

THE FRISCH–PARISI FORMALISM FOR FLUCTUATIONS OF THE SCHRÖDINGER EQUATION

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Dedicated to Prof. Tohru Ozawa on the occasion of his 60th birthday

ABSTRACT. We consider the solution of the Schrödinger equation u in \mathbb{R} when the initial datum tends to the Dirac comb. Let $h_{p,\delta}(t)$ be the fluctuations in time of $\int |x|^{2\delta} |u(x,t)|^2 dx$, for $0 < \delta < 1$, after removing a smooth background. We prove that the Frisch–Parisi formalism holds for $H_\delta(t) = \int_{[0,t]} h_{p,\delta}(2s) ds$, which is morally a simplification of the Riemann’s non-differentiable curve R . Our motivation is to understand the evolution of the vortex filament equation of polygonal filaments, which are related to R .

1. INTRODUCTION

The binormal curvature flow, also known as the vortex filament equation,

$$\chi_t = \chi_x \wedge \chi_{xx}, \tag{1}$$

is a model for the dynamics of vortex filaments in Euler equations. The function $\chi(t, x)$ describes a family of curves in 3d that move with time t and are parametrized by arclength x . Using the Frenet equations one easily concludes that the right-hand side of (1) is a vector whose modulus equals the curvature and whose direction is the binormal vector. By differentiating both sides by x , we get the one dimensional Schrödinger map

$$T_t = T \wedge T_{xx}, \quad \text{where } T := \chi_x \in \mathbb{S}^2. \tag{2}$$

Our interest in this paper is in curves that can develop corners in finite time. For that purpose, it is better to use the so-called parallel frame (T, e_1, e_2) instead of the usual Frenet frame, where the former is defined by

$$\begin{pmatrix} T_x \\ (e_1)_x \\ (e_2)_x \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}.$$

Hasimoto proved in [10] that for T to be a solution of (2), $u := \alpha + i\beta$ has to solve the 1d cubic non-linear Schrödinger equation

$$iu_t + u_{xx} + \frac{1}{2}(|u|^2 - A(t))u = 0,$$

for some real function $A(t)$; Hasimoto used the Frenet frame, but the proof admits more general frames.

In [6], de la Hoz and the fourth author studied the evolution of regular planar polygons χ_M , with M denoting the number of sides. In particular, they were interested in the trajectories described by

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any of the corners. That is to say, if we assume that at time $t = 0$ there is a corner at the origin, then they studied the curve in 3d

$$R_M(t) := \chi_M(t, 0). \quad (3)$$

These curves show a characteristic fractal behavior which is reminiscent of the so-called Riemann's non-differentiable function. In fact, they found compelling numerical evidence that $\lim_{M \rightarrow \infty} R_M(t) = R(t)$ with

$$R(t) := \int_0^t u_D(0, s) ds,$$

where u_D is the solution of the linear Schrödinger equation with initial datum $F_D = \sum_{n \in \mathbb{Z}} \delta_n$, that is, F_D is the Dirac's comb. It turns out that R is a small modification of the complex version of Riemann's function

$$\phi(t) := \sum_{n=1}^{\infty} \frac{e^{i\pi n^2 t}}{i\pi n^2} = 2\pi i R\left(\frac{-t}{4\pi}\right) - \frac{t}{2} - \frac{i\pi}{6}.$$

We notice that u_D exhibits the Talbot effect, that is, the appearance of rescaled and weighted Dirac combs at rational times, which easily justifies the fractal appearance of R . There is a rich literature about the Talbot effect; see for example [3, 7, 18, 19, 17].

Recently, Banica and the fourth author [2] tightened the connection between R and the binormal curvature flow. They proved that the evolution of a corner of a suitably chosen sequence of polygonal vortex filaments approaches $R(t)$ in the limit when the number of sides is infinite. Additionally, inspired by the work of Jaffard [11], they showed that the limiting behavior of the corners falls within the multifractal formalism of Frisch and Parisi, which is conjectured to govern turbulent fluids. By analogy with turbulence, we would expect that R could be understood as the outcome of some stochastic process; such an interpretation still seems to be missing.

Yet another interesting physical phenomenon, which is closely related to multifractality, is the intermittency. Roughly speaking, the idea is that the velocity of a fluid in fully developed turbulence may erratically change in very small distances, suggesting a very irregular structure. This phenomenon, called *intermittency in small scales* is related to the Frisch–Parisi multifractal formalism, but it does not seem to be well-defined in the literature. In [4], by adapting the physical concept of intermittency to the setting of functions and giving a precise definition, the authors gave quantitative estimates of the intermittency of the Riemann's non-differentiable function.

Within this circle of ideas, in [16], during an investigation of the dispersive properties of the free Schrödinger equation, an interesting behavior was discovered for the functional

$$h_\delta[f](t) := \int |x|^{2\delta} |u(x, t)|^2 dx, \quad x \in \mathbb{R}^d \quad \text{for } 0 < \delta < 1,$$

where u is the solution of the linear Schrödinger equation with initial datum f . By renormalization (removing an infinite and rescaling), the authors extended the definition of h_δ to periodic initial data like the Dirac comb F_D ; let us call $h_{p,\delta}[f]$ (p for periodic) to the renormalization.

During the renormalization of $h_\delta[f]$ a smooth function is removed, leaving behind small fluctuations that approach the point function in Figure 1 when f approaches the Dirac comb F_D . The function $h_{p,\delta}[F_D]$ is supported at rationals, so it is somehow a simplification of u_D , which has a complex structure at irrational times. This simplification offers the possibility of understanding hard questions associated with u_D by considering first $h_{p,\delta}[F_D]$.

In [16] the authors exposed evidences showing that

$$H_\delta(t) = \int_{[0,t]} h_{p,\delta}[F_D](s) ds, \quad \text{for } 0 < \delta < 1, \quad (4)$$

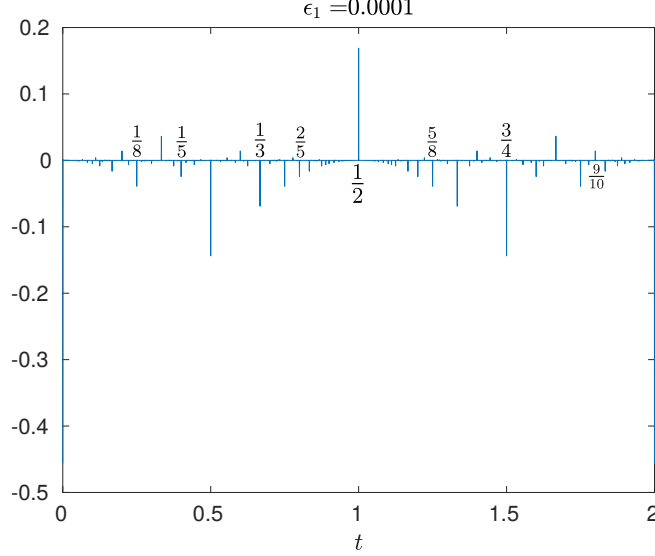


FIGURE 1. Evolution of $h_{p,\delta}[f_\epsilon]$, where f_ϵ is a smooth periodic function that approaches the Dirac comb in the sense of distributions as $\epsilon \rightarrow 0^+$.

can be seen as the outcome of a $(2/s)$ -Lévy process with $s := 2(1 + \delta)$. Unbeknownst to the authors, similarities between the Riemann non-differentiable function (which is behind u_D) and Lévy processes had already been pointed out by Jaffard in [12, Sections 2.3 and 4.4].

