Efficient 5-axis CNC trochoidal flank milling of 3D cavities using custom-shaped cutting tools

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Abstract

A novel method for trochoidal flank milling of 3D cavities bounded by free-form surfaces is proposed. Existing 3D trochoidal milling methods use on-market milling tools whose shape is typically cylindrical or conical, and is therefore not well-suited for meeting fine milling tolerances required for finishing of benchmark free-form surfaces like blades or blisks. In contrast, our variational framework incorporates the shape of the tool into the optimization cycle and looks not only for the trochoidal milling paths, but also for the shape of the tool itself. High precision quality is ensured by firstly designing flank milling paths for the side surfaces that delimit the motion space, in which the trochoidal milling paths are further computed. Additionally, the material removal rate is maximized with the cutter-workpiece engagement being constrained under a given tolerance. Our framework also supports multi-layer approach that is necessary to handle deep cavities. The ability and efficacy of the proposed method are demonstrated by several industrial benchmarks, showing that our approach meets fine machining tolerances using only a few trochoidal paths.

Keywords: 5-axis CNC machining, trochoidal milling, custom-shaped tools, roughing operations, tangential movability, free-form shape manufacturing

1. Introduction

Efficient manufacturing of curved objects is an essential step for many industrial sectors, automotive or aeronautical to name a few. Even though additive technologies like 3D printing are becoming more and more popular [1], there are objects that need to be, e.g. for stiffness reasons, manufactured from a single material block using traditional subtractive approaches and Computer Numerically Controlled (CNC) machining is the leading subtractive technology [2–8].

In conventional flank milling, the cutter is desired to be in tangential contact with the material block. This fact induces excessive cutting force but also accumulates heat and negatively affects the tool wear. This problem can be effectively avoided by trochoidal milling. A trochoidal milling path consists of two parts, the front-half path and the back-half path. Material is cut in the front-half path and the cutting heat is dissipated as the tool moves in the back-half path [12]. Therefore, in high-speed machining, trochoidal milling is increasingly used in slot milling and is also widely used in cutting hard materials, such as NiTi-based super alloy [13–15].

In this paper, we propose a novel method for efficient 5-axis CNC milling of 3D cavities bounded by free-form surfaces using custom-shaped cutting tools. The method is based on a variational framework that includes the shape of the tool in the optimization cycle.

In Fig. 1, we show an example of a blisk disc and the cavity between two blades of the blisk, delimited by two free-form surfaces. The side surface $S_1$ corresponds to the left (right) side of the cavity (rendered in green). Two positions of the cutting tool during one trochoidal cycle are shown. The path of the tool axis consists of two parts: the front-half path (red) and the back-half path (blue).

Suppose a cavity is delimited by two surfaces $S_1$ and $S_2$, see Fig. 1(b). Our purpose is to determine trochoidal milling paths as well as the shape of a milling tool such that high precision milling is obtained. The cutting tools we consider are not re-
stricted to cylindrical and/or conical, but are general surfaces of revolution. Important machining factors such as Material Removal Rate (MRR) and Cutter-Workpiece Engagement (CWE) are both incorporated in a unified variational framework. While high MRR is correlated with efficient milling, it is in contradiction with CWE, whose high values are closely related to the tool wear, and therefore required to be small. Our optimization framework looks for high MMR while keeping CWE under a given threshold. We refer the reader to [16] for more details on MRR and CWE.

A complete tool path of trochoidal milling is defined by the motion of tool axis which is a ruled surface. This ruled surface is composed of a sequence of cyclic path segments (TR cycles), and each TR cycle contains a front-half and a back-half path, see Fig. 1(b). The front-half path is also called an active part, along which the cutting tool keeps contacting with the raw material throughout the entire milling, while the back-half path is a transition path from current TR cycle to the next TR cycle. Typically, the cutting tool along a back-half path does not interact with the raw material at most of its positions.

For certain geometries, it is possible to mill the side surface with one single sweep using a large tool. However, for deep cavities using a single sweep, even of a custom-shaped tool, may not be sufficient to meet the fine machining tolerances. Therefore, one typically has to consider multi-layer milling approaches that allow the tool to move in several consecutive layers. In this work, we consider both, a single-layer milling for path generation of a cavity bounded by two surfaces in Sec. 3, and generalize it to multi-layer milling in Sec. 4.

In a summary, the problem to address in the TR path generation is to decide the front-half of every TR cycle such as key constraints are satisfied: 1) the motion of the tool along the path should be tangent to both sides $S_1$, $S_2$ and overcut should be avoided; 2) the MRR should be maximized; 3) the CWE is under a given tolerance. The last requirement comes from the observation that CWE is closely related to the cutting force during manufacturing, therefore, in order to reduce the tool wear, the cutter-workpiece engagement should be constrained to be under a given tolerance. The rest of the paper is organized as follows. Section 2 surveys the related research. Section 3 introduces the algorithm to compute single layer TR paths and Section 4 generalizes it to the multi-layer setup. The numerical examples are shown in Section 5 and the conclusions are drawn in Section 6.

2. Related Work

2.1. The cutting force and stability of trochoidal milling

The tool wear in trochoidal milling is strongly correlated to the cutting force, and there have been a lot of studies on cutting force analysis. Otkur et al. [17] propose a comprehensive analytical model to analyze the tool-workpiece engagement and predict the cutting force. Pleta et al. [13] find that the engagement angle has the highest correspondence with the component perpendicular to the feeding direction of the cutting force. The relationship between the cutting force and the cutting depth in trochoidal milling is further analyzed in [18]. Wu et al. [19] propose an improved model for analyzing the change of cutting force based on the typical linear milling force model. Niaki et al. [20] analyze the geometry of the in-process workpiece in TR milling in details and give a more accurate cutting force prediction model.

In addition to cutting force, stability is also particularly important in high-speed machining. Kardes et al. [21] analyze the property with the varying cutter immersion condition for suppressing the chatter during TR milling. Yan et al. [22] build a TR milling process stability prediction model. They consider the trochoidal step distance and the spindle speed for analyzing the stability of TR milling. Wang et al. [23] present an adaptive TR tool path generation method, in which machining stability is improved by maintaining the steady radial cutting depth.

2.2. Trochoidal milling of free-form surface

Traditional trochoidal milling methods mainly focus on the cutting force and milling stability of a cutting tool, and are usually used in simple slot milling. In recent years, some works considered the use trochoidal milling for curved slots or even 3D cavities. Xu et al. [24] propose a method based on polynomial curves, which realize the trochoidal machining of arbitrarily curved slot with constant width. They replace the traditional circular paths with polynomial curves as the base TR cycle to fit into the complex curved slots. Li et al.[25] propose an extension of [24], which supports curved slots with varying width by adjusting the polynomial curves. Even though these works can support more complex curved slots, they do not support general free-form cavities. Cavities bounded by free-form surfaces (such as a blisk groove in aerospace industry) are usually manufactured with traditional methods [26] which do not have the benefit of trochoidal milling.

To the best of our knowledge, the closest research to our work is on trochoidal milling of general free-form cavities by Li et al. [16]. They use a tangent sphere to generate the middle surface to design a guiding curve [27] of a TR path. In the latest work, they [28] propose an algorithm of variable-depth multi-layer 5-axis trochoidal milling. By introducing the concept of layered material removal rate (LMRR), the TR milling depth of each layer is optimized. These works apply trochoidal milling to free-form surfaces, but they target rough or semi-finishing operations and the shape of the cutting tool is a fixed input. In contrast, our approach looks also for the optimal shape of the tool.

