RESTRICTED TESTING FOR POSITIVE OPERATORS

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Abstract. We prove that for certain positive operators T, such as the Hardy-Littlewood maximal function and fractional integrals, there is a constant $D>1$, depending only on the dimension n, such that the two weight norm inequality

$$\int_{\mathbb{R}^n} T(f\sigma)^2 d\omega \leq C \int_{\mathbb{R}^n} f^2 d\sigma$$

holds for all $f \geq 0$ *if and only if* the (fractional) A_2 condition holds, and the restricted testing condition

$$\int_Q T(1_Q \sigma)^2 d\omega \leq C |Q|_\sigma$$

holds for all cubes Q satisfying $|2Q|_\sigma \leq D |Q|_\sigma$. If T is linear, we require as well that the dual restricted testing condition

$$\int_Q T^* (1_Q \omega)^2 d\sigma \leq C |Q|_\omega$$

holds for all cubes Q satisfying $|2Q|_\omega \leq D |Q|_\omega$.

1. Introduction

One of the earliest uses of testing conditions to characterize a weighted norm inequality occurs in 1982 in [20], where it was shown that for the Hardy-Littlewood maximal function M,

$$\int_{\mathbb{R}^n} Mf(x)^2 w(x)\, dx \leq C \int_{\mathbb{R}^n} f(x)^2 v(x)\, dx, \qquad \text{for all } f(x) \geq 0, \tag{1.1}$$

if and only if the following testing condition holds:

$$\int_Q M\left(\mathbf{1}_Q v^{-1}\right)(x)^2 w(x)\, dx \leq C \int_Q v(x)^{-1} dx, \qquad \text{for all cubes } Q \text{ in } \mathbb{R}^n.$$

Thus it suffices to test the weighted norm inequality over the simpler collection of test functions $f = \mathbf{1}_Q v^{-1}$ for cubes Q.

Two years later, David and Journé showed in their $T1$ theorem [3], that the unweighted inequality

$$\int_{\mathbb{R}^n} Tf(x)^2 dx \leq C \int_{\mathbb{R}^n} f(x)^2 dx, \qquad \text{for all } f \in L^2(\mathbb{R}^n),$$

holds if and only if the following pair of dual testing conditions hold:

$$\int_Q T(\mathbf{1}_Q)(x)^2 dx \leq C \int_Q dx \text{ and } \int_Q T^*(\mathbf{1}_Q)(x)^2 dx \leq C \int_Q dx, \qquad \text{for all cubes } Q \text{ in } \mathbb{R}^n.$$

Here T is a general Calderón-Zygmund singular integral on $\mathbb{R}^n$ and the testing functions are simply the indicators $\mathbf{1}_Q$ for cubes Q[1]. The following year David, Journé and Semmes extended the $T1$ theorem to a

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[1] The moniker '$T1$ theorem' refers to the equivalent formulation of the testing conditions as weak boundedness property, $T1 \in BMO$ and $T^*1 \in BMO$.

Tb theorem [4] in which the testing conditions become $b\mathbf{1}_Q$ and $b^*\mathbf{1}_Q$ for appropriately accretive functions b and b^* on $\mathbb{R}^n$.

A couple of decades later, and motivated by the Painlevé problem of characterizing removable singularities for bounded analytic functions, Nazarov, Treil and Volberg solved in 2003 a particular one-weight formulation of the norm inequality for Riesz transforms $\mathcal{R}$, including the Cauchy transform $\mathcal{C}g(z) \equiv \int_{\mathbb{C}} \frac{1}{w-z} g(w)\, dw$ [18],

$$\int_{\mathbb{R}^n} |\mathcal{R}(f\mu)(x)|^2 \, d\mu(x) \le C \int_{\mathbb{R}^n} f(x)^2 \, d\mu(x), \qquad \text{for all } f \in L^2(\mathbb{R}^n;\mu),$$

if and only if the following testing condition held:

$$\int_Q |\mathcal{R}(\mathbf{1}_Q\mu)(x)|^2 \, d\mu(x) \le C \int_Q d\mu(x), \qquad \text{for all cubes } Q \text{ in } \mathbb{R}^n.$$

Here the testing functions are $f = \mathbf{1}_Q$. The Painlevé problem was solved in the same year by Tolsa [27], a culmination of an impressive body of work by many mathematicians.

Finally, building on the work of Nazarov, Treil and Volberg in their 2004 paper [19] on the Hilbert transform, that in turn used the random dyadic grids of [18][2], and the weighted Haar wavelets of [18][3], the two weight norm inequality for the Hilbert transform was characterized in 2014 in Lacey, Sawyer, Shen and Uriarte-Tuero [12], Lacey [9] and Hytönen [7] as follows:

$$\int_{\mathbb{R}} H(f\sigma)(x)^2 \, d\omega(x) \le C \int_{\mathbb{R}} f(x)^2 \, d\sigma(x), \qquad \text{for all } f \in L^2(\sigma), \tag{1.2}$$

if and only if both the strong Muckenhoupt A_2 condition

$$\mathcal{A}_2(\sigma,\omega) \equiv \sup_{I:\ \text{intervals in } \mathbb{R}} \int_{\mathbb{R}} \frac{\ell(I)}{(\ell(I)+|x-c_I|)^2} d\omega(x) \cdot \int_{\mathbb{R}} \frac{\ell(I)}{(\ell(I)+|x-c_I|)^2} d\sigma(x) < \infty,$$

and the following dual testing conditions hold:

$$\int_I H(\mathbf{1}_I\sigma)(x)^2 \, d\omega(x) \le A \int_I d\sigma \text{ and } \int_I H(\mathbf{1}_I\omega)(x)^2 \, d\sigma(x) \le B \int_I d\omega, \qquad \text{for all intervals } I.$$

This is also referred as the two weight $T1$ theorem. Notice that here the two weight inequality (1.2) is written differently than (1.1), so that in the testing condition, one can avoid requesting the existence of the density of the measure. On the other hand, the main difference between the two weight $T1$ theorem and the unweighted $T1$ theorem is that, in the unweighted case, if one has L^2 boundedness for singular integral operators, one also get L^p boundedness for all $1 < p < \infty$, which is not the case in the two weight setting.

The two-weight inequality for the g function was then characterized by testing conditions in Lacey and Li [10], and a further extension to a local Tb theorem for the Hilbert transform is in [26].

Point of departure: The point of departure for the present paper begins with the observation that in the one-weight formulation above of the norm inequality for Riesz transforms by Nazarov, Treil and Volberg, their testing condition is

$$\int_Q |\mathcal{R}(\mathbf{1}_Q\mu)(x)|^2 \, d\mu(x) \le C \int_{2Q} d\mu(x), \qquad \text{for all cubes } Q \text{ in } \mathbb{R}^n,$$

where the double $2Q$ of the cube Q appears on the right hand side. Moreover, one may restrict the testing functions to those functions $f = \mathbf{1}_Q$ for which Q is a *μ-doubling cube* for some appropriate positive constant D[4]:

$$\int_{2Q} d\mu \le D \int_Q d\mu.$$

This then motivates the following problem.

Problem 1. *To what extent one can similarly restrict testing functions to doubling cubes for classical operators in the two-weight situations, including those discussed above?*

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[2]that in turn followed on those of Fefferman and Stein [5], Garnett and Jones [6], and Sawyer [20]

[3]that in turn followed on those of Coifman, Jones and Semmes [2]

[4]This philosophy was successfully carried out in the context of the one-weight Tb theorem for nonhomogeneous square functions by Martikainen, Mourgoglou and Vuorinen in [16].

Motivation: Besides the intrinsic interest in minimizing the functions over which an inequality must be tested in order to verify its validity, even a partial resolution of the question of restricted testing for singular integrals has the potential to characterize two weight norm inequalities for such operators - including Riesz transforms in higher dimensions, currently a very difficult open problem, see e.g. [24], [11] and [13]. Indeed, the *nondoubling cubes* have traditionally been viewed as the enemy in two weight inequalities for singular integrals, and (the techniques used in) the restriction of the testing conditions to just doubling cubes could help circumvent the difficulty that *energy conditions* fail to be necessary for two weight inequalities in higher dimensions [20] - the point being that a similar restricted energy condition could suffice. In the current paper, we only work on positive operators, such as the Hardy-Littlewood maximal function and fractional integrals, leaving square functions and the Hilbert transform for future work.

Let $\mathcal{P}^n$ be the collection of cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and with side lengths 2^ℓ for some $\ell \in \mathbb{Z}$. For $Q \in \mathcal{P}^n$ and $\Gamma \geq 1$, let ΓQ denote the cube concentric with Q with $\ell(\Gamma Q) = \Gamma \ell(Q)$. For a locally signed measure μ on $\mathbb{R}^n$ (meaning the total variation $|\mu|$ of μ is locally finite), we define $M(\mu)$ and $I_\alpha(\mu)$ at $x \in \mathbb{R}^n$ by[5]

$$M(\mu)(x) \equiv \sup_{Q \in \mathcal{P}^n:\ x \in Q} \frac{1}{|Q|} \int_Q d|\mu|, \quad I_\alpha(\mu)(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} d\mu(y).$$

Given a pair (σ, ω) of weights (i.e. positive Borel measures) in $\mathbb{R}^n$ and $\Gamma > 1$, we say that (σ, ω) satisfies the Γ-*testing condition* for the maximal function M if there is a constant $\mathfrak{T}_M(\Gamma)(\sigma, \omega)$ such that

$$\int_Q |M(\mathbf{1}_Q \sigma)|^2 d\omega \leq \mathfrak{T}_M(\Gamma)(\sigma, \omega)^2 |\Gamma Q|_\sigma , \qquad \text{for all } Q \in \mathcal{P}^n , \tag{1.3}$$

and we take $\mathfrak{T}_M(\Gamma)(\sigma, \omega)$ to be the least such constant. We define $\mathfrak{T}_{I_\alpha}(\Gamma)(\sigma, \omega)$ anagolously.

There is also the following weaker testing condition, in which one need only test the inequality over cubes that are 'doubling'. Given a pair (σ, ω) of weights in $\mathbb{R}^n$ and $D, \Gamma > 1$, we say that (σ, ω) satisfies the D-Γ-*testing condition* for the maximal function M if there is a constant $\mathfrak{T}_M^D(\Gamma)(\sigma, \omega)$ such that

$$\int_Q |M \mathbf{1}_Q \sigma|^2 d\omega \leq \mathfrak{T}_M^D(\Gamma)(\sigma, \omega)^2 |Q|_\sigma , \qquad \text{for all } Q \in \mathcal{P}^n \text{ with } |\Gamma Q|_\sigma \leq D |Q|_\sigma , \tag{1.4}$$

and again we take $\mathfrak{T}_M^D(\Gamma)(\sigma, \omega)$ to be the least such constant, and define $\mathfrak{T}_{I_\alpha}^D(\Gamma)(\sigma, \omega)$ similarly. Note that the Γ-testing condition implies the D-Γ-testing condition for all $D > 1$.

