Extended State Space for Describing Renormalized Fock Spaces in QFT

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Abstract

In quantum field theory (QFT) models, it often seems natural to use, instead of wave functions from Fock space, wave functions that are not square–integrable and have prefactors involving divergent integrals (known as infinite wave function renormalizations). Here, we rigorously construct vector spaces containing divergent integrals, as well as two extended state vector spaces, which contain a dense subspace of Fock space, but also incorporate non–square–integrable wave functions with infinite wave function renormalizations.
As a demonstration, we apply this construction to a non–perturbative renormalization of a class of simple non–relativistic QFT models, which are polaron models with resting fermions. The Hamiltonian without cutoffs, an infinite self–energy and a dressing transformation are defined as linear operators on certain subspaces of the two Fock space extensions. This way, we can obtain a renormalized Hamiltonian which can be realized as a densely defined self–adjoint operator on Fock space.

Key words: Non–perturbative QFT, renormalization, divergent integrals, Van Hove model, Fock space, Hamiltonian Formalism, dressing transformation, interior–boundary conditions (IBC)

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1 Introduction

In this work, we present new mathematical tools that are intended to help constructing dynamics of both relativistic and non–relativistic models with particle creation and annihilation (here called "quantum field theories" or QFTs). These tools include:

- A **vector space of divergent integrals** $\mathfrak{r} \in \mathrm{Ren}_1$.
- A **field of (possibly infinite) wave function renormalizations** eRen, containing exponentials $e^{\mathfrak{r}}$ and fractions of linear combinations thereof. In particular, eRen is a field extension of $\mathbb{C}$.
- An **extended state space (ESS)** $\overline{\mathscr{F}}$, which is an eRen–vector space over smooth, complex–valued functions on configuration space, which are not necessarily L^2–integrable. $\overline{\mathscr{F}}$ extends a dense subspace of the Fock space, $\mathscr{F}$.
- A **second ESS** $\overline{\mathscr{F}}_{\mathrm{ex}}$, which allows for multiplying $\Psi \in \overline{\mathscr{F}}$ by elements $\mathfrak{r} \in \mathrm{Ren}_1$ (and not only by elements of eRen).

We apply them to a class of formal Hamiltonians

$$H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty. \tag{1}$$

That means the following: We consider a species of $M \in \mathbb{N}_0$ fermions (associated with an index x) and a species of $N \in \mathbb{N}_0$ bosons (associated with an index y). The operators $A^\dagger(v), A(v)$ make each fermion create/annihilate a boson with some form factor v (in momentum space), $H_{0,y}$ describes the free evolution of the bosons and E_∞ is an **infinite self–energy counterterm**. We define E_∞ as a linear operator in Proposition 4.1 using the ESS framework. The Hamiltonian H is defined as a linear operator $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ following the same definition.
In Section 8, we will also encounter formal Hamiltonians of the kind

$$H_0 + A^\dagger(v) + A(v) - E_\infty, \tag{2}$$

with $H_0 = H_{0,x} + H_{0,y}$ including both a fermionic and a bosonic dispersion relation.
Apart from the flat fermion dispersion relation, our class of Hamiltonians (1) allows to include the same dispersion relations as are used in many non–relativistic QFT models, including the Van Hove model [1, 2], Nelson's model [3, 4, 5, 6], the Fröhlich polaron [7], as well as relativistic polarons [8, 9].

Physical meaning is assigned to H by a **dressing transformation** $W(s)$, which is a linear operator $W(s) : \overline{\mathscr{F}}_{\mathrm{ex}} \supset \mathcal{D}_W \to \overline{\mathscr{F}}_{\mathrm{ex}}$ in form of a Gross transformation. Its form factor $s(\boldsymbol{k})$ does only depend on the boson momentum $\boldsymbol{k}$ and not on the fermion momentum $\boldsymbol{p}$, so one can fiber–decompose it into several Weyl transformations, one for

each fermion configuration $\boldsymbol{X} \in \mathbb{R}^{Md}$. The "renormalized" or "undressed" Hamiltonian $\widetilde{H}$ can be defined without cutoffs by

$$W(s)\widetilde{H} = HW(s),$$

i.e., we require that the following diagram commutes:

$$\begin{array}{ccccc}
\overline{\mathscr{F}}_{\mathrm{ex}} \supset W(s)[\mathcal{D}_{\mathscr{F}}] \supset W(s)[\widetilde{\mathcal{D}}_{\mathscr{F}}] & \xrightarrow{\quad H \quad} & W(s)[\mathcal{D}_{\mathscr{F}}] \subset \overline{\mathscr{F}}_{\mathrm{ex}} \\
\Big\uparrow W(s) & & \Big\uparrow W(s) \\
\mathscr{F} \supset \mathcal{D}_{\mathscr{F}} \supset \widetilde{\mathcal{D}}_{\mathscr{F}} & \xrightarrow{\quad \widetilde{H} \quad} & \mathcal{D}_{\mathscr{F}} \subset \mathscr{F}
\end{array}$$

The large domain $\mathcal{D}_{\mathscr{F}}$, with the fermionic wave function being a Schwartz function, is defined in (132). It contains the small domain $\widetilde{\mathcal{D}}_{\mathscr{F}}$ defined in (147) via (146), with fermionic wave function in C_c^∞, and its support avoiding the collision configurations. It is necessary to introduce two domains, since $\widetilde{H}$ does neither map $\mathcal{D}_{\mathscr{F}}$ nor $\widetilde{\mathcal{D}}_{\mathscr{F}}$ to itself. Nevertheless, $\widetilde{H} : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$ can be defined. By means of Lemmas 5.6 and 7.1, both $\mathcal{D}_{\mathscr{F}}$ and $\widetilde{\mathcal{D}}_{\mathscr{F}}$ are dense in $\mathscr{F}$.

Our **main result** is Theorem 6.1, which establishes that $\widetilde{H}$ is indeed defined as a linear operator $\widetilde{H} : \mathscr{F} \supset \mathcal{D}_{WS} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ (with definition of the domain $\mathcal{D}_{WS}$ in (105)). By means of Lemma 7.1, under certain conditions, this $\widetilde{H}$ can indeed be interpreted as a Fock space operator $\widetilde{H} : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$, which allows for self–adjoint extensions by Corollary 7.1. More precisely, we have

$$\widetilde{H} = H_{0,y} + V, \tag{3}$$

with V being a pair potential interaction between fermions.

The result (3) after an "undressing" is not too surprising: Our formal Hamiltonian (1) actually just corresponds to a direct integral of uncoupled Van Hove Hamiltonians, one for each fermion position configuration $\boldsymbol{X} \in \mathbb{R}^{Md}$. It is well–known [1] that Van Hove Hamiltonians can be "diagonalized" by an algebraic Weyl transformation. That means, after replacing creation and annihilation operators like $a(v) \mapsto a(v) + \langle v, s\rangle$ and $a^\dagger(v) \mapsto a^\dagger(v) + \langle s, v\rangle$, as well as dropping an infinite self–energy constant, one arrives at a non–interacting Hamiltonian similar to $H_{0,y}$.
The main novelty in this paper is that we are able to define certain products consisting of dressing operators $W(s), s \notin L^2$, creation operators $A^\dagger(v), v \notin L^2$ and even divergent integrals $\mathfrak{r} \in \mathrm{Ren}_1$, as linear operators on suitable domains. The rigorous treatment of divergent integrals may also become useful in a purely algebraic approach, which does not refer to any Fock space extensions, but nevertheless produces divergent integrals when evaluating commutation relations.

Let us add some remarks about the dressing operator $W(s)$: Gross transformations are usually of the form $W(s) = e^{A^\dagger(s)-A(s)}$, where $A^\dagger(s) = \sum_{j=1}^M A_j^\dagger(s)$ describes a boson creation induced by all fermions $j \in \{1, \dots, M\}$. We will choose a slightly different dressing operator, which formally reads

$$W(s) = W_M(s) \dots W_1(s), \qquad W_j(s) = e^{A_j^\dagger(s)-A_j(s)}. \tag{4}$$

The reason is that, when applying the Baker–Campbell–Hausdorff formula (as in (115)), formal calculations with $e^{A^\dagger(s)-A(s)}$ would produce exponential operators of the kind $e^{V_{jj'}}$, with $V_{jj'}$ being a potential interaction between fermions j and j'. Those expressions $e^{V_{jj'}}$ cannot be defined as operators on the ESS $\overline{\mathscr{F}}_{\text{ex}}$, so we need to avoid their occurrence. In general, we will define the extended dressing operators $W_j(s)$ such that their action for $s \in \mathfrak{h}$ slightly differs from that of the usual Fock space dressing operators $W_{\mathscr{F},j}(s) : \mathscr{F} \to \mathscr{F}$. That is, we drop any terms of the kind $e^{V_{jj'}}$ in our definition. This ad–hoc modification is justified by the fact that, in a sufficiently regular case, the $V_{jj'}$–operators commute with $W(s)$ (Lemma 5.5). So even if we could define them as operators, they would just act as an independent factor that can be pulled to the left. So dropping all $e^{V_{jj'}}$–terms can essentially be seen as a simplification of the bookkeeping. Nevertheless, it would be interesting for future works to define $e^{V_{jj'}}$ as an operator, mapping to some ESS $\overline{\mathscr{F}}'_{\text{ex}}$, which is constructed differently than $\overline{\mathscr{F}}_{\text{ex}}$. The current ESS $\overline{\mathscr{F}}_{\text{ex}}$ also does not allow for defining general products of annihilation operators $A_{j_1}(v_1) \dots A_{j_n}(v_n)$ with $n \geq 2$. It is a further interesting question, how such products can be defined in the future by an alternative construction of some $\overline{\mathscr{F}}'_{\text{ex}}$.
The ESS construction is currently at an early stage of development. One may view the results of this paper rather as a "proof of concept", showing that the ESS construction generates reasonable outcomes in well–investigated environments. We do not yet attempt to produce renormalized Hamiltonians in models where non–perturbative renormalization has not succeeded before, although this is a clearly desirable objective for the future.

Although the spaces $\overline{\mathscr{F}}, \overline{\mathscr{F}}_{\text{ex}}$ do not have a topological structure, we may define a **renormalized scalar product** on $W(s)[\mathcal{D}_{\mathscr{F}}]$ via:

$$\langle W(s)\Psi, W(s)\Phi\rangle_{\text{ren}} := \langle \Psi, \Phi\rangle \qquad \forall \Psi, \Phi \in \mathcal{D}_{\mathscr{F}}. \tag{5}$$

The completion of $W(s)[\mathcal{D}_{\mathscr{F}}]$ with respect to $\langle \cdot, \cdot \rangle_{\text{ren}}$ defines a Hilbert space $\mathscr{F}_{\text{ren}}$, our **renormalized Fock space**. The map $W(s)$ then uniquely extends to an isometric isomorphism between $\mathscr{F}$ and $\mathscr{F}_{\text{ren}}$.

We remark that our result (3) is actually just the lowest–order approximation in a perturbation expansion within the weak–coupling regime, and completely decouples the fermions from the bosonic radiation field. The reason is that we both consider resting fermions (by excluding $H_{0,x}$ from H) and restrict to form factors $v(\boldsymbol{k})$, that only depend

on the momentum $\boldsymbol{k}$ of the emitted boson, and not on the momentum $\boldsymbol{p}$ of the fermion emitting it. It is physically expected and confirmed for the Nelson model with UV–cutoff [10, 11], that $W_\Lambda^* H_\Lambda W_\Lambda$ contains interactions between fermions and the radiation field. When changing to $v(\boldsymbol{p}, \boldsymbol{k})$, one may formally use a so–called "Lie–Schwinger series' [12]:

$$\tilde{H} = e^A B a^{-A} = \sum_{n=0}^{\infty} \frac{\mathrm{ad}^n(A)B}{n!},$$

where $A := -A^\dagger(s) + A(s),\ B := H$ and with the n–fold commutator

$$\mathrm{ad}^n(A)B := [A, [A, \ldots [A, B] \ldots]].$$

However, establishing well–definedness of $\mathrm{ad}^n(A)B$ for $H + H_{0,x}$ with v depending on $\boldsymbol{p}$ is a rather involved task, so we postpone it to future investigations.

Another interesting objective for future works would be to introduce a mass renormalization term δm. In constructive QFT (CQFT) [13], where the Hamiltonian is of a form different from (1), mass renormalization terms can also be found in some works [14, 15, 16, 17, 18], while other CQFT renormalization procedures work without a mass renormalization [19, 20, 21, 22, 23, 24, 25, 26].

The rest of this paper is structured as follows: After specifying some mathematical notation in Section 2, we conduct the ESS construction in Section 3, finally resulting in spaces $\overline{\mathscr{F}}, \overline{\mathscr{F}}_{\mathrm{ex}}$.
We then establish $H_0, A^\dagger, A$ and E_∞ as linear operators on $\overline{\mathscr{F}}$ or $\overline{\mathscr{F}}_{\mathrm{ex}}$ in Section 4.
In the following Section 5, we construct the dressing transformation $W(s)$. Further, we define an **extended Weyl algebra** $\overline{\mathcal{W}}$, which includes linear combinations of the operators $W_j(s) = e^{A_j^\dagger(s) - A_j(s)}$. $\overline{\mathcal{W}}$ allows for a multiplication with infinite wave function renormalization factors $\mathfrak{c} \in \mathrm{eRen}$, and the Weyl relations

$$\begin{aligned} W_j(s)^{-1} &= W_j(-s) \\ W_j(s_1)W_j(s_2) &= e^{\mathrm{Im}\langle s_1, s_2\rangle} W_j(s_1 + s_2), \end{aligned} \tag{6}$$

hold with $e^{\mathrm{Im}\langle s_1, s_2\rangle} \in \mathrm{eRen}$. However, note that $\overline{\mathcal{W}}$ is generated by $W_j(s)$ for a fixed j, instead of $W(s)$ and the dressing operator $W(s)$ is not contained in $\overline{\mathcal{W}}$.
In Section 6 we compute $\widetilde{H}$ such that $W(s)\widetilde{H} = HW(s)$. This section contains the main result, Theorem 6.1.
The proof that under certain conditions $\widetilde{H}$ is a Fock space operator and allows for self–adjoint extensions follows in Section 7. Thus, $\widetilde{H}$ generates well–defined quantum dynamics, although they are not necessarily unique.
Finally, Section 8 concerns dressing transformations different from $W(s)$, which are inspired by examples from the literature and can successfully be defined on the ESSs $\overline{\mathscr{F}}, \overline{\mathscr{F}}_{\mathrm{ex}}$. This includes dressings $(1 + H_0^{-1}A^\dagger)^{-1}$ appearing in a renormalization technique

called "interior–boundary conditions" (IBC), which recently gained some considerable attention [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37], and also served as a major source of inspiration for the ESS construction. Further, we will use the ESS framework to define cutoff–free dressing operators $T = e^{-H_0^{-1}A^\dagger}$, which is a strongly simplified version of dressings used in CQFT works like [18, 14].

2 The Mathematical Model

2.1 Formal Hamiltonian

In this article, we consider a class of non–perturbative QFT models. These involve two species of particles:

- There is one species of spinless fermionic particles (x).
- These fermions interact by exchange of spinless bosons (y).

By M and N, we denote the number of x– and y–particles, respectively. In the particle–position representation, a single particle is described by a Hilbert space vector

$$\varphi \in \mathfrak{h} = L^2(\mathbb{R}^d, \mathbb{C}), \tag{7}$$

with some coordinates denoted by a boldface symbol $\boldsymbol{x} \in \mathbb{R}^d$. So the system has d space dimensions and $d+1$ spacetime dimensions. The configuration of the entire system is given by an element $(\boldsymbol{X}, \boldsymbol{Y})$ of the configuration space $\mathcal{Q}$,

$$(\boldsymbol{X}, \boldsymbol{Y}) = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_N) \in \mathcal{Q}_x \times \mathcal{Q}_y =: \mathcal{Q}, \tag{8}$$

where the x–, and y–configuration spaces $\mathcal{Q}_x, \mathcal{Q}_y$ and its sectors $\mathcal{Q}_x^{(M)}, \mathcal{Q}_y^{(N)}$ are defined as

$$\bigsqcup_{M=0}^{\infty} \mathbb{R}^{Md} =: \bigsqcup_{M=0}^{\infty} \mathcal{Q}_x^{(M)} =: \mathcal{Q}_x \qquad \bigsqcup_{N=0}^{\infty} \mathbb{R}^{Nd} =: \bigsqcup_{N=0}^{\infty} \mathcal{Q}_y^{(N)} =: \mathcal{Q}_y. \tag{9}$$

The state of the system at time t is described by a Fock space vector

$$\Psi \in L^2(\mathcal{Q}, \mathbb{C}). \tag{10}$$

Note that the term "state" is also used to describe a complex–valued map ω on the bounded linear operators on $\mathscr{F}$ (possibly a mixed state). We will only refer to state vectors when using the term "state ".

For states, we assume the function $\Psi \in L^2(\mathcal{Q}, \mathbb{C})$ to be anti–symmetric under coordinate exchange for fermions x and symmetric under coordinate exchange for bosons y.

The corresponding symmetrization and antisymmetrization operators S_-, S_+ are defined on $L^2(\mathcal{Q}_x, \mathbb{C})$ and $L^2(\mathcal{Q}_y, \mathbb{C})$ as

$$\begin{aligned}(S_-\Psi)(\boldsymbol{x}_1, \dots, \boldsymbol{x}_M) &= \frac{1}{M!}\sum_\sigma \operatorname{sgn}(\sigma)\Psi(\boldsymbol{x}_{\sigma(1)}, \dots, \boldsymbol{x}_{\sigma(M)}) \\ (S_+\Psi)(\boldsymbol{y}_1, \dots, \boldsymbol{y}_N) &= \frac{1}{N!}\sum_\sigma \Psi(\boldsymbol{x}_{\sigma(1)}, \dots, \boldsymbol{x}_{\sigma(N)}),\end{aligned} \tag{11}$$

with the sum running over all permutations Σ of $\{1, \dots, M\}$ or $\{1, \dots, N\}$, respectively. The **fermionic/bosonic/total Fock space** is then

$$\mathscr{F}_x = S_-[L^2(\mathcal{Q}_x, \mathbb{C})], \qquad \mathscr{F}_y = S_+[L^2(\mathcal{Q}_y, \mathbb{C})], \qquad \mathscr{F} = \mathscr{F}_x \otimes \mathscr{F}_y. \tag{12}$$

It is convenient to describe the action of the formal Hamiltonian H in the particle–momentum representation: For any $\Psi \in \mathscr{F}$ with particle–position representation $\Psi(\boldsymbol{X}, \boldsymbol{Y})$, we define the Fourier transform $\Psi(\boldsymbol{P}, \boldsymbol{K})$ with momentum configuration

$$(\boldsymbol{P}, \boldsymbol{K}) = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_M, \boldsymbol{k}_1, \dots, \boldsymbol{k}_N) \in \mathbb{R}^{Md+Nd}, \tag{13}$$

via

$$\Psi(\boldsymbol{P}, \boldsymbol{K}) := (2\pi)^{-\frac{Md+Nd}{2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^M} \Psi(\boldsymbol{X}, \boldsymbol{Y})\, e^{-i\boldsymbol{P}\cdot\boldsymbol{X} - i\boldsymbol{K}\cdot\boldsymbol{Y}}\, d^{Md}\boldsymbol{X} d^{Nd}\boldsymbol{Y}. \tag{14}$$

In the following, the set of variables plugged into $\Psi(\cdot, \cdot)$, i.e., $(\boldsymbol{X}, \boldsymbol{Y})$ or $(\boldsymbol{P}, \boldsymbol{K})$ will specify, which representation is meant.