2.3. High precision 5-axis flank milling

There are a lot of works devoted to high-precision 5-axis flank milling [9, 29–35]. The shape and size of the milling tool are mostly fixed (usually cylindrical or conical [36]). Wang et al. propose a method to compute a composition of discrete ruled surfaces fitting to a given shape using the dynamic programming [37]. Elber et al. [38] approximate general free-form surface with segmented ruled surface by using a truncated conical milling tool. However, Elber et al.’s method requires a lot of subdivisions to well approximate a general free-form shape.
Redonnet et al. [39] use a cylindrical cutter for machining of ruled surfaces. They propose a three-tangential arrangements method to optimize the cutter position, which gets high precision compared with standard two-tangential arrangements.

In addition, from the general cutting tool, various methods have been proposed. Senatore et al. [40] analyze the size of a cylindrical cutter, which maximizes the radius while keeping the predefined geometric error. Zhu et al. [41] propose a method based on simultaneous optimization of the tool’s motion and shape. Based on this work, Lu et al. [42] consider additional constraints such as the stiffness of a cutter. These two methods focus on improving the stiffness to reduce the deflection and vibration of the tool. Bo et al. [43] propose an alternative formula of the optimization where the shape and motion of tool are both the unknowns and are simultaneously optimized to minimize the approximation error.

3. TR paths for cavities bounded by free-form surfaces

Our research focuses on 3D cavities like the one shown in Fig. 1(b), that typically consist of a bottom surface and two side surfaces. The bottom surface of the cavity is removed from our considerations as one cannot access it by flank-milling due to global collision anyway. We focus on the side surfaces which are to-be-milled by one (or several) sweeps of a single custom-shaped tool. In trochoidal milling, the cutting tool touches the side surfaces only at some discrete positions, called contact lines. The whole trochoidal path is divided by the contact lines into a sequence of TR cycles. Each TR cycle has a front-half part and a back-half part separated by the contact lines. The contact lines act as extreme positions of the motion space of the cutting tool and the positions of contact lines de facto govern the milling precision of the side surfaces.

It is therefore essential to find accurate positions of contact lines, with higher priority than other (intermediate) positions. Moreover, the motion directions of the cutting tool at the contact lines are also directly related to the finishing quality. However, it is non-trivial to define some contact lines in advance without considering the whole TR paths for milling 3D cavity. To achieve higher milling precision, we consider a variable (custom-shaped) tool which is represented as a one-parameter family of spheres centered on the axis, and whose radii are optimized in our framework. The position of the contact lines, as well as the TR paths inside cavities, are determined using a variational algorithm.

Firstly, the shape of the tool and its paths in the neighborhood of the two reference surfaces are computed. The paths are two ruled surfaces, traversed by the axis of the cutting tool, see Fig. 3(a). These two surfaces delimit the space that is further trochoidal-milled (Sec. 3.2). Secondly, we compute the front-half of the TR cycles one by one (Sec. 3.3) such that path planning will move the tool towards the two limiting surfaces with $G^1$ continuity, maximizing the MRR while satisfying the CWE constraints. Finally, we generate the back-half paths as a transition between the front-half paths to generate the entire TR cycles (Sec. 3.4).

3.1. Envelope surface fitting

Our objective is to compute an optimal shape of a cutting tool $T$ and its motion paths, i.e., the side paths for flank milling the side surfaces $S_1$ and $S_2$, recall Fig. 1. This goal is realized by an envelope surface fitting method which proceeds as follows. The motion of the axis of the cutting tool is a ruled surface $R$ which is defined as

$$R(s,t) = q^T(t) \cdot (1-s) + q^B(t) \cdot s, \quad s,t \in [0,1],$$

where $q^T(t)$ is the top boundary curve and $q^B(t)$ is the bottom boundary curve, which are both represented by B-spline curves in our work, see Fig. 2. In the following, we review the optimization method for computing $R$.

It was shown that one can approximate a single free-form surface by a motion of a custom-shaped tool [43]. In our current setup, the situation is a more complicated as we have two references surfaces, not just one. The cutting tool, a surface of revolution, is conceptualized as a one-parameter family of spheres centered along the tool axis. The behavior of the spheres is described by a radial function $r(s)$, $s$ being the arc length parametrization of the axis. The radial function can be thought of a smooth function, that describes the shape of the tool, but in our discrete optimization-based setup is represented by a discrete set of radii, see Fig. 2(b).

The motion path is obtained by minimizing the distance between the envelope surface of the tool along its paths and the target surfaces $S$. Due to the nonlinear distance function in surface approximation, an iterative procedure is employed and in each iteration a quadratic approximant function is minimized. We have

$$F_{\text{dist}}^1(q^T, q^B, r) = \frac{1}{MN} \sum_k \sum_l ||R(s_k,l) - f_{k,l} - r_k \cdot n_{k,l}||^2,$$
where \( f_{k,l} \) is the footprint of \( R(t_k, s_l) \) on the side surface \( S \) as shown in Fig. 4, \( q_{k,l}^x \) is the unit surface normal of \( S \) at \( f_{k,l} \) pointing to the machining side of \( S \), and \( M \) and \( N \) are the number of samples in \( s \) and in \( t \) directions, respectively. The optimization variables are the control points of the boundary curves \( q^T \) and \( q^B \) and a vector of radii \( r = (r_1, \ldots, r_N) \). The stability of the cutting tool during its motion is closely related to the acceleration of the tool’s motion, which can be expressed by the fairness of the motion. To control the fairness, we use a standard fairness term defined by the integral of norm of the second derivatives of the boundary curves, enriched by a fairness on the direction of the ruling (bottom term in (3)). Note that two fair rail curves do not imply a fair motion as the line between them may parse in a non-fair fashion. Therefore, we write

\[
F_{fair}^1(q^T, q^B) = f \|((q^T)'')^2(t)\|dt + f \|((q^B)'')^2(t)\|dt + \int f \|((q^T)'')^{(1)}(t) - (q^B)'')^{(1)}(t)\|dt.
\]

(3)

In addition, a rigid motion of the cutting tool is required which is guaranteed by a constraint on constant length of the tool axis, denoted by \( L \). This constraint has to be satisfied in every time instant, which in our implementation is controlled at \( N \) discrete positions, i.e., we write

\[
F_{rigid}^1 = \sum_{t} \left(\|q^T(t) - q^B(t)\|^2 - L^2\right)^2.
\]

(4)

The envelope fitting algorithm minimizes the following function in each iteration

\[
F_{prox}(q^T, q^B, r) = F_{dist} + \lambda_1 F_{fair}^1 + \lambda_2 F_{rigid}^1 \rightarrow \min
\]

where the optimization unknowns are the control points of the boundary curves of the ruled surfaces \( R \) and a vector of radii \( r \).

