it is known that f may not be bounded.

3 Dynamical, Fractional Uncertainty Principle

In this section we turn our attention to $h_\delta[f]$ in (2). We begin with a lower bound for $h_\delta[f]$ and then focus on the Fourier transform of $h_\delta[f]$ (sec. 3.1). In section 3.2 we determine the Hölder regularity of h_δ and the rate of decay of $\hat{h}_\delta$.

Theorem 2. *If* $f \in \Sigma_\delta(\mathbb{R}^n)$*, for* $0 < \delta < 1$*, and* $\|f\|_2 = 1$*, then*

$$h_\delta[f](t) \geqslant \Big(\frac{a_\delta^2}{\||x|^\delta f\|_2 \|D^\delta f\|_2}\Big)^2 \max\Big(\||x|^\delta f\|_2^2, \|D^\delta f\|_2^2 |t|^{2\delta}\Big), \tag{21}$$

where a_δ *is the constant in* (12)*. Furthermore, for any* $T \neq 0$

$$h_\delta[f](0) h_\delta[f](T) \geqslant a_\delta^4 |T|^{2\delta}, \tag{22}$$

with equality if and only if

$$f(x) = c e^{-\pi i |x|^2/T} \lambda^{n/2} Q_\delta(\lambda x) \tag{23}$$

for some $\lambda > 0$ *and* $|c| = 1$*.*

Proof. The solution u can be represented as

$$u(x,t) = \frac{1}{(it)^{\frac{n}{2}}} e^{\pi i |x|^2/t} \int f(y) e^{\pi i |y|^2/t - 2\pi i x\cdot y/t}\, dy, \qquad \text{where } \operatorname{Re}\sqrt{it} > 0.$$

If we define $g_t(y) := f(y) e^{\pi i |y|^2/t}$, then the solution can be written as

$$u(x,t) = \frac{1}{(it)^{\frac{n}{2}}} e^{\pi i |x|^2/t} \hat{g}_t(x/t).$$

By the uncertainty principle (16) we have

$$a_\delta^2 \leqslant \||x|^\delta g_t\|_2 \|D^\delta g_t\|_2 = |t|^{-\delta} h_\delta(0)^{\frac{1}{2}} h_\delta(t)^{\frac{1}{2}},$$

with equality if and only if $g_t(x) = c\lambda^{n/2} Q_\delta(\lambda x)$ for some $\lambda > 0$ and $|c| = 1$, so (22) and (23) hold. This inequality implies the lower bound

$$h_\delta(t) \geqslant \frac{a_\delta^4}{\||x|^\delta f\|_2^2} |t|^{2\delta}. \tag{24}$$

On the other hand, again by (16), we have

$$a_\delta^4 \leqslant h_\delta(t) \int |\xi|^{2\delta} |\hat{u}(\xi,t)|^2\, d\xi = h_\delta(t) \int |\xi|^{2\delta} |\hat{f}(\xi)|^2\, d\xi,$$

which implies the lower bound

$$h_\delta(t) \geqslant \frac{a_\delta^4}{\|D^\delta f\|_2^2}. \tag{25}$$

From (24) and (25) we conclude that

$$h_\delta(t) \geqslant \max\Big(\frac{a_\delta^4}{\|D^\delta f\|_2^2}, \frac{a_\delta^4}{\||x|^\delta f\|_2^2} |t|^{2\delta}\Big),$$

which is (21) after reordering. □

3.1 The Fourier Transform of h_δ

The computation of the Fourier transform of $h_\delta[f]$ is motivated by the oscillations that appear in numerical simulations when f approaches the Dirac comb.

Theorem 10. *If $f \in \Sigma_\delta(\mathbb{R}^n)$, then the Fourier transform of $h_\delta[f]$ in $\mathbb{R}\backslash\{0\}$ can be represented as*

$$\hat{h}_\delta(\tau) = -2b_{n,\delta} \int_{\mathbb{R}^{2n}} \hat{f}(\xi)\overline{\hat{f}}(\eta)\delta_0\Big(\tau - \frac{|\eta|^2 - |\xi|^2}{2}\Big)\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}, \tag{26}$$

where

$$b_{n,\delta} = \frac{1}{2\pi^{n/2+2\delta}}\frac{\Gamma(\frac{n+2\delta}{2})}{|\Gamma(-\delta)|}.$$

If $\varphi \in \mathcal{S}(\mathbb{R})$ is supported outside the interval $(-a, a)$, then

$$|\langle \hat{h}_\delta[f], \varphi\rangle| \leqslant C_a \|f\|^2_{\Sigma_\delta}\|\varphi\|_\infty. \tag{27}$$

Furthermore,

$$\|\hat{h}_\delta[f]\|_{L^1(\mathbb{R}\backslash[-a,a])} \leqslant C_a\|f\|^2_{\Sigma_\delta}. \tag{28}$$

Proof. The computation is based on the identity (see [12])

$$h_\delta(t) = b_{n,\delta}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|e^{-\pi i t|\xi|^2}\hat{f}(\xi) - e^{-\pi i t|\eta|^2}\hat{f}(\eta)|^2}{|\xi-\eta|^{n+2\delta}}\, d\xi d\eta, \quad \text{for } 0 < \delta < 1.$$

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a test function that vanishes in the interval $(-a, a)$. We apply Fubini to write the Fourier transform of h_δ as

$$\begin{aligned}\langle h_\delta, \hat{\varphi}\rangle = b_{n,\delta}\int_{\mathbb{R}^{2n}}\int \big[|\hat{f}(\xi)|^2 + |\hat{f}(\eta)|^2 - e^{-\pi i t(|\xi|^2-|\eta|^2)}\hat{f}(\xi)\overline{\hat{f}}(\eta) - \\ - e^{-\pi i t(|\eta|^2-|\xi|^2)}\overline{\hat{f}}(\xi)\hat{f}(\eta)\big]\hat{\varphi}(t)\,dt\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}\end{aligned} \tag{29}$$

$$\begin{aligned}= -b_{n,\delta}\int_{\mathbb{R}^{2n}}\Big[\hat{f}(\xi)\overline{\hat{f}}(\eta)\varphi\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big) + \\ + \overline{\hat{f}}(\xi)\hat{f}(\eta)\varphi\Big(\frac{|\xi|^2-|\eta|^2}{2}\Big)\Big]\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}},\end{aligned} \tag{30}$$

We have to show that this integral represents a bounded functional in $\mathcal{S}(\mathbb{R})$.

We can assume that $a \leqslant 1$. We bound the integral (30) as

$$\begin{aligned}|\langle h_\delta, \hat{\varphi}\rangle| &\leqslant C \int \Big|\hat{f}(\xi)\overline{\hat{f}}(\eta)\varphi\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\Big|\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}} \\ &\lesssim \int |\hat{f}(\xi)|^2\Big[\int \varphi\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{d\eta}{|\xi-\eta|^{n+2\delta}}\Big]d\xi+ \\ &\qquad + \int |\hat{f}(\eta)|^2\Big[\int \varphi\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{d\xi}{|\xi-\eta|^{n+2\delta}}\Big]d\eta \\ &:= C\int |\hat{f}(\xi)|^2 J(\xi)d\xi + C\int |\hat{f}(\eta)|^2 J'(\eta)d\eta, \end{aligned} \tag{31}$$

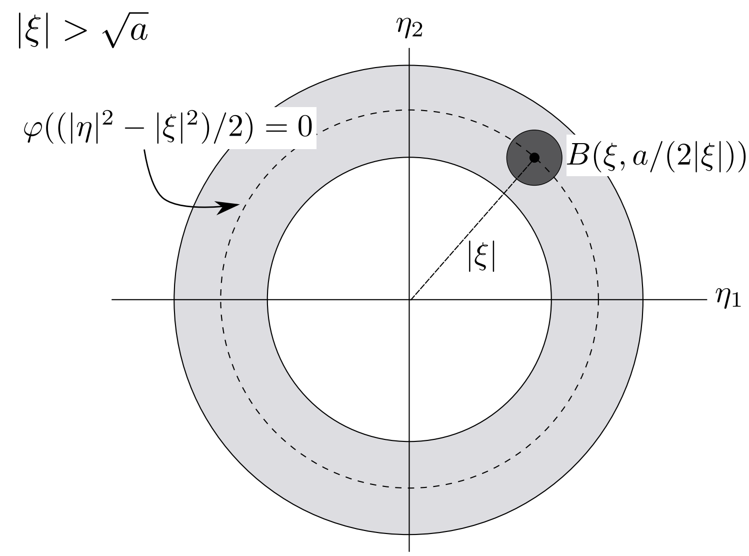
where J and J' are the integrals in square brackets.

We only bound the first integral in (31), the other being analogous; recall that $\varphi((|\eta|^2-|\xi|^2)/2)=0$ if $||\eta|^2-|\xi|^2|<2a$. When $|\xi|>\sqrt{a}$ we control J as

$$J(\xi) \leqslant \|\varphi\|_\infty \int_{\mathbb{R}^n\backslash B(\xi,a/(2|\xi|))} \frac{d\eta}{|\xi-\eta|^{n+2\delta}} \leqslant Ca^{-2\delta}\|\varphi\|_\infty |\xi|^{2\delta},$$

and when $|\xi|<\sqrt{a}$ we integrate instead over $\mathbb{R}^n\backslash B(\xi,\sqrt{a}/2)$. The final result is

$$J(\xi) \leqslant C\|\varphi\|_\infty \begin{cases} a^{-2\delta}|\xi|^{2\delta} & |\xi|>\sqrt{a}, \\ a^{-\delta} & |\xi|<\sqrt{a}. \end{cases} \tag{32}$$



We replace (32) in (31) and use the inclusion $\Sigma_\delta \hookrightarrow L^2$ to conclude that

$$|\langle h_\delta, \hat{\varphi}\rangle| \leqslant C a^{-2\delta} \|\varphi\|_\infty \|f\|_{\Sigma_\delta}^2,$$

which is (27).

Since $\mathcal{S}(\mathbb{R})$ is dense in the space of continuous functions that vanish at infinity, then from (27) and the Riesz-Markov Theorem we can see $\hat{h}_\delta$ as a (signed) regular measure in $\mathbb{R}\backslash[-a,a]$ with total variation $\leqslant C_a\|f\|_{\Sigma_\delta}^2$.

The measure $\hat{h}_\delta$ is actually a L^1-function away from the origin. If $U \subset \mathbb{R}\backslash[-a,a]$ is an open set, then we can approximate monotonically $\mathbb{1}_U$ with Schwartz functions φ such that $0 \leqslant \varphi \leqslant 1$ and $\operatorname{supp}\varphi \subset U$, so by dominated convergence we can write

$$\langle \hat{h}_\delta, \mathbb{1}_U\rangle = -2b_{n,\delta}\int_{\mathbb{R}^{2n}} \hat{f}(\xi)\overline{\hat{f}}(\eta)\mathbb{1}_U\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}.$$

Since $\hat{h}_\delta$ is a regular measure, we can actually extend this identity from $\mathbb{1}_U$ to all bounded, Borel measurable functions. If $A \subset \mathbb{R}\backslash[-a,a]$ is a bounded, Borel set with $|A| = 0$, then we can apply this identity to $\psi\mathbb{1}_A$, for $|\psi| \leqslant 1$, to conclude that $\hat{h}_\delta$ is absolutely continuous away from the origin. □

Corollary 11. *The function h_δ is continuous.*

Proof. We split h_δ into $P_{<1}h_\delta$ (an analytic function) and $P_{>1}h_\delta$. By (28) $(P_{>1}h_\delta)^\wedge \in L^1(\mathbb{R})$ and the claim follows. □

Theorem 10 only describes $\hat{h}_\delta$ away from the origin, so, for the record, we describe now the action of $\hat{h}_\delta$ on a general test function φ; even though this analysis is not crucial in the subsequent sections, it offers moral support when we remove the low frequencies of h_δ in Section 4.

We isolate the origin with a symmetric, positive function $\zeta_\varepsilon(t) := \zeta(t/\varepsilon)$, where $\zeta \in C_0^\infty(\mathbb{R})$ has support in $(-1,1)$ and $\zeta(t) = 1$ in a vicinity of zero. We develop $\varphi \in \mathcal{S}(\mathbb{R})$ as $\varphi(\tau) = \varphi(0) + \varphi'(0)\tau + r(\tau)$, and write $\hat{h}_\delta$ as

$$\langle \hat{h}_\delta, \varphi\rangle = \langle \hat{h}_\delta, (1-\zeta_\varepsilon)\varphi\rangle + \varphi(0)\langle \hat{h}_\delta, \zeta_\varepsilon\rangle + \varphi'(0)\langle \hat{h}_\delta, \tau\zeta_\varepsilon\rangle + \langle \hat{h}_\delta, \zeta_\varepsilon r\rangle. \qquad (33)$$

The first term at the right is well defined by Theorem 10, and in the next Theorem we show that we can neglect the last term at the right—and also the term $\varphi'(0)$ when $\delta < \frac{1}{2}$.

Theorem 12. *Let $\zeta \in C_0^\infty(\mathbb{R})$ be a function with support in $(-1,1)$ and such that $\zeta = 1$ around zero. If $f \in \Sigma_\delta(\mathbb{R}^n)$ and $r \in C^\infty(\mathbb{R})$ satisfies $r(0) = r'(0) = 0$, then*

$$\lim_{\varepsilon\to 0}\langle \hat{h}_\delta[f], \zeta_\varepsilon r\rangle = 0, \tag{34}$$

where $\zeta_\varepsilon(t) := \zeta(t/\varepsilon)$.

Furthermore, if $\delta < \frac{1}{2}$ and r only satisfies $r(0) = 0$, then the limit also vanishes.

Proof. Let us define $\varphi_\varepsilon := \zeta_\varepsilon r$ and test $\hat{h}_\delta$ against it:

$$\langle h_\delta, \hat{\varphi}_\varepsilon\rangle = -2b_{n,\delta}\int_{\mathbb{R}^{2n}} \hat{f}(\xi)\overline{\hat{f}}(\eta)\varphi_\varepsilon\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}.$$

We reuse (31) and bound J in $\{|\xi| > \sqrt{\varepsilon}\}$ as

$$J(\xi) \leqslant \int_{B(\xi,\varepsilon/|\xi|)} + \int_{\mathbb{R}^n\setminus B(\xi,\varepsilon/|\xi|)} |\varphi_\varepsilon\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)|\frac{d\eta}{|\xi-\eta|^{n+2\delta}}.$$

We exploit the conditions $r(0) = r'(0) = 0$ to control the first integral as

$$\begin{aligned} J_{B(\xi,\varepsilon/|\xi|)}(\xi) &\leqslant C\|r''\|_\infty \int_{B(\xi,\varepsilon/|\xi|)} ||\xi|^2-|\eta|^2|^2\frac{d\eta}{|\xi-\eta|^{n+2\delta}} \\ &\leqslant C\|r''\|_\infty|\xi|^2\int_{B(\xi,\varepsilon/|\xi|)}\frac{d\eta}{|\xi-\eta|^{n+2\delta-2}} \\ &\leqslant C\|r''\|_\infty\varepsilon^{2-2\delta}|\xi|^{2\delta}. \end{aligned}$$

We bound the remaining part of J as

$$J_{\mathbb{R}^n\setminus B(\xi,\varepsilon/|\xi|)} \leqslant C\varepsilon^2\|r''\|_\infty\int_{\mathbb{R}^n\setminus B(\xi,\varepsilon/|\xi|)}\frac{d\eta}{|\xi-\eta|^{n+2\delta}} \leqslant C\|r''\|_\infty\varepsilon^{2-2\delta}|\xi|^{2\delta}.$$

We have shown thus that $J(\xi) \leqslant C\varepsilon^{2-2\delta}|\xi|^{2\delta}$ for $|\xi| \geqslant \sqrt{\varepsilon}$.

To bound $J(\xi)$ in $\{|\xi| < \sqrt{\varepsilon}\}$, we notice that the support of φ_ε forces $|\eta| \leqslant C\sqrt{\varepsilon}$, so we can exploit again $r(0) = r'(0) = 0$ to reach

$$J(\xi) \leqslant C\|r''\|_\infty\varepsilon\int_{|\eta|<C\sqrt{\varepsilon}}\frac{d\eta}{|\xi-\eta|^{n+2\delta-2}} \leqslant C\|r''\|_\infty\varepsilon^{2-\delta}.$$

We replace our bounds for J in (31) to see that

$$|\langle h_\delta, \hat{\varphi}_\varepsilon\rangle| \lesssim \varepsilon^{2-\delta}\int_{|\xi|<\sqrt{\varepsilon}}|\hat{f}(\xi)|^2\,d\xi + \varepsilon^{2-2\delta}\int_{|\xi|>\sqrt{\varepsilon}}|\hat{f}(\xi)|^2|\xi|^{2\delta}\,d\xi \xrightarrow{\varepsilon\to 0} 0,$$

which shows that no term of order $\geqslant 2$ appears at the origin of $\hat{h}_\delta$.

A similar argument shows that the limit (34) vanishes if $\delta < \frac{1}{2}$ and $r(0) = 0$. $\square$

In the next Theorem we show that the term $\varphi'(0)$ also vanishes in the limit when $\delta \geqslant 1/2$, so no term of order $\geqslant 1$ appears in the limit.

Theorem 13. *Let $\zeta \in C_0^\infty(\mathbb{R})$ be a symmetric function with support in $(-1,1)$ and such that $\zeta = 1$ around zero. If $f \in \Sigma_\delta(\mathbb{R}^n)$ and $\delta \geqslant \frac{1}{2}$, then*

$$\lim_{\varepsilon\to 0}\langle \hat{h}_\delta, \tau\zeta_\varepsilon\rangle = 0, \tag{35}$$

where $\zeta_\varepsilon(t) := \zeta(t/\varepsilon)$.

Proof. From (30) we get

$$\langle \hat{h}_\delta, \tau\zeta_\varepsilon\rangle = -ib_{n,\delta}\int_{\mathbb{R}^{2n}} \operatorname{Im}[\hat{f}(\xi)\overline{\hat{f}}(\eta)]\zeta_\varepsilon\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{|\eta|^2-|\xi|^2}{|\xi-\eta|^{n+2\delta}}d\xi d\eta;$$

We use the identity $2i\operatorname{Im}(a\overline{b}) = |a|^2 - |b|^2 - (a+b)(\overline{a} - \overline{b})$ to get

$$|\langle \hat{h}_\delta, \tau\zeta_\varepsilon\rangle| \lesssim \int (|\hat{f}(\xi)| + |\hat{f}(\eta)|)|\hat{f}(\xi) - \hat{f}(\eta)|\zeta_\varepsilon\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{|\eta|^2-|\xi|^2}{|\xi-\eta|^{n+2\delta}}\, d\xi d\eta;$$

since the integrand tends to zero a.e. as $\varepsilon \to 0$, then, by dominated convergence, it suffices to show that the integral is finite when $\varepsilon = 1$.

