ON THE REGULARITY OF SOLUTIONS TO THE
k-GENERALIZED KORTEWEG-DE VRIES EQUATION

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ABSTRACT. This work is concerned with special regularity properties of solutions to the k-generalized Korteweg-de Vries equation. In [4] it was established that if the initial datum \( u_0 \in H^l((b, \infty)) \) for some \( l \in \mathbb{Z}^+ \) and \( b \in \mathbb{R} \), then the corresponding solution \( u(\cdot, t) \) belongs to \( H^l((\beta, \infty)) \) for any \( \beta \in \mathbb{R} \) and any \( t \in (0, T) \). Our goal here is to extend this result to the case where \( l > 3/4 \).

1. INTRODUCTION

In this note we study the regularity of solutions to the initial value problem (IVP) associated to the k-generalized Korteweg-de Vries equation

\[
\begin{aligned}
\partial_t u + \partial_x^3 u + u^k \partial_x u &= 0, \quad x, t \in \mathbb{R}, \ k \in \mathbb{Z}^+, \\
u(x, 0) &= u_0(x).
\end{aligned}
\tag{1.1}
\]

The starting point is a property found by Isaza, Linares and Ponce [4] concerning the propagation of smoothness in solutions of the IVP (1.1). To state it we first recall the following well-posedness (WP) result for the IVP (1.1):

**Theorem A1.** If \( u_0 \in H^{3/4+} \mathbb{R} \), then there exist \( T = T(\|u_0\|_{3/4+}^k) > 0 \) and a unique solution \( u = u(x,t) \) of the IVP (1.1) such that

\[
\begin{align*}
(i) & \quad u \in C([-T,T]: H^{3/4+}(\mathbb{R})), \\
(ii) & \quad \partial_x u \in L^4([-T,T]: L^\infty(\mathbb{R})), \quad (Strichartz), \\
(iii) & \quad \sup_x \int_{-T}^T |J^r \partial_x u(x,t)|^2 dt < \infty \quad \text{for} \quad r \in [0,3/4^+], \\
(iv) & \quad \int_{-\infty}^{\infty} \sup_{-T \leq t \leq T} |u(x,t)|^2 dx < \infty,
\end{align*}
\tag{1.2}
\]

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with \( J = (1 - \partial_x^2)^{1/2} \). Moreover, the map data-solution, \( u_0 \to u(x,t) \) is locally continuous (smooth) from \( H^{3/4+}(\mathbb{R}) \) into the class \( X^3_T \) defined in (1.2).

If \( k \geq 2 \), then the result holds in \( H^{3/4}(\mathbb{R}) \). If \( k = 1, 2, 3 \), then \( T \) can be taken arbitrarily large.

For the proof of Theorem A1 we refer to [6], [1] and [3]. The proof of our main result Theorem 1.1 is based on an energy estimate argument for which the estimate (ii) in (1.2) (i.e. the time integrability of \( \|\partial_x u(\cdot, t)\|_\infty \)) is essential. However, we remark that from the WP point of view is not optimal. For a detailed discussion on the WP of the IVP (1.1) we refer to [7], Chapters 7-8.

Now we enunciate the result obtained in [4] regarding propagation of regularities which motivates our study here:

**Theorem A2 ([4]).** Let \( u_0 \in H^{3/4+}(\mathbb{R}) \). If for some \( l \in \mathbb{Z}^+ \) and for some \( x_0 \in \mathbb{R} \)

\[
\|\partial_x^l u_0\|_{L^2((x_0, \infty))}^2 = \int_{x_0}^\infty |\partial_x^l u_0(x)|^2 dx < \infty, \tag{1.3}
\]

then the solution \( u = u(x,t) \) of the IVP (1.1) provided by Theorem A1 satisfies that for any \( v > 0 \) and \( \epsilon > 0 \)

\[
\sup_{0 \leq t \leq T} \int_{x_0 + \epsilon - vt}^{x_0 + \epsilon + vt} (\partial_x^j u)^2(x,t) \, dx < c, \tag{1.4}
\]

for \( j = 0, 1, \ldots, l \) with \( c = c(l; \|u_0\|_{3/4+}, \|\partial_x^l u_0\|_{L^2((x_0, \infty))}; v; \epsilon; T) \).