3.2 Side path computation for trochoidal milling with controlled overcutting

Our distance objective term \( F_{dist} \) as defined in (2) minimizes distances to \( S \) in the least square sense, which naturally results in errors with both positive and negative signs, and therefore overcutting. To penalize overcutting, we introduce a virtual surface of the side surface as a target, defined by

\[
\tilde{S}(s, t) = S(s, t) + \alpha(s, t) \cdot n(s, t)
\]

(6)

where \( n \) indicates the surface normal pointing to the inside of cavity, and \( \alpha(s, t) \) is a bivariate function that reflects the approximation error from the first optimization cycle achieved by Eq. (2). If \( \alpha(s, t) \equiv \text{const.} \), \( \tilde{S} \) would be an offset surface of \( S \). However, as \( \alpha(s, t) \) varies, see Fig. 3(b), the overcut compensation has to reflect this variance. In our discrete setup, the distance objective function is a sum of distance constraints over the set of samples of \( R(s, t) \), and becomes

\[
F_{dist}^2(q^T, q^B, r) = \frac{1}{MN} \sum_{k=1}^{M} \sum_{l=1}^{N} \|R(s, t) - (f_{k,l} + r_k \cdot n_{k,l} + \alpha_{k,l} \cdot n_{k,l})\|^2,
\]

(7)

where \( \alpha_{k,l} \) are the fitting errors at the sample point \( R(s, t) \) from the first optimization cycle. Observe that the optimization in Eq. (2) returns spheres (transparent) with radii \( r_k \) that, in the least square sense, approximate best the distances \( \|R(s, t) - f_{k,l}\| \). This results in defect (signed) distances \( \alpha_{k,l} \) which corresponds to overcut or undercut, and is corrected by incorporating \( \alpha_{k,l} \) in (7). Note that \( \alpha_{k,l} \) changes throughout optimization iterations, and it is therefore updated dynamically in each iteration in our framework.

It is also desirable that the same tool is used for both side surfaces. Therefore, when the shape of the cutting tool is considered as a variable, both tool paths \( R_1 \) and \( R_2 \) are optimized simultaneously with a single cutting tool. That is, the sphere radii \( r_k \) are the same for both \( R_1 \) and \( R_2 \), recall Fig. 3(a). The final objective distance function is

\[
F_{dist}(q^T, q^B, r) = \frac{1}{MN} \sum_{n=1}^{M} \sum_{l=1}^{N} \|R_n(s, t) - (f_{k,l} + r_k \cdot n_{k,l} + \alpha_{k,l} \cdot n_{k,l})\|^2,
\]

(8)

where \( \star \equiv T, B \), and the subscript \( n = 1, 2 \) indicates the left or the right side surface of the cavity. Similarly, energy terms of motion fairness and rigidity are defined respectively by

\[
F_{fair}(q^T) = \sum_{j=1,2} \int f \|((q^T)'')^2(t)\|dt + f \|((q^T)'')^2(t)\|dt + \int f \|((q^T)'')^{(1)}(t) - (q^B)'')^{(1)}(t)\|dt
\]

and

\[
F_{rigid}(q^T) = \sum_{j=1,2} \sum_{l=1}^{N} \left(\|q^T(t) - q^B(t)\|^2 - L^2\right)^2,
\]

(9)

(10)

where again \( \star \equiv T, B \), and \( j = 1, 2 \).

In summary, the computation of the tool shape as well as the tool paths close to \( S_1 \) and \( S_2 \) is done by iteratively minimizing the objective function

\[
\min_{\mathcal{P}, \mathfrak{R}} F_{dist} + \lambda_1 F_{fair} + \lambda_2 F_{rigid},
\]

(11)

where \( \mathcal{P} \) is a set of control points of the two ruled surfaces \( R_1 \) and \( R_2 \), \( \mathfrak{R} \) is the set of sphere radii uniformly distributed along the tool axis, and \( \lambda_1 \) and \( \lambda_2 \) are scalar weights set
empirically. Unless stated differently in a concrete example, we set \( \lambda_1 = 10^{-6} \) and \( \lambda_2 = 1 \) in our implementation. We solve the objective function in Eq. (11) with the Gauss-Newton method for the control points of \( R_1 \) and \( R_2 \), and the radii values \( r_k \). Note that the footpoints \( f_{k,t} \) and associated surface normals \( n_{k,t} \) on the target side surfaces are updated in each iteration. The solution of Eq. (11) results in two side paths \( R_1 \) and \( R_2 \) milling the side surfaces \( S_1 \) and \( S_2 \), respectively, and an optimal shape of the cutting tool represented by the vector of radii \( r \). In the following discussions, the tool shape (vector \( r \)) is fixed.

**Remark 1.** When there is no risk of confusion, from this point on, we omit the variables of the objective functions and list them as sets in the subscript of the minimization symbol as in Eq. (11).

### 3.3. Front-half paths computation

At this point, our algorithm computed two side paths \( R_1 \) and \( R_2 \) and a shape of the tool represented by the vector of radii \( r \). Now we are going to construct trochoidal paths that join \( R_1 \) with \( R_2 \), and start with the front-half paths.

Similarly to the side paths, we use B-spline surfaces for their representation. A front-half of a TR cycle, denoted by \( C_i \), is defined by

\[
C_i(s,t) = c^T_i(t) \cdot (1-s) + c^B_i(s) \cdot s, \quad s,t \in [0,1],
\]

with the boundary curves of \( C_i \) defined as

\[
c^T_i(t) = \sum_{k=0}^{m} a^*_{k,t} B_{k,d}(t), \quad * \in \{ T, B \},
\]

where \( a^*_{k,t} \) are the control points and \( B_{k,d}(t) \) the B-spline basis functions of degree \( d \). See Fig. 5 for two guiding paths \( (R_1, R_2) \) and two front-half paths. Since we need to interpolate boundary rulings, we use clamped uniform knot vectors for the B-spline surfaces. If not said differently, we use \( m+1 \) control points, \( m = 5 \) in our implementation.

The computation of the front-half paths should meet the following objectives: milling precision and milling efficiency. For milling precision, the cutting tool should touch the side surfaces with high degree of precision and the overcut to the side surface should meet fine manufacturing tolerances (which are tens of micrometers for objects with tens of centimeters large.) For milling efficiency, the MRR in each TR cycle should be maximized, while keeping the CWE under a pre-defined threshold. We emphasize that the CWE is closely related to physical entities such as cutting force, work load, or tool wear. We do not optimize directly these entities, as this goes beyond the scope of this paper, but we control CWE.

**Remark 2.** We consider several types of milling paths in our algorithm. They are all represented by ruled surfaces (motions of the milling axis) and parametrized by two variables: \( s \) (ruling direction) and \( t \) (time aka motion direction). If there is no risk of confusion, we use the same pair of parameter symbols \((s,t)\) for all these ruled surfaces, but if there are more surfaces involved, each surface has its own parameter domain and these variables are different.

Figure 5: Front-half paths of TR cycles. (a) \( C_i \) and \( C_{i+1} \) represent the front-half paths of ruled surfaces that join the side ruled surfaces, \( R_1 \) and \( R_2 \), in \( G^1 \) fashion. (b) This is achieved via the constraints (14) and (16), which is expressed in the terms of the control points of \( C_i \) and \( R_{1,2} \).