Unlike the cases of the classical two weight theorem for the maximal function and fractional integrals in [20, 22], where the testing condition is already sufficient for the boundedness of the maximal function and fractional integrals, these restricted testing conditions are not by themselves sufficient for the norm inequality - the classical Muckenhoupt condition is needed as well:

$$A_2(\sigma, \omega) := \sup_{Q \in \mathcal{P}^n} \frac{\sigma(Q)}{|Q|} \frac{\omega(Q)}{|Q|} < \infty, \quad A_2^\alpha(\sigma, \omega) := \sup_{Q \in \mathcal{P}^n} \frac{\sigma(Q)}{|Q|^{1-\frac{\alpha}{n}}} \frac{\omega(Q)}{|Q|^{1-\frac{\alpha}{n}}} < \infty$$

see the counterexamples in Section 6. Finally we let $\mathfrak{N}_M(\sigma, \omega)$ be the best constant (i.e., the $L^2(\sigma) \to L^2(\omega)$ norm of M) in the inequality

$$\int_{\mathbb{R}^n} |M(f\sigma)|^2 d\omega \leq \mathfrak{N}_M(\sigma, \omega)^2 \int_{\mathbb{R}^n} |f|^2 d\sigma, \qquad \text{for all } f \in L^2(\sigma) . \tag{1.5}$$

Again we define $\mathfrak{N}_{I_\alpha}(\sigma, \omega)$ analogously. Our main result for the maximal function is formulated as the following, which is the only improvement of the testing characterization of (1.5) by the third named author in 1982 (see [20]).

Theorem 1. *Let $\Gamma > 1$. Then there is $D > 1$ depending only on Γ and the dimension n such that*

$$\mathfrak{N}_M(\sigma, \omega) \approx \mathfrak{T}_M^D(\Gamma)(\sigma, \omega) + \sqrt{A_2(\sigma, \omega)},$$

for all locally finite positive Borel measures σ and ω on $\mathbb{R}^n$.

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[5] The supremum over $Q \in \mathcal{P}^n$ used here is pointwise equivalent to the usual supremum over all cubes Q with sides parallel to the coordinate axes.

Remark 1. *With probability one, a dyadic grid $\mathcal{D}^\beta$ in (2.1) below has the property that every $Q \in \mathcal{D}^\beta$ has* null boundary, *i.e.* $|\partial Q|_{\sigma+\omega} = 0$. *Thus the supremum over cubes Q in the testing constant $\mathfrak{T}_M^D(\Gamma)(\sigma,\omega)$ in Theorem 1 may be further restricted to cubes Q having null boundary (*cf. *the one-weight theorem in* [16] *where this type of reduction first appears).*

See also [14], [15] and [1] for earlier related work. For example, the following weaker 'parental' testing result for the maximal function was proved in [14], and subsequently given a particularly simple proof by Chen and Lacey in [1]: The two weight norm inequality for the maximal function M holds if and only if the following *D-parental testing condition* holds for some $D > 1$,

$$\int_Q |M\mathbf{1}_Q\sigma|^2\, d\omega \leq \mathfrak{P}_T^D(\sigma,\omega)^2 |Q|_\sigma\,, \quad \text{for all } Q \in \mathcal{P}^n \text{ with } \min_{P \text{ is a dyadic parent of } Q} |P|_\sigma \leq D\, |Q|_\sigma\ .$$

Let us say that a cube Q is *D-Γ-doubling* if it satisfies $|\Gamma Q|_\sigma \leq D|Q|_\sigma$ (as in (1.4)) and the Q is *D-parental doubling* if it satisfies the condition at the end of the previous display. It is easy to see that every D-3-doubling cube is D-parental doubling, but the converse is false in general: D-parental doubling means that the measure σ is under control on *at least one side of* Q, while D-Γ-doubling means that it is under control on *all sides*. Thus there are many more cubes that one needs to test in the D-parental testing condition than in the D-Γ-testing condition (1.4).

The proof of Theorem 1 splits neatly into two steps. In the first step, we prove the sufficiency of the Γ-testing condition (1.3), which we record as the following theorem

Theorem 2. *For $\Gamma > 1$ we have*

$$\mathfrak{N}_M(\sigma,\omega) \approx \mathfrak{T}_M(\Gamma)(\sigma,\omega) + \sqrt{A_2(\sigma,\omega)},$$

for all pairs (σ,ω) of locally finite positive Borel measures on $\mathbb{R}^n$, and where the implicit constants of comparability depend on both Γ and dimension n.

This step requires a careful application of a probabilistic argument of the type pioneered by Nazarov, Treil and Volberg ([18]), and refined in [7]. In the second step we use this interim result to establish an *a priori* bound on the operator norm $\mathfrak{N}_M(\sigma,\omega)$ in order to absorb additional terms arising from the absence of any testing condition at all in (1.4) when the cubes are not doubling. As a consequence of this splitting, we will give the proof in two stages, beginning with the proof of the following weaker theorem, which requires probability, and which is then used to prove our main result Theorem 1. We emphasize that this paper is self-contained.

Our main result for fractional integrals is the following theorem, whose proof involves Whitney decompositions at each threshold that deal with the tails of the kernels of the operators. And multiples of Whitney cubes have bounded overlap, a key to restricted testing. This is a phenomenon quite different from the above maximal function case, which is naturally handled by taking maximal cubes at each threshold, whose multiples fail to have bounded overlap and results in a number of delicate estimates explained in the above.

Theorem 3. *Let $\Gamma > 1$. Then there is $D > 1$ depending only on Γ and the dimension n such that*

$$\mathfrak{N}_{I_\alpha}(\sigma,\omega) \approx \mathfrak{T}_{I_\alpha}^D(\Gamma)(\sigma,\omega) + \mathfrak{T}_{I_\alpha}^D(\Gamma)(\omega,\sigma) + \sqrt{A_2^\alpha(\sigma,\omega)},$$

for all locally finite positive Borel measures σ and ω on $\mathbb{R}^n$.

For convenience we will restrict our proof of the above results to the case $\Gamma = 3$, the general case of Γ large being an easy modification of this one.

2. Preliminaries

Here we introduce some standard tools we will use in the proof of Theorem 2.

2.1. **Random dyadic grids.** In this subsection we introduce the usual random dyadic grids. Let

$$\mathcal{D}_0 := \{2^{-j}([0,1)^n + k), j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

Then for $\beta = \{\beta_j\}_{j=-\infty}^{\infty} \in (\{0,1\}^n)^{\mathbb{Z}}$, define

$$\mathcal{D}^\beta := \left\{ Q + \sum_{j:2^{-j}<\ell(Q)} 2^{-j}\beta_j, Q \in \mathcal{D}_0 \right\}. \tag{2.1}$$

Let $\mathbb{P}$ be the natural product probability measure on $\Omega := (\{0,1\}^n)^{\mathbb{Z}}$. We have the following estimate, which goes back to Fefferman and Stein [5, page 112] and also [20, Lemma 2]. For any $\beta \in \Omega$, we denote the associated *dyadic* maximal operator by

$$M^{\mathcal{D}^\beta} f(x) := \sup_{Q \ni x, Q \in \mathcal{D}^\beta} \frac{1}{|Q|} \int_Q |f| \,.$$

Lemma 1. *For $x \in \mathbb{R}^n$ and a positive Borel measure $f \geq 0$ on $\mathbb{R}^n$ we have*

$$Mf(x) \leq 2^{4n+1} \mathbb{E}_\beta M^{\mathcal{D}^\beta} f(x) \,. \tag{2.2}$$

Proof. Fix $x \in \mathbb{R}^n$, and let Q be a cube such that $x \in Q$ and

$$\frac{1}{|Q|} \int_Q f > \frac{1}{2} Mf(x) \,.$$

Then there exists $Q \subset \widetilde{Q} \in \mathcal{D}^\gamma$ for some $\gamma \in \Omega$ such that

$$\frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} f > \frac{1}{2^{n+1}} Mf(x) \,.$$

Since γ is fixed, denote $j_0 = -\log_2 \ell(\widetilde{Q}) + 1$, we have

$$\mathbb{P}(\{\beta \in \Omega : \gamma_{j_0} + \beta_{j_0} = (1, \cdots, 1)\}) = 2^{-n}.$$

The key is when $\gamma_{j_0} + \beta_{j_0} = (1, \cdots, 1)$, suppose

$$\widetilde{Q} - \sum_{j : 2^{-j} < \ell(\widetilde{Q})} 2^{-j} \gamma_j + \sum_{j : 2^{-j} < \ell(\widetilde{Q})} 2^{-j} \beta_j = [a_1, b_1) \times \cdots \times [a_n, b_n),$$

let $\widetilde{Q}_\beta = J_1 \times \cdots \times J_n$ be the cube satisfying that $J_i = [a_i, 2b_i - a_i)$ if $(\gamma_{j_0})_i = 1$ and $J_i = [2a_i - b_i, b_i)$ if $(\gamma_{j_0})_i = 0$, then $\widetilde{Q}_\beta \supset \widetilde{Q}$ and $\widetilde{Q}_\beta \in \mathcal{D}^\beta$ with some fixed $\beta_{j_0 - 1}$ (whose precise value depends on γ_{j_0} and the standard dyadic cube $\widetilde{Q} - \sum_{j : 2^{-j} < \ell(\widetilde{Q})} 2^{-j} \gamma_j$). This implies

$$\mathbb{P}(\{\beta \in \Omega : M^{\mathcal{D}^\beta} f > \frac{1}{2^{2n+1}} Mf\}) \geq 4^{-n}$$

which completes the proof of (2.2). $\square$

2.2. **Whitney decompositions.** In order to apply probabilistic method, here we use a slightly weaker version of Whitney decomposition, which adapts to the probabilistic method quite naturally. Indeed, given an open set Ω we let $\{Q_j\}_j$ be the collection of dyadic cubes such that

(1) $Q_j \subset \Omega$;
(2) $10\sqrt{n}\ell(Q_j) < \mathrm{dist}(Q_j, \Omega^c) \leq 21\sqrt{n}\ell(Q_j)$.

Notice that we do not request Q_j to be the maximal dyadic cube such that the above properties holds. With this definition, we prove the following proposition.

Proposition 1. *Let Ω be an open set and $\{Q_j\}$ be the related Whitney cubes defined as the above. Then there holds*

(i) $\Omega \subset \cup_j Q_j$;
(ii) $\sum_j \chi_{Q_j} \leq 2$;
(iii) $\sum_j \chi_{3Q_j} \leq c_n$.

