BILINEAR SPHERICAL MAXIMAL FUNCTIONS OF PRODUCT TYPE

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Abstract. In this paper we introduce and study a bilinear spherical maximal function of product type in the spirit of bilinear Calderón–Zygmund theory. This operator is different from the bilinear spherical maximal function considered by Geba et al. in [14]. We deal with lacunary and full versions of this operator, and prove weighted estimates with respect to genuine bilinear weights beyond the Banach range. Our results are implied by sharp sparse domination for both the operators, following ideas by Lacey [20]. In the case of the lacunary maximal operator we also use interpolation of analytic families of operators to address the weighted boundedness for the whole range of tuples.

1. Introduction

The theory of multilinear operators has been an active area of research for the past two decades in harmonic analysis. It finds its roots in the pioneer work by Coifman and Meyer [7], although it was the remarkable proof of the boundedness of the bilinear Hilbert transform by Lacey and Thiele [22, 23] that renewed the motivation for the study of multilinear singular integrals. The multilinear Calderón-Zygmund operators were systematically treated in [16] and later on, in [25], Lerner et al. developed an appropriate theory of multilinear maximal functions and multilinear weights. In particular, they established weighted boundedness for multilinear Calderón-Zygmund operators. Since then there have been several developments in the weighted theory of multilinear weights, we emphasize the recent works [27, 30] and references therein.

For notational convenience we shall restrict ourselves to the bilinear setting in this paper. Given locally integrable functions f_1 and f_2 defined on $\mathbb{R}^n$, the bilinear maximal function $\mathcal{M}(f_1, f_2)$ is defined by

$$\mathcal{M}(f_1, f_2)(x) := \sup_{Q \ni x} \prod_{i=1}^{2} \frac{1}{|Q|} \int_Q |f_i(y_i)|\, dy_i, \tag{1}$$

where the supremum in the above is taken over all cubes Q in $\mathbb{R}^n$ containing the point x. The cubes are always assumed to have their sides parallel to coordinate axes.

Note that the bilinear maximal operator $\mathcal{M}$ is dominated by the product of the classical Hardy-Littlewood maximal functions in a pointwise manner, i.e.,

$$\mathcal{M}(f_1, f_2) \leq M(f_1)M(f_2),$$

where M denotes the Hardy-Littlewood maximal operator given by

$$M(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

Let $1 < p_1, p_2 < \infty$ and p be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Hölder's inequality yields that the operator $\mathcal{M}$ is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(w)$ for all $w_i \in A_{p_i}$, $i = 1, 2$, and $w = \prod_{j=1}^{2} w_j^{p/p_j}$. Here A_p denotes the class of Muckenhoupt weights, see Subsection 3.1.

In [25], the authors showed that the bilinear maximal operator $\mathcal{M}$ is the appropriate analogue of the classical Hardy–Littlewood maximal operator. They introduced a suitable analogue of Muckenhoupt weights in the bilinear setting, the class $A_{\vec{P}}$ (see Subsection 3.1), and showed that the class $A_{\vec{P}}$ is bigger than the product of corresponding linear A_p classes. The class $A_{\vec{P}}$ characterizes the weighted boundedness of the bilinear maximal operator $\mathcal{M}$. Moreover, the

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bilinear Calderón-Zygmund operators possess weighted boundedness with respect to bilinear weights in $A_{\vec{P}}$. We refer the reader to [10, 24, 25] for more details.

Later on, first in [27] and then in [26, 30], the notion of bilinear (or multilinear) weights was further generalised and extrapolation results were proved, see Subsection 3.1.

Motivated from the discussions above, in this paper we introduce a bilinear spherical maximal function of product type in the spirit of Calderón–Zygmund theory and investigate its weighted boundedness with respect to the bilinear weights just mentioned.

1.1. **Linear spherical maximal functions and bilinear product-type analogues.** Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function. Consider the average of f over the sphere of radius $0 < r < \infty$ given by

$$\mathcal{A}_r f(x) = \int_{\mathbb{S}^{n-1}} f(x - ry)\, d\sigma_{n-1}(y),$$

where $d\sigma_{n-1}$ is the normalized rotation invariant surface measure on the sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$. The spherical maximal function was introduced by Stein [32] and is defined as

$$M_{\text{full}}(f)(x) := \sup_{r>0} \mathcal{A}_r f(x), \quad x \in \mathbb{R}^n.$$

Stein proved that M_{full} is bounded in $L^p(\mathbb{R}^n)$ if and only if $\frac{n}{n-1} < p \leq \infty$ for all $n \geq 3$. The problem in dimension $n = 2$ was settled later by Bourgain [4] (we refer to [29] for a different proof of Bourgain's result).

The dyadic or lacunary version of the spherical maximal function results by taking the supremum over the set $\{2^j : j \in \mathbb{Z}\}$, i.e.,

$$M_{\text{lac}}(f)(x) = \sup_{j\in\mathbb{Z}} \mathcal{A}_{2^j} f(x).$$

The lacunary spherical maximal operator M_{lac} is bounded in $L^p(\mathbb{R}^n)$ for all $1 < p \leq \infty$ and $n \geq 2$, see [6, 8] for details. Weighted boundedness properties of the spherical maximal operators have been studied in [9, 12, 13, 28].

In a recent article, Lacey [20] revisited the spherical maximal function and, using a new approach that unified the lacunary and full versions, he managed to prove sparse bounds for these operators which led him to obtain new weighted norm inequalities. We also refer to [20] for a discussion about the suitability of A_p weights in the context of the spherical maximal function.

In this paper we introduce a bilinear analogue of the spherical maximal function in the spirit of the bilinear Hardy-Littlewood maximal function (1), which plays a key role in the theory of bilinear Calderón–Zygmund operators. Define

$$\mathcal{M}_{\text{full}}(f_1, f_2)(x) := \sup_{t>0} \mathcal{A}_t f_1 \mathcal{A}_t f_2(x).$$

As earlier, if we take the supremum in the above over the dyadic numbers, we get bilinear analogue of the lacunary spherical maximal function. This way, the bilinear lacunary spherical maximal operator $\mathcal{M}_{\text{lac}}$ is defined as

$$\mathcal{M}_{\text{lac}}(f_1, f_2)(x) := \sup_{j\in\mathbb{Z}} \mathcal{A}_{2^j} f_1 \mathcal{A}_{2^j} f_2(x).$$

We refer to these operators as *bilinear spherical maximal functions of product type*[1].

Note that $\mathcal{M}_{\text{full}}(f_1, f_2)$ (and $\mathcal{M}_{\text{lac}}(f_1, f_2)$) is dominated by the product of the linear full (respectively lacunary) spherical maximal functions in a pointwise sense. Therefore, Hölder's inequality immediately yields the $L^{p_1} \times L^{p_2} \to L^p$ estimates for the operators $\mathcal{M}_{\text{full}}$ and $\mathcal{M}_{\text{lac}}$. In fact, we also get the weighted estimates for the operator with respect to product weights, see Theorem 5.3. We will prove new weighted estimates for the bilinear spherical maximal functions with respect to bilinear weights that are beyond the type of weights as described in Theorem 5.3. This result is stated in Theorem 2.1. We exploit the ideas from [20] and establish a sparse domination principle for the bilinear spherical maximal functions in Theorem 2.3 so

[1] In a private communication with the second and third authors, Loukas Grafakos suggested that the terminology for this operator should be instead *maximal product of spherical averages* to better portray the nature of the operators.

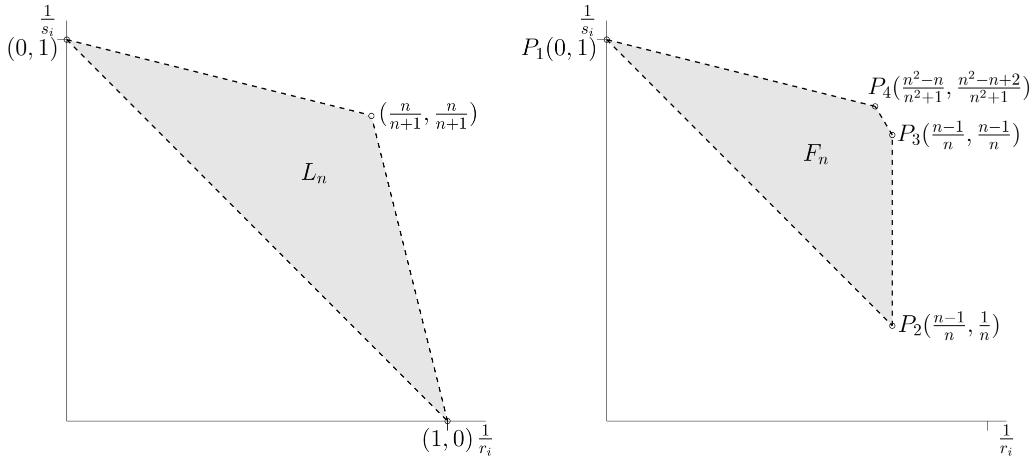

FIGURE 1. Triangle L_n on the left and trapezium F_n on the right.

that we deduce weighted estimates as a consequence of known results in the literature. These weighted estimates cover the best possible range of tuples (p_1, p_2, p) for the L^p boundedness of $\mathcal{M}_{\text{full}}$. Nevertheless, there is a dimensional constraint in this range of (p_1, p_2, p) that prevents the operator $\mathcal{M}_{\text{lac}}$ from getting weighted boundedness in the range $1 < p_1, p_2 < \infty$ via Theorem 2.3. This restriction is overcome in Theorems 2.4 and 2.5. We also provide weighted boundedness of operators $\mathcal{M}_{\text{full}}$ and $\mathcal{M}_{\text{lac}}$ for the case of power weights in Corollaries 5.4, 5.5, and Theorem 2.4. This makes it easier to compare our results with the Hölder type results, see Section 5.

A different analogue of the spherical maximal function in the bilinear setting has been studied in the literature. It was introduced in [14] and is defined as follows:

$$\mathcal{M}_{\text{sph}}(f_1, f_2)(x) := \sup_{t>0} \int_{\mathbb{S}^{2n-1}} |f_1(x-ty)f_2(x-tz)|\, d\sigma_{2n-1}(y,z). \tag{2}$$

In [3, 15, 17] the authors proved partial results obtaining $L^{p_1} \times L^{p_2} \to L^p$ estimates for the operator $\mathcal{M}_{\text{sph}}$ for a certain range of p_1, p_2 and p and some assumptions on the dimension n. In [18] the authors proved the following pointwise domination result

$$\mathcal{M}_{\text{sph}}(f_1, f_2)(x) \lesssim M_{\text{full}}(f_1)(x) M(f_2)(x), \tag{3}$$

and extended the $L^{p_1} \times L^{p_2} \to L^p$ estimates for the operator $\mathcal{M}_{\text{sph}}$ to the best possible range of exponents p_1, p_2 and p for all $n \geq 2$ (note that an estimate similar to (3) holds with the roles of M_{full} and M interchanged due to symmetry). We also refer to the recent papers [1, 11] for the generalisation of the bilinear spherical maximal function to the multilinear setting. Weighted estimates for the bilinear maximal operator $\mathcal{M}_{\text{sph}}$ defined in (2) beyond the ones that can be obtained trivially from the pointwise estimate (3) remain as an open problem.

The paper is organised as follows. We state the main results in the next section, then in Section 3 we recall necessary definitions and results and also set notation that we use in the paper. Section 4 is devoted to prove weighted estimates for the operators under consideration and we complete the proofs of Theorems 2.1, 2.4, and 2.5 in this section. In Section 5 we discuss some examples comparing the weighted results obtained in Theorem 2.1 with the Hölder type results. Next, in Section 6 we give the proof of sparse domination result Theorem 2.3. Finally, in Section 7 we provide the necessity of some conditions for such a sparse domination.

2. MAIN RESULTS

Our first main result is the following theorem containing weighted estimates for the product type operators with bilinear weights in the class $A_{\vec{q},\vec{r}}$ (see Definition 3.2). In what follows, we will denote by L_n the triangle with vertexes $(0,1), (1,0)$ and $\left(\frac{n}{n+1}, \frac{n}{n+1}\right)$ and by F_n the trapezium with vertexes $(0,1)$, $\left(\frac{n-1}{n}, \frac{1}{n}\right)$, $\left(\frac{n-1}{n}, \frac{n-1}{n}\right)$ and $\left(\frac{n^2-n}{n^2+1}, \frac{n^2-n+2}{n^2+1}\right)$, see Figure 1.

Theorem 2.1. *Let $n \geq 2$. For $i = 1, 2$, let $(\frac{1}{r_i}, \frac{1}{s_i})$ be in the interior of L_n (respectively F_n). Assume that $t := \frac{s_1 s_2}{s_1+s_2-s_1 s_2} > 1$. Then for all $\vec{q} = (q_1, q_2)$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ with $r_i \leq q_i$, $i = 1, 2$, and $t' > q$, the operator $\mathcal{M}_{\mathrm{lac}}$ (respectively $\mathcal{M}_{\mathrm{full}}$) extends to a bounded operator from $L^{q_1}(w_1) \times L^{q_2}(w_2) \to L^q(w)$, i.e.,*

$$\|\mathcal{M}(f_1, f_2)\|_{L^q(w)} \leq C([\vec{w}]_{A_{\vec{q},\vec{r}}}) \prod_{i=1}^{2} \|f_i\|_{L^{q_i}(w_i)},$$

where $\mathcal{M} := \mathcal{M}_{\mathrm{lac}}$ (respectively $\mathcal{M}_{\mathrm{full}}$), $\vec{w} = (w_1, w_2) \in A_{\vec{q},\vec{r}}$ with $\vec{r} = (r_1, r_2, t)$ defined as in Definition 3.2 and $C([\vec{w}]_{A_{\vec{q},\vec{r}}})$ is a constant depending on the characteristic of the weight.

Remark 2.2. *Observe that, in Theorem 2.1, we can consider $\vec{q} = (q_1, q_2)$ with $q < 1$ as well, which means that the weighted inequalities hold beyond the Banach range.*

The weighted estimates in Theorem 2.1 are indeed consequence of a sparse domination principle for the bilinear spherical maximal functions shown in Theorem 2.3 below and the extrapolation result in [27, Theorem 1.1]. Actually, one could state an improved result, providing the quantitative bounds (i.e., giving more explicit information on $C([\vec{w}]_{A_{\vec{q},\vec{r}}})$), including end-points, and vector-valued inequalities, see Theorem 4.3 and Remark 4.4. For these consequences we appeal to [26, 27, 30].

Before stating the sparse domination result let us set up the notation. A collection of cubes $\mathcal{S}$ in $\mathbb{R}^n$ is said to be η-sparse, $0 < \eta < 1$, if there are sets $\{E_S \subset S : S \in \mathcal{S}\}$ which are pairwise disjoint and satisfy $|E_S| > \eta|S|$ for all $S \in \mathcal{S}$. By the term (p, q, r)-sparse form we mean:

$$\Lambda_{\mathcal{S}_{p,q,r}}(f, g, h) := \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,p} \langle g \rangle_{S,q} \langle h \rangle_{S,r},$$

see Section 3 for notations.

Theorem 2.3. *Let $n \geq 2$. For $i = 1, 2$, let $(\frac{1}{r_i}, \frac{1}{s_i})$ be in the interior of L_n (respectively F_n) and $\rho_i > r_i$. Then for any non-negative compactly supported bounded functions f_1, f_2 and h, there exists a sparse collection $\mathcal{S} = \mathcal{S}_{\rho_1,\rho_2,t}$ such that*

$$\langle \mathcal{M}(f_1, f_2), h \rangle \leq C \Lambda_{\mathcal{S}_{\rho_1,\rho_2,t}}(f_1, f_2, h),$$

where $t := \frac{s_1 s_2}{s_1+s_2-s_1 s_2} > 1$ and $\mathcal{M} := \mathcal{M}_{\mathrm{lac}}$ (respectively $\mathcal{M}_{\mathrm{full}}$).

Furthermore, the ranges of $(\frac{1}{r_i}, \frac{1}{s_i})$, $i = 1, 2$, are sharp up to endpoints in the sense that no such result can hold if both $(\frac{1}{r_i}, \frac{1}{s_i})$, $i = 1, 2$, do not lie in the closure of L_n (respectively F_n). The condition $1 < t < \infty$ is also a necessary condition.

We prove the sufficiency part of this theorem in two steps. First, we shall establish an analogous result for characteristic functions. Then we consider the theorem for general functions. In this second stage, we use a recursive argument in which both functions involved in the bilinear sparse form are decomposed into simple functions. We perform this step by taking one of the functions to be a characteristic function, then keeping this function fixed we decompose the other function into suitable simple functions. This process along with an application of Carleson embedding theorem allows us to obtain sparse domination for general compactly supported bounded function in one place whereas the other function is taken to be a characteristic function. We repeat the procedure with the second function. The proof of these results and of Theorem 2.3 will be given in Section 6 and in Section 7 (in the latter we prove the necessity of the conditions for the sparse domination.

Note that in Theorem 2.3 we have the necessary condition $1 < t := \frac{s_1 s_2}{s_1+s_2-s_1 s_2} < \infty$. This condition translates into the fact that we cannot choose both $r_1, r_2 \leq \frac{2n}{2n-1}$. Thus, in the case of lacunary spherical maximal operator $\mathcal{M}_{\mathrm{lac}}$ in Theorem 2.1, we cannot consider both $q_1, q_2 \leq \frac{2n}{2n-1}$ simultaneously. Nevertheless, we cover the complete range of exponents for the operator $\mathcal{M}_{\mathrm{lac}}$ using a different method. We establish non-trivial weighted estimates for $\mathcal{M}_{\mathrm{lac}}$ for tuples (q_1, q_2, q) with $1 < q_1, q_2 \leq \frac{2n}{2n-1}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, see Theorem 2.4 and Theorem 2.5. We exploit ideas from [19, 31] to obtain these results, based on interpolation of analytic families of linear operators in [5].