In particular, for all \( t \in (0, T] \), the restriction of \( u(\cdot, t) \) to any interval of the form \((a, \infty)\) belongs to \( H^l((a, \infty)) \).

Moreover, for any \( v \geq 0 \), \( \epsilon > 0 \) and \( R > 0 \)

\[
\int_0^T \int_{x_0 + R - vt}^{x_0 + R + vt} (\partial_x^{l+1} u)^2(x,t) \, dx \, dt < c, \tag{1.5}
\]

with \( c = c(l; \|u_0\|_{3/4+}, \|\partial_x^l u_0\|_{L^2((x_0, \infty))}; v; \epsilon; R; T) \).

Theorem A2 tells us that the \( H^l \)-regularity \((l \in \mathbb{Z}^+)\) on the right hand side of the data travels forward in time with infinite speed. Notice that since the equation is reversible in time a gain of regularity in \( H^s(\mathbb{R}) \) cannot occur so at \( t > 0 \), so \( u(\cdot, t) \) fails to be in \( H^l(\mathbb{R}) \) due to its decay at \( -\infty \). In this regard, it was also shown in [4] that for any \( \delta > 0 \) and \( t \in (0, T) \) and \( j = 1, \ldots, l \)

\[
\int_{-\infty}^{\infty} \frac{1}{(x-\delta)^{j+\delta}} (\partial_x^j u)^2(x,t) \, dx \leq \frac{c}{t^j},
\]
with $c = c(\|u_0\|_{3/4^+}; \|\partial_x^j u_0\|_{L^2((x_0,\infty))}; x_0; \delta)$, $x_- = \max\{0; -x\}$ and 
$$\langle x \rangle = (1 + x^2)^{1/2}.$$ 

The aim of this note is to extend Theorem A2 to the case where the local regularity of the datum $u_0$ in (1.3) is measure with a fractional exponent. Thus, our main result is:

**Theorem 1.1.** Let $u_0 \in H^{3/4^+}(\mathbb{R})$. If for some $s \in \mathbb{R}$, $s > 3/4$, and for some $x_0 \in \mathbb{R}$

$$\| J^s u_0 \|_{L^2((x_0,\infty))}^2 = \int_{x_0}^\infty |J^s u_0(x)|^2 dx < \infty,$$  
(1.6)

then the solution $u = u(x,t)$ of the IVP (1.1) provided by Theorem A1 satisfies that for any $v > 0$ and $\epsilon > 0$

$$\sup_{0 \leq t \leq T} \int_{x_0 + \epsilon - vt}^\infty (J^r u)^2(x,t) dx < c,$$  
(1.7)

for $r \in (3/4, s]$ with $c = c(l; \|u_0\|_{3/4^+}; \|J^r u_0\|_{L^2((x_0,\infty))}; v; \epsilon; T)$.

Moreover, for any $v > 0$, $\epsilon > 0$ and $R > 0$

$$\int_0^T \int_{x_0 + R - vt}^{x_0 + R - vt} (J^{s+1} u)^2(x,t) dx dt < c,$$  
(1.8)

with $c = c(l; \|u_0\|_{3/4^+}; \|J^s u_0\|_{L^2((x_0,\infty))}; v; \epsilon; R; T)$.

From the results in section 2 it will be clear that we need only consider the case $s \in (3/4, \infty) - \mathbb{Z}^+$. The rest of this paper is organized as follows: section 2 contains some preliminary estimates required for Theorem 1.1 whose proof will be given in section 3.

2. Preliminary estimates

Let $T_a$ be a pseudo-differential operator whose symbol

$$\sigma(T_a) = a(x, \xi) \in S^r, \ r \in \mathbb{R},$$  
(2.1)

so that

$$T_a f(x) = \int_{\mathbb{R}^n} a(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$  
(2.2)

The following result is the singular integral realization of a pseudo-differential operator, whose proof can be found in [8] Chapter 4.