Proof. To prove the first assertion, given x and a dyadic cube Q with $x \in Q \subsetneq \Omega$, notice that

$$\frac{1}{2} \cdot \frac{\mathrm{dist}(Q, \Omega^c)}{\ell(Q)} - \frac{\sqrt{n}}{2} \leq \frac{\mathrm{dist}(Q^{(1)}, \Omega^c)}{2\ell(Q)} \leq \frac{1}{2} \cdot \frac{\mathrm{dist}(Q, \Omega^c)}{\ell(Q)}.$$

Obviously, we can take a dyadic cube $Q_0 \ni x$ such that $\frac{\mathrm{dist}(Q_0, \Omega^c)}{\ell(Q_0)}$ is sufficiently big, with the above estimates, there must exists a cube $Q_x \ni x$ such that

$$10\sqrt{n}\ell(Q_x) < \mathrm{dist}(Q_x, \Omega^c) \leq 21\sqrt{n}\ell(Q_x).$$

Now we turn to prove (ii). Fix $x \in \Omega$, let $d = \mathrm{dist}(x, \Omega^c)$ and $x \in Q_j$, then

$$d \geq \mathrm{dist}(Q_j, \Omega^c) > 10\sqrt{n}\ell(Q_j).$$

On the other hand,

$$d \le \operatorname{dist}(Q_j, \Omega^c) + \sqrt{n}\ell(Q_j) \le 22\sqrt{n}\ell(Q_j).$$

Combing the above two estimates one obtains that

$$\frac{d}{22\sqrt{n}} \le \ell(Q_j) < \frac{d}{10\sqrt{n}}.$$

Now since Q_j is a dyadic cube, we get (ii). Finally, let us fix x and $3Q_j \ni x$. If Q_l is a Whitney cube such that $x \in 3Q_l$, then we have

$$21\sqrt{n}\ell(Q_l) \ge \operatorname{dist}(Q_l, \Omega^c) \ge \operatorname{dist}(Q_j, \Omega^c) - 2\sqrt{n}(\ell(Q_j) + \ell(Q_l)) > 8\sqrt{n}\ell(Q_j) - 2\sqrt{n}\ell(Q_l).$$

Likewise, we also have $23\ell(Q_j) > 8\ell(Q_l)$. Since $\operatorname{dist}(Q_j, Q_l) \le 2\sqrt{n}(\ell(Q_j) + \ell(Q_l))$, we completes the proof of (iii). $\square$

2.3. **Good/bad cubes.** We call a cube $Q \in \mathcal{D}^\gamma$ is good, if for any $R \in \mathcal{D}^\gamma$ with $\ell(R) \ge 2^r \ell(Q)$, there holds

$$\operatorname{dist}(Q, \partial R) > \ell(Q)^{\frac{1}{2}} \ell(R)^{\frac{1}{2}}.$$

Otherwise we call Q is bad. We note that $\pi_{\text{good}} := \mathbb{P}_\gamma(Q + \sum_{j:2^{-j}<\ell(Q)} 2^{-j}\gamma_j$ is good) is independent of $Q \in \mathcal{D}_0$, this is because by definition the goodness is determined by how the parent and ancestors of $Q + \sum_{j:2^{-j}<\ell(Q)} 2^{-j}\gamma_j$ are constructed(i.e. related with $j : 2^{-j} \ge \ell(Q)$). It is well-known that if r is large enough, then $\pi_{\text{good}} > 0$. So without loss of generality, we can assume $r > 4$.

3. Strong triple testing

Now we begin the proof of Theorem 2, the ‘only if’ part is trivial, so we only focus on the ‘if’ part. We shall prove

$$\mathfrak{N}_M(\sigma, \omega) \lesssim \mathfrak{T}_M(3)(\sigma, \omega) + \sqrt{A_2(\sigma, \omega)}. \tag{3.1}$$

Fix f nonnegative and bounded with compact support, say $\operatorname{supp} f \subset Q(0,R) = [-R,R]^n$. Since $M(f\sigma)$ is lower semicontinuous, the set $\Omega_k := \left\{M(f\sigma) > 2^k\right\}$ is open and we can consider the Whitney decomposition of the open set Ω_k into the union $\bigcup_{j\in\mathbb{N}} Q_j^k$ of $\mathcal{D}^\gamma$-dyadic intervals Q_j^k with the properties as in Proposition 1, where $\gamma \in \Omega$. In the sequel, we will use $\mathcal{W}_k^\gamma$ to denote the Whitney cubes of Ω_k in $\mathcal{D}^\gamma$. We now use random grids to obtain from Lemma 1 that

$$M(f\sigma)(x) \lesssim \mathbb{E}_\gamma M^{\mathcal{D}^\gamma}(f\sigma)(x) \ , \qquad x \in \mathbb{R}^n.$$

Notice that if we replace ω by $\omega_N = \omega \mathbf{1}_{Q(0,N)}$ with $N > R$, we have

$$\int M(f\sigma)^2 \, d\omega_N \le \|f\|_{L^\infty}^2 \int_{Q(0,N)} M(\mathbf{1}_{Q(0,N)}\sigma)^2 d\omega \le \|f\|_{L^\infty}^2 \mathfrak{T}_M^2 \left|3Q(0,N)\right|_\sigma < \infty,$$

and therefore, without loss of generality, we can assume

$$\int M(f\sigma)^2 d\omega < \infty.$$

We now have

$$\begin{aligned}
\mathbb{E}_\gamma \int_{\mathbb{R}^n} \left[M^{\mathcal{D}^\gamma}(f\sigma)(x)\right]^2 d\omega(x) &\le \mathbb{E}_\gamma C_n \sum_{k\in\mathbb{Z}} 2^{2(k+m)} \left|\left\{M^{\mathcal{D}^\gamma}(f\sigma) > 2^{k+m}\right\}\right|_\omega \\
&= \mathbb{E}_\gamma C_n \sum_{k\in\mathbb{Z},\ j\in\mathbb{N}} 2^{2(k+m)} \left|Q_j^k \cap \Omega_{k+m}^\gamma\right|_\omega \\
&\le C_{n,m}\mathbb{E}_\gamma \sum_{k\in\mathbb{Z},\ j\in\mathbb{N}} 2^{2k} \left|E_{j,\gamma}^k\right|_\omega + 3^n C_n 2^{-2m_0} \int \left[M(f\sigma)\right]^2 d\omega \ ,
\end{aligned}$$

where

$$E_{j,\gamma}^k := Q_j^k \cap \left(\Omega_{k+m}^\gamma \setminus \Omega_{k+m+m_0}\right), \ \Omega_{k+m}^\gamma = \left\{x : M^{\mathcal{D}^\gamma}(f\sigma) > 2^{k+m}\right\},$$

and we shall choose m_0 to be sufficiently large so that the second term can be absorbed (since it is finite). So the goal is to prove

$$\mathbb{E}_\gamma \sum_{k\in\mathbb{Z},\ j\in\mathbb{N}} 2^{2k} \left|E^k_{j,\gamma}\right|_\omega \lesssim \left(\mathfrak{T}_M\left(3\right)(\sigma,\omega)^2 + A_2(\sigma,\omega)\right)\|f\|^2_{L^2(\sigma)}.$$

We claim the maximum principle,

$$2^{k+m-1} < M^{\mathcal{D}^\gamma}_{Q^k_j}(f\sigma)(x) := \sup_{Q\in\mathcal{D}^\gamma, x\in Q\subseteq Q^k_j} \frac{1}{|Q|}\int_Q f\sigma, \qquad x\in E^k_{j,\gamma}. \tag{3.2}$$

Indeed,

$$\begin{aligned} M^{\mathcal{D}^\gamma}(f\sigma)(x) &\le M^{\mathcal{D}^\gamma}(1_{(Q^k_j)^c}f\sigma)(x) + M^{\mathcal{D}^\gamma}(1_{Q^k_j}f\sigma)(x) \\ &\le M^{\mathcal{D}^\gamma}(1_{(Q^k_j)^c}f\sigma)(x) + \sup_{Q\in\mathcal{D}^\gamma: Q\supsetneq Q^k_j}\frac{1}{|Q|}\int_Q f\sigma + M^{\mathcal{D}^\gamma}_{Q^k_j}(f\sigma)(x). \end{aligned} \tag{3.3}$$

Given $x\in E^k_{j,\gamma}$, there is $Q\in\mathcal{D}^\gamma$ with $x\in Q$ and $Q\cap\left(Q^k_j\right)^c\neq\emptyset$ (which implies that $Q^k_j\subset Q$), and also $z\in\Omega^c_k$, such that

$$\begin{aligned} M^{\mathcal{D}^\gamma}\left(\mathbf{1}_{(Q^k_j)^c}f\sigma\right)(x) &\le 2\frac{1}{|Q|}\int_{Q\setminus Q^k_j} f\sigma \le 2\frac{1}{|Q|}\int_{50\sqrt{n}Q} f\sigma \\ &= \frac{2(50\sqrt{n})^n}{|50\sqrt{n}Q|}\int_{50\sqrt{n}Q} f\sigma \le 2(50\sqrt{n})^n M\left(f\sigma\right)(z) \le 2^{k+m-2} \end{aligned} \tag{3.4}$$

if we choose $m>1$ large enough.

This same computation shows that the cubes $Q\supsetneq Q^k_j$ in the second term on the right of (3.3) satisfy

$$\frac{1}{|Q|}\int_Q f\sigma \le 2^{k+m-2}.$$

Now we use $2^{k+m} < M^{\mathcal{D}^\gamma}\left(f\sigma\right)(x)$ for $x\in E^k_j$ to obtain

$$\begin{aligned} 2^{k+m} < M^{\mathcal{D}^\gamma}(f\sigma)(x) &\le M^{\mathcal{D}^\gamma}(1_{(Q^k_j)^c}f\sigma)(x) + \sup_{Q\in\mathcal{D}^\gamma: Q\supsetneq Q^k_j}\frac{1}{|Q|}\int_Q f\sigma + M^{\mathcal{D}^\gamma}_{Q^k_j}(f\sigma)(x) \\ &\le 2^{k+m-2} + 2^{k+m-2} + M^{\mathcal{D}^\gamma}_{Q^k_j}(f\sigma)(x) = 2^{k+m-1} + M^{\mathcal{D}^\gamma}_{Q^k_j}(f\sigma)(x). \end{aligned} \tag{3.5}$$

Hence we have proved the maximum principle. Thus set

$$\widetilde{\Omega}^\gamma_{k+m} := \{M^{D^\gamma}_{Q^k_j}(f\sigma) > 2^{k+m-1}\}, \quad \widetilde{E}^k_{j,\gamma} := Q^k_j \cap (\widetilde{\Omega}^\gamma_{k+m}\setminus\Omega_{k+m+m_0})$$

then $E^k_{j,\gamma}\subset\widetilde{E}^k_{j,\gamma}$ and the latter depends only on the cube Q^k_j (not on its parents or ancestors in the dyadic system $\mathcal{D}^\gamma$) so in particular it is independent of the goodness of Q^k_j. Thus (see e.g. [8])

$$\mathbb{E}_\gamma\sum_{k,j}2^{2k}|E^k_{j,\gamma}|_\omega \le \mathbb{E}_\gamma\sum_{k,j}2^{2k}|\widetilde{E}^k_{j,\gamma}|_\omega = \pi^{-1}_{\mathrm{good}}\mathbb{E}_\gamma\sum_{k,j}2^{2k}|\widetilde{E}^k_{j,\gamma}|_\omega 1_{Q^k_j \text{ good}}.$$

We now introduce some further notation which will play a crucial role below. Let

$$\begin{aligned} \mathcal{H}^k_j &:= \widetilde{\Omega}^\gamma_{k+m}, \\ \mathcal{H}^k_{j,\mathrm{out}} &:= \left\{M^{\mathcal{D}^\gamma}_{Q^k_j}\left(\mathbf{1}_{Q^k_j\setminus\Omega_{k+m+m_0}}f\sigma\right) > 2^{k+m-2}\right\}, \\ \mathcal{H}^k_{j,\mathrm{in}} &:= \left\{M^{\mathcal{D}^\gamma}_{Q^k_j}\left(\mathbf{1}_{Q^k_j\cap\Omega_{k+m+m_0}}f\sigma\right) > 2^{k+m-2}\right\}, \end{aligned}$$

so that $\mathcal{H}^k_j\subset\mathcal{H}^k_{j,\mathrm{out}}\cup\mathcal{H}^k_{j,\mathrm{in}}$. We are here suppressing the dependence of $\mathcal{H}^k_j$ on $\gamma\in\Omega$. In particular in below a dyadic cube always means a dyadic cube in $\mathcal{D}^\gamma$.