Observe that this scenario does not occur in the case of $\mathcal{M}_{\text{full}}$. Indeed, the condition $r_1, r_2 > \frac{n}{n-1}$ holds necessarily for the operator $\mathcal{M}_{\text{full}}$. Therefore, Theorem 2.3 addresses the question of weighted boundedness of the operator $\mathcal{M}_{\text{full}}$ with respect to genuine weights $\vec{w} = (w_1, w_2) \in A_{\vec{q},\vec{r}}$ for the entire possible range of tuples (q_1, q_2, q).

As mentioned previously, we shall establish non-trivial weighted estimates for $\mathcal{M}_{\text{lac}}$ for the tuple (p_1, p_2, p) with $1 < p_1, p_2 \le \frac{2n}{2n-1}$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. For the sake of simplicity we consider $p_1 = p_2 = p$ in the following theorem, since the statement of the result for general tuples turns out to be rather cumbersome.

Theorem 2.4. *Let $n \ge 2$. The operator $\mathcal{M}_{\text{lac}}$ is bounded from $L^p(|x|^\alpha) \times L^p(|x|^\beta)$ to $L^{p/2}(|x|^{\frac{\alpha+\beta}{2}})$ with $1 < p \le \frac{2n}{2n-1}$ for α, β satisfying:*

$$(1-n)p < \alpha, \beta < (n-1)(p-1) \quad \textit{and} \quad \alpha + \beta > 2(1-n)\big(n - (n-1)p\big), \quad n \ge 2.$$

We would like to remark that while proving Theorem 2.4, we actually get weighted boundedness of operators $\mathcal{M}_{\text{lac}}$ for the triplet (p_1, p_2, p) for more general weights than considered in Theorem 2.4 above (see Theorem 2.5). Moreover, these weights do not come from the product type bilinear weights. This point is discussed in detail in Section 5.

Let $\frac{1}{\phi_{\text{lac}}(\frac{1}{r})}$ denote the piecewise linear function on the interval $(0, 1)$ whose graph connects the points $(0, 1), (\frac{n}{n+1}, \frac{n}{n+1})$ and $(1, 0)$, i.e.,

$$\frac{1}{\phi_{\text{lac}}(\frac{1}{r})} = \begin{cases} 1 - \frac{1}{rn}, & \text{if } \ 0 < \frac{1}{r} \le \frac{n}{n+1} \\ n(1 - \frac{1}{r}), & \text{if } \ \frac{n}{n+1} < \frac{1}{r} < 1. \end{cases} \tag{4}$$

An inspection of the proof of Theorem 2.4 delivers the following (indeed, this is a byproduct of Step I in the proof), see Section 3 for the definitions of weights.

Theorem 2.5. *Let $n \ge 2$. The operator $\mathcal{M}_{\text{lac}}$ is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(w)$, where $1 < p_1, p_2 \le \frac{2n}{2n-1}$ and for certain weights $\vec{w} = (w_1, w_2)$ which do not belong to product type weights*

$$\bigcup_{1<r_i<p_i} \Big(\prod_{i=1}^{2} A_{\frac{p_i}{r_i}} \cap \mathrm{RH}_{\left(\frac{\phi'_{\text{lac}}(\frac{1}{r_i})}{\tilde{p}_i}\right)'} \Big) \bigcup (\mathcal{R}_{p_1} \times \mathcal{R}_{p_2}),$$

where

$$\mathcal{R}_p = \{|x|^b : 1 - n \le b < (n-1)(p-1)\}, \quad n \ge 2. \tag{5}$$

3. Notations and definitions

In this section we collect some of the notations and definitions that we use in this paper. With the letters $c, C \dots$ we denote structural constants that depend only on the dimension and on parameters. Their values might vary from one occurrence to another, and in most of the cases we will not track the explicit dependence. We will write $\gamma_1 \lesssim \gamma_2$ if $\gamma_1 \le c\gamma_2$ for a structural constant c. Given $p \ge 1$, the conjugate exponent of p will be denoted by p', i.e., $1/p + 1/p' = 1$.

For any cube Q and $1 < p < \infty$, we define

$$\langle f \rangle_{Q,p} := \left(\frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p}, \qquad \langle f \rangle_Q := \frac{1}{|Q|} \int_Q |f(x)| \, dx,$$

where $|Q|$ denotes the Lebesgue measure of Q.

A weight is a non-negative locally Lebesgue integrable function that is non-zero in a set of positive measure. We say that a weight w belongs to the Muckenhoupt class A_p if

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|} \int_Q w \, dx \Big) \Big(\frac{1}{|Q|} \int_Q w^{1-p'} \, dx \Big)^{p-1} < \infty, \quad 1 < p < \infty.$$

The quantity $[w]_{A_p}$ is referred to as the A_p characteristic of $w \in A_p$. For $p = 1$ the class A_1 consists of all w such that

$$[w]_{A_1} := \operatorname{ess\,sup} \frac{M(w)}{w} < \infty.$$

Given $s > 1$, a weight belongs to the reverse Hölder RH_s if there exists a constant C such that, for every cube Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes,

$$\Big(\frac{1}{|Q|}\int_Q w^s\,dx\Big)^{1/s} \leq \frac{C}{|Q|}\int_Q w\,dx < \infty.$$

3.1. **Bilinear weights.** Let $1 \leqslant p_1, p_2 < \infty$ and p be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{6}$$

Definition 3.1. [25, Definition 3.5] *Let $\vec{p} = (p_1, p_2)$. For a given pair of weights $\vec{w} = (w_1, w_2)$, set $w := \prod_{i=1}^2 w_i^{p/p_i}$. We say that $\vec{w} \in A_{\vec{P}}$ if*

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w\,dx\Big)\prod_{j=1}^{2}\Big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\,dx\Big)^{p/p_j'} < \infty.$$

When $p_j = 1$, $\big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\big)^{1/p_j'}$ is understood as $(\inf_Q w_j)^{-1}$. The quantity $[\vec{w}]_{A_{\vec{P}}}$ is referred to as the bilinear $A_{\vec{P}}$ characteristic of the bilinear weight $\vec{w}$.

The bilinear $A_{\vec{P}}$ class was further generalised recently in [27].

Definition 3.2. [27, Section 1] *Let $\vec{p} = (p_1, p_2)$ and p be as in (6). For a tuple $\vec{r} = (r_1, r_2, r_3)$ with $r_i \leq p_i$, $i = 1, 2$, and $r_3' > p$, where $1 \leq r_1, r_2, r_3 < \infty$, we say that $\vec{w} = (w_1, w_2) \in A_{\vec{p},\vec{r}}$ if $0 < w_i < \infty$ a.e. for $i = 1, 2$ and*

$$[\vec{w}]_{A_{\vec{p},\vec{r}}} := \sup_{Q\subset\mathbb{R}^n} \langle w^{\frac{r_3'}{r_3'-p}}\rangle_Q^{\frac{1}{p}-\frac{1}{r_3'}} \prod_{i=1}^{2}\langle w_i^{\frac{r_i}{r_i-p_i}}\rangle_Q^{\frac{1}{r_i}-\frac{1}{p_i}} < \infty,$$

where $w := \prod_{i=1}^2 w_i^{p/p_i}$. When $r_3 = 1$, the term corresponding to w needs to be replaced by $\langle w\rangle_Q^{1/p}$. Analogously, when $p_i = r_i$, the term corresponding to w_i needs to be replaced by $\operatorname{ess\,sup}_Q w_i^{-1/p_i}$.

Remark 3.3. *Note that $A_{\vec{p},(1,1,1)}$ agrees with the class $A_{\vec{P}}$.*

The following result describes the bilinear weights $A_{\vec{p},\vec{r}}$ in terms of the classical A_p weights. This provides a useful tool in the study of weighted estimates with respect to bilinear weights.

Lemma 3.4. [27, Lemma 5.3] *Let $\vec{p} = (p_1, p_2)$ with $1 < p_1, p_2 < \infty$ and $\vec{r} = (r_1, r_2, r_3)$ with $1 \leq r_1, r_2, r_3 < \infty$. Let $p' := p_3$ and $\frac{1}{r} := \sum_{i=1}^3 \frac{1}{r_i}$. Assume that $r_i \leq p_i$ for $i = 1, 2$ and $r_3' > p$. Consider*

$$\frac{1}{\delta_i} = \frac{1}{r_i} - \frac{1}{p_i} \quad \text{and} \quad \frac{1}{\theta_i} = \frac{1-r}{r} - \frac{1}{\delta_i}, \qquad i = 1, 2, 3.$$

Then $\vec{w} = (w_1, w_2) \in A_{\vec{p},\vec{r}}$ if and only if

$$w_i^{\frac{\theta_i}{p_i}} \in A_{\frac{1-r}{r}\theta_i} \quad \text{with} \quad [w_i^{\frac{\theta_i}{p_i}}]_{A_{\frac{1-r}{r}\theta_i}} \leq [\vec{w}]_{A_{\vec{p},\vec{r}}}^{\theta_i}, \qquad i = 1, 2$$

and

$$w^{\frac{\delta_3}{p}} \in A_{\frac{1-r}{r}\delta_3} \quad \text{with} \quad [w^{\frac{\delta_3}{p}}]_{A_{\frac{1-r}{r}\delta_3}} \leq [\vec{w}]_{A_{\vec{p},\vec{r}}}^{\delta_3}.$$

In [30], Nieraeth presented an alternative approach to describe the bilinear weights $A_{\vec{p},\vec{r}}$ and defined yet another class of weights that is equivalent to the class defined in [27]. Nieraeth extended the extrapolation results contained in [27] in several directions.

Definition 3.5. [30, Definition 2.1] *Let $\vec{p} = (p_1, p_2)$, $\vec{q} = (q_1, q_2)$ with $p_1, p_2 \in (0, \infty)$ and $q_1, q_2 \in (0, \infty]$. Let q be given by $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. We say $(\vec{p}, s) \leq \vec{q}$ if $\vec{p} \leq \vec{q}$ and $q \leq s$ where $s \in (0, \infty]$. Here $\vec{p} \leq \vec{q}$ means that $p_i \leq q_i$, $i = 1, 2$. For weights w_1, w_2 write $w = \prod_{i=1}^2 w_i$. We say that $\vec{w} = (w_1, w_2) \in A_{\vec{q},(\vec{p},s)}$ if*

$$[\vec{w}]_{\vec{q},(\vec{p},s)} := \sup_Q \Big(\prod_{i=1}^{2}\langle w_i^{-1}\rangle_{\frac{1}{\frac{1}{p_i}-\frac{1}{q_i}},Q}\langle w\rangle_{\frac{1}{\frac{1}{q}-\frac{1}{s}},Q}\Big) < \infty,$$

where the supremum is taken over all cubes (with sides parallel to coordinate axes) in $\mathbb{R}^n$.

Remark 3.6. *Note that the definition above includes the case $q_j = \infty$. In this case the norm is interpreted as $\|f_j\|_{L^{q_j}(w_j^{q_j})} = \|f_j w_j\|_{L^\infty}$. Also, the definition is used with $\frac{1}{q_j} = 0$ when $q_j = \infty$. We refer to [30] for more details on this. We also would like to refer the reader to [26], where authors consider a slightly different approach to include the end-points cases which allows one or more indices to take value infinity. Further, note that when q_j are finite, the following relation holds: $(w_1^{q_1}, w_2^{q_2}) \in A_{\vec{q},(r_1,r_2,t)}$ if and only if $\vec{w} \in A_{\vec{q},(\vec{r},t')}$.*

4. Proofs of weighted estimates

4.1. Proof of Theorem 2.1.

As pointed out earlier, the proof of Theorem 2.1 follows from the sparse domination result Theorem 2.3 and the already well-known consequences in the literature.

Theorem 4.1. [27, Corollary 2.15] *Fix $\vec{r} = (r_1, r_2, r_3)$, with $r_i \geq 1$ and $\sum_{i=1}^{3} \frac{1}{r_i} > 1$, and a sparsity constant $\eta \in (0,1)$. Let T be an operator so that for every $f_1, f_2, h \in C_c^\infty(\mathbb{R}^n)$*

$$\Big| \int_{\mathbb{R}^n} T(f_1,f_2)(x)h(x)\,dx \Big| \lesssim \sup_{\mathcal{S}} \Lambda_{\mathcal{S},\vec{r}}(f_1,f_2,h),$$

where the supremum runs over all sparse families with sparsity constant η. Then for all exponents $\vec{q} = (q_1, q_2)$, with $r_i < q_i$ for $i = 1,2$ and $r_3' > q$ and all the weights $\vec{v} = (v_1, v_2) \in A_{\vec{q},\vec{r}}$, and for all $f_1, f_2, h \in C_c^\infty(\mathbb{R}^n)$, we have

$$\|T(f_1,f_2)\|_{L^q(v)} \lesssim \prod_{i=1}^{2} \|f_i\|_{L^{q_i}(v_i)},$$

where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $v = v_1^{\frac{q}{q_1}} v_2^{\frac{q}{q_2}}$.

The sparse domination result contained in Theorem 2.3 yields the weighted estimates in Theorem 2.1 by using Theorem 4.1 and the extrapolation result in [27, Theorem 1.1].

4.2. Quantitative bounds in Theorem 2.1.

In [30], an improvement of the quantitative bounds obtained from sparse domination in multilinear forms was achieved. Indeed, the results in [27] missed the quantitative weighted bounds for the range $q < 1$.

Theorem 4.2. [30, Corollary 4.2] *Let T be a bilinear or positive valued bi-sublinear operator and assume that for some $p_1, p_2 \in (0,\infty)$ and $t \in [1,\infty]$, we have the sparse domination of the bilinear operator for every $f_1, f_2, h \in C_c^\infty(\mathbb{R}^n)$, i.e.,*

$$\Big| \int_{\mathbb{R}^n} T(f_1,f_2)(x)h(x)\,dx \Big| \lesssim \sup_{\mathcal{S}} \Lambda_{\mathcal{S},(p_1,p_2,t)}(f_1,f_2,h),$$

then for all $\vec{q} = (q_1, q_2)$ with $q_1, q_2 \in (0,\infty]$ such that $(\vec{p}, t') < \vec{q}$ and all weights $\vec{w} \in A_{\vec{q},(\vec{p},t')}$, the operator T extends to a bounded operator $L^{q_1}(w_1^{q_1}) \times L^{q_2}(w_2^{q_2}) \to L^q(w^q)$, where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, with the bound

$$\|T\|_{L^{q_1}(w_1^{q_1}) \times L^{q_2}(w_2^{q_2}) \to L^q(w^q)} \lesssim [\vec{w}]_{A_{\vec{q},(\vec{p},t')}}^{\max\left(\frac{\frac{1}{p_1}}{\frac{1}{p_1}-\frac{1}{q_1}}, \frac{\frac{1}{p_2}}{\frac{1}{p_2}-\frac{1}{q_2}}, \frac{1-\frac{1}{t'}}{\frac{1}{q}-\frac{1}{t'}}\right)}.$$

In view of the theorem above, the sparse domination results obtained in Theorem 2.3 together with the extrapolation in [27, Theorem 1.1] yield the following improved weighted estimates for the operators $\mathcal{M}_{\mathrm{lac}}$ and $\mathcal{M}_{\mathrm{full}}$.

Theorem 4.3. *Let $n \geq 2$ and $(\frac{1}{r_i}, \frac{1}{s_i})$, $i = 1,2$, be in the interior of L_n (respectively F_n). Assume that $t = \frac{s_1 s_2}{s_1 + s_2 - s_1 s_2} > 1$. Then for all $\vec{q} = (q_1, q_2)$ with $q_1, q_2 \in (0,\infty]$ such that $\vec{q} > (\vec{r}, t')$ the operator $\mathcal{M}_{\mathrm{lac}}$ (respectively $\mathcal{M}_{\mathrm{full}}$) extends to a bounded operator $L^{q_1}(w_1^{q_1}) \times L^{q_2}(w_2^{q_2}) \to L^q(w^q)$, where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, with the bound*

$$\|\mathcal{M}\|_{L^{q_1}(w_1^{q_1}) \times L^{q_2}(w_2^{q_2}) \to L^q(w^q)} \lesssim [\vec{w}]_{A_{\vec{q},(\vec{r},t')}}^{\max\left(\frac{\frac{1}{r_1}}{\frac{1}{r_1}-\frac{1}{q_1}}, \frac{\frac{1}{r_2}}{\frac{1}{r_2}-\frac{1}{q_2}}, \frac{1-\frac{1}{t'}}{\frac{1}{q}-\frac{1}{t'}}\right)},$$

where $\mathcal{M} := \mathcal{M}_{\mathrm{lac}}$ (respectively $\mathcal{M}_{\mathrm{full}}$) and $A_{\vec{q},(\vec{r},t')}$, with $(\vec{r}, t') = (r_1, r_2, t')$, is defined as in Definition 3.5.

Remark 4.4. *Note that the end-point extrapolation in* [26, 30] *allows the index* q_j *above to take value infinity. Moreover, the original Theorem 4.1 contained in* [27] *includes vector-valued results. These apply to our sparse domination in Theorem 2.1, so that vector-valued inequalities are immediately obtained from* [27, Corollary 2.15]*, see also* [30, Corollary 4.6].

4.3. **Weighted boundedness for the triplet** (p_1, p_2, p)**.**

Proof of Theorem 2.4. The proof of Theorem 2.4 is done in two steps. The first step is to establish a more general result by using analytic interpolation for a family of bilinear operators. Then in the second step we use this general result with a suitable choice of exponents to deduce the theorem.