As a formal Hamiltonian, we consider the following expression with zero fermion dispersion relation:

$$H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty. \tag{15}$$

In Section 8, we will also consider similar Hamiltonians of this kind that feature a nonzero fermion dispersion relation, i.e., where $H_{0,y}$ is replaced by $H_0 = H_{0,x} + H_{0,y}$. The formal definitions of the relevant expressions read as follows:

- The kinetic term H_0 is characterized by two dispersion relations, i.e., by real–valued functions $\theta(\boldsymbol{p}), \omega(\boldsymbol{k}) \in C^\infty(\mathbb{R}^d \setminus \{0\})$ for fermions and bosons, respectively. We can decompose

$$\begin{aligned}H_0 =& H_{0,x} + H_{0,y}, \\ (H_{0,x}\Psi)(\boldsymbol{P}, \boldsymbol{K}) =& \sum_{j=1}^M \theta(\boldsymbol{p}_j)\Psi(\boldsymbol{P}, \boldsymbol{K}), \\ (H_{0,y}\Psi)(\boldsymbol{P}, \boldsymbol{K}) =& \sum_{\ell=1}^N \omega(\boldsymbol{k}_\ell)\Psi(\boldsymbol{P}, \boldsymbol{K}).\end{aligned} \tag{16}$$

A shorthand notation is to use the symbols $d\Gamma_x(\cdot)$ and $d\Gamma_y(\cdot)$ for second quantization:

$$H_{0,x} = d\Gamma_x(\theta), \qquad H_{0,y} = d\Gamma_y(\omega), \tag{17}$$

with multiplication operators

$$\begin{aligned} &\theta : \mathfrak{h} \supseteq \mathrm{dom}(\theta) \to \mathfrak{h} && \omega : \mathfrak{h} \supseteq \mathrm{dom}(\omega) \to \mathfrak{h} \\ &\phi(\boldsymbol{p}) \mapsto \theta(\boldsymbol{p})\phi(\boldsymbol{p}) && \phi(\boldsymbol{k}) \mapsto \omega(\boldsymbol{k})\phi(\boldsymbol{k}). \end{aligned} \tag{18}$$

- The creation part $A^\dagger(v)$ makes each fermion create a boson. It is specified by a form factor v. In this article, we will restrict to $v \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and put some further assumptions on the scalings of v for $|\boldsymbol{k}| \to 0$ (IR–regime) and $|\boldsymbol{k}| \to \infty$ (UV–regime). These assumptions are described in Section 2.2.

We may write $A^\dagger(v)$ as a sum over operators $A_j^\dagger(v), j \in \{1, \ldots, M\}$, which only make fermion j create a boson. The formal definition in particle–momentum representation reads

$$\begin{aligned} (A^\dagger(v)\Psi)(\boldsymbol{P}, \boldsymbol{K}) &= \left(\sum_{j=1}^M A_j^\dagger \Psi \right)(\boldsymbol{P}, \boldsymbol{K}) \\ &= \sum_{j=1}^M \frac{1}{\sqrt{N}} \sum_{\ell=1}^N v(\boldsymbol{k}_\ell) \Psi(\boldsymbol{P} + e_j \boldsymbol{k}_\ell, \boldsymbol{K} \setminus \boldsymbol{k}_\ell), \end{aligned} \tag{19}$$

where

- $\boldsymbol{K} \setminus \boldsymbol{k}_\ell = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\ell-1}, \boldsymbol{k}_{\ell+1}, \ldots, \boldsymbol{k}_N)$ denotes $\boldsymbol{K}$ without $\boldsymbol{k}_\ell$
- $\boldsymbol{P} + e_j \boldsymbol{k}_\ell = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{j-1}, \boldsymbol{p}_j + \boldsymbol{k}_\ell, \boldsymbol{p}_{j+1}, \ldots, \boldsymbol{p}_M)$ is the shifted fermion momentum.

Here, e_j is meant to denote the j–th unit vector and, by slight abuse of notation, $e_j \boldsymbol{k}_\ell = (0, \ldots, \boldsymbol{k}_\ell, \ldots, 0) \in \mathbb{R}^{Md}$ is used to denote the assignment of an additional momentum $\boldsymbol{k}_\ell$ to fermion j, before boson ℓ is emitted.

The corresponding definition in particle–position representation uses the Fourier inverse $\check{v}$ of the form factor v and reads

$$(A^\dagger(v)\Psi)(\boldsymbol{X}, \boldsymbol{Y}) = \left(\sum_{j=1}^M A_j^\dagger(v) \Psi \right)(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{j=1}^M \frac{1}{\sqrt{N}} \sum_{\ell=1}^N \check{v}(\boldsymbol{y}_\ell - \boldsymbol{x}_j) \Psi(\boldsymbol{X}, \boldsymbol{Y} \setminus \boldsymbol{y}_\ell). \tag{20}$$

Without the cutoff, the form factor is $v \notin L^2$ in many physically relevant models. Its Fourier inverse $\check{v}$ is therefore $\notin L^2$, as well, if it is even defined.

Note that there is a more general class of physically interesting models, which we do not treat here: The form factor may depend on the fermion momentum $\boldsymbol{p}$, i.e., it may read $v(\boldsymbol{p}, \boldsymbol{k})$ with $v : \mathbb{R}^{2d} \to \mathbb{C}$, such as in [38, 39, 40].

- The annihilation part $A(v)$ in particle–momentum representation reads

$$\begin{aligned}(A(v)\Psi)(\boldsymbol{P}, \boldsymbol{K}) &= \left(\sum_{j=1}^{M} A_j(v)\Psi\right)(\boldsymbol{P}, \boldsymbol{K}) \\ &= \sum_{j=1}^{M} \sqrt{N+1} \int v(\tilde{\boldsymbol{k}})^* \Psi(\boldsymbol{P} - e_j\tilde{\boldsymbol{k}}, \boldsymbol{K}, \tilde{\boldsymbol{k}})\, d\tilde{\boldsymbol{k}}.\end{aligned} \tag{21}$$

In particle–position representation,

$$\begin{aligned}(A(v)\Psi)(\boldsymbol{X}, \boldsymbol{Y}) &= \left(\sum_{j=1}^{M} A_j(v)\Psi\right)(\boldsymbol{X}, \boldsymbol{Y}) \\ &= \sum_{j=1}^{M} \sqrt{N+1} \int \check{v}(\tilde{\boldsymbol{y}} - \boldsymbol{x}_j)^* \Psi(\boldsymbol{X}, \boldsymbol{Y}, \tilde{\boldsymbol{y}})\, d\tilde{\boldsymbol{y}}.\end{aligned} \tag{22}$$

Note that for $v \in \mathfrak{h}$, both $A^\dagger(v)$ and $A(v)$ can be defined as operators on a dense domain in $\mathscr{F}$, where $A^\dagger(v)$ is the adjoint of $A(v)$.

- The self–energy E_∞ is a formal multiplication operator of the form $E_\infty = d\Gamma_x(E_1)$ with $E_1 : \mathbb{R}^d \to \mathbb{R}$. In particle–momentum representation,

$$(E_\infty\Psi)(\boldsymbol{P}, \boldsymbol{K}) = \sum_{j=1}^{M} E_1(\boldsymbol{p}_j)\Psi(\boldsymbol{P}, \boldsymbol{K}). \tag{23}$$

We will consider the expression

$$E_1(\boldsymbol{p}) = \int -\frac{v(\boldsymbol{k})^* v(\boldsymbol{k})}{\omega(\boldsymbol{k})}\, d\boldsymbol{k}. \tag{24}$$

With cutoffs σ, Λ applied to v, this integral would be finite in many physically interesting situations. However, without cutoffs it is typically divergent and hence the operator E_∞ becomes a formal expression. We will define it as a map $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$ in Proposition 4.1.

2.2 Scaling Degrees

The convergence of integrals appearing in formal calculations depends on how $\theta(\boldsymbol{p}), \omega(\boldsymbol{k})$ and $v(\boldsymbol{k})$ scale at $|\boldsymbol{p}|, |\boldsymbol{k}|$ going to ∞ or 0. We assume a polynomial scaling, of which we keep track using scaling degrees m (UV) and β (IR). The scaling degree m in the UV regime is commonly used for symbols $s \in \mathcal{S}^m$. As we choose $\theta, \omega, v \in C^\infty(\mathbb{R}^d \setminus \{0\})$, an additional pole might appear at $|\boldsymbol{k}| \to 0$, which we assume to be of order β. Both scalings are assumed to be exact, i.e., polynomial bounds exist from above and below:

Definition 2.1 (scaling)**.** Consider $s \in C^\infty(\mathbb{R}^d \setminus \{0\})$. We say that s has a **polynomial scaling** if there are some scaling degrees $m_s \leq \beta_s \in \mathbb{R}$ and constants $C_1, C_2 \in \mathbb{R}$ such that

$$|s(\boldsymbol{k})| \leq C_1 |\boldsymbol{k}|^{\beta_s} + C_2 |\boldsymbol{k}|^{m_s} \qquad \forall \boldsymbol{k} \in \mathbb{R}^d \setminus \{0\}. \tag{25}$$

The space of all **symbols with polynomial scaling** is denoted

$$\dot{\mathcal{S}}_1^\infty := \big\{ s \in C^\infty(\mathbb{R}^d \setminus \{0\}) \ \mid \ (25) \text{ holds} \big\}. \tag{26}$$

Definition 2.2 (exact scaling)**.** Consider a symbol $s \in C^\infty(\mathbb{R}^d \setminus \{0\})$. We say that the symbol has an **exact polynomial scaling** (see Figure 1) if there are some $0 < \underline{\varepsilon} < \overline{\varepsilon} \in \mathbb{R}$ such that

- There are a UV scaling degree $m_s \in \mathbb{R}$ and $c_1, C_1 > 0$ with

$$c_1 |\boldsymbol{k}|^{m_s} \leq |s(\boldsymbol{k})| \leq C_1 |\boldsymbol{k}|^{m_s} \qquad \forall |\boldsymbol{k}| > \overline{\varepsilon}. \tag{27}$$

- There are an IR scaling degree $\beta_s \in \mathbb{R}$ and $c_2, C_2 > 0$ with

$$c_2 |\boldsymbol{k}|^{\beta_s} \leq |s(\boldsymbol{k})| \leq C_2 |\boldsymbol{k}|^{\beta_s} \qquad \forall 0 < |\boldsymbol{k}| < \underline{\varepsilon}. \tag{28}$$

- There is a constant $c_3 > 0$ with

$$c_3 \leq |s(\boldsymbol{k})| \qquad \forall \underline{\varepsilon} \leq |\boldsymbol{k}| \leq \overline{\varepsilon}. \tag{29}$$

The space of all **symbols with exact polynomial scaling** is denoted

$$\dot{\mathcal{S}}_{1,>}^\infty := \big\{ s \in C^\infty(\mathbb{R}^d \setminus \{0\}) \ \mid \ (27), (28) \text{ and } (29) \text{ hold} \big\} \subseteq \dot{\mathcal{S}}_1^\infty. \tag{30}$$

The "$>$" refers to the fact that $|s(\boldsymbol{k})| > 0$, so $\frac{1}{s}$ is defined everywhere except from $\boldsymbol{k} = 0$ and also scales polynomially.

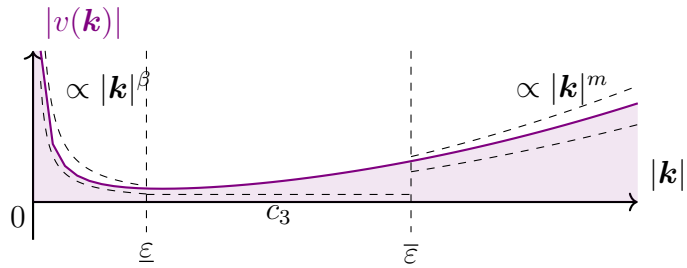

Figure 1: Scaling degrees in the special case of a radially symmetric, positive function v with $|v(\boldsymbol{k})| > 0$. Color online.

Obviously, $C_c^\infty(\mathbb{R}^d) \subset \dot{\mathcal{S}}_1^\infty$, so $\dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$ is dense in $\mathfrak{h}$.

For obtaining a well–defined renormalized Hamiltonian $\widetilde{H}$ in Lemma 7.1, we will assume

$$\theta, \omega, v \in \dot{\mathcal{S}}_{1,>}^\infty, \tag{31}$$

(see Figure 2) and denote their scaling degrees with m_θ, m_ω, m_v and $\beta_\theta, \beta_\omega, \beta_v$, respectively. Further, we will assume in Lemma 7.1:

$$m_\theta, m_\omega, \beta_\theta, \beta_\omega \geq 0. \tag{32}$$

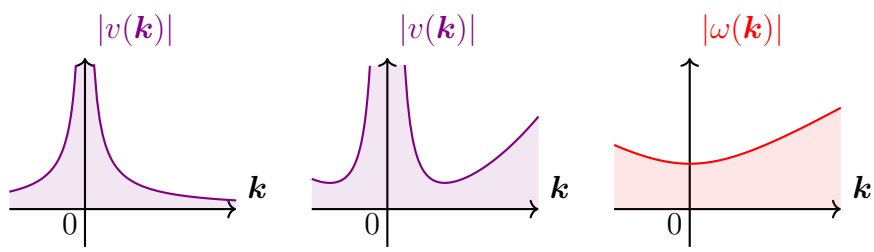

Figure 2: Examples for absolute values of functions $v, \omega \in \dot{\mathcal{S}}_{1,>}^\infty$. The functions v, ω are complex–valued. Note that $\beta_\omega, m_\omega \geq 0$. Color online.

This implies, that for θ and ω, there is no pole at the origin. QFT models often use dispersion relations based on symbols with $m_\theta, m_\omega, \beta_\theta, \beta_\omega \in \{0, 1, 2\}$, which all satisfy this condition.

In order to obtain symmetric operators, we will also need to impose a symmetry condition in Lemma 7.1:

$$\theta(\boldsymbol{p}) = \theta(-\boldsymbol{p}), \qquad \omega(\boldsymbol{k}) = \omega(-\boldsymbol{k}), \qquad v(\boldsymbol{k}) = v(-\boldsymbol{k}). \tag{33}$$

3 Construction of the Extended State Space

3.1 Wave Function Renormalization Factors

The motivation behind the introduction of an extended state space is to rigorously define the formal dressing operator

$$W(s) = W_M(s) \dots W_1(s), \qquad \text{with } W_j(s) = e^{A_j^\dagger(s) - A_j(s)},\ s = -\frac{v}{\omega}, \tag{34}$$

in the case where $s \in \dot{\mathcal{S}}_1^\infty \subseteq C^\infty(\mathbb{R}^d \setminus \{0\})$, but $s \notin \mathfrak{h} = L^2(\mathbb{R}^d, \mathbb{C})$. Here, the number M corresponds to the fermion sector, so the second term in (34) is a rather symbolic expression requiring us to adapt M in dependence of the fermion sector (similar to (16)). Formally, if we apply $W(s)$ to a state vector Ψ_1 where the boson field is in the vacuum Ω_y and $\Psi_x \in \mathscr{F}_x^{(1)}$ is a one–fermion state,

$$\Psi_1 = \Psi_x \otimes \Omega_y, \tag{35}$$

then we obtain the following expression in momentum space:

$$(W(s)\Psi_1)(\boldsymbol{p}, \boldsymbol{K}) = e^{-\frac{\|s\|^2}{2}} \frac{1}{\sqrt{N!}} \left(\prod_{\ell=1}^N s(\boldsymbol{k}_\ell) \right) \Psi_x \left(\boldsymbol{p} + \sum_{\ell=1}^N \boldsymbol{k}_\ell \right) =: e^{-\frac{\|s\|^2}{2}} \Psi_0(\boldsymbol{p}, \boldsymbol{K}). \tag{36}$$

For $s \in \mathfrak{h}$, the expression $W(s)\Psi_1$ is a Fock space vector with norm $\|W(s)\Psi_1\| = \|\Psi_1\|$. However, if we set $s \notin \mathfrak{h}$, two problems arise:

- $\|s\|^2 = \langle s, s \rangle$ is formally a divergent integral. So the wave function renormalization factor $e^{-\frac{\|s\|^2}{2}}$ is not well–defined.
- $\left(\prod_{\ell=1}^N s(\boldsymbol{k}_\ell) \right)$ (on every sector $N \geq 1$) is a non–square integrable function, so even without the infinite renormalization factor $e^{-\frac{\|s\|^2}{2}}$, (36) does not describe an element of $\mathscr{F}$.

The second problem is tackled by defining a space $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$ containing non–square integrable functions, including the above product. As $s \in \dot{\mathcal{S}}_1^\infty$, the product above is a smooth function apart from the zero boson momentum configuration set

$$\exists(\boldsymbol{k} = 0) := \big\{ q = (\boldsymbol{P}, \boldsymbol{K}) \in \mathcal{Q} \,\big|\, \exists \ell \in \{1, \dots, N\} :\ \boldsymbol{k}_\ell = 0 \big\}. \tag{37}$$

The restriction of $\exists(\boldsymbol{k} = 0)$ to any sector is a union of hyperplanes (see Figure 3), which have Lebesgue measure 0 and hence, also $\exists(\boldsymbol{k} = 0) \subseteq \mathcal{Q}$ is null. Excluding this set from configuration space, we obtain

$$\dot{\mathcal{Q}} := \mathcal{Q} \setminus \exists(\boldsymbol{k} = 0), \tag{38}$$

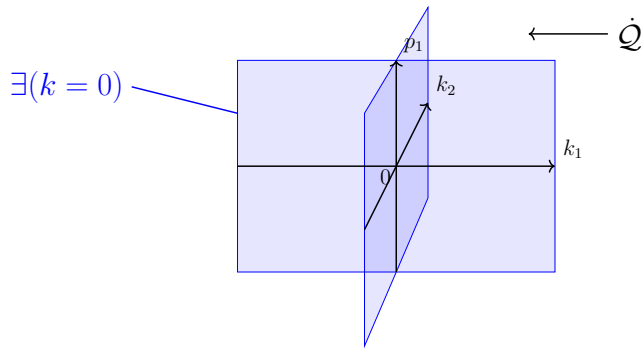

Figure 3: The set $\exists(\boldsymbol{k}=0)$ on the $(M,N)=(1,2)$–sector in configuration space $\mathcal{Q}$ for $d=1$. Color online.

and on $q \in \dot{\mathcal{Q}}$, the function Ψ_0 in (36) is smooth. For $|\boldsymbol{k}| \to 0$ or $|\boldsymbol{p}|, |\boldsymbol{k}| \to \infty$ we require the following scaling conditions to hold:

$$\lim_{|\boldsymbol{k}_\ell|\to 0} \frac{\Psi_0(\boldsymbol{P},\boldsymbol{K})}{|\boldsymbol{k}_\ell|^\beta} = \lim_{|\boldsymbol{k}_\ell|\to\infty} \frac{\Psi_0(\boldsymbol{P},\boldsymbol{K})}{|\boldsymbol{k}_\ell|^m} = \lim_{|\boldsymbol{p}_j|\to\infty} \frac{\Psi_0(\boldsymbol{P},\boldsymbol{K})}{|\boldsymbol{p}_j|^m} = 0, \tag{39}$$

for all fixed $q = (\boldsymbol{P},\boldsymbol{K}) \in \dot{\mathcal{Q}}$ and some $\beta, m \in \mathbb{R}$. We hence interpret Ψ_0 as an element of the following space:

$$\dot{\mathcal{S}}^\infty_{\mathscr{F}} := \left\{ \Psi \in C^\infty(\dot{\mathcal{Q}}) \;\middle|\; (39) \text{ holds} \right\}. \tag{40}$$

Note that $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap \mathscr{F}$ contains $C^\infty_c(\mathcal{Q})$, which is a dense set in $\mathscr{F}$. So $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap \mathscr{F}$ is dense in $\mathscr{F}$. The Fourier transform of some $\Psi_0 \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ can be taken, if it is an element of the space

$$\dot{\mathcal{S}}^\infty_{\mathscr{F},\mathrm{loc}} := \dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap L^1_{\mathrm{loc}}(\mathcal{Q}). \tag{41}$$

In order to address the first problem, it is necessary to assign mathematical meaning to the expression $\langle s, s\rangle$ (called **renormalization factor**). It may be convenient to take a Fourier transform of $s \in \dot{\mathcal{S}}^\infty_1$, for instance, in order to define a particle–position representation. This can be done whenever $s \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, since then $s \in \mathcal{S}'(\mathbb{R}^d)$, i.e., it is a tempered distribution. We may hence distinguish two interesting function spaces for s:

- The generic case is given by $s \in \dot{\mathcal{S}}^\infty_1$
- The special case is given by $s \in \dot{\mathcal{S}}^\infty_1 \cap L^1_{\mathrm{loc}}(\mathbb{R}^d) =: \dot{\mathcal{S}}^\infty_{1,\mathrm{loc}}$, which allows for taking the Fourier transform.