We will now follow the main idea for fractional integrals in [22], but with two main changes:

(1) **Sublinearizations**: Since M is not linear, the duality arguments in [22] require that we construct symmetric linearizations L that are dominated by M, and

(2) **Tripling decompositions**: In order to exploit the triple testing conditions we introduce Whitney grids, and construct stopping times for tripling cubes, which entails some combinatorics. In particular, most of our effort is spent on decomposing and controlling the analogue of term IV from [22] using good and bad cubes.

Now take $0 < \beta < 1$ to be chosen later, and consider the following three exhaustive cases for Q_j^k and $\widetilde{E}_{j,\gamma}^k$.
(**1**): Q_j^k is good and $|\widetilde{E}_{j,\gamma}^k|_\omega < \beta|3Q_j^k|_\omega$, in which case we say $(k,j) \in \Pi_1$,
(**2**): Q_j^k is good and $|\widetilde{E}_{j,\gamma}^k|_\omega \geq \beta|3Q_j^k|_\omega$ and $|\widetilde{E}_{j,\gamma}^k \cap \mathcal{H}_{j,\mathrm{out}}^k|_\omega \geq \frac{1}{2}|\widetilde{E}_{j,\gamma}^k|_\omega$, say $(k,j) \in \Pi_2$,
(**3**): Q_j^k is good and $|\widetilde{E}_{j,\gamma}^k|_\omega \geq \beta|3Q_j^k|_\omega$ and $|\widetilde{E}_{j,\gamma}^k \cap \mathcal{H}_{j,\mathrm{in}}^k|_\omega \geq \frac{1}{2}|\widetilde{E}_{j,\gamma}^k|_\omega$, say $(k,j) \in \Pi_3$.

3.1. **The three cases.** The first case is trivially handled, the second case is easy, and the third case consumes most of our effort.

Case (1): The treatment of case (1) is easy by absorption. Indeed,

$$\sum_{(k,j)\in\Pi_1} 2^{2k}|\widetilde{E}_{j,\gamma}^k|_\omega \lesssim \sum_{k\in\mathbb{Z},\ j\in\mathbb{N}} 2^{2k}\beta\left|3Q_j^k\right|_\omega \lesssim \beta \int M\left(f\sigma\right)^2 d\omega, \tag{3.6}$$

and then it suffices to take β sufficiently small at the end of the proof.

Case (2): In case (2) we have

$$\sum_{(k,j)\in\Pi_2} 2^{2k}|\widetilde{E}_{j,\gamma}^k|_\omega \lesssim \sum_{(k,j)\in\Pi_2} 2^k \int \mathbf{1}_{\widetilde{E}_{j,\gamma}^k} \mathcal{L}_j^k\left(\mathbf{1}_{Q_j^k\setminus\Omega_{k+m+m_0}} f\sigma\right) d\omega. \tag{3.7}$$

Here the positive linear operator $\mathcal{L}_j^k$ given by

$$\mathcal{L}_j^k\left(h\sigma\right)(x) := \sum_{\ell=1}^\infty \frac{1}{|I_j^k(\ell)|}\int_{I_j^k(\ell)} hd\sigma \mathbf{1}_{I_j^k(\ell)}(x),$$

where $I_j^k(\ell) \in \mathcal{D}^\gamma(Q_j^k)$ are the maximal dyadic cubes contained in $\mathcal{H}_{j,\mathrm{out}}^k$, which implies that $\mathcal{L}_j^k(\mathbf{1}_{Q_j^k\setminus\Omega_{k+m+m_0}} f\sigma) \approx 2^k\mathbf{1}_{\mathcal{H}_{j,\mathrm{out}}^k}$. Indeed, as we have calculated,

$$\frac{1}{|Q_j^k|}\int_{Q_j^k} f\sigma \leq 2^{k+m-2}, \tag{3.8}$$

so in particular, $I_j^k(\ell)$'s are proper subcubes of Q_j^k and the claim follows. Now we can continue from (3.7) as follows:

$$\begin{aligned}
\sum_{(k,j)\in\Pi_2} 2^k \int_{\widetilde{E}_{j,\gamma}^k} \mathcal{L}_j^k\big(\mathbf{1}_{Q_j^k\setminus\Omega_{k+m+m_0}} f\sigma\big)d\omega &= \sum_{(k,j)\in\Pi_2} 2^k \int_{Q_j^k\setminus\Omega_{k+m+m_0}} \mathcal{L}_j^k\big(\mathbf{1}_{\widetilde{E}_{j,\gamma}^k}\omega\big) f d\sigma \\
&\leq \sum_{(k,j)\in\Pi_2} 2^k \Big(\int_{Q_j^k\setminus\Omega_{k+m+m_0}} \mathcal{L}_j^k\big(\mathbf{1}_{\widetilde{E}_{j,\gamma}^k}\omega\big)^2 d\sigma\Big)^{\frac{1}{2}} \Big(\int_{Q_j^k\setminus\Omega_{k+m+m_0}} f^2 d\sigma\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{(k,j)\in\Pi_2} 2^{2k} \int_{Q_j^k\setminus\Omega_{k+m+m_0}} \mathcal{L}_j^k\big(\mathbf{1}_{\widetilde{E}_{j,\gamma}^k}\omega\big)^2 d\sigma\Big)^{\frac{1}{2}} \Big(\sum_{(k,j)\in\Pi_2}\int_{Q_j^k\setminus\Omega_{k+m+m_0}} f^2 d\sigma\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{(k,j)\in\Pi_2} 2^{2k} \int_{Q_j^k} \mathcal{L}_j^k\big(\mathbf{1}_{Q_j^k}\omega\big)^2 d\sigma\Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{Z}}\int_{\Omega_k\setminus\Omega_{k+m+m_0}} 2f^2 d\sigma\Big)^{\frac{1}{2}} \\
&\leq C_{m,m_0} A_2^{\frac{1}{2}} \Big(\sum_{(k,j)\in\Pi_2} 2^{2k}\left|Q_j^k\right|_\omega\Big)^{\frac{1}{2}} \|f\|_{L^2(\sigma)} \\
&\leq \beta^{-\frac{1}{2}} C_{m,m_0} A_2^{\frac{1}{2}} \Big(\sum_{(k,j)\in\Pi_2} 2^{2k}|\widetilde{E}_{j,\gamma}^k|_\omega\Big)^{\frac{1}{2}} \|f\|_{L^2(\sigma)},
\end{aligned}$$

where we have used the following trivial estimate

$$\int_{Q_j^k} \mathcal{L}_j^k\big(\mathbf{1}_{Q_j^k}\omega\big)^2 d\sigma \leq \sum_{\ell=1}^\infty \frac{|I_j^k(\ell)|_\omega |I_j^k(\ell)|_\sigma}{|I_j^k(\ell)|^2} |I_j^k(\ell)\cap Q_j^k|_\omega \leq A_2|Q_j^k|_\omega. \tag{3.9}$$

Then immediately we get

$$\sum_{(k,j)\in\Pi_2} 2^{2k}|\widetilde{E}^k_{j,\gamma}|_\omega \le \beta^{-1}C^2_{m+m_0}A_2\|f\|^2_{L^2(\sigma)}. \tag{3.10}$$

Case (3): For this case, we let $\{I^k_j(\ell)\}_\ell$ be the collection of the maximal dyadic cubes in $\mathcal{H}^k_{j,\text{in}}$ and define $\mathcal{L}^k_j$ similarly. Then likewise, $\mathcal{L}^k_j(\mathbf{1}_{Q^k_j\cap\Omega_{k+m+m_0}}f\sigma) \approx 2^k\mathbf{1}_{\mathcal{H}^k_{j,\text{in}}}$ and therefore,

$$\begin{aligned}\sum_{(k,j)\in\Pi_3} 2^{2k}|\widetilde{E}^k_{j,\gamma}|_\omega &\lesssim \sum_{(k,j)\in\Pi_3} 2^k\int_{\widetilde{E}^k_{j,\gamma}} \mathcal{L}^k_j\left(\mathbf{1}_{Q^k_j\cap\Omega_{k+m+m_0}}f\sigma\right)d\omega\\ &= \sum_{(k,j)\in\Pi_3} 2^k\int_{Q^k_j\cap\Omega_{k+m+m_0}} \mathcal{L}^k_j\left(\mathbf{1}_{\widetilde{E}^k_{j,\gamma}}\omega\right)fd\sigma\\ &= \sum_{(k,j)\in\Pi_3} 2^k \sum_{i\in\mathbb{N}:\ Q^{k+m+m_0}_i\subset Q^k_j}\int_{Q^{k+m+m_0}_i} \mathcal{L}^k_j\left(\mathbf{1}_{\widetilde{E}^k_{j,\gamma}}\omega\right)fd\sigma.\end{aligned}$$

Before moving on, let us make some observations. Since we only need to consider $I^k_j(\ell)$ such that $I^k_j(\ell)\cap\widetilde{E}^k_{j,\gamma}\ne\emptyset$, we have $I^k_j(\ell)\not\subset\Omega_{k+m+m_0}$. Therefore, if we fix $Q^{k+m+m_0}_i$, only those $I^k_j(\ell)$ such that $Q^{k+m+m_0}_i\subset I^k_j(\ell)$ contribute to $\mathcal{L}^k_j$. In other words, $\mathcal{L}^k_j\big(\mathbf{1}_{\widetilde{E}^k_{j,\gamma}}\omega\big)$ is constant on $Q^{k+m+m_0}_i$. Set

$$A^k_j := \frac{1}{\left|Q^k_j\right|_\sigma}\int_{Q^k_j} fd\sigma. \tag{3.11}$$

We have

$$\begin{aligned}\sum_{(k,j)\in\Pi_3} 2^{2k}\left|\widetilde{E}^k_{j,\gamma}\right|_\omega &\lesssim \sum_{(k,j)\in\Pi_3} 2^k \sum_{i\in\mathbb{N}:\ Q^{k+m+m_0}_i\subset Q^k_j} A^{k+m+m_0}_i\int_{Q^{k+m+m_0}_i}\mathcal{L}^k_j\big(\mathbf{1}_{\widetilde{E}^k_{j,\gamma}}\omega\big)\sigma\\ &= \lim_{N_0\to-\infty}\sum_{\substack{k\in\mathbb{Z},k\ge N_0\\ j\in\mathbb{N},(k,j)\in\Pi_3}} 2^k \sum_{i\in\mathbb{N}:\ Q^{k+m+m_0}_i\subset Q^k_j} A^{k+m+m_0}_i\int_{Q^{k+m+m_0}_i}\mathcal{L}^k_j\big(\mathbf{1}_{\widetilde{E}^k_{j,\gamma}}\omega\big)\sigma.\end{aligned}$$

We make a convention that the summation over k is understood as $k\equiv k_0 \operatorname{mod}(m+m_0)$ for some fixed $0\le k_0\le m+m_0-1$, and since we are summing over products with factor $|\widetilde{E}^k_{j,\gamma}|_\omega$, without loss of generality we only consider Q^k_j for the largest k if it is repeated, and define

$$\mathscr{W}^\gamma := \{Q^k_j\in\mathcal{D}^\gamma : k\equiv k_0 \operatorname{mod}\,(m+m_0)\,, k\ge N_0, (k,j)\in\Pi_3\}.$$