The velocity of turbulent flows differs widely from point to point, so in this context it has been introduced the spectrum of singularities of a function, which measures the size of the sets with different Hölder exponents.

Definition 1.1 (Hölder exponent). Let f be a function and $t \in \mathbb{R}$. A function $f \in C^l(t)$, for real $l \geq 0$, if there is a polynomial P_t of degree at most $[l]$ such that in a neighborhood of t

$$|f(s) - P_t(s)| \lesssim |t - s|^l.$$

The Hölder exponent of f at t is

$$h_f(t) := \sup\{l \mid f \in C^l(t)\}.$$

To measure the size of a set, we use the concept of Hausdorff dimension.

Definition 1.2 (Hausdorff dimension). Let $A \subset \mathbb{R}^n$ and R_ϵ be the set of all coverings of A by sets A_i of diameter at most ϵ . Let

$$\mathcal{H}_\epsilon^d(A) := \inf_{r \in R_\epsilon} \sum_{A_i \in r} (\text{diam } A_i)^d.$$

Then,

$$\mathcal{H}^d(A) := \lim_{\epsilon \rightarrow 0} \mathcal{H}_\epsilon^d(A)$$

is the d -dimensional Hausdorff content of A . The *Hausdorff dimension* of A is

$$\dim_{\mathcal{H}} A := \inf\{d : \mathcal{H}^d(A) = 0\} = \sup\{d : \mathcal{H}^d(A) = +\infty\}.$$

Now we can define the spectrum of singularities of a function.

Definition 1.3 (Spectrum of singularities). Let f be a function and define the set

$$\Gamma_h := \{t \in \mathbb{R} \mid f \text{ has Hölder exponent } h \text{ at } t\}.$$

The spectrum of singularities is the function

$$D_f(h) = \dim_{\mathcal{H}} \Gamma_h.$$

If $\Gamma_h = \emptyset$, then $D_f(h) = -\infty$.

As we mentioned, in [2, Theorem 1(iii)] it was proved that the spectrum of singularities of R (and modifications of it) is

$$D_R(h) = 4h - 2, \quad \text{for all } h \in \left[\frac{1}{2}, \frac{3}{4}\right].$$

Concerning H_δ , it was proved in [16, Theorem 4] that the spectrum of singularities of H_δ is

$$D_{H_\delta}(h) = \begin{cases} \alpha h, & \text{if } h \in [0, 1/\alpha], \\ -\infty, & \text{if } h > 1/\alpha, \end{cases} \quad (5)$$

where $\alpha = 2/s$ and $s = 2(1 + \delta)$, for $0 < \delta < 1$. Surprisingly, for α -Lévy processes Jaffard proved in [14] that the spectrum of singularities is almost surely equal to (5). Before stating our main result, we describe briefly the Frisch–Parisi formalism.

1.1. Frisch–Parisi formalism. The so-called multifractal formalism for functions relates some functional norms of a function to its spectrum of singularities. This formalism was introduced by Frisch and Parisi in order to numerically determine the spectrum of fully turbulent fluids [8]. Even though the Frisch-Parisi formalism has several versions, we decided to use the *wavelet-transform integral method* as described at the introduction of [13]. First we must define the wavelet transform of a function.

Definition 1.4. The *wavelet transform* of a function f is defined as $\psi_N * f$, where $\psi_N(x) := N\psi(Nx)$. The wavelet ψ is a function whose smoothness and decay are adjusted depending on the problem, and such that

$$\int x^k \psi(x) dx = 0, \quad \text{for } k = 0, \dots, L \text{ and some suitable } L.$$

The Frisch–Parisi formalism suggests that the spectrum of singularities can be computed through the *scaling exponent* η_f , which is defined by

$$\eta_f(p) := -\liminf_{N \rightarrow \infty} \frac{\log \|\psi_N * f\|_{L^p}^p}{\log N}. \quad (6)$$

The Legendre transform provides a link between D_f and η_f through the conjectured relationship:

$$D_f(h) = \inf_{p > 0} (ph - \eta_f(p) + 1).$$

1.2. Main result. Since $H_\delta[F_D]$ might be seen as a simplification of the Riemann’s non-differentiable function, for which the Frisch-Parisi formalism has been proved (see [11] or [2]), then we should be able to prove the Frisch–Parisi formalism for $H_\delta[F_D]$ in the range $[0, s/2]$. We confirm this in our main theorem below.

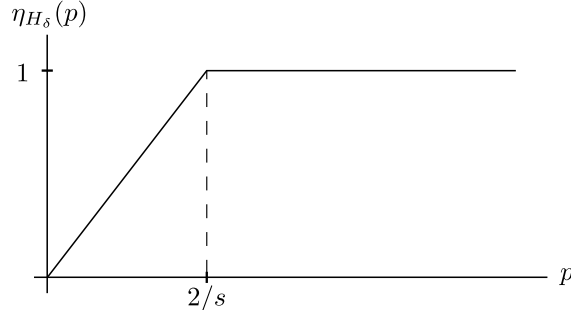
Theorem 1.5 (Frisch–Parisi formalism). *Let $0 < \delta < 1$ and $s := 2(1 + \delta)$. Let ψ be an integrable function such that:*

- (i) $\int_{\mathbb{R}} \psi = 0$.
- (ii) $|\psi(x)| \lesssim x^{-\beta}$, for $\beta > 1 + s$.

Let η_{H_δ} be the scaling exponent defined in (6) for the function H_δ . Then,

$$\eta_{H_\delta}(p) = \begin{cases} sp/2, & \text{if } 0 < p \leq 2/s, \\ 1, & \text{if } p \geq 2/s. \end{cases}$$

In particular, the Frisch–Parisi formalism holds in the range $[0, s/2]$.



As already said, we see $H_\delta[F_D]$ as a simplification of the Riemann’s non-differentiable function R . Therefore, a natural question is to determine whether the multifractal formalism holds true for the non-linear trajectories R_M given in (3). This seems to be a very challenging question at the theoretical level, so, to gain insight into the subject, we computed numerically the spectrum of singularities of R_M for several values of M ; see Figure 2.A. As a matter of comparison, we also computed numerically the spectrum of R (Fig. 2.A) and H_δ (Fig. 2.B), for which the theoretical values are known. In Appendix A we describe the methods used to compute the spectrum of singularities. Although more careful experiments are needed, they suggest that the spectrum of singularities of R_M should be equal to that of R .

We wonder whether it is possible to define H_δ replacing u_D by the solution of the non-linear Schrödinger equation, and in that case, whether the resulting H_δ and its spectrum of singularities is more amenable to theoretical studies.

Notation. We write $A \lesssim B$ if $A \leq CB$ for some constant $C > 0$; the relations \gtrsim and \simeq are similar. We also write $\|f\|_p = \|f\|_{L^p([0,1])}$.

The Hölder exponent is given in Definition 1.1; the Hausdorff dimension is Definition 1.2; the spectrum of singularities is Definition 1.3; and the scaling exponent is (6).