In the following, we will present a construction based on the generic case, since the special one can be treated by the same means. Clearly, $\mathfrak{h} \cap \dot{\mathcal{S}}^\infty_1 \subseteq \dot{\mathcal{S}}^\infty_{1,\mathrm{loc}}$, so even $\dot{\mathcal{S}}^\infty_{1,\mathrm{loc}}$ contains a dense subspace of $\mathfrak{h}$. If $s \in \mathfrak{h}$, then

$$\langle s, s\rangle = \int |s(\boldsymbol{k})|^2 \, d\boldsymbol{k} \in \mathbb{C}. \tag{42}$$

Otherwise, the renormalization factor $\langle s, s\rangle$ is a symbolic expression, corresponding to a divergent integral. We interpret it as an element of the following vector space:

Definition 3.1. Consider the free $\mathbb{C}$–vector space $F(\dot{\mathcal{S}}_1^\infty \times \dot{\mathcal{S}}_1^\infty)$. By a free vector space, we mean the set of all finite $\mathbb{C}$–linear combinations of pairings, denoted $\mathfrak{r} = \sum_{m=1}^M c_m\langle s_m, t_m\rangle$, with $s_m, t_m \in \dot{\mathcal{S}}_1^\infty$, $c \in \mathbb{C}$ and the sum $\sum_{m=1}^M$ being commutative. The **space of renormalization integrals** is defined as the quotient space

$$\mathrm{Ren}_{01} := F\left(\dot{\mathcal{S}}_1^\infty \times \dot{\mathcal{S}}_1^\infty\right) /_{\sim_{\mathrm{Ren}_{01}}}, \tag{43}$$

where the equivalence relation $\sim_{\mathrm{Ren}_{01}}$ of formal equality is given by

$$\begin{aligned} &\sum_{m=1}^M c_m\langle s_m, t_m\rangle \sim_{\mathrm{Ren}_{01}} \sum_{m=1}^{\tilde{M}} \tilde{c}_m\langle \tilde{s}_m, \tilde{t}_m\rangle \\ :\Leftrightarrow \quad &\sum_{m=1}^M c_m\overline{s_m(\boldsymbol{k})}t_m(\boldsymbol{k}) = \sum_{m=1}^{\tilde{M}} \tilde{c}_m\overline{\tilde{s}_m(\boldsymbol{k})}\tilde{t}_m(\boldsymbol{k}) \quad \forall \boldsymbol{k} \in \mathbb{R}^d \setminus \{0\}. \end{aligned} \tag{44}$$

There is a natural one–to–one identification of renormalization integrals with functions $\mathrm{Ren}_{01} \cong \dot{\mathcal{S}}_1^\infty$: It is easy to see that the following map is an embedding (by definition of $\sim_{\mathrm{Ren}_{01}}$):

$$\begin{aligned} \iota_1 : \mathrm{Ren}_{01} &\to \dot{\mathcal{S}}_1^\infty \\ \sum_{m=1}^M c_m\langle s_m, t_m\rangle = \mathfrak{r} &\mapsto r(\boldsymbol{k}) = \sum_{m=1}^M c_m\overline{s_m(\boldsymbol{k})}t_m(\boldsymbol{k}). \end{aligned} \tag{45}$$

Conversely, for a given $r \in \dot{\mathcal{S}}_1^\infty$, the element $\langle r, \chi_{\mathbb{R}^d\setminus\{0\}}\rangle \in \mathrm{Ren}_1$ (with χ being the indicator function) is identified with r, so ι_1 is indeed an isomorphism.

However, it can be misleading to think of $\mathfrak{r} \in \mathrm{Ren}_1$ as a function r. Renormalization integrals are formally $\mathfrak{r} = \int r(\boldsymbol{k})\ d\boldsymbol{k}$, so they can be added but not directly multiplied. Hence the notation $\langle\cdot,\cdot\rangle$, resembling an L^2 scalar product. By contrast, functions $r_1, r_2 \in \dot{\mathcal{S}}_1^\infty$ can be multiplied which leads again to a function in $\dot{\mathcal{S}}_1^\infty$.

In case $r \in L^1(\mathbb{R}^d)$, we can identify $\mathfrak{r} \in \mathrm{Ren}_{01}$ with the $\mathbb{C}$–number

$$\mathfrak{r} = \int_{\mathbb{R}^d} \sum_{m=1}^M c_m\overline{s_m(\boldsymbol{k})}t_m(\boldsymbol{k})\ d\boldsymbol{k} = \int_{\mathbb{R}^d} r(\boldsymbol{k})\ d\boldsymbol{k} \in \mathbb{C}. \tag{46}$$

Now, several $\mathfrak{r} \in \mathrm{Ren}_{01}$ get identified with the same $\mathbb{C}$–number, e.g., all $\mathfrak{r}$ corresponding to a function r with $\int r(\boldsymbol{k})\ d\boldsymbol{k} = 0$ are identified with 0. We remove this ambiguity by modding out another equivalence relation:

Definition 3.2. The renormalization factor space Ren_1 is defined as

$$\mathrm{Ren}_1 := \mathrm{Ren}_{01}/_{\sim_{\mathrm{Ren}_1}}, \tag{47}$$

where for $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathrm{Ren}_{01}$ we define

$$\mathfrak{r}_1 \sim_{\mathrm{Ren}_1} \mathfrak{r}_2 \;:\Leftrightarrow\; \iota_1(\mathfrak{r}_1) - \iota_1(\mathfrak{r}_2) \in L^1 \quad \text{and} \quad \int (\iota_1(\mathfrak{r}_1) - \iota_1(\mathfrak{r}_2))(\boldsymbol{k})\, d\boldsymbol{k} = 0. \tag{48}$$

Elements of Ren_1 will be denoted equally to a representative $\mathfrak{r}$. Note that we can identify Ren_1 with the quotient space

$$\mathrm{Ren}_1 \cong (\mathbb{C} \oplus \mathrm{Ren}_{01})/D \quad \text{with} \quad D := \left\{ \left(-\textstyle\int \iota_1(\mathfrak{r})(\boldsymbol{k})\, d\boldsymbol{k}, \mathfrak{r}\right) \,\middle|\, \iota_1(\mathfrak{r}) \in L^1(\mathbb{R}^d) \right\}, \tag{49}$$

where the isomorphism is given by identification of $(c, \mathfrak{r})$ with that class $[\mathfrak{r}'] \in \mathrm{Ren}_1$ where $\int (\iota_1(\mathfrak{r}')(\boldsymbol{k}) - \iota_1(\mathfrak{r})(\boldsymbol{k}))\, d\boldsymbol{k} = c$.

We will encounter the special case where $r(\boldsymbol{k})$ only takes values in $[0, \infty)$:

Definition 3.3. The **positive renormalization factor cone** Ren_{1+} is the set of all $[\mathfrak{r}] \in \mathrm{Ren}_1$, where at least one representative $\mathfrak{r} \in \mathrm{Ren}_{01}$ is identified with a positive–valued function $\iota_1(\mathfrak{r}) : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$.

Scalar multiplication by $c \in [0, \infty)$ is well–defined, making this indeed a cone.

It is also convenient to make sense of products and polynomials of factors $\mathfrak{r} \in \mathrm{Ren}_1$. We hence define the following vector spaces:

Definition 3.4. Consider the free $\mathbb{C}$–vector space of all finite linear combinations of products of up to $P \in \mathbb{N}_0$ renormalization factors (i.e., formal polynomials of degree P)

$$\mathrm{Pol}_P := \left\{ \mathfrak{R} = \sum_{m=1}^{M} c_m \mathfrak{r}_{m,1} \cdot \ldots \cdot \mathfrak{r}_{m,p_m} \,\middle|\, 0 \le p_m \le P,\ c_m \in \mathbb{C},\ \mathfrak{r}_{m,p} \in \mathrm{Ren}_1 \right\}, \tag{50}$$

with the sum $\sum_{m=1}^{M}$ and products being commutative. Then, the space of **renormalization factor polynomials** of order $\le P$ is the quotient space

$$\mathrm{Ren}_P = \mathrm{Pol}_P/_{\sim_{\mathrm{Ren}_P}}, \tag{51}$$

with the equivalence relation $\sim_{\mathrm{Ren}_P}$ of formal equality generated by

$$\begin{aligned} \mathfrak{r}_1\mathfrak{r}_2 \ldots \mathfrak{r}_{p_1} + \mathfrak{R} \;&\sim_{\mathrm{Ren}_P}\; c\mathfrak{r}_2 \ldots \mathfrak{r}_{p_1} + \mathfrak{R} \qquad \text{if } \mathfrak{r}_1 = c \in \mathbb{C} \\ (c_1c_2)\mathfrak{r}_1 \ldots \mathfrak{r}_{p_1} + \mathfrak{R} \;&\sim_{\mathrm{Ren}_P}\; c_1(c_2\mathfrak{r}_1) \ldots \mathfrak{r}_{p_1} + \mathfrak{R}, \end{aligned} \tag{52}$$

with $P_1 \le P$, $\mathfrak{R} \in \mathrm{Pol}_P$, $c_1, c_2 \in \mathbb{C}$ and $\mathfrak{r}_m \in \mathrm{Ren}_1$.

The bound P on the polynomial order is removed by taking the union over all orders.

Definition 3.5. The space of **renormalization factor polynomials** is given by

$$\mathrm{Ren} = \bigcup_{P=1}^{\infty} \mathrm{Ren}_P. \tag{53}$$

The spaces defined in this Section follow the hierarchy

$$\mathrm{Ren}_{1+} \subseteq \mathrm{Ren}_1 \subseteq \mathrm{Ren}_2 \subseteq \ldots \subseteq \mathrm{Ren}_P \subseteq \ldots \subseteq \mathrm{Ren}. \tag{54}$$

3.2 Exponentials of Renormalization Factors and the Field eRen

The state vector $W(s)\Psi_1$ in (36) contains an exponential of a renormalization factor, i.e., an expression $\mathfrak{c} = e^{\mathfrak{r}}$ with $\mathfrak{r} \in \mathrm{Ren}_1$. More generally, we would like to give a meaning to sums of exponentials with different (perhaps infinite) renormalization factors $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathrm{Ren}_1$. Formally, we would like to identify

$$e^{\mathfrak{r}_1} + e^{\mathfrak{r}_2} = e^{\mathfrak{r}_1}(1 + e^{\mathfrak{r}_2 - \mathfrak{r}_1}). \tag{55}$$

The bracket is a $\mathbb{C}$–number, whenever $\mathfrak{r}_1 - \mathfrak{r}_2 \in \mathbb{C}$, which defines an equivalence relation

$$\mathfrak{r}_1 \sim_1 \mathfrak{r}_2 \quad :\Leftrightarrow \quad \mathfrak{r}_1 - \mathfrak{r}_2 \in \mathbb{C}. \tag{56}$$

Definition 3.6. The space of **renormalization factor classes** is then given by the quotient space

$$\mathrm{Clas}_1 = \mathrm{Ren}_1/_{\sim_1}. \tag{57}$$

Whenever two elements $\mathfrak{r}_1, \mathfrak{r}_2$ are of two different classes, we have $\mathfrak{c} = e^{\mathfrak{r}_1 - \mathfrak{r}_2} \notin \mathbb{C}$ and may think of $\mathfrak{c}$ as an "infinite constant".

We may also interpret Clas_1 as a subspace of Ren_1: The elements in the class of zero, $b \in \mathrm{Ren}_1 : b \sim_1 0$, form a subspace $V \subset \mathrm{Ren}_1$, $V \cong \mathbb{C}$. Now we can find a basis B (containing one element) of V and, by the axiom of choice [41], extend it to a basis $B \cup B'$ of Ren_1. By defining $W := \mathrm{span}(B')$, we obtain a decomposition

$$\mathrm{Ren}_1 = V \oplus W \qquad \Rightarrow \qquad W \cong \mathrm{Ren}_1/V. \tag{58}$$

Now, the following map is a bijection from W to Clas_1:

$$W \ni w \mapsto [w]_{\sim_{\mathrm{Ren}_1}} \in \mathrm{Clas}_1. \tag{59}$$

Note that this bijection is not unique, since it depends on the choice of B', i.e., of representatives within each class.

We now consider elements of the group algebra $\mathbb{C}[\mathrm{Ren}_1] \ni c_1 e^{\mathfrak{r}_1} + \ldots + c_M e^{\mathfrak{r}_M}$ of the additive group of the vector space Ren_1. The formal exponentials make an addition in the group $e^{\mathfrak{r}_1+\mathfrak{r}_2} = e^{\mathfrak{r}_1} e^{\mathfrak{r}_2}$ appear as a multiplication. Here, we would like to consider two summands as equal, if "parts of the complex number can be pulled into the exponent", i.e., $ce^{c'+\mathfrak{r}} = (ce^{c'})e^{\mathfrak{r}}$. This is done by defining an ideal $\mathcal{I} \subset \mathbb{C}[\mathrm{Ren}_1]$ generated by all elements of the form

$$e^{c+\mathfrak{r}} - e^c e^{\mathfrak{r}} \qquad \text{with} \qquad c \in \mathbb{C},\ \mathfrak{r} \in \mathrm{Ren}_1. \tag{60}$$

Note that this ideal gives rise to an equivalence relation

$$u \sim_{\mathcal{I}} v \qquad :\Leftrightarrow (u - v) \in \mathcal{I}. \tag{61}$$

Proposition 3.1.

$$\mathbb{C}[\mathrm{Ren}_1]/\mathcal{I} \cong \mathbb{C}[W]. \tag{62}$$

Proof. We define a map $\pi : \mathrm{Ren}_1 \to W$, which assigns to $\mathfrak{r} \in \mathrm{Ren}_1$ the vector w within the decomposition $\mathrm{Ren}_1 = V \oplus W$ above. So $\mathfrak{r} - \pi(\mathfrak{r}) \in \mathbb{C}$. Now within, $c_1 e^{\mathfrak{r}_1} + \ldots + c_M e^{\mathfrak{r}_M} \in \mathbb{C}[\mathrm{Ren}_1]$, we can re–write each summand as

$$c_m e^{\mathfrak{r}_m} \sim_{\mathcal{I}} (c_m e^{\mathfrak{r}_m - \pi(\mathfrak{r}_m)}) e^{\pi(\mathfrak{r}_m)}. \tag{63}$$

This gives rise to a re–writing map

$$\begin{aligned} \Pi : \mathbb{C}[\mathrm{Ren}_1] &\to \mathbb{C}[\mathrm{Ren}_1] \\ c_1 e^{\mathfrak{r}_1} + \ldots + c_M e^{\mathfrak{r}_M} &\mapsto (c_1 e^{\mathfrak{r}_1 - \pi(\mathfrak{r}_1)}) e^{\pi(\mathfrak{r}_1)} + \ldots + (c_M e^{\mathfrak{r}_M - \pi(\mathfrak{r}_M)}) e^{\pi(\mathfrak{r}_M)}. \end{aligned} \tag{64}$$

Note that by the computation rules for group algebras, whenever $e^{\pi(\mathfrak{r}_m)} = e^{\pi(\mathfrak{r}_{m'})}$ for two summands, they can be combined into one.

The map Π is an algebra homomorphism: it is linear and respects the multiplication:

$$\Pi(c_1 e^{\mathfrak{r}_1} + \ldots + c_M e^{\mathfrak{r}_M}) \Pi(c_1' e^{\mathfrak{r}_1'} + \ldots + c_M' e^{\mathfrak{r}_M'}) = \Pi(c_1 e^{\mathfrak{r}_1} + \ldots + c_M e^{\mathfrak{r}_M} + c_1' e^{\mathfrak{r}_1'} + \ldots + c_M' e^{\mathfrak{r}_M'}). \tag{65}$$

The latter is a consequence of $\pi(\mathfrak{r}_1 + \mathfrak{r}_2) = \pi(\mathfrak{r}_1) + \pi(\mathfrak{r}_2)$ which does only hold true because we have chosen W as a vector space. Further, Π does not change the equivalence class with respect to $\sim_{\mathcal{I}}$. So $\Pi(I) = 0$ implies $I \in \mathcal{I}$. Conversely, if $I \in \mathcal{I}$ then $I = I_1 + \ldots + I_M$ with each $I_i = A_i B_i C_i$ with $A_i, C_i \in \mathbb{C}[\mathrm{Ren}_1]$ and $B_i = e^{c_i + \mathfrak{r}_i} - e^{c_i} e^{\mathfrak{r}_i}$, so $\Pi(B_i) = 0$ and $\Pi(I_i) = 0$ and hence $\Pi(I) = 0$. So the kernel of Π is exactly $\mathrm{Ker}(\Pi) = \mathcal{I}$.
Moreover, the image is $\Pi[\mathbb{C}[\mathrm{Ren}_1]] = \mathbb{C}[W]$, since only elements of $\mathbb{C}[W]$ appear in it

and any element of $\mathbb{C}[W] \subset \mathbb{C}[\mathrm{Ren}_1]$ is mapped to itself.
By the isomorphism theorems, we now have that

$$\mathbb{C}[\mathrm{Ren}_1]/\mathrm{Ker}(\Pi) \cong \Pi[\mathbb{C}[\mathrm{Ren}_1]] \qquad :\Leftrightarrow \qquad \mathbb{C}[\mathrm{Ren}_1]/\mathcal{I} \cong \mathbb{C}[W], \tag{66}$$

as claimed. □

Proposition 3.2. $\mathbb{C}[\mathrm{Ren}_1]/\mathcal{I}$ *has no proper zero divisors.*

($a, b \in \mathbb{C}[\mathrm{Ren}_1]/\mathcal{I}, a, b \neq 0$ are called proper zero divisors if and only if $ab = 0$.)

Proof. By Proposition 3.1, it suffices to show that $\mathbb{C}[W]$ has no proper zero divisors.

Now, the additive group W is an Abelian group that is torsion–free, i.e., for $g \in W$ and $n \in \mathbb{N}$:

$$\underbrace{g + g + \ldots + g}_{n \text{ times}} = 0 \quad \Rightarrow \quad g = 0. \tag{67}$$

Now, by Lemma 26.6 [42], the group $G = W$ is ordered, so by Lemma 26.4 it is "t.u.p.", so by Lemma 26.2 it is "u.p." and $K[G] = \mathbb{C}[W]$ has no proper zero divisors, i.e., it is an entire ring.

□

Following [43, II. §3], it is a theorem that for every ring without proper zero divisors, the quotients form a field. So by Proposition 3.2 the following field extension of $\mathbb{C}$ is well–defined:

Definition 3.7 (and Corollary)**.** The **field of (exponential) wave function renormalizations** is given by all fractions of linear combinations

$$\mathrm{eRen} := \left\{ \mathfrak{c} = \frac{a_1}{a_2} \;\middle|\; a_1, a_2 \in \mathbb{C}[\mathrm{Ren}_1]/\mathcal{I} \right\}. \tag{68}$$

By using representatives of $\mathbb{C}[\mathrm{Ren}_1]/\mathcal{I}$, we can write any $\mathfrak{c} \in \mathrm{eRen}$ as

$$\mathfrak{c} = \frac{\sum_{m=1}^{M} c_m e^{\mathfrak{r}_m}}{\sum_{m'=1}^{M'} c_{m'} e^{\mathfrak{r}_{m'}}} \quad \text{with} \quad c_m, c_{m'} \in \mathbb{C}, \mathfrak{r}_m, \mathfrak{r}_{m'} \in \mathrm{Ren}_1. \tag{69}$$

In particular, we can view a wave function renormalization $\mathfrak{c} \in \mathrm{eRen}$ as an "extended complex number".