By the Cauchy-Schwarz inequality

$$|\langle \hat{h}_\delta, \tau\zeta\rangle| \lesssim \Big(\int (|\hat{f}(\xi)| + |\hat{f}(\eta)|)^2\zeta^2\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{(|\eta|^2-|\xi|^2)^2}{|\xi-\eta|^{n+2\delta}}\, d\xi d\eta\Big)^{\frac{1}{2}} \Big(\int \frac{|\hat{f}(\xi) - \hat{f}(\eta)|^2}{|\xi-\eta|^{n+2\delta}} d\xi d\eta\Big)^{\frac{1}{2}},$$

and the last integral is finite because it equals $\||x|^\delta f\|_2 < \infty$, so we are left with the first integral, or

$$I := \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\Big[\int_{\mathbb{R}^n} \zeta\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{(|\eta|^2-|\xi|^2)^2}{|\xi-\eta|^{n+2\delta}}\, d\eta\Big]\, d\xi.$$

We can control I in the region $\{|\xi| < 2\}$ as

$$I_{\{|\xi|<2\}} \leqslant C\int_{|\xi|<2} |\hat{f}(\xi)|^2\Big[\int_{|\eta|<4} \frac{d\eta}{|\xi-\eta|^{n+2\delta-2}}\Big]\, d\xi < \infty. \tag{36}$$

The region $\{|\xi|>2\}$ is harder, and we begin with definitions

$$I_{\{|\xi|>2\}} := \int_{|\xi|>2} |\hat{f}(\xi)|^2 \Big[\int_{\mathbb{R}^n} \zeta\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big) \frac{(|\eta|^2-|\xi|^2)^2}{|\xi-\eta|^{n+2\delta}}\, d\eta \Big]\, d\xi$$
$$:= C\int_{|\xi|>2} |\hat{f}(\xi)|^2 L(|\xi|)\, d\xi,$$

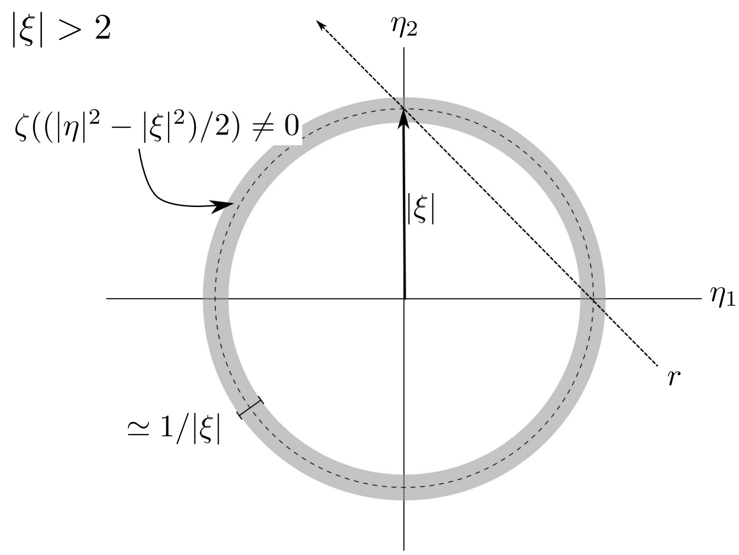
so the goal is to prove $L(|\xi|) \leqslant C|\xi|^{2\delta}$.

We assume that $\xi = |\xi| e_n$, so, passing to spherical coordinates centered at ξ,

$$L(|\xi|) = \int_{S_+^{n-1}} \int_{\mathbb{R}} \zeta\Big(\frac{r^2+2r|\xi|\theta_n}{2}\Big) \frac{(r+2|\xi|\theta_n)^2}{|r|^{2\delta-1}}\, dr dS(\theta),$$

where $dS(\theta) = \omega_{n-2}(1-\theta_n^2)^{\frac{n-3}{2}} d\theta_n$ when $n \geqslant 2$, and $dS(\theta) = \delta(\theta_n - 1)$ when $n=1$. We can verify that

$$\zeta\Big(\frac{r^2+2r|\xi|\theta_n}{2}\Big) \leqslant \begin{cases} \mathbb{1}_{\{|r|<2/(|\xi|\theta_n)\}} + \mathbb{1}_{|\{r+2|\xi|\theta_n|<2/(|\xi|\theta_n)\}} & \text{if } \theta_n > 2/|\xi| \\ \mathbb{1}_{|r|<5} & \text{if } \theta_n < 2/|\xi|. \end{cases}$$



When $n=1$, we get the bound (recall that $|\xi|>2$)

$$L(|\xi|) \lesssim \int_0^{2/|\xi|} \frac{(r+2|\xi|)^2}{r^{2\delta-1}}\, dr + \int_{-2|\xi|-2/|\xi|}^{-2|\xi|+2/|\xi|} \frac{(r+2|\xi|)^2}{|r|^{2\delta-1}}\, dr \lesssim |\xi|^{2\delta},$$

which shows that $I_{\{|\xi|>2\}} \leqslant C\||\xi|^\delta \hat{f}\|_2^2$, so this bound and (36) imply that I is finite, and the Theorem follows in this case.

When $n \geqslant 2$, we get the bound

$$L(|\xi|) \lesssim \int_{\theta_n < 2/|\xi|} 1\, dS(\theta) + \int_{\theta_n > 2/|\xi|} (|\xi|\theta_n)^{2\delta}\, dS(\theta) \lesssim \frac{1}{|\xi|} + |\xi|^{2\delta} \lesssim |\xi|^{2\delta}$$

which shows that $I_{\{|\xi|>2\}} \leqslant C\||\xi|^\delta \hat{f}\|_2^2$, so this bound and (36) imply that I is finite, and the Theorem follows. □

Theorems 12 and 13 simplify (33) to

$$\langle \hat{h}_\delta, \varphi \rangle = \lim_{\varepsilon \to 0^+} (\langle \hat{h}_\delta, (1-\zeta_\varepsilon)\varphi \rangle + \varphi'(0)\langle \hat{h}_\delta, \zeta_\varepsilon \rangle).$$

The term $\langle \hat{h}_\delta, \zeta_\varepsilon \rangle$ describes the mean size of $h_\delta(t)$ for times $|t| \lesssim 1/\varepsilon$, and its analysis is considerably more laborious, demanding more careful estimates of integrals already appearing in Theorem 13.

We begin with (29) and write $\langle \hat{h}_\delta, \zeta_\varepsilon \rangle$ as (recall $\zeta(0) = 1$)

$$\begin{aligned}
\langle \hat{h}_\delta, \zeta_\varepsilon \rangle &= 2b_{n,\delta} \int |\hat{f}(\xi)|^2 \zeta(0) - \mathrm{Re}(\hat{f}(\xi)\overline{\hat{f}(\eta)}\,)\zeta_\varepsilon\Big(\frac{|\eta|^2 - |\xi|^2}{2}\Big) \frac{d\xi d\eta}{|\xi - \eta|^{n+2\delta}} \\
&= 2b_{n,\delta} \int |\hat{f}(\xi)|^2 \int \Big[1 - \zeta_\varepsilon\Big(\frac{|\eta|^2 - |\xi|^2}{2}\Big)\Big] \frac{d\eta}{|\xi - \eta|^{n+2\delta}} d\xi + \\
&\qquad + b_{n,\delta} \int |\hat{f}(\xi) - \hat{f}(\eta)|^2 \zeta_\varepsilon\Big(\frac{|\eta|^2 - |\xi|^2}{2}\Big) \frac{d\eta d\xi}{|\xi - \eta|^{n+2\delta}}; \quad (37)
\end{aligned}$$

the last integral goes to zero as $\varepsilon \to 0$ by dominated convergence, so we can ignore it in the limit. Therefore, we only have to understand the first term, or the function

$$\begin{aligned}
K_\varepsilon(\xi) &:= \int \Big[1 - \zeta_\varepsilon\Big(\frac{|\eta|^2 - |\xi|^2}{2}\Big)\Big] \frac{d\eta}{|\xi - \eta|^{n+2\delta}} \\
&= \int_{\theta_n > 0} \int \Big[1 - \zeta\Big(\frac{r^2 + 2|\xi|\theta_n r}{2\varepsilon}\Big)\Big] \frac{dr}{|r|^{1+2\delta}} dS(\theta), \qquad (38)
\end{aligned}$$

where $dS(\theta) = \omega_{n-2}(1 - \theta_n^2)^{\frac{n-3}{2}} d\theta_n$ for $n \geqslant 2$, and $dS(\theta) = \delta(\theta_n - 1)$ for $n = 1$; compare with the function L in the proof of Theorem 13. We observe that $K_\varepsilon(\xi) = \varepsilon^{-\delta} K(\xi/\sqrt{\varepsilon})$, so we fix $\varepsilon = 1$ in (38).

When $|\xi| < \sqrt{2}$ the integrand in (38), with $\varepsilon = 1$, is zero if $|r| < \frac{1}{2}$, so

$$|K(\xi)| \leqslant C.$$

When $|\xi| > \sqrt{2}$, we integrate by parts in r using the identity $|r|^{-1-2\delta} = -(\operatorname{sgn}(r)|r|^{-2\delta})'/(2\delta)$, which holds outside the origin, so that

$$K(\xi) = -\frac{1}{2\delta} \int_{\theta_n > 0} \int \zeta'\Big(\frac{r^2 + 2|\xi|\theta_n r}{2}\Big) \operatorname{sgn}(r) \frac{(r + |\xi|\theta_n)}{|r|^{2\delta}}\, dr dS(\theta).$$

We apply the change of variables $t = (r^2 + 2|\xi|\theta_n r)/2$:

$$\begin{aligned} K(\xi) = -\frac{1}{2\delta}|\xi|^{-2\delta} \int_{\theta_n>0} \Big[\int_{-\theta_n^2/a}^{\infty} \zeta'(t) \frac{dt}{(\theta_n + \sqrt{\theta_n^2 + at})^{2\delta}} + \\ + \int_{-\theta_n^2/a}^{\infty} \zeta'(t) \operatorname{sgn}(t) \frac{(\theta_n + \sqrt{\theta_n^2 + at})^{2\delta}}{|at|^{2\delta}}\, dt \Big]\, dS(\theta), \end{aligned}$$

where $a := 2/|\xi|^2 < 1$. After the dilation $t \mapsto t/a$ and exchange of integrals, we reach

$$\begin{aligned} K(\xi) = \frac{1}{2\delta}|\xi|^{-2\delta} \int \Big[\frac{-1}{a}\zeta'\Big(\frac{t}{a}\Big) \int_{\theta_n>0} \mathbb{1}_{\{\theta_n^2 > -t\}} \frac{dS(\theta)}{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}} + \\ + \frac{-\operatorname{sgn}(t)}{a}\zeta'\Big(\frac{t}{a}\Big) \int_{\theta_n>0} \mathbb{1}_{\{\theta_n^2 > -t\}} \frac{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}}{|t|^{2\delta}}\, dS(\theta) \Big]\, dt \end{aligned}$$

We split K into the terms

$$\begin{aligned} K_1(\xi) &:= \frac{1}{2\delta}|\xi|^{-2\delta} \int_{\mathbb{R}} \frac{-1}{a}\zeta'\Big(\frac{t}{a}\Big) \Big[\int_{\theta_n>0} \mathbb{1}_{\{\theta_n^2 > t\}} \frac{dS(\theta)}{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}} \Big]\, dt \\ &:= \frac{1}{2\delta}|\xi|^{-2\delta} \int_{\mathbb{R}} \frac{-1}{a}\zeta'\Big(\frac{t}{a}\Big) J_1(t)\, dt \end{aligned} \tag{39}$$

and

$$\begin{aligned} K_2(\xi) &:= \frac{1}{2\delta}|\xi|^{-2\delta} \int_{\mathbb{R}} \frac{-\operatorname{sgn} t}{a}\zeta'\Big(\frac{t}{a}\Big) \Big[\int_{\theta_n>0} \mathbb{1}_{\{\theta_n^2 > -t\}} (\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}\, dS(\theta) \Big] \frac{dt}{|t|^{2\delta}} \\ &:= \frac{1}{2\delta}|\xi|^{-2\delta} \int_{\mathbb{R}} \frac{-\operatorname{sgn} t}{a}\zeta'\Big(\frac{t}{a}\Big) J_2(t) \frac{dt}{|t|^{2\delta}}. \end{aligned} \tag{40}$$

Before studying the asymptotic expansion of $\langle \hat{h}_\delta, \zeta_\varepsilon \rangle$, we need an auxiliary result.

Lemma 14. *If $f \in \Sigma_\delta(\mathbb{R}^n)$, for $n > 2\delta$, and $R(\xi) \lesssim \langle \xi \rangle^{-\alpha}$, for $\alpha > 2\delta$, then*

$$\lim_{\varepsilon \to 0} \int |\hat{f}(\xi)|^2 \varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon})\, d\xi = 0 \tag{41}$$

Proof. Since $\alpha > 2\delta$, then $\varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon}) \to 0$ pointwise outside zero, so, by dominated convergence,

$$\int |\hat{f}(\xi)|^2 \varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon})\, d\xi = \int_{|\xi|<1} |\hat{f}(\xi)|^2 \varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon})\, d\xi + o(1).$$

If $\hat{f}$ is smooth, then

$$\int_{|\xi|<1} |\hat{f}(\xi)|^2 \varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon})\, d\xi \lesssim \varepsilon^{\frac{n}{2}-\delta} \|\hat{f}\|_\infty^2 \int_{|\xi|<\varepsilon^{-\frac{1}{2}}} \frac{d\xi}{\langle \xi \rangle^{\alpha}}.$$

When $\alpha > n$, the last integral is bounded by a constant and the right hand side tends to zero because $n > 2\delta$. When $\alpha < n$, we have

$$\int_{|\xi|<1} |\hat{f}(\xi)|^2 \varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon})\, d\xi \lesssim \varepsilon^{\frac{\alpha}{2}-\delta} \|\hat{f}\|_\infty^2 \to 0$$

because $\alpha > 2\delta$. Therefore, the Lemma holds for smooth functions $\hat{f}$.

For general f, we use Theorem 9 with $\frac{1}{p} = \frac{1}{2} - \frac{\delta}{n}$ to control the integral as

$$\int_{|\xi|<1} |\hat{f}(\xi)|^2 \varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon})\, d\xi \leqslant \varepsilon^{-\delta} \|\hat{f}\|_p^2 \Big(\int_{|\xi|<1} \frac{d\xi}{\langle \xi/\sqrt{\varepsilon} \rangle^{\frac{\alpha n}{2\delta}}} \Big)^{\frac{2\delta}{n}} \leqslant C \|\hat{f}\|_{\Sigma_\delta}^2;$$

here, we used again $\alpha > 2\delta$. We decompose f into a smooth part $g \in C_0^\infty(\mathbb{R}^n)$ and a small part $\|h\|_{\Sigma_\delta}$ (Theorem 7), so we deduce the limit in (41) goes to zero. □

Having paved the way, we are ready for the analysis of $\langle \hat{h}_\delta, \zeta_\varepsilon \rangle$, and we warm-up with the analysis at dimension one.

Theorem 15. *Let $\zeta \in C_0^\infty(\mathbb{R})$ be a symmetric, positive function with support in $(-1,1)$ and such that $\zeta(t) = 1$ around zero; recall $\zeta_\varepsilon(t) := \zeta(t/\varepsilon)$. If $f \in \Sigma_\delta(\mathbb{R})$, then $\langle \hat{h}_\delta[f], \zeta_\varepsilon \rangle$ admits the following asymptotic expansion in ε:*

$\delta < 1/2$

$$\langle \hat{h}_\delta[f], \zeta_\varepsilon \rangle = A \varepsilon^{-2\delta} \| |\xi|^\delta \hat{f} \|_2^2 + o(1). \tag{42}$$

$\delta = 1/2$

$$\begin{aligned}\langle \hat{h}_\delta[f], \zeta_\varepsilon \rangle &= A\varepsilon^{-1} \| |\xi|^\delta \hat{f} \|_2^2 + \int |\hat{f}(\xi)|^2 \varepsilon^{-\frac{1}{2}} R(\xi/\sqrt{\varepsilon})\, d\xi + o(1) \\ &= A\varepsilon^{-1} \| |\xi|^\delta \hat{f} \|_2^2 + \mathcal{O}_\alpha(\varepsilon^{-\alpha}), \quad \textit{for every } 0 < \alpha \ll 1,\end{aligned} \tag{43}$$

where $R(\xi) \leqslant C\langle \xi \rangle^{-3}$.

$\delta > 1/2$

$$\begin{aligned}\langle \hat{h}_\delta[f], \zeta_\varepsilon \rangle &= A\varepsilon^{-2\delta} \| |\xi|^\delta \hat{f} \|_2^2 + B\varepsilon^{\frac{1}{2}-\delta} |\hat{f}(0)|^2 + \\ &\qquad + \int (|\hat{f}(\xi)|^2 - |\hat{f}(0)|^2) \varepsilon^{-\frac{1}{2}} R(\xi/\sqrt{\varepsilon})\, d\xi + o(1) \\ &= A\varepsilon^{-2\delta} \| |\xi|^\delta \hat{f} \|_2^2 + B\varepsilon^{\frac{1}{2}-\delta} |\hat{f}(0)|^2 + \mathcal{O}(\varepsilon^{-\frac{1}{2}(\delta-\frac{1}{2})}),\end{aligned} \tag{44}$$

where $R(\xi) \leqslant C\langle \xi \rangle^{2\delta-4}$.

The constants A and B, as well as the function R, depend on ζ and δ.

Proof. From (37) and (38) we have

$$\langle \hat{h}_\delta, \zeta_\varepsilon \rangle = \int |\hat{f}(\xi)|^2 \varepsilon^{-\delta} K(\xi/\sqrt{\varepsilon})\, d\xi + o(1),$$

so we split K into K_1 (39) and K_2 (40).

To estimate K_1 we notice that

$$J_1(t) := \frac{1}{(1+\sqrt{1+t})^{2\delta}} = J_1(0) + \mathcal{O}(t),$$

where t is restricted to $\operatorname{supp} \zeta \subset [-1,1]$; since ζ' is anti-symmetric, then (recall $a := 2/|\xi|^2$)

$$K_1(\xi) \leqslant C|\xi|^{-2\delta} \int \frac{1}{a} |\zeta'(t/a) t|\, dt \leqslant C|\xi|^{-2-2\delta}. \tag{45}$$

To estimate K_2 we notice that

$$J_2(t) := (1+\sqrt{1+t})^{2\delta} = J_2(0) + J_2'(0)t + \mathcal{O}(t^2);$$

since $\operatorname{sgn}(t)\zeta'(t/a)|t|^{-2\delta}$ is symmetric, then

$$\begin{aligned}K_2(\xi) &= \frac{1}{2\delta}|\xi|^{-2\delta}\int \frac{-\operatorname{sgn}(t)}{a}\zeta'(t/a)(J_2(0)+\mathcal{O}(t^2))\frac{dt}{|t|^{2\delta}} \\ &:= A|\xi|^{2\delta}+\mathcal{O}(|\xi|^{2\delta-4}).\end{aligned} \tag{46}$$

If we define $R(\xi) := K(\xi) - A|\xi|^{2\delta}$, then we have

$$\langle \hat{h}_\delta, \zeta_\varepsilon\rangle = A\varepsilon^{-2\delta}\int |\xi|^{2\delta}|\hat{f}(\xi)|^2\,d\xi + \int |\hat{f}(\xi)|^2\varepsilon^{-\delta}R(\xi/\sqrt{\varepsilon})\,d\xi + o(1). \tag{47}$$

When $\delta < \frac{1}{2}$, we see from (45) and (46) that the residue satisfies $R(\xi) \leqslant C\langle\xi\rangle^{-2-2\delta}$. We apply Lemma 14 to conclude that the residual integral goes to zero, and then (42) holds.