Step I: Let $1 < \tilde{p}_1 = \tilde{p}_2 \leq \frac{2n}{2n-1}$. Now consider $\frac{2n}{2n-1} < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $(\frac{1}{r_i}, \frac{1}{s_i}) \in L_n$, $i = 1, 2$ with $t = \frac{s_1 s_2}{s_1 + s_2 - s_1 s_2} > 1$. For $\vec{r} = (r_1, r_2, t) < \vec{p} := (p_1, p_2, p)$, let $\vec{w} = (w_1, w_2) \in A_{\vec{p}, \vec{r}}$. By Theorem 2.1 we have

$$\|\mathcal{M}_{\text{lac}}(f_1, f_2)\|_{L^p(w)} \leq C_1 \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}. \tag{7}$$

Also, note that by Theorem 5.3 we have the following weighted estimates for the product weights.

$$\|\mathcal{M}_{\text{lac}}(f_1, f_2)\|_{L^q(v)} \leq C_2 \|f_1\|_{L^{q_1}(v_1)} \|f_2\|_{L^{q_2}(v_2)}, \tag{8}$$

for $1 < q_i < \tilde{p}_i$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $v_i \in \left(A_{\frac{q_i}{t_i}} \cap \text{RH}_{\left(\frac{\phi'_{\text{lac}}(\frac{1}{t_i})}{q_i}\right)'} \right) \cup \mathcal{R}_{q_i}$, $v = v_1^{\frac{q}{q_1}} v_2^{\frac{q}{q_2}}$ and $(\frac{1}{t_i}, \frac{1}{\eta_i}) \in L_n$ for some $\eta_i \in (1, \infty)$ and $1 < t_i < q_i < \eta_i'$, for $i = 1, 2$.

We consider the linearised operator $\mathcal{M}_{\text{lac}}$ as follows

$$\mathcal{M}_{\text{lac}}(f_1, f_2)(x) = \mathcal{A}_{\tau(x)} f_1(x) \mathcal{A}_{\tau(x)} f_2(x),$$

where τ is a measurable function from $\mathbb{R}^n$ to $[0, \infty)$. For $z \in \mathcal{S} := \{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1\}$, consider the functions

$$\frac{1}{l(z)} := \frac{1-z}{p} + \frac{z}{q}, \qquad \frac{1}{l_i(z)} := \frac{1-z}{p_i} + \frac{z}{q_i}, \qquad i = 1, 2.$$

Choose $\theta \in (0, 1)$ such that

$$\frac{1}{l(\theta)} := \frac{1-\theta}{p} + \frac{\theta}{q} = \frac{1}{\tilde{p}}, \qquad \frac{1}{l_i(\theta)} := \frac{1-\theta}{p_i} + \frac{\theta}{q_i} = \frac{1}{\tilde{p}_i}, \qquad i = 1, 2.$$

Note that for any linear operator T and a positive number $k \in (0, 1)$ satisfying $\frac{k}{p} + \frac{k}{q} < 1$ and $k < \tilde{p}$, we can write the following

$$\|Tf\|_{L^{\tilde{p}}}^k = \||Tf|^k\|_{L^{\frac{\tilde{p}}{k}}} = \sup_{\substack{g \in L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n) \\ \|g\|_{L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n)} = 1}} \left| \int_{\mathbb{R}^n} |Tf|^k g \right|.$$

Consider

$$\widetilde{v}_N(x) = v(x), \quad \text{if} \quad v(x) \leq N \quad \text{and} \quad \widetilde{v}_N(x) = N, \quad \text{if} \quad v(x) > N,$$
$$\widetilde{w}_N(x) = w(x), \quad \text{if} \quad w(x) \leq N \quad \text{and} \quad \widetilde{w}_N(x) = N, \quad \text{if} \quad w(x) > N.$$

Let f_1, f_2 be finite simple functions and g be a non-negative finite simple function such that $\|f_i\|_{L^{\tilde{p}_1}(\mathbb{R}^n)} = 1$, for $i = 1, 2$ and $\|g\|_{L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n)} = 1$.

With the notations introduced as above, consider the following function

$$\psi(z) := \int_{\mathbb{R}^n} \left| \mathcal{A}_{\tau(x)} f_{1,z}(x) \mathcal{A}_{\tau(x)} f_{2,z}(x) \widetilde{v}_N^{\frac{z}{q}} \widetilde{w}_N^{\frac{1-z}{p}} g^{\frac{(1-\frac{k}{l(z)})}{k(1-\frac{k}{\tilde{p}})}} \right|^k dx, \tag{9}$$

where

$$f_{j,z}(x) := |f_j(x)|^{\frac{\tilde{p}_j}{l_j(z)}} e^{iu_j} (v_j + \epsilon)^{\frac{-z}{q_j}} (w_j + \epsilon)^{\frac{z-1}{p_j}}, \quad j = 1, 2,$$

for $z \in \mathcal{S}$, $\epsilon > 0$ and $u_j \in [0, 2\pi]$. Note that we have the following expression for $\psi(\theta)$, $\theta \in (0,1)$,

$$\psi(\theta) = \int_{\mathbb{R}^n} |\prod_{j=1}^{2} \mathcal{A}_{\tau(x)}(f_j(v_j+\epsilon)^{-\frac{\theta}{q_j}}(w_j+\epsilon)^{\frac{\theta-1}{p_j}})(x)\tilde{v}_N^{\frac{\theta}{q}}\tilde{w}_N^{\frac{1-\theta}{p}}|^k g(x)dx.$$

For each $x \in \mathbb{R}^n$, the functions $\mathcal{A}_{\tau(x)}f_{j,z}(x)$, $\tilde{v}_N^{\frac{z}{q}}(x)$, $\tilde{w}_N^{\frac{1-z}{p}}(x)$ and $g^{\frac{(1-\frac{k}{l(z)})}{k(1-\frac{k}{p})}}(x)$ are analytic in the domain $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$. Therefore the integrand in (9) is a continuous in $z \in \mathcal{S}$ and subharmonic in the domain $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$. Also, using the Hölder's inequality with exponents $\frac{p}{k}$ and $\frac{p}{p-k}$, it is easy to see that ψ is a bounded function. Moreover, the Hölder's inequality with exponents $\frac{p}{k}$ and $\frac{p}{p-k}$ and the fact that $\|f_i\|_{L^{\tilde{p}_i}(\mathbb{R}^n)} = 1$, $i = 1,2$ and $\|g\|_{L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n)} = 1$, yield that

$$|\psi(it)| \le C_1^k.$$

Similarly, using the Hölder's inequality with exponents $\frac{q}{k}$ and $\frac{q}{q-k}$, we get

$$|\psi(1+it)| \le C_2^k.$$

The constants C_1, C_2 are independent of ϵ, N and τ.

We invoke the maximum modulus principle for subharmonic functions to deduce that

$$|\psi(\theta)| = \int_{\mathbb{R}^n} |\prod_{j=1}^{2} \mathcal{A}_{\tau(x)}(f_j v_{j,\epsilon}^{-\frac{\theta}{q_j}} w_{j,\epsilon}^{\frac{\theta-1}{p_j}})(x)\tilde{v}_N^{\frac{\theta}{q}}\tilde{w}_N^{\frac{1-\theta}{p}}|^k g(x)dx \le C_1^{k(1-\theta)}C_2^{k\theta}.$$

Here we have used the notation $v_{j,\epsilon} = v_j + \epsilon$ and $w_{j,\epsilon} = w_j + \epsilon$ for $j = 1,2$. Therefore, using a duality argument we obtain that

$$\Big(\int_{\mathbb{R}^n}\Big(|\mathcal{A}_{\tau(x)}(f_1 v_{1,\epsilon}^{-\frac{\theta}{q_1}} w_{1,\epsilon}^{\frac{\theta-1}{p_1}})(x)\mathcal{A}_{\tau(x)}(f_2 v_{2,\epsilon}^{-\frac{\theta}{q_2}} w_{2,\epsilon}^{\frac{\theta-1}{p_2}})(x)|\tilde{v}_N^{\frac{\theta}{q}}\tilde{w}_N^{\frac{1-\theta}{p}}\Big)^{\tilde{p}}dx\Big)^{\frac{1}{\tilde{p}}}$$
$$\le C\Big(\int_{\mathbb{R}^n}|f_1|^{\tilde{p}_1}\Big)^{\frac{1}{\tilde{p}_1}}\Big(\int_{\mathbb{R}^n}|f_2|^{\tilde{p}_2}\Big)^{\frac{1}{\tilde{p}_2}}.$$

Since the set of finite simple functions is dense in $L^s(\mathbb{R}^n)$, $1 \le s < \infty$, we get the estimate above for all $L^{\tilde{p}_1}(\mathbb{R}^n)$ functions f_1 and f_2 (note that we have assumed $\tilde{p}_1 = \tilde{p}_2$). Next, recall that the constants C_1, C_2 are independent of ϵ, N and τ. Let $\epsilon \to 0$ and $N \to \infty$ and replace f_i by $f_i v_i^{\frac{\theta}{q_i}} w_i^{\frac{1-\theta}{p_i}}$, $i = 1,2$, in the above to get that

$$\Big(\int_{\mathbb{R}^n}\Big(|\mathcal{A}_{\tau(x)}(f_1)(x)\mathcal{A}_{\tau(x)}(f_2)(x)|v^{\frac{\theta}{q}}w^{\frac{1-\theta}{p}}\Big)^{\tilde{p}}dx\Big)^{\frac{1}{\tilde{p}}}$$
$$\le C\Big(\int_{\mathbb{R}^n}|f_1|^{\tilde{p}_1}(v_1^{\frac{\theta}{q_1}}w_1^{\frac{1-\theta}{p_1}})^{\tilde{p}_1}\Big)^{\frac{1}{\tilde{p}_1}}\Big(\int_{\mathbb{R}^n}|f_2|^{\tilde{p}_2}(v_2^{\frac{\theta}{q_2}}w_2^{\frac{1-\theta}{p_2}})^{\tilde{p}_2}\Big)^{\frac{1}{\tilde{p}_2}}.$$

Since the constant C is independent of τ, therefore we get the boundedness of the operator $\mathcal{M}_{\mathrm{lac}}$, i.e.

$$(10)\quad \Big(\int_{\mathbb{R}^n}\Big(|\mathcal{M}_{\mathrm{lac}}(f_1,f_2)(x)|v^{\frac{\theta}{q}}w^{\frac{1-\theta}{p}}\Big)^{\tilde{p}}dx\Big)^{\frac{1}{\tilde{p}}}$$
$$\le C\Big(\int_{\mathbb{R}^n}|f_1|^{\tilde{p}_1}(v_1^{\frac{\theta}{q_1}}w_1^{\frac{1-\theta}{p_1}})^{\tilde{p}_1}\Big)^{\frac{1}{\tilde{p}_1}}\Big(\int_{\mathbb{R}^n}|f_2|^{\tilde{p}_2}(v_2^{\frac{\theta}{q_2}}w_2^{\frac{1-\theta}{p_2}})^{\tilde{p}_2}\Big)^{\frac{1}{\tilde{p}_2}}.$$

Step II: We shall use the estimate (10) for radial weights with a suitable choice of exponents to conclude the proof of Theorem 2.4. Indeed, we make the following choice. For $\epsilon > 0$, let $p_1 = p_2 = \frac{2n}{2n-1} + 2\epsilon$, $r_1 = r_2 = \frac{2n}{2n-1} + \epsilon$ and $(\frac{1}{r_i}, \frac{1}{s_i}) \in L_n$. Check that for this choice of $(\frac{1}{r_i}, \frac{1}{s_i})$, $t = \frac{s_1 s_2}{s_1+s_2-s_1 s_2} > 1$ and set $\vec{r} = (r_1, r_2, r_3)$ with $r_3 = t$. Let $\vec{w} = (|x|^{\alpha'}, |x|^{\beta'}) \in A_{\vec{p},\vec{r}}$ and note that the estimate (7) holds for bilinear $A_{\vec{p},\vec{r}}$ weights. Next, for $0 < \delta < \tilde{p}_1 - 1$, choose $q_1 = q_2 = \tilde{p}_1 - \delta$ and $\vec{v} = (|x|^a, |x|^b)$ with $1 - n \le a, b < (n-1)(\tilde{p}_1 - \delta - 1)$. Then we know that the estimate (8) holds for $\mathcal{M}_{\text{lac}}$. Therefore, by the previous steps the operator $\mathcal{M}_{\text{lac}}$ satisfies the estimate (10) for the above choice of exponents and we get

$$\Big(\int_{\mathbb{R}^n}\Big|\mathcal{M}_{\text{lac}}(f_1,f_2)\Big|^{\tilde{p}}(|x|^{\frac{(a+b)\theta}{\tilde{p}_1-\delta}+\frac{(\alpha'+\beta')(1-\theta)(2n-1)}{2(n+\epsilon(2n-1))}})^{\tilde{p}}\Big)^{\frac{1}{\tilde{p}}} \tag{11}$$
$$\leq C\Big(\int_{\mathbb{R}^n}|f_1|^{\tilde{p}_1}(|x|^{\frac{a\theta}{\tilde{p}_1-\delta}+\frac{\alpha'(1-\theta)(2n-1)}{2n+2\epsilon(2n-1)}})^{\tilde{p}_1}\Big)^{\frac{1}{\tilde{p}_1}}\Big(\int_{\mathbb{R}^n}|f_2|^{\tilde{p}_2}(|x|^{\frac{b\theta}{\tilde{p}_2-\delta}+\frac{\beta'(1-\theta)(2n-1)}{2n+2\epsilon(2n-1)}})^{\tilde{p}_2}\Big)^{\frac{1}{\tilde{p}_2}}$$

with $\theta\in(0,1)$ and $\frac{(1-\theta)(2n-1)}{n+\epsilon(2n-1)}+\frac{2\theta}{\tilde{p}_1-\delta}=\frac{1}{\tilde{p}}=\frac{2}{\tilde{p}_1}$.

Now, we show that the exponents of weights in the estimate above may be chosen suitably so that they satisfy the hypothesis of Theorem 2.4. Observe that by Lemma 3.4 we have that $\vec{w}=(|x|^{\alpha'},|x|^{\beta'})\in A_{\vec{p},\vec{r}}$ implies that

$$|x|^{\frac{\alpha'\theta_1(2n-1)}{2n+2\epsilon(2n-1)}}\in A_{(\frac{1-r}{r})\theta_1},\quad |x|^{\frac{\beta'\theta_2(2n-1)}{2n+2\epsilon(2n-1)}}\in A_{(\frac{1-r}{r})\theta_2},\ \text{and}\ |x|^{\frac{(\alpha'+\beta')\delta_3(2n-1)}{2n+2\epsilon(2n-1)}}\in A_{\frac{1-r}{r}\delta_3},$$

where

$$\frac{1}{\delta_i}=\frac{1}{r_i}-\frac{1}{p_i},\quad \frac{1}{\theta_i}=\frac{1-r}{r}-\frac{1}{\delta_i}\quad\text{and}\quad p_3=p',\quad i=1,2,3.$$

Substituting the values of the various parameters, we obtain

$$\frac{1-r}{r}=\frac{(2n-2)(1+\epsilon(2n-1))}{2n+\epsilon(2n-1)},$$
$$\frac{1}{\theta_i}=\frac{\epsilon(2n-1)(2n-2)-1}{2n+\epsilon(2n-1)}+\frac{2n-1}{2n+2\epsilon(2n-1)},\quad\text{for}\quad i=1,2,$$
$$\frac{1}{\delta_3}=\frac{\epsilon(2n-1)^2}{2n+\epsilon(2n-1)}+\frac{2n-1}{n+\epsilon(2n-1)}-1.$$

Since ϵ can be chosen arbitrarily small, therefore taking $\epsilon\to0$ we get $(1-n)\frac{2n}{2n-1}<\alpha',\beta'<0$ and $(1-n)\frac{2n}{2n-1}<\alpha'+\beta'<0$.

Now taking $\delta\to\tilde{p}_1-1$, we get $\theta=\frac{2n}{\tilde{p}_1}-(2n-1)$. Since the range of α' and β' is an open set, we get that $\mathcal{M}_{\text{lac}}$ is bounded from $L^{\tilde{p}_1}(|x|^{\alpha})\times L^{\tilde{p}_2}(|x|^{\beta})$ to $L^{\tilde{p}}(|x|^{\frac{\alpha+\beta}{2}})$ for α,β satisfying satisfying

$$(1-n)\tilde{p}_1<\alpha,\beta<0\quad\text{and}\quad\alpha+\beta>2(1-n)\big(n-(n-1)\tilde{p}_1\big).$$

Further, using the product-type weighted boundedness of $\mathcal{M}_{\text{lac}}$ for $\tilde{p}_1=\tilde{p}_2$, we get $\mathcal{M}_{\text{lac}}$ is bounded from $L^{\tilde{p}_1}(|x|^{a})\times L^{\tilde{p}_2}(|x|^{b})$ to $L^{\tilde{p}}(|x|^{\frac{a+b}{2}})$ for $1-n\leq a,b<(n-1)(\tilde{p}_1-1)$.

This proves the desired result for the operator $\mathcal{M}_{\text{lac}}$. □

5. Comparing Theorem 2.1 with Hölder type results

In this section we compare the weighted estimates obtained in Theorem 2.1 with the estimates that can be deduced using Hölder's inequality for both the operator $\mathcal{M}_{\text{lac}}$ and $\mathcal{M}_{\text{full}}$. For $1<p<\infty$, define the sets $\mathcal{L}_p:=\{w: M_{\text{lac}}\text{ maps }L^p(w)\text{ to }L^p(w)\}$ and $\mathcal{F}_p:=\{w: M_{\text{full}}\text{ maps }L^p(w)\text{ to }L^p(w)\}$. Recall also the definition of $\mathcal{R}_p$ in (5) and let $\widetilde{\mathcal{R}}_p$ be defined as

$$\widetilde{\mathcal{R}}_p:=\{|x|^b:1-n<b<(n-1)(p-1)-1\},\quad n\geq2. \tag{12}$$

In [13], Duoandikoetxea and Vega proved the following weighted estimates for spherical maximal functions with respect to radial weights.