3.3 First Extended State Space

With the above definitions, we are able to give meaning to expressions like

$$\Psi = e^{-\mathfrak{r}}\Psi_0 \qquad \text{or even} \qquad \Psi = \mathfrak{c}\Psi_0, \tag{70}$$

with $\mathfrak{r} \in \mathrm{Ren}_1, \mathfrak{c} \in \mathrm{eRen}$ and $\Psi_0 \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$, as they appear in (36). We would like to take linear combinations of them and even handle expressions like $\mathfrak{c}\mathfrak{R}\Psi_0$ with $\mathfrak{R} \in \mathrm{Ren}$. This is done by defining eRen–vector spaces including such expression, either without $\mathfrak{R}$ (this will be the first ESS $\overline{\mathscr{F}}$) or with $\mathfrak{R}$ (this will be the second ESS $\overline{\mathscr{F}}_{\mathrm{ex}}$).

Definition 3.8. Consider the free eRen–vector space of all finite (commutative) sums of the form

$$\overline{\mathscr{F}}_0 := \left\{ \Psi = \sum_{m=1}^{M} \mathfrak{c}_m \Psi_m \;\middle|\; \Psi_m \in \dot{\mathcal{S}}^\infty_{\mathscr{F}},\ \mathfrak{c}_m \in \mathrm{eRen} \right\}. \tag{71}$$

Then, the **first extended state space (ESS)** is the eRen–quotient space

$$\overline{\mathscr{F}} = \overline{\mathscr{F}}_0 / \sim_{\mathrm{F}}, \tag{72}$$

with equivalence relation $\sim_{\mathrm{F}}$ generated by

$$(c\mathfrak{c})\Psi_a + \Psi \ \sim_{\mathrm{F}}\ \mathfrak{c}(c\Psi_a) + \Psi \qquad \text{if } c \in \mathbb{C}, \tag{73}$$

for any $\mathfrak{c} \in \mathrm{eRen},\ \Psi \in \overline{\mathscr{F}}_0$ and $\Psi_a \in \dot{\mathcal{S}}^\infty_{\mathscr{F}}$.

The dense subspace $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap \mathscr{F}$ of the Fock space can be naturally embedded into $\overline{\mathscr{F}}$: Every $\Psi_{\mathscr{F}} \in \dot{\mathcal{S}}^\infty_{\mathscr{F}} \cap \mathscr{F}$, can be identified with

$$\Psi_{\mathscr{F}} = e^0 \Psi_{\mathscr{F}} \in \overline{\mathscr{F}}. \tag{74}$$

However, elements of $\overline{\mathscr{F}}$ do not necessarily satisfy any symmetry conditions.

The coherent state in (36) can now be seen as an ESS element:

$$(W(s)\Psi_1)(\boldsymbol{p}, \boldsymbol{K}) = \underbrace{e^{-\frac{\|s\|^2}{2}}}_{=e^{\mathfrak{r}}} \underbrace{\frac{1}{\sqrt{N!}} \left(\prod_{\ell=1}^{N} s(\boldsymbol{k}_\ell) \right) \Psi_x \left(\boldsymbol{p} + \sum_{\ell} \boldsymbol{k}_\ell \right)}_{\in \dot{\mathcal{S}}^\infty_{\mathscr{F}}}.$$

For $s \in \dot{\mathcal{S}}^\infty_{1,\mathrm{loc}}$ and $\Psi_x \in C_c^\infty$, the second factor is even in $\dot{\mathcal{S}}^\infty_{\mathscr{F},\mathrm{loc}}$.

3.4 Second Extended State Space

Later in this work, we will encounter expressions like

$$\Psi(q) = e^{-\mathfrak{r}}\mathfrak{R}(q)\Psi_0(q), \tag{75}$$

with $q \in \dot{\mathcal{Q}}$, $\mathfrak{R}(q) \in \mathrm{Ren}, \Psi_0(q) \in \mathbb{C}$, or linear combinations of them. We interpret $\Psi_{\mathfrak{R}} := \mathfrak{R}\Psi_0$ as a function in the function space

$$\Psi_{\mathfrak{R}} \in \mathrm{Ren}^{\dot{\mathcal{Q}}} := \left\{\Psi_{\mathfrak{R}} : \dot{\mathcal{Q}} \to \mathrm{Ren}\right\}. \tag{76}$$

The second ESS then covers expressions (75) and linear combinations of them:

Definition 3.9. Consider the free eRen–vector space of all finite (commutative) sums of the form

$$\overline{\mathscr{F}}_{\mathrm{ex},0} := \left\{ \Psi = \sum_{m=1}^{M} \mathfrak{c}_m \Psi_m \;\middle|\; \mathfrak{c}_m \in \mathrm{eRen}, \Psi_m \in \mathrm{Ren}^{\dot{\mathcal{Q}}} \right\}. \tag{77}$$

Then, the **second extended state space (ESS)** is the eRen–quotient space

$$\overline{\mathscr{F}}_{\mathrm{ex}} = \overline{\mathscr{F}}_{\mathrm{ex},0}/_{\sim_{\mathrm{Fex}}}, \tag{78}$$

with equivalence relation $\sim_{\mathrm{Fex}}$ generated by

$$(c\mathfrak{c})\Psi_a + \Psi \;\sim_{\mathrm{Fex}}\; \mathfrak{c}(c\Psi_a) + \Psi \qquad \text{if } c \in \mathbb{C}, \tag{79}$$

where $\mathfrak{c} \in \mathrm{eRen}$, $\Psi \in \overline{\mathscr{F}}_{\mathrm{ex},0}$ and $\Psi_a \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

The first ESS can be embedded into the second ESS $\overline{\mathscr{F}} \hookrightarrow \overline{\mathscr{F}}_{\mathrm{ex}}$ by interpreting all $\Psi_m \in \dot{\mathcal{S}}^{\infty}_{\mathscr{F}}$ as elements $\Psi_m \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

4 Operators on the Extended State Space

We first prove that creation and annihilation terms $A^\dagger(v), A(v)$ as in (19) and (21) can be defined as operators using extended state spaces. Here, we even permit form factors $v(\boldsymbol{p}, \boldsymbol{k})$, that are allowed to depend on the fermion momentum $\boldsymbol{p}$. The momentum space definition reads

$$
\begin{aligned}
(A_j^\dagger(v)\Psi)(\boldsymbol{P}, \boldsymbol{K}) &= \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} v(\boldsymbol{p}_j, \boldsymbol{k}_\ell)\Psi(\boldsymbol{P} + e_j\boldsymbol{k}_\ell, \boldsymbol{K} \setminus \boldsymbol{k}_\ell), \\
(A_j(v)\Psi)(\boldsymbol{P}, \boldsymbol{K}) &= \sqrt{N+1} \int v(\boldsymbol{p}_j - \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}})^* \Psi(\boldsymbol{P} - e_j\tilde{\boldsymbol{k}}, \boldsymbol{K}, \tilde{\boldsymbol{k}})\, d\tilde{\boldsymbol{k}}, \\
A^\dagger(v) = \sum_{j=1}^{M} A_j^\dagger(v), &\qquad A(v) = \sum_{j=1}^{M} A_j(v),
\end{aligned}
\tag{80}
$$

which can be seen as a generalization of (19) and (21).

Lemma 4.1 ($A^\dagger, A$ are well–defined for $v(\boldsymbol{p}, \boldsymbol{k})$)**.** *Let $v : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{C}$ be smooth and satisfying the scaling condition* (39)*. Then,* (80) *entails well–defined operators*

$$
A_j^\dagger(v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}, \qquad A_j(v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}, \tag{81}
$$

which may be restricted[1] *to*

$$
A_j^\dagger(v) : \dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \dot{\mathcal{S}}_{\mathscr{F}}^\infty, \qquad A_j(v) : \dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \mathrm{Ren}^{\dot{\mathcal{Q}}}. \tag{82}
$$

We may even extend $A_j^\dagger(v) : \overline{\mathscr{F}}_{\mathrm{ex}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$.

Proof. Suppose, $\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$ and consider the expression $A_j^\dagger(v)\Psi$ in (80). This is a finite sum over products consisting of two factors. By definition of $\dot{\mathcal{S}}_1^\infty$, the first factor $v(\boldsymbol{p}_j, \boldsymbol{k}_\ell)$ in each product is smooth everywhere except where $\boldsymbol{k}_\ell = 0$. By definition of $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$, the second factor $\Psi(\boldsymbol{P} + e_j\boldsymbol{k}_\ell, \boldsymbol{K} \setminus \boldsymbol{k}_\ell)$ is smooth at all configurations, at which $\boldsymbol{K} \setminus \boldsymbol{k}_\ell$ contains no coordinate $\boldsymbol{k}_{\ell'} = 0$. So the product is smooth on $\dot{\mathcal{Q}}$. In addition, the factors v and Ψ scale polynomially as in (39), so $A^\dagger(v)\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$.

The expression for $A(v)\Psi$ in (21) is a (possibly divergent) integral for each fixed $(\boldsymbol{P}, \boldsymbol{K}) \in \dot{\mathcal{Q}}$. Since both the functions $\tilde{\boldsymbol{k}} \mapsto v(\boldsymbol{p}_j - \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}})$ and $\tilde{\boldsymbol{k}} \mapsto \Psi(\boldsymbol{P} - e_j\tilde{\boldsymbol{k}}, \boldsymbol{K}, \tilde{\boldsymbol{k}})$ are in $\dot{\mathcal{S}}_1^\infty$, the integral defines an element $\mathfrak{R} \in \mathrm{Ren}_1 \subset \mathrm{Ren}$ for each fixed $(\boldsymbol{P}, \boldsymbol{K})$. Thus, $A(v)\Psi \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

Both operators can be extended to $\overline{\mathscr{F}}$ by taking eRen–linear combinations.
For $\Psi \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$, the expression $A_j^\dagger(v)\Psi$ in (80) is again a sum of products, that are all in

[1] We use the same notation for all extended or restricted versions of operators, here.

$\mathrm{Ren}^{\dot{\mathcal{Q}}}$. By eRen–linearity, we can then extend $A_j^\dagger(v)$ to $\overline{\mathscr{F}}_{\mathrm{ex}}$.

□

We may also define the constituent operators of $H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty$, as well as $H_{0,x}$ on $\overline{\mathscr{F}}$:

Proposition 4.1 (Constituents of H are well–defined)**.** *Consider the momentum space definitions of H_0 (16), $A^\dagger(v)$ (19), $A(v)$ (21) and E_∞ (23), where we only assume $\theta, \omega, v \in \dot{\mathcal{S}}_1^\infty$ and an arbitrary self–energy function $E_1 : \mathbb{R}^d \to \mathrm{Ren}$. Then, the above four momentum space definitions entail well–defined operators*

$$\begin{aligned} H_0 :&\overline{\mathscr{F}} \to \overline{\mathscr{F}}, & A^\dagger(v) :&\overline{\mathscr{F}} \to \overline{\mathscr{F}}, \\ A(v) :&\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}, & E_\infty :&\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}. \end{aligned} \tag{83}$$

It is also possible to restrict

$$\begin{aligned} H_0 :&\dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \dot{\mathcal{S}}_{\mathscr{F}}^\infty, & A^\dagger(v) :&\dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \dot{\mathcal{S}}_{\mathscr{F}}^\infty, \\ A(v) :&\dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \mathrm{Ren}^{\dot{\mathcal{Q}}}, & E_\infty :&\dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \mathrm{Ren}^{\dot{\mathcal{Q}}}, \end{aligned} \tag{84}$$

or to extend

$$H_0 : \overline{\mathscr{F}}_{\mathrm{ex}} \to \overline{\mathscr{F}}_{\mathrm{ex}}, \qquad A^\dagger(v) : \overline{\mathscr{F}}_{\mathrm{ex}} \to \overline{\mathscr{F}}_{\mathrm{ex}}. \tag{85}$$

Further, the statements for H_0 equally hold true for $H_{0,x}$ and $H_{0,y}$.

Proof. Well–definedness and the mapping properties of $A^\dagger(v)$ and $A(v)$ are a consequence of Lemma 4.1: The function $v(\boldsymbol{k})$ in (19) and (21) can be seen as a special case of $v(\boldsymbol{p}, \boldsymbol{k})$ in (80). Taking the finite linear sum over $j \in \{1, \ldots, M\}$ sustains the mapping properties.

The operator H_0 as in (16) just multiplies with a function in $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$ in momentum space, so for $\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$, we also have $H_0\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$. The same holds true for $H_{0,x}$ and $H_{0,y}$. By an analogous argument, H_0 can be extended to $H_0 : \mathrm{Ren}^{\dot{\mathcal{Q}}} \to \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

Finally, we consider $E_\infty\Psi$ for $\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$. By (23), at each fixed $(\boldsymbol{P}, \boldsymbol{K}) \in \dot{\mathcal{Q}}$ the expression $E_\infty\Psi$ is defined as a finite sum over terms $E_1(\boldsymbol{p})\Psi(\boldsymbol{P}, \boldsymbol{K}) \in \mathrm{Ren}$. So indeed, $E_\infty\Psi \in \mathrm{Ren}^{\dot{\mathcal{Q}}}$.

Extensions to $\overline{\mathscr{F}}$ or $\overline{\mathscr{F}}_{\mathrm{ex}}$ can again be done by eRen–linear combination.

□

Thus, also the linear operator $H : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ is well–defined.

Finally, we prove that the momentum space definition (80) indeed entails certain canonical commutation relations on the extended state space:

Lemma 4.2 (Extended CCR)**.** *For $\phi, \varphi \in \dot{\mathcal{S}}_1^\infty$, the definitions* (19) *and* (21) *imply the commutation relations*

$$[A_j^\dagger(\varphi), A_{j'}^\dagger(\phi)] = 0, \qquad [A_j(\varphi), A_{j'}^\dagger(\phi)] = \begin{cases} \langle \varphi, \phi \rangle & \text{for } j = j' \\ V_{jj'}(\varphi^* \phi) & \text{for } j \neq j' \end{cases}, \tag{86}$$

as a strong operator identity. That is, we have operators

$$[A_j^\dagger(\varphi), A_{j'}^\dagger(\phi)] : \overline{\mathscr{F}} \to \overline{\mathscr{F}}, \qquad [A_j(\varphi), A_{j'}^\dagger(\phi)] : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}. \tag{87}$$

Here, the interaction potential $V_{jj'} : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$ for momentum transfer from fermion j' to j is given by

$$V_{jj'}(\varphi^* \phi)(\boldsymbol{P}, \boldsymbol{K}) := \int \varphi^*(\tilde{\boldsymbol{k}}) \phi(\tilde{\boldsymbol{k}}) \Psi(\boldsymbol{P} + (e_{j'} - e_j)\tilde{\boldsymbol{k}}, \boldsymbol{K}) \, d\tilde{\boldsymbol{k}}. \tag{88}$$

Proof. By Lemma 4.1, the products $A_j(\varphi) A_{j'}^\dagger(\phi)$ and $A_{j'}^\dagger(\phi) A_j(\varphi)$ are well–defined as operators $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$. A momentum space evaluation renders

$$\begin{aligned} &\big([A_j(\varphi), A_{j'}^\dagger(\phi)]\Psi\big)(\boldsymbol{P}, \boldsymbol{K}) \\ &= \int \varphi^*(\tilde{\boldsymbol{k}}) \phi(\tilde{\boldsymbol{k}}) \Psi(\boldsymbol{P} + (e_{j'} - e_j)\tilde{\boldsymbol{k}}, \boldsymbol{K}) \, d\tilde{\boldsymbol{k}} \\ &= \begin{cases} \langle \varphi, \phi \rangle \Psi(\boldsymbol{P}, \boldsymbol{K}) & \text{for } j = j' \\ \int \varphi^*(\tilde{\boldsymbol{k}}) \phi(\tilde{\boldsymbol{k}}) \Psi(\boldsymbol{P} + (e_{j'} - e_j)\tilde{\boldsymbol{k}}, \boldsymbol{K}) \, d\tilde{\boldsymbol{k}} & \text{for } j \neq j' \end{cases}. \end{aligned} \tag{89}$$

Similarly, Lemma 4.1 establishes that $A_j^\dagger(\varphi) A_{j'}^\dagger(\phi)$ and $A_{j'}^\dagger(\phi) A_j^\dagger(\varphi)$ are well–defined operators $\overline{\mathscr{F}} \to \overline{\mathscr{F}}$ and a short momentum space calculation verifies that they are equal.

□

The operator $V_{jj'}$ defined above can be seen as an interaction potential operator. Under an inverse Fourier transform $\mathcal{F}^{-1}$, it amounts to a multiplication operator in position space via

$$(V_{jj'}(\varphi^* \phi)\Psi)(\boldsymbol{X}, \boldsymbol{Y}) = \mathcal{F}^{-1}(\varphi^* s)(\boldsymbol{x}_j - \boldsymbol{x}_{j'}) \Psi(\boldsymbol{X}, \boldsymbol{Y}), \tag{90}$$

provided that $\mathcal{F}^{-1}(\varphi^* \phi)$ exists (e.g., for exact scaling degrees $\beta_\varphi + \beta_s > -d \Rightarrow \varphi^* s \in \mathcal{S}'(\mathbb{R}^d)$).
The following property about $V_{jj'}$ will become useful in later proofs:

Lemma 4.3. *If either of the functions $\phi, \varphi \in \dot{\mathcal{S}}_1^\infty$ is an element of $C_c^\infty(\mathbb{R}^d \setminus \{0\})$, then we even have*

$$V_{jj'}(\varphi^* \phi) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}. \tag{91}$$

Proof. First, let us consider $\Psi \in \dot{\mathcal{S}}^{\infty}_{\mathscr{F}}$. Without loss of generality, assume that $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, so φ is compactly supported, and the function

$$\tilde{\boldsymbol{k}} \mapsto \phi(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P} + (e_{j'} - e_j)\tilde{\boldsymbol{k}}, \boldsymbol{K}),$$

is smooth everywhere on that support. So the function

$$\tilde{\boldsymbol{k}} \mapsto \varphi^*(\tilde{\boldsymbol{k}})\phi(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P} + (e_{j'} - e_j)\tilde{\boldsymbol{k}}, \boldsymbol{K}) \quad \text{is in} \quad C_c^\infty,$$

and the integral over it converges to a $\mathbb{C}$–number.
We now show that this number depends smoothly on $(\boldsymbol{P}, \boldsymbol{K}) \in \dot{\mathcal{Q}}$: Consider any multi–index α corresponding to a derivative ∂^α composed of arbitrarily many partial derivatives $\partial_{\boldsymbol{p}_j}, \partial_{\boldsymbol{k}_\ell}$ with $j, \ell \in \mathbb{N}$. Then also $\partial^\alpha \Psi \in \dot{\mathcal{S}}^{\infty}_{\mathscr{F}}$ and by the same arguments as above,

$$\tilde{\boldsymbol{k}} \mapsto \varphi^*(\tilde{\boldsymbol{k}})\phi(\tilde{\boldsymbol{k}})\partial^\alpha\Psi(\boldsymbol{P} + (e_{j'} - e_j)\tilde{\boldsymbol{k}}, \boldsymbol{K}) \quad \text{is in} \quad C_c^\infty. \tag{92}$$

So the integral converges absolutely, derivative and integral commute, and we obtain $\partial^\alpha(V_{jj'}(\varphi^*\phi)\Psi)(\boldsymbol{P}, \boldsymbol{K}) \in \mathbb{C}$ for any multi–index α. Hence, $(V_{jj'}(\varphi^*\phi)\Psi)(\boldsymbol{P}, \boldsymbol{K})$ is smooth at $(\boldsymbol{P}, \boldsymbol{K}) \in \dot{\mathcal{Q}}$.