### 3.3.1. Milling precision

The side paths \( R_1 \), and \( R_2 \), obtained using the method described in Sec. 3.2, provide essential guiding information for computing \( C_i(s,t) \) which is required to touch the side surfaces with high degree of precision with overcut control. The rulings of the side paths define the position and orientation of the cutting tool moving along the side paths. Therefore, it is desired that the front-half path segments (and also the back-half path segments) interpolate some specific rulings of the side ruled surfaces. Moreover, at the contact lines, the motion direction of points on the tool axis should be parallel to the instantaneous motion direction of the same points on the side paths. Therefore we speak about \( G^1 \) interpolation of certain rulings of \( R_1 \) and \( R_2 \). In addition to these boundary constraints, the whole front-half path should stay inside the space enclosed by the side paths. The considerations on milling precision lead to the following specific constraints in the representation of B-spline surfaces.

**Position constraints (\( G^0 \) constraints).** The end rulings of \( C_i(s,t) \), which define the tool positions at both ends of the half path, are required to be some specific rulings of the side paths \( R_1(s,t) \) or \( R_2(s,t) \), i.e., \( C_i(s,0) = R_i(s,t_1) \), \( s \in [0,1] \) and \( C_i(s,1) = R_i(s,t_2) \), \( s \in [0,1] \). Using the clamped knot vector in the B-spline surface \( C_i(s,t) \), this is easily met if the end control line of \( C_i(s,t) \) connects two points lying on the top and bottom boundary curves of the side path at the same parameter \( t \), respectively, see Fig. 6 for an illustration. In particular, we have

\[
\begin{align*}
\hat{c}^{1}_{pos} &= a_0^* - q_0^T(t_1) = 0, \\
\hat{c}^{2}_{pos} &= a_m^* - q_0^T(t_2) = 0,
\end{align*}
\]
where \( * \in \{T, B\} \) with \( T \) and \( B \) indicate the top curve and the bottom curve of \( C_i \), respectively. Note that the parameters on \( R_1 \) and \( R_2 \) are in general not identical, i.e., \( i1 \neq i2 \). The parameters \( \mathcal{T} = \{t_{i1}, t_{i2}\} \) are treated as unknowns in our optimization and their computation will be discussed later in Sec. 3.3.3.

Motion direction constraints (\( G^2 \) constraints). The instantaneous motion vectors of points at the tool axis, which are the contact lines, should be proportional to the motion vectors of the same points on the side path. This \( G^2 \) continuity constraint of the motion of the tool axis is expressed as

\[
\begin{align*}
C_i'(s, 0) &= \alpha_1 R_1'(s, t_{i1}), \quad s = 0, 1 \\
C_i'(s, 1) &= \alpha_2 R_2'(s, t_{i2}), \quad s = 0, 1
\end{align*}
\]

where the prime symbol indicates the derivative w.r.t. \( t \). Note that \( R_1 \) (and \( R_2 \)) is a rigid body motion of the tool axis and therefore the instantaneous vector field that moves the points of the ruling is linear in \( s \). Consequently, satisfying the constraints in Eq. (15) at the parameters \( s = 0 \) and \( s = 1 \) implies it for all \( s \in [0, 1] \). Since we work with clamped B-spline curves, Eq. (15) can be reformulated in terms of control points of \( a_i^s \) as follows

\[
\begin{align*}
\epsilon_{1\text{tan}}^s &= \| (a_i^s - a_0^s) - \alpha_1 \cdot (q_i^s)'(t_{i1}) \|^2 = 0 \\
\epsilon_{2\text{tan}}^s &= \| (a_i^s - a_0^s) - \alpha_2 \cdot (q_i^s)'(t_{i2}) \|^2 = 0
\end{align*}
\]

where \( \alpha_1, \alpha_2 \) are positive constants and positive sign corresponds to the correct direction of the motion. Note that these constants can be set arbitrarily as they correspond just to a reparametrization of the ruled surface \( C_i \). Since we work with cubic splines, we set \( \alpha_1 = \alpha_2 = \frac{1}{3} \) as this corresponds to control point match in the case of the boundary positions of the ruling \( R_1(s, 0) \) and \( R_2(s, 1), i = 1, 2 \), see Fig. 6.

The constraints in Eq. (14) and Eq. (16) concerning milling precision of the side surfaces will also be used later in our algorithm of TR path computation.

3.3.2. MRR and CWE control

Except for the precision objectives discussed above, it is also desired to maximize MRR while controlling the CWE throughout the milling paths. This goal can be formulated as a constrained optimization problem. Towards this end, we now formally define the MRR and the CWE quantities.

Definition of MRR. The Material Removal Rate (MRR) is defined as the removed volume between two paths \( C_i \) and \( C_{i+1} \) over a unit of time. Let the front patch of the envelope of the tool motion along \( C_{i+1} \) be denoted by \( E_{i+1} \). The volume \( Y_{i+1} \) milled by the tool along \( C_{i+1} \) is the space enclosed by two consecutive envelope surfaces \( E_i \) and \( E_{i+1} \), as shown in Fig. 7. The machining time of \( C_{i+1} \) is approximately proportional to the average of the lengths of the top and bottom curves of \( C_{i+1} \), denoted by \( L_{i+1} \). Therefore, MRR can be approximately represented by

\[
M_{i+1} = a \cdot \frac{Y_{i+1}}{L_{i+1}}.
\]

where \( a \) is a constant value that corresponds to the average velocity needed for milling \( Y_{i+1} \). In the end, the units of (17) are \( \text{mm}^3/\text{s} \) (volume per time).

Definition of CWE. For real-life 3D trochoidal milling, the CWE area continuously changes over time in a complicated manner, as shown in Fig. 7(b) and it is a common practice to make the argument easier by considering a 2D simplification, see e.g. [16]. Denote the CWE area by \( \Sigma(t) \) which is time-dependent. For a time instant of the cutting tool corresponding to the parameter \( t_j \), \( \Sigma(t_j) \) is the contact area of the tool with the raw material. This area on the tool’s surface is bounded by two curves: i) the intersection curve of the tool surface with \( E_i \), and ii) the characteristic curve on \( E_{i+1} \). To represent this area, we consider the circles on the tool surface which are perpendicular to the tool axis. The axis is parametrized by \( s \) and the circle at the parameter \( s_l \) of the tool touching the material is called a touching circle. The part of the touching circle in the engagement area is called an engagement arc. The interior angle of the engagement arc is called an engagement angle which generally varies along the tool axis and can be defined by \( \epsilon_{ad} = \epsilon_{a}(s) \), see Fig. 8. In physical machining, \( \epsilon_{ad} \) is required to be restricted under a specific tolerance, i.e., \( \epsilon_{ad} < \epsilon^* \), for all the 2D cuts, i.e., for all \( l \).

Problem formulation. Our aim is to compute a set of front-half paths and we do this in an iterative fashion. That is, once \( C_i \) is computed, we construct \( C_{i+1} \) with MRR and CWE control,
which is possible because both CWE and MRR depend only on \( C_i \) and \( C_{i+1} \). Assuming \( C_i \) is computed (and fixed), the computation of \( C_{i+1} \) considering the control of MRR and CWE can be formulated as a constrained optimization problem as follows

\[
\max_{\beta_{i+1}, l_1, l_2} \mathcal{M}_{i+1} \quad \text{subject to} \quad \begin{cases} 
\epsilon_{kl} < \epsilon^*, \text{for samples } k, \text{ and } l \\
C_{i+1} \text{ meets the constraints in Eq.(14) and Eq.(16)}
\end{cases}
\]

We recall that the parameters \( t_{i1}, t_{i2} \) of the end rulings of \( C_{i+1} \), in addition to the control points of \( C_{i+1} \), are also unknowns. The solution of the problem in Eq.(18) is a ruled surface \( C_{i+1} \) which is the path of the tool along which the CWE meets the tolerance constraints. However, due to the complexity of the functions involved, solving Eq.(18) with standard numerical optimization method is difficult and an optimal solution can hardly be found efficiently and robustly. In the following, we propose a practical algorithm to obtain a reasonably good path that is robust and easy to implement.