When $\delta = \frac{1}{2}$, the best we can say using Hölder and Theorem 9 is

$$\int |\hat{f}(\xi)|^2\varepsilon^{-\frac{1}{2}}R(\xi/\sqrt{\varepsilon})\,d\xi = \mathcal{O}_\alpha(\varepsilon^{-\alpha}), \qquad \text{for every } 0<\alpha\ll 1,$$

where $R(\xi) \leqslant 1/\langle\xi\rangle^3$; this is (43).

When $\delta > \frac{1}{2}$, the residue satisfies $R(\xi) \leqslant C\langle\xi\rangle^{2\delta-4}$. By the Sobolev embedding Theorem $\hat{f} \in W^{\delta,2}(\mathbb{R}) \hookrightarrow C^{\delta-\frac{1}{2}}(\mathbb{R})$, so

$$\begin{aligned}\int |\hat{f}(\xi)|^2\varepsilon^{-\delta}R(\xi/\sqrt{\varepsilon})\,d\xi &= B\varepsilon^{\frac{1}{2}-\delta}|\hat{f}(0)|^2 + \\ &\qquad + \int (|\hat{f}(\xi)|^2 - |\hat{f}(0)|^2)\varepsilon^{-\delta}R(\xi/\sqrt{\varepsilon})\,d\xi;\end{aligned}$$

the last integral at the right has the upper bound

$$\begin{aligned}\int (|\hat{f}(\xi)|^2 - |\hat{f}(0)|^2)\varepsilon^{-\delta}R(\xi/\sqrt{\varepsilon})\,d\xi &\lesssim \varepsilon^{-\delta}\|\hat{f}\|_\infty \int |\hat{f}(\xi)-\hat{f}(0)|\langle\xi/\sqrt{\varepsilon}\rangle^{2\delta-4}\,d\xi \\ &\leqslant C\varepsilon^{-\delta}\|\hat{f}\|_\infty\|\hat{f}\|_{C^{\delta-\frac{1}{2}}}\int |\xi|^{\delta-\frac{1}{2}}\langle\xi/\sqrt{\varepsilon}\rangle^{2\delta-4}\,d\xi \\ &= C\varepsilon^{-\frac{1}{2}(\delta-\frac{1}{2})}\|\hat{f}\|_\infty\|\hat{f}\|_{C^{\delta-\frac{1}{2}}},\end{aligned}$$

which concludes the proof of (44) . □

Not surprisingly, the leading term $A\varepsilon^{-2\delta}\||\xi|^\delta\hat{f}\|_2^2$ is consistent with the L^2–limit

$$e^{it\hbar\Delta/2}f(x) \xrightarrow{t\to\infty} \frac{1}{(it)^{\frac{n}{2}}}e^{\pi i|x|^2/t}\hat{f}(x/t);$$

which leads to $\lim_{t\to\infty} t^{-2\delta} h_\delta[f](t) = \||\xi|^\delta \hat{f}\|_2^2$—to prove this, use the continuity of $h_\delta[\hat{f}](\tau) = \tau^{2\delta} h_\delta[f](1/\tau)$ at $\tau = 0$.

In Theorem 15, very small frequencies play a distinctive role; in fact, the residue $\varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon}) \to 0$ pointwise outside the origin, so only very small frequencies, or momenta, contribute to the residual term, and actually all the lower order terms in ε^{-1} disappear if $\hat{f} = 0$ in a neighborhood of zero.

In the next Theorem, we continue our analysis of $\langle \hat{h}_\delta, \zeta_\varepsilon \rangle$ in higher dimensions, but now computations are more demanding than in $\mathbb{R}$.

Theorem 16. *Let $\zeta \in C_0^\infty(\mathbb{R})$ be a symmetric, positive function with support in $(-1, 1)$ and such that $\zeta(t) = 1$ around zero; recall $\zeta_\varepsilon(t) := \zeta(t/\varepsilon)$. If $f \in \Sigma_\delta(\mathbb{R}^n)$, for $n \geqslant 2$, then $\langle \hat{h}_\delta[f], \zeta_\varepsilon \rangle$ admits the following asymptotic expansion in ε:*

$\delta < 1/2$

$$\langle \hat{h}_\delta, \zeta_\varepsilon \rangle = A\varepsilon^{-2\delta} \||\xi|^\delta \hat{f}\|_2^2 + o(1). \tag{48}$$

$\delta = 1/2$

$$\begin{aligned} \langle \hat{h}_\delta, \zeta_\varepsilon \rangle &= A\varepsilon^{-1} \||\xi|^{\frac{1}{2}} \hat{f}\|_2^2 + B \int_{|\xi| > \sqrt{2\varepsilon}} |\hat{f}(\xi)|^2 \frac{d\xi}{|\xi|} + o(1) \\ &= A\varepsilon^{-1} \||\xi|^{\frac{1}{2}} \hat{f}\|_2^2 + \mathcal{O}(-\log \varepsilon). \end{aligned} \tag{49}$$

$\delta > 1/2$

$$\langle \hat{h}_\delta, \zeta_\varepsilon \rangle = A\varepsilon^{-2\delta} \||\xi|^\delta \hat{f}\|_2^2 + B\varepsilon^{\frac{1}{2}-\delta} \||\xi|^{-\frac{1}{2}} \hat{f}\|_2^2 + o(1). \tag{50}$$

The constants A and B depend on ζ, n and δ.

Proof. From (37) and (38) we have

$$\langle \hat{h}_\delta, \zeta_\varepsilon \rangle = \int |\hat{f}(\xi)|^2 \varepsilon^{-\delta} K(\xi/\sqrt{\varepsilon})\, d\xi + o(1), \tag{51}$$

so we split K into K_1 (39) and K_2 (40). Since $K(\xi) \lesssim 1$ for $|\xi| < \sqrt{2}$, then we only have to estimate K for $|\xi| > \sqrt{2}$.

The heart of the matter lies in the analysis of J_1 and J_2, so we paste their definition here for reference:

$$J_1(t) = \begin{cases} \displaystyle\int_0^1 \frac{dS(\theta)}{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}}, & \text{for } t > 0, \\ \displaystyle\int_{\theta_n > \sqrt{-t}}^1 \frac{dS(\theta)}{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}}, & \text{for } t < 0, \end{cases}$$

and

$$J_2(t) = \begin{cases} \displaystyle\int_0^1 (\theta_n + \sqrt{\theta_n^2 + t})^{2\delta} \, dS(\theta), & \text{for } t > 0, \\ \displaystyle\int_{\theta_n > \sqrt{-t}}^1 (\theta_n + \sqrt{\theta_n^2 + t})^{2\delta} \, dS(\theta), & \text{for } t < 0, \end{cases}$$

where $dS(\theta) = \omega_{n-2}(1-\theta_n^2)^{\frac{n-3}{2}} \, d\theta_n$.

Case $\delta < 1/2$

To estimate K_1 we have to estimate J_1, especially around the origin. The function J_1 is continuous at zero, and for $t > 0$

$$\begin{aligned} J_1(t) - J_1(0) &= \int_0^1 \Big[\frac{1}{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}} - \frac{1}{(2\theta_n)^{2\delta}} \Big] \, dS(\theta) \\ &\lesssim \int_0^{\sqrt{2t}} \frac{dS(\theta)}{\theta_n^{2\delta}} + t \int_{\sqrt{2t}}^1 \frac{dS(\theta)}{\theta_n^{2+2\delta}} \\ &\leqslant C t^{\frac{1}{2}-\delta}; \end{aligned}$$

the same bound holds for $t < 0$. Hence, $J_1(t) = J_1(0) + \mathcal{O}(|t|^{\frac{1}{2}-\delta})$ and

$$K_1(\xi) = \frac{1}{2\delta} |\xi|^{-2\delta} \int_{\mathbb{R}} \frac{-1}{a} \zeta'(t/a) \, \mathcal{O}(|t|^{\frac{1}{2}-\delta}) \, dt \lesssim |\xi|^{-1};$$

recall $a := 2/|\xi|^2$. This model computation will repeat itself during the proof.

To estimate K_2 we have to study J_2, which is also continuous. For $t > 0$,

$$J_2(t) - J_2(0) = \int_0^1 \Big[(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta} - (2\theta_n)^{2\delta} \Big] \, dS(\theta) \leqslant C t^{\frac{1}{2}+\delta};$$

the same bound holds for $t < 0$. Hence, $J_2(t) = J_2(0) + \mathcal{O}(|t|^{\frac{1}{2}+\delta})$ and

$$\begin{aligned} K_2(\xi) &= \frac{1}{2\delta} |\xi|^{-2\delta} \int_{\mathbb{R}} \frac{-\operatorname{sgn}(t)}{a} \zeta'(t/a) (J_2(0) + \mathcal{O}(|t|^{\frac{1}{2}+\delta})) \frac{dt}{|t|^{2\delta}} \\ &= A|\xi|^{2\delta} + \mathcal{O}(|\xi|^{-1}). \end{aligned}$$

If we define $R(\xi) := K(\xi) - A|\xi|^{2\delta}$, then we have

$$\langle \hat{h}_\delta, \zeta_\varepsilon \rangle = A\varepsilon^{-2\delta} \int |\xi|^{2\delta} |\hat{f}(\xi)|^2 \, d\xi + \int |\hat{f}(\xi)|^2 \varepsilon^{-\delta} R(\xi/\sqrt{\varepsilon}) \, d\xi + o(1).$$

From the estimates for K_1 and K_2 we conclude $R(\xi) \lesssim |\xi|^{-1}$, which is in the scope of Lemma 14, and (48) holds true.

Case $\delta = \frac{1}{2}$

When $|t|$ is small, $J_1(t)$ approaches the *even function*

$$J_{1,0}(t) := \frac{1}{2} \int_{\sqrt{|t|}}^{1} dS(\theta)/\theta_n \simeq -\log|t|.$$

We remove $J_{1,0}$ from J_1 to get, for $t > 0$,

$$\begin{aligned} J_1(t) - J_{1,0}(t) &= \int_0^{\sqrt{t}} \frac{dS(\theta)}{\theta_n + \sqrt{\theta_n^2 + t}} + \int_{\sqrt{t}}^{1} \Big[\frac{1}{\theta_n + \sqrt{\theta_n^2 + t}} - \frac{1}{2\theta_n}\Big] dS(\theta) \\ &:= I_1 + I_2. \end{aligned}$$

The first integral is

$$I_1 = \omega_{n-2} \int_0^1 \frac{(1 - t\theta_n^2)^{\frac{n-3}{2}} \, d\theta_n}{\theta_n + \sqrt{\theta_n^2 + 1}} = \omega_{n-2} \int_0^1 \frac{d\theta_n}{\theta_n + \sqrt{\theta_n^2 + 1}} + \mathcal{O}(t)$$

The second integral is

$$\begin{aligned} I_2 &= \omega_{n-2} \int_{\sqrt{t}}^{1} \Big[\frac{1}{\theta_n + \sqrt{\theta_n^2 + t}} - \frac{1}{2\theta_n}\Big] d\theta_n + \\ &\qquad + \omega_{n-2} \int_{\sqrt{t}}^{1} \Big[\frac{1}{\theta_n + \sqrt{\theta_n^2 + t}} - \frac{1}{2\theta_n}\Big] ((1 - \theta_n^2)^{\frac{n-3}{2}} - 1) \, d\theta_n \\ &= \omega_{n-2} \int_1^{1/\sqrt{t}} \Big[\frac{1}{\theta_n + \sqrt{\theta_n^2 + 1}} - \frac{1}{2\theta_n}\Big] d\theta_n + \mathcal{O}(t(-\log t + 1)) \\ &= \omega_{n-2} \int_1^{\infty} \Big[\frac{1}{\theta_n + \sqrt{\theta_n^2 + 1}} - \frac{1}{2\theta_n}\Big] d\theta_n + \mathcal{O}(t(-\log t + 1)) \end{aligned}$$

We arrive so at

$$\begin{aligned} (J_1 - J_{1,0})(t) &= \omega_{n-2} \int_0^{\infty} \Big[\frac{1}{\theta_n + \sqrt{\theta_n^2 + 1}} - \mathbb{1}_{\{\theta_n > 1\}} \frac{1}{2\theta_n}\Big] d\theta_n + \mathcal{O}(t(-\log t + 1)) \\ &:= a_+ + \mathcal{O}(t(-\log t + 1)). \end{aligned}$$

In the same way, for $t < 0$, we get

$$(J_1 - J_{1,0})(t) = \omega_{n-2} \int_1^\infty \Big[\frac{1}{\theta_n + \sqrt{\theta_n^2 - 1}} - \frac{1}{2\theta_n} \Big] \, d\theta_n + \mathcal{O}(t(-\log|t| + 1))$$
$$:= a_- + \mathcal{O}(t(-\log t + 1)).$$

The estimates above carry us to

$$J_1(t) = \int_{\sqrt{t}}^1 \frac{dS(\theta)}{2\theta_n} + a_+ \mathbb{1}_{\{t>0\}} + a_- \mathbb{1}_{\{t<0\}} + \mathcal{O}(-t(-\log|t| + 1))$$
$$:= J_{1,0}(t) + J_{1,1}(t) + \mathcal{O}(t(-\log|t| + 1)),$$

and then

$$K_1(\xi) = |\xi|^{-1} \int_{\mathbb{R}} \frac{-1}{a} \zeta'(t/a) [J_{1,1}(t) + \mathcal{O}(t(-\log|t| + 1))] \, dt$$
$$= B|\xi|^{-1} + \mathcal{O}(|\xi|^{-3}(\log|\xi| + 1));$$

recall $a := 2/|\xi|^2$.

The function J_2 is continuous at zero, and for $t > 0$

$$J_2(t) - J_2(0) = t \int_0^1 \frac{dS(\theta)}{\theta_n + \sqrt{\theta_n^2 + t}} = t J_1(t);$$

for $t < 0$

$$J_2(t) - J_2(0) = t \int_{\theta_n > \sqrt{-t}}^1 \frac{dS(\theta)}{\theta_n + \sqrt{\theta_n^2 + t}} + \omega_{n-2} t + \mathcal{O}(t^2) = t J_1(t) + \omega_{n-2} t + \mathcal{O}(t^2).$$

Hence,

$$K_2(\xi) = |\xi|^{-1} \int_{\mathbb{R}} \frac{-\operatorname{sgn}(t)}{a} \zeta'(t/a) (J_2(0) + t J_1(t) + \omega_{n-2} t \mathbb{1}_{\{t<0\}} + \mathcal{O}(t^2)) \frac{dt}{|t|}$$
$$= A|\xi| + B|\xi|^{-1} + \mathcal{O}(|\xi|^{-3} \log|\xi|).$$

Therefore, for $|\xi| > \sqrt{2}$,

$$K(\xi) = K_1(\xi) + K_2(\xi) = A|\xi| + B|\xi|^{-1} + \mathcal{O}(|\xi|^{-3} \log|\xi|),$$

and (49) follows after inserting this estimate in (51) and applying Lemma 14 to the residue.

Case $\delta > \frac{1}{2}$

We remove from J_1 the *even function* $J_{1,0}(t) := \int_{\sqrt{|t|}}^1 dS(\theta)/(2\theta_n)^{2\delta} \simeq |t|^{\frac{1}{2}-\delta}$, so that, for $t > 0$,

$$J_1(t) - J_{0,1}(t) = \int_0^{\sqrt{t}} \frac{dS(\theta)}{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}} + \int_{\sqrt{t}}^1 \Big[\frac{1}{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}} - \frac{1}{(2\theta_n)^{2\delta}}\Big] dS(\theta)$$
$$:= I_1 + I_2.$$

The first integral is

$$I_1 = \omega_{n-2} t^{\frac{1}{2}-\delta} \int_0^1 \frac{d\theta_n}{(\theta_n + \sqrt{\theta_n^2 + 1})^{2\delta}} + \mathcal{O}(t^{\frac{3}{2}-\delta}).$$

The second integral is

$$I_2 = \omega_{n-2} \int_{\sqrt{t}}^1 \Big[\frac{1}{(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta}} - \frac{1}{(2\theta_n)^{2\delta}}\Big] d\theta_n + \mathcal{O}(t^{\frac{3}{2}-\delta})$$
$$= \omega_{n-2} t^{\frac{1}{2}-\delta} \int_1^\infty \Big[\frac{1}{(\theta_n + \sqrt{\theta_n^2 + 1})^{2\delta}} - \frac{1}{(2\theta_n)^{2\delta}}\Big] d\theta_n + \mathcal{O}(t^{\frac{3}{2}-\delta}).$$

We arrive so at

$$(J_1 - J_{1,0})(t) = \omega_{n-2} t^{\frac{1}{2}-\delta} \int_0^\infty \Big[\frac{1}{(\theta_n + \sqrt{\theta_n^2 + 1})^{2\delta}} - \mathbb{1}_{\{\theta_n > 1\}} \frac{1}{(2\theta_n)^{2\delta}}\Big] d\theta_n + \mathcal{O}(t^{\frac{3}{2}-\delta})$$
$$:= a_+ t^{\frac{1}{2}-\delta} + \mathcal{O}(t^{\frac{3}{2}-\delta})$$

In the same way, for $t < 0$, we get

$$(J_1 - J_{1,0})(t) = \omega_{n-2} t^{\frac{1}{2}-\delta} \int_1^\infty \Big[\frac{1}{(\theta_n + \sqrt{\theta_n^2 - 1})^{2\delta}} - \frac{1}{(2\theta_n)^{2\delta}}\Big] d\theta_n + \mathcal{O}(|t|^{\frac{3}{2}-\delta})$$
$$:= a_- |t|^{\frac{1}{2}-\delta} + \mathcal{O}(|t|^{\frac{3}{2}-\delta})$$

Therefore,

$$J_1(t) = \int_{\sqrt{|t|}}^1 \frac{dS(\theta)}{(2\theta_n)^{2\delta}} + |t|^{\frac{1}{2}-\delta}(a_+ \mathbb{1}_{\{t>0\}} + a_- \mathbb{1}_{\{t<0\}}) + \mathcal{O}(|t|^{\frac{3}{2}-\delta})$$
$$:= J_{1,0}(t) + J_{1,1}(t) + \mathcal{O}(|t|^{\frac{3}{2}-\delta}).$$

Finally, as we did in the case $\delta = 1/2$, we reach

$$K_1(\xi) = B|\xi|^{-1} + \mathcal{O}(|\xi|^{-3}). \tag{52}$$

It remains to estimate K_2.