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2. PROOF OF THEOREM 1.5

To prove Theorem 1.5 it is more convenient to work with $h_{p,\delta}$ rather than directly with H_δ . Since ψ has vanishing mean, we can write it as $\psi = \phi'$. Thus, the operator $\psi_N * H_\delta$ is expressed as

$$\int \psi_N(x - y)H_\delta(y) dy = - \int \frac{d}{dy} \phi(N(x - y)) H_\delta(y) dy = \int \phi(N(x - y))h_{p,\delta}(y) dy. \quad (7)$$

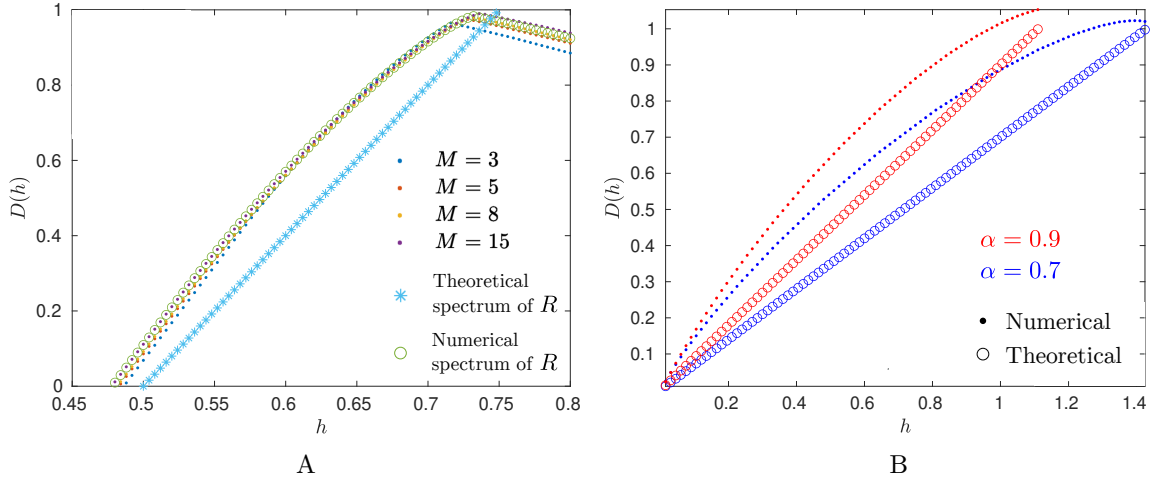


FIGURE 2. The spectrum of singularities $D(h)$ estimated using the *wavelet-transform modulus maxima* method for: A) M -sided polygons R_M with different M values and Riemann's function R , and B) H_δ with two values of $\alpha := 1/(1 + \delta)$. Clearly, up to the numerical errors, it captures the support of $D(h)$ very well in both cases. See Appendix A for more details about the numerical methods.

Here, the wavelet ϕ has the following properties: for some $c_1, c_2 > 0$,

- $|\phi(x)| \lesssim 1$, for $|x| \leq c_1$, because ψ is integrable;
- decay of the tails: for some $\alpha > s$,

$$|x|^\alpha |\phi(x)| \leq c_2, \quad \text{for } |x| \geq c_1,$$

because of the decay of ψ ;

- the L^p -norm is concentrated

$$\int_{|x| \leq c_1} |\phi|^p dx \geq \frac{1}{2} \int |\phi|^p dx.$$

Along the paper, we will be using systematically the properties of ϕ without further comment.

Now let us write out the distribution $h_{p,\delta}[F_D] \in \mathcal{S}'(\mathbb{T})$:

$$h_{p,\delta}[F_D](x) = \sum_{(p,q)=1} \frac{a_{q,\delta}}{q^s} \delta_{p/q}(x), \quad (8)$$

where $s = 2(1 + \delta)$ and

$$a_{q,\delta} = -2b_{1,\delta}\zeta(2(1 + \delta)) \begin{cases} 1, & \text{if } q \text{ is odd,} \\ -2(2^{1+2\delta} - 1), & \text{if } q \equiv 2 \pmod{4}, \\ 2^{2(1+\delta)}, & \text{if } q \equiv 0 \pmod{4}. \end{cases}$$

Here ζ is the Riemann zeta function and

$$b_{1,\delta} = \frac{1}{(2\pi)^{2\delta}} \frac{\Gamma(2\delta)}{|\Gamma(-\delta)|\Gamma(\delta)}.$$

At the end, the only property of $a_{q,\delta}$ we will make use of is $|a_{q,\delta}| \simeq_\delta 1$.

Let us denote the last integral in (7) by $P_N h_{p,\delta}$, so in view of (8)

$$P_N h_{p,\delta}(x) = \sum_{p/q} \frac{a_{q,\delta}}{q^s} \phi(N(x - p/q)),$$

where from here onwards, the notation $\sum_{p/q}$ stands for the sum over all pairs of integers p, q such that $(p, q) = 1$. Without loss of generality, we will assume that q is nonnegative. There should not be confusion between the appearance of p as an integer and in the L^p norms. We aim to prove the next theorem, from which our main theorems follow.

Theorem 2.1. *Let $0 < p \leq \infty$, $0 < \delta < 1$ and $s = 2(1 + \delta)$. Then, for $N \gg 1$,*

$$\|P_N h_{p,\delta}\|_{L^p([0,1])} \simeq \delta \begin{cases} N^{-1/p}, & \text{if } 2/s \leq p \leq \infty, \\ N^{-s/2} (\log N)^{s/2}, & \text{if } p = 2/s, \\ N^{-s/2}, & \text{if } 0 < p \leq 2/s. \end{cases}$$

To estimate the L^p norm of $P_N h_{p,\delta}$, the idea is to split this function as

$$\begin{aligned} P_N h_{p,\delta}(x) &= \sum_{\substack{p/q \\ q \leq c_0 \sqrt{N}}} \frac{a_{q,\delta}}{q^s} \phi(N(x - p/q)) + && \text{(main term } \mathbf{M}(x)) \\ &+ \sum_{\substack{p/q \\ q > c_0 \sqrt{N}}} \frac{a_{q,\delta}}{q^s} \phi(N(x - p/q)). && \text{(error term } \mathbf{E}(x)) \end{aligned}$$

Here, $0 < c_0 \ll 1$ is a constant to be fixed later. In light of the inequality

$$\| \|P_N h_{p,\delta}\|_p - \|\mathbf{M}\|_p \| \leq \|\mathbf{E}\|_p,$$

the goal is to get an estimate for $\|\mathbf{M}\|_p$ and a suitable upper bound of $\|\mathbf{E}\|_p$.

2.1. The role of \mathbf{M} . This subsection is philosophical in nature. We want to discuss what is the relationship between \mathbf{M} and the spectrum of singularities of H_δ .

In [16, Theorem 4], during the proof of (5), it is actually shown that the Hölder exponent of H_δ is s/μ , for $\mu > 2$, exactly in the set of numbers Γ_μ with irrationality μ . Let us recall the definition of irrationality.

Definition 2.2 (Irrationality Measure). A number x has irrationality μ if for every $\eta < \mu$ there are infinitely many rationals p/q such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^\eta},$$

but for $\eta > \mu$ there are at most finitely many.