So in particular, there are no repeated cubes in $\mathscr{W}^\gamma$. We have, using $|\widetilde{E}^k_{j,\gamma}|_\omega\approx|3Q^k_j|_\omega$ and $\mathcal{L}^k_j(\mathbf{1}_{Q^k_j\cap\Omega_{k+m+m_0}}f\sigma)\approx 2^k\mathbf{1}_{\mathcal{H}^k_{j,\text{in}}}$ for $(k,j)\in\Pi_3$ again, it suffices to prove that

(3.12)

$$\begin{aligned}&\sum_{Q^k_j\in\mathscr{W}^\gamma} 2^k \sum_{i\in\mathbb{N}:\ Q^{k+m+m_0}_i\subset Q^k_j} A^{k+m+m_0}_i\int_{Q^{k+m+m_0}_i}\mathcal{L}^k_j\left(\mathbf{1}_{\widetilde{E}^k_{j,\gamma}}\omega\right)\sigma\\ &\lesssim \sum_{Q^k_j\in\mathscr{W}^\gamma}\frac{|\widetilde{E}^k_{j,\gamma}|_\omega}{|3Q^k_j|^2_\omega}\Big[\sum_{i\in\mathbb{N}:\ Q^{k+m+m_0}_i\subset Q^k_j} A^{k+m+m_0}_i\int_{Q^{k+m+m_0}_i}\mathcal{L}^k_j\left(\mathbf{1}_{\widetilde{E}^k_{j,\gamma}}\omega\right)\sigma\Big]^2 \lesssim (\mathfrak{T}_M\,(3)\,(\sigma,\omega)+\sqrt{A_2\,(\sigma,\omega)})^2\|f\|^2_{L^2(\sigma)}.\end{aligned}$$

For notational convenience, set

$$\begin{aligned}III^* &:= \sum_{Q^k_j\in\mathscr{W}^\gamma} III^*\left(Q^k_j\right);\\ III^*\left(Q^k_j\right) &:= \frac{|\widetilde{E}^k_{j,\gamma}|_\omega}{|3Q^k_j|^2_\omega}\Big[\sum_{i\in\mathbb{N}:\ Q^{k+m+m_0}_i\subset Q^k_j} A^{k+m+m_0}_i\int_{Q^{k+m+m_0}_i}\mathcal{L}^k_j\left(\mathbf{1}_{\widetilde{E}^k_{j,\gamma}}\omega\right)\sigma\Big]^2.\end{aligned}$$

3.2. **Principal cube decomposition.** The subsection is to define principal cubes. Although we have reduced the summation over good cubes, we still define the stopping collection which admits bad cubes. To be precise, denote

$$\mathcal{W}^\gamma := \{Q_j^k \in \mathcal{W}_k^\gamma : k \equiv k_0 \operatorname{mod} (m+m_0), k \geq N_0\}.$$

With the grid $\mathcal{W} = \mathcal{W}^\gamma$ in hand, we now introduce principal cubes as in [17, page 804] (note that we are suppressing the dependence of $\mathcal{W}$ on γ for reduction of notation). Define G_0 to consist of the maximal cubes in $\mathcal{W}$. If G_n has been defined, let G_{n+1} consist of those indices (k,j) for which $Q_j^k \in \mathcal{W}$, there is an index $(t,u) \in G_n$ with $k \geq t$ and $Q_j^k \subset Q_u^t$, and

(**i**) $A_j^k > \eta A_u^t$,

(**ii**) $A_i^\ell \leq \eta A_u^t$ whenever $Q_j^k \subsetneqq Q_i^\ell \subset Q_u^t$.

Here η is any constant larger than 1, for example $\eta = 4$ works fine. Now define $\Gamma \equiv \bigcup_{n=0}^{\infty} G_n$ and for each index (k,j) define $P\left(Q_j^k\right)$ to be the smallest dyadic cube Q_u^t containing Q_j^k and with $(t,u) \in \Gamma$. Then we have

$$\begin{aligned} &(\mathbf{i})\ P\left(Q_j^k\right) = Q_u^t \Longrightarrow A_j^k \leq \eta A_u^t \ , \\ &(\mathbf{ii})\ Q_j^k \subsetneqq Q_u^t \text{ with } (k,j),(t,u) \in \Gamma \Longrightarrow A_j^k > \eta A_u^t \ . \end{aligned} \tag{3.13}$$

Now we return to the estimate of III^*. Splitting the sum over i inside $III^*\left(Q_j^k\right)$ according to whether $(k+m+m_0, i) \in \Gamma$ or $P(Q_i^{k+m+m_0}) = P\left(Q_j^k\right)$:

$$\begin{aligned} III^* &\lesssim \sum_{Q_j^k \in \mathscr{W}^\gamma} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2} \Big[\sum_{i\in\mathbb{N}:\ P\left(Q_i^{k+m+m_0}\right) = P\left(Q_j^k\right)} A_i^{k+m+m_0} \int_{Q_i^{k+m+m_0}} \mathcal{L}_j^k \left(\mathbf{1}_{\widetilde{E}_{j,\gamma}^k}\omega\right)\sigma\Big]^2 \\ &+ \sum_{Q_j^k \in \mathscr{W}^\gamma} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2} \Big[\sum_{\substack{i\in\mathbb{N}:\ (k+m+m_0,i)\in\Gamma \\ Q_i^{k+m+m_0} \subset Q_j^k}} A_i^{k+m+m_0} \int_{Q_i^{k+m+m_0}} \mathcal{L}_j^k \left(\mathbf{1}_{\widetilde{E}_{j,\gamma}^k}\omega\right)\sigma\Big]^2 \\ &=: IV + V. \end{aligned}$$

It is relatively easy to estimate term V by the Cauchy-Schwarz inequality and (3.9),

$$\begin{aligned} V &= \sum_{Q_j^k \in \mathscr{W}^\gamma} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2} \Big[\sum_{\substack{i\in\mathbb{N}:\ (k+m+m_0,i)\in\Gamma \\ Q_i^{k+m+m_0} \subset Q_j^k}} A_i^{k+m+m_0} \int_{Q_i^{k+m+m_0}} \mathcal{L}_j^k \big(\mathbf{1}_{\widetilde{E}_{j,\gamma}^k}\omega\big)\sigma\Big]^2 \\ &\leq \sum_{Q_j^k \in \mathscr{W}^\gamma} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{\left|3Q_j^k\right|_\omega^2} \Big[\sum_{\substack{i\in\mathbb{N}:\ (k+m+m_0,i)\in\Gamma \\ Q_i^{k+m+m_0} \subset Q_j^k}} |Q_i^{k+m+m_0}|_\sigma \big(A_i^{k+m+m_0}\big)^2\Big] \\ &\times \Big[\sum_{\substack{i\in\mathbb{N}:\ (k+m+m_0,i)\in\Gamma \\ Q_i^{k+m+m_0} \subset Q_j^k}} \Big(\int_{Q_i^{k+m+m_0}} \mathcal{L}_j^k \big(\mathbf{1}_{\widetilde{E}_{j,\gamma}^k}\omega\big) d\sigma\Big)^2 |Q_i^{k+m+m_0}|_\sigma^{-1}\Big] \\ &\leq \sum_{Q_j^k \in \mathscr{W}^\gamma} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{\left|3Q_j^k\right|_\omega^2} \Big[\sum_{\substack{i\in\mathbb{N}:\ (k+m+m_0,i)\in\Gamma \\ Q_i^{k+m+m_0} \subset Q_j^k}} |Q_i^{k+m+m_0}|_\sigma \big(A_i^{k+m+m_0}\big)^2\Big] \int_{Q_j^k} \big[\mathcal{L}_j^k \big(\mathbf{1}_{Q_j^k}\omega\big)\big]^2 d\sigma \\ &\lesssim A_2 \sum_{(t,u)\in\Gamma} (A_u^t)^2 |Q_u^t|_\sigma \lesssim A_2 \|f\|_{L^2(\sigma)}^2. \end{aligned} \tag{3.14}$$

Thus we are left to estimate term IV. Fix (t,u), and consider the sum

$$IV^{t,u} \quad := \quad \sum_{Q_j^k \in \mathscr{W}^\gamma : P(Q_j^k) = Q_u^t} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2} \Big[\sum_{i\in\mathbb{N}:\ P(Q_i^{k+m+m_0}) = P(Q_j^k)} A_i^{k+m+m_0} \int_{Q_i^{k+m+m_0}} \mathcal{L}_j^k \left(\mathbf{1}_{\widetilde{E}_{j,\gamma}^k}\omega\right)\sigma\Big]^2$$

$$\begin{aligned}&\lesssim \quad \left(A_u^t\right)^2 \sum_{Q_j^k\in\mathscr{W}^\gamma:P(Q_j^k)=Q_u^t} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2}\Big[\int_{Q_j^k}\mathcal{L}_j^k(\mathbf{1}_{Q_j^k}\sigma)\omega\Big]^2\\ &=: \quad \left(A_u^t\right)^2\mathcal{S}^{t,u},\end{aligned}$$

where

$$\mathcal{S}^{t,u}:=\sum_{Q_j^k\in\mathscr{W}^\gamma:P(Q_j^k)=Q_u^t} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2}\Big[\int_{Q_j^k}\mathcal{L}_j^k(\mathbf{1}_{Q_j^k}\sigma)\omega\Big]^2.$$

It is here in estimating $\mathcal{S}^{t,u}$, that the only quantitative use of the triple testing condition occurs.

Lemma 2. *We claim that*

$$\mathcal{S}^{t,u}\le C\left(\left(\mathfrak{T}_M\left(3\right)\right)^2+A_2\right)\left|Q_u^t\right|_\sigma \ . \tag{3.15}$$

Proof. Let $\{K_i\}_{i\in\mathcal{I}}$ be the collection of maximal $\mathcal{D}^\gamma$-children K_i satisfying $5K_i\subset Q_u^t$. If $\ell(Q_j^k)<2^{-r}\ell(Q_u^t)$, since Q_j^k is good, we have $\operatorname{dist}(Q_j^k,\partial Q_u^t)>4\ell(Q_j^k)$. Hence $5Q_j^k\subset Q_u^t$ and in particular $Q_j^k\subset K_i$ for some $i\in\mathcal{I}$. For all cubes K_i we have

$$\begin{aligned}\sum_{Q_j^k\in\mathscr{W}^\gamma:Q_j^k\subset K_i} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2}\Big[\int_{Q_j^k}\mathcal{L}_j^k(\mathbf{1}_{Q_j^k}\sigma)\omega\Big]^2 &\le \sum_{Q_j^k\in\mathscr{W}^\gamma,Q_j^k\subset K_i}|\widetilde{E}_{j,\gamma}^k|_\omega\left[\frac{1}{\left|Q_j^k\right|_\omega}\int_{Q_j^k}\mathbf{1}_{K_i}M\left(\mathbf{1}_{K_i}\sigma\right)d\omega\right]^2\\ &\le C\int\Big[M_\omega^{\mathcal{D}^\gamma}\big(\mathbf{1}_{K_i}M\left(\mathbf{1}_{K_i}\sigma\right)\big)\Big]^2d\omega\\ &\le C\int_{K_i}M\left(\mathbf{1}_{K_i}\sigma\right)^2d\omega\lesssim\left(\mathfrak{T}_M\left(3\right)\right)^2|3K_i|_\sigma\ .\end{aligned}$$