**Theorem A3.** Using the above notation (2.1)-(2.2) one has that

$$T_a f(x) = \int_{\mathbb{R}^n} k(x, x - y) f(y) dy, \text{ if } x \notin \text{supp}(f)$$  
(2.3)
where $k \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n - \{0\})$ satisfies:

$$|\partial^\alpha_x \partial^\beta_z k(x, z)| \leq A_{\alpha, \beta, N, \delta} |z|^{-(n+m+|\beta|+N)}, \quad |z| \geq \delta,$$

if $n + m + |\beta| + N > 0$ uniformly in $x \in \mathbb{R}^n$. \hfill (2.4)

To simplify the exposition from now on we restrict ourselves to the one-dimensional case $x \in \mathbb{R}$ where in the next section these results will be applied.

As a direct consequence of Theorem A3 one has:

**Corollary 2.1.** Let $m \in \mathbb{Z}^+$ and $l \in \mathbb{R}$. If $g \in L^2(\mathbb{R})$ and $f \in L^p(\mathbb{R})$, $p \in [2, \infty]$, with
distance(supp$(f)$; supp$(g)$) $\geq \delta > 0$,

then

$$\|f \partial^m x^l g\|_2 \leq c\|f\|_p \|g\|_2.$$ \hfill (2.5)

Next, let $\theta_j \in C_0^\infty(\mathbb{R}) - \{0\}$ with $\theta'_j \in C_0^\infty(\mathbb{R})$ for $j = 1, 2$ and
distance(supp$(1 - \theta_1)$; supp$(\theta_2)$) $\geq \delta > 0$. \hfill (2.6)

**Lemma 2.2.** Let $f \in H^s(\mathbb{R})$, $s < 0$, and $T_\alpha$ be a pseudo-differential operator of order zero ($\alpha \in S^0$). If $\theta_1 f \in L^2(\mathbb{R})$, then

$$\theta_2 T_\alpha f \in L^2(\mathbb{R}).$$ \hfill (2.7)

**Proof of Lemma 2.2.** Since

$$\theta_2 T_\alpha f = \theta_2 T_\alpha (\theta_1 f) + \theta_2 T_\alpha ((1 - \theta_1) f),$$

combining the hypothesis and the continuity of $T_\alpha$ in $L^2(\mathbb{R})$ it follows that $\theta_2 T_\alpha (\theta_1 f) \in L^2(\mathbb{R})$). Also

$$\theta_2(x) T_\alpha ((1 - \theta_1) f)(x)$$

$$= \int_{-\infty}^{\infty} \theta_2(x) a(x, \xi)((1 - \theta_1) f)(\xi) e^{2\pi i x \xi} d\xi$$

$$= \int (\int \theta_2(x) a(x, \xi_1 + \xi_2)(1 - \theta_1)(\xi_1) e^{2\pi i x \xi_1} d\xi_1)\hat{f}(\xi_2) e^{2\pi i x \xi_2} d\xi_2$$

$$= T_b f(x) = \int \theta_2(x) k(x, x - z)(1 - \theta_1(z)) f(z) dz$$

$$= \int \theta_2(x) k(x, x - z)(1 - \theta_1(z)) J^{2m} f(z) dz$$

with $-2m < s$, $m \in \mathbb{Z}^+$ and $k(\cdot, \cdot)$ as in (2.4), so integration by parts and Theorem A3 yield the desired result.
Proposition 2.3. Let \( f \in L^2(\mathbb{R}) \) and
\[
J^s f = (1 - \partial_x^2)^{s/2} f \in L^2(\{x > a\}) \quad s > 0,
\]
then for any \( \epsilon > 0 \) and any \( r \in (0, s] \)
\[
J^r f \in L^2(\{x > a + \epsilon\}). \quad (2.9)
\]

Proof of Proposition 2.3. Define
\[
g = J^s f \in L^2(\{x > a\}),
\]
thus \( J^s f \in H^{-s}(\mathbb{R}) \). Let \( \theta_j \in C^\infty(\mathbb{R}), j = 1, 2 \), with \( \theta_1(x) = 1 \) for \( x \geq a + \epsilon/4 \), \( \text{supp} \theta_1 \subset \{x > a\} \), and \( \theta_2(x) = 1 \) for \( x \geq a + \epsilon \) and \( \text{supp} \theta_2 \subset \{x > a + \epsilon/2\} \), therefore \( \theta_1 g \in L^2(\mathbb{R}) \). Let \( T = J^{i\beta}, \beta \in \mathbb{R} \).