The function J_2 is differentiable, so we consider $J_{2,2}(t) := J_2(t) - J_2(0) - tJ_2'(0)$. For $t > 0$,

$$J_{2,2}(t) = \int_0^1 [(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta} - (2\theta_n)^{2\delta} - 2\delta(2\theta_n)^{2\delta-2}t]\, dS(\theta).$$

As we have done until now, we replace $dS(\theta)$ by $\omega_{n-2} d\theta_n$:

$$J_{2,2}(t) = \omega_{n-2} \int_0^1 [(\theta_n + \sqrt{\theta_n^2 + t})^{2\delta} - (2\theta_n)^{2\delta} - 2\delta(2\theta_n)^{2\delta-2}t]\, d\theta_n + \mathcal{O}(t^{\frac{3}{2}+\delta}),$$

and then we dilate:

$$\begin{aligned} J_{2,2}(t) &= t^{\frac{1}{2}+\delta} \int_0^{\frac{1}{\sqrt{t}}} [(\theta_n + \sqrt{\theta_n^2 + 1})^{2\delta} - (2\theta_n)^{2\delta} - 2\delta(2\theta_n)^{2\delta-2}] d\theta_n + \mathcal{O}(t^{\frac{3}{2}+\delta}) \\ &= t^{\frac{1}{2}+\delta} \int_0^{\infty} [(\theta_n + \sqrt{\theta_n^2 + 1})^{2\delta} - (2\theta_n)^{2\delta} - 2\delta(2\theta_n)^{2\delta-2}] d\theta_n + \mathcal{O}(t^{\frac{3}{2}+\delta}) \\ &:= b_+ t^{\frac{1}{2}+\delta} + \mathcal{O}(t^{\frac{3}{2}+\delta}). \end{aligned}$$

In the same way, for $t < 0$, we have

$$\begin{aligned} J_{2,2}(t) &= |t|^{\frac{1}{2}+\delta} \int_1^{\infty} [(\theta_n + \sqrt{\theta_n^2 - 1})^{2\delta} - (2\theta_n)^{2\delta} + 2\delta(2\theta_n)^{2\delta-2}] d\theta_n + \mathcal{O}(|t|^{\frac{3}{2}+\delta}) \\ &:= b_- |t|^{\frac{1}{2}+\delta} + \mathcal{O}(|t|^{\frac{3}{2}+\delta}). \end{aligned}$$

Therefore,

$$J_2(t) = J_2(0) + J_2'(0)t + |t|^{\frac{1}{2}+\delta}(b_+ \mathbb{1}_{\{t>0\}} + b_- \mathbb{1}_{\{t<0\}}) + \mathcal{O}(|t|^{\frac{3}{2}+\delta}),$$

which leads, as before, to

$$K_2(\xi) = A|\xi|^{2\delta} + B|\xi|^{-1} + \mathcal{O}(|\xi|^{-3}). \tag{53}$$

We join both estimates, (52) and (53), to get

$$K(\xi) = A|\xi|^{2\delta} + B|\xi|^{-1} + \mathcal{O}(|\xi|^{-3}),$$

which implies the last asymptotic expansion (50) after replacing it in (51); recall Lemma 14. □

3.2 Regularity of h_δ

Corollary 11 says that h_δ is continuous, however we can improve our estimates and refine the information about regularity.

The space of Lipschitz functions $\Lambda^\alpha(\mathbb{R}^n)$, for $\alpha > 0$, is

$$\Lambda^\alpha(\mathbb{R}^n) := \{f \in L^\infty(\mathbb{R}^n) \mid \\ \|P_{[2^k,2^{k+1}]}f\|_\infty \leqslant C2^{-\alpha k}, \text{ for } k \geqslant 0, \text{ and } \|P_{[0,1]}f\|_\infty \leqslant C\}. \tag{54}$$

If $f \in \Lambda^\alpha(\mathbb{R})$, for $0 < \alpha < 1$, then $|f(x) - f(y)| \leqslant C|x-y|^\alpha$; see Ch. V.4 of [23].

Theorem 17. *If $f \in \Sigma_\delta$, for $0 < \delta < 1$, then*

$$\|\psi h_\delta[f]\|_{\Lambda^\alpha} \lesssim C_\psi \|f\|_{\Sigma_\delta}^2 \tag{55}$$

where $\psi \in C_0^\infty(\mathbb{R})$ and

$$\alpha = \begin{cases} 2\delta & \text{for } n \geqslant 2, \text{ or for } n = 1 \text{ and } \delta < \frac{1}{2}, \\ 1- & \text{for } n = 1 \text{ and } \delta = \frac{1}{2}, \\ \frac{1}{4} + \frac{3}{2}\delta & \text{for } n = 1 \text{ and } \delta > \frac{1}{2}. \end{cases}$$

The result is best possible—up to the end point in the case $n = 1$ and $\delta = \frac{1}{2}$. In particular, $h_\delta \in C^1_{loc}(\mathbb{R})$ when $\delta > \frac{1}{2}$.

Proof. Since $P_{\leqslant 1}h_\delta$ and its derivatives are bounded in compact sets by the Nahas-Ponce inequality (3), then it suffices to prove that $P_{\geqslant 1}h_\delta \in \Lambda^\alpha(\mathbb{R})$. Since h_δ is real, then $\hat{h}_\delta(\tau) = \overline{\hat{h}}_\delta(-\tau)$ and we only need to work with positive frequencies. Hence, by the Hausdorff-Young inequality, it suffices to prove

$$\|\hat{h}_\delta\|_{L^1(\tau \simeq 2^k)} \leqslant C\|f\|^2_{\Sigma_\delta} \begin{cases} 2^{-2\delta k} & \text{for } n \geqslant 2, \text{ or for } n = 1 \text{ and } \delta \leqslant \frac{1}{2} \\ 2^{-(\frac{1}{4}+\frac{3}{2}\delta)k} & \text{for } n = 1 \text{ and } \delta > \frac{1}{2}. \end{cases}$$

We define $I_\lambda := [\lambda, 2\lambda]$, for $\lambda \geqslant 1$, and re-scale (30) so as to get, for $|g| \leqslant 1$,

$$|\langle \hat{h}_\delta, g\mathbb{1}_{I_\lambda}\rangle| \leqslant C\lambda^{\frac{n}{2}-\delta} \int_{\mathbb{R}^{2n}} |\hat{f}(\sqrt{\lambda}\xi)\hat{f}(\sqrt{\lambda}\eta)| \mathbb{1}_I\Big(\frac{|\eta|^2 - |\xi|^2}{2}\Big) \frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}, \tag{56}$$

where $I = [1,2]$.

To bound the integral over the region $\{|\xi| > 1\}$, we begin with

$$|\langle \hat{h}_\delta, g\mathbb{1}_{I_\lambda}\rangle|_{\{|\xi|>1\}} \leqslant C\lambda^{\frac{n}{2}-\delta} \int_{|\xi|>1} |\hat{f}(\sqrt{\lambda}\xi)|^2 \Big[\int \mathbb{1}_{I\cup -I}\Big(\frac{|\eta|^2 - |\xi|^2}{2}\Big) \frac{d\eta}{|\xi-\eta|^{n+2\delta}} \Big]\, d\xi$$
$$:= C\lambda^{\frac{n}{2}-\delta} \int_{|\xi|>1} |\hat{f}(\sqrt{\lambda}\xi)|^2 J(\xi)\, d\xi;$$

compare with (31). We use (32), for $a = 1$, to find out

$$\begin{aligned} |\langle \hat{h}_\delta, g\mathbb{1}_{I_\lambda}\rangle|_{\{|\xi|>1\}} &\leqslant C\lambda^{\frac{n}{2}-\delta} \int |\hat{f}(\sqrt{\lambda}\xi)|^2 |\xi|^{2\delta}\, d\xi \\ &\leqslant C\lambda^{-2\delta} \|f\|^2_{\Sigma_\delta}. \end{aligned} \tag{57}$$

To bound the integral over the region $\{|\xi| < 1\}$, we begin with (56) and notice that the factor $\mathbb{1}_I((|\eta|^2 - |\xi|^2)/2)$ forces $|\eta| \simeq 1$. Hence,

$$\begin{aligned} |\langle \hat{h}_\delta, g\mathbb{1}_{I_\lambda}\rangle|_{\{|\xi|<1\}} &\leqslant C\lambda^{\frac{n}{2}-\delta} \int_{|\xi|<1, |\eta|\simeq 1} |\hat{f}(\sqrt{\lambda}\xi)\hat{f}(\sqrt{\lambda}\eta)|\, d\xi d\eta \\ &\leqslant C\lambda^{-\frac{n}{2}-\delta} \int_{|\xi|<\sqrt{\lambda}} |\hat{f}(\xi)|\, d\xi \int_{|\eta|\simeq\sqrt{\lambda}} |\hat{f}(\eta)|\, d\eta \\ &\leqslant C\lambda^{-\frac{n}{4}-\frac{3}{2}\delta} \Big(\int_{|\xi|<\sqrt{\lambda}} |\hat{f}(\xi)|\, d\xi \Big) \||\eta|^\delta \hat{f}\|_2. \end{aligned} \tag{58}$$

We control the term in parentheses as

$$\int_{|\xi|<\sqrt{\lambda}} |\hat{f}(\xi)|\, d\xi \lesssim \Big(\int_{|\xi|<1} |\hat{f}(\xi)|^2\, d\xi \Big)^{\frac{1}{2}} +$$
$$+ \Big(\int_{1<|\xi|<\sqrt{\lambda}} |\xi|^{-2\delta}\, d\xi \Big)^{\frac{1}{2}} \Big(\int_{1<|\xi|<\sqrt{\lambda}} |\xi|^{2\delta} |\hat{f}(\xi)|^2\, d\xi \Big)^{\frac{1}{2}};$$

after replacing in (58) we arrive to

$$|\langle \hat{h}_\delta, g\mathbb{1}_{I_\lambda}\rangle|_{\{|\xi|<1\}} \lesssim \|f\|^2_{\Sigma_\delta} \begin{cases} \lambda^{-2\delta} & \text{for } n \geqslant 2, \text{ or } n = 1 \text{ and } \delta < \frac{1}{2}, \\ \lambda^{-1}\sqrt{\log\lambda} & \text{for } n = 1 \text{ and } \delta = \frac{1}{2}, \\ \lambda^{-\frac{1}{4}-\frac{3}{2}\delta} & \text{for } n = 1 \text{ and } \delta > \frac{1}{2}, \end{cases}$$

which together with (57) implies (55)—notice that $\frac{1}{4} + \frac{3}{2}\delta < 2\delta$.

Sharpness of the regularity

We consider functions $\hat{f}_\alpha(\xi) := \langle\xi\rangle^{-\alpha}$, for $\alpha = \frac{n}{2} + \delta +$. The Fourier transform of $h_\delta[f]$ is symmetric and, for $\tau > 0$, it equals

$$\hat{h}_\delta(\tau) = -2b_{n,\delta}\tau^{\frac{n}{2}-1-\delta} \int \hat{f}_\alpha(\sqrt{\tau}\xi)\hat{f}_\alpha(\sqrt{\tau}\eta)\delta\Big(1 - \frac{|\eta|^2 - |\xi|^2}{2}\Big)\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}$$

Since $\hat{h}_\delta \leqslant 0$, it is enough to prove that $|\hat{h}_\delta(\tau)| \geqslant c\tau^{-1-\beta}$ for $|\tau| \gg 1$, where

$$\beta = \begin{cases} 2\delta+ & \text{for } n \geqslant 2, \text{ or for } n = 1 \text{ and } \delta < \frac{1}{2} \\ \frac{1}{4} + \frac{3}{2}\delta+ & \text{for } n = 1 \text{ and } \delta \geqslant \frac{1}{2}. \end{cases} \tag{59}$$

In fact, if $\{\zeta_I\}$ is a cut-off function of $I := \{2^k \leqslant |\tau| \leqslant 2^{k+1}\}$, then

$$c2^{-\beta k} \leqslant \|\zeta_I \hat{h}_\delta\|_1 = |P_I h_\delta(0)| \leqslant \|P_I h_\delta\|_\infty,$$

and $\hat{h}_\delta \notin \Lambda^{\beta+}(\mathbb{R})_{\mathrm{loc}}$.

We use spherical coordinates and bound $\hat{h}_\delta$ from below as

$$\begin{aligned} |\hat{h}_\delta(\tau)| \geqslant c\tau^{\frac{n}{2}-1-\delta} \int_{\tau^{-\frac{1}{2}}<r_1<1} & \hat{f}_\alpha(\sqrt{\tau}r_1)\hat{f}_\alpha(\sqrt{\tau}r_2)\delta\Big(1 - \frac{r_2^2 - r_1^2}{2}\Big) \\ & \Big[\int_{S^{n-1}\times S^{n-1}} \frac{d\theta_1 d\theta_2}{|r_1\theta_1 - r_2\theta_2|^{n+2\delta}}\Big] r_1^{n-1} r_2^{n-1} dr_1 dr_2. \end{aligned} \tag{60}$$

We denote by $J(r_1, r_2)$ the term inside parentheses; by rotational symmetry

$$J(r_1, r_2) = c\int_{S^{n-1}} \frac{d\theta}{|r_1 e_n - r_2\theta|^{n+2\delta}}.$$

The term $\delta(1 - (r_2^2 - r_1^2)/2)$ forces $r_2 \simeq 1$, so $J(r_1, r_2) \gtrsim 1$, and from (60) we deduce

$$\begin{aligned} |\hat{h}_\delta(\tau)| &\geqslant c\tau^{\frac{n}{2}-1-\delta} \int_{\tau^{-\frac{1}{2}}}^1 \frac{1}{\tau^\alpha r_1^\alpha} \int_0^\infty \delta\Big(1 - \frac{r_2^2 - r_1^2}{2}\Big)\, dr_2\, r_1^{n-1} dr_1 \\ &\geqslant c\tau^{\frac{n}{2}-1-\delta-\alpha} \int_{\tau^{-\frac{1}{2}}}^1 r_1^{n-1-\alpha}\, dr_1 \end{aligned}$$

Since $\alpha = \frac{n}{2} + \delta+$, we conclude, for $|\tau| \gg 1$, that

$$|\hat{h}_\delta(\tau)| \geqslant c \begin{cases} \tau^{-1-2\delta-} & \text{for } n \geqslant 2, \text{ or for } n = 1 \text{ and } \delta < \frac{1}{2} \\ \tau^{-1-\frac{1}{4}-\frac{3}{2}\delta-} & \text{for } n = 1 \text{ and } \delta \geqslant \frac{1}{2}, \end{cases}$$

which implies (59).

As a final remark, if f is one of the examples we used, then $h_\delta[f]$, which is an even function, has a singularity at zero of the form $|t|^\rho$. By translation in time, we can place the singularity at any other time. □

We may compare the regularity of h_δ with its classical counterpart $h^c_\delta[x,\xi](t) := |x+t\xi|^{2\delta}$, which belongs to $\Lambda^{2\delta}_{\rm loc}(\mathbb{R})$. If $n \geqslant 2$ then h^c_δ is smooth in general, but if $n=1$ then h^c_δ is singular in general, which agrees with the loss of regularity in Theorem 17 when $n=1$.

When $\delta > 1/2$ we can give an alternative proof of Theorem 17, which we sketch below.

Alternative proof of Thm. 17 when $\delta < 1/2$. Suppose we have proved (55) in the simpler case $n=1$. We write $|x|^{2\delta} = c\int_{S^{n-1}} |\omega\cdot x|^{2\delta}\,dS(\omega)$ so that

$$h_\delta[f](t) = c\int_{S^{n-1}}\int |x_1|^{2\delta}|u(R_\omega x,t)|^2\,dx dS(\omega),$$

where R_ω is a rotation. We write $f_{R_\omega}(x) := f(R_\omega x)$ so that

$$u(R_\omega x,t) = \int\Big[\int \hat f_{R_\omega}(\xi)e^{2\pi i x_1\xi_1 - \pi i t\xi_1^2}\,d\xi_1\Big]e^{2\pi i x'\cdot\xi' - \pi i t|\xi'|^2}\,d\xi'$$

By Plancherel

$$\|u(R_\omega x,t)\|_{L^2_{x'}} = \|e^{it\hbar\partial_1^2/2} f_{R_\omega}(x_1,x')\|_{L^2_{x'}},$$

then we can apply (55) to the function $x_1 \mapsto f_{R_\omega}(x_1,x')$ and arrive at

$$\|\psi h_\delta[f]\|_{\Lambda^\alpha} \lesssim \int_{S^{n-1}}\int_{\mathbb{R}^{n-1}} \|\psi h_\delta[f_{R_\omega}(\cdot,x')]\|_{\Lambda^{2\delta}}\,dx' dS(\omega) \lesssim_\psi \|f\|^2_{\Sigma_\delta}.$$

□

In the following theorem we investigate the rate of decay of $\hat h_\delta$; however, first we have to prove an auxiliary result.

Lemma 18. *Let $n \geqslant 1$ and let r_1 and r_2 be different, positive numbers. If $\alpha > n-1$ and $A, B \in L^2(S^{n-1})$, then*

$$\int_{S^{n-1}\times S^{n-1}} \frac{A(\theta_1)B(\theta_2)\,d\theta_1 d\theta_2}{|r_1\theta_1 - r_2\theta_2|^\alpha} \leqslant C_{r_1,r_2}\|A\|_{L^2(S^{n-1})}\|B\|_{L^2(S^{n-1})} \tag{61}$$

where

$$C_{r_1,r_2} \lesssim \begin{cases} 1/r_1^\alpha & \text{for } r_1 > 2r_2, \\ 1/r_2^\alpha & \text{for } r_2 > 2r_1, \\ (r_1r_2)^{-\frac{n-1}{2}}|r_1 - r_2|^{n-1-\alpha} & \text{for } \frac{1}{2}r_2 < r_1 < 2r_2. \end{cases}$$

Proof. We assume that $r_2 < r_1$ and that $\|A\|_2 = \|B\|_2 = 1$, so

$$\begin{aligned}\int \frac{A(\theta_1)B(\theta_2)\,d\theta_1 d\theta_2}{|r_1\theta_1 - r_2\theta_2|^\alpha} &\leqslant \frac{1}{2}\sup_{\theta_1}\int \frac{d\theta_2}{|r_1\theta_1 - r_2\theta_2|^\alpha} + \frac{1}{2}\sup_{\theta_2}\int \frac{d\theta_1}{|r_1\theta_1 - r_2\theta_2|^\alpha} \\ &= \frac{1}{r_2^\alpha}\int_{S^{n-1}} \frac{d\theta}{|\rho\theta - e_n|^\alpha},\end{aligned}$$

where $\rho := r_1/r_2 > 1$.

When $\rho \geqslant 2$, we notice that $|\rho\theta - e_n| \geqslant \rho/2$, so

$$\int \frac{d\theta}{|\rho\theta - e_n|^\alpha} \leqslant C\frac{1}{\rho^\alpha},$$

which implies $C_{r_1,r_2} \lesssim 1/r_1^\alpha$, for $r_1 > 2r_2$.