3.3.3. Optimization algorithm

In order to find a reasonably good front-half path \( C_{i+1} \) as an approximate solution of Eq.(18), we have to deal with \( \mathcal{L}_{i+1} \), \( \mathcal{Z}_{i+1} \) and the CWE generated by \( C_{i+1} \). It is non-trivial to find an optimal \( C_{i+1} \) meeting all objectives since the quantities depend on \( C_{i+1} \) in a complicated, highly non-linear, manner. Notice that both \( \mathcal{L}_{i+1} \) and the CWE are directly defined by \( E_{i+1} \) and therefore instead of looking for \( C_{i+1} \), we focus on the corresponding envelope \( E_{i+1} \) first, and compute \( C_{i+1} \) from it by the surface fitting algorithm. Observe that \( \mathcal{Z}_{i+1} \) directly depends on \( C_{i+1} \) and can be integrated into the envelope surface fitting algorithm with a function term penalizing the length of the path.

Path optimization. Firstly, we propose a sub-algorithm which computes a ruled surface between two rulings on the left and right side path, respectively. This is achieved via envelope fitting with the \( G^1 \) constraints defined by Eq.(14) and Eq.(16). Notice that the machining time is proportional to the length of the curve traversed by a particular tool point, assuming a constant speed of the tool. Consequently, in order to reduce the machining time, the length the tool path should be penalized and this is achieved by a boundary curve’s penalization term as

\[
F_{\text{length}} = \int \| (\mathbf{c}_I^f)'(t)\|^2 dt + \int \| (\mathbf{c}_I^b)'(t)\|^2 dt,
\]

rec Fig. 6. The algorithm of computing a ruled surface between two side paths is formally defined as follows

\[
\min_{\mathcal{P}_{\text{Inner}}} \beta F_{\text{dist}} + \lambda_1 F_{\text{fair}} + \lambda_2 F_{\text{rigid}} + \lambda_3 F_{\text{length}}
\]

subject to \( \phi_{\text{pos}}^1 = 0, \phi_{\text{pos}}^2 = 0, \phi_{\text{tan}}^1 = 0, \phi_{\text{tan}}^2 = 0, * \in \{T, B\} \)

where the variables \( \mathcal{P}_{\text{Inner}} \) in the optimization are the control points of the ruled surface excluding the end control points. That is, the end rulings of the ruled surface are fixed in the optimization. The lengths of the boundary curves can be controlled with the energy term \( F_{\text{length}} \) which has the effect of shortening the milling time. This sub-algorithm also works when there is no reference surface by setting \( \beta = 0 \) in the objective function (20). For the cases with a reference surface, \( \beta = 1 \) is used.

Path searching. Assuming \( C_i \) being fixed, our method to compute \( C_{i+1} \) consists of an initialization stage and an adjustment stage. In the initialization stage, we look for a front-half path that maximizes MRR, by moving a candidate path in the forward direction, until the CWE constraint is violated. In the adjustment stage, we move backwards the path obtained in the initialization step to satisfy the CWE constraint globally.

- Initialization stage. We need to determine the two end rulings of \( C_{i+1} \) which coincide with some rulings of \( R_1 \) and \( R_2 \). First, the end rulings are set to the end ruling lines of \( C_i \) by setting \( t_{i+1,1} = t_{i1} \) and \( t_{i+1,2} = t_{i2} \), where \( t_{i1} \) and \( t_{i2} \) are parameters of \( R_1 \) and \( R_2 \), respectively. We then increase \( t_{i+1,1} \) and \( t_{i+1,2} \) iteratively with identical incremental value \( \Delta t \) for both sides, resulting in a pair of lines \( R_1(s, t_{i+1,1}) \) and \( R_2(s, t_{i+1,2}) \) stepping forward.

For every time instant during this iterative process, we construct a ruled surface \( C_{i+1}^0 \) between \( R_1(s, t_{i+1,1}) \) and \( R_2(s, t_{i+1,2}) \), and check the CWE constraint by measuring \( \epsilon_{kl} \) at some sampled points; we sample the ruled surface by a quad mesh with 20 × 100 quads, 100 in time \( t \) and 20 in the ruling \( s \) directions. \( C_{i+1}^0 \) is computed by solving the path optimization algorithm (see Eq. (20)) by setting \( \beta = 0 \).

To maximize the MRR, the inner control points are moved in the direction of the \( t \)-derivatives of the side surfaces, i.e., their initial values are set as \( a_{i+1,j} = a_{i,j} + v^t \), for \( j = 1, ..., m - 1 \), where \( v^t = (a_{i+1,1,0} - a_{i,1,0} + a_{i+1,1,m} - a_{i,1,m})/2 \), \( * \in \{T, B\} \) and moved forward by an iterative process which stops once the CWE constraint is violated. The ruled surface at this moment is denoted by \( C_{i+1}^1 \) and is the input for the adjustment stage.

- Adjustment stage. In this stage, we aim to move back some parts of the path \( C_{i+1}^1 \) obtained in the initialization stage where the CWE is violated. It is difficult to directly modify \( C_{i+1} \) to ensure its envelope \( E_{i+1}^1 \) meeting the CWE constraint (due to the fact that the envelope is a one parameter family of characteristics that change dynamically in time). Therefore, we approach the problem in the reverse order, i.e., we update the envelope first, such that the CWE constraint is met, and consequently compute \( C_{i+1}^2 \) via envelope fitting. Since the CWE is directly related to the envelope surface, modifying the envelope to satisfy the CWE constraint is much easier than modifying the path. To modify the envelope, we measure again \( \epsilon_{kl} \) at the samples and update \( E_{i+1}^2 \) where \( \epsilon_{kl} < \epsilon^* \) is violated. The process iteratively executes two steps as follows. See Fig. 8(b).

(1) Let \( p \) be a contact point of the envelope \( E_{i+1}^1 \) and the tool \( T \), and \( c \) be the center of the circle of the tool touching \( p \). Then the vector \( V_D = p - c \) is perpendicular to both \( E_{i+1}^1 \) and \( T \) which is the direction along the fastest increase of the step-over distance. Therefore, if the engagement angle associated with \( p \) is larger than the tolerance, we move back the point \( p \) on \( E_{i+1}^1 \) by \(-d \cdot \frac{V_D}{|V_D|} \) to decrease...
the CWE. To decide $d$, we first find the position $a$ on the circle whose engagement angle is equal to $\epsilon^*$. Let $b$ be the intersection point of the line defined by $a$ and $V_a$ and the envelope in previous cycle $E_i$. Then $d$ is the distance between $a$ and $b$, i.e. $d = ||a-b||$.