Theorem 5.1. [13] *$\mathcal{R}_p\subseteq\mathcal{L}_p$, $1<p<\infty$ and $\widetilde{\mathcal{R}}_p\subseteq\mathcal{F}_p$, $\frac{n}{n-1}<p<\infty$.*

Let $\frac{1}{\phi_{\text{full}}(\frac{1}{r})}$ denote the piecewise linear function on $(0,\frac{n-1}{n})$ whose graph connects the points $(0,1)$, $(\frac{n^2-n}{n^2+1},\frac{n^2-n+2}{n^2+1})$ and $(\frac{n-1}{n},\frac{n-1}{n})$, defined similarly as (4). Recently, in [20] Lacey proved the weighted estimates for the operators with respect to general weights using sparse domination principle.

Theorem 5.2. [20] *The following estimates hold.*

- *Let* $1 < r < p < \phi'_{\text{lac}}(\frac{1}{r})$, *then* $A_{\frac{p}{r}} \cap \text{RH}_{\left(\frac{\phi'_{\text{lac}}(\frac{1}{r})}{p}\right)'} \subseteq \mathcal{L}_p$.
- *Let* $\frac{n}{n-1} < r < p < \phi'_{\text{full}}(\frac{1}{r})$, *then* $A_{\frac{p}{r}} \cap \text{RH}_{\left(\frac{\phi'_{\text{full}}(\frac{1}{r})}{p}\right)'} \subseteq \mathcal{F}_p$.

For $\vec{p} = (p_1, p_2, p)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, define

$$\mathcal{L}_{\vec{p}} := \left\{\vec{w} = (w_1, w_2) : \mathcal{M}_{\text{lac}} \ \text{ maps } \ L^{p_1}(w_1) \times L^{p_2}(w_2) \text{ to } L^p(w)\right\}$$

and

$$\mathcal{F}_{\vec{p}} := \left\{\vec{w} = (w_1, w_2) : \mathcal{M}_{\text{full}} \ \text{ maps } \ L^{p_1}(w_1) \times L^{p_2}(w_2) \text{ to } L^p(w)\right\}.$$

In view of Theorems 5.1 and 5.2, Hölder's inequality yields the following weighted estimates for bilinear spherical maximal functions with respect to product type bilinear weights.

Theorem 5.3. *The following holds:*

- $\prod_{i=1}^2 A_{\frac{p_i}{r_i}} \cap \text{RH}_{\left(\frac{\phi'_{\text{lac}}(\frac{1}{r_i})}{p_i}\right)'} \subseteq \mathcal{L}_{\vec{p}}$ *for all* $1 < r_i < p_i < \phi'_{\text{lac}}(\frac{1}{r_i})$ *and* $\prod_{i=1}^2 \mathcal{R}_{p_i} \subseteq \mathcal{L}_{\vec{p}}$, *where* $p_i > 1,\ i = 1, 2$.
- $\prod_{i=1}^2 A_{\frac{p_i}{r_i}} \cap \text{RH}_{\left(\frac{\phi'_{\text{full}}(\frac{1}{r_i})}{p_i}\right)'} \subseteq \mathcal{F}_{\vec{p}}$ *for all* $\frac{n}{n-1} < r_i < p_i < \phi'_{\text{full}}(\frac{1}{r_i})$ *and* $\prod_{i=1}^2 \widetilde{\mathcal{R}}_{p_i} \subseteq \mathcal{F}_{\vec{p}}$, *where* $p_i > \frac{n}{n-1},\ i = 1, 2$.

We show that Theorem 2.1 addresses the weighted boundedness of bilinear operators $\mathcal{M}_{\text{lac}}$ and $\mathcal{M}_{\text{full}}$ with respect to bilinear weights that are not of product type as covered by Theorem 5.3 above. Note that it is enough to consider the case of $p_1 = p_2 = p$ and point out the difference between the two weighted estimates for power weights.

5.1. **The case of full spherical maximal operator $\mathcal{M}_{\text{full}}$.** The following result may be deduced from Theorem 2.1.

Corollary 5.4. *The operator* $\mathcal{M}_{\text{full}}$ *is bounded from* $L^p(|x|^\alpha) \times L^p(|x|^\beta)$ *to* $L^{\frac{p}{2}}(|x|^{\frac{\alpha+\beta}{2}})$, *for* α, β *satisfying the following conditions:*

(i) *For* $n \geq 2$ *and* $\frac{n}{n-1} < p \leq \frac{n^2+1}{n^2-n}$,

$$\frac{-2n}{n-1}\big(p(n-1) - 2\big) < \alpha, \beta < 0 \quad \text{with} \quad \frac{-2n}{n-1}\big(p(n-1) - 2\big) < \alpha + \beta < 0$$

or, for $n \geq 3$, *and* $\frac{n}{n-1} < p \leq \frac{n^2+1}{n^2-n}$

$$-\big(n(p+1) - 3p\big) < \alpha, \beta < n(p-1) - p \quad \text{with} \quad -2(n-p) < \alpha + \beta < 2\big(n(p-1) - p\big)$$

or, for $n = 2$, *and* $2 < p < \frac{5}{2}$

$$\frac{-2}{3}(p-2) < \alpha, \beta < \frac{2}{3}(p-2) \quad \text{with} \quad 0 < \alpha + \beta < \frac{4}{3}(p-2).$$

(ii) *For* $n \geq 2$ *and* $\frac{n^2+1}{n^2-n} < p < \infty$,

$$-2(n-1) < \alpha, \beta < 0 \quad \text{with} \quad -2(n-1) < \alpha + \beta < 0.$$

(iii) *For* $n \geq 2$ *and* $\frac{n^2+1}{n^2-n} < p \leq \frac{n^2+1}{n-1}$,

$$\frac{-n}{n+1}\big(p(n-1) - 2\big) < \alpha, \beta < \frac{n}{n+1}\big(p(n-1) - 2\big) \quad \text{with} \quad 0 < \alpha + \beta < \frac{2n}{n+1}\big(p(n-1) - 2\big)$$

or

$$\frac{-n}{n^2+1}\big((n-1)(n-2)p + n^2 + 1\big) < \alpha, \beta < \frac{n}{n^2+1}\big((n^2-n)p - n^2 - 1\big)$$

with

$$\frac{-2n}{n^2+1}\big(n^2 + 1 - p(n-1)\big) < \alpha + \beta < \frac{2n}{n^2+1}\big((n^2-n)p - n^2 - 1\big).$$

(iv) *For* $n \geq 2$ *and* $\frac{n^2+1}{n-1} < p < \infty$,

$$-n(n-1) < \alpha, \beta < n(n-1) \quad \text{with} \quad 0 < \alpha + \beta < 2n(n-1).$$

Proof. **Proof of (i).** Consider $p_1 = p_2 = p$ with $\frac{n}{n-1} < p \le \frac{n^2+1}{n^2-n}$. Choose $r_1 = r_2 = p - \epsilon$, where $0 < \epsilon < p - \frac{n}{n-1}$. For this choice of r_i we have $\frac{1}{s_i} = 2 - \frac{n+1}{(n-1)(p-\epsilon)}$. Therefore,

$$\frac{1}{t} = \frac{1}{s_1} + \frac{1}{s_2} - 1 = 4 - \frac{2(n+1)}{(n-1)(p-\epsilon)} - 1 = \frac{3(n-1)(p-\epsilon) - 2(n+1)}{(n-1)(p-\epsilon)}.$$

Recall that we need to make sure of the condition $t' > \frac{p}{2}$. This gives us that $\epsilon < \frac{p[(p+1)(n-1)-(n+1)]}{(n-1)(p+1)}$. Therefore, we have that

$$0 < \epsilon < \min\Big\{p - \frac{n}{n-1}, \frac{p[(p+1)(n-1)-(n+1)]}{(n-1)(p+1)}\Big\}.$$

We shall use Lemma 3.4 to get the condition on the exponent of power weights. Following the notation therein, we can write down the parameters r, θ_i and δ_3 with the above choice of r_i, s_i as follows

$$\begin{aligned} \frac{1-r}{r} &= \frac{2(n-1)(p-\epsilon) - 4}{(n-1)(p-\epsilon)} \\ \frac{1}{\theta_i} &= \frac{(2p+1)(n-1)(p-\epsilon) - p(n+3)}{p(p-\epsilon)(n-1)}, \quad \text{for} \quad i = 1,2 \\ \frac{1}{\delta_3} &= \frac{2(n-1)p(p-\epsilon) - 2p(n+1) + 2(n-1)(p-\epsilon)}{p(p-\epsilon)(n-1)}. \end{aligned}$$

We know that $(|x|^{\alpha})^{\frac{\theta_1}{p}} \in A_{\frac{1-r}{r}\theta_1}$ and $(|x|^{\frac{\alpha+\beta}{2}})^{\frac{2\delta_3}{p}} \in A_{\frac{1-r}{r}\delta_3}$ imply

$$\frac{-n\big((2p+1)(n-1)(p-\epsilon) - p(n+3)\big)}{(n-1)(p-\epsilon)} < \alpha < \frac{n\epsilon}{p-\epsilon}$$

and

$$\frac{-n\big(2(n-1)p(p-\epsilon) - 2p(n+1) + 2(n-1)(p-\epsilon)\big)}{(n-1)(p-\epsilon)} < \alpha + \beta < \frac{2n\epsilon}{p-\epsilon}$$

Since the conditions above hold for arbitrarily small ϵ, taking $\epsilon \to 0$, we get that

$$\frac{-2n\big(p(n-1)-2\big)}{n-1} < \alpha < 0 \quad \text{with} \quad \frac{-2n\big(p(n-1)-2\big)}{n-1} < \alpha + \beta < 0.$$

Observe that due to symmetry of the bilinear operator we get the same condition on β as given above for α.

This proves the first condition in (i). The proof for the other cases in (i) as well as the other parts (ii)-(iv) may be completed by using the same idea, therefore we skip it. ☐

Now it is easy to compare the exponents of power weights obtained in Corollary 5.4 with that of the Hölder type power weights as given in Theorem 5.3. Recall that in Theorem 5.3 we have that $\mathcal{M}_{\text{full}}$ maps $L^p(|x|^a) \times L^p(|x|^b)$ to $L^{\frac{p}{2}}(|x|^{\frac{a+b}{2}})$ for $1 - n < a, b < (n-1)(p-1) - 1$. Further, note that this range of exponents a and b is the best possible, except possibly at the point $1 - n$, that can be obtained through the classical full spherical maximal function M_{full} along with the Hölder inequality. On the other hand observe that in Corollary 5.4 we get a significantly better range of α and β for which the operator $\mathcal{M}_{\text{full}}$ maps $L^p(|x|^{\alpha}) \times L^p(|x|^{\beta})$ to $L^{\frac{p}{2}}(|x|^{\frac{\alpha+\beta}{2}})$ for $\frac{n}{n-1} < p < \infty$. This is the advantage of our method, that give us improved weighted estimates for the operator $\mathcal{M}_{\text{full}}$ with respect to genuine bilinear weights that are not possible using the Hölder's inequality.

5.2. **The case of lacunary spherical maximal operator $\mathcal{M}_{\text{lac}}$.** As in the previous case, we can deduce the following weighted estimates operator $\mathcal{M}_{\text{lac}}$ with respect to power weights.

Corollary 5.5. *Let $n \ge 2$. The operator $\mathcal{M}_{\text{lac}}$ is bounded from $L^p(|x|^{\alpha}) \times L^p(|x|^{\beta})$ to $L^{\frac{p}{2}}(|x|^{\frac{\alpha+\beta}{2}})$ for α, β satisfying the following conditions:*

(i) *For* $\frac{2n}{2n-1} < p \le \frac{n+1}{n}$,

$$-2n(n-1)(p-1) < \alpha, \beta < 0 \quad \textit{with} \quad -2n(n-1)(p-1) < \alpha+\beta < 0$$

or

$$-\Big(n - \frac{p}{2}\Big) < \alpha, \beta < n(p-1) - \frac{p}{2} \quad \textit{with} \quad -n(2-p) < \alpha+\beta < 2n(p-1) - p.$$

(ii) *For* $\frac{n+1}{n} < p < \infty$,

$$-2(n-1) < \alpha, \beta < 0 \quad \textit{with} \quad -2(n-1) < \alpha+\beta < 0.$$

(iii) *For* $\frac{n+1}{n} < p \le n+1$,

$$\frac{-n(n-2)p}{n+1} - n < \alpha, \beta < \frac{n^2 p}{n+1} - n \quad \textit{with} \quad -2n\Big(1 - \frac{p}{n+1}\Big) < \alpha+\beta < \frac{2n^2 p}{n+1} - 2n$$

or for $2 < p \le n+1$

$$-(n-1)(p-1) < \alpha, \beta < (n-1)(p-1) \quad \textit{with} \quad 0 < \alpha+\beta < 2(n-1)(p-1).$$

(iv) *For* $n+1 \le p < \infty$,

$$-n(n-1) < \alpha, \beta < n(n-1) \quad \textit{with} \quad 0 < \alpha+\beta < 2n(n-1).$$

We skip the proof of the corollary above because it uses similar computations as in the case of Corollary 5.4. Moreover, it can be verified that the range of exponents covered in Corollary 5.5 is better than that given in Theorem 5.3 for the $L^p(|x|^\alpha) \times L^p(|x|^\beta)$ to $L^{\frac{p}{2}}(|x|^{\frac{\alpha+\beta}{2}})$ boundedness of the operator $\mathcal{M}_{\mathrm{lac}}$ for all $1 < p < \infty$. As a final remark, observe that part (ii) is not comparable to parts (iii) and (iv) in both corollaries. For example, in the case of $\mathcal{M}_{\mathrm{lac}}$, the individual range of α, β in (iv) is better than the individual range of α, β in (ii) for $n+1 < p < \infty$, but we cannot choose both α, β negative due to the condition $0 < \alpha+\beta < 2n(n-1)$. However, in (ii) we could do that.

6. Sparse domination: proof of Theorem 2.3 (sufficiency)

In this section we prove the sufficient part of Theorem 2.3. We have exploited the corresponding ideas for the linear case from [20]. As announced, we will proceed proving the sparse domination for characteristic functions and later extend it to general functions. The argument when passing from characteristic functions to general functions is a bit tricky and we use a two-step procedure without which we would obtain an extra condition restricting the result to the Banach range. We follow a unified approach, stating as simultaneously as possible the results for both $\mathcal{M}_{\mathrm{lac}}$ and $\mathcal{M}_{\mathrm{full}}$.

Theorem 6.1. *Let* $n \ge 2$. *For* $i = 1, 2$, *let* $(\frac{1}{r_i}, \frac{1}{s_i})$ *be in the interior of the triangle* L_n *(respectively the trapezium* F_n*). Then for characteristic functions* $f_1 = \chi_{F_1}$, $f_2 = \chi_{F_2}$ *and compactly supported bounded function* h, *where* F_1, F_2 *are bounded measurable subsets of* $\mathbb{R}^n$, *there exists a sparse collection* $\mathcal{S} = \mathcal{S}_{r_1,r_2,t}$ *such that*

$$\langle \mathcal{M}(f_1, f_2), h \rangle \le C \Lambda_{\mathcal{S}_{r_1,r_2,t}}(f_1, f_2, h),$$

where $t := \frac{s_1 s_2}{s_1+s_2-s_1 s_2} > 1$ *and* $\mathcal{M} := \mathcal{M}_{\mathrm{lac}}$ *(respectively* $\mathcal{M}_{\mathrm{full}}$*).*

For a cube $Q \subset \mathbb{R}^n$ with side-length $l_Q = 2^q$, define $\mathcal{A}_Q f_i(x) = \mathcal{A}_{2^{q-3}}(f_i \chi_{\frac{1}{3}Q})(x)$, $i = 1, 2$. Note that $\mathcal{A}_Q f_i$ is supported in $\frac{2}{3}Q$. The following lemma involving stopping time arguments is the key result in the proof of Theorem 6.1.

Lemma 6.2. *Let* $n \ge 2$. *For* $i = 1, 2$, *let* $(\frac{1}{r_i}, \frac{1}{s_i})$ *be in the interior of the triangle* L_n *(respectively the trapezium* F_n*), with the additional condition* $\frac{1}{s_1} + \frac{1}{s_2} > 1$. *Let* $f_1 = \chi_{F_1}$, $f_2 = \chi_{F_2}$, *where* F_1, F_2 *are measurable subsets of* Q_0 *and* h *be a bounded function supported in* Q_0. *Let* $C_0 > 1$ *be a constant and let* $\mathcal{D}_0$ *be a collection of dyadic subcubes of* Q_0 *such that*

$$\sup_{Q' \in \mathcal{D}_0} \sup_{Q: Q' \subset Q \subset Q_0} \Big(\frac{\langle f_i \rangle_{Q, r_i}}{\langle f_i \rangle_{Q_0, r_i}} + \frac{\langle h \rangle_{Q,t}}{\langle h \rangle_{Q_0,t}} \Big) \le C_0, \quad \textit{for} \quad i = 1, 2,$$

where $t = \frac{s_1 s_2}{s_1+s_2-s_1 s_2} > 1$. *Then,*

(i) *If* $(\frac{1}{r_i}, \frac{1}{s_i})$ *are in the interior of* L_n*, with the additional condition* $\frac{1}{s_1} + \frac{1}{s_2} > 1$*,*

$$|\langle \sup_{Q\in\mathcal{D}_0} \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\rangle| \lesssim |Q_0| \langle f_1\rangle_{Q_0,r_1} \langle f_2\rangle_{Q_0,r_2} \langle h\rangle_{Q_0,t},$$

(ii) *If* $(\frac{1}{r_i}, \frac{1}{s_i})$ *are in the interior of* F_n*, with the additional condition* $\frac{1}{s_1} + \frac{1}{s_2} > 1$*,*

$$|\langle \sup_{Q\in\mathcal{D}_0} \sup_{2^{q-4}\leq t\leq 2^{q-3}} \mathcal{A}_t(f_1\chi_{\frac{1}{3}Q})(x)\mathcal{A}_t(f_2\chi_{\frac{1}{3}Q}), h\rangle| \lesssim |Q_0| \langle f_1\rangle_{Q_0,r_1} \langle f_2\rangle_{Q_0,r_2} \langle h\rangle_{Q_0,t}.$$

We assume Lemma 6.2 for a moment and complete the proof of Theorems 2.3 and 6.1.