When $\rho < 2$, we notice that

$$\begin{aligned}|\rho(\theta - e_n) + (\rho-1)e_n|^2 &= 2\rho^2(\theta_n - 1)^2 + (\rho-1)^2 + 2\rho(\rho-1)(\theta_n - 1) \\ &\geqslant 2\rho^2(\theta_n - 1)^2 + (\rho-1)^2 - a\rho^2(\theta_n-1)^2 - a^{-1}(\rho-1)^2,\end{aligned}$$

so we can take either $a = 2$ or $a = 1$ to see $|\rho\theta - e_n| \geqslant c\max\{\rho|\theta - e_n|, \rho - 1\}$. Hence,

$$\begin{aligned}\int \frac{d\theta}{|\rho\theta - e_n|^\alpha} &\lesssim \rho^{-\alpha}\int_{\rho|\theta - e_n| > \rho - 1} \frac{d\theta}{|\theta - e_n|^\alpha} + (\rho-1)^{-\alpha}\int_{\rho|\theta - e_n| < \rho - 1} d\theta \\ &\simeq (\rho-1)^{n-1-\alpha}/\rho^{n-1},\end{aligned}$$

which implies $C_{r_1,r_2} \lesssim |r_1 - r_2|^{n-1-\alpha}(r_1r_2)^{-\frac{n-1}{2}}$, for $r_2 < r_1 < 2r_2$. □

Theorem 19. *If $f \in \Sigma_\delta$, for $0 < \delta < 1$, then for $|\tau| \geqslant 1$ it holds*

$$|\hat{h}_\delta[f](\tau)| \leqslant C\|f\|_{\Sigma_\delta}^2 \begin{cases} \tau^{-1-2\delta} & \text{for } n \geqslant 3 \text{ and } \delta \leqslant \frac{n}{2} - 1, \\ \tau^{-\frac{n+2}{4} - \frac{3}{2}\delta} & \text{for } n = 2, 3 \text{ and } \delta > \frac{n}{2} - 1, \\ \tau^{-\frac{3}{4} - \frac{3}{2}\delta +} & \text{for } n = 1. \end{cases} \quad a.e. \qquad (62)$$

The rate of decay is best possible—up to the end point when $n = 1$.

Theorem 19 provides an alternative proof of the Theorem 17 when $n \geqslant 3$ and $\delta \leqslant \frac{n}{2} - 1$.

Proof. We can assume that $f \in C_0^\infty(\mathbb{R}^n)$. In fact, for general $f \in \Sigma_\delta(\mathbb{R}^n)$ we can take a sequence of functions $\{f_n\}_n$ in $C_0^\infty(\mathbb{R}^n)$ (Lemma 7) converging to f in $\Sigma_\delta(\mathbb{R}^n)$. If we reprise the arguments in the proof of (28) we can see that

$$\|\hat{h}[f] - \hat{h}[g]\|_{L^1(\mathbb{R}\setminus[-1,1])} \leqslant C\|f - g\|_{\Sigma_\delta}(\|f\|_{\Sigma_\delta} + \|g\|_{\Sigma_\delta}).$$

Thus, we can assume that $h_\delta[f_n] \to h_\delta[f]$ a.e. and we are done.

Since $\hat{h}_\delta(-\tau) = \overline{\hat{h}}_\delta(\tau)$, we assume $\tau > 0$. We re-scale (26) to write

$$\hat{h}_\delta(\tau) = -2b_{n,\delta}\tau^{\frac{n}{2}-\delta-1}\int \hat{f}(\sqrt{\tau}\xi)\overline{\hat{f}}(\sqrt{\tau}\eta)\delta\Big(1 - \frac{|\eta|^2 - |\xi|^2}{2}\Big)\frac{d\xi d\eta}{|\xi - \eta|^{n+2\delta}};$$

passing to spherical coordinates we have

$$|\hat{h}_\delta(\tau)| \leqslant C\tau^{\frac{n}{2}-\delta-1}\int \delta\Big(1 - \frac{r_2^2 - r_1^2}{2}\Big)r_1^{n-1}r_2^{n-1} \Big[\int_{S^{n-1}\times S^{n-1}} \frac{|\hat{f}(\sqrt{\tau}r_1\theta_1)\hat{f}(\sqrt{\tau}r_2\theta_2)|\, d\theta_1 d\theta_2}{|r_1\theta_1 - r_2\theta_2|^{n+2\delta}}\Big]\, dr_1 dr_2.$$

The term $\delta(1 - (r_2^2 - r_1^2)/2)$ forces $|r_2/r_1| = \sqrt{1 + 2/r_1^2}$.

We apply Lemma 18 to the term in parentheses to deduce

$$\begin{aligned} |\hat{h}_\delta(\tau)| &\leqslant C\tau^{\frac{n}{2}-\delta-1}\int \delta\Big(1 - \frac{r_2^2 - r_1^2}{2}\Big)r_1^{n-1}r_2^{n-1}\|\hat{f}(\sqrt{\tau}r_1\cdot)\|_2\|\hat{f}(\sqrt{\tau}r_2\cdot)\|_2 \\ &\qquad \Big[\mathbb{1}_{\{r_1>\sqrt{2/3}\}}\frac{1}{(r_1r_2)^{\frac{n}{2}-1-\delta}} + \mathbb{1}_{\{r_1<\sqrt{2/3}\}}\frac{1}{r_2^{n+2\delta}}\Big]\, dr_1dr_2 \\ &:= I_{\{r_1>\sqrt{2/3}\}} + I_{\{r_1<\sqrt{2/3}\}}. \end{aligned} \tag{63}$$

We bound the contribution over the region $\{r_1 > \sqrt{2/3}\}$ as

$$\begin{aligned} I_{\{r_1>\sqrt{2/3}\}} &\leqslant C\tau^{\frac{n}{2}-\delta-1}\int_{r_1>\sqrt{2/3}} \delta\Big(1 - \frac{r_2^2 - r_1^2}{2}\Big) \\ &\qquad \Big[r_1^{n+2\delta}\|\hat{f}(\sqrt{\tau}r_1\cdot)\|_2^2 + r_2^{n+2\delta}\|\hat{f}(\sqrt{\tau}r_2\cdot)\|_2^2\Big]\, dr_1dr_2 \\ &\leqslant C\tau^{\frac{n}{2}-\delta-1}\int_{r>c} r^{n-1+2\delta}\|\hat{f}(\sqrt{\tau}r\cdot)\|_2^2\, dr \\ &\leqslant C\tau^{-1-2\delta}\||\xi|^\delta\hat{f}\|_2^2. \end{aligned} \tag{64}$$

It remains to control the integral over $\{r_1 < \sqrt{2/3}\}$.

When $r_1 < c$, the term $\delta(1-(r_2^2-r_1^2)/2)$ forces $r_2 \simeq 1$, so

$$I_{\{r_1<\sqrt{2/3}\}} \leqslant C\tau^{\frac{n}{2}-\delta-1}\int_{r_1<\sqrt{2/3}} r_1^{n-1}\|\hat{f}(\sqrt{\tau}r_1\cdot)\|_2\|\hat{f}(\sqrt{\tau}(2+r_1^2)^{\frac{1}{2}}\cdot)\|_2\,dr_1$$

We leave aside momentarily the case $n=1$. We use Hölder to get (we write $r=r_1$)

$$I_{\{r_1<\sqrt{2/3}\}} \leqslant C\tau^{\frac{n}{2}-\delta-1}\Big(\int_{r<c} r^{2(n-1)-1}\|\hat{f}(\sqrt{\tau}r\cdot)\|_2^2\,dr\Big)^{\frac{1}{2}} \Big(\int_{r<c} r\|\hat{f}(\sqrt{\tau}(2+r^2)^{\frac{1}{2}}\cdot)\|_2^2\,dr\Big)^{\frac{1}{2}},$$

after the change of variable $t=\sqrt{2+r^2}$ we get

$$\begin{aligned} I_{\{r_1<\sqrt{2/3}\}} &\leqslant C\tau^{\frac{n}{2}-\delta-1}\Big(\int_{|\xi|<c}|\xi|^{n-2}|\hat{f}(\sqrt{\tau}\xi)|^2\,d\xi\Big)^{\frac{1}{2}}\Big(\int_{|\xi|\simeq 1}|\hat{f}(\sqrt{\tau}\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \\ &\leqslant C\tau^{\frac{n}{4}-\frac{3}{2}\delta-1}\Big(\int_{|\xi|<c}|\xi|^{n-2}|\hat{f}(\sqrt{\tau}\xi)|^2\,d\xi\Big)^{\frac{1}{2}}\||\xi|^\delta\hat{f}\|_2. \end{aligned}$$

If $n-2\geqslant 2\delta$, then we bound the last integral in parentheses as

$$\int_{|\xi|<c}|\xi|^{n-2}|\hat{f}(\sqrt{\tau}\xi)|^2\,d\xi \leqslant C\int_{|\xi|<c}|\xi|^{2\delta}|\hat{f}(\sqrt{\tau}\xi)|^2\,d\xi \leqslant C\tau^{-\frac{n}{2}-\delta}\||\xi|^\delta\hat{f}\|_2^2,$$

so $I_{\{r_1<\sqrt{2/3}\}} \leqslant C\tau^{-2\delta-1}$, which together with (63) and (64) implies the first case in (62).

If $0\leqslant n-2<2\delta$, then

$$\begin{aligned} \int_{|\xi|<c}|\xi|^{n-2}|\hat{f}(\sqrt{\tau}\xi)|^2\,d\xi &\leqslant \tau^{-n+1}\int_{|\xi|<\sqrt{\tau}c}|\xi|^{n-2}|\hat{f}(\xi)|^2\,d\xi \\ &\leqslant C\tau^{-n+1}\|f\|_{\Sigma_\delta}^2. \end{aligned}$$

so $I_{\{r_1<\sqrt{2/3}\}} \leqslant C\tau^{-\frac{n+2}{4}-\frac{3}{2}\delta}$, which together with (63) and (64) implies the second case in (62).

Now we consider the case $n=1$, so we have to bound the integral

$$I'_{\{r<\sqrt{2/3}\}} := C\tau^{-\frac{1}{2}-\delta}\int_{r<c}|\hat{f}(\sqrt{\tau}r)||\hat{f}(\sqrt{\tau}(2+r^2)^{\frac{1}{2}})|\,dr.$$

We intend to use the embedding $\Sigma_\delta \hookrightarrow L^p$ in Lemma 9. We apply Hölder inequality (twice) and the change of variables $t = \sqrt{2+r^2}$ to get

$$\begin{aligned} I'_{\{r<\sqrt{2/3}\}} &\leqslant C\tau^{-\frac{1}{2}-\delta}\Big(\int_{r<c}|\hat{f}(\sqrt{\tau}r)|^p\,dr\Big)^{\frac{1}{p}}\Big(\int_{\sqrt{2}}^{\sqrt{2+c^2}}|\hat{f}(\sqrt{\tau}t)|^{p'}\frac{dt}{\sqrt{t^2-2}}\Big)^{\frac{1}{p'}} \\ &\leqslant C_\varepsilon\tau^{-\frac{1}{2}-\delta-\frac{1}{2p}}\|\hat{f}\|_p\Big(\int_{t\simeq 1}|\hat{f}(\sqrt{\tau}t)|^{(2+\varepsilon)p'}\,dt\Big)^{\frac{1}{(2+\varepsilon)p'}}. \end{aligned}$$

If $\delta < \frac{1}{2}$, then we take $\frac{1}{(2+\varepsilon)p'} = \frac{1}{2}-\delta$, in which case $\frac{1}{p} = (2+\varepsilon)\delta - \varepsilon/2 < \frac{1}{2}+\delta$, so that

$$I'_{\{r<\sqrt{2/3}\}} \leqslant C_\varepsilon\tau^{-\frac{1}{2}-\delta-\frac{1}{2p}-\frac{1}{2(2+)p'}}\|\hat{f}\|_p\|\hat{f}\|_{(2+)p'} \leqslant C\tau^{-\frac{3}{4}-\frac{3}{2}\delta+}\|f\|^2_{\Sigma_\delta},$$

which together with (63) and (64) implies the third case in (62), for $\delta < \frac{1}{2}$.

If $\delta \geqslant \frac{1}{2}$, then we take p very large and notice that

$$\Big(\int_{t\simeq 1}|\hat{f}(\sqrt{\tau}t)|^{(2+\varepsilon)p'}\,dt\Big)^{\frac{1}{(2+\varepsilon)p'}} \leqslant \Big(\int_{t\simeq 1}|\hat{f}(\sqrt{\tau}t)|^2\,dt\Big)^{\frac{\theta}{2}}\Big(\int|\hat{f}(\sqrt{\tau}t)|^p\,dt\Big)^{\frac{1-\theta}{p}},$$

where $\frac{1}{(2+)p'} = \frac{\theta}{2}+\frac{1-\theta}{p}$, so $0<\theta<1$ can be made arbitrarily close to 1 if $p \gg 1$. Hence,

$$\begin{aligned} I'_{\{r<\sqrt{2/3}\}} &\leqslant C_\varepsilon\tau^{-\frac{1}{2}-\delta-\frac{2-\theta}{2p}}\|\hat{f}\|_p^{2-\theta}\Big(\int_{t\simeq 1}|\hat{f}(\sqrt{\tau}t)|^2\,dt\Big)^{\frac{\theta}{2}} \\ &\leqslant C_\varepsilon\tau^{-\frac{3}{4}-\frac{3}{2}\delta+}\|\hat{f}\|_p^{2-\theta}\||\xi|^\delta\hat{f}\|_2^\theta \\ &\leqslant C\tau^{-\frac{3}{4}-\frac{3}{2}\delta+}\|f\|^2_{\Sigma_\delta}, \end{aligned}$$

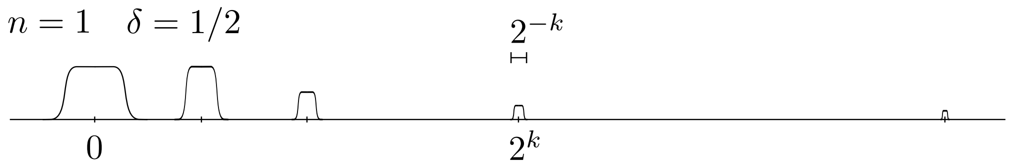
which together with (63) and (64) concludes the proof of the last case in (62).

Sharpness of the rate of decay

The example used in Theorem 17 shows that the decay $|\tau|^{-1-2\delta}$, for $f\in\Sigma_\delta$, cannot be improved, so we turn to the case $n\leqslant 3$.

Let $\zeta \in C_0^\infty(\mathbb{R})$ be a symmetric cut-off of B_1, and let dS_k denote the standard measure on the sphere with radius 2^k and center at the origin. To construct the example, we define $\zeta_k(\xi) := 2^{k(n-1)}\zeta(2^k\xi)$ and set

$$\hat{f}(\xi) := \zeta(\xi) + \sum_{k\geqslant 1} 2^{-k(\frac{n-2}{2}+\delta)}\frac{1}{k^2}(\zeta_k * dS_k)(\xi).$$



Direct computation shows that $\| |\xi|^\delta \hat{f} \|_2 < \infty$, so we must show that $\| |x|^\delta f \|_2 < \infty$; we only consider the harder case $n \geqslant 2$.

By the triangle inequality

$$\| |x|^\delta f \|_2 \leqslant \| |x|^\delta \check{\zeta} \|_2 + \sum_{k \geqslant 1} 2^{-k(\frac{n-2}{2}+\delta)} \frac{1}{k^2} \| |x|^\delta \check{\zeta}_k (dS_k)^\vee \|_2,$$

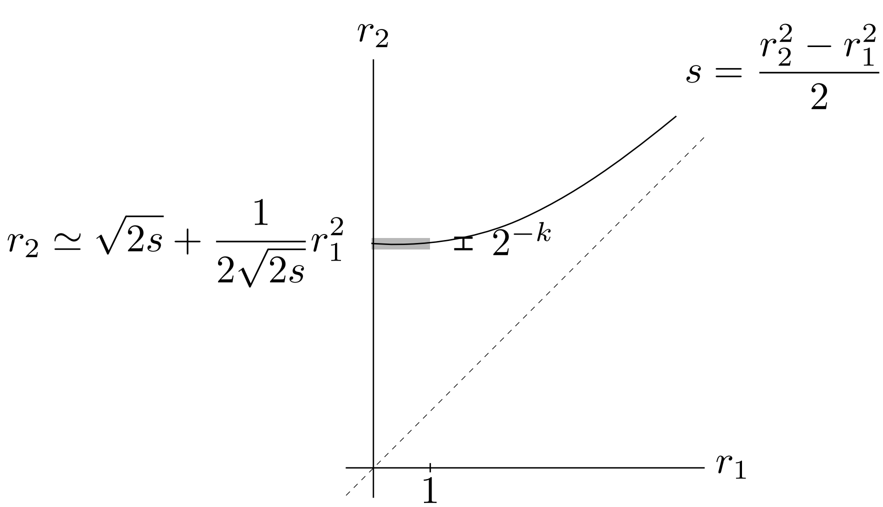
After the dilation $x \mapsto 2^{-k} x$, each term in the sum gets into

$$\| |x|^\delta \check{\zeta}_k (dS_k)^\vee \|_2 = 2^{k(\frac{n}{2}-2-\delta)} \| |x|^\delta \check{\zeta}(2^{-2k} x)(dS)^\vee \|_2,$$

From the inequality $|(dS)^\vee(\xi)| \lesssim \langle \xi \rangle^{-\frac{n-1}{2}}$ [24, Ch. VIII-3] we deduce that $\| |x|^\delta \check{\zeta}_k (dS_k)^\vee \|_2 \lesssim 2^{k(\frac{n-2}{2}+\delta)}$, which leads to $\| |x|^\delta f \|_2 < \infty$.

We estimate now $|\hat{h}_\delta(\tau)|$ for $\tau = 2^{2k-1}$ and $k \gg 1$:

$$\begin{aligned}
|\hat{h}_\delta(\tau)| &\geqslant c \frac{1}{k^2} 2^{-k(\frac{n-2}{2}+\delta)} \int_{\mathbb{R}^{2n}} \zeta(\xi)(\zeta_k * dS_k)(\eta) \delta\Big(\tau - \frac{\eta^2 - \xi^2}{2}\Big) \frac{d\xi d\eta}{|\xi - \eta|^{n+2\delta}} \\
&\geqslant c \frac{1}{k^2} 2^{-k(\frac{n-2}{2}+\delta)} \int_{\mathbb{R}^2} \zeta(r_1) \zeta(2^k(r_2 - \sqrt{2s})) \delta\Big(\tau - \frac{r_2^2 - r_1^2}{2}\Big) \frac{r_1^{n-1} dr_1 dr_2}{r_2^{1+2\delta}} \\
&\geqslant c \frac{1}{k^2} 2^{-k(\frac{n-2}{2}+\delta)} \tau^{-1-\delta} \int_{\mathbb{R}^2} \zeta(r_1) \zeta(2^k(\sqrt{2s + r_1^2} - \sqrt{2s})) r_1^{n-1} \, dr_1.
\end{aligned}$$



Since $\zeta(2^k(\sqrt{2s+r_1^2}-\sqrt{2s}))\gtrsim 1$ for $|\xi|<c$, then

$$|\hat{h}_\delta(\tau)|\geqslant c\frac{1}{k^2}\tau^{-\frac{n+2}{4}-\frac{3}{2}\delta}.$$

Hence, if $|\hat{h}_\delta(\tau)|\leqslant C\tau^{-\alpha}$, then $\alpha\leqslant\frac{n+2}{4}+\frac{3}{2}\delta$ and the rate of decay in (62) cannot be improved. □

4 Periodic Data

In the section we extend the definition of h_δ to solutions of the Schrödinger equation with periodic initial data, with the aim to define $h_\delta[f]$ when f is the Dirac comb.