(2) After moving back all sample points on $E_i^{1} \ + \ 1$ violating the CWE constraint, a ruled surface $C_{i+1}$ is computed whose envelope fits to the updated envelope $E_i^{1} \ + \ 1$. This is done by solving Eq.(20) with $\beta = 1$. Then, the envelope of $C_{i+1}^{1}$, $E_{i+1}^{1}$, is generated and tested again.

The two above described steps are wrapped into a single loop of a process to update $E_{i+1}^{1}$ with $q$ being the number of iterations. One loop of this procedure is shown in Fig. 9.

The maximum number of iterations is set $q_{\text{max}} = 10$, but this limit has not been reached in any of our examples. The pseudocode of path searching is given in Algorithm 1.

**Algorithm 1 Path Searching**

1: procedure ALGORITHM($C_i$, $C_{i+1}$)
2:  Initialize $t_{i+1,1}$ and $t_{i+1,2}$
3:  repeat
4:   Increase $t_{i+1,1}$ and $t_{i+1,2}$ by $\Delta t_{i+1}$
5:   Generate $C_{i+1}$ via path optimization
6:   Measure $\epsilon_{i+1}$ at some samples
7:   $\Delta t_{i+1} := \Delta t_{i+1} \cdot (1 - \max\{\epsilon_{i+1}\}) / \epsilon^*$
8: until $\epsilon_{i+1} \ > \ \epsilon^*$ for some sample
9:  $C_{i+1} := C_{i+1}$, $q := q + 1$
10: while $\max\{\epsilon_{i+1}\} > \epsilon^*$ and $q \ < \ q_{\text{max}}$ do
11:     for all sample points $p$ do
12:       if $\epsilon_{i+1}$ associated with $p$ is bigger than $\epsilon^*$ then
13:         Move back the point on $E_{i+1}^{1}$ by $-V_{p} / ||V_{p}|| \cdot d$
14:       end if
15:     end for
16:  $q := q + 1$
17:  Generate $C_{i+1}$ with adjusted envelope as reference
18: end while
19: $C_{i+1} := C_{i+1}^{1}$
20: end procedure

**First front-half path.** The above procedure for computing a sequence of front-half paths $C_i$ depends on $C_0$, which is the front-half path of the first TR cycle. In our implementation, $C_0$ is generated by the path optimization algorithm with $\beta = 0$, with the initial control points uniformly distributed. The bisector surface $H$ of $S_1$ and $S_2$ is computed and fitted with a ruled surface $R_{H}(s,t)$ as in [9] with $s$ being the parameter in ruling direction and $t$ in motion direction. To compute the bisector surface, we seek 3D points $x$ that are equally distant from the side surfaces, therefore we compute the signed minimal distances to $S_1$ and $S_2$ and define their difference as

\[
\epsilon(x) = d(x,S_1) - d(x,S_2)
\]

which measures the deviation of $x$ from $H$. We look for $x$ which satisfy $\epsilon(x) = 0$ and use a variant of the marching cubes algorithm which returns a triangular mesh that approximates $H$. To define the end rulings of $C_0$, the starting ruling line of $R_{H}$ is projected to $R_1$ and $R_2$, and the end rulings of $C_0$ are defined to be the best fitting lines to the projection points on $R_1$ and $R_2$, respectively.

**3.4. Back-half path computation and entire path generation**

Once the front-half paths of the TR cycles are computed, the back-half paths are constructed to form the transition between two consecutive front-half paths. The construction of the back-half paths is easier than the computation of the front-half paths since the cutting tool does not touch the raw material, except for the region near the end of paths where the tool touches the side surfaces. A back-half path $D_i$ is defined as a B-spline ruled surface

\[
D_i(t,s) = d_{i}^{T}(t) \cdot (1-s) + d_{i}^{B}(t) \cdot s, \quad t,s \in [0,1],
\]

where $d_{i}^{T}(t) = \sum_{p=0}^{q_{\text{max}}} \mathbf{b}_{p}^{T} B_{p}(t), \quad p \in \{T,B\}$ are B-spline curves. In order to make $D_i$ the transition from the last ruling of $C_i$ to the start ruling of $C_{i+1}$, $D_i(t,s)$ is required to connect two lines $C_i(1,s)$ and $C_{i+1}(0,s)$. Analogously to Eq. (14), the $G^0$ connection constraints are defined as

\[
\left\{ \begin{array}{ll}
\epsilon_{i+1}^{1,\text{pos}} = b_{i}^{1} - q_{1}^{i}(t_{1}) = 0 \\
\epsilon_{i+1}^{2,\text{pos}} = b_{m}^{2} - q_{2}^{i}(t_{2}) = 0 
\end{array} \right.
\]

In addition to the connection constraints, the motion of the tool along the back-half paths should also join the side surfaces tangentially (in $G^1$ fashion) to maintain a high precision milling guaranteed by the construction of $R_1$ and $R_2$. These $G^1$-continuity constraints are formulated analogously to Eq.(16) as

\[
\left\{ \begin{array}{ll}
\epsilon_{i+1}^{1,\text{tan}} = ||(\mathbf{b}_{m}^{1} - \mathbf{b}_{i}^{1}) + \lambda_{1} \cdot (q_{1}^{i})'(t_{1})||^2 = 0 \\
\epsilon_{i+1}^{2,\text{tan}} = ||(\mathbf{b}_{m}^{2} - \mathbf{b}_{i}^{1}) + \lambda_{2} \cdot (q_{2}^{i})'(t_{2})||^2 = 0 
\end{array} \right.
\]

where $* \in \{T,B\}$ with $T$ and $B$ indicate the top or the bottom curve, and $\lambda_{1}$ and $\lambda_{2}$ are positive constants. Note that the velocity vectors have opposite signs than their counterparts on the ruled surfaces $R_1$ and $R_2$ due to the fact that $D_i$ are the back-half paths.

**4. Multi-layer paths for deep cavities**

For trochoidal milling of deep cavities bounded by free-form surfaces, it is typically not possible to finish the milling with one single sweep because a long cutting tool is not practical, e.g. for the chattering reasons. In such a case, a multi-layer milling strategy is needed which divides the side surfaces into layers and each layer is processed with one trochoidal milling path. The milling regions of different layers may have overlapping patches, but the composition of milled regions should cover the whole side surfaces. The milling procedure starts from the top layer from where the cutting tool removes material from the material block, and processes the other layers from the top of the cavity to the bottom. It is also desired that the same tool is used for all the layers for the sake of time needed for the tool exchange and the cost of custom-shaped tools.