The set Γ_μ of numbers with irrationality μ has Hausdorff dimension $2/\mu$, which is consequence of Jarník's [15, Theorem 1], that is,

$$\dim W = 2/\mu \quad \text{and} \quad \mathcal{H}^{2/\mu}(W) = +\infty,$$

$$\text{where } W = \left\{ x \mid \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu} \text{ for infinitely many rationals } p/q \right\}.$$

Now if we consider \mathbf{M} , it is essentially supported around fractions p/q with $q \leq \sqrt{N}$; let us forget about the parameter c_0 , which is introduced for technical reasons. We can decompose \mathbf{M} in a dyadic parameter λ as

$$\mathbf{M}(x) = \sum_{\lambda \leq \sqrt{N}} \sum_{q \simeq \lambda} \frac{a_{q,\delta}}{q^s} \phi(N(x - p/q)) =: \sum_{\lambda \leq \sqrt{N}} \mathbf{M}_\lambda(x).$$

Hence, ignoring the tail of ϕ , \mathbf{M}_λ is supported in a set V_λ which is union of $\simeq \lambda^2$ pairwise disjoint intervals of length $1/N$. For each $x \in V_\lambda$ we can find a fraction p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{N},$$

where $q \simeq \lambda \simeq N^{1/\mu}$ for some $\mu \geq 2$. We might see V_λ as a blurring of Γ_μ at scale $1/N$, so let us rename V_λ as “ Γ_μ ” and notice that $|\Gamma_\mu| \simeq N^{2/\mu-1}$, which is what we would expect of a blurring at scale $1/N$ of a set of dimension $2/\mu$.

We can compute heuristically the Hölder dimension of $P_N H_\delta$ in “ Γ_μ ”, where H_δ was defined in (4). Neglecting \mathbf{E} (we are being overbold here), for $x, y \in \Gamma_\mu$ with $|x - y| \simeq 1/N$ we would have

$$|H_\delta(x) - H_\delta(y)| \simeq \mathbf{M}(x) \simeq N^{-s/\mu}.$$

Hence, in “ Γ_μ ” the Hölder exponent would be s/μ , which agrees heuristically with [16, Theorem 4]. The Frisch–Parisi formalism is thus reflected in the L^p norm of \mathbf{M} , see for instance (10). The strength of the relationship between \mathbf{M} and D_{H_δ} fades away as $\mu \rightarrow 2$, and at $\mu = 2$ the “error term” \mathbf{E} takes the main role.

2.2. Proof of Theorem 2.1: the upper bound. First, we bound $P_N h_{p,\delta}$ pointwise with a simpler function. Since ϕ decays strongly, then to control it pointwise we can tile \mathbb{R} with intervals J with some suitable length $|J| = c_1$ so that

$$\phi(x) \leq \sum_J b_J \mathbb{1}_J(x),$$

where b_J are coefficients decaying very fast; the tiling is so that one of the intervals, call it J_0 , is centered at the origin. Hence, we can write

$$\|P_N h_{p,\delta}(\cdot)\|_p \lesssim \sum_J b_J \left\| \sum_{p/q} \frac{1}{q^s} \mathbb{1}_J(N(\cdot - p/q)) \right\|_p.$$

By translation symmetry, it suffices to consider $J = J_0$.

Let $\{K\}$ be a tiling of \mathbb{R} with intervals of length $4c_1/N$, one of them centered at zero, and let $\{K'\}$ be another tiling equal to $\{K\}$ but shifted by $2c_1/N$. Since every interval $I_{p/q}$ of length $2c_1/N$ and centered at p/q is contained in one interval either from $\{K\}$ or from $\{K'\}$, then

$$\sum_{p/q} \frac{1}{q^s} \mathbb{1}_{I_{p/q}}(x) \leq \sum_K \sum_{p/q \in K} \frac{1}{q^s} \mathbb{1}_K(x) + \sum_{K'} \sum_{p/q \in K'} \frac{1}{q^s} \mathbb{1}_{K'}(x).$$

Let us assume that $c_0 < (2c_1)^{-1/2}$. Hence, it suffices to control the L^p norm of

$$\sum_K \sum_{p/q \in K} \frac{1}{q^s} \mathbb{1}_K = \sum_K \sum_{\substack{p/q \in K \\ q \leq c_0 \sqrt{N}}} \frac{1}{q^s} \mathbb{1}_K + \sum_K \sum_{\substack{p/q \in K \\ q > c_0 \sqrt{N}}} \frac{1}{q^s} \mathbb{1}_K =: \mathbf{M} + \mathbf{E}.$$

Now we prove an upper bound for \mathbf{E} .

Lemma 2.3. *Let $0 < p \leq \infty$. Let $0 < \delta < 1$ and $s = 2(1 + \delta)$. Then, for $N \gg 1$,*

$$\|\mathbf{E}\|_p \lesssim_\delta \begin{cases} N^{-\frac{s}{2} + \frac{1}{2p}}, & \text{if } 1 \leq p \leq \infty \\ N^{-\frac{s}{2}}, & \text{if } 0 < p < 1. \end{cases}$$

Proof. We begin with the range $1 \leq p \leq \infty$. We first estimate the L^1 norm. Let φ be the Euler's totient function¹, then

$$\begin{aligned}
\|\mathbf{E}\|_1 &\lesssim_\delta \sum_K \sum_{\substack{p/q \in K \\ q > c_0 \sqrt{N}}} \frac{1}{q^s} |K| \\
&\lesssim_\delta \frac{1}{N} \sum_{q > c_0 \sqrt{N}} \frac{\varphi(q)}{q^s} \\
&\lesssim_\delta \frac{1}{N} \sum_{q > c_0 \sqrt{N}} \frac{1}{q^{s-1}} \quad (\text{by } \varphi(q) \leq q) \\
&\lesssim_\delta \frac{1}{N} (\sqrt{N})^{-s+2} = N^{-s/2}.
\end{aligned}$$

On the other hand,

$$\|\mathbf{E}\|_\infty \lesssim_\delta \sup_{|K|=2c_1/N} \sum_{\substack{p/q \in K \\ q > c_0 \sqrt{N}}} \frac{1}{q^s}.$$

Fix an interval K . If $kN/(2c_1) \leq q < (k+1)N/(2c_1)$, then there are at most $k+1$ rationals $p/q \in K$, so

$$\begin{aligned}
\sum_{\substack{p/q \in K \\ q > c_0 \sqrt{N}}} \frac{1}{q^s} &\leq \sum_{c_0 \sqrt{N} < q \leq N/(2c_1)} \frac{1}{q^s} + \sum_{k \geq 1} \sum_{k \leq 2c_1 q/N < k+1} \frac{k+1}{q^s} \\
&\lesssim N^{(1-s)/2} + N^{1-s} \\
&\lesssim N^{(1-s)/2}.
\end{aligned}$$

Now, interpolate the two above estimates: for $\theta = 1/p$, $p > 1$,

$$\begin{aligned}
\|\mathbf{E}\|_p &\lesssim \|\mathbf{E}\|_1^\theta \|\mathbf{E}\|_\infty^{1-\theta} \\
&\lesssim (N^{-s/2})^\theta (N^{(1-s)/2})^{1-\theta} \\
&= N^{-\frac{s}{2} + \frac{1-\theta}{2}} = N^{-\frac{s}{2} + \frac{1}{2p}},
\end{aligned}$$

as desired.