We choose a real, symmetric function $\psi\in\mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\hat{\psi}\subset B_1$ and $\psi(0)=1$. Now we approach a periodic function F in $\mathbb{R}^n/\mathbb{Z}^n$ as

$$f_\varepsilon(x):=N_\varepsilon^{-1}\psi(\varepsilon x)F(x)=N_\varepsilon^{-1}\psi(\varepsilon x)\sum_{\nu\in\mathbb{Z}^n}\hat{F}(\nu)e(2\pi ix\cdot\nu),\tag{65}$$

where $N_\varepsilon^2=\varepsilon^{-n}\|\psi\|_2^2\|F\|_{L^2(\mathbb{T})}^2$ is the normalization constant; henceforth, we will assume that $\|F\|_{L^2(\mathbb{T})}=1$. The Fourier transform is

$$\hat{f}_\varepsilon(\xi)=N_\varepsilon^{-1}\frac{1}{\varepsilon^n}\sum_{\nu\in\mathbb{Z}^n}\hat{F}(\nu)\hat{\psi}((\xi-\nu)/\varepsilon).$$

We want to study how $h_\delta[f_\varepsilon]$ evolves as $\varepsilon\to 0$.

The Fourier transform of $h_\delta[f_\varepsilon]$ away from the origin is

$$\begin{aligned}\langle\hat{h}_\delta,\varphi\rangle&=-2b_{n,\delta}\int_{\mathbb{R}^{2n}}\hat{f}_\varepsilon(\xi)\overline{\hat{f}}_\varepsilon(\eta)\varphi\Big(\frac{|\eta|^2-|\xi|^2}{2}\Big)\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}\\&=-2b_{n,\delta}\frac{N_\varepsilon^{-2}}{\varepsilon^{2n}}\sum_{\nu_1,\nu_2}\hat{F}(\nu_1)\overline{\hat{F}}(\nu_2)\\&\qquad\int\hat{\psi}((\xi-\nu_1)/\varepsilon)\hat{\psi}((\eta-\nu_2)/\varepsilon)\varphi\Big(\frac{|\xi|^2-|\eta|^2}{2}\Big)\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}},\end{aligned}$$

where $\varphi\in\mathcal{S}(\mathbb{R})$ is supported away from the origin. In this expression we can distinguish two types of terms: diagonal ($\nu_1=\nu_2$) and off-diagonal ($\nu_1\neq\nu_2$). Diagonal terms are more related to the behavior of h_δ in the large, and off-diagonal terms are more related to the local phenomena we are interested in.

Definition 20 (Decomposition of h_δ)**.** Let F be a normalized periodic function in $\mathbb{R}^n/\mathbb{Z}^n$. The ε-periodic part $h_{\mathrm{p},\varepsilon,\delta}[F]$ (off-diagonal part) is given by

$$\langle \hat{h}_{\mathrm{p},\varepsilon,\delta}, \varphi\rangle := -\frac{2b_{n,\delta}}{\varepsilon^{2n}\|\psi\|_2^2}\sum_{\nu_1\neq\nu_2}\hat{F}(\nu_1)\overline{\hat{F}}(\nu_2) \int \hat{\psi}((\xi-\nu_1)/\varepsilon)\hat{\psi}((\eta-\nu_2)/\varepsilon)\varphi\Big(\frac{|\xi|^2-|\eta|^2}{2}\Big)\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}, \tag{66}$$

where $\varphi \in \mathcal{S}(\mathbb{R})$ is a test function. The ε-background part $h_{\mathrm{b},\varepsilon,\delta}$ (diagonal part) is given by

$$h_{\mathrm{b},\varepsilon,\delta}[F] := h_\delta[f_\varepsilon] - \varepsilon^n h_{\mathrm{p},\varepsilon,\delta}[F]. \tag{67}$$

Once we have defined the decomposition of h_δ, we concentrate for the moment on the behavior of the ε-periodic part $h_{\mathrm{p},\varepsilon,\delta}$ as ε tends to zero, but we need first a definition.

Definition 21. Let F be a normalized periodic function in $\mathbb{R}^n/\mathbb{Z}^n$. The periodic limit $h_{\mathrm{p},\delta}[F]$ is given by

$$\langle \hat{h}_{\mathrm{p},\delta}, \varphi\rangle := -\frac{2b_{n,\delta}}{\|\psi\|_2^2}\sum_{\nu_1\neq\nu_2}\hat{F}(\nu_1)\overline{\hat{F}}(\nu_2)\varphi\Big(\frac{|\nu_1|^2-|\nu_2|^2}{2}\Big)\frac{1}{|\nu_1-\nu_2|^{n+2\delta}}, \tag{68}$$

where $\varphi \in \mathcal{S}(\mathbb{R})$ is a test function.

Lemma 22. *Let F be a normalized periodic function such that $\hat{F} \in \ell^2(|\nu|^{2\delta})$. If $h_{p,\varepsilon,\delta}[F]$ and $h_{p,\delta}[F]$ are the distributions in* (66) *and* (68)*, respectively, then $h_{p,\varepsilon,\delta}[F]$ converges uniformly in compact sets to $h_{p,\delta}[F]$, and $\|\hat{h}_{p,\delta}[F]\|_{L^1} \lesssim 1$.*

Proof. The distribution $\hat{h}_{\mathrm{p},\varepsilon,\delta}[F]$ is an integrable function. In fact, we can bound $|\langle \hat{h}_{\mathrm{p},\varepsilon,\delta}, \varphi\rangle|$ as

$$|\langle \hat{h}_{\mathrm{p},\varepsilon,\delta}, \varphi\rangle| \leqslant C\|\varphi\|_\infty,$$

where C is independent from ε. The same arguments used in Theorem 10 to prove (28) show that $\|\hat{h}_{\mathrm{p},\varepsilon,\delta}\|_{L^1} \lesssim 1$, so there exists a measure μ and a sequence $\{h_{\mathrm{p},\varepsilon_k,\delta}\}_k$, with $\varepsilon_k \to 0$, that converges weakly* to μ with $|\mu|(\mathbb{R}) \lesssim 1$.

To evaluate the integral in (66) we fix a number $R \geqslant 1$ and notice that for (ξ,η) at distance ε from (ν_1,ν_2) we have two bounds: if $\{\max|\nu_i| > R\}$

then

$$\varphi\Big(\frac{|\xi|^2-|\eta|^2}{2}\Big)\frac{1}{|\xi-\eta|^{n+2\delta}}-\varphi\Big(\frac{|\nu_1|^2-|\nu_2|^2}{2}\Big)\frac{1}{|\nu_1-\nu_2|^{n+2\delta}}=\mathcal{O}\left(\frac{\|\varphi\|_\infty}{|\nu_1-\nu_2|^{n+2\delta}}\right),$$

and if $\{\max|\nu_i|<R\}$ then

$$\varphi\Big(\frac{|\xi|^2-|\eta|^2}{2}\Big)\frac{1}{|\xi-\eta|^{n+2\delta}}-\varphi\Big(\frac{|\nu_1|^2-|\nu_2|^2}{2}\Big)\frac{1}{|\nu_1-\nu_2|^{n+2\delta}}=\mathcal{O}\left(\frac{\|\varphi\|_\infty\varepsilon}{|\nu_1-\nu_2|^{n+2\delta+1}}+\frac{\|\varphi'\|_\infty R\varepsilon}{|\nu_1-\nu_2|^{n+2\delta}}\right).$$

With these two bounds we get the following estimate of (66):

$$\begin{aligned}\langle\hat{h}_{\mathrm{p},\varepsilon,\delta},\varphi\rangle=-\frac{2b_{n,\delta}}{\|\psi\|_2^2}\sum_{\nu_1\neq\nu_2}\hat{F}(\nu_1)\overline{\hat{F}}(\nu_2)\Big[\varphi\Big(\frac{|\nu_1|^2-|\nu_2|^2}{2}\Big)\frac{1}{|\nu_1-\nu_2|^{n+2\delta}}+\\ +\mathbb{1}_{\{\max|\nu_i|>R\}}\,\mathcal{O}\left(\frac{\|\varphi\|_\infty}{|\nu_1-\nu_2|^{n+2\delta}}\right)+\\ +\mathbb{1}_{\{\max|\nu_i|<R\}}\,\mathcal{O}\left(\frac{\|\varphi\|_\infty\varepsilon}{|\nu_1-\nu_2|^{n+2\delta+1}}+\frac{\|\varphi'\|_\infty R\varepsilon}{|\nu_1-\nu_2|^{n+2\delta}}\right)\Big]\\ :=\langle\hat{h}_{\mathrm{p},\delta},\varphi\rangle+\mathrm{E}_1+\mathrm{E}_2.\end{aligned}$$

We bound the first error term as

$$\begin{aligned}\mathrm{E}_1&\leqslant C\|\varphi\|_\infty\sum_{\nu_1}|\hat{F}(\nu_1)|^2\sum_{\max|\nu_i|>R}\frac{1}{|\nu_1-\nu_2|^{n+2\delta}}\\ &\leqslant C\|\varphi\|_\infty\sum_{\nu_1}|\hat{F}(\nu_1)|^2\Big(\mathbb{1}_{|\nu_1|<R/2}R^{-2\delta}+\mathbb{1}_{|\nu_1|>R/2}\Big)\\ &\leqslant C\|\varphi\|_\infty R^{-2\delta}(1+\|\hat{F}\|^2_{\ell^2(|\nu|^{2\delta})});\end{aligned}\tag{69}$$

we bound the second error term as

$$\mathrm{E}_2\leqslant C(\|\varphi\|_\infty\varepsilon+\|\varphi'\|_\infty R\varepsilon).\tag{70}$$

We have thus

$$\langle\hat{h}_{\mathrm{p},\varepsilon,\delta},\varphi\rangle=-\frac{2b_{n,\delta}}{\|\psi\|_2^2}\sum_{\nu_1\neq\nu_2}\hat{F}(\nu_1)\overline{\hat{F}}(\nu_2)\varphi\Big(\frac{|\nu_1|^2-|\nu_2|^2}{2}\Big)\frac{1}{|\nu_1-\nu_2|^{n+2\delta}}+\\ +\mathcal{O}(\|\varphi\|_\infty R^{-2\delta}+\|\varphi\|_\infty\varepsilon+\|\varphi'\|_\infty R\varepsilon).$$

Taking $R = \varepsilon^{-\frac{1}{2}}$ we see that $\langle \hat{h}_{\mathrm{p},\varepsilon,\delta}, \varphi \rangle \xrightarrow{\varepsilon \to 0} \langle \hat{h}_{\mathrm{p},\delta}, \varphi \rangle$ when $\varphi \in \mathcal{S}(\mathbb{R})$, which implies that $\mu = \hat{h}_{\mathrm{p},\delta}$ is unique and that $\|\hat{h}_{\mathrm{p},\delta}\|_{L^1} \lesssim 1$.

To compute $h_{\mathrm{p},\delta}$ we set $\varphi(\tau) = e^{2\pi i \tau t}$, and since the error terms (69) and (70) are uniform in t when $|t| \leqslant T$, for any $T > 0$, then we conclude that $h_{p,\varepsilon,\delta}$ converges uniformly in compact sets to $h_{\mathrm{p},\delta}$. $\square$

The function $h_{\mathrm{p},\delta}[F]$ is our desired extension of $h_\delta[f]$ to periodic functions, and we may write it as

$$\hat{h}_{p,\delta}[F](\tau) = -\frac{2b_{n,\delta}}{\|\psi\|_2^2} \sum_{k \in \mathbb{Z}} \delta_{\frac{k}{2}}(\tau) \sum_{\substack{\nu_1 \neq \nu_2 \\ |\nu_1|^2 - |\nu_2|^2 = k}} \hat{F}(\nu_1) \overline{\hat{F}}(\nu_2) \frac{1}{|\nu_1 - \nu_2|^{n+2\delta}}. \tag{71}$$

We observe that $h_{p,\delta}$ is a periodic function with period 2.

Lemma 22 says that we can recover $h_{\mathrm{p},\delta}[F]$ if we remove $h_{\mathrm{b},\varepsilon,\delta}[F]$ from $h_\delta[f_\varepsilon]$—recall (65)—multiply by ε^{-n} and then take the limit as $\varepsilon \to 0$, *i.e.* we can recover $h_{\mathrm{p},\delta}[F]$ if we renormalize $h_\delta[f_\varepsilon]$.

We investigate now the background of $h_\delta[f_\varepsilon]$; this function contains the information of $\hat{h}_\delta[f_\varepsilon]$ around zero, which we already described in Section 3.1.

Lemma 23. *Let F be a normalized periodic function in $\mathbb{R}^n/\mathbb{Z}^n$. If $\hat{F} \in \ell^2(|\nu|^{2\delta})$, then the background $h_{b,\varepsilon,\delta}[F]$ is*

$$h_{b,\varepsilon,\delta}(t) = \frac{\varepsilon^n}{\|\psi\|_2^2} \sum_{\nu} |\hat{F}(\nu)|^2 \int |x|^{2\delta} |e^{-i\varepsilon^2 t \hbar \Delta/2} \psi(\varepsilon(x - t\nu))|^2 \, dx, \tag{72}$$

$$= \frac{\varepsilon^{-2\delta}}{\|\psi\|_2^2} \int |x|^{2\delta} |\psi(x)|^2 \, dx + o(1), \tag{73}$$

where the error term $o(1)$ is uniform in compact sets.

If $\hat{F} \in \ell^2(|\nu|^{n+2\delta})$, then

$$\|P_{>\frac{1}{4}} h_{b,\varepsilon,\delta}\|_\infty = o(\varepsilon^n). \tag{74}$$

If we assume further that $n = 1$ and $\delta < \frac{1}{2}$, then

$$h_{b,\varepsilon,\delta}(t) = \frac{\varepsilon^{-2\delta}}{\|\psi\|_2^2} \int |x|^{2\delta} |\psi(x)|^2 \, dx + o(\varepsilon), \tag{75}$$

where the error term $o(\varepsilon)$ is uniform in compact sets.

We can understand (72) also as a self-interaction term because the evolution of $e^{it\hbar\Delta/2}f_\varepsilon$ is

$$e^{it\hbar\Delta/2}f_\varepsilon = N_\varepsilon^{-1}\sum_{\nu\in\mathbb{Z}^n}\hat{F}(\nu)e^{2\pi i x\cdot\nu-\pi it|\nu|^2}e^{i\varepsilon^2 t\hbar\Delta/2}\psi(\varepsilon(x-t\nu)).$$

Hence, $h_{\mathrm{p},\delta}[F]$ represents the sum of the pairwise interaction of different waves.

Equation (73) says that $\varepsilon^{2\delta}h_{\mathrm{b},\varepsilon,\delta}$ tends to a constant function as $\varepsilon\to 0$; unfortunately, the rate of convergence is not fast enough, so the ε-periodic part $\varepsilon^n h_{\mathrm{p},\varepsilon,\delta}$ may be thwarted by the noise in the limit. However, if F is smooth, *i.e.* $\hat{F}\in\ell^2(|\nu|^{n+2\delta})$, then the high frequencies $P_{>\frac{1}{4}}h_{\mathrm{b},\varepsilon,\delta}$ are smaller than $\varepsilon^n h_{\mathrm{p},\delta}$ and, in the limit, we can think of $h_\delta[f_\varepsilon]$ as

$$h_\delta[f_\varepsilon]\approx P_{<\frac{1}{4}}h_{\mathrm{b},\varepsilon,\delta}+\varepsilon^n h_{\mathrm{p},\delta},$$

where $P_{<\frac{1}{4}}h_{\mathrm{b},\varepsilon,\delta}$ is an analytic function, essentially constant at scale 2, while $h_{\mathrm{p},\delta}$ is periodic with period 2. This representation offers the possibility of "watching" $h_{\mathrm{p},\delta}$ numerically as tiny oscillations over a smooth background.

Proof of Lemma 23. We begin with the proof of (72). Since $\operatorname{supp}\hat{\psi}\subset B_1$, we can write $h_\delta[f_\varepsilon]$ as

$$\begin{aligned}
h_\delta(t) &= b_{n,\delta}\int\frac{|e^{-\pi it|\xi|^2}\hat{f}_\varepsilon(\xi)-e^{-\pi it|\eta|^2}\hat{f}_\varepsilon(\eta)|^2}{|\xi-\eta|^{n+2\delta}}\,d\xi d\eta\\
&= \frac{b_{n,\delta}}{\varepsilon^n\|\psi\|_2^2}\sum_\nu|\hat{F}(\nu)|^2\\
&\quad\int\Big|e^{-\pi it|\xi|^2}\hat{\psi}\Big(\frac{\xi-\nu}{\varepsilon}\Big)-e^{-\pi it|\eta|^2}\hat{\psi}\Big(\frac{\eta-\nu}{\varepsilon}\Big)\Big|^2\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}+\varepsilon^n h_{p,\varepsilon,\delta}(t)\\
&= \frac{1}{\varepsilon^n\|\psi\|_2^2}\sum_\nu|\hat{F}(\nu)|^2\int|x|^{2\delta}|e^{-it\hbar\Delta/2}(e^{2\pi i\nu\cdot y}\psi_\varepsilon)(x)|^2\,dx+\varepsilon^n h_{p,\varepsilon,\delta}(t),
\end{aligned}$$

where $e^{2\pi i\nu\cdot y}\psi_\varepsilon(y):=\varepsilon^n e^{2\pi i\nu\cdot y}\psi(\varepsilon y)$, so

$$|e^{-it\hbar\Delta/2}(e^{2\pi i\nu\cdot y}\psi_\varepsilon)(x)|=\varepsilon^n|(e^{-i\varepsilon^2 t\hbar\Delta/2}\psi)(\varepsilon(x-t\nu))|.$$

We replace it above to get

$$h_\delta(t)=\frac{\varepsilon^n}{\|\psi\|_2^2}\sum_\nu|\hat{F}(\nu)|^2\int|x|^{2\delta}|(e^{-i\varepsilon^2 t\hbar\Delta/2}\psi)(\varepsilon(x-t\nu))|^2\,dx+\varepsilon^n h_{p,\varepsilon,\delta}(t),$$

which implies (72) by the definition of $h_{\mathrm{b},\varepsilon,\delta}$—see (67).