6.1. **Proof of Theorem 6.1.** We will present the proof for $\mathcal{M}_{\text{lac}}$, and after that we will point out the main differences in the proof for $\mathcal{M}_{\text{full}}$. First note that using standard arguments we can reduce our work to proving analogous results for the dyadic version of the maximal functions under consideration. Indeed, let f_1 and f_2 be positive functions with their support contained inside a cube Q_0. Fix a dyadic lattice $\mathcal{D}$ and consider the maximal function

$$\mathcal{M}_{\mathcal{D}}(f_1, f_2)(x) := \sup_{Q\in\mathcal{D}} |\mathcal{A}_Q f_1(x)\mathcal{A}_Q f_2(x)|.$$

Since $\text{supp}(f_i) \subset Q_0$, we get that $\mathcal{A}_Q f_i = 0$ if $Q \cap Q_0 = \emptyset$ and also $\mathcal{A}_Q f_i = 0$ for large enough cubes. To justify this, note that we are assuming that h is supported in the cube Q_0. Therefore, we need to check that for $x \in Q_0$, $\mathcal{A}_Q f_i(x) = 0$, for large enough cube Q (with respect to Q_0). From the definition of $\mathcal{A}_Q f_i$, we have

$$\mathcal{A}_Q f_i(x) = \int_{\mathbb{S}^{n-1}} f_i(x - 2^{q-3}y)\chi_{\frac{1}{3}Q}(x - 2^{q-3}y)\, d\sigma(y).$$

Observe that for large enough cube Q with side length $l_Q > 16 l_{Q_0}$, $x - 2^{q-3}y \notin Q_0$. Since $\text{supp}(f_i) \subset Q_0$, then $\mathcal{A}_Q f_i(x) = 0$.

All in all, in view of the above, it is enough to prove corresponding sparse domination for the bilinear maximal operator

$$\mathcal{M}_{\mathcal{D}\cap Q_0}(f_1, f_2)(x) = \sup_{Q\in\mathcal{D}\cap Q_0} |\mathcal{A}_Q f_1(x)\mathcal{A}_Q f_2(x)|.$$

Then, $\langle \mathcal{M}_{\text{lac}}(f_1, f_2), h\rangle$ can be dominated by the sum of finitely many sparse forms. By the definition of supremum, one can find a universal sparse form in the sparse domination.

We proceed to prove the sparse domination result for the operator $\mathcal{M}_{\mathcal{D}\cap Q_0}$. Let C_0 be a constant and $\mathcal{E}_{Q_0}$ denote the collection of maximal dyadic subcubes of Q_0 satisfying

$$\langle f_1\rangle_{Q,r_1} > C_0\langle f_1\rangle_{Q_0,r_1} \quad \text{or} \quad \langle f_2\rangle_{Q,r_2} > C_0\langle f_2\rangle_{Q_0,r_2} \quad \text{or} \quad \langle h\rangle_{Q,t} > C_0\langle h\rangle_{Q_0,t}. \tag{13}$$

Let $E_{Q_0} = \cup_{P\in\mathcal{E}_{Q_0}} P$. Note that we can choose $C_0 > 1$ so that $|E_{Q_0}| < \frac{1}{2}|Q_0|$. Writing $F_{Q_0} = Q_0 \setminus E_{Q_0}$, we have that $|F_{Q_0}| \geq \frac{1}{2}|Q_0|$.

Next, denote

$$\mathcal{D}_0 := \{Q \in \mathcal{D} \cap Q_0 : Q \cap E_{Q_0} = \emptyset\} \tag{14}$$

and observe that for $Q \in \mathcal{D}_0$ we get that

$$\langle f_1\rangle_{Q,r_1} \leq C_0\langle f_1\rangle_{Q_0,r_1} \quad \text{and} \quad \langle f_2\rangle_{Q,r_2} \leq C_0\langle f_2\rangle_{Q_0,r_2} \quad \text{and} \quad \langle h\rangle_{Q,t} \leq C_0\langle h\rangle_{Q_0,t}. \tag{15}$$

For, if (13) holds, then there exists $P \in \mathcal{E}_{Q_0}$ such that $P \supset Q$. This will contradict the definition of $\mathcal{D}_0$. In a similar way, note that if $Q' \in \mathcal{D}_0$ and $Q' \subset Q \subset Q_0$, then we also have (15). These two observations together give us that, for $i = 1, 2$,

$$\sup_{Q'\in\mathcal{D}_0} \sup_{Q:Q'\subset Q\subset Q_0} \langle f_i\rangle_{Q,r_i} \leq C_0\langle f_i\rangle_{Q_0,r_i} \quad \text{and} \quad \sup_{Q'\in\mathcal{D}_0} \sup_{Q:Q'\subset Q\subset Q_0} \langle h\rangle_{Q,t} \leq C_0\langle h\rangle_{Q_0,t}. \tag{16}$$

Now we claim, using a standard linearisation argument, that it is enough to prove sparse domination for a suitable linearised form. For, let $\mathcal{Q}$ be the collection of all dyadic subcubes of Q_0. Given $Q \in \mathcal{Q}$, consider the set

$$H_Q := \Big\{x \in Q : \mathcal{A}_Q f_1(x)\mathcal{A}_Q f_2(x) \geq \frac{1}{2} \sup_{P\in\mathcal{Q}} \mathcal{A}_P f_1(x)\mathcal{A}_P f_2(x)\Big\}.$$

Note that for any $x \in Q_0$, there exists a cube $Q \in \mathcal{Q}$ such that $x \in H_Q$. Set $B_Q = H_Q \setminus \bigcup_{Q' \supseteq Q} H_{Q'}$. Observe that $\{B_Q\}_{Q \in \mathcal{Q}}$ are pairwise disjoint and $\bigcup_{Q \in \mathcal{Q}} B_Q = \bigcup_{Q \in \mathcal{Q}} H_Q$. Then

$$\begin{aligned}
\langle \sup_{P \in \mathcal{Q}} \mathcal{A}_P f_1 \mathcal{A}_P f_2, h\rangle &= \sum_{Q \in \mathcal{Q}} \int_{B_Q} \sup_{P \in \mathcal{Q}} \left(\mathcal{A}_P f_1(x) \mathcal{A}_P f_2(x)\right) h(x) dx \\
&\leq 2 \sum_{Q \in \mathcal{Q}} \int_{B_Q} \mathcal{A}_Q f_1(x) \mathcal{A}_Q f_2(x) h(x) dx \\
&\leq 2 \sum_{Q \in \mathcal{Q}} \int_{\mathbb{R}^n} \mathcal{A}_Q f_1(x) \mathcal{A}_Q f_2(x) h(x) \chi_{B_Q}(x) dx \\
&= 2 \sum_{Q \in \mathcal{Q}} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h_Q\rangle \\
&\leq 2 \sum_{Q \in \mathcal{D} \cap Q_0} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle
\end{aligned}$$

where $h_Q = h\chi_{B_Q}$. The estimate above allows us to work with a linearised form instead of the supremum. Notice that this argument uses the full collection of dyadic subcubes of the given cube Q_0. So, in particular,

$$\begin{aligned}
\big|\langle \sup_{Q \in \mathcal{D}_0} \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\rangle\big| \leq \big|\langle \sup_{Q \in \mathcal{Q}} \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\rangle\big| &\leq \Big|2 \sum_{Q \in \mathcal{Q}} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h_Q\rangle\Big| \\
&\leq 2\Big| \sum_{Q \in \mathcal{D} \cap Q_0} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle\Big|.
\end{aligned}$$

Therefore, it suffices to prove the sparse domination for

$$\sum_{Q \in \mathcal{D} \cap Q_0} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle.$$

Next, observe that for any cube $Q \in \mathcal{D} \cap Q_0$ we either have $Q \in \mathcal{D}_0$ or $Q \subset P$ for some $P \in \mathcal{E}_{Q_0}$. Therefore,

$$\sum_{Q \in \mathcal{D} \cap Q_0} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle = \sum_{Q \in \mathcal{D}_0} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle + \sum_{P \in \mathcal{E}_{Q_0}} \sum_{Q \subset P} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle. \tag{17}$$

We would like to remark here that so far we have not required that f_1 and f_2 are characteristic functions. Now we invoke Lemma 6.2 to get that

$$\sum_{Q \in \mathcal{D}_0} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle \lesssim |Q_0| \langle f_1\rangle_{Q_0,r_1} \langle f_2\rangle_{Q_0,r_2} \langle h\rangle_{Q_0,t}. \tag{18}$$

Let $\{P_j\}$ be an enumeration of cubes in $\mathcal{E}_{Q_0}$. Then we can rewrite the remaining term as

$$\sum_{P \in \mathcal{E}_{Q_0}} \sum_{Q \subset P} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle = \sum_{j=1}^{\infty} \sum_{Q \in P_j \cap \mathcal{D}} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle.$$

We repeatedly use the estimate above for each j and put all the terms together to get a sparse collection $\mathcal{S}$ so that the following holds

$$\sum_{Q \in \mathcal{D} \cap Q_0} \langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h\chi_{B_Q}\rangle \lesssim \sum_{S \in \mathcal{S}} |S| \langle f_1\rangle_{S,r_1} \langle f_2\rangle_{S,r_2} \langle h\rangle_{S,t}.$$

This completes the proof of Theorem 6.1 for $\mathcal{M}_{\text{lac}}$.

In order to prove the corresponding results for the operator $\mathcal{M}_{\text{full}}$, we require a bilinear analogue of local spherical maximal functions. It is defined as follows

$$\widetilde{\mathcal{M}}(f_1, f_2)(x) := \sup_{t \in [1,2]} \mathcal{A}_t f_1(x) \mathcal{A}_t f_2(x). \tag{19}$$

Again standard arguments reduce the task to consider a dyadic version with the maximal function

$$\sup_{Q\in\mathcal{D}\cap Q_0} |\widetilde{\mathcal{M}}_Q(f_1,f_2)(x)|,$$

where

$$\widetilde{\mathcal{M}}_Q(f_1,f_2)(x) := \sup_{2^{q-4}\le t\le 2^{q-3}} \mathcal{A}_t(f_1\chi_{\frac{1}{3}Q})(x)\mathcal{A}_t(f_2\chi_{\frac{1}{3}Q})(x).$$

Note that a linearisation trick as earlier tells us that it suffices to replace the supremum (19) with the form

$$\Big|\sum_{Q\in\mathcal{D}_0}\langle \widetilde{\mathcal{M}}_Q(f_1,f_2),h_Q\rangle\Big|,$$

with $h_Q = h\chi_{B_Q}$ and $B_Q = E_Q\setminus\bigcup_{Q'\supset Q}E_{Q'}$, where

$$E_Q = \Big\{x\in Q\in\mathcal{Q} : \widetilde{\mathcal{M}}_Q(f_1,f_2)(x)\ge \frac{1}{2}\sup_{P\in\mathcal{Q}}\widetilde{\mathcal{M}}_P(f_1,f_2)(x)\Big\}.$$

The remaining part of the proof can be completed following the lacunary case.

6.2. **Proof of Theorem 2.3.** We will make use of Theorem 6.1 in proving Theorem 2.3. The proof is unified in both lacunary and full cases.

Step I: Let f_1 and h be non-negative compactly supported bounded functions with support in the cube Q_0 and $f_2 = \chi_{F_2}$, where $F_2\subset Q_0$. We use the same argument as in the proof of Theorem 6.1 up to the estimate (17) with the same notation and here $\mathcal{D}_0$ is defined with respect to f_1, χ_{F_2} and h. In fact, it is enough to prove an analogue of estimate (18) for the setting under consideration, i.e., we need to show that

$$\sum_{Q\in\mathcal{D}_0}\langle \mathcal{A}_Qf_1\mathcal{A}_Q\chi_{F_2},h_Q\rangle \lesssim |Q_0|\langle f_1\rangle_{Q_0,\rho_1}\langle\chi_{F_2}\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t},$$

where $\rho_1 > r_1$.

In order to use Theorem 6.1, we first need to decompose the function f_1 into suitable characteristic functions. Consider $E_m = \{x\in Q_0 : 2^m\le f_1(x)\le 2^{m+1}\}$. Then there exists $m_0 > 1$ such that $E_m=\emptyset$ for all $m > m_0$. Denote $f_1^m = f_1\chi_{E_m}$. Thus, we use Theorem 6.1 for each pair of characteristic functions χ_{E_m} and χ_{F_2} and obtain the sparse domination for the functions f_1^m and χ_{F_2} as follows

$$\begin{aligned}\sum_{Q\in\mathcal{D}_0}\langle \mathcal{A}_Qf_1^m\mathcal{A}_Q\chi_{F_2},h_Q\rangle &\le 2^{m+1}\sum_{Q\in\mathcal{D}_0}\langle \mathcal{A}_Q\chi_{E_m}\mathcal{A}_Q\chi_{F_2},h_Q\rangle\\ &\lesssim 2^{m+1}\sum_{Q\in\mathcal{D}\cap Q_0}\langle \mathcal{A}_Q\chi_{E_m}\mathcal{A}_Q\chi_{F_2},h_Q\chi_{F_{Q_0}}\rangle\\ &\lesssim 2^{m+1}\sum_{S\in\mathcal{S}_m}|S|\langle\chi_{E_m}\rangle_{S,r_1}\langle\chi_{F_2}\rangle_{S,r_2}\langle h\chi_{F_{Q_0}}\rangle_{S,t}\end{aligned}$$

where $\mathcal{S}_m$ is the sparse family corresponding to characteristic functions χ_{E_m} and χ_{F_2}.

Next, using the stopping time condition on the functions h and $f_2=\chi_{F_2}$ as given in (16), we get

$$\sum_{S\in\mathcal{S}_m}|S|\langle\chi_{E_m}\rangle_{S,r_1}\langle\chi_{F_2}\rangle_{S,r_2}\langle h\chi_{F_{Q_0}}\rangle_{S,t}\lesssim \langle h\rangle_{Q_0,t}\langle\chi_{F_2}\rangle_{Q_0,r_2}\sum_{S\in\mathcal{S}_m}|S|\langle\chi_{E_m}\rangle_{S,r_1}.$$

Choose $\widetilde{\rho_1} > r_1$ and consider $\widetilde{\rho_1}' > 1$ such that $\frac{1}{\widetilde{\rho_1}}+\frac{1}{\widetilde{\rho_1}'}=1$. Now, using an easy consequence of the Carleson embedding theorem (see [21]) we get that

$$\begin{aligned}\sum_{S\in\mathcal{S}_m}|S|\langle\chi_{E_m}\rangle_{S,r_1} &= \sum_{S\in\mathcal{S}_m}|S|^{\frac{1}{\widetilde{\rho}_1}}\langle\chi_{E_m}\rangle_{S,r_1}|S|^{\frac{1}{\widetilde{\rho}_1'}}\\ &\le \Big(\sum_{S\in\mathcal{S}_m}|S|\langle\chi_{E_m}\rangle_{S,r_1}^{\widetilde{\rho}_1}\Big)^{\frac{1}{\widetilde{\rho}_1}}\Big(\sum_{S\in\mathcal{S}_m}|S|\Big)^{\frac{1}{\widetilde{\rho}_1'}}\\ &\le \langle\chi_{E_m}\rangle_{Q_0,\widetilde{\rho}_1}|Q_0|.\end{aligned}$$

Finally, using [2, Lemma 4.6 and Lemma 4.7], we obtain

$$|\langle \sup_{Q\in\mathcal{D}_0} \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h_Q\rangle| \lesssim |Q_0|\langle f_1\rangle_{Q_0,\rho_1}\langle \chi_{F_2}\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}, \tag{20}$$

where $\rho_1 > \widetilde{\rho_1}$.

Note that we can decompose $\sum_{Q\in\mathcal{D}\cap Q_0}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h\chi_{B_Q}\rangle$ as

(21)

$$\sum_{Q\in\mathcal{D}\cap Q_0}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h\chi_{B_Q}\rangle = \sum_{Q\in\mathcal{D}_0}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h\chi_{B_Q}\rangle + \sum_{P\in\mathcal{E}_{Q_0}}\sum_{Q\subset P}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h\chi_{B_Q}\rangle.$$

Now from (20) we get that

$$\sum_{Q\in\mathcal{D}_0}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h\chi_{B_Q}\rangle \lesssim |Q_0|\langle f_1\rangle_{Q_0,\rho_1}\langle \chi_{F_2}\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}. \tag{22}$$

Let $\{P_j\}$ be an enumeration of cubes in $\mathcal{E}_{Q_0}$ defined with f_1, χ_{F_2} and h. Then we can rewrite the remaining term as

$$\sum_{P\in\mathcal{E}_{Q_0}}\sum_{Q\subset P}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h\chi_{B_Q}\rangle = \sum_{j=1}^{\infty}\sum_{Q\in P_j\cap\mathcal{D}}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h\chi_{B_Q}\rangle.$$

We repeatedly use the estimate above for each j and put all the terms together to get a sparse collection $\mathcal{S}$ so that the following holds

$$\sum_{Q\in\mathcal{D}\cap Q_0}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q \chi_{F_2}, h\chi_{B_Q}\rangle \lesssim \sum_{S\in\mathcal{S}}|S|\langle f_1\rangle_{S,\rho_1}\langle \chi_{F_2}\rangle_{S,r_2}\langle h\rangle_{S,t}.$$

Step II: Now consider f_1, f_2 and h are non-negative compactly supported bounded function with support in the cube Q_0. We use the same argument as in the proof of Theorem 6.1 up to the estimate (17) with the same notation and $\mathcal{D}_0$ is defined with respect to f_1, f_2 and h. It suffices to prove an analogue of estimate (18) for the setting under consideration, i.e., we need to show that

$$\sum_{Q\in\mathcal{D}_0}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h_Q\rangle \lesssim |Q_0|\langle f_1\rangle_{Q_0,\rho_1}\langle f_2\rangle_{Q_0,\rho_2}\langle h\rangle_{Q_0,t},$$

where $\rho_1 > r_1$ and $\rho_2 > r_2$.