For the range $0 < p < 1$ we use Hölder with $r = 1/p$ so that $\|\mathbf{E}\|_p = \|\mathbf{E}\|_{L^p([0,1])} \leq \|\mathbf{E}\|_1 \lesssim N^{-s/2}$. \square

It remains to bound the main term \mathbf{M} . Notice that each interval K with $|K| = \frac{2c_1}{N}$ contains at most one rational p/q with $q \leq c_0 \sqrt{N}$ whenever

$$c_0 < (2c_1)^{-1/2}. \quad (9)$$

Indeed, this follows by contradiction, since $\left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right| \geq \frac{1}{q_1 q_2} \geq \frac{1}{c_0^2 N}$, assuming $q_1, q_2 \leq c_0 \sqrt{N}$. Hence, we can control the L^p norm as

$$\|\mathbf{M}\|_p^p = \sum_K \sum_{\substack{p/q \in K \\ q \leq c_0 \sqrt{N}}} \frac{1}{q^{ps}} |K| \lesssim \frac{1}{N} \sum_{\substack{p/q \\ q \leq c_0 \sqrt{N}}} \frac{1}{q^{ps}} \lesssim \frac{1}{N} \sum_{q \leq c_0 \sqrt{N}} \frac{1}{q^{ps-1}},$$

¹The Euler's totient function $\varphi(n)$ is the number of integers k , $1 \leq k \leq n$, such that $(n, k) = 1$

where we used $\varphi(q) \leq q$. We compute the last sum to get

$$\|\mathbf{M}\|_p \lesssim \begin{cases} N^{-1/p}, & \text{if } p > 2/s, \\ N^{-s/2}(\log N)^{s/2}, & \text{if } p = 2/s, \\ N^{-s/2}, & \text{if } p < 2/s. \end{cases} \quad (10)$$

Inequality (10) and Lemma 2.3 imply the upper bound in Theorem 2.1.

2.3. Proof of Theorem 2.1: the lower bound. Before getting to the proof of the lower bound, we need a quite technical lemma that says that the tail of ϕ can be safely ignored.

Recall that for $q \leq c_0\sqrt{N}$ with $c_0 \ll 1$, we denote by $I_{p/q}$ the interval of length $2c_1/N$ centered at p/q ; we assume that $c_0 < (2c_1)^{-1/2}$ to ensure that the intervals are disjoint. We pick a rational p_0/q_0 and define the error function

$$e(x) = \sum_{p/q \notin 2I_{p_0/q_0}} \frac{a_{q,\delta}}{q^s} \phi(N(x - p/q)), \quad x \in I_{p_0/q_0}.$$

We will show that e , which is the sum of the tails in I_{p_0/q_0} , is small. The reader can skip the next lemma under the assumption $\text{supp } \phi \subset [-c_1, c_1]$.

Lemma 2.4 (Tails are negligible). *Let $s > 2$ and $q_0 \leq c_0\sqrt{N}$ for $c_0 \ll 1$. Then,*

$$|e(x)| \lesssim \frac{q_0^{s-2}}{N^{s-1}}, \quad \text{for } x \in I_{p_0/q_0}.$$

Proof. Since $|x - p/q| \geq c_1/N$ then

$$|e(x)| \leq C \frac{c_2}{N^\alpha} \sum_{\substack{p/q \\ p/q \notin 2I_{p_0/q_0}}} \frac{1}{q^s |x - p/q|^\alpha}. \quad (11)$$

Moreover, we can write $x = p_0/q_0 + \delta x$, with $|\delta x| \leq c_1/N$, then

$$\left| x - \frac{p}{q} \right| \geq \left| \frac{p_0}{q_0} - \frac{p}{q} \right| - \frac{c_1}{N} \geq \frac{1}{2} \left| \frac{p_0}{q_0} - \frac{p}{q} \right|.$$

We replace the above in (11) so that

$$|e(x)| \leq C \frac{q_0^\alpha}{N^\alpha} \sum_{\substack{p/q \\ p/q \notin 2I_{p_0/q_0}}} \frac{q^{\alpha-s}}{|qp_0 - q_0p|^\alpha}.$$

Since $p/q \notin 2I_{p_0/q_0}$ is the same as

$$|qp_0 - q_0p| \geq 2c_1 \frac{q_0q}{N},$$

then it is sensible to break the sum above as

$$|e(x)| \leq C \frac{q_0^\alpha}{N^\alpha} \sum_{k \geq 0} \sum_{\{q : k < 2c_1 q_0 q / N \leq k+1\}} q^{\alpha-s} \sum_{\{p : |qp_0 - q_0p| > k\}} \frac{1}{|qp_0 - q_0p|^\alpha}. \quad (12)$$

To estimate the very last sum in (12), let \bar{a} denote a residue of $a \bmod q_0$ such that $|\bar{a}| \leq q_0/2$, so if $|\bar{a}p_0| > k$ then $\{l : |\bar{a}p_0 + lq_0| > k\} = \mathbb{Z}$ and

$$\sum_{\{p : |qp_0 - q_0p| > k\}} \frac{1}{|qp_0 - q_0p|^\alpha} = \sum_{\{l : |\bar{a}p_0 + lq_0| > k\}} \frac{1}{|\bar{a}p_0 + lq_0|^\alpha} \lesssim \frac{1}{|\bar{a}p_0|^\alpha} \quad (\text{assume } \alpha > 1).$$

If $|\overline{qp_0}| \leq k < q_0/2$ then

$$\sum_{\{l: |\overline{qp_0} + lq_0| > k\}} \frac{1}{|\overline{qp_0} + lq_0|^\alpha} \lesssim \sum_{l \neq 0} \frac{1}{|lq_0|^\alpha} \leq \frac{1}{q_0^\alpha}.$$

If $|\overline{qp_0}| \leq q_0/2 \leq k$ then

$$\sum_{\{l: |\overline{qp_0} + lq_0| > k\}} \frac{1}{|\overline{qp_0} + lq_0|^\alpha} \lesssim \sum_{l \geq k/q_0} \frac{1}{|lq_0|^\alpha} \lesssim \frac{1}{q_0 k^{\alpha-1}}.$$

Now we have to compute the contribution of each case to the sum in (12)

The contribution of the case $|\overline{qp_0}| > k$ is less than

$$A_1 := \frac{q_0^\alpha}{N^\alpha} \sum_{0 \leq k < q_0/2} \sum_{\{q: k < 2c_1 q_0 q / N \leq k+1\}} \mathbb{1}_{\{|\overline{qp_0}| > k\}}(q) \frac{q^{\alpha-s}}{|\overline{qp_0}|^\alpha}. \quad (13)$$

The last sum in q runs over an interval of length $N/(2c_1 q_0) \geq q_0$ if $c_0 \ll 1$ (recall that $q_0 \leq c_0 \sqrt{N}$), so we can break it into a number $\simeq N/(2c_1 q_0^2)$ of blocks K of length q_0 so that

$$\sum_{\{q: k < 2c_1 q_0 q / N \leq k+1\}} \mathbb{1}_{\{|\overline{qp_0}| > k\}}(q) \frac{q^{\alpha-s}}{|\overline{qp_0}|^\alpha} \leq \sum_K \sum_{q \in K} \mathbb{1}_{\{|\overline{qp_0}| > k\}}(q) \frac{q^{\alpha-s}}{|\overline{qp_0}|^\alpha};$$

see Figure 3.