**4.1. Side path initialization**

For trochoidal milling of deep cavities bounded by free-form surfaces, it is typically not possible to finish the milling with one single sweep because a long cutting tool is not practical, e.g. for the chattering reasons. In such a case, a multi-layer milling strategy is needed which divides the side surfaces into layers and each layer is processed with one trochoidal milling path. The milling regions of different layers may have overlapping patches, but the composition of milled regions should cover the whole side surfaces. The milling procedure starts from the top layer from where the cutting tool removes material from the material block, and processes the other layers from the top of the cavity to the bottom. It is also desired that the same tool is used for all the layers for the sake of time needed for the tool exchange and the cost of custom-shaped tools.
For the initialization of the side paths, we use a bisector surface $H$ of the two side surfaces $S_1$ and $S_2$. We further define a scalar function $f(x) = d(x,S_1)$, $x \in H$. To compute a boundary curve of a ruled surface, an iso-line $f(x) = \text{const.}$ on $H$ is extracted and projected to the side surface $S_i (i = 1, 2)$. The iso-lines are approximately the iso-parametric lines at some particular parameter $s$ of the ruled surface $R^H(s,t)$ fitting to $H$. The projected curve is then moved along the surface normal of $S_i (i = 1, 2)$ in a direction towards the inside of the cavity with a certain distance, which is decided by the preferred tool size which must be less than half of the width of the cavity. In order to create multiple paths, we uniformly select several iso-lines $L_i$ on $H$ and generate boundary curves $L^1_i$ and $L^2_i$ associated with $S_1$ and $S_2$, respectively. The distance between iso-lines is decided assuming the tool length is roughly known. This constructions forces the tool to have the first and the last radius of the same size (for a tool with two different limit radii, see discussion later in Section 5.4).

After we have a sequence of lines $L^1_i$ and $L^2_i$ ($i = 0, ..., k$) on $S_1$ and $S_2$, respectively, each pair of neighbouring lines can be used as boundary curves to define the initial paths. In order to avoid gaps between neighboring paths, the corresponding lines lying on the ruled surfaces are scaled up to either direction by a certain factor, which is set 5% in our implementation.

**Special treatment of the top layer.** For the top layer, it is common that a part of the tool lies outside the cavity. The sampling points on the tool axis which is located outside the cavity needs to be ignored during the side path calculation. These points $p$ can be recognized by checking their closest points on the freeform side surface. If the closest point of $p$ is located on the boundary of the side surface, $p$ is considered to be outside the cavity and does not contribute to the side path calculation. Note that for the part of the tool with the top layer laying outside the cavity, its top boundary curve cannot be found using the above discussed iso-line method. To resolve this issue, in our implementation, we modify the top ruled surface generated with our iso-line method by moving its top boundary curve in the direction of the tool axis to the outside of the cavity. However, we conclude that handling perfectly boundaries is a separate issue in many CNC projects, and goes also beyond the scope of this paper.

### 4.2. Side path computation

Once the side surface is divided into layers, we generate side paths for all layers. Since it is required that all layers are processed with the same cutting tool, we look for the motion paths for all layers simultaneously with shape parameters of a single cutting tool. Let $F^k_{\text{dist}}$ and $F^k_{\text{fair}}$ be the distance function term and fairness term, respectively, corresponding to the layer $k$. We solve the following minimization problem for side path computation for multi-layer milling

$$
\min_{\mathcal{P}^1, \ldots, \mathcal{P}^k} \sum_{k=1}^{K} F^k_{\text{dist}} + \lambda_1 \sum_{k=1}^{K} F^k_{\text{fair}} + \lambda_2 \sum_{k=1}^{K} F^k_{\text{rigid}},
$$

where $K$ is the number of layers, $\mathcal{P}^k$ is a set of control points of the ruled surfaces in the $k$-th layer, and $r$ is the vector of radii set uniformly distributed along the tool axis. Note that only one single tool shape is used for all layers. $F^k_{\text{dist}}$ and $F^k_{\text{fair}}$ are defined similarly to Eq.(8) and Eq.(3), respectively. To avoid problems with surface patch boundaries and to make the motion path computation highly accurate, whole side surfaces are used as reference instead of their layer counterparts.

### 4.3. Trochoidal path computation for multi-layer milling

Once the side paths are obtained for all layers and the tool shape is determined, we can compute trochoidal path for each layer. We start with the top layer and process the remaining layers one by one. Each layer can be regarded as an independent TR cycle problem and all the front-half and the back-half paths of the single layer can be generated by the methods described in Sec. 3.3 and Sec. 3.4.

### 5. Experimental results

In this section, we verify the proposed 5-axis trochoidal milling methods described in Sec. 3 and Sec. 4 by performing experiments on three test cases. The algorithms are implemented in C++ language. The experimental environment is a desktop with CPU i7-10700K 3.80 GHz and 16G RAM.

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**Figure 9:** One iteration of the front envelope adjustment. (a) Two consecutive front envelopes $E_i$ and $E_{i+1}$, connecting two side surfaces $S_1$ and $S_2$, are shown. $E_i$ is taken for granted and $E_{i+1}$ gets optimized; the $q$-th iteration $E_{i+1}^q$ is shown. (b) $E_{i+1}$ violates the CWE test at 268 points (white dots). (c) $E_{i+1}^q$ gets adjusted (see Section 3.3.3) and becomes closer to $E_i$ (red envelope) to reduce the CWE violation. (d) $E_{i+1}^{q+1}$ represents the optimized envelope, now the CWE is violated at only 127 points (white dots).
We measure the error of the envelope surface generated by cutting tool and the target surface in a discrete way, at samples of the ruled surfaces. The error $\sigma_{kl}$ is the larger of the distance error between the points on envelope surface and the target side surfaces $S_n$, $n=1,2$, that is

$$\sigma_{kl} = \max_{n=1,2} \left| \text{dist}(R(t_k,s_l), \phi_{kl}) - r_l \right|$$

(26)

where $R(t_k,s_l)$ are the samples on the ruled surface, $\phi_{kl}$ are the corresponding footpoints on the target surface, and $r_l$ are the radii of cutting tool corresponding to the $l$-sample in the ruling direction. All the color-coded examples shown in the paper reflect this absolute (non-signed) error. Finally, the total error is defined as

$$\sigma_{\text{max}} = \max_{k,l} \sigma_{kl}.$$  

(27)

5.1. Single-layer TR paths for two free-form surfaces

We start with a test case where we consider a single-layer trochoidal milling, described in Sec. 3, for two free-form side surfaces. The side surfaces form a synthetic (symmetric) cavity that admits highly accurate approximation. The parameters in Eq.(16) that control the overcut direction are all set to 0.1.

Fig. 11 shows the optimization results. The error of the initial side path and the optimized side path are shown in Fig.11(a). For a cavity with a bounding box $BB = 35 \times 42 \times 50$, the error $\sigma_{\text{max}}$ is optimized from the initial $\sigma_{\text{ini}} = 2.1950$ to $\sigma_{\text{opt}} = 0.0869$, which meets the accuracy required for semi-finishing operations. Fig. 11(c)-(d) show the final envelopes of the trochoidal paths computed by our method. A total of 22 trochoidal cycle paths are generated between the two free-form surfaces.

To give comparisons with traditional tools, we firstly give a comparison with conical tools. We employ our method with an additional constraint enforcing the tool to be conical. Fig.10 shows the shape and error of the conical tool where the machining error of the optimized paths reaches $\sigma_{\text{opt}} = 0.6493$, which is by order of magnitude worse than using a custom-shaped tool shown in Fig. 11.