By Lemma 2.2
\[
\theta_2 T fg = \theta_2 J^{s+i\beta} f \in L^2(\mathbb{R}),
\]
and since \( f \in L^2(\mathbb{R}) \)
\[
\theta_2 J^{s+i\beta} f \in L^2(\mathbb{R}).
\]
Hence, by the Three Lines Theorem it follows that
\[
\theta_2 J^z f \in L^2(\mathbb{R}), \quad z = r + i\beta, \quad r \in [0, s], \quad \beta \in \mathbb{R},
\]
which completes the proof.

Remark 2.4. In a similar manner one has: for \( \epsilon > 0 \) let \( \varphi_\epsilon \in C^\infty(\mathbb{R}) \) with \( \varphi_\epsilon(x) = 1, \ x \geq \epsilon, \ \text{supp} \varphi_\epsilon \subset \{x > \epsilon/2\} \) and \( \varphi'_\epsilon(x) \geq 0 \). Then
(I) If \( m \in \mathbb{Z}^+ \) and \( \varphi_\epsilon J^m f \in L^2(\mathbb{R}) \), then \( \forall \epsilon' > 2\epsilon \)
\[
\varphi_{\epsilon'} \partial^j_x f \in L^2(\mathbb{R}), \quad j = 0, 1, ..., m.
\]
(II) If \( m \in \mathbb{Z}^+ \) and \( \varphi_\epsilon \partial^j_x f \in L^2(\mathbb{R}) \), \( j = 0, 1, ..., m \), then \( \forall \epsilon' > 2\epsilon \)
\[
\varphi_{\epsilon'} J^m f \in L^2(\mathbb{R}).
\]
(III) If \( s > 0 \), and \( J^s(\varphi_\epsilon f), f \in L^2(\mathbb{R}) \), then \( \forall \epsilon' > 2\epsilon \)
\[
\varphi_{\epsilon'} J^s f \in L^2(\mathbb{R}).
\]
(IV) If \( s > 0 \), and \( \varphi_\epsilon J^s f, f \in L^2(\mathbb{R}) \), then \( \forall \epsilon' > 2\epsilon \)
\[
J^s(\varphi_{\epsilon'} f) \in L^2(\mathbb{R}).
\]
The same results hold with \( \theta_1, \theta_2 \), as in (2.6), instead of \( \chi_\epsilon, \chi'_{\epsilon'} \).

Next, we recall some inequalities obtained in [5] :

Theorem A4 ([5]). If \( s > 0 \) and \( p \in (1, \infty) \), then
\[
\| J^s(fg) \|_p \leq c(\|f\|_\infty \|J^s g\|_p + \|J^s f\|_p \|g\|_\infty), \quad (2.10)
\]
and

\[ \| [J^s; f]g \|_p = \| J^s(fg) - fJ^s g \|_p \leq c(\| \partial f \|_{\infty} \| J^{s-1}g \|_p + \| J^s f \|_p \| g \|_{\infty}). \] (2.11)

Also we shall use the following elementary estimate whose proof is similar to that found in [2], Chapter 6.

**Lemma 2.** Let \( \phi \in C^\infty(\mathbb{R}) \) with \( \phi' \in C^\infty_0(\mathbb{R}) \). If \( s \in \mathbb{R} \), then for any \( l > |s - 1| + 1/2 \)

\[ \| [J^s; \phi] f \|_2 + \| [J^{s-1}; \phi] \partial_x f \|_2 \leq c \| J^l \phi' \|_2 \| J^{s-1} f \|_2. \] (2.12)