Thus we have

$$\sum_{i\in\mathcal{I}}\sum_{Q_j^k\in\mathscr{W}^\gamma:Q_j^k\subset K_i} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2}\Big[\int_{Q_j^k}\mathcal{L}_j^k(\mathbf{1}_{Q_j^k}\sigma)\omega\Big]^2\le\sum_{i\in\mathcal{I}}\left(\mathfrak{T}_M\left(3\right)\right)^2|3K_i|_\sigma\le C_{\text{bound}}\left(\mathfrak{T}_M\left(3\right)\right)^2\left|Q_u^t\right|_\sigma\ ,$$

where C_{bound} is a constant such that $\sum_{i\in\mathcal{I}}\mathbf{1}_{3K_i}\le C_{\text{bound}}\mathbf{1}_{Q_u^t}$ (due to a similar argument as (iii) of Proposition 1). We also have

$$\sum_{\substack{Q_j^k\in\mathscr{W}^\gamma:\ Q_j^k\subset Q_u^t\\ \ell(Q_j^k)\ge2^{-r}\ell(Q_u^t)}} \frac{|\widetilde{E}_{j,\gamma}^k|_\omega}{|3Q_j^k|_\omega^2}\Big[\int_{Q_j^k}\mathcal{L}_j^k(\mathbf{1}_{Q_j^k}\sigma)\omega\Big]^2\le C\sum_{\substack{Q_j^k\in\mathscr{W}^\gamma:\ Q_j^k\subset Q_u^t\\ \ell(Q_j^k)\ge2^{-r}\ell(Q_u^t)}}A_2|Q_j^k|_\sigma\le CrA_2\left|Q_u^t\right|_\sigma\ .$$

Combining the above estimates we conclude the proof. □

Then summing over $(t,u)\in\Gamma$ we obtain

$$IV\quad\lesssim\quad\left(\left(\mathfrak{T}_M\left(3\right)\right)^2+A_2\right)\sum_{(t,u)\in\Gamma}|Q_u^t|_\sigma\left(A_u^t\right)^2\lesssim\left(\left(\mathfrak{T}_M\left(3\right)\right)^2+A_2\right)\|f\|_{L^2(\sigma)}^2,$$

which combined with (3.14) gives

$$\sum_{(k,j)\in\Pi_3}2^{2k}\left|E_j^k\right|_\omega\le\left(\left(\mathfrak{T}_M\left(3\right)\right)^2+A_2\right)\|f\|_{L^2(\sigma)}^2. \tag{3.16}$$

3.2.1. *Wrapup of the proof.* Now letting the integer $N_0\to-\infty$ in the construction of principal cubes, and summing over $0\le k_0\le m+m_0-1$ in our convention regarding distinguished index pairs, we obtain from (2.2) that

$$\begin{aligned}&\int_{\mathbb{R}^n}\left[M\left(f\sigma\right)(x)\right]^2d\omega\left(x\right)\lesssim\mathbb{E}_\gamma\int_{\mathbb{R}^n}\Big[M^{\mathcal{D}^\gamma}\left(f\sigma\right)(x)\Big]^2d\omega\left(x\right) \qquad (3.17)\\ &\lesssim\quad\mathbb{E}_\gamma\Big(\sum_{\text{all }(k,j)}2^{2k}|E_{j,\gamma}^k|_\omega\Big)+3^nC_n2^{-2m_0}\int\left[M\left(f\sigma\right)\right]^2d\omega\end{aligned}$$

$$\leq \quad \pi_{\mathrm{good}}^{-1}\mathbb{E}_\gamma\Big(\sum_{(k,j)\in\Pi_1} 2^{2k}|\widetilde{E}^k_{j,\gamma}|_\omega + \sum_{(k,j)\in\Pi_2} 2^{2k}|\widetilde{E}^k_{j,\gamma}|_\omega + \sum_{(k,j)\in\Pi_3} 2^{2k}|\widetilde{E}^k_{j,\gamma}|_\omega\Big) + 3^n C_n 2^{-2m_0}\int [M(f\sigma)]^2\, d\omega,$$

which by the estimates (3.6), (3.10) and (3.16) gives

$$\int_{\mathbb{R}^n}[M(f\sigma)(x)]^2\, d\omega(x) \tag{3.18}$$
$$\lesssim \quad \left(\beta + 2^{-2m_0}\right)\int M(f\sigma)^2\, d\omega + \beta^{-1}C^2_{m+m_0}A_2\|f\|^2_{L^2(\sigma)} + \Big((\mathfrak{T}_M(3))^2 + A_2\Big)\|f\|^2_{L^2(\sigma)}$$

Now we can absorb the first term on the right hand side by choosing $\beta > 0$ sufficiently small and m_0 sufficiently large since the integral $\int M(f\sigma)^2\, d\omega$ is finite. Then we take the supremum over $f \in L^2(\sigma)$ with $\|f\|_{L^2(\sigma)} = 1$ to obtain

$$\mathfrak{N}_M \leq C\left(\mathfrak{T}_M(3) + \sqrt{A_2}\right) .$$

As the opposite inequality is trivial, this completes the proof of Theorem 2.

4. Weak triple testing: proof of Theorem 1

This section is devoted to proving Theorem 1. We begin with a basic observation, that is, the places where we need $\mathfrak{T}_M(3) < \infty$ are the following

(1) qualitatively, at the beginning of the argument, in order to assume without loss of generality that $\int M(f\sigma)^2\, d\omega < \infty$,
(2) and quantitatively, near the end of the argument, in the proof of Lemma 2.

The qualitative use of the triple testing condition is easily handled using D-triple testing as follows. As before, we can replace ω by $\omega_N = \omega \mathbf{1}_{B(0,N)}$ with $N > R$, and f is supported on $Q(0,R)$. Moreover, the *testing condition* for the cube $Q_m = Q(0, 3^m N)$ must hold for some $m \geq 0$, since otherwise iteration of the inequality $|Q_m|_\sigma \leq \frac{1}{D}|Q_{m+1}|_\sigma$ eventually violates the A_2 condition,

$$A_2(\sigma,\omega) \geq \frac{|Q_m|_\sigma |Q_m|_{\omega_N}}{|Q_m|^2} \geq \frac{D^m |Q_0|_\sigma |Q_0|_\omega}{3^{2mn}|Q_0|^2} = \left(\frac{D}{3^{2n}}\right)^m \frac{|Q_0|_\sigma |Q_0|_\omega}{|Q_0|^2},$$

if D is chosen greater than 3^{2n+1}. Thus if the *testing condition* holds for the cube Q_m we have

$$\int M(f\sigma)^2\, d\omega_N \leq \|f\|_{L^\infty}\int_{Q(0,N)} M(\mathbf{1}_{Q_m}\sigma)^2 d\omega < \infty,$$

and therefore, without loss of generality, we can assume $\int M(f\sigma)^2 d\omega < \infty$. The second point is delicate. Notice that, if we *a priori* have the usual testing, i.e., $\mathfrak{T}_M < \infty$, then we trivially have

$$\mathfrak{T}_M(3) \leq \mathfrak{T}^D_M(3) + D^{-1}\mathfrak{T}_M.$$

Then let D be sufficiently large and use the trivial fact that $\mathfrak{T}_M \leq \mathfrak{N}_M$ we get

$$\mathfrak{N}_M \leq C\left(\mathfrak{T}^D_M(3) + \sqrt{A_2}\right).$$

Therefore, the main goal is to remove the a priori assumption. To this end, consider the truncated maximal function

$$M^t f(x) := \sup_{Q\ni x, \ell(Q)\geq t}\frac{1}{|Q|}\int_Q |f|.$$

Obviously, $M^t \leq M$ and therefore, the related testing condition for M^t also holds. By the monotone convergence theorem, it suffices to prove

$$\|M^t(f\sigma)\|_{L^2(\omega)} \leq C\|f\|_{L^2(\sigma)}.$$

The argument for M^t is basically identical with that for M above. Indeed, with the same proof, we have the following analogy of Lemma 1:

$$M^t f(x) \leq 2^{4n+1}\mathbb{E}_\gamma M^{\mathcal{D}^\gamma,t} f(x), \quad M^{\mathcal{D}^\gamma,t}f(x) := \sup_{x\in Q\in\mathcal{D}^\gamma, \ell(Q)\geq t}\frac{1}{|Q|}\int_Q |f|.$$

Then everything is same with M^t and $M^{\mathcal{D}^\gamma,t}$ in place of M and $M^{\mathcal{D}^\gamma}$, respectively. We only remark that when $\ell(Q_j^k) < t$ we do not have the analogy of the maximal principle (3.2), instead, the same arguments will imply that $E_j^{k,\gamma}$ (defined according to M^t and $M^{\mathcal{D}^\gamma,t}$) is empty, which does not affect the proof.

Hence, for fixed $t > 0$, it remains to check that the *a priori* assumption $\mathfrak{T}_{M^t}(\sigma, \omega_N) < \infty$ holds automatically. Indeed, fix a cube Q and denote $K := Q(0, N)$, we have

$$\begin{aligned}\int_Q M^t(1_Q\sigma)^2 d\omega_N &\le 2\int_Q M^t(1_{Q\cap 3K}\sigma)^2 d\omega_N + 2\int_Q M^t(1_{Q\setminus 3K}\sigma)^2 d\omega_N \\ &\le 2\int_K M^t(1_{Q\cap 3K}\sigma)^2 d\omega + 2\int_{Q\cap K} M(1_{Q\setminus 3K}\sigma)^2 d\omega.\end{aligned}$$

To continue, notice that trivially

$$M^t(1_{Q\cap 3K}\sigma) \le t^{-n}\sigma(Q\cap 3K) \quad \text{and} \quad M(1_{Q\setminus 3K}\sigma)1_K \approx \text{constant}.$$

Therefore,

$$\begin{aligned}\int_Q M^t(1_Q\sigma)^2 d\omega_N &\lesssim t^{-2n}N^{2n}A_2\sigma(Q) + w(K)M(1_{Q\setminus 3K}\sigma)(c_K)^2 \\ &\lesssim (t^{-2n}N^{2n}+1)A_2\sigma(Q),\end{aligned}$$

which affirms the claim and therefore the proof of Theorem 1 is complete.

5. Proof of Theorem 3

This section is devoted to proving Theorem 3. We start with the weak type inequalities for fractional integrals. Let $\mathfrak{N}_{I_\alpha}^{\text{weak}}(\sigma,\omega)$ denote the best constant in the weak type $(2,2)$ inequality for the fractional integral I_α:

$$\sup_{\lambda>0} \lambda^2 \left|\{I_\alpha(f\sigma) > \lambda\}\right|_\omega \le \mathfrak{N}_{I_\alpha}^{\text{weak}}(\sigma,\omega)^2 \int |f|^2 \, d\sigma, \qquad f \in L^2(\sigma).$$

It is known from [21] that the weak type norm is equivalent to the dual testing condition, $\mathfrak{N}_{I_\alpha}^{\text{weak}}(\sigma,\omega) \approx \mathfrak{T}_{I_\alpha}(\omega,\sigma)$, and also that $A_2^\alpha(\sigma,\omega) \le \mathfrak{T}_{I_\alpha}(\omega,\sigma)$ where the α-fractional Muckenhoupt condition is given by

$$A_2^\alpha(\sigma,\omega) \equiv \sup_{Q\in\mathcal{P}^n} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}.$$

Now we are ready to record the characterization of weak type inequalities for the fractional integrals.

Theorem 4. *For $D > 1$ sufficiently large we have*

$$\mathfrak{N}_{I_\alpha}^{\text{weak}}(\sigma,\omega) \approx \mathfrak{T}_{I_\alpha}^D(3)(\omega,\sigma) + A_2^\alpha(\sigma,\omega),$$

for all pairs (σ,ω) of locally finite positive Borel measures on $\mathbb{R}^n$.