Polynomial scaling of $V_{jj'}(\varphi^*\phi)\Psi$ can be seen as follows: Under the coordinate rotation $\boldsymbol{p}_+ := \boldsymbol{p}_j + \boldsymbol{p}_{j'}$ and $\boldsymbol{p}_- := \boldsymbol{p}_j - \boldsymbol{p}_{j'}$, the expression $V_{jj'}\Psi$ becomes a convolution in $\boldsymbol{p}_-$ of a polynomially scaling function with the function $\varphi^*\phi \in C_c^\infty$. Polynomial scaling bounds are neither affected by the coordinate rotation nor by the convolution with a C_c^∞–function. So indeed, $V_{jj'}(\varphi^*\phi)\Psi \in \dot{\mathcal{S}}^{\infty}_{\mathscr{F}}$, and we have that $V_{jj'}(\varphi^*\phi) : \dot{\mathcal{S}}^{\infty}_{\mathscr{F}} \to \dot{\mathcal{S}}^{\infty}_{\mathscr{F}}$.

This mapping property extends to $V_{j'j}(\varphi^*\phi) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ by eRen–linearity.

□

Remarks.

1. In (86), we have not included the commutation relation for annihilation operators $[A_j(\varphi), A_{j'}(\phi)] = 0$. The reason is that products of two or more annihilation operators are not necessarily defined, since we only have $A_j(\varphi) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$. An arbitrary product $A_j(\varphi)A_{j'}(\phi)\Psi$ with $\Psi \in \dot{\mathcal{S}}^{\infty}_{\mathscr{F}}$ contains a double integral

$$\begin{aligned} &\big(A_j(\varphi)A_{j'}(\phi)\Psi\big)(\boldsymbol{P}, \boldsymbol{K}) \\ =&\sqrt{(N+1)(N+2)} \int\int \varphi(\tilde{\boldsymbol{k}})^*\phi(\tilde{\boldsymbol{k}}')^*\Psi(\boldsymbol{P} - e_j\tilde{\boldsymbol{k}} - e_{j'}\tilde{\boldsymbol{k}}', \boldsymbol{K}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}')\, d\tilde{\boldsymbol{k}} d\tilde{\boldsymbol{k}}', \end{aligned} \tag{93}$$

   where the first integral produces a configuration space function $\dot{\mathcal{Q}} \to \text{Ren}$. And a second integral over such a function can generally not be interpreted as an element in Ren.

A definition of such operator products would require a modification of $\overline{\mathscr{F}}_{\text{ex}}$ such that it also accommodates general divergent integrals over multiple coordinates of a $\dot{\mathcal{S}}^{\infty}_{\mathscr{F}}$–function as in (93). We postpone the investigation of such choices for $\overline{\mathscr{F}}_{\text{ex}}$ to future investigations.

5 Dressing on the Extended State Space

Our next step is to define a dressing operator $W(s)$ with $s \in \dot{\mathcal{S}}^{\infty}_1$. To do so, a naive approach would be to start from the expression $W(s) = e^{A^\dagger_M(s)-A_M(s)} \dots e^{A^\dagger_1(s)-A_1(s)}$ (with M depending on the fermion sector) and expand the exponentials into series

$$e^{A^\dagger_j(s)-A_j(s)} = \sum_{n\in\mathbb{N}_0} \frac{(A^\dagger_j(s) - A_j(s))^n}{n!},$$

which can be multiplied out. There are two difficulties with this approach:

- Some terms in the resulting sum contain two or more annihilation operators $A(s)$ (see Remark 1).
- There is an infinite number of such terms.

So with the current definition of $\overline{\mathscr{F}}_{\text{ex}}$ we cannot simply define $W(s)$ as an operator $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$. Instead, we pursue a different approach and define $W(s) : \mathcal{D}_W \to \mathscr{F}_{\text{ex}}$. Here, we choose $\mathcal{D}_W \subset \overline{\mathscr{F}}_{\text{ex}}$ in (131), such that $\mathcal{D}_{\mathscr{F}}$ (which is a symmetrized version of $\mathcal{D}_W \cap L^2$) is dense in $\mathscr{F}$.

If we consider $W_{\mathscr{F}}(\varphi) : \mathscr{F} \to \mathscr{F}, \varphi \in \mathfrak{h}$ as a unitary operator on Fock space, together with some suitable $\Psi \in \mathscr{F}$, then there is a well–defined expression (similar to (36)) for $W_{\mathscr{F}}(\varphi)\Psi$ as an L^2–function on momentum–configuration space. For φ replaced by $s \in \dot{\mathcal{S}}^{\infty}_1$, we may then define $W(s)\Psi \in \overline{\mathscr{F}}$ based on the momentum space expression of $W_{\mathscr{F}}(\varphi)\Psi$.

The domain $\mathcal{D}_W$ in (131) is generated by vectors Ψ_C of the form

$$\Psi_C = W_1(\varphi)A^\dagger_j(v)\Psi_m \qquad \text{or} \qquad \Psi_C = XW_1(\varphi)\Psi_m,$$

where:

- $\Psi_m = \Psi_{mx} \otimes \Omega_y \in \mathscr{F}$ with $\Psi_{mx} \in \mathcal{S}(\mathcal{Q}_x)$ (i.e., we have a Schwartz function) and Ω_y describing the boson field in the vacuum.
- $W_1(\varphi) = e^{A^\dagger_1(\varphi)-A_1(\varphi)}, \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}^{\infty}_1$ describes a dressing induced by only the first fermion.

- $A_j^\dagger(v), v \in \dot{\mathcal{S}}_1^\infty$ describes creation by only the fermion with number $j \in \{1, \dots, M\}$.
- X is a linear combination of Ren_1–constants and operators $V_{jj'}(v^* s)$ that formally commute with $W(s)$.

When setting $X = 1$ (which formally commutes with $W(s)$), we see that $\mathcal{D}_W$ contains vectors of the kind $W_1(\varphi)\Psi_m$. We show in Lemma 5.4, that these are equal to $W_{\mathscr{F},1}(\varphi)\Psi_m$ and, after symmetrization, span a dense subspace of $\mathscr{F}$ (Lemma 5.1). This will later allow for a dense definition of $\widetilde{H}$.

The definition of $W(s)\Psi_C$ now exactly works as explained above: We establish a momentum space expression in case $s, v \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ using Lemma 5.2. Then, we generalize to $s, v \in \dot{\mathcal{S}}_1^\infty$ by a suitable definition. As discussed in the introduction, we remove certain exponential factors of the form $e^{V_{jj'}}$ in an ad–hoc modification. Thus, $W_j(\varphi_j)$ differs from $W_{\mathscr{F},j}(\varphi_j)$ for $\varphi_j \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$. However, we show in Lemma 5.5 that any $V_{j'j''}$ commutes with $W_{\mathscr{F},j}(\varphi_j)$, so the omitted factor $e^{V_{jj'}}$ can heuristically be "pulled into any position". Heuristically, this factor disappears when performing an undressing, which justifies the omission within the computation of $\widetilde{H}$.

5.1 Bosonic Dressing $W_y(\varphi)$

The upcoming proofs are based on some well–known facts about coherent states, where only bosons are present, i.e., $\Psi_y \in \mathscr{F}_y$. The momentum space representation of the bosonic creation and annihilation operators $a^\dagger(v), a(v)$ with form factor $v \in \mathfrak{h}$ given by

$$\begin{aligned} (a^\dagger(v)\Psi_y)(\boldsymbol{K}) &:= \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} v(\boldsymbol{k}_\ell)\Psi_y(\boldsymbol{K} \setminus \boldsymbol{k}_\ell) \\ (a(v)\Psi_y)(\boldsymbol{K}) &:= \sqrt{N+1} \int v(\tilde{\boldsymbol{k}})^* \Psi_y(\boldsymbol{K}, \tilde{\boldsymbol{k}})\, d\tilde{\boldsymbol{k}}. \end{aligned} \tag{94}$$

This definition implies that the commutation relations $[a(v_1), a^\dagger(v_2)] = \langle v_1, v_2 \rangle$ hold as a strong operator identity on a dense domain in $\mathscr{F}_y$. These operators $a^\dagger(v), a(v)$ substantially differ from $A^\dagger(v), A(v)$ defined in (19), (21), which create one boson at each position of a fermion, whereas in $\mathscr{F}_y$, there are no fermions.

Using $a^\dagger(v), a(v)$, we may define a set of displacement operators

$$W_y(\varphi) = e^{a^\dagger(\varphi) - a(\varphi)}, \tag{95}$$

and coherent states $\Psi_y(\varphi) := W_y(\varphi)\Omega_y$. Indeed, $W_y(\varphi)$ is well–defined, since for $\varphi \in \mathfrak{h}$, we have the bounds

$$\left\| a^\dagger(\varphi)\Psi_y \right\| \leq \left\| (N+1)^{1/2}\Psi_y \right\| \, \|\varphi\|, \qquad \left\| a(\varphi)\Psi_y \right\| \leq \left\| N^{1/2}\Psi_y \right\| \, \|\varphi\|, \tag{96}$$

so the exponential series (95) converges in norm:

$$\begin{aligned} \Psi_y(\varphi) = W_y(\varphi)\Omega_y &:= \sum_{k=0}^{\infty} \frac{1}{k!}(a^\dagger(\varphi) - a(\varphi))^k \Omega_y \\ \text{where} \quad \left\| \frac{1}{k!}(a^\dagger(\varphi) - a(\varphi))^k \Omega_y \right\| &\leq \frac{1}{k!} \left\| 2^k (k!)^{1/2} \Omega_y \right\| \, \|\varphi\|^k = (k!)^{-1/2} \|2\varphi\|^k. \end{aligned} \tag{97}$$

Here, we used in the second line that $(a^\dagger(\varphi) - a(\varphi))^k \Omega_y$ occupies only sectors in Fock space with $\leq k$ particles, so we can set $N \leq (k-1)$ in (96). Subsequent application of (96) leads to the factor $(k!)^{1/2}$.
In momentum space representation and in terms of tensor products,

$$\Psi_y(\varphi)(\boldsymbol{K}) = e^{-\frac{\|\varphi\|^2}{2}} \frac{1}{\sqrt{N!}} \left(\prod_{\ell=1}^{N} \varphi(\boldsymbol{k}_\ell) \right) \quad \Leftrightarrow \quad \Psi_y(\varphi) = \sum_{N=0}^{\infty} \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \underbrace{\varphi \otimes_S \ldots \otimes_S \varphi}_{N \text{ times}}. \tag{98}$$

A calculation similar to (97) verifies that $W_y(\varphi)$ can be defined on all Ψ_y with finite particle number, i.e., $\Psi_y \in \mathscr{F}_{\text{fin},y}$ with

$$\mathscr{F}_{\text{fin},y} := \left\{ \Psi_y \in \mathscr{F}_y \;\middle|\; \exists N_{\max} \in \mathbb{N} : \; \Psi_y^{(N)} = 0 \; \forall N > N_{\max} \right\}. \tag{99}$$

And by continuity, we can thus define $W_y(\varphi)$ on all of $\mathscr{F}_y$.

Moreover, the $W_y(\varphi)$ are unitary, so $\left\|\Psi_y(\varphi)\right\| = 1$, and they satisfy the Weyl relations. Further, it is a well–known fact that the span of the set of coherent states $\{\Psi_y(\varphi) \mid \varphi \in \mathfrak{h}\}$ is dense in $\mathscr{F}_y$ [29, Prop. 12]. In addition,

$$\langle \Psi_y(\varphi_1), \Psi_y(\varphi_2) \rangle_{\mathscr{F}_y} = e^{-\frac{\|\varphi_1\|^2 + \|\varphi_2\|^2}{2}} e^{\langle \varphi_1, \varphi_2 \rangle}, \tag{100}$$

so

$$\left\|\Psi_y(\varphi_1) - \Psi_y(\varphi_2)\right\|^2 = \left\|\Psi_y(\varphi_1)\right\|^2 + \left\|\Psi_y(\varphi_2)\right\|^2 - 2e^{-\frac{\|\varphi_1\|^2 + \|\varphi_2\|^2}{2}} \mathrm{Re}\big(e^{\langle \varphi_1, \varphi_2 \rangle}\big). \tag{101}$$

As $\mathfrak{h}$ is separable, we can find a countable dense set $(\varphi_n)_{n \in \mathbb{N}}$ in $\mathfrak{h}$, such that $(\Psi_y(\varphi_n))_{n \in \mathbb{N}}$ is also dense in the coherent states. So

$$\mathrm{span}\big\{ \Psi_y(\varphi_n) \;\big|\; n \in \mathbb{N} \big\} \quad \text{is dense in } \mathscr{F}_y. \tag{102}$$

Since $\mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty \supset C_c^\infty$ is dense in $\mathfrak{h}$ the above statements hold true, if we replace $\varphi_n \in \mathfrak{h}$ by $\varphi_n \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$.

5.2 Dressing Induced by Fermions $W_1(\varphi)$

5.2.1 Density

Now, let us turn to the case with two particle species, i.e., $\mathscr{F} = \mathscr{F}_x \otimes \mathscr{F}_y$ and $A^\dagger, A$ instead of $a^\dagger, a$. In order to make an analogous statement to (102) work, we restrict from $A^\dagger(\varphi) = \sum_{j=1}^M A_j^\dagger(\varphi)$ to $A_1^\dagger(\varphi)$, i.e., creation by only the first fermion. Just as $W_y(\varphi)$, the Fock space operator $W_{\mathscr{F},1}(\varphi) = e^{A_1^\dagger(\varphi) - A_1^\dagger(\varphi)}$ can be defined in analogy to $W_y(\varphi)$. The operators $A_1^\dagger(\varphi)$ and $W_{\mathscr{F},1}(\varphi)$ break the fermionic symmetry, so they map $\mathscr{F} \to L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ (instead of $\mathscr{F} \to \mathscr{F}$). We will therefore proceed by considering vectors $\Psi \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$, i.e., with only bosonic exchange symmetry.

As a "cyclic set" of vectors Ψ_m, used for generating further domains, we choose

$$\mathcal{C}_{WS} := \mathcal{S}(\mathcal{Q}_x) \otimes \{\Omega_y\} \subset L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y. \tag{103}$$

Since the boson field is in the vacuum, $\mathcal{C}_{WS}$ is obviously not dense in $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$. However, it generates a dense subspace by applying operators $W_{\mathscr{F},1}(\varphi)$ to it. The momentum representation after such an application is given by

$$(W_{\mathscr{F},1}(\varphi)\Psi_m)(\boldsymbol{P}, \boldsymbol{K}) = \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \left(\prod_{\ell=1}^N \varphi(\boldsymbol{k}_\ell) \right) \Psi_{mx} \left(\boldsymbol{p}_1 + \sum_{\ell=1}^N \boldsymbol{k}_\ell, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_M \right). \tag{104}$$

Definition 5.1. The **span of coherent states created by the first fermion** is given by

$$\mathcal{D}_{WS} := \operatorname{span}\big\{ W_{\mathscr{F},1}(\varphi)\Psi_m \;\big|\; \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty,\ \Psi_m \in \mathcal{C}_{WS} \big\}. \tag{105}$$

With (105), it is true that

Lemma 5.1. *$(S_- \otimes I)[\mathcal{D}_{WS}]$ is dense in $\mathscr{F} = \mathscr{F}_x \otimes \mathscr{F}_y$*

The proof is based on denseness of coherent states in $\mathscr{F}_y$ and can be found in Appendix A.

5.2.2 Dressed One–Boson States

Just as $W_{\mathscr{F},1}(\varphi)$, we may define $W_{\mathscr{F},j}(\varphi), j \in \mathbb{N}$ and $W_{\mathscr{F}}(\varphi) = W_{\mathscr{F},M}(\varphi) \ldots W_{\mathscr{F},1}(\varphi)$ with $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ on each M–fermion sector. These operators are all unitary on $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ and well–defined on $\mathscr{F}$.
We will now establish some useful commutation relations on the dense subspace

$$\mathscr{F}_{\text{fin}} := \mathscr{F}_x \otimes \mathscr{F}_{\text{fin},y} \subset \mathscr{F}, \tag{106}$$

with $\mathscr{F}_{\text{fin},y}$ defined in (99).

Lemma 5.2 (Commutation relations for $W_{\mathscr{F}}$)**.** *For $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$, we have the following strong operator identities*[2] *on $\mathscr{F}_{\text{fin}}$:*

$$\begin{aligned} W_{\mathscr{F},j}(\varphi)A^\dagger_{j'}(\phi) &= \begin{cases} \big(A^\dagger_{j'}(\phi) - \langle \varphi, \phi\rangle\big)W_{\mathscr{F},j}(\varphi) & \text{if } j = j' \\ \big(A^\dagger_{j'}(\phi) - V_{jj'}(\varphi^*\phi)\big)W_{\mathscr{F},j}(\varphi) & \text{if } j \neq j' \end{cases}, \\ W_{\mathscr{F}}(\varphi)A^\dagger_{j'}(\phi) &= \big(A^\dagger_{j'}(\phi) - \langle \varphi, \phi\rangle - V_{\bullet j'}(\varphi^*\phi)\big)W_{\mathscr{F}}(\varphi), \\ W_{\mathscr{F},j}(\varphi)A^\dagger(\phi) &= \big(A^\dagger(\phi) - \langle \varphi, \phi\rangle - V_{j\bullet}(\varphi^*\phi)\big)W_{\mathscr{F},j}(\varphi), \\ W_{\mathscr{F}}(\varphi)A^\dagger(\phi) &= \big(A^\dagger(\phi) - M\langle \varphi, \phi\rangle - V(\varphi^*\phi)\big)W_{\mathscr{F}}(\varphi), \end{aligned} \tag{107}$$

as well as

$$\begin{aligned} W_{\mathscr{F},j'}(\varphi)A_{j}(\phi) &= \begin{cases} \big(A_{j}(\phi) - \langle \phi, \varphi\rangle\big)W_{\mathscr{F},j'}(\varphi) & \text{if } j = j' \\ \big(A_{j}(\phi) - V_{jj'}(\phi^*\varphi)\big)W_{\mathscr{F},j'}(\varphi) & \text{if } j \neq j' \end{cases}, \\ W_{\mathscr{F}}(\varphi)A_{j}(\phi) &= \big(A_{j}(\phi) - \langle \phi, \varphi\rangle - V_{j\bullet}(\phi^*\varphi)\big)W_{\mathscr{F}}(\varphi), \\ W_{\mathscr{F},j'}(\varphi)A(\phi) &= \big(A(\phi) - \langle \phi, \varphi\rangle - V_{\bullet j'}(\phi^*\varphi)\big)W_{\mathscr{F},j'}(\varphi), \\ W_{\mathscr{F}}(\varphi)A(\phi) &= \big(A(\phi) - M\langle \phi, \varphi\rangle - V(\phi^*\varphi)\big)W_{\mathscr{F}}(\varphi), \end{aligned} \tag{108}$$

with $V_{jj'}$ defined in (88) *and*[3]

$$\begin{aligned} V_{\bullet j'}(\varphi^*\phi) &:= \sum_{j:j\neq j'} V_{jj'}(\varphi^*\phi), \qquad V_{j\bullet}(\varphi^*\phi) := \sum_{j':j\neq j'} V_{jj'}(\varphi^*\phi), \\ V(\varphi^*\phi) &:= \sum_{j\neq j'} V_{jj'}(\varphi^*\phi). \end{aligned} \tag{109}$$

The proof of Lemma 5.2 is straightforward by applying the CCR. We present it in Appendix B.

5.3 Extended Dressing $W(s)$

We would now like the relations in (107) to also hold true if we replace $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ by $s, v \in \dot{\mathcal{S}}_1^\infty$. In that case, the Fock space operators $W_{\mathscr{F},j}$ turn into extended operators W_j. More precisely, it would be desirable to have

$$W_j(s)A^\dagger_{j'}(v)\Psi_m = \begin{cases} \big(A^\dagger_{j'}(v) - \langle s, v\rangle\big)W_j(s)\Psi_m & \text{if } j = j' \\ \big(A^\dagger_{j'}(v) - V_{jj'}(s^*v)\big)W_j(s)\Psi_m & \text{if } j \neq j' \end{cases}, \tag{110}$$

[2]By a strong operator identity $A = B$ for $A, B : \mathscr{F} \to L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$, we mean that $A\Psi = B\Psi\ \forall \Psi \in \mathscr{F}$, even if possibly $A\Psi, B\Psi \notin \mathscr{F}$.