3. **Proof of Theorem 1.1**

Without loss of generality \( x_0 = 0 \). For \( \epsilon > 0 \) and \( b \geq 5\epsilon \) define the families of functions

\[ \chi_{\epsilon,b}, \phi_{\epsilon,b}, \bar{\phi}_{\epsilon,b}, \psi_{\epsilon} \in C^\infty(\mathbb{R}), \]

with \( \chi'_{\epsilon,b} \geq 0, \chi_{\epsilon,b}(x) = 0, \ x \leq \epsilon, \chi_{\epsilon,b}(x) = 1, \ x \geq b, \)

\[ \chi'_{\epsilon,b}(x) \geq \frac{1}{10(b - \epsilon)} 1_{[2\epsilon, b-2\epsilon]}(x), \]

\[ \text{supp}(\psi_{\epsilon,b}), \text{supp}(\bar{\psi}_{\epsilon,b}) \subset [\epsilon/4, b], \]

\[ \phi_{\epsilon,b}(x) = \bar{\phi}_{\epsilon,b}(x) = 1, \ x \in [\epsilon/2, \epsilon], \]

\[ \text{supp}(\psi_{\epsilon}) \subset (-\infty, \epsilon/2] \]

\[ \chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_{\epsilon}(x) = 1, \ x \in \mathbb{R}, \]

\[ \chi^2_{\epsilon,b}(x) + \bar{\phi}^2_{\epsilon,b}(x) + \psi_{\epsilon}(x) = 1, \ x \in \mathbb{R}. \]

Hence,

\[ \text{distance}(\text{supp}(\chi_{\epsilon,b}); \text{supp}(\psi_{\epsilon})) \geq \epsilon/2. \]

Formally, we apply the operator \( J^s \) to the equation in (1.1) and multiply by \( J^s u \chi^2_{\epsilon}(x + vt) \) to obtain after integration by parts the
identity
\[
\frac{1}{2} \frac{d}{dt} \int (J^s u)^2(x, t) \chi^2(x + vt) \, dx
\]
\[
- v \int (J^s u)^2(x, t) \chi \chi'(x + vt) \, dx
\]
\[
+ \frac{3}{2} \int (\partial_x J^s u)^2(x, t) \chi \chi'(x + vt) \, dx
\]
\[
- \frac{1}{2} \int (J^s u)^2(x, t) \partial_x^3(\chi^2(x + vt)) \, dx
\]
\[
+ \int J^s(u \partial_x u) J^s u(x, t) \chi^2(x + vt) \, dx = 0
\]
(3.2)

where in \( \chi \) the index \( \epsilon, b \) were omitted. We shall do that now on.

Case: \( s \in (3/4, 1) \).

First, we observe that combining (1.2) and the results in section 2 it follows that for any \( R > 0 \)
\[
\int_0^T \int_{-R}^R |J^r u(x, t)|^2 \, dx \, dt < \infty \quad \forall r \in [0, 7/4^+]. \quad (3.3)
\]
Thus, after integration in time the terms \( A_1 \) and \( A_2 \) in (3.2) are bounded. So it only remains to handle \( A_3 \).

Thus, \( J^s(u \partial_x u) \chi = J^s(u \partial_x u \chi) - [J^s; \chi](u \partial_x u) \)
\[
= u \chi J^s \partial_x u + [J^s; u \chi] \partial_x u - [J^s; \chi](u \partial_x u)
\]
\[
= u \chi J^s \partial_x u + [J^s; u \chi] \partial_x (u(\chi + \phi + \psi)) - [J^s; \chi](u \partial_x u)
\]
\[
= B_1 + B_2 + B_3 + B_4 + B_5. \quad (3.4)
\]
Inserting \( B_1 \) in (3.2) one obtains a term which can be estimated by integration by parts, Gronwall’s inequality and (1.2). Using (2.11) it follows that
\[
\|B_2\|_2 \leq c \|\partial_x(u \chi)\|_\infty \|J^s(u \chi)\|_2, \quad (3.5)
\]
and
\[
\|B_3\|_2 \leq c(\|\partial_x(u \chi)\|_\infty \|J^s(u \phi)\|_2 + \|\partial_x(u \phi)\|_\infty \|J^s(u \chi)\|_2). \quad (3.6)
\]
To bound $B_4$ and $B_5$ we apply Corollary 2.1 and (2.12), respectively, to get
\[
\|B_4\|_2 = \|u \chi J^s \partial_x (u \psi)\|_2 \leq c\|u\|_\infty \|u\|_2 \quad \text{(3.7)}
\]
and
\[
\|B_5\|_2 \leq c\|u\|_\infty \|u\|_2. \quad \text{(3.8)}
\]
Collecting the above information (3.4)-(3.8) in (3.2) we obtain (1.7) for any $r \in (3/4, 1)$, $\nu > 0$ and $\epsilon > 0$, and that for any $\nu > 0$, $\epsilon > 0$,
\[
\int_0^T \int_{\epsilon - \nu t}^{R - \nu t} (J^s \partial_x u)^2 \, dx \, dt < \infty,
\]
from which using the results, Remark 2.4, one obtains (1.8).