Proof. We modify the proof of the weak type characterization in [21]. For f bounded nonnegative and having compact support, define

$$\Omega_\lambda \equiv \{I_\alpha(f\sigma) > \lambda\} = \bigcup_j Q_k$$

as in the standard Whitney decomposition with $N = 9$. Then we have the well known maximum principle,

$$I_\alpha(f\mathbf{1}_{(3Q_k)^c}\sigma)(x) \le \gamma\lambda, \qquad \text{for } x \in Q_k \cap \{M_\alpha(f\sigma) \le \varepsilon(\gamma)\lambda\}.$$

Denote by E the set of indices k such that

$$|9Q_k|_\omega > D\,|Q_k|_\omega\ , \tag{5.1}$$

by F the set of indices k such that (5.1) fails and

$$\frac{1}{|Q_k|_\omega}\int_{Q_k} I_\alpha(\mathbf{1}_{3Q_k} f d\sigma)\, d\omega > \beta\lambda, \tag{5.2}$$

and by G the set of indices k such that (5.1) and (5.2) fails. Then for k in F we have

$$\lambda^2 |Q_k|_\omega < \beta^{-2}|Q_k|_\omega^{-1}\left(\int_{Q_k} I_\alpha(\mathbf{1}_{3Q_k} f d\sigma)\, d\omega\right)^2$$

$$\begin{aligned}
&= \beta^{-2} \left|Q_k\right|_\omega^{-1} \left(\int_{3Q_k} I_\alpha \left(\mathbf{1}_{Q_k} d\omega \right) \ f d\sigma \right)^2 \\
&\leq \beta^{-2} \left|Q_k\right|_\omega^{-1} \left(\int_{3Q_k} I_\alpha \left(\mathbf{1}_{Q_k} d\omega \right)^2 d\sigma \right) \left(\int_{Q_k} f^2 d\sigma \right) \\
&\leq \beta^{-2} \left(\mathfrak{T}_{I_\alpha}^D(3) \right)^2 D \left(\int_{Q_k} f^2 d\sigma \right),
\end{aligned}$$

where we have used D-restricted testing $\int_{3Q_k} I_\alpha \left(\mathbf{1}_{3Q_k} d\omega \right)^2 d\sigma \leq (\mathfrak{T}_{I_\alpha}^D(3))^2 D \left|Q_k\right|_\omega$ since (5.1) fails for $k \in F$. On the other hand, for $k \in G$, we have by the maximum principle that

$$\begin{aligned}
&\left|Q_k \cap \left\{ I_\alpha(f\sigma) > (\gamma+1)\lambda \right\}\right|_\omega \\
&\leq \left|Q_k \cap \left\{ I_\alpha \left(\mathbf{1}_{3Q_k} f\sigma \right) > \lambda \right\}\right|_\omega + \left|Q_k \cap \left\{ M_\alpha (f\sigma) > \varepsilon(\gamma)\lambda \right\}\right|_\omega \\
&\leq \beta \left|Q_k\right|_\omega + \left|Q_k \cap \left\{ M_\alpha (f\sigma) > \varepsilon(\gamma)\lambda \right\}\right|_\omega \ ,
\end{aligned}$$

since (5.1) and (5.2) fails. Altogether this gives the 'good λ-inequality'

$$\begin{aligned}
&(3\lambda)^2 \left|\left\{ I_\alpha(f\sigma) > 3\lambda \right\}\right|_\omega \\
&= \sum_k (3\lambda)^2 \left|Q_k \cap \left\{ I_\alpha(f\sigma) > 3\lambda \right\}\right|_\omega \\
&\leq 9 \sum_{k \in E} \lambda^2 \left|Q_k\right|_\omega + 9 \sum_{k \in F} \lambda^2 \left|Q_k\right|_\omega + 9\lambda^2 \sum_{k \in G} \left|Q_k \cap \left\{ I_\alpha(f\sigma) > 3\lambda \right\}\right|_\omega \\
&\leq \frac{9}{D} \sum_{k \in E} \lambda^2 \left|9Q_k\right|_\omega + \left(\frac{3}{\beta} \mathfrak{T}_{I_\alpha}^D(3) \right)^2 D \sum_{k \in F} \left(\int_{Q_k} f^2 d\sigma \right) + 9\lambda^2 \beta \sum_{k \in G} \left|Q_k\right|_\omega \\
&\qquad + (3\lambda)^2 \left|\left\{ M_\alpha (f\sigma) > \varepsilon\lambda \right\}\right|_\omega \\
&\leq \frac{9C_W}{D} \lambda^2 \left|\left\{ I_\alpha(f\sigma) > \lambda \right\}\right|_\omega + \left(\frac{3}{\beta} \mathfrak{T}_{I_\alpha}^D(3) \right)^2 D \left(\int f^2 d\sigma \right) + 9\lambda^2 \beta \left|\left\{ I_\alpha(f\sigma) > \lambda \right\}\right|_\omega \\
&\qquad + C(\varepsilon) A_2^\alpha(\sigma, \omega) \int f^2 d\sigma,
\end{aligned}$$

where C_W is a dimensional constant. If we now choose $\beta = \frac{1}{27}$ and $D = 27C_W$, then we obtain for each $t > 0$ that

$$\begin{aligned}
&\sup_{t \geq \lambda > 0} \lambda^2 \left|\left\{ I_\alpha(f\sigma) > \lambda \right\}\right|_\omega \leq \sup_{t \geq \lambda > 0} (3\lambda)^2 \left|\left\{ I_\alpha f\sigma > 3\lambda \right\}\right|_\omega \\
&\leq \left(\left(81 \mathfrak{T}_{I_\alpha}^D(3) \right)^2 D + C(\varepsilon) A_2^\alpha(\sigma, \omega) \right) \left(\int f^2 d\sigma \right) + \frac{2}{3} \sup_{t \geq \lambda > 0} \lambda^2 \left|\left\{ I_\alpha(f\sigma) > \lambda \right\}\right|_\omega \ ,
\end{aligned}$$

We now claim that $\sup_{t \geq \lambda > 0} \lambda^2 \left|\left\{ I_\alpha(f\sigma) > \lambda \right\}\right|_\omega$ is finite for all $t > 0$, which will complete the proof of the theorem after subtracting the last term on the right hand side from both sides, and then letting $t \to \infty$. To prove the claim we recall that f is bounded and supported in $B(0, R)$, so that

$$I_\alpha(f\sigma)(x) \approx \frac{1}{|x|^{n-\alpha}} \int_{B(0,R)} f d\sigma, \qquad x \notin 3B(0, R).$$

In other words,

$$(3B(0, R))^c \cap \left\{ I_\alpha(f\sigma) > \lambda \right\} \subset B\Big(0, \Big(\frac{c_{n,\alpha}}{\lambda} \int_{B(0,R)} f d\sigma \Big)^{\frac{1}{n-\alpha}} \Big) =: B(0, r_\lambda).$$

Then for $0 < \lambda \leq t$ we have

$$\begin{aligned}
\lambda^2 \left|\left\{ I_\alpha(f\sigma) > \lambda \right\}\right|_\omega &= \lambda^2 \left|3B(0, R) \cap \left\{ I_\alpha(f\sigma) > \lambda \right\}\right|_\omega + \lambda^2 \left|(3B(0, R))^c \cap \left\{ I_\alpha(f\sigma) > \lambda \right\}\right|_\omega \\
&\leq t^2 \left|3B(0, R)\right|_\omega + \frac{c_{n-\alpha}^2}{r_\lambda^{2(n-\alpha)}} \Big(\int_{B(0,R)} f d\sigma \Big)^2 |B(0, r_\lambda)|_\omega \\
&\leq t^2 \left|3B(0, R)\right|_\omega + c'_{n,\alpha} A_2^\alpha(\sigma, \omega) \int f^2 d\ \sigma,
\end{aligned}$$

where in the last step we have used the fact that $r_\lambda > R$. This proves the claim.

This completes the proof of the theorem since

$$A_2^\alpha(\sigma,\omega)+\mathfrak{T}_{I_\alpha}^D(3)(\omega,\sigma)\lesssim A_2^\alpha(\sigma,\omega)+\mathfrak{T}_{I_\alpha}(\omega,\sigma)\lesssim \mathfrak{N}_{I_\alpha}^{\mathrm{weak}}(\sigma,\omega)$$

is trivial. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. By the classical characterization of the weak and strong type two weight inequality for the fractional integrals (see [21] and [22]), we see that

$$\mathfrak{N}_{I_\alpha}(\sigma,\omega)\approx \mathfrak{N}_{I_\alpha}^{\mathrm{weak}}(\sigma,\omega)+\mathfrak{N}_{I_\alpha}^{\mathrm{weak}}(\omega,\sigma).$$

Then Theorem 3 follows immediately from Theorem 4. □

6. Counterexamples

In this section we provide two counterexamples which showing that the classical Muckenhoupt condition is needed in the characterization. We begin the following counterexample, which shows that the Muckenhoupt A_2 condition cannot be removed in Theorem 2.

Example 1. *Define*

$$\begin{aligned} d\sigma(y) &= e^y dy,\\ d\omega(x) &= \mathbf{1}_{[0,1]}(x)\,dx. \end{aligned}$$

Then

$$\mathfrak{N}_M(\sigma,\omega)\geq A_2(\sigma,\omega)\geq \sup_{R>1}\frac{|[0,R]|_\omega}{|R|}\frac{|[0,R]|_\sigma}{|R|}=\sup_{R>1}\frac{1}{R}\frac{e^R-1}{R}=\infty,$$

and

$$\mathfrak{T}_M(3)(\sigma,\omega)\lesssim 1.$$

Indeed, without loss of generality, $I=[a,b]$ with $I\cap[0,1]\neq\emptyset$ (since otherwise $\mathbf{1}_I\omega=0$ and $\int_I|M(\mathbf{1}_I\sigma)|^2d\omega=0$) and so

$$a<1 \text{ and } b>0. \tag{6.1}$$

Now we assume (6.1) and compute $\frac{1}{|3I|_\sigma}\int_I|M(\mathbf{1}_I\sigma)|^2d\omega$ in two cases.