[3]Here, $\sum_{j:j\neq j'}$ is to be understood as a sum over only j, while $\sum_{j\neq j'}$ is a sum over both j and j'. The second kind of sum will appear more often, so we give it a shorter notation.

for $\Psi_m \in \mathcal{C}_{WS}$. By Lemma 4.2, and since $\langle s, v\rangle \in \mathrm{Ren}_1$, we may obviously interpret

$$V_{jj'}(s^*v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}, \qquad \langle s, v\rangle : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}. \tag{111}$$

So if we can define $W_j(s)\Psi_m \in \overline{\mathscr{F}}$, then the right–hand side of (110) serves as a definition for $W_j(s)A^\dagger_{j'}(v)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}$. And if we can further define products like $W(s)W_1(\varphi)\Psi_m = W_M(s)\dots W_1(s)W_1(\varphi)\Psi_m \in \overline{\mathscr{F}}$, then a generalization of (107) may even be used to define $W(s)W_1(\varphi)A^\dagger_{j'}(v)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}$.
However, before doing so, it is first necessary to specify what $W_1(s)W_1(\varphi)$ is, which we will do by introducing some "extended Weyl relations".

5.3.1 Extended Weyl Relations

In order to treat products of factors $W_j(s), s \in \dot{\mathcal{S}}_1^\infty$, we introduce an extended Weyl algebra $\overline{\mathcal{W}}$ that is generated by all $W_j(s)$ and taken over the field eRen (as in Definition 3.7). Recall that $\mathfrak{c} \in \mathrm{eRen}$ is a fraction of linear combinations of exponentials $e^{\mathfrak{r}}$, with $\mathfrak{r} \in \mathrm{Ren}_1$ being a possibly divergent integral (see Definition 3.2). Multiplication on $\overline{\mathcal{W}}$ is defined by the Weyl relations

$$\begin{aligned} W_j(s)^{-1} &= W_j(-s) \\ W_j(s_1)W_j(s_2) &= e^{-\frac{i}{2}\sigma(s_1,s_2)}W_j(s_1+s_2), \end{aligned} \tag{112}$$

with symplectic form

$$\begin{aligned} \sigma = \dot{\mathcal{S}}_1^\infty \times \dot{\mathcal{S}}_1^\infty &\to \mathrm{Ren}_1 \\ (s_1, s_2) &\mapsto \langle s_1, s_2\rangle - \langle s_2, s_1\rangle. \end{aligned} \tag{113}$$

Note that $e^{-\frac{i}{2}\sigma(s_1,s_2)} = e^{\mathrm{Im}\langle s_1,s_2\rangle} \in \mathrm{eRen}$ is not necessarily a complex number.

This $\overline{\mathcal{W}}$ can be seen as an "almost–extension" of the usual Weyl algebra generated by $\{W_j(s) \mid s \in \mathfrak{h}\}$ (strictly speaking, it only extends some Weyl algebra $\mathcal{W}_0$ generated by $\{W_j(s) \mid s \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty\}$).

The definition of the extended Weyl algebra now allows us to write

$$\begin{aligned} W(s)W_1(\varphi) =& W_M(s)\dots W_2(s)W_1(s)W_1(\varphi) \\ =& e^{\mathrm{Im}\langle s,\varphi\rangle}W_M(s)\dots W_2(s)W_1(s+\varphi). \end{aligned} \tag{114}$$

So $W(s)W_1(\varphi)$ can be brought into the form $W_M(s_M)\dots W_1(s_1)$ times an eRen–factor, which is the same on each M–fermion sector.

5.3.2 Extended Dressing on Coherent States

In order to define vectors of the kind $W_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}$, we make use of the momentum space definition of $W_{\mathscr{F},M}(\varphi_M)\dots W_{\mathscr{F},1}(\varphi_1)\Psi_m$ for $\varphi_\ell \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$. For two dressing operators with $j \neq j'$, the Baker–Campbell–Hausdorff formula implies

$$\begin{aligned} W_{\mathscr{F},j}(\varphi_j)W_{\mathscr{F},j'}(\varphi_{j'})\Psi_m &= e^{A_j^\dagger(\varphi_j)-A_j(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})-A_{j'}(\varphi_{j'})}\Psi_m \\ &= e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}}e^{A_j^\dagger(\varphi_j)}e^{-A_j(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})}\underbrace{e^{-A_{j'}(\varphi_{j'})}\Psi_m}_{=\Psi_m} \\ &= e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}-[A_j(\varphi_j),A_{j'}^\dagger(\varphi_{j'})]}e^{A_j^\dagger(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})}e^{-A_j(\varphi_j)}\Psi_m \\ &= e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}-V_{jj'}(\varphi_j^*\varphi_{j'})}e^{A_j^\dagger(\varphi_j)+A_{j'}^\dagger(\varphi_{j'})}\Psi_m. \end{aligned} \tag{115}$$

Here, we used that $V_{jj'}$ commutes with all $A_{j''}^\dagger$ and $A_{j''}$, which follows by the same arguments as in the proof of Lemma 5.5 below. Thus, all double commutators between A– and $A^\dagger$–operators vanish. The generalization to arbitrarily many factors, $W_{\mathscr{F},M}(\varphi_M)\dots W_{\mathscr{F},1}(\varphi_1)\Psi_m$ is straightforward, and we obtain an exponential of constants and $V_{jj'}$–terms, followed by $e^{A_M^\dagger(\varphi_M)+\dots+A_1^\dagger(\varphi_1)}\Psi_m$.
We now define $W_M(\varphi_M)\dots W_1(\varphi_1)$ by dropping the $V_{jj'}$–terms in (115), which yields the following momentum space expression for $\Psi_m = \Psi_{mx}\otimes\Omega_y \in \mathcal{C}_{WS}$

$$(W_M(\varphi_M)\dots W_1(\varphi_1)\Psi_m)(\boldsymbol{P},\boldsymbol{K}) := \frac{1}{\sqrt{N!}}e^{-\sum_{j=1}^M\frac{\|\varphi_j\|^2}{2}}\sum_\sigma\left(\prod_{\ell=1}^N\varphi_{\sigma(\ell)}(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P}'), \tag{116}$$

where the sum over σ runs over all M^N maps

$$\sigma : \{1,\dots,N\} \to \{1,\dots,M\}, \tag{117}$$

assigning each boson ℓ to a fermion $j = \sigma(\ell)$. The shifted momentum, illustrated in Figure 4, is then

$$\boldsymbol{P}' := \boldsymbol{P} + \sum_\ell e_{\sigma(\ell)}\boldsymbol{k}_\ell. \tag{118}$$

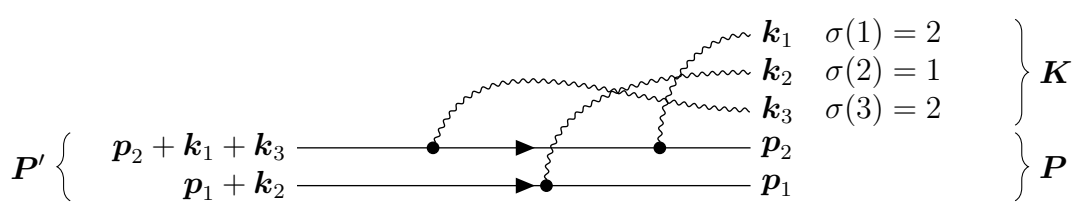

Figure 4: An example for momentum shift within the dressing.

Lemma 5.3 (Products of W are well–defined)**.** *Consider a sequence* $(s_j)_{j\in\mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$ *and* $\Psi_m \in \mathcal{C}_{WS}$*. Then, the momentum space definition* (116) *renders a well-defined vector*

$$W_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}, \tag{119}$$

where (119) *is to be interpreted as a sector–wise definition in* $M \in \mathbb{N}$*.*

Proof. Copying the momentum space definition (116), we obtain

$$(W_M(s_M)\dots W_1(s_1)\Psi_m)(\boldsymbol{P},\boldsymbol{K}) := \frac{1}{\sqrt{N!}} e^{-\sum_{j=1}^M \frac{\|s_j\|^2}{2}} \sum_\sigma \left(\prod_{\ell=1}^N s_{\sigma(\ell)}(\boldsymbol{k}_\ell)\right) \Psi_{mx}(\boldsymbol{P}'). \tag{120}$$

Obviously, $\left(\prod_{\ell=1}^N s_{\sigma(\ell)}(\boldsymbol{k}_\ell)\right) \Psi_{mx}(\boldsymbol{P}')$ defines a function in $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$, which is still true after taking the finite sum over σ.

Further, we have $\|s_j\|^2 = \langle s_j, s_j\rangle \in \mathrm{Ren}_1$, so $e^{-\sum_{j=1}^M \frac{\|s_j\|^2}{2}} \in \mathrm{eRen}$.

Therefore, the expression (116) defines an element of $\overline{\mathscr{F}}$.

□

This already allows us to define $W_M(s_M)\dots W_1(s_1)$ on vectors $\Psi_m \in \mathcal{C}_{WS}$ with the boson field in the vacuum. In order to define $W_M(s_M)\dots W_1(s_1)$ also on a dense domain in $\mathscr{F}$, we extend the definition to vectors $W_{\mathscr{F},1}(\varphi)\Psi_m \in \mathcal{D}_{WS}$, whose symmetrized span, by Lemma 5.1, is dense in $\mathscr{F}$. This extension is done by assuming that $W_{\mathscr{F},1}(\varphi)$ can be merged into $W_1(s_1)$, just as $W_1(\varphi)$ in (114).
We will also allow for a treatment of state vectors by using the operator $(S_- \otimes I)$, which can obviously be extended to $(S_- \otimes I) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ or $(S_- \otimes I) : \overline{\mathscr{F}}_{\mathrm{ex}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$, using the momentum space definition (11).

Definition 5.2. Let $(s_j)_{j\in\mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$. Then, by Lemma 5.3, copying the momentum space definition (116) results in a well–defined **product of dressing operators**

$$\begin{aligned} W_M(s_M)\dots W_1(s_1) &: \mathcal{D}_{WS} \to \overline{\mathscr{F}}, \\ W_M(s_M)\dots W_1(s_1) W_{\mathscr{F},1}(\varphi)\Psi_m &:= e^{\mathrm{Im}\langle s_1,\varphi\rangle} W_M(s_M)\dots W_1(s_1+\varphi)\Psi_m, \end{aligned} \tag{121}$$

where M is the respective fermion number on each sector. Further, we define the extension to **symmetrized vectors**

$$W_M(s_M)\dots W_1(s_1) : (S_- \otimes I)[\mathcal{D}_{WS}] \cup \mathcal{D}_{WS} \to \overline{\mathscr{F}}, \tag{122}$$

by imposing that $W_M(s_M)\dots W_1(s_1)$ shall commute with the symmetrization operator $(S_- \otimes I)$.

With this definition, it is true that

Lemma 5.4. *For all $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$, it holds that*

$$W_{\mathscr{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m, \tag{123}$$

in terms of momentum space functions.

Proof. Consider (115) with $j = 1$ and $\varphi_{j'} = 0$. Then, the $V_{jj'}$–term vanishes, so no $V_{jj'}$–terms are dropped when copying momentum space expressions in the transition $W_{\mathscr{F},1} \to W_1$ and indeed $W_{\mathscr{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m$.

□

Remarks.

2. It may seem natural to extend Definition 5.2 to a general $\Psi \in \mathscr{F}$. By Lemma 5.1, we can write Ψ as a symmetrized version of $\Psi' = \sum_{nn'} W_1(\varphi_n)\Psi_{n'}$ with $\varphi_n \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and $\Psi_{n'} \in \mathcal{C}_{WS}$. In that case, $W(s)\Psi = \sum_{nn'} W(s)W_1(\varphi_n)\Psi_{n'}$ contains a possibly infinite sum over functions $\dot{\mathcal{Q}} \to \mathbb{C}$, which may not converge.
However, our aim is to give a dense definition of $\widetilde{H} : \mathscr{F} \supset \mathrm{dom}(\widetilde{H}) \to \mathscr{F}$, so it suffices to consider the action of $W(s)$, and $HW(s)$ on a dense subset of $\mathscr{F}$, such as $(S_- \otimes I)[\mathcal{D}_{WS}]$.

3. Concerning the renormalization classes: Two ESS vectors $W(s)W_1(\varphi)\Psi_m$ and $W(s)W_1(\tilde{\varphi})\Psi_{m'}$ with Ψ_m and $\Psi_{m'}$ concentrated on the same M–fermion sector can be added if the wave function renormalizations $\mathfrak{c} = e^{\mathfrak{r}}, \mathfrak{r} = -\frac{\|s\|^2}{2}$ belong to the same renormalization factor class, i.e.,

$$\begin{aligned} \mathfrak{r} - \tilde{\mathfrak{r}} \in \mathbb{C} \quad &\Leftrightarrow \quad \big|\, \|s+\varphi\|^2 - \|s+\tilde{\varphi}\|^2 + 2\mathrm{Im}\langle s, \varphi - \tilde{\varphi}\rangle \,\big| < \infty \\ &\Leftrightarrow \quad \left| 2\mathrm{Re}\langle s, \varphi - \tilde{\varphi}\rangle + 2\mathrm{Im}\langle s, \varphi - \tilde{\varphi}\rangle + \underbrace{\|\varphi\|^2}_{<\infty} - \underbrace{\|\tilde{\varphi}\|^2}_{<\infty} \right| < \infty \\ &\Leftarrow \quad |\langle s, \varphi - \tilde{\varphi}\rangle| < \infty. \end{aligned} \tag{124}$$

That means, convergence of the integral $\int s(\boldsymbol{k})^*(\varphi(\boldsymbol{k}) - \tilde{\varphi}(\boldsymbol{k}))\, d\boldsymbol{k}$ ensures that the renormalization classes coincide. Note that both Re and Im above may be infinite, but cancel each other out.

5.3.3 Extended Dressing on One–Boson States

Now, as announced, when replacing $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ by $v, s \in \dot{\mathcal{S}}_1^\infty$ in (107), we obtain a well–defined right–hand side. This allows for the following extension of dressing operator products

Definition 5.3. Let $v \in \dot{\mathcal{S}}_1^\infty, (s_j)_{j\in\mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$. We extend the **product of dressing operators** to one–boson states via

$$W_M(s_M)\dots W_1(s_1)A^\dagger_{j'}(v)\Psi_m \in \overline{\mathscr{F}}_{\text{ex}}, \tag{125}$$

where M is the respective fermion number on each sector, via

$$\begin{aligned} W_M(s_M)\dots W_1(s_1)A^\dagger_{j'}(v)\Psi_m &:= \left(A^\dagger_{j'}(v) - \sum_{j=1}^M X_j\right) W_M(s_M)\dots W_1(s_1)\Psi_m, \\ \text{with} \quad X_j &= \begin{cases} \langle s, v\rangle & \text{if } j = j' \\ V_{jj'}(s^*v) & \text{if } j \neq j' \end{cases}. \end{aligned} \tag{126}$$

This operator can further be extended to **symmetrized vectors** by imposing that $W_M(s_M)\dots W_1(s_1)$ shall commute with the symmetrization operator $(S_- \otimes I)$.

It is easy to see that the right–hand side of (126) makes sense: By Lemma 5.3, we have $W_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}$. Lemma 4.1 implies that $A^\dagger_{j'}(v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ and by Lemma 4.2 and $\langle s, v\rangle \in \text{Ren}_1$, we have that $X_j : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$.

Heuristically, the factors X_j now commute with $W_{j'}(s)$, since

Lemma 5.5. *For $\varphi, \varphi', \phi \in \mathfrak{h}$ it is true that*

$$[W_{\mathscr{F},j}(\varphi'), V_{j'j''}(\varphi^*\phi)] = 0, \qquad \textit{and} \qquad [W_{\mathscr{F},j}(\varphi'), \langle\varphi, \phi\rangle] = 0, \tag{127}$$

as a strong operator identity on $\mathscr{F}$.

Proof. Since $\varphi, \phi \in L^2$, we have $\varphi^*\phi \in L^1$, so after a Fourier transform, the operator $V_{j'j''}(\varphi^*\phi)$ amounts to a multiplication by an L^∞–function, and is hence bounded. Further, $W_{\mathscr{F},j}(\varphi')$ is unitary on $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ (and hence bounded). So the commutator is defined on all of $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ and hence $\mathscr{F}$.
Now, in position space, both $A^\dagger_j(\varphi')$ and $A_j(\varphi')$ can be decomposed into a fiber integral by fiber–decomposing $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y = \int_{\mathcal{Q}_x} \mathscr{F}_y \, d\boldsymbol{X}$ (see (20) and (22)). So we can also decompose $W_{\mathscr{F},j}(\varphi') = e^{A^\dagger_j(\varphi') - A_j(\varphi')}$ into a fiber integral. And by (90), the operator $V_{j'j''}(\varphi^*\phi)$ just amounts to a multiplication by a complex constant on each fiber Hilbert space. So the fiber operators commute on all fibers and hence the original operators commute on $L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ and $\mathscr{F}$.

The expression $\langle\varphi, \phi\rangle$ is just a constant, so it trivially commutes with $W_{\mathscr{F},j}(\varphi')$.

□

Mathematically, if we replace $\varphi, \varphi', \phi \in \mathfrak{h}$ by $s, s', v \in \dot{\mathcal{S}}_1^\infty$, then the commutation relations (127) are not a priori valid, since $W_j(s)$ is not necessarily defined on vectors of the kind $V_{j'j''}(\varphi^*\phi)\Psi_m$ or $\langle\varphi, \phi\rangle\Psi_m$. We enforce their validity by taking (127) as a definition for an extension of $W_j(s)$:

Definition 5.4. Let $(s_j)_{j\in\mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$, $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$, and let X be an element of the set of operators

$$\mathcal{X} := \mathrm{span}_{\mathrm{eRen}}\big\{\langle s, v\rangle,\ V_{jj'}(s^*v) \ \big|\ s, v \in \dot{\mathcal{S}}_1^\infty\big\}, \tag{128}$$

so X formally commutes with all $W_j(s_j)$. Then we extend the **product of dressing operators** via

$$\begin{aligned} W_M(s_M)\dots W_1(s_1)X\Psi_m :=& XW_M(s_M)\dots W_1(s_1)\Psi_m, \\ W_M(s_M)\dots W_1(s_1)XW_{\mathscr{F},1}(\varphi)\Psi_m :=& XW_M(s_M)\dots W_1(s_1)W_{\mathscr{F},1}(\varphi)\Psi_m \\ =& XW_M(s_M)\dots W_1(s_1)W_1(\varphi)\Psi_m, \end{aligned} \tag{129}$$

with M being the respective fermion number on each sector and where the last equality in (129) holds by Lemma 5.5. Again, we may extend the definition to **symmetrized vectors** by imposing that $W_M(s_M)\dots W_1(s_1)$ shall commute with the symmetrization operator $(S_- \otimes I)$.

Again, it is easy to see that this definition makes sense: By Lemma 5.3, we have $W_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}$. And since $X \in \mathcal{X}$ maps $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$, indeed

$$XW_M(s_M)\dots W_1(s_1)\Psi_m \in \overline{\mathscr{F}}_{\mathrm{ex}}, \tag{130}$$

so the right–hand sides of (129) are well–defined.

Remarks.