Case: $s \in (m, m + 1)$, $m \in \mathbb{Z}^+$. We assume (1.7) and (1.8) with $s \leq m$. Hence, from the results in section it follows that for any $\epsilon > 0$, $R > 0$ and $r \in [0, m]$
\[
\int_0^T \int_{\epsilon - \nu t}^{R - \nu t} (J^r \partial_x u)^2 \, dx \, dt < \infty. \quad \text{(3.9)}
\]
Again the starting point is the energy estimate identity (3.2). After integrating in time the terms $A_1$ and $A_2$ can be easily bounded using (3.9). So it suffices to consider $A_3$. Thus, using the notation introduced in (3.1) we have
\[
\chi J^s (u \partial_x u) = J^s (u \chi \partial_x u) - \frac{1}{2} [J^s; \chi] \partial_x (u^2)
\]
\[
= u \chi J^s \partial_x u + [J^s; u \chi] \partial_x u - \frac{1}{2} [J^s; \chi] \partial_x (u^2)
\]
\[
= u \chi J^s \partial_x u + [J^s; u \chi] \partial_x (u (\chi + \phi + \psi))
\]
\[
- \frac{1}{2} [J^s; \chi] \partial_x ((u^2) (\chi^2 + (\tilde{\phi})^2 + \psi))
\]
\[
= E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7. \quad \text{(3.10)}
\]
Inserting $E_1$ in (3.2) one obtains a term which can be estimated by integration by parts, Gronwall’s inequality and (1.2). From (2.11) we see that
\[
\|E_2\|_2 \leq c \|\partial_x (u \chi)\|_\infty \|J^s (u \chi)\|_2 \quad \text{(3.11)}
\]
and
\[
\|E_3\|_2 \leq c(\|\partial_x (u \chi)\|_\infty \|J^s (u \phi)\|_2 + \|\partial_x (u \phi)\|_\infty \|J^s (u \chi)\|_2). \quad \text{(3.12)}
\]
For $E_4$ it follows that from Corollary 2.1 that
\[
\|E_4\|_2 = \|u \chi J^s \partial_x (u \psi)\|_2 \leq c\|u\|_\infty \|u\|_2. \quad \text{(3.13)}
\]
For $E_5$ and $E_6$ a combination of (2.10) and (2.12) yields the estimates
\[
\|E_5\|_2 \leq \|[J^s; \chi] \partial_x ((u\chi)^2)\|_2 \leq c\|J^s ((u\chi)^2)\|_2 \leq c\|u\|_\infty \|J^s (u\chi)\|_2,
\]
(3.14)
and
\[
\|E_6\|_2 \leq \|[J^s; \chi] \partial_x ((u\tilde{\phi})^2)\|_2 = \|J^s ((u\tilde{\phi})^2)\|_2 \leq c\|u\|_\infty \|J^s (u\tilde{\phi})\|_2.
\]
(3.15)

Finally, using Corollary 2.1 we write
\[
\|E_7\|_2 \leq \|[J^s; \chi] \partial_x ((u^2 \psi))\|_2 = \|\chi J^s \partial_x (u^2 \psi)\|_2 \leq c\|u\|_\infty \|u\|_2.
\]
(3.16)

To complete the estimates in (3.11), (3.12), (3.14) and (3.15) we observe that
\[
J^s (u\chi) = J^s u\chi + [J^s; \chi] (u(\chi + \phi + \psi)) = G_1 + G_2,
\]
where $G_1$ is the term whose $L^2$-norm we are estimating and $G_2$ is of lower order, (hence bounded by assumption), and $\|J^2 (u\phi)\|_2$ is bounded by (1.8) (assumption).

Collecting the above information in (3.2) we obtain the desired result.

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