Another comparison is made against a fixed barrel tool. The TR paths are optimized with our method, however, the shape of the tool stays constant. The barrel tool is defined by the radius function $R_\phi$ [44]

$$R_\phi = R_e - R_t (1 - \cos \phi) \cos \phi, \phi \in [-\arcsin(L_e/2R_e), \arcsin(L_e/2R_e)],$$

where $\phi$ is the angle of a circular arc generatrix. We choose reasonable shape parameters for the barrel tool suggested by the side surfaces to be milled, i.e. $R_e = 15$, $R_t = 6$. The length of the tool axis is set $L_e = 16$. Fig.12 shows the results where

Figure 10: Trochoidal milling with conical vs. custom-shaped tools. (a) In the optimization of the side paths, we constraint the meridian (radial function) to be linear for a pair of side surfaces with a bounding box BB. (b) This results in the best conical cutting tool; its envelopes are color-coded by the error $\sigma_{kl}$, see Eq. (26), with the maximum error $\sigma_{\text{max}} = 0.6493$. (c) The envelopes of a single (side) path using the custom-shaped tool shown in Fig. 11(b), color-coded by $\sigma_{kl}$ with $\sigma_{\text{max}} = 0.0869$. (d) The complete trochoidal paths, with almost the same error as for the side surface.

Figure 11: (a) Two free-form surfaces $S_1$ and $S_2$ define a cavity and their approximation by the envelopes of a custom-shaped tool $T$, before and after optimization, are shown framed. (b) The motion of $T$: side paths and the tool are optimized by Eq.(11). (c) Front-half envelopes. There are 22 front-half envelopes computed by the approach described in Sec. 3.3 and color-coded by the number of the TR cycle. (d) The whole cavity filled by the TR envelopes of $T$.

Figure 12: Trochoidal milling with a fixed barrel tool. (a) TR milling with a fixed barrel tool where only the milling path is optimized. The color coding of machining error is shown in (b). While the maximum error is better than for the best conical tool, Fig. 10(b), it is more than six times worse than using a custom-shaped tool, see Fig. 10(d).
5.2. Multi-layer TR paths for the industrial blisk model

The second experiment is the cavity of the blisk model shown in Fig. 1. Due to the depth of the cavity, we adopt the multi-layer TR milling strategy described in Section 4 and consider three layers. We use Eq. (25) to optimize the cutter shape (represented by the vector of radii \( r \)), simultaneously for the three-layer side paths. The weights of \( F_{\text{fair}} \) and \( F_{\text{rigid}} \) in Eq. (25) are set to \( 1.0 \times e^{-5} \) and 0.1, respectively. We chose 10 control points for the single-side motion path of each layer, and the number of sampling points on the tool axis is set to 21. The optimization results of the side path and tool shape are shown in Fig.15. In this experiment, the weights of \( F_{\text{fair}} \) and \( F_{\text{length}} \) in Eq. (20) were both set to \( 1.0 \times e^{-5} \) and the weight of \( F_{\text{rigid}} \) was set to 0.1. The parameters in Eq. (16) were all set to 1/3.

Fig. 15 shows the results of the multi-layer approach, compared with a single layer; in both approaches the shape of the tool is computed (≈ optimized). The multi-layer approach generates in turn (from the bottom to top of the cavity) 17, 19, and 21 trochoidal paths. The difference is related to the fact that the size of the cavity is smallest at the bottom. The multi-layer approach returns finer approximation, see the color-maps in Fig. 15(a) and (b). The optimal tool is close to a conical, yet it is curvature-varying, see Fig. 15(d) top-framed.

Equation (16) was set to 1.0 \( \times \) \( e^{-5} \) and the weight of \( F_{\text{rigid}} \) was set to 0.1. The parameters in Eq. (16) were all set to 1/3.

Fig. 15 shows the results of the multi-layer approach, compared with a single layer; in both approaches the shape of the tool is computed (≈ optimized). The multi-layer approach generates in turn (from the bottom to top of the cavity) 17, 19, and 21 trochoidal paths. The difference is related to the fact that the size of the cavity is smallest at the bottom. The multi-layer approach returns finer approximation, see the color-maps in Fig. 15(a) and (b). The optimal tool is close to a conical, yet it is curvature-varying, see Fig. 15(d) top-framed.

Figure 16: (a) CWE angle and MRR for a TR path with 3 layers, c.f. Fig. 15(b). (b) The MRR information of each TR cycle. (c) The polyline shows the cost time of each round path generation. The histogram shows the number of optimizations for each iteration, which is positively correlated with the cost time of each iteration.

Figure 15: Layer refinement. Single-layer (left) vs. multi-layer (right) trochoidal milling are compared. The side paths (framed) are color coded by the distance error \( \sigma_{\text{max}} \). While the error in the case of a single layer \( \sigma_{\text{max}} = 0.3837 \), using three layers the error gets reduced to \( \sigma_{\text{max}} = 0.0751 \). (c+d) The final TR paths of our algorithm. The color coding here shows the number of trochoidal cycle. (d) top-framed. The optimized \( T \) color-coded by the Gaussian curvature and the curvature plot of the meridian curve of \( T \).

5.3. Multi-layer TR paths for second real blisk model

The third test example uses another industrial blisk dataset. The optimization parameters and the number of control points were set equally to those in Sec. 5.2. Since this cavity is deeper, we chose five instead of three layers, each one generated 27 ~ 31 rounds of trochoidal paths. The side paths are
shown in Fig. 17(a) and the complete TR envelopes are shown in Fig. 17(b). The successive optimization process is visualized in Fig. 17(c+d) and a comparison against a single-layer approach is shown in Fig. 17(e). Again, the errors are by the order of magnitude better in favor of the multi-layer strategy. The computation times and other statistics of all three test cases are listed in Table 1.

5.4. Discussion & limitations

**Optimal tool selection.** We initialized the tool such that its side (flank) motion fits the side surfaces, however, its size, in the terms of thickness, remained uncontrolled in our optimization and one could also look for the thickest tool that fits the cavity, as this should further reduce the machining time. We also slightly restricted the space of tools as we unified the \( L_i \) paths in the initialization (Section 4.1), which resulted in equal limit radii \( r_1 = r_M \). However, these radii were not constrained to be equal in the optimization stage.

**Global collision detection.** Our algorithm tests only local collision, i.e., the collision of the custom-shaped tool and the side surfaces. As the tool is typically mounted on a handle/shank, there could be, however, collision of that part with the cavity. We checked the global collisions of the tools’ axis as a post-process, but a more thorough global collision test shall be done in the case of physical experiments.

**Global optimization.** Our approach looks for a tool that minimizes the distance error between the motion of the tool and two side surfaces of the cavity. The problem is formulated as an optimization problem and we look for a minimizer. There is no guarantee that our method finds the global minimizer (which may not be unique), however, our results show that for cavities of industrial benchmarks, such as the blisk geometry, sufficiently accurate approximation (tens of micrometers) exists.

**Existence of an exact tool.** An exact tool exists iff the two side surfaces are exact envelopes of a motion of such a tool. One can construct a (counter-)example of a cavity where a single tool cannot give sufficiently good results; e.g. such a cavity can be formed by one convex elliptic side surface and other one concave elliptic.

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### 6. Conclusions

In this work, a method for trochoidal milling of 3D cavities bounded by free-form surfaces has been proposed. Our method computes not only the milling paths, but also the shape of the cutting tool itself, both in a single- and multi-layer setup. Material removal rate and cutter-workpiece engagement are also incorporated inside our variational framework and easy to control. The proposed method is validated on both synthetic and industrial benchmarks, and returns highly accurate milling paths that meet fine machining tolerances.

### 7. References