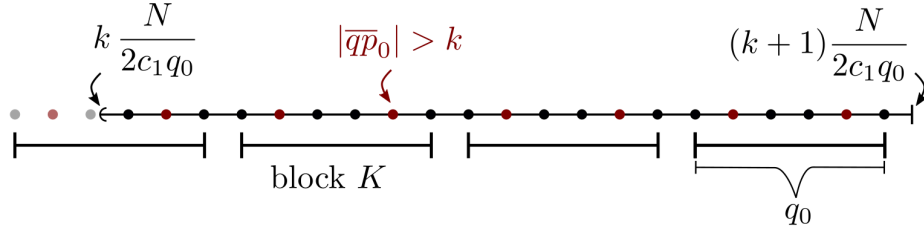


FIGURE 3. Blocks K of length q_0

Since $(p_0, q_0) = 1$, then $\overline{qp_0}$ runs over all residues $r \pmod{q_0}$, so we may bound this case as

$$\begin{aligned} \sum_{\{q: k < 2c_1 q_0 q / N \leq k+1\}} \mathbb{1}_{\{|\overline{qp_0}| > k\}}(q) \frac{q^{\alpha-s}}{|\overline{qp_0}|^\alpha} &\lesssim \left((k+1) \frac{N}{c_1 q_0} \right)^{\alpha-s} \sum_K \sum_{\substack{r \in K \\ |r| > k}} \frac{1}{|r|^\alpha} \\ &\lesssim \frac{1}{q_0 (k+1)^{s-1}} \left(\frac{N}{c_1 q_0} \right)^{\alpha+1-s} \end{aligned}$$

We replace it in (13) so that

$$A_1 \lesssim \frac{q_0^\alpha}{N^\alpha} \sum_{0 \leq k < q_0/2} \frac{1}{q_0 (k+1)^{s-1}} \left(\frac{N}{c_1 q_0} \right)^{\alpha+1-s} \lesssim \frac{q_0^{s-2}}{N^{s-1}}. \quad (14)$$

The contribution of the case $|\overline{qp_0}| \leq k < q_0/2$ is less than

$$\begin{aligned} A_2 &:= \frac{q_0^\alpha}{N^\alpha} \sum_{0 \leq k < q_0/2} \sum_{\{q: k < 2c_1 q_0 q / N \leq k+1\}} \mathbb{1}_{\{|\overline{qp_0}| \leq k\}}(q) \frac{q^{\alpha-s}}{q_0^\alpha} \\ &\lesssim \frac{1}{N^\alpha} \sum_{0 \leq k < q_0/2} \left[(k+1) \frac{N}{q_0} \right]^{\alpha-s} \sum_{\{q: k < 2c_1 q_0 q / N \leq k+1\}} \mathbb{1}_{\{|\overline{qp_0}| \leq k\}}(q). \end{aligned}$$

We estimate the last sum in q as before, breaking the sum into blocks of length q_0 , so that

$$\begin{aligned}
A_2 &\lesssim \frac{1}{N^s q_0^{\alpha-s}} \sum_{0 \leq k < q_0/2} (k+1)^{\alpha-s} \sum_K \sum_{\substack{r \in K \\ |r| \leq k}} \mathbb{1}_{\{|\overline{qp}_0| \leq k\}}(q) \\
&\lesssim \frac{1}{N^s q_0^{\alpha-s}} \sum_{0 \leq k < q_0/2} (k+1)^{\alpha-s} (k+1) \frac{N}{2c_1 q_0^2} \\
&\lesssim \frac{1}{N^{s-1}}.
\end{aligned} \tag{15}$$

The contribution of the case $k \geq q_0/2$ is less than

$$\begin{aligned}
A_3 &:= \frac{q_0^\alpha}{N^\alpha} \sum_{k \geq q_0/2} \sum_{\{q: k < 2c_1 q_0 q / N \leq k+1\}} \frac{q^{\alpha-s}}{q_0 k^{\alpha-1}} \\
&\lesssim \frac{q_0^\alpha}{N^\alpha} \cdot \frac{N^{\alpha+1-s}}{q_0^{\alpha+2-s}} \sum_{k \geq q_0/2} \frac{1}{k^{s-1}} \\
&\lesssim \frac{1}{N^{s-1}}.
\end{aligned} \tag{16}$$

We sum up all the contributions (14), (15) and (16) to the error term (12) to find out

$$|e(x)| \lesssim A_1 + A_2 + A_3 \lesssim \frac{q_0^{s-2}}{N^{s-1}}, \quad \text{for } x \in I_{p_0/q_0},$$

which is what we wanted. \square

After this lemma, for each p_0/q_0 with $q_0 \leq c_0 \sqrt{N}$, we can split $P_N h_{p,\delta}$ as

$$P_N h_{p,\delta}(x) = \sum_{\substack{p/q \in 2I_{p_0/q_0} \\ q \leq c_0 \sqrt{N}}} \frac{a_{q,\delta}}{q^s} \phi(N(x-p/q)) + \sum_{\substack{p/q \in 2I_{p_0/q_0} \\ q > c_0 \sqrt{N}}} \frac{a_{q,\delta}}{q^s} \phi(N(x-p/q)) + e(x), \quad \text{if } x \in I_{p_0/q_0}.$$

Since the only fraction $p/q \in 2I_{p_0/q_0}$ with $q \leq c_0 \sqrt{N}$, for $c_0 \ll 1$, is p_0/q_0 itself, then we can write this decomposition as

$$\begin{aligned}
P_N h_{p,\delta}(x) &= \frac{a_{q_0,\delta}}{q_0^s} \phi(N(x-p_0/q_0)) + \sum_{\substack{p/q \in 2I_{p_0/q_0} \\ q > c_0 \sqrt{N}}} \frac{a_{q,\delta}}{q^s} \phi(N(x-p/q)) + e(x), \quad \text{if } x \in I_{p_0/q_0}, \\
&=: \mathbf{M}(x) + \mathbf{E}(x) + e(x).
\end{aligned} \tag{17}$$

2.3.1. *The range $p > 2/s$.* We estimate the L^p norm of \mathbf{M} by integrating in the interval I_0 , that is, $p_0/q_0 = 0$, so

$$\|\mathbf{M}\|_{L^p(I_0)} \gtrsim N^{-1/p}.$$

The L^p norm of \mathbf{E} is

$$\|\mathbf{E}\|_{L^p(I_0)} \lesssim \frac{1}{N^{1/p}} \sum_{\substack{p/q \in 2I_0 \\ q > c_0 \sqrt{N}}} \frac{1}{q^s}.$$

Since $|p/q| \leq 2c_1/N$ then necessarily $q \geq N/(2c_1) > c_0 \sqrt{N}$, and the number of fractions with denominator q in $2I_0$ is $\leq 2c_1 q/N$, so

$$\|\mathbf{E}\|_{L^p(I_0)} \lesssim \frac{1}{N^{1/p}} \sum_{q \geq N/(2c_1)} \frac{1}{N q^{s-1}} \lesssim N^{-1/p-s+1}.$$

By Lemma 2.4 we have

$$\|e\|_{L^p(I_0)} \lesssim N^{-1/p-s+1}.$$

This leads us to the conclusion

$$\|P_N h_{p,\delta}\|_p \geq \|P_N h_{p,\delta}\|_{L^p(I_0)} \gtrsim N^{-1/p},$$

which proves the lower bound in Theorem 2.1 for the range $p > 2/s$.