Let us define

$$A_{\varepsilon,\nu}(t) := b_{n,\delta}\int |e^{-\pi i\varepsilon^2 t|\xi|^2}\hat{\psi}(\xi-\varepsilon^{-1}\nu) - e^{-\pi i\varepsilon^2 t|\eta|^2}\hat{\psi}(\eta-\varepsilon^{-1}\nu)|^2\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}$$

so that

$$h_{\mathrm{b},\varepsilon,\delta}(t) = \frac{\varepsilon^{-2\delta}}{\|\psi\|_2^2}\sum_{\nu}|\hat{F}(\nu)|^2 A_{\varepsilon,\nu}(t). \tag{76}$$

Since ψ is real and symmetric, the Fourier transform of $A_{\varepsilon,\nu}$ is symmetric and then we can restrict ourselves to symmetric test functions, so

$$\begin{aligned}\langle \hat{A}_{\varepsilon,\nu},\varphi\rangle &= b_{n,\delta}\int \Big[|\hat{\psi}(\xi-\varepsilon^{-1}\nu)|^2\varphi(0) + |\hat{\psi}(\eta-\varepsilon^{-1}\nu)|^2\varphi(0) - \\ &\qquad - 2\varphi\Big(\varepsilon^2\frac{|\eta|^2-|\xi|^2}{2}\Big)\hat{\psi}(\xi-\varepsilon^{-1}\nu)\hat{\psi}(\eta-\varepsilon^{-1}\nu)\Big]\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}} \\ &= \varphi(0)\int |x|^{2\delta}|\psi|^2\,dx + \\ &\quad + 2b_{n,\delta}\int\Big[\varphi(0) - \varphi\Big(\big(\varepsilon\frac{\eta+\xi}{2}+\nu\big)\cdot\varepsilon(\eta-\xi)\Big)\Big]\hat{\psi}(\xi)\hat{\psi}(\eta)\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}\end{aligned} \tag{77}$$

We use a test function supported in $\mathbb{R}\backslash[-a,a]$ to bound $\hat{A}_{\varepsilon,\nu}$ away from the origin as

$$|\langle \hat{A}_{\varepsilon,\nu},\varphi\rangle| \leqslant C\int |\hat{\psi}(\xi)\hat{\psi}(\eta)\varphi\Big(\big(\varepsilon\frac{\eta+\xi}{2}+\nu\big)\cdot\varepsilon(\eta-\xi)\Big)|\frac{d\xi d\eta}{|\xi-\eta|^{n+2\delta}}.$$

If $|\nu|\leqslant a\varepsilon^{-1}/4$ then $\langle \hat{A}_{\varepsilon,\nu},\varphi\rangle = 0$ because $\varphi(t)=0$ when $|t|\leqslant a$; otherwise, we change variables and bound the integral as

$$\begin{aligned}|\langle \hat{A}_{\varepsilon,\nu},\varphi\rangle| &\leqslant C\int |\hat{\psi}(\xi)\hat{\psi}(\eta)\varphi\Big(\big(\varepsilon u+\nu\big)\cdot\varepsilon v\Big)|\frac{du dv}{|v|^{n+2\delta}} \\ &\leqslant C\|\varphi\|_\infty\int_{|v|>a/(2|\nu|\varepsilon)}\frac{dv}{|v|^{n+2\delta}} \\ &\leqslant C\|\varphi\|_\infty\Big(\frac{|\nu|\varepsilon}{a}\Big)^{2\delta}.\end{aligned}$$

Hence, by Hausdorff-Young inequality, $\|P_{>a}A_{\varepsilon,\nu}\|_\infty \lesssim_a (|\nu|\varepsilon)^{2\delta}$ and then, by (76),

$$\|P_{>a}h_{\mathrm{b},\varepsilon,\delta}\|_\infty \lesssim_a \sum_{|\nu|>a\varepsilon^{-1/2}} |\hat{F}(\nu)|^2|\nu|^{2\delta} = o_a(1);$$

to prove (73) it remains to estimate $P_{<\frac{1}{4}}h_{\mathrm{b},\varepsilon,\delta}$. If we assume further that $\hat{F} \in \ell^2(|\nu|^{n+2\delta})$, then we can state the stronger upper bound$\|P_{>a}h_{\mathrm{b},\varepsilon,\delta}\|_\infty = o_a(\varepsilon^n)$, which is (74).

We turn now to the term $P_{<\frac{1}{4}}h_{\mathrm{b},\varepsilon,\delta}$. We will prove that

$$\langle \hat{h}_{\mathrm{b},\varepsilon,\delta}, \varphi\rangle - \varphi(0)\varepsilon^{-2\delta}\|\psi\|_2^{-2}\int |x|^{2\delta}|\psi|^2\,dx \to 0, \tag{78}$$

which implies (73) after replacing φ by the test function $\tau \mapsto \psi(\tau)\cos(2\pi t\tau)$, where ψ is a symmetric cut-off of $[-1/4, 1/4]$; the bounds will be uniform in t if $|t| \leqslant T$, so the convergence is uniform in compact sets.

From (77) we see that $\varepsilon^{-2\delta}(\langle \hat{A}_{\varepsilon,\nu}, \varphi\rangle - \varphi(0)\int |x|^{2\delta}|\psi|^2\,dx) \to 0$ as $\varepsilon \to 0$. In fact,

$$\begin{aligned}\varepsilon^{-2\delta}|I_{\varepsilon,\nu}| &:= \varepsilon^{-2\delta}|\langle \hat{A}_{\varepsilon,\nu}, \varphi\rangle - \varphi(0)\int |x|^{2\delta}|\psi|^2\,dx| \\ &\leqslant C\varepsilon^{-2\delta}\int_{|u|,|v|<1} |\varphi(0) - \varphi\Big(\big(\varepsilon\frac{\eta+\xi}{2} + \nu\big)\cdot\varepsilon(\eta-\xi)\Big)|\frac{d\xi d\eta}{|\eta-\xi|^{n+2\delta}} \\ &\leqslant C\varepsilon^{2-2\delta}\|\varphi''\|_\infty\end{aligned}$$

and the last term tends to zero, so the claim follows.

To prove (78), and so (73), it suffices to show that $\varepsilon^{-2\delta}|\hat{F}(\nu)|^2|I_{\varepsilon,\nu}|$ is uniformly dominated in ε by an integrable (summable) function; recall (76) and $\|F\|_2 = 1$. To control $I_{\varepsilon,\nu}$, we change variables and bound the integral as

$$\begin{aligned}|I_{\varepsilon,\nu}| &\leqslant C\int_{|u|,|v|<1} |\varphi(0) - \varphi((\varepsilon u + \nu)\cdot\varepsilon v)|\frac{du dv}{|v|^{n+2\delta}} \\ &\lesssim \|\varphi''\|_\infty\varepsilon^2\langle\nu\rangle^2\int_{|v|<r}\frac{dv}{|v|^{n-2+2\delta}} + \|\varphi\|_\infty\int_{r<|v|<1}\frac{dv}{|v|^{n+2\delta}} \\ &\lesssim \|\varphi''\|_\infty\varepsilon^2\langle\nu\rangle^2\min\{r^{2(1-\delta)}, 1\} + \|\varphi\|_\infty r^{-2\delta}\mathbb{1}_{r<1}.\end{aligned}$$

When $(\varepsilon\langle\nu\rangle)^2 < \|\varphi\|_\infty/\|\varphi''\|_\infty$ we choose $r = 1$ and we get

$$|I_{\varepsilon,\nu}| \leqslant C\|\varphi''\|_\infty\varepsilon^2\langle\nu\rangle^2 \leqslant C(\varepsilon\langle\nu\rangle)^{2\delta}\|\varphi\|_\infty^{1-\delta}\|\varphi''\|_\infty^\delta.$$

When $(\varepsilon\langle\nu\rangle)^2 > \|\varphi\|_\infty/\|\varphi''\|_\infty$ we choose $r^2 = \|\varphi\|_\infty/(\varepsilon^2\langle\nu\rangle^2\|\varphi''\|_\infty)$ and we get

$$|I_{\varepsilon,\nu}| \leqslant C(\varepsilon\langle\nu\rangle)^{2\delta}\|\varphi\|_\infty^{1-\delta}\|\varphi''\|_\infty^{\delta}.$$

Thus, we have that $\varepsilon^{-2\delta}|\hat{F}(\nu)|^2|I_{\varepsilon,\nu}| \leqslant C\|\varphi\|_\infty^{1-\delta}\|\varphi''\|_\infty^{\delta}|\hat{F}(\nu)|^2\langle\nu\rangle^{2\delta}$, and (78) follows by dominated convergence.

The proof of (75) goes along the same lines, but the new hypotheses are $n = 1$, $\delta < \frac{1}{2}$ and $\||\nu|^{\frac{1}{2}+\delta}\hat{F}\|_{\ell^2} < \infty$. Since $\delta < \frac{1}{2}$ we have that $\varepsilon^{-1-2\delta}(\langle\hat{A}_{\varepsilon,\nu},\varphi\rangle - \varphi(0)\int|x|^{2\delta}|\psi|^2\,dx) \to 0$ as $\varepsilon \to 0$, so it suffices to show that $\varepsilon^{-1-2\delta}|\hat{F}(\nu)|^2|I_{\varepsilon,\nu}|$ is uniformly dominated in ε by an integrable function.

The previous bounds of $|I_{\varepsilon,\nu}|$ lead to

$$|I_{\varepsilon,\nu}| \leqslant C\|\varphi''\|_\infty\varepsilon^2\langle\nu\rangle^2 \leqslant C(\varepsilon\langle\nu\rangle)^{1+2\delta}\|\varphi\|_\infty^{\frac{1}{2}-\delta}\|\varphi''\|_\infty^{\frac{1}{2}+\delta},$$

when $(\varepsilon\langle\nu\rangle)^2 < \|\varphi\|_\infty/\|\varphi''\|_\infty$, and

$$|I_{\varepsilon,\nu}| \leqslant C(\varepsilon\langle\nu\rangle)^{2\delta}\|\varphi\|_\infty^{1-\delta}\|\varphi''\|_\infty^{\delta} \leqslant C(\varepsilon\langle\nu\rangle)^{1+2\delta}\|\varphi\|_\infty^{\frac{1}{2}-\delta}\|\varphi''\|_\infty^{\frac{1}{2}+\delta},$$

when $(\varepsilon\langle\nu\rangle)^2 > \|\varphi\|_\infty/\|\varphi''\|_\infty$. Therefore,

$$\varepsilon^{-1-2\delta}|\hat{F}(\nu)|^2|I_{\varepsilon,\nu}| \leqslant C\langle\nu\rangle^{1+2\delta}\|\varphi\|_\infty^{\frac{1}{2}-\delta}\|\varphi''\|_\infty^{\frac{1}{2}+\delta}.$$

By dominated convergence again we get (75). □

Recall that our main interest is the Talbot effect in $n = 1$, so (75) in Lemma 23 provides the convenient asymptotic representation

$$h_\delta(t) = \frac{\varepsilon^{-2\delta}}{\|\psi\|_2^2}\int|x|^{2\delta}|\psi(x)|^2\,dx + \varepsilon h_{p,\delta}(t) + o(\varepsilon),$$

as long as $\hat{F} \in \ell^2(|\nu|^{1+2\delta})$ and $\delta < \frac{1}{2}$.

We summarize our main findings in the following theorem.

Theorem 24. *Let F be a normalized periodic function with period 1 in $\mathbb{R}^n$ — recall the definition of f_ε in* (65)*.*

If $\hat{F} \in \ell^2(|\nu|^{2\delta})$, then

$$h_\delta[f_\varepsilon](t) = \frac{\varepsilon^{-2\delta}}{\|\psi\|_2^2}\int|x|^{2\delta}|\psi(x)|^2\,dx + o(1). \tag{79}$$

If $\hat{F} \in \ell^2(|\nu|^{n+2\delta})$, *then*

$$h_\delta[f_\varepsilon](t) = P_{<\frac{1}{4}} h_{b,\varepsilon,\delta}(t) + \varepsilon^n h_{p,\delta}(t) + o(\varepsilon^n). \tag{80}$$

If $n = 1$, $\delta < \frac{1}{2}$ *and* $\hat{F} \in \ell^2(|\nu|^{1+2\delta})$, *then*

$$h_\delta[f_\varepsilon](t) = \frac{\varepsilon^{-2\delta}}{\|\psi\|_2^2} \int |x|^{2\delta} |\psi(x)|^2 \, dx + \varepsilon h_{p,\delta}(t) + o(\varepsilon). \tag{81}$$

The error terms in all the limits are uniform in compact sets of $\mathbb{R}$.

In Figure 1 we saw how convenient is (81) to visualize $h_{\mathrm{p},\delta}$ numerically.

4.1 The Dirac comb

Now that we have succeeded in defining a functional $h_{\mathrm{p},\delta}[F]$ for a periodic function F, we want to pass again to the limit to study the Dirac comb, *i.e.* the periodic distribution $F_D(x) := \sum_{m\in\mathbb{Z}} \delta(x-m)$ in $\mathbb{R}$.

To approach the Dirac comb in $\mathbb{R}$ we use the function

$$F_{\varepsilon_1} := \sum_{m\in\mathbb{Z}} \varepsilon_1^{-1} e^{-\pi((x-m)/\varepsilon_1)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi(\varepsilon_1 m)^2} e^{2\pi i x m},$$

and define so the approximation

$$f_{\varepsilon_1,\varepsilon_2}(x) := N_{\varepsilon_2}^{-1} \psi(\varepsilon_2 x) \|F_{\varepsilon_1}\|_2^{-1} F_{\varepsilon_1},$$

where N_{ε_2} is the normalization constant of $f_{\varepsilon_1,\varepsilon_2}$. Since the periodic function F_{ε_1}—ε_1 fixed—is smooth, then from (80) we see that $h_\delta[f_{\varepsilon_1,\varepsilon_2}]$ splits, in the limit $\varepsilon_2 \to 0$, into a smooth background and a oscillating, periodic function $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$.

We use (68), or (71), to see that

$$\hat{h}_{\mathrm{p},\delta}[F_{\varepsilon_1}](\tau) = -\frac{2b_{1,\delta}}{\|\psi\|_2^2} \sum_{k\in\mathbb{Z}} \delta_{\frac{k}{2}}(\tau) \sum_{\substack{m_1 \neq m_2 \\ |m_1|^2 - |m_2|^2 = k}} \hat{F}_{\varepsilon_1}(m_1) \hat{F}_{\varepsilon_1}(m_2) \frac{1}{|m_1 - m_2|^{n+2\delta}}. \tag{82}$$

At this stage, we let ε_1 go to zero and take the weak limit of $\hat{h}_{\mathrm{p},\delta}[F_{\varepsilon_1}]$ to get the distribution

$$\hat{h}_{\mathrm{p},\delta}[F_D](\tau) := -\frac{2b_{1,\delta}}{\|\psi\|_2^2} \sum_{k} \delta_{\frac{k}{2}}(\tau) \sum_{\substack{m_1 \neq m_2 \\ m_1^2 - m_2^2 = k}} \frac{1}{|m_1 - m_2|^{1+2\delta}},$$

which is our definition of periodic h_δ for the Dirac comb F_D. Surprisingly, $h_{\mathrm{p},\delta}[F_D]$ is a pure point measure; to see this, we compute first the coefficients of $\hat{h}_{\mathrm{p},\delta}[F_D]$.

Lemma 25.

$$\sum_{\substack{m_1\neq m_2\\ m_1^2-m_2^2=k}} \frac{1}{|m_1-m_2|^{1+2\delta}} = \begin{cases} 2\displaystyle\sum_{\substack{d|k\\ d>0}} \frac{1}{d^{1+2\delta}} & \text{for } k\in\mathbb{Z} \text{ odd}\\ \dfrac{1}{2^{2\delta}}\displaystyle\sum_{\substack{4d|k\\ d>0}} \frac{1}{d^{1+2\delta}} & \text{for } k\equiv 0\,(mod\, 4)\\ 0 & \text{for } k\equiv 2\,(mod\, 4)\end{cases} \tag{83}$$

Remark. In number theory notation, for k odd the coefficients are $2\sigma_{-1-2\delta}(k)$, and for $k\equiv 0\,(\mathrm{mod}\ 4)$ the coefficients are $2^{-2\delta}\sigma_{-1-2\delta}(k/4)$.

Proof. We write $m_1^2-m_2^2=(m_1-m_2)(m_1+m_2):=de=k$, so necessarily $d\mid k$. On the other hand, we have $m_1=\frac{1}{2}(e+d)$ and $m_2=\frac{1}{2}(e-d)$, so d and e have the same parity, *i.e.* $d\equiv e\,(\mathrm{mod}\ 2)$. Consequently,

$$\sum_{\substack{m_1\neq m_2\\ m_1^2-m_2^2=k}} \frac{1}{|m_1-m_2|^{1+2\delta}} = 2\sum_{\substack{d\equiv e\,(\mathrm{mod}\ 2)\\ de=k,d>0}} \frac{1}{d^{1+2\delta}},$$

from which the Lemma follows. □

Theorem 3.

$$\begin{aligned} h_{p,\delta}[F_D](2t) &= -\frac{2b_{1,\delta}}{\|\psi\|_2^2}\zeta(2(1+\delta))\Big[\sum_{\substack{(p,q)=1\\ q>0\ odd}} \frac{1}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t)-\\ &\quad -\sum_{\substack{(p,q)=1\\ q\equiv 2\,(mod\,4)}} \frac{2(2^{1+2\delta}-1)}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t)+\sum_{\substack{(p,q)=1\\ q\equiv 0\,(mod\,4)}} \frac{2^{2(1+\delta)}}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t)\Big], \end{aligned} \tag{84}$$

$$:= \sum_{\substack{(p,q)=1\\ q>0}} \frac{a_{\delta,q}}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t) \tag{85}$$

where $\zeta(z)$ is the Riemann zeta function.