Note that we already have an analogue of estimate (18) for general compactly supported bounded function f_1 and $f_2 = \chi_{F_2}$ a characteristic function in (20). Therefore, in order to prove the required estimate for general compactly supported bounded functions f_1 and f_2 we decompose the function f_2 such that $f_2 = \sum_{n=1}^{n_0} f_2^n$ with $f_2^n = f_2\chi_{F_n}$, where $F_n = \{x \in Q_0 : 2^n \le f_2(x) \le 2^{n+1}\}$ and $F_n = \emptyset$ for $n > n_0$. From this point onward the proof follows in a similar fashion as in Step I. Finally, we get the following estimate

$$|\langle \sup_{Q\in\mathcal{D}_0} \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h_Q\rangle| \lesssim |Q_0|\langle f_1\rangle_{Q_0,\rho_1}\langle f_2\rangle_{Q_0,\rho_2}\langle h\rangle_{Q_0,t},$$

where $\rho_1 > r_1$ and $\rho_2 > r_2$.

Remark 6.3. *Observe that the argument in which we freeze one of the characteristic functions when passing from characteristic functions to general functions is crucial. Dealing with both characteristic functions simultaneously yields the extra condition* $\frac{1}{r_1} + \frac{1}{r_2} < 1$, *restricting the result to the Banach range.*

6.3. **Proof of Lemma 6.2.** Finally we provide the proof of Lemma 6.2. This is the most technical and tedious part of the paper. We will begin by giving the proof in the lacunary case, and after that we will sketch the significantly different parts in case of $\mathcal{M}_{\mathrm{full}}$.

First note that one can use the same linearisation trick as in the proof of Theorem 6.1. This would mean that it is enough to prove the following estimate

$$\sum_{Q\in\mathcal{D}_0}\langle \mathcal{A}_Q f_1 \mathcal{A}_Q f_2, h_Q\rangle \lesssim |Q_0|\langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t},$$

where $\mathcal{A}_Q f_i(x) = \mathcal{A}_{2^{q-3}}(f_i \chi_{\frac{1}{3}Q})(x)$, and so

$$\mathcal{A}_Q f_i(x) = \int_{\mathbb{S}^{n-1}} f_i(x - 2^{q-3}y)\chi_{\frac{1}{3}Q}(x - 2^{q-3}y)\, d\sigma(y).$$

Since $l_Q = 2^q$, we see that for $y \in \mathbb{S}^{n-1}$ we have $\frac{1}{3}Q + 2^{q-3}y \subset \frac{2}{3}Q$ and hence $\operatorname{supp}(\mathcal{A}_Q f_i) \subset \frac{2}{3}Q$. Now define

$$\widetilde{\mathcal{A}}_Q f_i(x) = \mathcal{A}_{2^{q-3}}(f_i \chi_{\frac{1}{2}Q})(x).$$

Observe that $\operatorname{supp}(\widetilde{\mathcal{A}}_Q f_i) \subset Q$ and since f_i's are non-negative functions, therefore

$$\begin{aligned}\mathcal{A}_Q f_i(x) = \int_{\mathbb{S}^{n-1}} & f_i(x - 2^{q-3}y)\chi_{\frac{1}{3}Q}(x - 2^{q-3}y)\, d\sigma(y) \\ &\leq \int_{\mathbb{S}^{n-1}} f_i(x - 2^{q-3}y)\chi_{\frac{1}{2}Q}(x - 2^{q-3}y)\, d\sigma(y) = \widetilde{\mathcal{A}}_Q f_i(x),\end{aligned}$$

for a.e. x and $i = 1, 2$. In view of these observations, it is enough to prove that

$$\sum_{Q \in \mathcal{D}_0} \langle \widetilde{\mathcal{A}}_Q f_1 \widetilde{\mathcal{A}}_Q f_2, h_Q \rangle \lesssim |Q_0| \langle f_1 \rangle_{Q_0, r_1} \langle f_2 \rangle_{Q_0, r_2} \langle h \rangle_{Q_0, t}.$$

The important observation here is that with this localization we will be able to separately distinguish the dyadic (bad) cubes in which the cancellation property can be applied. Here we have used the same notation as in Theorem 6.1. For $i = 1, 2$, let

$$\gamma_{f_i} = \{\text{collection of maximal dyadic subcubes } P \subset Q_0 : \langle f_i \rangle_{P, r_i} > 2C_0 \langle f_i \rangle_{Q_0, r_i}\}.$$

Applying the Calderón-Zygmund decomposition to each f_i at the height $\alpha_i = 2C_0 \langle f_i \rangle_{Q_0, r_i}$, $i = 1, 2$, we can decompose

$$f_i = b_i + g_i,$$

where $\|g_i\|_{L^\infty} \lesssim \langle f_i \rangle_{Q_0, r_i}$ and

$$b_i = \sum_{P \in \gamma_{f_i}} (f_i - \langle f_i \rangle_P)\chi_P = \sum_{k=-\infty}^{q_0 - 1} \sum_{P \in B_i(k)} (f_i - \langle f_i \rangle_P)\chi_P =: \sum_{k=-\infty}^{q_0 - 1} B_{i,k},$$

with $l_{Q_0} = 2^{q_0}$ and $B_i(k) = \{P \in \gamma_{f_i} : l_P = 2^k\}$. Now,

$$\begin{aligned}\Big|\sum_{Q \in \mathcal{D}_0} \langle \widetilde{\mathcal{A}}_Q f_1 \widetilde{\mathcal{A}}_Q f_2, h_Q \rangle\Big| &\leq \Big|\sum_{Q \in \mathcal{D}_0} \langle \widetilde{\mathcal{A}}_Q g_1 \widetilde{\mathcal{A}}_Q g_2, h_Q \rangle\Big| + \Big|\sum_{Q \in \mathcal{D}_0} \langle \widetilde{\mathcal{A}}_Q g_1 \widetilde{\mathcal{A}}_Q b_2, h_Q \rangle\Big| \\ &\quad + \Big|\sum_{Q \in \mathcal{D}_0} \langle \widetilde{\mathcal{A}}_Q b_1 \widetilde{\mathcal{A}}_Q g_2, h_Q \rangle\Big| + \Big|\sum_{Q \in \mathcal{D}_0} \langle \widetilde{\mathcal{A}}_Q b_1 \widetilde{\mathcal{A}}_Q b_2, h_Q \rangle\Big| \\ &=: GG + GB + BG + BB.\end{aligned}$$

We estimate all the four parts separately. Note that in view of the symmetry in GB and BG parts, it is enough to estimate one of them.

Estimate for GG (both functions good) part. Using the fact that $t > 1$, we have

$$\begin{aligned}GG \leq \sum_{Q \in \mathcal{D}_0} \|\widetilde{\mathcal{A}}_Q g_1 \widetilde{\mathcal{A}}_Q g_2\|_{L^\infty} \|h_Q\|_{L^1} &\lesssim \langle f_1 \rangle_{Q_0, r_1} \langle f_2 \rangle_{Q_0, r_2} \sum_{Q \in \mathcal{D}_0} \int |h(x)\chi_{B_Q}(x)| dx \\ &\lesssim \langle f_1 \rangle_{Q_0, r_1} \langle f_2 \rangle_{Q_0, r_2} \langle h \rangle_{Q_0} |Q_0| \\ &\lesssim \langle f_1 \rangle_{Q_0, r_1} \langle f_2 \rangle_{Q_0, r_2} \langle h \rangle_{Q_0, t} |Q_0|.\end{aligned}$$

Estimate for BG (one function bad and one function good) part. Arguing with a similar argument as in the proof of Theorem 6.1 in our paper we note that, for all $Q \in \mathcal{D}_0$ and $P \in \gamma_{f_1}$, if $P \cap Q \neq \emptyset$, then $P \subsetneq Q$, by the stopping argument. Therefore, for any $Q \in \mathcal{D}_0$ with $l_Q = 2^q$, we have

$$\langle \widetilde{\mathcal{A}}_Q b_1 \widetilde{\mathcal{A}}_Q g_2, h_Q \rangle = \sum_{k < q} \langle \widetilde{\mathcal{A}}_Q B_{1,k} \widetilde{\mathcal{A}}_Q g_2, h_Q \rangle = \sum_{k=1}^{\infty} \langle \widetilde{\mathcal{A}}_Q B_{1,q-k} \widetilde{\mathcal{A}}_Q g_2, h_Q \rangle.$$

Thus,

$$|\sum_{Q\in\mathcal{D}_0}\langle\widetilde{\mathcal{A}}_Q b_1\widetilde{\mathcal{A}}_Q g_2, h_Q\rangle| = |\sum_{k=1}^{\infty}\sum_{Q\in\mathcal{D}_0}\langle\widetilde{\mathcal{A}}_Q B_{1,q-k}\widetilde{\mathcal{A}}_Q g_2, h_Q\rangle|.$$

Note that $(\widetilde{\mathcal{A}}_Q)^*f(x) = \chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}f(x)$. Indeed, for compactly supported bounded functions f and g, we have

$$\begin{aligned}\langle\widetilde{\mathcal{A}}_Q f, g\rangle &= \int_{\mathbb{R}^n}\widetilde{\mathcal{A}}_Q f(x)g(x)\,dx\\ &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{S}^{n-1}} f\chi_{\frac{1}{2}Q}(x-2^{q-3}y)\,d\sigma(y)\Big)g(x)\,dx\\ &= \int_{\mathbb{R}^n} f(x)\chi_{\frac{1}{2}Q}(x)\Big(\int_{\mathbb{S}^{n-1}} g(x+2^{q-3}y)\,d\sigma(y)\Big)\,dx\\ &=: \langle f, (\widetilde{\mathcal{A}}_Q)^*g\rangle,\end{aligned}$$

where

$$(\widetilde{\mathcal{A}}_Q)^*g(x) = \chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}g(x).$$

Hence,

$$\begin{aligned}BG &= |\sum_{Q\in\mathcal{D}_0}\int b_1(x)(\widetilde{\mathcal{A}}_Q)^*(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)dx|\\ &= |\sum_{Q\in\mathcal{D}_0}\sum_{k=1}^{\infty}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)(\widetilde{\mathcal{A}}_Q)^*(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx|\\ &= |\sum_{Q\in\mathcal{D}_0}\sum_{k=1}^{\infty}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx|.\end{aligned}$$

Now, observe that in the above line only those P will survive for which $P\cap\frac{1}{2}Q\neq\emptyset$. Further, we can write

$$|\sum_{Q\in\mathcal{D}_0}\sum_{k=1}^{\infty}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx| \le I + II,$$

where

$$I := |\sum_{Q\in\mathcal{D}_0}\sum_{k=1}^{3}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx|$$

and

$$II := |\sum_{Q\in\mathcal{D}_0}\sum_{k=4}^{\infty}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx|.$$

Here we have separated subcubes of kth generation with $k\ge 4$ in part II so that we can use [20, Lemma 2.3]. For the part I we do not need to use the cancellation condition on $B_{1,q-k}$ as there are only finitely many terms with respect to k and the sizes of subcubes P in these generations are comparable with the size of original cube Q with an absolute constant.

Recall that if $P\cap Q\neq\emptyset$, then $P\subsetneq Q$ (by stopping time argument). Therefore, all the P's are dyadic subcubes of Q. Also, observe that all the dyadic subcubes of Q of second and higher generations which intersect (except with edges) with $\frac{1}{2}Q$, are contained in $\frac{1}{2}Q$. In fact, all the second generation subcubes of Q which are around the center of Q, are contained in $\frac{1}{2}Q$, and the other second generation subcubes of Q along the boundary of Q do not intersect with $\frac{1}{2}Q$, i.e, for any $P\in B_1(q-k)$, $k\ge 2$, with $P\cap\frac{1}{2}Q\neq\emptyset$, implies $P\subset\frac{1}{2}Q$. In particular, this property will be used in estimating part II for all $k\ge 4$.

Now, we first deal with part I, in which we cannot use the cancellation property (since the dyadic cubes of first generation P are all intersecting $\frac{1}{2}Q$, but $P \not\subset \frac{1}{2}Q$). Nevertheless, we will not need the decay to conclude the estimate in this case[2]:

$$\begin{aligned} I &:= \Big|\sum_{Q\in\mathcal{D}_0}\sum_{k=1}^{3}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\Big| \\ &= \Big|\sum_{k=1}^{3}\sum_{Q\in\mathcal{D}_0}\int_{\mathbb{P}_k} B_{1,q-k}(x)(\widetilde{\mathcal{A}}_Q)^*(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\Big|, \end{aligned}$$

where $\bigcup\limits_{P\in B_1(q-k)} P = \mathbb{P}_k$. Since, $P\in B_1(q-k)$ are disjoint subcubes of Q, we get $\mathbb{P}_k\subseteq Q$. Now,

$$\begin{aligned} I &= \Big|\sum_{k=1}^{3}\sum_{Q\in\mathcal{D}_0}\int_{\mathbb{R}^n}\widetilde{\mathcal{A}}_Q(B_{1,q-k}\chi_{\mathbb{P}_k})(x)\widetilde{\mathcal{A}}_Q g_2(x)h_Q(x)\;dx\Big| \\ &\le \sum_{k=1}^{3}\sum_{Q\in\mathcal{D}_0}\|\widetilde{\mathcal{A}}_Q(B_{1,q-k}\chi_{\mathbb{P}_k})\|_{L^{s_1'}}\|\widetilde{\mathcal{A}}_Q g_2\cdot h_Q\|_{L^{s_1}} \\ &\lesssim \sum_{k=1}^{3}\sum_{Q\in\mathcal{D}_0}|Q|\langle B_{1,q-k}\rangle_{Q,r_1}\langle\widetilde{\mathcal{A}}_Q g_2\cdot h_Q\rangle_{Q,s_1}. \end{aligned}$$

In the second last line we have applied Hölder's inequality with respect to s_1' and s_1, where $\frac{1}{s_1}+\frac{1}{s_1'}=1$ and in the last line we have applied $L^{r_1}\to L^{s_1'}$ boundedness of the averaging operator $\mathcal{A}_{2^{q-3}}$, for $(\frac{1}{r_1},\frac{1}{s_1'})$ belonging to the interior of L_n'.

Let us now estimate the part II. The key point here is that, for small enough cubes, we first write the dual and then use the cancellation:

$$\begin{aligned} II :&= \Big|\sum_{Q\in\mathcal{D}_0}\sum_{k=4}^{\infty}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\Big| \\ &\le \sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\sum_{P\in B_1(q-k)}\frac{1}{|P|} \\ &\quad\times\Big|\int_P\int_P B_{1,q-k}(x)\chi_{\frac{1}{2}Q}(x)[\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)-\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x')]\,dxdx'\Big|. \end{aligned}$$

Observe that in the last line we have used the fact that if $P\in B_1(q-k)$, $k\ge 4$ with $P\cap\frac{1}{2}Q\neq\emptyset$, then $P\subset\frac{1}{2}Q$. Therefore $\int_P B_{1,q-k}\chi_{\frac{1}{2}Q}(x)\,dx=0$. Now, write $x'=x-y$, then $y\in P-P$. Applying this change of variable we get

$$\begin{aligned} II &\le \sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\sum_{P\in B_1(q-k)}\frac{1}{|P|} \\ &\quad\times\Big|\int_{P-P}\int_P B_{1,q-k}\chi_{\frac{1}{2}Q}(x)[\mathcal{A}_{2^{q-3}}-\tau_y\mathcal{A}_{2^{q-3}}](\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\,dy\Big| \\ &\lesssim \sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\sum_{P\in B_1(q-k)}\frac{1}{|P_0|} \\ &\quad\times\int_{P_0}\Big|\int_P B_{1,q-k}\chi_{\frac{1}{2}Q}(x)[\mathcal{A}_{2^{q-3}}-\tau_y\mathcal{A}_{2^{q-3}}](\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\Big|\,dy, \end{aligned} \tag{23}$$

[2]Note that we are applying the Calderón-Zygmund decomposition for the functions f_1, f_2. Therefore, $\int_P B_{1,q-k}(x)dx=0$ may not imply $\int_P B_{1,q-k}(x)\chi_{\frac{1}{2}Q}(x)\;dx=0$, unless $P\subseteq\frac{1}{2}Q$.

where $P - P \subset P_0$ and $l_{P_0} = 2l_P$. Observe that the quantity within the modulus sign in (23) is a function of y. We apply duality for the $L^1(P_0)$-norm. For $H \in L^\infty(P_0)$ we write (23) as

$$\begin{aligned}
&\sum_{P\in B_1(q-k)} \frac{1}{|P_0|}\int_{P_0}\int_P B_{1,q-k}\chi_{\frac{1}{2}Q}(x)[\mathcal{A}_{2^{q-3}} - \tau_y\mathcal{A}_{2^{q-3}}](\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\,H(y)\,dy\\
&= \frac{1}{|P_0|}\int_{P_0}\int_{\mathbb{P}_k} B_{1,q-k}\chi_{\frac{1}{2}Q}(x)[\mathcal{A}_{2^{q-3}} - \tau_y\mathcal{A}_{2^{q-3}}](\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\,H(y)\,dy\\
&\leq \|H\|_{L^\infty}\frac{1}{|P_0|}\int_{P_0}\Big|\int_{\mathbb{P}_k} B_{1,q-k}\chi_{\frac{1}{2}Q}(x)[\mathcal{A}_{2^{q-3}} - \tau_y\mathcal{A}_{2^{q-3}}](\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\Big|dy\\
&= \|H\|_{L^\infty}\frac{1}{|P_0|}\int_{P_0}\Big|\int_{\mathbb{R}^n} B_{1,q-k}\chi_{\frac{1}{2}Q\cap\mathbb{P}_k}(x)[\mathcal{A}_{2^{q-3}} - \tau_y\mathcal{A}_{2^{q-3}}](\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\Big|dy\\
&= \|H\|_{L^\infty}\frac{1}{|P_0|}\int_{P_0}\Big|\int_{\mathbb{R}^n} [\mathcal{A}_{2^{q-3}} - \mathcal{A}_{2^{q-3}}\tau_{-y}]B_{1,q-k}\chi_{\frac{1}{2}Q\cap\mathbb{P}_k}(x)(\widetilde{\mathcal{A}}_Q g_2\cdot h_Q)(x)\,dx\Big|\,dy.
\end{aligned}$$