2.3.2. *The range $0 < p \leq 2/s$.* The lower bound will be estimated by integrating $P_N h_{p,\delta}$ over

$$U := \bigcup_{\substack{p/q \\ q \leq c_0 \sqrt{N}}} I_{p/q}. \quad (18)$$

Since the intervals are pairwise disjoint (see (9))

$$|U| = \sum_{\substack{p/q \\ q \leq c_0 \sqrt{N}}} |I_{p/q}| \leq \frac{2c_1}{N} \sum_{\substack{p/q \\ q \leq c_0 \sqrt{N}}} 1 \leq \frac{2c_1}{N} \sum_{q \leq c_0 \sqrt{N}} \varphi(q) \leq \frac{2c_1}{N} \sum_{q \leq c_0 \sqrt{N}} q \leq 2c_0^2 c_1. \quad (19)$$

By Lemma 2.4 the L^p norm of e is small

$$\|e\|_{L^p(U)}^p \lesssim \sum_{\substack{p/q \\ q \leq c_0 \sqrt{N}}} \frac{q^{p(s-2)}}{N^{p(s-1)}} |I_{p/q}| \lesssim \frac{1}{N^{p(s-1)+1}} \sum_{q \leq c_0 \sqrt{N}} q^{p(s-2)+1} \lesssim c_0^{p(s-2)+2} N^{-ps/2}. \quad (20)$$

In the decomposition (17), the L^p norm of the main term is

$$\begin{aligned} \|\mathbf{M}\|_{L^p(U)} &\gtrsim \left(\sum_{\substack{p/q \\ q \leq c_0 \sqrt{N}}} \int_{x \in I_{p/q}} \frac{1}{q^{ps}} |\phi(N(x - p/q))|^p dx \right)^{1/p} \\ &\gtrsim \left(\frac{1}{N} \sum_{q \leq c_0 \sqrt{N}} \frac{\varphi(q)}{q^{ps}} \right)^{1/p}, \end{aligned} \quad (21)$$

by using the properties of ϕ . We estimate the last sum in the next lemma.

Lemma 2.5. *Let $0 < \alpha \leq 2$ and $M \gg 1$. Then,*

$$\sum_{1 \leq q \leq M} \frac{\varphi(q)}{q^\alpha} \gtrsim \begin{cases} \log M & \text{if } \alpha = 2, \\ M^{2-\alpha} & \text{if } 0 < \alpha < 2. \end{cases} \quad (22)$$

Proof. Recall the identity

$$\varphi(q) = q \sum_{d|q} \frac{\mu(d)}{d}, \quad (23)$$

where μ is the Möbius function; see [9, Section 16.3].

Let $M_0 \gg 1$ and replace (23) into the left-hand side of (22) so that

$$\begin{aligned} \sum_{1 \leq q \leq M} \frac{\varphi(q)}{q^\alpha} &\geq \sum_{M_0 \leq q \leq M} \frac{\varphi(q)}{q^\alpha} \\ &\simeq \sum_{M_0 \leq m \leq M} \frac{1}{m^{\alpha-1}} \sum_{m/2 \leq q \leq m} \sum_{d|q} \frac{\mu(d)}{d}, \end{aligned} \quad (24)$$

where $m \in 2^{\mathbb{N}}$. For each dyadic block we have

$$\begin{aligned} \sum_{m/2 \leq q \leq m} \sum_{d|q} \frac{\mu(d)}{d} &= \sum_{d \geq 1} \frac{\mu(d)}{d} \sum_{m/2 \leq q \leq m} \mathbf{1}_{d|q}(q) \\ &= \sum_{1 \leq d \leq m} \frac{\mu(d)}{d} \sum_{m/(2d) \leq k \leq m/d} 1 \\ &= \sum_{1 \leq d \leq m} \frac{\mu(d)}{d} \left(\frac{m}{2d} \right) + \sum_{1 \leq d \leq m} \frac{\mu(d)}{d} \left(-\frac{m}{2d} + \sum_{m/(2d) \leq k \leq m/d} 1 \right) \end{aligned}$$

Since $|m/(2d) - |\{k \mid m/(2d) \leq k \leq m/d\}|| \lesssim 1$ and $|\mu(d)| \leq 1$, then

$$\begin{aligned} \sum_{m/2 \leq q \leq m} \sum_{d|q} \frac{\mu(d)}{d} &= \frac{m}{2} \sum_{1 \leq d \leq m} \frac{\mu(d)}{d^2} + \mathcal{O}(\log m) \\ &= \frac{m}{2\zeta(2)} - \frac{m}{2} \sum_{d > m} \frac{\mu(d)}{d^2} + \mathcal{O}(\log m) \\ &= \frac{m}{2\zeta(2)} + \mathcal{O}(\log m). \end{aligned}$$

Here, we used the identity $\sum_{d \geq 1} \mu(d)/d^2 = 1/\zeta(2)$, where ζ is the Riemann zeta function, see [9, Theorem 287].

Going back to (24), for $M_0 \gg 1$ we get

$$\sum_{1 \leq q \leq M} \frac{\varphi(q)}{q^\alpha} \gtrsim \sum_{M_0 \leq m \leq M} \frac{1}{m^{\alpha-2}},$$

which yields (22). □

We apply Lemma 2.5 to (21) with $c_0 \ll 1$ to find out that for $N \gg 1$ we have

$$\|\mathbf{M}\|_p \gtrsim \begin{cases} N^{-s/2} (\log c_0 N)^{1/p}, & \text{if } p = 2/s \\ c_0^{2/p-s} N^{-s/2}, & \text{if } p < 2/s. \end{cases} \quad (25)$$

It remains to bound the error term

$$\mathbf{E}(x) = \sum_{\substack{p/q \in 2I_{p_0/q_0} \\ q > c_0 \sqrt{N}}} \frac{a_{q,\delta}}{q^s} \phi(N(x - p/q)), \quad \text{if } x \in I_{p_0/q_0}. \quad (26)$$

To compute the L^p norm we use Hölder with exponent $r = 1/p$ so that, in view of (19),

$$\|\mathbf{E}\|_{L^p(U)} \leq |U|^{1/(pr')} \|\mathbf{E}\|_{L^1(U)} \lesssim c_0^{2/p-2} \|\mathbf{E}\|_{L^1(U)}.$$

The L^1 norm is

$$\|\mathbf{E}\|_{L^1(U)} \lesssim \sum_{\substack{p'/q' \\ q' \leq c_0 \sqrt{N}}} \sum_{\substack{p/q \in 2I_{p'/q'} \\ q > c_0 \sqrt{N}}} \frac{1}{q^s} |I_{p'/q'}| \lesssim \frac{1}{N} \sum_{\substack{p/q \\ q > c_0 \sqrt{N}}} \frac{1}{q^s} \sum_{\substack{p'/q' \\ q' \leq c_0 \sqrt{N}}} \mathbf{1}_{\{|p/q - p'/q'| \leq 2c_1/N\}}(p'/q').$$

The last sum in p'/q' is at most one because of the restriction $q' \leq c_0 \sqrt{N}$. Indeed, if there were at least two, say $p_1/q_1, p_2/q_2$ with $|p/q - p_i/q_i| \leq 2c_1/N$, then $1/(q_1 q_2) \leq |p_1/q_2 - p_1/q_2| \leq 4c_1/N$ and $1/(q_1 q_2) \geq 1/(c_0^2 N)$, which implies $c_0 > 1/(2c_1^{1/2})$, contradiction to (9). If the last sum in p'/q'

is not empty (which happens when $q > \sqrt{N}/(2c_0c_1)$), then necessarily $p'/q' \neq p/q$, which implies that $1 \leq 2c_1qq'/N$ or $q \geq N/(2c_1q') \geq \sqrt{N}/(2c_0c_1)$. Hence,

$$\|\mathbf{E}\|_{L^1(U)} \lesssim \frac{1}{N} \sum_{\substack{p/q \\ q \geq \sqrt{N}/(2c_0c_1)}} \frac{1}{q^s} \lesssim \frac{1}{N} \sum_{q \geq \sqrt{N}/(2c_0c_1)} \frac{\varphi(q)}{q^s} \lesssim c_0^{s-2} N^{-s/2}. \quad (27)$$