4. It seems natural to define (129) for all operators X which commute with $A_j^\dagger(s')$ in a sufficiently regular case and $A_j(s')$. However, since we have only defined $A_j(s') : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$ and $V_{j'j''}(s^*v) : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$, it is not clear how to interpret the commutator $[A_j(s'), V_{j'j''}(s^*v)]$. So $V_{j'j''}(s^*v)$ would then not be a valid X–operator, although it commutes with $A_j^\dagger(s')$ and $A_j(s')$ for $s, s', v \in \mathfrak{h}$.
   If one succeeded to modify the definition of $\overline{\mathscr{F}}, \overline{\mathscr{F}}_{\mathrm{ex}}$ such that commutators as $[A_j(s'), V_{jj'}(s^*v)]$ are well–defined operators, then it seems reasonable to change the set of allowed X in Definition 5.4 to all X with $[A_j^\dagger(s'), X] = [A_j(s'), X] = 0$.

5.3.4 Final definition of $W(s)$

With Definitions 5.3 and 5.4, we may now provide the final domains for the product $W_M(s_M)\dots W_1(s_1)$: The **extended dressing domain** $\mathcal{D}_W$ is defined as

$$\begin{aligned} &\mathcal{D}_W := \\ &\mathrm{span}_{\mathrm{eRen}}\big\{W_1(\varphi)A_j^\dagger(v)\Psi_m,\ XW_1(\varphi)\Psi_m \ \big|\ \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty,\ v \in \dot{\mathcal{S}}_1^\infty,\ X \in \mathcal{X},\ \Psi_m \in \mathcal{C}_{WS}\big\}, \end{aligned} \tag{131}$$

with $\dot{\mathcal{S}}_1^\infty$ defined in (26), $\mathcal{X}$ defined in (128) and $\mathcal{C}_{WS}$ defined in (103). Well–definedness of $W(s)$ on $\mathcal{D}_W$ can be seen by combining (114) with Definitions 5.3 and 5.4. By imposing that $W_M(s_M)\dots W_1(s_1)$ shall commute with $(S_- \otimes I)$, we extend $W_M(s_M)\dots W_1(s_1)$ to $(S_- \otimes I)[\mathcal{D}_W] \cup \mathcal{D}_W$.
The maximal domain of $W(s)$ in Fock space is now given by the **large domain**

$$\mathcal{D}_{\mathscr{F}} := (S_- \otimes I)\big[\mathcal{D}_W \cap (L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y)\big]. \tag{132}$$

The symmetrization operator $(S_- \otimes I)$ ensures that indeed $\mathcal{D}_{\mathscr{F}} \subset \mathscr{F}$. With this definition, it holds true that

Lemma 5.6. *We have the inclusion*

$$\mathcal{D}_{WS} \subset \mathcal{D}_W \cap (L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y), \tag{133}$$

and in particular, $\mathcal{D}_{\mathscr{F}}$ is dense in $\mathscr{F}$.

Proof. Setting $X = 1$ and using Lemma 5.4, we see that $W_{\mathscr{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m \in \mathcal{D}_{WS}$ with $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$ is also an element of $\mathcal{D}_W$. Further, $W_{\mathscr{F},1}(\varphi)\Psi_m \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$, which yields the inclusion relation (133).

Hence, the symmetrized version $(S_- \otimes I)[\mathcal{D}_{WS}]$ is included in $\mathcal{D}_{\mathscr{F}}$. And since the former is dense in $\mathscr{F}$ (Lemma 5.1), also the latter is.

□

Remarks.

5. In essence, we just transferred the commutation relations (107) for creation operators $A_j^\dagger$ from Lemma 5.2 in a certain sense to extended dressing operators $W_j(s)$. This was done by imposing definitions such that these commutation relations still hold true. What about the commutation relations (108) for annihilation operators A_j?
   In fact, these relations cannot be imposed by definition, but one may show that they are an immediate consequence of Definition 5.2. This is proved in the following lemma:

Lemma 5.7. *Let $s, v \in \dot{\mathcal{S}}_1^\infty$ and $\Psi_m \in \mathcal{C}_{WS}$. Then, we have the commutation relations*

$$\begin{aligned}
W_{j'}(s)A_j(v)\Psi_m &= \begin{cases} \big(A_j(v) - \langle v, s\rangle\big)W_{j'}(s)\Psi_m & \text{if } j = j' \\ \big(A_j(v) - V_{jj'}(v^*s)\big)W_{j'}(s)\Psi_m & \text{if } j \neq j' \end{cases} \\
W(s)A_j(v)\Psi_m &= \big(A_j(v) - \langle v, s\rangle - V_{j\bullet}(v^*s)\big)W(s)\Psi_m \\
W_{j'}(s)A(v)\Psi_m &= \big(A(v) - \langle v, s\rangle - V_{\bullet j'}(v^*s)\big)W_{j'}(s)\Psi_m \\
W(s)A(v)\Psi_m &= \big(A(v) - M\langle v, s\rangle - V(v^*s)\big)W(s)\Psi_m.
\end{aligned} \tag{134}$$

Proof. First, note that $A_j(v)\Psi_m = 0$. The first line in (134) then follows by momentum space definitions (21) and (120):

$$\begin{aligned}
&\big(A_j(v)W_{j'}(s)\Psi_m\big)(\boldsymbol{P},\boldsymbol{K}) \\
=&\frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}}\int v(\tilde{\boldsymbol{k}})^* s(\tilde{\boldsymbol{k}})\left(\prod_{\ell=1}^N s(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P}'+(e_{j'}-e_j)\tilde{\boldsymbol{k}})\,d\tilde{\boldsymbol{k}} \\
=&\begin{cases}\langle v,s\rangle\Psi_m(\boldsymbol{P},\boldsymbol{K}) & \text{if } j=j' \\ \big(V_{jj'}(v^*s)\Psi_m\big)(\boldsymbol{P},\boldsymbol{K}) & \text{if } j\neq j'\end{cases},
\end{aligned} \tag{135}$$

with $\boldsymbol{P}' = \boldsymbol{P} + e_{j'}\sum_{\ell=1}^N \boldsymbol{k}_\ell$.

The second line in (134) is established similarly. We use again (21) and (120), yielding:

$$\begin{aligned}
&\big(A_j(v)W(s)\Psi_m\big)(\boldsymbol{P},\boldsymbol{K}) \\
=&\frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}}\sum_{\tilde\sigma}\int v(\tilde{\boldsymbol{k}})^* s(\tilde{\boldsymbol{k}})\left(\prod_{\ell=1}^N s(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P}'+(e_{\tilde\sigma(N+1)}-e_j)\tilde{\boldsymbol{k}})\,d\tilde{\boldsymbol{k}},
\end{aligned} \tag{136}$$

where the sum runs over all $\tilde\sigma : \{1,\dots,N+1\} \to \{1,\dots,M\}$ and we have set $\boldsymbol{P}' = \boldsymbol{P} + \sum_{\ell=1}^N e_{\tilde\sigma(\ell)}\boldsymbol{k}_\ell$, as well as $\boldsymbol{k}_{N+1} = \tilde{\boldsymbol{k}}$. We can split this sum into a sum over (σ, j) with $\sigma : \{1,\dots,N\} \to \{1,\dots,M\}, \sigma(\ell) = \tilde\sigma(\ell)$ and $j' \in \{1,\dots,M\}, j' = \tilde\sigma(N+1)$:

$$\begin{aligned}
&\big(A_j(v)W(s)\Psi_m\big)(\boldsymbol{P},\boldsymbol{K}) \\
=&\frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}}\sum_{j'}\sum_{\sigma}\int v(\tilde{\boldsymbol{k}})^* s(\tilde{\boldsymbol{k}})\left(\prod_{\ell=1}^N s(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P}'+(e_{j'}-e_j)\tilde{\boldsymbol{k}})\,d\tilde{\boldsymbol{k}}.
\end{aligned} \tag{137}$$

Now, the term with $j' = j$ renders the contribution $\langle s, v\rangle\Psi_m$ and all other $M-1$ terms add up to $\sum_{j':j\neq j'} V_{jj'}(v^*s)\Psi_m = V_{j\bullet}(v^*s)\Psi_m$, which is exactly the desired contribution.

Lines three and four of (134) just follow by summing over $j \in \{1,\dots,M\}$ in the first two lines.

□

6 Pulling Back the Hamiltonian

This section concerns taking a formal Hamiltonian

$$H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty,$$

and pulling it back under the dressing transformation $W(s)$, i.e., we compute

$$\widetilde{H} : \mathscr{F}_{\text{ex}} \supset \mathcal{D}_{WS} \to \overline{\mathscr{F}}_{\text{ex}} \qquad \text{with} \qquad W(s)\widetilde{H} = HW(s).$$

The computation is split into two steps. In Section 6.1, we compute the pullback of $(A(v) - E_\infty)$. Pulling back only $A(v)$ will result in divergences which are canceled by E_∞.
The pullback of $(H_{0,y} + A^\dagger(v))$ is then computed in Section 6.2. Combining $H_{0,y}$ and $A^\dagger(v)$ yields a particularly easy result.

Our main theorem is the following:

Theorem 6.1. *Let $s = -\frac{v}{\omega}$ with $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$. Then the pullback of the self–energy renormalized Hamiltonian*

$$\widetilde{H} := H_{0,y} + V(v^* s) \qquad \textit{satisfies} \qquad W(s)\widetilde{H} = \big(H_{0,y} + A^\dagger(v) + A(v) - E_\infty\big)W(s), \tag{138}$$

which holds as a strong operator identity on $\mathcal{D}_{WS}$ (defined in (105)*), as well as on $(S_- \otimes I)[\mathcal{D}_{WS}]$.*

Note that, the potential interaction V defined in (109) via (88) acts as

$$(V\Psi)(\boldsymbol{P}, \boldsymbol{K}) = \sum_{j \neq j'} \int v(\tilde{\boldsymbol{k}})^* s(\tilde{\boldsymbol{k}}) \Psi(\boldsymbol{P} + (e_j - e_{j'})\tilde{\boldsymbol{k}}, \boldsymbol{K})\, d\tilde{\boldsymbol{k}}.$$

Remarks.

6. We have not so far shown that $W(s)$ is invertible on $\mathcal{D}_{\mathscr{F}}$. While we conjecture that this is true, we do not immediately see how to prove it, here. If this conjecture was true, we could write

$$W(s)\widetilde{H} = HW(s) \qquad \Leftrightarrow \qquad \widetilde{H} = W(s)^{-1} H W(s), \tag{139}$$

and for a given H and $W(s)$, the renormalized Hamiltonian $\widetilde{H}$ would be unique. Since we do not have an inverse $W(s)^{-1}$ here, there might be several $\widetilde{H}$, which satisfy $W(s)\widetilde{H} = HW(s)$. So the direct renormalization procedure is yet not guaranteed to be unique, although it produces a well–defined renormalized Hamiltonian $\widetilde{H}$ that is physically reasonable.

6.1 Pulling Back $A - E_\infty$

We recall that by Proposition 4.1, one can define $E_\infty : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$ with

$$(E_\infty \Psi)(\boldsymbol{P}, \boldsymbol{K}) = M\langle v, s\rangle\, \Psi(\boldsymbol{P}, \boldsymbol{K}) = \sum_{j=1}^{M} \int -\frac{v(\boldsymbol{k})^* v(\boldsymbol{k})}{\omega(\boldsymbol{k})}\, d\boldsymbol{k}\, \Psi(\boldsymbol{P}, \boldsymbol{K}), \tag{140}$$

even if $\langle v, s\rangle \notin \mathbb{C}$, but $\langle v, s\rangle \in \text{Ren}_1$.

Lemma 6.1. *Let $\Psi_m \in \mathcal{C}_{WS}$ and $s = -\frac{v}{\omega}$ with $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$. Then for $\varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$,*

$$\big(A(v) - E_\infty\big)W(s)W_{\mathscr{F},1}(\varphi)\Psi_m = W(s)\big(\mathrm{res}_1(\varphi) + V(v^*s)\big)W_{\mathscr{F},1}(\varphi)\Psi_m \quad \in \overline{\mathscr{F}}_{\mathrm{ex}}, \tag{141}$$

*where the **residual operator***

$$\mathrm{res}_1(\varphi) = \underbrace{\langle v, \varphi\rangle}_{\in \mathrm{Ren}_1} + V_{\bullet 1}(v^*\varphi), \tag{142}$$

is by Lemma 4.2 a well–defined mapping $\overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\mathrm{ex}}$.

The proof of Lemma 6.1 is given in Appendix C.

6.2 Pulling Back $H_{0,y} + A^\dagger$

Lemma 6.2. *Let $\Psi_m \in \mathcal{C}_{WS}$ and $s = -\frac{v}{\omega}$ with $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$. Then for $\varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$,*

$$\big(H_{0,y} + A^\dagger(v)\big)W(s)W_{\mathscr{F},1}(\varphi)\Psi_m = W(s)\big(H_{0,y} - \mathrm{res}_1(\varphi)\big)W_{\mathscr{F},1}(\varphi)\Psi_m \quad \in \overline{\mathscr{F}}_{\mathrm{ex}}, \tag{143}$$

with the same residual operator $\mathrm{res}_1 = \langle v, \varphi\rangle + V_{\bullet 1}(v^*\varphi)$ as in Lemma 6.1.

As for Lemma 6.1, the proof of Lemma 6.2 is rather technical. It can be found in Appendix D. With both lemmas at hand, Theorem 6.1 can directly be proved.

Proof of Theorem 6.1. This is a simple consequence of Lemmas 6.1 and 6.2. We put together (141) and (143) which yields

$$\begin{aligned} W(s)\widetilde{H} = \big(H_{0,y} + A^\dagger(v) + A(v) - E_\infty\big)W(s) =& W(s)\big(H_{0,y} + V(v^*s) + \mathrm{res}_1(\varphi) - \mathrm{res}_1(\varphi)\big) \\ =& W(s)\big(H_{0,y} + V(v^*s)\big), \end{aligned} \tag{144}$$

as a strong operator identity on all $\Psi = W_{\mathscr{F},1}(\varphi)\Psi_m, \varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$. And these Ψ span $\mathcal{D}_{WS}$.

Since we imposed in Definitions 5.2, 5.3 and 5.4 that symmetrization $(S_- \otimes I)$ shall commute with $W(s)$, the strong operator identity is also valid on $(S_- \otimes I)[\mathcal{D}_{WS}]$.

□

7 Self–Adjointness

In this section, we prove that in certain cases, $\widetilde{H}$ can indeed be defined as a self–adjoint operator $\widetilde{H} : \mathscr{F} \supset \mathrm{dom}(\widetilde{H}) \to \mathcal{D}_{\mathscr{F}}$. So far we have by Theorem 6.1 that $\widetilde{H} : (S_- \otimes I)[\mathcal{D}_{WS}] \to \overline{\mathscr{F}}_{\mathrm{ex}}$ is well–defined.

In order for the image of $\widetilde{H}$ to be in $\mathcal{D}_{\mathscr{F}}$, we need to restrict the domain of $\widetilde{H}$ even further to some subspace $\widetilde{\mathcal{D}}_{\mathscr{F}} \subset \mathcal{D}_{\mathscr{F}}$, defined in (147), and prove well–definedness of $\widetilde{H} : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$ (Lemma 7.1). The existence of a self–adjoint extension on some $\mathrm{dom}(\widetilde{H}) \supset \widetilde{\mathcal{D}}_{\mathscr{F}}$ is then a simple consequence (Corollary 7.1).

7.1 Existence of Self–adjoint Extensions

First, we verify that $\widetilde{H} = H_{0,y} + V$ is well–defined and symmetric on a dense domain in $\mathscr{F}$.

Definition 7.1. Let $\mathcal{Q}_{\mathrm{col},x}$ be the set of **collision configurations**, i.e., all fermion position space configurations

$$\mathcal{Q}_{\mathrm{col},x} := \big\{ \boldsymbol{X} \in \mathcal{Q}_x \;\big|\; \exists j \neq j' : \; \boldsymbol{x}_j = \boldsymbol{x}_{j'} \big\}. \tag{145}$$

Denote by

$$\widetilde{\mathcal{C}}_{WS} := \big\{ \Psi_m = \Psi_{mx} \otimes \Omega_y \;\big|\; \mathcal{F}^{-1}(\Psi_{mx}) \in C_c^\infty(\mathcal{Q}_x \setminus \mathcal{Q}_{\mathrm{col},x}) \big\}, \tag{146}$$

the "cyclic set" of functions whose support avoids the collision configurations (where $\mathcal{F}^{-1}$ is the inverse Fourier transform). We define the **small domain**, on which $\widetilde{H}$ is initially defined as a Fock space operator as

$$\widetilde{\mathcal{D}}_{\mathscr{F}} := (S_- \otimes I)\big[\mathrm{span}\big\{ W_{\mathscr{F},1}(\varphi)\Psi_m \;\big|\; \varphi \in C_c^\infty(\mathbb{R}^d),\; \Psi_m \in \widetilde{\mathcal{C}}_{WS} \big\}\big], \tag{147}$$

see Figure 5. It is easy to see that $\widetilde{\mathcal{D}}_{\mathscr{F}} \subseteq \mathcal{D}_{\mathscr{F}}$ and $\widetilde{\mathcal{D}}_{\mathscr{F}} \subseteq (S_- \otimes I)[\mathcal{D}_{WS}]$.

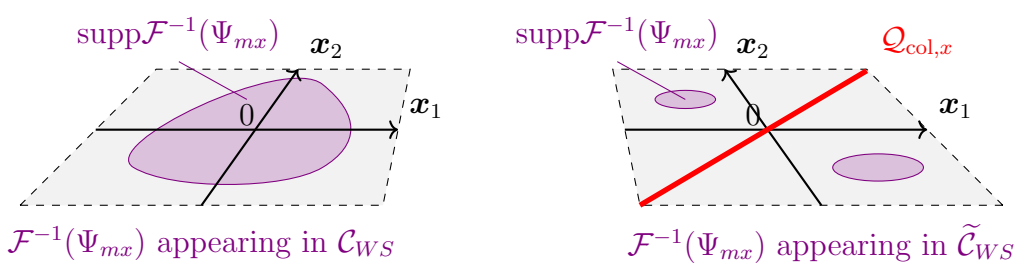

Figure 5: Within $\mathcal{C}_{WS}$, the fermionic wave functions must be in $\mathcal{S}$, allowing for any support (including compact ones). Within $\widetilde{\mathcal{C}}_{WS}$, only C_c^∞–functions are allowed with support avoiding the collision configurations $\mathcal{Q}_{\mathrm{col},x}$. Color online.

In the following Lemma, we will use that if $\omega, v \in \dot{\mathcal{S}}_1^\infty$ satisfy (31) and (32) (so they scale polynomially), then the potential function

$$\hat{V} := v^* s = -\tfrac{v^* v}{\omega}, \tag{148}$$

also scales polynomially with

$$m_V = 2m_v - m_\omega \qquad \beta_V = 2\beta_v - \beta_\omega. \tag{149}$$

Further, if $\beta_V > -d$, then the inverse Fourier transform $V = \mathcal{F}^{-1}(\hat{V}) \in S'(\mathbb{R}^d)$ also exists, so we can make statements about the singular support of V.