Observe that for all $k \geq 4$, the side length $l_P \leq 2^{q-4}$ and hence we have $|y| \leq 2^{q-3}$. Now, applying [20, Lemma 2.3] we get

$$\begin{aligned}
II &\lesssim \sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\frac{1}{|P_0|}\int_{P_0}\Big(\frac{|y|}{l_Q}\Big)^\eta|Q|\langle B_{1,q-k}\chi_{\mathbb{P}_k}\rangle_{Q,r_1}\langle\widetilde{\mathcal{A}}_Q g_2\cdot h_Q\rangle_{Q,s_1}\,dy\\
&\leq \sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\frac{1}{|P_0|}\int_{P_0}\Big(\frac{|y|}{l_Q}\Big)^\eta|Q|\langle B_{1,q-k}\rangle_{Q,r_1}\langle\widetilde{\mathcal{A}}_Q g_2\cdot h_Q\rangle_{Q,s_1}\,dy.
\end{aligned}$$

Further, since $y \in P_0$ we have $|y| \lesssim 2^{q-k+1}$. This implies that

$$BG \lesssim \sum_{k=1}^{\infty}2^{-\eta k}\sum_{Q\in\mathcal{D}_0}|Q|\langle B_{1,q-k}\rangle_{Q,r_1}\langle\widetilde{\mathcal{A}}_Q g_2\cdot h_Q\rangle_{Q,s_1}.$$

Further, note that

$$\begin{aligned}
\langle\widetilde{\mathcal{A}}_Q g_2\cdot h_Q\rangle_{Q,s_1} = \Big(\frac{1}{|Q|}\int_Q(\widetilde{\mathcal{A}}_Q g_2)^{s_1}(x)h_Q^{s_1}(x)dx\Big)^{\frac{1}{s_1}} &\leq \|\widetilde{\mathcal{A}}_Q g_2\|_{L^\infty}\langle h_Q\rangle_{Q,s_1}\\
&\lesssim \langle f_2\rangle_{Q_0,r_2}\langle h_Q\rangle_{Q,s_1}
\end{aligned}$$

and $\langle B_{1,q-k}\rangle_{Q,r_1} \lesssim \langle\chi_{F_{1,q,k}}\rangle_{Q,r_1} + \langle f_1\rangle_{Q_0,r_1}\langle\chi_{E_{1,q,k}}\rangle_{Q,r_1}$, where $F_{1,q,k}$ are disjoint subsets of F_1 and $E_{1,q,k}$ are disjoint subsets of Q_0. Putting these estimates together we get

$$\begin{aligned}
BG &\lesssim \langle f_2\rangle_{Q_0,r_2}\sum_{k=1}^{\infty}2^{-\eta k}\sum_{Q\in\mathcal{D}_0}|Q|\langle\chi_{F_{1,q,k}}\rangle_{Q,r_1}\langle h_Q\rangle_{Q,s_1}\\
&\quad + \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\sum_{k=1}^{\infty}2^{-\eta k}\sum_{Q\in\mathcal{D}_0}|Q|\langle\chi_{E_{1,q,k}}\rangle_{Q,r_1}\langle h_Q\rangle_{Q,s_1} =: BG_1 + BG_2.
\end{aligned}$$

We estimate both the terms separately. For the term BG_1, note that $(\frac{1}{r_1},\frac{1}{s_1})$ in the interior of L_n, which implies that $\frac{1}{r_1}+\frac{1}{s_1} > 1$. Choose $\tau > 0$ such that $\frac{1}{r_1}-\tau+\frac{1}{s_1} = 1$. Write $\frac{1}{r_1}-\tau = \frac{1}{\dot{r}_1}$ and note that $\dot{r}_1 > r_1$. We have, by using that $F_{1,q,k} \subseteq F_1$ and the stopping time condition for the function f_1,

$$\begin{aligned}
\langle\chi_{F_{1,q,k}}\rangle_{Q,r_1} = \Big(\frac{1}{|Q|}\int_Q\chi_{F_{1,q,k}}^{r_1}\Big)^{\frac{1}{\dot{r}_1}}\Big(\frac{1}{|Q|}\int_Q\chi_{F_{1,q,k}}^{r_1}\Big)^{\tau} &\leq \Big(\frac{1}{|Q|}\int_Q\chi_{F_{1,q,k}}^{r_1}\Big)^{\frac{1}{\dot{r}_1}}\Big(\frac{1}{|Q|}\int_Q f_1^{r_1}\Big)^{\tau}\\
&\lesssim \Big(\frac{1}{|Q|}\int_Q\chi_{F_{1,q,k}}^{r_1}\Big)^{\frac{1}{\dot{r}_1}}\langle f_1\rangle_{Q_0,r_1}^{\tau r_1}.
\end{aligned}$$

Therefore, as $t \geq s_1$,

$$BG_1 = \langle f_2\rangle_{Q_0,r_2}\sum_{k=1}^{\infty}2^{-\eta k}\sum_{Q\in\mathcal{D}_0}|Q|\langle\chi_{F_{1,q,k}}\rangle_{Q,r_1}\langle h_Q\rangle_{Q,s_1}$$

$$\lesssim \langle f_1\rangle_{Q_0,r_1}^{\tau r_1}\langle f_2\rangle_{Q_0,r_2}\sum_{k=1}^{\infty}2^{-\eta k}\sum_{Q\in\mathcal{D}_0}|Q|\Big(\frac{1}{|Q|}\int_Q\chi_{F_{1,q,k}}\Big)^{\frac{1}{\dot r_1}}\Big(\frac{1}{|Q|}\int_Q h_Q^{s_1}\Big)^{\frac{1}{s_1}}$$

$$\lesssim \langle f_1\rangle_{Q_0,r_1}^{\tau r_1}\langle f_2\rangle_{Q_0,r_2}\sum_{k=1}^{\infty}2^{-\eta k}\Big(\sum_{Q\in\mathcal{D}_0}\int_Q\chi_{F_{1,q,k}}\Big)^{\frac{1}{\dot r_1}}\Big(\sum_{Q\in\mathcal{D}_0}\int_Q h^{s_1}\chi_{B_Q}\Big)^{\frac{1}{s_1}}$$

$$\lesssim \langle f_1\rangle_{Q_0,r_1}^{\tau r_1}\langle f_2\rangle_{Q_0,r_2}\langle f_1\rangle_{Q_0,r_1}^{\frac{r_1}{\dot r_1}}|Q_0|^{\frac{1}{\dot r_1}}\langle h\rangle_{Q_0,s_1}|Q_0|^{\frac{1}{s_1}}$$

$$\le \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}|Q_0|. \tag{24}$$

Next, the term BG_2 may be estimated as follows. Since $\dot r_1 > r_1$, we have $\langle\chi_{E_{1,q,k}}\rangle_{Q,r_1}\le\langle\chi_{E_{1,q,k}}\rangle_{Q,\dot r_1}$. Consider

$$BG_2 = \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\sum_{k=1}^{\infty}2^{-\eta k}\sum_{Q\in\mathcal{D}_0}|Q|\langle\chi_{E_{1,q,k}}\rangle_{Q,r_1}\langle h_Q\rangle_{Q,s_1}$$

$$\le \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\sum_{k=1}^{\infty}2^{-\eta k}\sum_{Q\in\mathcal{D}_0}|Q|\langle\chi_{E_{1,q,k}}\rangle_{Q,\dot r_1}\langle h_Q\rangle_{Q,s_1}$$

$$\le \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\sum_{k=1}^{\infty}2^{-\eta k}\Big(\sum_{Q\in\mathcal{D}_0}\int_Q\chi_{E_{1,q,k}}\Big)^{\frac{1}{\dot r_1}}\Big(\sum_{Q\in\mathcal{D}_0}\int_Q h^{s_1}\chi_{B_Q}\Big)^{\frac{1}{s_1}}$$

$$\lesssim \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}|Q_0|^{\frac{1}{\dot r_1}}\langle h\rangle_{Q_0,s_1}|Q_0|^{\frac{1}{s_1}}$$

$$\le \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}|Q_0| \quad \text{as} \quad t\ge s_1. \tag{25}$$

Estimates (24) and (25) yield the desired result for the term BG. The estimate for the third term GB follows similarly.

Estimate for BB (both functions are bad) part: We have

$$BB = \Big|\sum_{Q\in\mathcal{D}_0}\langle\widetilde{\mathcal{A}}_Q b_1\widetilde{\mathcal{A}}_Q b_2, h_Q\rangle\Big| = \Big|\sum_{Q\in\mathcal{D}_0}\int b_1(x)(\widetilde{\mathcal{A}}_Q)^*(\widetilde{\mathcal{A}}_Q b_2\cdot h_Q)(x)dx\Big|$$

$$= \Big|\sum_{Q\in\mathcal{D}_0}\sum_{k=1}^{\infty}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac12 Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q b_2\cdot h_Q)(x)dx\Big|$$

$$\le \Big|\sum_{Q\in\mathcal{D}_0}\sum_{k=1}^{3}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac12 Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q b_2\cdot h_Q)(x)dx\Big|$$

$$+ \Big|\sum_{Q\in\mathcal{D}_0}\sum_{k=4}^{\infty}\sum_{P\in B_1(q-k)}\int_P B_{1,q-k}(x)\chi_{\frac12 Q}(x)\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q b_2\cdot h_Q)(x)dx\Big|$$

$$= I_{bb} + II_{bb}.$$

Now I_{bb} can be handled similarly as I. On the other hand, we estimate II_{bb} as follows

$$II_{bb} \le \sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\sum_{P\in B_1(q-k)}\frac{1}{|P|}$$

$$\times\Big|\int_P\int_P B_{1,q-k}\chi_{\frac12 Q}(x)[\mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q b_2\cdot h_Q)(x) - \mathcal{A}_{2^{q-3}}(\widetilde{\mathcal{A}}_Q b_2\cdot h_Q)(x')]\,dx\,dx'\Big|$$

$$\lesssim \sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\sum_{P\in B_1(q-k)}\frac{1}{|P|}$$

$$\times\Big|\int_{P-P}\int_P B_{1,q-k}\chi_{\frac12 Q}(x)[\mathcal{A}_{2^{q-3}} - \tau_y\mathcal{A}_{2^{q-3}}](\widetilde{\mathcal{A}}_Q b_2\cdot h_Q)(x)\,dx\,dy\Big|.$$

Now, we proceed as for BG, but at the last line apply Hölder's inequality. Then the term is dominated by

$$\sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\frac{1}{|P_0|}\int_{P_0}\|[\mathcal{A}_{2^{q-3}}-\mathcal{A}_{2^{q-3}}\tau_{-y}]B_{1,q-k}\chi_{\mathbb{P}_k}\|_{L^{s_1'}}\|\widetilde{\mathcal{A}}_Q b_2\|_{L^{s_2'}}\|h_Q\|_{L^t}$$
$$\lesssim\sum_{k=4}^{\infty}\sum_{Q\in\mathcal{D}_0}\frac{1}{|P_0|}\int_{P_0}\Big(\frac{|y|}{l_Q}\Big)^{\eta}|Q|^{1-\frac{1}{s_1}}\langle B_{1,q-k}\rangle_{Q,r_1}\|\widetilde{\mathcal{A}}_Q b_2\|_{L^{s_2'}}\|h_Q\|_{L^t}.$$

where, in the last inequality, we have used [20, Theorem 2.1] for $\big(\frac{1}{r_1},\frac{1}{s_1'}\big)$ in the interior of $L_n'=\big\{\big(\frac{1}{r},\frac{1}{s}\big):\big(\frac{1}{r},1-\frac{1}{s}\big)\in L_n\big\}$ and Hölder's inequality with exponents $\frac{1}{t'}=\frac{1}{s_1'}+\frac{1}{s_2'}$. This yields

$$BB\le\sum_{k=1}^{\infty}2^{-\eta k}\sum_{Q\in\mathcal{D}_0}|Q|^{1-\frac{1}{s_1}}\langle B_{1,q-k}\rangle_{Q,r_1}\|\widetilde{\mathcal{A}}_Q b_2\|_{L^{s_2'}}\|h_Q\|_{L^t}.$$

Next, we make use of [20, Lemma 2.3] to estimate the quantity

$$\|\widetilde{\mathcal{A}}_Q b_2\|_{L^{s_2'}}=\sup_{\|\psi\|_{L^{s_2}}=1}|\langle\widetilde{\mathcal{A}}_Q b_2,\psi\rangle|.$$

Indeed, with a similar computation as the one for BG, we can conclude

$$\begin{aligned}|\langle\widetilde{\mathcal{A}}_Q b_2,\psi\rangle|&=\Big|\int\widetilde{\mathcal{A}}_Q b_2(x)\psi(x)dx\Big|=\Big|\sum_{j=1}^{\infty}\sum_{P\in B_2(q-j)}\int_P B_{2,q-j}(x)(\widetilde{\mathcal{A}}_Q)^*\psi(x)\,dx\Big|\\&=\Big|\sum_{j=1}^{\infty}\sum_{P\in B_2(q-j)}\int_P B_{2,q-j}(x)\chi_{\frac{1}{2}Q}(x)\mathcal{A}_{2^{q-3}}\psi(x)\,dx\Big|\\&\lesssim\sum_{j=1}^{3}|Q|^{1-\frac{1}{s_2}}\langle B_{2,q-j}\rangle_{Q,r_2}\|\psi\|_{L^{s_2}}+\sum_{j=4}^{\infty}\frac{1}{|P_0|}\int_{P_0}|Q|^{1-\frac{1}{s_2}}\Big(\frac{|y|}{l_Q}\Big)^{\eta}\langle B_{2,q-j}\rangle_{Q,r_2}\|\psi\|_{L^{s_2}}dy.\end{aligned}$$

Thus we obtain the following estimate.

$$BB\lesssim\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}\frac{|Q|^2}{|Q|^{\frac{1}{s_1}+\frac{1}{s_2}}}\langle B_{1,q-k}\rangle_{Q,r_1}\langle B_{2,q-j}\rangle_{Q,r_2}\Big(\int_Q h_Q^t\Big)^{\frac{1}{t}},$$

where we know that

(26) $$\langle B_{1,q-k}\rangle_{Q,r_1}\lesssim\langle\chi_{F_{1,q,k}}\rangle_{Q,r_1}+\langle f_1\rangle_{Q_0,r_1}\langle\chi_{E_{1,q,k}}\rangle_{Q,r_1}$$

and

(27) $$\langle B_{2,q-j}\rangle_{Q,r_2}\lesssim\langle\chi_{F_{2,q,j}}\rangle_{Q,r_2}+\langle f_2\rangle_{Q_0,r_2}\langle\chi_{E_{2,q,j}}\rangle_{Q,r_2}.$$

Here, $E_{1,q,k}$, $E_{2,q,j}$ are disjoint subsets of Q_0 and $F_{1,q,k}$, $F_{2,q,j}$ are disjoint subsets of F_1, F_2, respectively.

Substituting (26) and (27) into (6.3), we get the following four terms, which will be estimated separately.

$$\begin{aligned}BB&\lesssim\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}\frac{|Q|^2}{|Q|^{\frac{1}{s_1}+\frac{1}{s_2}}}\langle\chi_{F_{1,q,k}}\rangle_{Q,r_1}\langle\chi_{F_{2,q,j}}\rangle_{Q,r_2}\Big(\int_Q h_Q^t\Big)^{\frac{1}{t}}\\&\quad+\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}\frac{|Q|^2}{|Q|^{\frac{1}{s_1}+\frac{1}{s_2}}}\langle\chi_{F_{1,q,k}}\rangle_{Q,r_1}\langle f_2\rangle_{Q_0,r_2}\langle\chi_{E_{2,q,j}}\rangle_{Q,r_2}\Big(\int_Q h_Q^t\Big)^{\frac{1}{t}}\\&\quad+\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}\frac{|Q|^2}{|Q|^{\frac{1}{s_1}+\frac{1}{s_2}}}\langle f_1\rangle_{Q_0,r_1}\langle\chi_{E_{1,q,k}}\rangle_{Q,r_1}\langle\chi_{F_{2,q,j}}\rangle_{Q,r_2}\Big(\int_Q h_Q^t\Big)^{\frac{1}{t}}\\&\quad+\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}\frac{|Q|^2}{|Q|^{\frac{1}{s_1}+\frac{1}{s_2}}}\langle f_1\rangle_{Q_0,r_1}\langle\chi_{E_{1,q,k}}\rangle_{Q,r_1}\langle f_2\rangle_{Q_0,r_2}\langle\chi_{E_{2,q,j}}\rangle_{Q,r_2}\Big(\int_Q h_Q^t\Big)^{\frac{1}{t}}\\&=:BB_1+BB_2+BB_3+BB_4.\end{aligned}$$

Estimate for the first term BB_1. At this point, one has to deal with the cases $\frac{1}{r_1}+\frac{1}{r_2}>1$ and $\frac{1}{r_1}+\frac{1}{r_2}\leq 1$ separately. Let us start with the case $\frac{1}{r_1}+\frac{1}{r_2}>1$. Choose positive numbers τ_1 and τ_2 such that $\frac{1}{r_1}+\frac{1}{r_2}=1+\tau_1+\tau_2$ and denote $\frac{1}{r_i}-\tau_i=\frac{1}{\dot r_i}$, $i=1,2$. Note that $\frac{1}{\dot r_1}+\frac{1}{\dot r_2}=1$. We have

$$BB_1 \lesssim \langle h\rangle_{Q_0,t}\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}|Q|^{1-\frac{1}{r_1}-\frac{1}{r_2}}\Big(\int_Q \chi_{F_{1,q,k}}\Big)^{\frac{1}{r_1}}\Big(\int_Q \chi_{F_{2,q,j}}\Big)^{\frac{1}{r_2}}$$

$$\lesssim \langle h\rangle_{Q_0,t}\langle f_1\rangle_{Q_0,r_1}^{\tau_1 r_1}\langle f_2\rangle_{Q_0,r_2}^{\tau_2 r_2}\langle f_1\rangle_{Q_0,r_1}^{\frac{r_1}{\dot r_1}}\langle f_2\rangle_{Q_0,r_2}^{\frac{r_2}{\dot r_2}}|Q_0| = \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}|Q_0|.$$

The case $\frac{1}{r_1}+\frac{1}{r_2}=1$ follows similarly with $\tau_1=\tau_2=0$.