In Appendix B we prove a better upper bound, but for the present purposes this is enough. We get thus

$$\|\mathbf{E}\|_{L^p(U)} \lesssim c_0^{2/p+s-4} N^{-s/2}. \quad (28)$$

We can now conclude the lower bound. We have

$$\|P_N h_{p,\delta}\|_{L^p(U)} \geq \|\mathbf{M}\|_{L^p(U)} - \|\mathbf{E}\|_{L^p(U)} - \|e\|_{L^p(U)},$$

so, when $p < 2/s$, from (25), (28) and (20) we get

$$\|P_N h_{p,\delta}\|_p \geq \|P_N h_{p,\delta}\|_{L^p(U)} \gtrsim c_0^{2/p} (c_0^{-s} - Cc_0^s - Cc_0^{s-4}) N^{-s/2} \gtrsim N^{-s/2} \quad (\text{by } s > 2).$$

For the critical exponent $p = 2/s$ we get (observe that the only logarithmic term below comes from \mathbf{M})

$$\|P_N h_{p,\delta}\|_p \geq \|P_N h_{p,\delta}\|_{L^p(U)} \gtrsim N^{-s/2} (\log c_0 N)^{s/2},$$

which concludes the proof of Theorem 2.1 when $0 < p \leq 2/s$.

APPENDIX A. NUMERICAL SIMULATIONS

For a given signal/function, we calculate its spectrum of singularities $D(h)$ numerically using the *wavelet transform modulus maxima (WTMM) method* implemented in MATLAB using the Wavelab 850 toolbox [5]. With a suitable choice of a wavelet, through the wavelet coefficients, we compute the partition function, scaling exponent $\eta(p)$ and thus, $D(h)$ is estimated using the Legendre transform (see [20] for their precise definition as they are different from the ones mentioned earlier). The input parameters consist of the signal X with length $N = 2^J$, the number of scales, range of parameters p and h . Thus, for $X = H_\delta$ in Figure 2.B, we choose $J = 13$, $\alpha = 0.7, 0.9$, $p \in [-5, 5]$, $h \in [h_{\min}, h_{\max}]$, where $h_{\min} = 0$, $h_{\max} = 1/\alpha$, that is, the support of H_δ , and the wavelet used is the first derivative of a Gaussian. To further compare them quantitatively, we calculate the error as defined in [20, (32)] and obtain the values 0.0956 and 0.1166 for $\alpha = 0.7$ and $\alpha = 0.9$, respectively. We notice that these results are indeed comparable with the ones obtained in [20] and can be reduced further with a larger N .

Next, we estimate $D(h)$ for R and R_M in the context of vortex filament equation. More precisely, we consider the input signal X as the trajectory of the third component (without the vertical height) of the M -sided filament curve [6]. With $p \in [-5, 5]$, $h_{\min} = 0.4$ and $h_{\max} = 0.8$, we plot the Riemann's function R and R_M , for $M = 3, 5, 8, 15$, in Figure 2.A. The plots show the estimated values of $D(h)$ where its maximum value varies with M and converges to that of R (circled points). Indeed, for $M = 15$, the agreement is remarkable and deviations from the theoretical values (starred points) are a result of a numerical error, which is minimum when the wavelet chosen is the second derivative of a Gaussian. The support of $D(h)$ in each case is very close to 0.25.

APPENDIX B. COUNTING RATIONALS

In the next proposition we improve the upper bound $\|\mathbf{E}\|_{L^1(U)} \lesssim c_0^{s-2} N^{-s/2}$ we proved in (27).

Proposition B.1. *Let U be the set (18) and \mathbf{E} the function (26). Then,*

$$\|\mathbf{E}\|_{L^1(U)} \lesssim c_0^s N^{-s/2}.$$

Proof. The L^1 norm is

$$\|\mathbf{E}\|_{L^1(U)} \lesssim \frac{1}{N} \sum_{\substack{p/q \\ q > c_0\sqrt{N}}} \sum_{\substack{p'/q' \\ q' \leq c_0\sqrt{N}}} \frac{1}{q^s} \mathbb{1}_{\{|p/q - p'/q'| \leq 2c_1/N\}}(p/q, p'/q').$$

Now we break the ranges $q > c_0\sqrt{N}$ and $q' \leq c_0\sqrt{N}$ dyadically into parameters $\lambda, \mu \in 2^{\mathbb{N}}$, respectively, so that

$$\|\mathbf{E}\|_{L^1(U)} \lesssim \frac{1}{N} \sum_{\substack{\lambda < c_0\sqrt{N} \\ \mu \geq c_0\sqrt{N}}} \mu^{-s} \sum_{\substack{p/q, p'/q' \\ q \simeq \lambda, q' \simeq \mu}} \mathbb{1}_{\{|p/q - p'/q'| \leq 2c_1/N\}}(p/q, p'/q').$$

Equivalently, we have to count at most how many pairs of rationals $(p/q, p'/q')$ satisfy

$$0 < |q'p - qp'| \leq 2c_1 \frac{\lambda\mu}{N}; \quad (29)$$

for that, we use the arguments in [1, Proposition 4.2].

For $m \in \mathbb{Z} \setminus \{0\}$, the goal is to count how many representations has m as $m = q'p - qp'$ with $0 \leq p < q$ and $0 \leq p' < q'$, so let us fix q and q' . If $(q, q') = d$ then necessarily $m = d\tilde{m}$, so let us clear out d from the representation of m and write $\tilde{m} = \tilde{q}'p - \tilde{q}p'$, where $q' = d\tilde{q}'$, $q = d\tilde{q}$ and $(\tilde{q}, \tilde{q}') = 1$. Now assume that $\tilde{q}'p - \tilde{q}p' = \tilde{q}'r - \tilde{q}r'$, or after reordering $\tilde{q}'(p - r) = \tilde{q}(p' - r')$. This implies that $\tilde{q} \mid (p - r)$ so, for some $l \in \mathbb{Z}$, we have $0 \leq r = p + l\tilde{q} < q$ (recall that $0 \leq p < q$ and $q = d\tilde{q}$) and then the number of different r 's is at most d . In conclusion, for every m divisible by d there are d representations $m = q'p - qp'$ with $0 \leq p < q$ and $0 \leq p' < q'$.

The above paragraph shows that, for fixed q and q' , the number of choice of pairs (p, p') satisfying (29) is $\leq 2c_1\lambda\mu/N$, so the total number of fractions satisfying (29) is $\lesssim 2c_1\lambda^2\mu^2/N$. Notice that the collection of representations is empty unless $\lambda\mu \geq N/(2c_1)$, so

$$\begin{aligned} \|\mathbf{E}\|_{L^1(U)} &\lesssim \frac{1}{N^2} \sum_{\substack{\lambda < c_0\sqrt{N} \\ \mu \geq c_0\sqrt{N}}} \mu^{2-s} \lambda^2 \mathbb{1}_{\{\lambda\mu \geq N/(2c_1)\}}(\lambda, \mu) \\ &= \frac{1}{N^2} \sum_{\lambda < c_0\sqrt{N}} \lambda^2 \sum_{\mu \geq N/(2c_1\lambda)} \mu^{2-s} \quad (\text{by } N/(2c_1\lambda) \geq c_0\sqrt{N} \text{ for } c_0 \ll 1) \\ &\lesssim \frac{1}{N^s} \sum_{\lambda < c_0\sqrt{N}} \lambda^s \lesssim c_0^s N^{-s/2}, \end{aligned}$$

where we used that μ and λ are dyadic. The proof is completed. \square

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