(1) *Case $b>2$: In this case we have $M(\mathbf{1}_I\sigma)(x)=\frac{\int_x^b e^y dy}{b-x}\leq\frac{e^b-1}{b-1}$ for $0\leq x\leq 1$, and so*

$$\begin{aligned} \int_I|M(\mathbf{1}_I\sigma)|^2d\omega &\leq \int_0^1\left(\frac{e^b-1}{b-1}\right)^2dx\approx\frac{e^{2b}}{b^2},\\ |3I|_\sigma &= \int_{2a-b}^{2b-a}d\sigma\geq\int_{2b-a-1}^{2b-a}e^y dy\approx e^{2b-a},\\ \implies \frac{1}{|3I|_\sigma}\int_I|M(\mathbf{1}_I\sigma)|^2d\omega &\lesssim\frac{\frac{e^{2b}}{b^2}}{e^{2b-a}}=\frac{e^a}{b^2}\leq 1. \end{aligned}$$

(2) *Case $b\leq 2$: In this case we have $M(\mathbf{1}_I\sigma)(x)=\frac{\int_x^b e^y dy}{b-x}\leq e^2$ for $0\leq x\leq 1$, and so we consider two subcases.*
   (a) *Subcase $a\geq -1$:*

$$\begin{aligned} \int_I|M(\mathbf{1}_I\sigma)|^2d\omega &\leq e^2|I\cap[0,1]|\\ |3I|_\sigma &= \int_{2a-b}^{2b-a}e^y dy\geq e^{2a-b}3(b-a)\geq 3e^{-4}(b-a),\\ \implies \frac{1}{|3I|_\sigma}\int_I|M(\mathbf{1}_I\sigma)|^2d\omega &\leq\frac{e^2|I\cap[0,1]|}{3e^{-4}(b-a)}\leq\frac{e^6}{3}. \end{aligned}$$

(b) *Subcase $a<-1$: In this subcase we also have $b-a>0-(-1)=1$ and so*

$$\begin{aligned}
&\int_I |M(\mathbf{1}_I\sigma)|^2 d\omega \le e^2 |I\cap[0,1]| \\
&|3I|_\sigma = \int_{2a-b}^{2b-a} e^y dy \ge \int_{b-1}^{b} e^y dy = e^b - e^{b-1} \ge 1-e^{-1}, \\
\Longrightarrow \quad &\frac{1}{|3I|_\sigma}\int_I |M(\mathbf{1}_I\sigma)|^2 d\omega \le \frac{e^2|I\cap[0,1]|}{1-e^{-1}} \le \frac{e^2}{1-e^{-1}}.
\end{aligned}$$

On the other hand, if we interchange these measures, then we have $\mathfrak{T}_M(3)(\omega,\sigma)=\infty$ since with $I=[0,R]$ and $R>2$, we have

$$\frac{1}{|3I|_\omega}\int_I |M(\mathbf{1}_I\omega)|^2 d\sigma \gtrsim \int_{[1,R]} \left|\frac{\int_0^1 dx}{y}\right|^2 e^y dy = \int_1^R \frac{e^y}{y^2}dy \approx \frac{e^R}{R^2} .$$

We now show that the weight pair (σ,ω) in Example 1 above also satisfies $\mathfrak{T}_{I_\alpha}(3)(\sigma,\omega)<\infty$ and $A_2^\alpha(\sigma,\omega)=\infty$ for $0<\alpha<1$. Hence it is necessary to assume $A_2^\alpha(\sigma,\omega)$ condition in Theorem 4 as well.

Example 2. *Define*

$$\begin{aligned}
d\sigma(y) &= e^y dy, \\
d\omega(x) &= \mathbf{1}_{[0,1]}(x)\,dx.
\end{aligned}$$

Then

$$\mathfrak{N}_{I_\alpha}^{\text{weak}}(\omega,\sigma) \ge A_2^\alpha(\sigma,\omega) \ge \sup_{R>1} \frac{|[0,R]|_\omega}{|R|^{1-\frac{\alpha}{n}}}\frac{|[0,R]|_\sigma}{|R|^{1-\frac{\alpha}{n}}} = \sup_{R>1}\frac{1}{R^{1-\frac{\alpha}{n}}}\frac{e^R-1}{R^{1-\frac{\alpha}{n}}} = \infty,$$

and

$$\mathfrak{T}_{I_\alpha}(3)(\sigma,\omega) \lesssim 1.$$

Indeed, without loss of generality, $I=[a,b]$ with $I\cap[0,1]\ne\emptyset$ (since otherwise $\mathbf{1}_I\omega=0$ and $\int_I |I_\alpha(\mathbf{1}_I\sigma)|^2 d\omega = 0$) and so

$$a<1 \text{ and } b>0.$$

Now we assume this and compute $\frac{1}{|3I|_\sigma}\int_I |I_\alpha(\mathbf{1}_I\sigma)|^2 d\omega$ in two cases.

(1) *Case $b>2$: In this case we have*

$$I_\alpha(\mathbf{1}_I\sigma)(x) = \int_a^b |x-y|^{\alpha-1} e^y dy \le e^x \int_{-\infty}^b |y|^{\alpha-1}e^y dy \lesssim b^{\alpha-1}e^b$$

for $0\le x\le 1$, and so

$$\begin{aligned}
&\int_I |I_\alpha(\mathbf{1}_I\sigma)|^2 d\omega \lesssim \int_0^1 \left(b^{\alpha-1}e^b\right)^2 dx \approx \frac{e^{2b}}{b^{2(1-\alpha)}}, \\
&|3I|_\sigma = \int_{2a-b}^{2b-a} d\sigma \ge \int_{2b-a-1}^{2b-a} e^y dy \approx e^{2b-a}, \\
\Longrightarrow \quad &\frac{1}{|3I|_\sigma}\int_I |I_\alpha(\mathbf{1}_I\sigma)|^2 d\omega \lesssim \frac{\frac{e^{2b}}{b^{2(1-\alpha)}}}{e^{2b-a}} = \frac{e^a}{b^{2(1-\alpha)}} \lesssim 1.
\end{aligned}$$

(2) *Case $b\le 2$: In this case we have $I_\alpha(\mathbf{1}_I\sigma)(x)=\int_a^b |x-y|^{\alpha-1}e^y dy \lesssim e^2$ for $0\le x\le 1$, and so we consider two subcases.*

(a) *Subcase $a\ge -1$:*

$$\begin{aligned}
&\int_I |I_\alpha(\mathbf{1}_I\sigma)|^2 d\omega \lesssim e^2|I\cap[0,1]| \\
&|3I|_\sigma = \int_{2a-b}^{2b-a} e^y dy \ge e^{2a-b}3(b-a) \ge 3e^{-4}(b-a), \\
\Longrightarrow \quad &\frac{1}{|3I|_\sigma}\int_I |I_\alpha(\mathbf{1}_I\sigma)|^2 d\omega \le \frac{e^2|I\cap[0,1]|}{3e^{-4}(b-a)} \le \frac{e^6}{3}.
\end{aligned}$$

(b) *Subcase $a<-1$: In this subcase we also have $b-a>0-(-1)=1$ and so*

$$\begin{aligned} &\int_I |I_\alpha(\mathbf{1}_I\sigma)|^2 \, d\omega \lesssim e^2 |I\cap[0,1]| \\ &|3I|_\sigma = \int_{2a-b}^{2b-a} e^y dy \geq \int_{b-1}^{b} e^y dy = e^b - e^{b-1} \geq 1-e^{-1}, \\ \Longrightarrow \quad &\frac{1}{|3I|_\sigma}\int_I |I_\alpha(\mathbf{1}_I\sigma)|^2 \, d\omega \lesssim \frac{e^2|I\cap[0,1]|}{1-e^{-1}} \leq \frac{e^2}{1-e^{-1}}. \end{aligned}$$

References

[1] Wei Chen, Michael T. Lacey, *Two weight inequalities for positive operators: doubling cubes,* Rev. Mat. Iberoam. **36** (2020), 2209–2216.

[2] Ronald R. Coifman, Peter W. Jones, Stephen Semmes, *Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves,* J. Amer. Math. Soc., **2** (1989), 553-564.

[3] Guy David, Jean-Lin Journé, *A boundedness criterion for generalized Calderón-Zygmund operators,* Ann. of Math. (2) **120** (1984), 371–397.

[4] Guy David, Jean-Lin Journé, Stephen Semmes, Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation. Rev. Mat. Iberoamericana 1 (1985), 1–56.

[5] Charles Fefferman and Elias M. Stein, *Some maximal inequalities,* Amer. J. Math. ***93** (1971), 107-115.*

[6] John B. Garnett, Peter W. Jones, *BMO from dyadic BMO*, Pacific J. Math., **99** (1982), 351-371.

[7] Tuomas Hytönen,*The two weight inequality for the Hilbert transform with general measures,* Proc. London. Math. Soc. (3) **117** (2018), no. 3, 483–526.

[8] Tuomas Hytönen, Carlos Pérez, Sergei Treil and Alexander Volberg, *Sharp weighted estimates for dyadic shifts and the A_2 conjecture,* J. Reine Angew. Math. **687** (2014), 43–86.

[9] Michael T. Lacey, *Two weight inequality for the Hilbert transform: A real variable characterization, II*, Duke Math. J. **163**(2014), 2821-2840.

[10] Michael T. Lacey, Kangwei Li *Two weight norm inequalities for the g function*, Math. Res. Lett., **21**(2014), 521-536.

[11] Michael T. Lacey, Brett D. Wick, *Two weight inequalities for Riesz transforms: uniformly full dimension weights*, *arXiv:1312.6163v3*.

[12] Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen, Ignacio Uriarte-Tuero, *Two weight inequality for the Hilbert transform: A real variable characterization I*, Duke Math. J, **163**(2014), 2795-2820.

[13] Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen, Ignacio Uriarte-Tuero, Brett D. Wick, *Two weight inequalities for the Cauchy transform from* $\mathbb{R}$ *to* $\mathbb{C}_+$, *arXiv:1310.4820v4*.

[14] Kangwei Li, Eric T. Sawyer, *Restricted testing for positive operators,* arXiv:1809.04873.

[15] Kangwei Li, Eric T. Sawyer, *Restricted testing for the Hardy-Littlewood maximal function,* arXiv:1811.11032v1.

[16] Henri Martikainen, Mihalis Mourgoglou, Emil Vuorinen, *A new approach to non-homogenous local Tb theorems*, to appear in J. Anal. Math., arxiv:1511.00528v1.

[17] Benjamin Muckenhoupt, Richard L. Wheeden, *Some weighted weak type inequalities for the Hardy Littlewood maximal function and the Hilbert transform*, Indiana Univ. Math. J. **26** (1977), 801-816.

[18] Fedor Nazarov, Sergei Treil, Alexander Volberg *The Tb-theorem on non-homogeneous spaces,* Acta Math. **190** (2003), no. 2, MR 1998349 (2005d:30053).

[19] Fedor Nazarov, Sergei Treil, Alexander Volberg, *Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures*, preprint (2004) arxiv:1003.1596.

[20] Eric T. Sawyer *A characterization of a two weight norm inequality for maximal operators*, Studia Math., 75 (1982), 1–11.

[21] Eric T. Sawyer, *A two weight weak-type inequality for fractional integrals*, Trans. A.M.S. **281** (1984), 339-345, MR{719674 (85j:26010)}.

[22] Eric T. Sawyer *A characterization of two weight norm inequalities for fractional and Poisson integrals*, Trans. A.M.S. **308** (1988), 533-545, MR{930072 (89d:26009)}.

[23] Eric T. Sawyer *Energy conditions and twisted localizations of operators,* arXiv:1801.03706v2.

[24] Eric T. Sawyer, Chun-Yen Shen, Ignacio Uriarte-Tuero A *two weight theorem for α-fractional singular integrals with an energy side condition*, Revista Mat. Iberoam. **32** (2016), no. 1, 79-174.

[25] Eric T. Sawyer, Chun-Yen Shen, Ignacio Uriarte-Tuero *A good-λ lemma, two weight $T1$ theorems without weak boundedness, and a two weight accretive global Tb theorem,* Harmonic Analysis, Partial Differential Equations and Applications (In Honor of Richard L. Wheeden), Birkhäuser 2017 (see also arXiv:1609.08125v2).

[26] Eric T. Sawyer, Chun-Yen Shen, Ignacio Uriarte-Tuero *A two weight local Tb theorem for the Hilbert transform,* arXiv:1709.09595v11.

[27] Xavier Tolsa *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. **190** (1) (2003), 105–149.

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