Let us turn to the case when $\frac{1}{r_1}+\frac{1}{r_2}<1$. Observe that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{t}>1$. Now, choose $\tau_1,\tau_2>0$ such that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{t}=1+\tau_1+\tau_2$. This implies $\frac{1}{\dot r_1}+\frac{1}{\dot r_2}+\frac{1}{t}=1$, where $\frac{1}{\dot r_i}=\frac{1}{r_i}-\tau_i$, for $i=1,2$.

$$BB_1 = \sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}|Q|^{1-\frac{1}{t}}\langle \chi_{F_{1,q,k}}\rangle_{Q,r_1}\langle \chi_{F_{2,q,j}}\rangle_{Q,r_2}\Big(\int_Q h_Q^t\Big)^{\frac{1}{t}}$$

$$\lesssim \langle f_1\rangle_{Q_0,r_1}^{\tau_1 r_1}\langle f_2\rangle_{Q_0,r_2}^{\tau_2 r_2}\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}\Big(\int_Q \chi_{F_{1,q,k}}\Big)^{\frac{1}{\dot r_1}}\Big(\int_Q \chi_{F_{2,q,j}}\Big)^{\frac{1}{\dot r_2}}\Big(\int_Q h_Q^t\Big)^{\frac{1}{t}}$$

$$\lesssim \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}|Q_0|.$$

In the last inequality we have used the Hölder's inequality with respect to $\dot r_1, \dot r_2$ and t.

The latter case is analogous for the remaining three terms BB_2, BB_3 and BB_4, hence we will focus on the estimates only for the case when $\frac{1}{r_1}+\frac{1}{r_2}>1$.

Estimate for the second and third terms BB_2 and BB_3. The estimates for BB_2 and BB_3 may be obtained in a similar fashion. We provide here the argument for the term BB_3.

Since $\frac{1}{r_1}+\frac{1}{r_2}>1$, we can choose a positive number τ such that $\frac{1}{r_1}-\tau+\frac{1}{r_2}=1$. Denote $\frac{1}{r_1}-\tau=\frac{1}{\dot r_1}$ and note that $\frac{1}{\dot r_1}+\frac{1}{r_2}=1$ and $r_1<\dot r_1$. Then we have,

$$BB_3 \lesssim \langle f_1\rangle_{Q_0,r_1}\langle h\rangle_{Q_0,t}\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}|Q|^{1-\frac{1}{r_1}-\frac{1}{r_2}+\tau}\Big(\int_Q \chi_{E_{1,q,k}}\Big)^{\frac{1}{\dot r_1}}\Big(\int_Q \chi_{F_{2,q,j}}\Big)^{\frac{1}{r_2}}$$

$$\leq \langle f_1\rangle_{Q_0,r_1}\langle h\rangle_{Q_0,t}\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\Big(\sum_{Q\in\mathcal{D}_0}\int_Q \chi_{E_{1,q,k}}\Big)^{\frac{1}{\dot r_1}}\Big(\sum_{Q\in\mathcal{D}_0}\int_Q \chi_{F_{2,q,j}}\Big)^{\frac{1}{r_2}}$$

$$\lesssim \langle f_1\rangle_{Q_0,r_1}\langle h\rangle_{Q_0,t}|Q_0|^{\frac{1}{\dot r_1}}\langle f_2\rangle_{Q_0,r_2}|Q_0|^{\frac{1}{r_2}} = \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}|Q_0|.$$

Estimate for the fourth term BB_4. Choose τ_1 and τ_2 as in the case BB_1. Consider

$$BB_4 \lesssim \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\sum_{Q\in\mathcal{D}_0}|Q|^{1-\frac{1}{r_1}-\frac{1}{r_2}}\Big(\int_Q \chi_{E_{1,q,k}}\Big)^{\frac{1}{r_1}}\Big(\int_Q \chi_{E_{2,q,j}}\Big)^{\frac{1}{r_2}}$$

$$\leq \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}\sum_{k,j=1}^{\infty}2^{-\eta(k+j)}\Big(\sum_{Q\in\mathcal{D}_0}\int_Q \chi_{E_{1,q,k}}\Big)^{\frac{1}{\dot r_1}}\Big(\sum_{Q\in\mathcal{D}_0}\int_Q \chi_{E_{2,q,j}}\Big)^{\frac{1}{\dot r_2}}$$

$$\lesssim \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0,t}|Q_0|.$$

This completes the proof of Lemma 6.2 for the lacunary bilinear spherical maximal operator.

For the case of the full bilinear spherical maximal operator, we proceed analogously as in the proof of the lacunary maximal operator. In this case, we have to deal with

$$\widetilde{\widetilde{\mathcal{M}}}_Q(f_1,f_2)(x) := \sup_{2^{q-4}\leq t\leq 2^{q-3}}\mathcal{A}_t(f_1\chi_{\frac{1}{2}Q})(x)\mathcal{A}_t(f_2\chi_{\frac{1}{2}Q})(x)$$

and we call

$$\widetilde{\mathcal{A}}_t f_i(x) = \mathcal{A}_t(f_i\chi_{\frac{1}{2}Q})(x)$$

We use the Calderón-Zygmund decomposition to write $f_i = g_i + b_i, \quad i = 1, 2$ to get the following

$$\begin{aligned}|\sum_{Q\in\mathcal{D}_0} \langle \widetilde{\widetilde{\mathcal{M}}}_Q(f_1,f_2),h_Q\rangle| &\le |\sum_{Q\in\mathcal{D}_0} \langle \widetilde{\widetilde{\mathcal{M}}}_Q(g_1,g_2),h_Q\rangle| + |\sum_{Q\in\mathcal{D}_0} \langle \widetilde{\widetilde{\mathcal{M}}}_Q(g_1,b_2),h_Q\rangle| \\ &\quad + |\sum_{Q\in\mathcal{D}_0} \langle \widetilde{\widetilde{\mathcal{M}}}_Q(b_1,g_2),h_Q\rangle| + |\sum_{Q\in\mathcal{D}_0} \langle \widetilde{\widetilde{\mathcal{M}}}_Q(b_1,b_2),h_Q\rangle| \\ &=: GG + GB + BG + BB.\end{aligned}$$

Estimate for GG (both functions good). In this case we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}_0} |\langle \widetilde{\widetilde{\mathcal{M}}}_Q(g_1,g_2),h_Q\rangle| &\le \sum_{Q\in\mathcal{D}_0} \|\widetilde{\widetilde{M}}_Q g_1\|_{L^\infty}\|\widetilde{\widetilde{M}}_Q g_2\|_{L^\infty}\|h_Q\|_{L^1} \\ &\lesssim \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2} \sum_{Q\in\mathcal{D}_0}\int |h(x)|\chi_{B_Q}(x)dx \\ &\lesssim \langle f_1\rangle_{Q_0,r_1}\langle f_2\rangle_{Q_0,r_2}\langle h\rangle_{Q_0}|Q_0|.\end{aligned}$$

Estimate for BG (one function good and one function bad). We have, proceeding as in the case of BG for the lacunary spherical maximal function. Let $t_Q : Q \to [2^{q-4}, 2^{q-3}]$ be a measurable function, then

$$\begin{aligned}|\sum_{Q\in\mathcal{D}_0} \langle \widetilde{\widetilde{\mathcal{M}}}_Q(b_1,g_2),h_Q\rangle| &\le \sum_{Q\in\mathcal{D}_0} \Big|\int b_1(x)(\widetilde{\mathcal{A}}_{t_Q})^*(\widetilde{\mathcal{A}}_{t_Q} g_2\cdot h_Q)(x)\,dx\Big| \\ \lesssim \sum_{k=1}^{3}\sum_{Q\in\mathcal{D}_0} |Q|\langle B_{1,q-k}\rangle_{Q,r_1}\langle \widetilde{\mathcal{A}}_{t_Q} g_2\cdot h_Q\rangle_{Q,s_1} \\ + \sum_{k\ge 4}\sum_{Q\in\mathcal{D}_0} \frac{1}{|P_0|}\int_{P_0}\Big(\frac{|y|}{l_Q}\Big)^\eta |Q|\langle B_{1,q-k}\rangle_{Q,r_1}\langle \widetilde{\mathcal{A}}_{t_Q} g_2\cdot h_Q\rangle_{Q,s_1}\,dy \\ \lesssim \sum_{k\ge1} 2^{-k\eta}\sum_{Q\in\mathcal{D}_0} |Q|\langle B_{1,q-k}\rangle_{Q,r_1}\langle \widetilde{\mathcal{A}}_{t_Q} g_2\cdot h_Q\rangle_{Q,s_1},\end{aligned}$$

where we have used [20, Theorem 3.2] in the second to last inequality. Next, observe that

$$\langle \widetilde{\mathcal{A}}_{t_Q} g_2\cdot h_Q\rangle_{Q,s_1} \lesssim \langle f_2\rangle_{Q_0,r_2}\langle h_Q\rangle_{Q,s_1}.$$

This point onward, we can follow the proof in the case of bilinear lacunary spherical maximal function and get the desired estimates. We skip the details.

This completes the proof of Lemma 6.2.

7. Necessary conditions for the sparse domination

In this section we prove the necessity part of Theorem 2.3. Indeed, we discuss the relations involving the exponents r_1, r_2, s_1, s_2 and t and show that they are necessary conditions for the validity of the sparse domination of the bilinear (both lacunary and full) spherical maximal functions. We make use of examples in the spirit of Knapp and Stein [32]. The approach in the (linear) sparse domination setting is developed in [20].

7.1. **Sparse form for $\mathcal{M}_{\text{lac}}$.** Let $f_1 = f_2 = \chi_{||x|-1|<\delta}$ and $h = \chi_{|x|\le c\delta}$ for some $0 < \delta < 1/4$ and $c \in (0, \frac{1}{2})$. Then we get that $\mathcal{A}_1 f_i(x) \ge c\, h(x)$, $i = 1, 2$. Therefore, the sparse domination for the operator $\mathcal{M}_{\text{lac}}$ implies that

$$\delta^n \lesssim \int_{\mathbb{R}^n} \mathcal{A}_1 f_1(x)\mathcal{A}_1 f_2(x)h(x)\,dx \le C_0 \sum_{Q\in\mathcal{S}} |Q|\langle f_1\rangle_{Q,r_1}\langle f_2\rangle_{Q,r_2}\langle h\rangle_{Q,t},$$

where $\mathcal{S}$ is a sparse collection. Observe that in the estimate above, in order to make non-trivial contribution to the term on the right hand side, the cube $Q \in \mathcal{S}$ must necessarily intersect with the supports of f_1, f_2 and h. Moreover, it is crucial that

$$\mathrm{dist}(\mathrm{supp}(f_i), \mathrm{supp}(h)) \geq 1/2,$$

which implies that all the cubes contributing to the sum have side length at least $1/2$. Further, since the contribution of a cube decreases as its size increases, it suffices to assume that $\mathcal{S}$ consists of one such cube Q. We have the estimate

$$\delta^n \lesssim \|f_1\|_{L^{r_1}}\|f_2\|_{L^{r_2}}\|h\|_{L^t} \lesssim |Q|^{1-\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{t}}\delta^{\frac{1}{r_1}+\frac{1}{r_2}+\frac{n}{t}}.$$

Since $\delta > 0$ can be chosen arbitrarily small, we get that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{n}{t} \leq n.$$

Note that the estimate above forces the condition $t > 1$. Substituting the value of t in terms of s_1 and s_2, we get the following necessary condition

$$\frac{1}{r_1}+\frac{n}{s_1}+\frac{1}{r_2}+\frac{n}{s_2} \leq 2n. \tag{28}$$

In a similar fashion, one can show that if $f_1 = f_2 = \chi_{|x|<\delta}$ and $h = \chi_{||x|-1|<c\delta}$ for some $0 < \delta < 1/4$ and $0 < c < \frac{1}{2}$, then we get that $\mathcal{A}_1 f_1(x) \geq c\delta^{n-1}h(x)$. This gives us another necessary condition, namely $\frac{n}{r_1}+\frac{n}{r_2}+\frac{1}{t} \leq 2n-1$. This would mean that

$$\frac{n}{r_1}+\frac{1}{s_1}+\frac{n}{r_2}+\frac{1}{s_2} \leq 2n. \tag{29}$$

The conditions (28) and (29) imply that both of $(\frac{1}{r_i}, \frac{1}{s_i})$, $i = 1, 2$, cannot lie outside of the triangle L_n.

Next, take $f_1 = \chi_{|x|<\delta}$, $f_2 = \chi_{|x|<2}$ (also interchanging f_1 and f_2) and $h = \chi_{||x|-1|<c\delta}$ for some $0 < \delta < 1$ and $0 < c < \frac{1}{2}$ and observe that

$$\delta^{n-1}\delta \lesssim \delta^{\frac{n}{r_1}}\delta^{\frac{1}{t}}.$$

This yields that $\frac{n}{r_i}+\frac{1}{t} \leq n$, $i = 1, 2$. Similarly, by taking $f_1 = \chi_{||x|-1|<\delta}$, $f_2 = \chi_{|x|<2}$, $h = \chi_{|x|<c\delta}$ for some $0 < \delta < 1/4$ and $0 < c < \frac{1}{2}$ and interchanging the roles of f_1 and f_2 we get that $\frac{1}{r_i}+\frac{n}{t} \leq n$, $i = 1, 2$.

Putting the above two conditions together we get the following condition.

$$\max\left\{\frac{n}{r_i}+\frac{1}{t}, \frac{1}{r_i}+\frac{n}{t}\right\} \leq n, \quad i = 1, 2. \tag{30}$$

The conditions (28), (29) and (30) must necessarily be satisfied for the sparse domination of the operator $\mathcal{M}_{\mathrm{lac}}$ to hold.

7.2. **Sparse form for $\mathcal{M}_{\mathrm{full}}$.** Consider $f_1 = |x|^{1-n}(\log\frac{1}{|x|})^{-1}\chi_{|x|<\frac{3}{4}}$ and $f_2 = \chi_{|x|<1}$ and note that $f_1 \in L^{r_1}(\mathbb{R}^n)$ for $1 < r_1 \leq \frac{n}{n-1}$. It is easy to verify that $\mathcal{M}_{\mathrm{full}}(f_1, f_2)$ is infinite on a set of positive measure. This gives us the condition that $\frac{1}{r_1} < \frac{n-1}{n}$. Using the symmetry between f_1 and f_2, we also have that $\frac{1}{r_2} < \frac{n-1}{n}$. Next, we observe that both of $(\frac{1}{r_i}, \frac{1}{s_i})$, $i = 1, 2$ cannot lie above the line segment P_1P_4 in F_n, see Figure 1. This can be proved by considering the functions $f_1 = f_2 = \chi_{||x|-1|<\delta}$ and $h = \chi_{|x|\leq c\delta}$ for some $0 < \delta < 1/4$ and $c \in (0, \frac{1}{2})$. This is same as in the case of lacunary maximal function. Note that the support-separation property also holds. We omit the details.

Consider $f_1 = f_2 = \chi_{R_1}$ and $h = \chi_{R_2}$, where $R_1 = [-C\sqrt{\delta}, C\sqrt{\delta}]^{n-1} \times [-C\delta, C\delta]$ and $R_2 = [-\sqrt{\delta}, \sqrt{\delta}]^{n-1} \times [\frac{4}{3}, \frac{5}{3}]$. This yields

$$\langle\widetilde{\mathcal{M}}(f_1, f_2), h\rangle \gtrsim \delta^{\frac{3(n-1)}{2}}.$$

The sparse domination of $\langle\widetilde{\mathcal{M}}(f_1, f_2), h\rangle$ yields

$$\delta^{\frac{3(n-1)}{2}} \leq \delta^{\frac{n+1}{2r_1}}\delta^{\frac{n+1}{2r_2}}\delta^{\frac{n-1}{2t}}.$$

This gives us the condition

$$\frac{n+1}{r_1}+\frac{n-1}{s_1}+\frac{n+1}{r_2}+\frac{n-1}{s_2}\le 4(n-1). \tag{31}$$

Therefore, both of $(\frac{1}{r_i},\frac{1}{s_i})$, $i=1,2$, cannot lie above the line segment P_3P_4 in Figure 1.

Also, the conditions $\frac{1}{r_i}+\frac{n}{t}\le n$, $i=1,2$, must be satisfied for the sparse domination of the full maximal function as they hold for the lacunary maximal function. Further, by considering $f_1=\chi_{R_1}$, $f_2=\chi_{B((0,0,\dots,\frac{4}{3}),2)}$ and $h=\chi_{R_2}$, we obtain that

$$\delta^{n-1}\lesssim\langle\mathcal{M}_{\text{full}}(f_1,f_2),h\rangle\lesssim\delta^{\frac{n+1}{2r_1}}\delta^{\frac{n-1}{2t}}.$$

Therefore, we get that

$$\frac{n+1}{r_1}+\frac{n-1}{t}\le 2(n-1).$$

Interchanging the roles of f_1 and f_2, we also have that

$$\frac{n+1}{r_2}+\frac{n-1}{t}\le 2(n-1).$$

These are necessary conditions on various parameters in order the sparse domination to hold for the full bilinear spherical maximal function.

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