Proof. We split $\hat{h}_{\mathrm{p},\delta}[F_D]$ into

$$\hat{h}^{\mathrm{odd}}_{\mathrm{p},\delta}(\tau) := 2\sum_{k \text{ odd}} \sigma_{-1-2\delta}(k)\delta_{\frac{k}{2}}(\tau)$$
$$\hat{h}^{\mathrm{even}}_{\mathrm{p},\delta}(\tau) := \frac{1}{2^{2\delta}}\sum_{k\equiv 0 \,(\mathrm{mod}\ 4)} \sigma_{-1-2\delta}(k/4)\delta_{\frac{k}{2}}(\tau).$$

We rearrange the terms in the sum of the odd part so that

$$\hat{h}^{\mathrm{odd}}_{\mathrm{p},\delta}(\tau) = 2\sum_{k \text{ odd}}\Big(\sum_{d|k}\frac{1}{d^{1+2\delta}}\Big)\delta_{\frac{k}{2}}(\tau) = 2\sum_{d>0 \text{ odd}}\frac{1}{d^{1+2\delta}}\sum_{l \text{ odd}}\delta_{\frac{dl}{2}}(\tau).$$

The very last sum is a Dirac comb supported on the arithmetic progression $\{l \text{ odd} \mid dl/2\}$, so the inverse Fourier transform of $\hat{h}^{\mathrm{odd}}_{\mathrm{p},\delta}$ is

$$\begin{aligned} h^{\mathrm{odd}}_{\mathrm{p},\delta}(t) &= 2\sum_{d>0 \text{ odd}}\frac{1}{d^{2(1+\delta)}}\sum_{l\in\mathbb{Z}}(-1)^l\delta_{\frac{l}{d}}(t) \\ &= 2\sum_{\substack{(p,q)=1\\ q>0}}\delta_{\frac{p}{q}}(t)\sum_{\substack{d>0 \text{ odd},l\\ l/d=p/q}}\frac{(-1)^l}{d^{2(1+\delta)}}. \end{aligned}$$

Since $q \mid d$ and $p \mid l$, then

$$h^{\mathrm{odd}}_{\mathrm{p},\delta}(t) = 2\sum_{\substack{(p,q)=1\\ q>0 \text{ odd}}}\delta_{\frac{p}{q}}(t)\frac{(-1)^p}{q^{2(1+\delta)}}\sum_{r>0 \text{ odd}}\frac{1}{r^{2(1+\delta)}}.$$

We follow a similar argument to evaluate the even part

$$\hat{h}^{\mathrm{even}}_{\mathrm{p},\delta}(\tau) = \frac{1}{2^{2\delta}}\sum_{k\in\mathbb{Z},\, d|k}\delta_{2k}(\tau)\frac{1}{d^{1+2\delta}} = \frac{1}{2^{2\delta}}\sum_{d>0}\frac{1}{d^{1+2\delta}}\sum_{l\in\mathbb{Z}}\delta_{2dl}(\tau);$$

hence, the inverse Fourier transform of $\hat{h}^{\text{even}}_{\text{p},\delta}$ is

$$\begin{aligned}
h^{\text{even}}_{\text{p},\delta}(t) &= \frac{1}{2^{1+2\delta}} \sum_{d>0} \frac{1}{d^{2(1+\delta)}} \sum_{l\in\mathbb{Z}} \delta_{\frac{l}{2d}}(t) \\
&= 2 \sum_{\substack{(p,q)=1\\ q>0}} \delta_{\frac{p}{q}}(t) \sum_{\substack{d>0 \text{ even},l\\ l/d=p/q}} \frac{1}{d^{2(1+\delta)}} \\
&= 2\Big[\sum_{\substack{(p,q)=1\\ q>0 \text{ odd}}} \delta_{\frac{p}{q}}(t) \frac{1}{q^{2(1+\delta)}} \sum_{r>0 \text{ even}} \frac{1}{r^{2(1+\delta)}} + \\
&\qquad + \sum_{\substack{(p,q)=1\\ q>0 \text{ even}}} \delta_{\frac{p}{q}}(t) \frac{1}{q^{2(1+\delta)}} \sum_{r>0} \frac{1}{r^{2(1+\delta)}} \Big].
\end{aligned}$$

We sum $h^{\text{even}}_{\text{p},\delta}$ and $h^{\text{odd}}_{\text{p},\delta}$ to conclude that

$$\begin{aligned}
h_{\text{p},\delta}(t) = -\frac{4b_{1,\delta}}{\|\psi\|_2^2} \Big[\sum_{\substack{p \text{ even}\\ q>0 \text{ odd}}} \frac{\zeta(2(1+\delta))}{q^{2(1+\delta)}} \delta_{\frac{p}{q}}(t) + \sum_{\substack{(p,q)=1\\ q>0, p \text{ odds}}} \frac{-\eta(2(1+\delta))}{q^{2(1+\delta)}} \delta_{\frac{p}{q}}(t) + \\
+ \sum_{\substack{(p,q)=1\\ q>0 \text{ even}}} \frac{\zeta(2(1+\delta))}{q^{2(1+\delta)}} \delta_{\frac{p}{q}}(t) \Big].
\end{aligned}$$

where $\eta(z) = -\sum_{n>0}(-1)^n/n^z = (1-2^{1-z})\zeta(z)$ is the Dirichlet eta function. Finally, we dilate and rearrange the terms so as to get (84). □

To study the function $h_{\text{p},\delta}[F_D]$ we deem appropriate to consider its primitive

$$H_\delta(t) := \int_{[0,t]} h_{\text{p},\delta}[F_D](2s)\, ds; \tag{86}$$

This function is right-continuous, the limits from the left exist, has jumps at rational times and is continuous elsewhere. We think that H_δ can be seen as a realization of some stochastic process when $t \in [0,1)$; we do not consider $t > 1$ because the derivative of a random process is almost surely non-periodic. We will review briefly some aspects of Lévy processes.

We start defining Poisson point processes, so we have to consider first point functions

$$p : D_p \subset (0,\infty) \to X,$$

where D_p is countable, and X is some measure space—$X = \mathbb{R}\backslash\{0\}$ in our case. We denote by Π the set of all point functions. To every interval $I \subset (0,\infty)$ and measurable set $U \subset X$ we assign the counting function

$$N_p(I,U) := |\{t \in D_p \cap I \mid p(t) \in U\}|.$$

We endow Π with the minimal σ-field $\mathcal{B}$ generated by the functions $p \mapsto N_p(I,U)$.

A (stationary) Poisson point process is a random variable $\boldsymbol{p}$ from some probability space $(\Omega,\mathcal{F},P)$ into the space of point functions $(\Pi,\mathcal{B})$, which satisfies, among other properties,

$$\mathbf{E}[N_p(I,U)] := \int_\Omega N_{\mathbf{p}(\omega)}(I,U)\, dP(\omega) = |I|n(U),$$

where n the characteristic measure of the process. We refer the reader to Ch. I.9 of [17] for details. The point function p_δ attached to H_δ represents the location and size of the jumps:

$$p_\delta : \mathbb{Q} \cap [0,1) \to X = \mathbb{R}\backslash\{0\}.$$

The following theorem shows that, in a weak sense, $\mathbf{E}[N_p(I,U)] \approx N_{p_\delta}(I,U)$ for some measure n on $\mathbb{R}\backslash\{0\}$, *i.e.* p_δ resembles an outcome of some Poisson point process $\boldsymbol{p}$.

Theorem 26. *For $I \subset [0,1)$, the function*

$$|N|_{p_\delta}(I,r) := N_{p_\delta}(I,(-\infty,-r] \cup [r,\infty)), \quad \text{for } r > 0, \tag{87}$$

satisfies the bounds

$$|N|_{p_\delta}(I,r) \leqslant C_\delta |I| r^{-1/(1+\delta)} + 1, \qquad \text{all } r \lesssim_\delta 1, \tag{88}$$

$$|N|_{p_\delta}(I,r) \gtrsim_\delta \frac{|I|}{\log(c_\delta/r)} r^{-1/(1+\delta)}, \qquad \text{all } r \lesssim_\delta |I|^{2(1+\delta)}. \tag{89}$$

The theorem is consequence of the following lemma.

Lemma 27. *For $I \subset [0,1)$, the function*

$$M(I,N) := |\{p/q \in I \subset \mathbb{R} \mid q \leqslant N \text{ and } (p,q) = 1\}|, \quad \text{for } N \geqslant 1,$$

satisfies the bounds

$$M(I,N) \leqslant |I|N^2 + 1, \qquad \text{all } N \geqslant 1, \tag{90}$$

$$M(I,N) \gtrsim |I|\frac{N^2}{\log N}, \qquad \text{all } N > 2/|I|. \tag{91}$$

Proof. We arrange the rationals inside I in increasing order $p_1/q_1 < \cdots < p_M/q_M$ and then use the fact that $p_{i+1}/q_{i+1} - p_i/q_i = 1/(q_{i+1}q_i)$, see Thm. 28 in [15], to get

$$|I| \geqslant \sum_{i=1}^{M-1} \frac{1}{q_{i+1}q_i} > \frac{M-1}{N^2},$$

which is (90).

For the lower bound, we only count fractions p/q with prime denominator. Given a prime $q \leqslant N$ such that $q|I| > 1$, the number of fractions $p/q \in I$ is $\geqslant q|I|/2$, so for $N > 2/|I|$ we have

$$M(I,N) \geqslant \frac{1}{2}|I| \sum_{|I|^{-1} < q \leqslant N} q \geqslant \frac{1}{4}|I|N\,|\{N/2 \leqslant q \leqslant N \mid q \text{ prime}\}|.$$

Using the prime number theorem and the Bertrand's postulate we arrive at (91). □

We expect the bounds in the lemma can be improved, in particular, the $\log N$-loss in (91) should be removable. It is interesting to investigate the behavior of $M(I,N)$ when $N \leqslant 2/|I|$. For example, in the interval $I = (0, 1/N)$ there is no rational p/q with $q \leqslant N$, so $M(I,N)$ can be zero when $q \leqslant 1/|I|$, but $(0,1/N)$ is a very special interval, can we do any better for other type of intervals?

Proof of Theorem 26. According to (84) the value of the point function p_δ at a rational time $t = p/q$ is $p_\delta(t) = a_{\delta,q}/q^{2(1+\delta)}$, where $|a_{\delta,q}| \sim_\delta 1$, so

$$M(I, c_\delta r^{-\frac{1}{2(1+\delta)}}) \leqslant |N|_{p_\delta}(I,r) \leqslant M(I, C_\delta r^{-\frac{1}{2(1+\delta)}}),$$

and the bounds in the theorem follow from the Lemma. □

Theorem 26 suggests $N_{p_\delta}(I,U) \approx |I| n(U)$ with characteristic measure $dn(r) \approx r^{-1-1/(1+\delta)}\,dr$. We can write H_δ in (86) in terms of N_{p_δ} as

$$H_\delta(t) = \int_{[0,t]} \int_{\mathbb{R}\setminus\{0\}} y\, N_{p_\delta}(ds dy).$$

We recognize here a "realization" of an (asymmetric) α-Lévy process with exponent $\alpha := 1/(1+\delta)$. We ignore the compensator term because it would add a linear term in t, and we can always think of H_δ as a Lévy process with drift; see Ch. II.3-4 of [17].

This connection between H_δ and Lévy processes also suggests that H_δ behaves intermittently, with bursts at rational times with small denominator. It is worth mentioning that α-Lévy processes, with $1 < \alpha < 2$,[1] have already been studied and described as strongly intermittent; see Sec. 3.3 in [5].

Yet another evidence of intermittency lies in the variability of the Hölder exponent of H_δ or multifractality, which is the content of the next theorem, but first we introduce a few definitions and a lemma.

Definition 28 (Hölder exponent)**.** Let $t_0 \in \mathbb{R}$. A function f is in $C^l(t_0)$, for $l \in \mathbb{R}_+$, if there is a polynomial P_{t_0} of degree at most $\lfloor l \rfloor$ such that in a neighborhood of t_0

$$|f(t) - P_{t_0}(t)| \lesssim |t - t_0|^l.$$

The Hölder exponent of f at t_0 is

$$h_f(t_0) := \sup\{l \mid f \in C^l(t_0)\}. \tag{92}$$

Definition 29 (Irrationality measure)**.** Fix $t \in \mathbb{R}$ and let $A \subset \mathbb{R}_+$ be the set of exponents $m \in \mathbb{R}_+$ such that

$$0 < \left| t - \frac{p}{q} \right| < \frac{1}{q^m},$$

has infinitely many solutions. The irrationality measure $\mu(t)$ of $t \in \mathbb{R}$ is

$$\mu(t) := \sup A. \tag{93}$$

If t is rational, then $\mu(t) = 1$; if t is irrational, then by the Dirichlet's approximation theorem $\mu(t) \geqslant 2$; if t is an irrational algebraic number, then $\mu(t) = 2$ by Roth's theorem; and t is a Liouville number if and only if $\mu(t) = \infty$.

Lemma 30. *Let t be an irrational number with finite $\mu(t)$ and let $\varepsilon > 0$. If P/Q is the fraction with the smallest denominator among all fractions $|t - p/q| < h$, for $h \ll_\varepsilon 1$, then $h^{-1/(\mu+\varepsilon)} < Q \leqslant h^{-1+1/(\mu+\varepsilon)}$.*

For $t = p_0/q_0$, if $h > 0$ and $0 < |t - p/q| \leqslant h$, then $q \geqslant 1/(q_0 h)$.

[1] The larger the exponent, the lower the probability of very large jumps.

Proof. We only consider the case when t is irrational. Suppose on the contrary that there is p/q with $q \leqslant h^{-1/(\mu+\varepsilon)}$ such that $0 < |t - p/q| < h$, then $0 < |t - p/q| < 1/q^{\mu+\varepsilon}$, but this can only happen for finitely many fractions, so taking $h \ll_{\varepsilon} 1$ we can avoid those fractions and necessarily $h^{-1/(\mu+\varepsilon)} < Q$.

The bound $Q \leqslant h^{-1+1/(\mu+\varepsilon)}$ is consequence of the Dirichlet's approximation theorem and our bound $h^{-1/(\mu+\varepsilon)} < Q$. In fact, for $q \leqslant N = h^{-1+1/(\mu+\varepsilon)}$ we can always find a fraction such that

$$\left|t - \frac{p}{q}\right| \leqslant \frac{1}{qN} < h,$$

so $Q \leqslant N = h^{-1+1/(\mu+\varepsilon)}$. □

Theorem 4. *Let* $\alpha := 1/(1+\delta)$, *then*

$$d_{H_\delta}(\gamma) = \alpha\gamma, \quad \textit{for } \gamma \in [0, 1/\alpha). \tag{94}$$

If t is rational, then $|H_\delta(t+h) - H_\delta(t)| \leqslant C_\delta(t) h^{1+2\delta}$ *for all* $h > 0$.

Proof. Let t be irrational. We prove first that for every $\varepsilon > 0$

$$|H_\delta(t+h) - H_\delta(t)| \leqslant C|h|^{2(1+\delta)/(\mu+\varepsilon)}, \qquad \text{for } h \ll_{t,\varepsilon} 1, \tag{95}$$

and the exponent has to be $\leqslant 2(1+\delta)/\mu$ if $2 < \mu \leqslant \infty$.

We can assume that $h > 0$. We integrate by parts to write the difference as

$$\begin{aligned} H_\delta(t+h) - H_\delta(t) &= \int_{\mathbb{R}\setminus\{0\}} y N_{p_\delta}(I, dy) \\ &= \int_0^\infty [N_{p_\delta}(I, [y,\infty)) - N_{p_\delta}(I, [-y, -\infty))]\, dy. \end{aligned} \tag{96}$$

Among all $p/q \in I = (t, t+h]$, let P/Q be the rational with the smallest denominator, so

$$\begin{aligned} H_\delta(t+h) - H_\delta(t) &= \frac{a_{\delta,Q}}{Q^{2(1+\delta)}} + \\ &+ \int_0^{|a_{\delta,Q}|/Q^{2(1+\delta)}} [N_{p_\delta}(I, [y,\infty)) - N_{p_\delta}(I, [-y,-\infty)) - \frac{a_{\delta,Q}}{|a_{\delta,Q}|}]\, dy \\ &= \frac{a_{\delta,Q}}{Q^{2(1+\delta)}} + J_1. \end{aligned}$$

To control the integral J_1 we recall the definition of $|N|_{p_\delta}(I,r)$ in (87) and write

$$|J_1| \leqslant \int_0^{|a_{\delta,Q}|/Q^{2(1+\delta)}} |N|_{p_\delta}(I,r) - 1\,dr = \int_0^{|a_{\delta,Q_*}|/Q_*^{2(1+\delta)}} |N|_{p_\delta}(I,r) - 1\,dr,$$

where $Q_* > Q$ is the next to the smallest denominator in $I := (t, t+h]$. Since P/Q and P_*/Q_* have to be successive in a Farey sequence, then $1/Q_*^2 < 1/(QQ_*) < h$, so, using (90), we have that $|J_1| \leqslant C_\delta h Q_*^{-2\delta} \leqslant C_\delta h^{1+\delta}$. Hence,

$$H_\delta(t+h) - H_\delta(t) = a_{\delta,Q}/Q^{2(1+\delta)} + \mathcal{O}(h^{1+\delta}) \tag{97}$$

and from Lemma 30 we get (95), so $h_{H_\delta}(t) \geqslant 2(1+\delta)/\mu$; recall Def. 28.

To see that the exponent in (95) is best possible when $2 < \mu \leqslant \infty$, let $\{q_i\}_i$ be an infinite list of numbers such that $|t - p_i/q_i| < 1/q_i^{\mu-\varepsilon}$ for some $\varepsilon > 0$. If we take $h_i = 1/q_i^{\mu-\varepsilon}$, then the smallest denominator in $(t, t+h_i]$ is $Q = q_i$ and then, by (97), $|H_\delta(t+h) - H_\delta(t)| \gtrsim h_i^{2(1+\delta)/(\mu-\varepsilon)}$ if $h_i \ll 1$.

Up to now, we know that for $2 < \mu(t) \leqslant \infty$ there is a sequence $h_i \to 0$ such that $|h_i|^{2(1+\delta)/(\mu-\varepsilon)} \lesssim |H_\delta(t+h_i) - H_\delta(t)| \lesssim |h_i|^{2(1+\delta)/(\mu+\varepsilon)}$, so necessarily $h_{H_\delta}(t) = 2(1+\delta)/\mu$ as long as $2(1+\delta)/\mu \neq 1$. On the other hand, when $\mu(t) = 2$ we have that $h_{H_\delta}(t) \geqslant 1+\delta$, so to prove the theorem we still need to settle the case $2(1+\delta)/\mu = 1$.

From (97) we see that $H_\delta(t+h) - H_\delta(t) - Ah = a_{\delta,Q}/Q^{2(1+\delta)} - Ah + \mathcal{O}(h^{1+\delta})$, where A is any constant. Again, $|H_\delta(t+h) - H_\delta(t) - Ah| \lesssim |h|^{2(1+\delta)/(\mu+\varepsilon)}$, but now, to see that it is best possible, we use Dirichlet's theorem to find a sequence $\{q_i\}$ such that $|t - p_i/q_i| < 1/q_i^2$. We choose $h_i = 1/q_i^2$ and use again a Farey sequence to see that $Q = q_i$ is the smallest denominator among all fractions in $(t, t+h_i]$. Thus, $|H_\delta(t+h) - H_\delta(t) - Ah| \gtrsim |h|$ and $h_{H_\delta}(t) = 1$.

When t is rational $h_{H_\delta}(t) = 0$, but we can still measure the Hölder exponent from the right using (96), (88) and Lemma 30.

To conclude the theorem we use a deep result of Güting [14], which asserts that the Hausdorff dimension of the set of numbers with irrationality μ is $2/\mu$; see [4] for a shorter proof of Güting's theorem. This result refines Jarník's theorem; see *e.g.* Thm 10.3 of [10]. The set of numbers where H_δ has Hölder exponent $\gamma < 1+\delta := 1/\alpha$ coincides with the set of numbers with irrationality $2(1+\delta)/\gamma$, and the dimension of the latter is $\gamma/(1+\delta)$, which is (94). □

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