Lemma 7.1 ($\widetilde{H}$ is densely defined and symmetric)**.** *The set $\widetilde{\mathcal{D}}_{\mathscr{F}}$ is dense in $\mathscr{F}$. Assume, that $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$ with ω, v satisfying* (31) *and* (32)*, as well as $\beta_V > -d$. If now*

$$\operatorname{singsupp}(V) \subseteq \{0\}, \tag{150}$$

then $\widetilde{H}$ maps $\widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$ and is thus densely defined. If in addition the symmetry condition (33) *holds, then $\widetilde{H}$ is symmetric.*

Proof. Density of $\widetilde{\mathcal{D}}_{\mathscr{F}}$ in $\mathscr{F}$ is established as density of $(S_- \otimes I)\mathcal{D}_{WS}$ in Lemma 5.1 (proof in Appendix A). We recall that by the last line of (174),

$$D_1 W_{\mathscr{F},1}(\varphi)\Psi_m = \Psi_{mx} \otimes W_y(\varphi)\Omega_y.$$

In the proof of Lemma 5.1, we argue that $(S_- \otimes I)\mathcal{D}_{WS}$ is dense in $\mathscr{F}$, since $\Psi_{mx} \otimes W_y(\varphi)\Omega_y$ approximates any $\Psi \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ arbitrarily well. The transition from $(S_- \otimes I)\mathcal{D}_{WS}$ to $\widetilde{\mathcal{D}}_{\mathscr{F}}$ is achieved by a restriction to $\Psi_{mx} \in \mathcal{F}[C_c^\infty(\mathcal{Q}_x \setminus \mathcal{Q}_{\mathrm{col},x})]$. The set $\mathcal{Q}_{\mathrm{col},x}$ is a union of hyperplanes on each sector in the fermionic configuration space, so $C_c^\infty(\mathcal{Q}_x \setminus \mathcal{Q}_{\mathrm{col},x})$ is dense in $L^2(\mathcal{Q}_x)$. The Fourier transform $\mathcal{F}$ is an isometry, so the allowed set for Ψ_{mx} is dense in $L^2(\mathcal{Q}_x)$. Thus, we can still approximate any $\Psi \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$ arbitrarily well by $\Psi_{mx} \otimes W_y(\varphi)\Omega_y$ and by the same arguments as in the proof of Lemma 5.1, $\widetilde{\mathcal{D}}_{\mathscr{F}}$ is dense in $\mathscr{F}$.

Now we verify that $H_{0,y}$ maps $\widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$. By linearity, it suffices to show well–definedness on all vectors of the form $W_{\mathscr{F},1}(\varphi)\Psi_m, \varphi \in C_c^\infty(\mathbb{R}^d)$. Denote by $P_y^{(N)}$ the projection of $\mathscr{F}_y$ to the N–boson sector $\mathscr{F}_y^{(N)}$, so $\sum_{N\in\mathbb{N}_0} P_y^{(N)} = 1$. The L^2–norm squares of $W_{\mathscr{F},1}(\varphi)\Psi_m$ are Poisson–distributed over N, i.e.,

$$\left\| P_y^{(N)} W_{\mathscr{F},1}(\varphi)\Psi_m \right\|^2 = e^{-\|\varphi\|^2} \frac{\|\varphi\|^{2N}}{N!}, \tag{151}$$

so for any $0 < q < 1$ they decay faster than q^N in N–direction. Now, define

$$\lambda := \max_{\boldsymbol{k}\in\operatorname{supp}(\varphi)} |\omega(\boldsymbol{k})|. \tag{152}$$

Then,

$$\left\| P_y^{(N)} H_{0,y} W_{\mathscr{F},1}(\varphi)\Psi_m \right\|^2 \leq N^2\lambda^2 \left\| P_y^{(N)} W_{\mathscr{F},1}(\varphi)\Psi_m \right\|^2, \tag{153}$$

which still decays faster than q^N in N–direction. Thus, $\left\|H_{0,y}W_{\mathscr{F},1}(\varphi)\Psi_m\right\| < \infty$ and we have that $H_{0,y}W_{\mathscr{F},1}(\varphi)\Psi_m \in L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$.

It remains to be shown that $V = V(v^*s)$ is well–defined, which amounts to proving that

$$(V\Psi)(\boldsymbol{P},\boldsymbol{K}) = \sum_{j\neq j'}\int \hat{V}(\tilde{\boldsymbol{k}})\Psi(\boldsymbol{P}+(e_j-e_{j'})\tilde{\boldsymbol{k}},\boldsymbol{K})\,d\tilde{\boldsymbol{k}},$$

defines an L^2–function on $\mathcal{Q}$. Since $\beta_V > -d$, we have that $\hat{V}\in L^1_{\mathrm{loc}} \Rightarrow \hat{V}\in\mathcal{S}'$, so we can take the Fourier transform as in (90):

$$(V\Psi)(\boldsymbol{X},\boldsymbol{Y}) = \sum_{j\neq j'} V(\boldsymbol{x}_j-\boldsymbol{x}_{j'})\Psi(\boldsymbol{X},\boldsymbol{Y}),$$

with $V(\boldsymbol{x}) = \mathcal{F}^{-1}(\hat{V})(\boldsymbol{x})$.

Now, for $\Psi = W_{\mathscr{F},1}(\varphi)\Psi_m$ we obtain the position space representation by Fourier–transforming (104):

$$\Psi(\boldsymbol{X},\boldsymbol{Y}) = \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\left(\prod_{\ell=1}^{N}\check{\varphi}(\boldsymbol{y}_\ell-\boldsymbol{x}_1)\right)\Psi_{mx}(\boldsymbol{X}), \tag{154}$$

where $\check{\varphi} = \mathcal{F}^{-1}(\varphi)$ is a Schwartz function, as $\varphi\in C_c^\infty$ is Schwartz. So as $\Psi_{mx}(\boldsymbol{X})$ is a smooth function with compact support apart from collision configurations in $\mathcal{Q}_x$, also Ψ is smooth, and it is zero at fermion collision configurations in $\mathcal{Q}$. Since the singular support of $V(\boldsymbol{x})$ is at most $\{0\}$, the multiplication function $\sum_{j\neq j'}V(\boldsymbol{x}_j-\boldsymbol{x}_{j'})$ is smooth on $\mathrm{supp}(\Psi_{mx})$ (which excludes collision configurations). And as $\mathrm{supp}(\Psi_{mx})$ is compact, there is some $C_\Psi\in\mathbb{R}$ with

$$\max_{\boldsymbol{X}\in\mathrm{supp}(\Psi_{mx})}\left|\sum_{j\neq j'}V(\boldsymbol{x}_j-\boldsymbol{x}_{j'})\right| \leq C_\Psi. \tag{155}$$

Further, by compactness of support, a maximum occupied fermion sector $\overline{M}$ exists. So

$$\|V\Psi\|^2 \leq \overline{M}^4 C_\Psi^2\|\Psi\|^2 < \infty,$$

for $\Psi\in\widetilde{\mathcal{D}}_{\mathscr{F}}$. Thus, $V\Psi\in L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$.

Symmetrization for fermions by $(S_-\otimes I)$ is preserved by $H_{0,y}$ and V. Hence, indeed $\widetilde{H}\Psi\in\mathscr{F}$. And by Theorem 6.1, we have that $\widetilde{H}\Psi\in(S_-\otimes I)[\mathcal{D}_W]$ (otherwise, we could not apply $W(s)$ to it).

Symmetry of $\widetilde{H}$ is an obvious consequence of the symmetry condition (33). And since $\widetilde{H}$ preserves symmetry, it maps $\widetilde{\mathcal{D}}_{\mathscr{F}}\to\mathcal{D}_{\mathscr{F}} = (S_-\otimes I)[\mathcal{D}_W\cap(L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y)]$ (compare (132)).

□

Corollary 7.1 (Existence of a self–adjoint extension)**.** *$\widetilde{H} : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \mathcal{D}_{\mathscr{F}}$ as in Lemma 7.1 allows for a self–adjoint extension.*

Proof. This is a direct consequence of [44, Thm. X.3] (von Neumann's theorem): For a symmetric operator $\widetilde{H}$ (called A within [44]), this theorem asserts that there is a self–adjoint extension, provided that a conjugation operator $C : \widetilde{\mathcal{D}}_{\mathscr{F}} \to \widetilde{\mathcal{D}}_{\mathscr{F}}$ can be found, such that

$$C\widetilde{H} = \widetilde{H}C. \tag{156}$$

As a conjugation, we choose $(C\Psi)(\boldsymbol{K}) = \Psi(-\boldsymbol{K})^*$, which amounts to complex conjugation in particle–position representation. By symmetry (33) and since ω is real–valued, $\hat{V}(\boldsymbol{k}) = \hat{V}(-\boldsymbol{k})^*$, so $VC = CV$. And analogously, $CH_{0,y} = H_{0,y}C$. Thus, (156) holds, and we have at least one self–adjoint extension. □

8 Further Dressing Types

There exist also other types of (non–unitary) dressing operators replacing $W(s)$. An example is given by the IBC construction mentioned, mentioned in the introduction, where essentially a dressing operator $W = W_{\text{IBC}}^{-1} = (1+H_0^{-1}A^\dagger)^{-1}$ defined on a subspace of $\mathscr{F}$ is used. In this section, we extend this operator to $\overline{\mathscr{F}} \to \overline{\mathscr{F}}$.
Another example is the dressing operator $T = e^{-H_0^{-1}A^\dagger}$, which is a strongly simplified version of certain operators used in CQFT.

8.1 IBC on the Extended State Space

In certain cases, the IBC renormalization renders a self–adjoint operator H with dense domain in $\mathscr{F}$, by using a formal undressing operator of the kind $W_{\text{IBC}} = (1 + H_0^{-1}A^\dagger)$. Within the construction, several divergent integrals appear, which get combined to convergent ones. Using the ESS construction, one can directly make sense of the divergent expressions and, in certain cases, perform the IBC renormalization in a particularly convenient way. Suppose, $\theta(\boldsymbol{p}) > 0$ and $\omega(\boldsymbol{k}) > 0$. By Proposition 4.1, $AH_0^{-1}A^\dagger : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$ is a well–defined operator. Using a self–energy operator $E : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$, we can define $H_{\text{IBC}} : \overline{\mathscr{F}} \to \overline{\mathscr{F}}_{\text{ex}}$ with $H_{\text{IBC}} = H_0 + A^\dagger + A - E$ via

$$\begin{aligned} H_{\text{IBC}} =& H_0 + A^\dagger + A && - E \\ =& H_0 + A^\dagger + A + AH_0^{-1}A^\dagger && - AH_0^{-1}A^\dagger - E \\ =& \underbrace{(1 + AH_0^{-1})H_0^{1/2}}_{=:S^*}\underbrace{H_0^{1/2}(1 + H_0^{-1}A^\dagger)}_{=:S} && \underbrace{-AH_0^{-1}A^\dagger - E}_{T} \\ =& S^*S + T. \end{aligned} \tag{157}$$

Clearly, S^*S is formally a symmetric and positive operator. If it can be densely defined on $\mathscr{F}$ as a closed operator, then by [44, X.25] we have self–adjointness of S^*S.

Using this argument, it is shown in [30, 31, 33, 34] that for certain dispersion relations and form factors, S^*S is self–adjoint on the domain

$$\mathrm{dom}(S^*S) = \big\{\Psi \in \mathscr{F} \;\big|\; (1 + H_0^{-1}A^\dagger)\Psi \in \mathrm{dom}(H_0)\big\}. \tag{158}$$

The condition $(1 + H_0^{-1}A^\dagger)\Psi \in \mathrm{dom}(H_0)$ is called **interior**– or **abstract boundary condition**.

Now, suppose there is a suitable E such that $T : \mathscr{F} \supset \mathcal{D}(T) \to \mathscr{F}$ is a densely defined Kato–perturbation of S^*S, that is,

$$\|T\Psi\| \leq a\|S^*S\Psi\| + b\|\Psi\| \qquad \forall \Psi \in \mathrm{dom}(S^*S), \tag{159}$$

with $a < 1$. Then by the Kato–Rellich Theorem [44, X.12] we immediately obtain a self–adjoint H_{IBC} on the same domain $\mathrm{dom}(S^*S)$.

Using the ESS construction, we may now rigorously define W_{IBC} and W_{IBC}^{-1}, even if they formally map out of Fock space:

Proposition 8.1. *The operator $W_{\mathrm{IBC}} : \overline{\mathscr{F}} \to \overline{\mathscr{F}}$ is well–defined and bijective.*

Proof. We show that $W_{\mathrm{IBC}} : \dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ is bijective. The extension to a bijective operator on $\overline{\mathscr{F}}$ is then done by linearity with respect to the field eRen.

The operator $H_0^{-1}A^\dagger$ maps $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$ to itself: $A^\dagger$ maps $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$ by Proposition 4.1 and H_0^{-1} is just a multiplication by a function that is smooth on $\dot{\mathcal{Q}}$ and polynomially scaling. So W_{IBC} is well–defined on all of $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$.

Bijectivity is shown by directly constructing W_{IBC}^{-1}. Formally, the inverse is given by a Neumann series:

$$W_{\mathrm{IBC}}^{-1} = (1 + H_0^{-1}A^\dagger)^{-1} := \sum_{k=0}^{\infty}(-H_0^{-1}A^\dagger)^k. \tag{160}$$

<u>Claim:</u> The Neumann series (160) defines an operator $\dot{\mathcal{S}}^\infty_{\mathscr{F}} \to \dot{\mathcal{S}}^\infty_{\mathscr{F}}$.

<u>Proof of the Claim:</u> Each $-H_0^{-1}A^\dagger$ increases the boson number by 1. So $(-H_0^{-1}A^\dagger)^k\Psi$ is only supported on configuration space sectors with $N \geq k$. Hence, on each $(\boldsymbol{P}, \boldsymbol{K}) \in \dot{\mathcal{Q}}$ with $\boldsymbol{K} \in \mathbb{R}^{Nd}$, we have that

$$\begin{aligned} &\Big((-H_0^{-1}A^\dagger)^k\Psi\Big)(\boldsymbol{P}, \boldsymbol{K}) = 0 \qquad \text{for } k > N \\ \Rightarrow \quad &\left(\sum_{k=0}^{\infty}(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P}, \boldsymbol{K}) = \left(\sum_{k=0}^{N}(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P}, \boldsymbol{K}). \end{aligned} \tag{161}$$

Since $H_0^{-1}A^\dagger$ maps $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$ to itself, also all sums $\sum_{k=0}^n(-H_0^{-1}A^\dagger)^k$ with $n\in\mathbb{N}$ map $\dot{\mathcal{S}}^\infty_{\mathscr{F}}\to\dot{\mathcal{S}}^\infty_{\mathscr{F}}$. So the Neumann series is defined on each N–boson sector and hence on all of $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$. $\Diamond$

<u>Claim:</u> The Neumann series (160) is the inverse of $(1+H_0^{-1}A^\dagger)$.

<u>Proof of the Claim:</u> We use the first line of (161) and perform a sector–wise verification:

$$\begin{aligned}&(1+H_0^{-1}A^\dagger)\left(\sum_{k=0}^\infty(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P},\boldsymbol{K})\\ =&\left(\sum_{k=0}^\infty(-H_0^{-1}A^\dagger)^k\Psi-\sum_{k=1}^\infty(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P},\boldsymbol{K})\\ \overset{(161)}{=}&\left(\sum_{k=0}^N(-H_0^{-1}A^\dagger)^k\Psi-\sum_{k=1}^N(-H_0^{-1}A^\dagger)^k\Psi\right)(\boldsymbol{P},\boldsymbol{K})\\ =&\Psi(\boldsymbol{P},\boldsymbol{K}).\end{aligned}\tag{162}$$

So indeed $(1+H_0^{-1}A^\dagger)\left(\sum_{k=0}^\infty(-H_0^{-1}A^\dagger)^k\right)=1$. $\Diamond$

As there is a well–defined inverse of $W_{\mathrm{IBC}}=(1+H_0^{-1}A^\dagger)$ on all of $\dot{\mathcal{S}}^\infty_{\mathscr{F}}$, the operator $W_{\mathrm{IBC}}:\dot{\mathcal{S}}^\infty_{\mathscr{F}}\to\dot{\mathcal{S}}^\infty_{\mathscr{F}}$ must be bijective.

$\square$

With Proposition 8.1, we have a well–defined linear space $W_{\mathrm{IBC}}^{-1}[\dot{\mathcal{S}}^\infty_{\mathscr{F}}\cap\mathscr{F}]$ that can be equipped with a scalar product

$$\langle W_{\mathrm{IBC}}^{-1}\Psi,W_{\mathrm{IBC}}^{-1}\Phi\rangle_{\mathrm{renI}}:=\langle\Psi,\Phi\rangle\qquad\text{for }\Psi,\Phi\in\dot{\mathcal{S}}^\infty_{\mathscr{F}}\cap\mathscr{F}.\tag{163}$$

The completion of $W_{\mathrm{IBC}}^{-1}[\dot{\mathcal{S}}^\infty_{\mathscr{F}}\cap\mathscr{F}]$ with respect $\langle\cdot,\cdot\rangle_{\mathrm{renI}}$ is a Hilbert space $\mathscr{F}_{\mathrm{renI}}$, which we call the **IBC–renormalized Fock space**. H_{IBC} is then defined on $\mathscr{F}_{\mathrm{renI}}$. The pullback to $\mathscr{F}$ reads

$$\widetilde{H}_{\mathrm{IBC}}=W_{\mathrm{IBC}}H_{\mathrm{IBC}}W_{\mathrm{IBC}}^{-1}\tag{164}$$

Whenever the expression (164) extends to a self–adjoint operator, H_{IBC} extends to a self–adjoint operator on $\mathscr{F}_{\mathrm{renI}}$.

8.2 The $e^{-H_0^{-1}A^\dagger}$–Transformation Inspired by CQFT

An operator $T\propto e^{-H_0^{-1}A^\dagger}$ appears in a similar form in the CQFT literature [18, 14]. Within the latter reference, a renormalized scalar product is constructed by a procedure of the kind

$$\langle T\Psi,T\Phi\rangle_{\mathrm{ren}}:=\lim_{\Lambda\to\infty}\langle T_\Lambda\Psi,T_\Lambda\Phi\rangle e^{-\boldsymbol{\Lambda}_\Lambda}\qquad\forall T\Psi,T\Phi\in T[\mathcal{D}],\quad\mathcal{D}\subset\mathscr{F},\tag{165}$$

where the operators T, T_Λ in [14] are, however, much more involved. Here,

$$\mathbf{\Lambda}_\Lambda = 4!\|\omega v_\Lambda\|_2^2, \tag{166}$$

and Λ is a UV–cutoff. The renormalized Hamiltonian is then constructed by a limiting procedure [14, Thm. 4.1.1]:

$$\langle T\Psi, H_{\mathrm{ren}} T\Phi\rangle_{\mathrm{ren}} = \lim_{\Lambda\to\infty} \langle T_\Lambda\Psi, H_{\mathrm{ren}}(\Lambda) T_\Lambda\Phi\rangle e^{-\mathbf{\Lambda}_\Lambda}, \tag{167}$$

with $H_{\mathrm{ren}}(\Lambda)$ containing counterterms. The limit $\Lambda \to \infty$ formally leads to an infinite wave function renormalization with

$$\mathbf{\Lambda} = 4!\|\omega v\|_2^2. \tag{168}$$

The ESS construction now allows to directly define $\mathbf{\Lambda} \in \mathrm{Ren}_1$. In the simplified case $T \propto e^{-H_0^{-1}A^\dagger}$, we can even define the dressing transformation directly on $\overline{\mathscr{F}}$:

Proposition 8.2. *For $\Psi \in \dot{\mathcal{S}}_{\mathscr{F}}^\infty$, we have*

$$T\Psi := e^{-\mathbf{\Lambda}/2} e^{-H_0^{-1}A^\dagger}\Psi \in \overline{\mathscr{F}}, \tag{169}$$

with $e^{-\mathbf{\Lambda}/2} \in \mathrm{eRen}$. In particular, $e^{-H_0^{-1}A^\dagger}$ and T are well–defined linear operators

$$e^{-H_0^{-1}A^\dagger} : \dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \dot{\mathcal{S}}_{\mathscr{F}}^\infty, \qquad T : \dot{\mathcal{S}}_{\mathscr{F}}^\infty \to \overline{\mathscr{F}}.$$

Proof. As argued in the proof of Proposition 8.1, $H_0^{-1}A^\dagger$ is well–defined on $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$ and maps Fock space vectors supported on the N–particle sector to those supported on the $N+1$–particle sector.

We can write the exponential series as

$$e^{-H_0^{-1}A^\dagger} = \sum_{k=0}^\infty \frac{(-H_0^{-1}A^\dagger)^k}{k!} = 1 - H_0^{-1}A^\dagger + \frac{1}{2}H_0^{-1}A^\dagger H_0^{-1}A^\dagger - \dots. \tag{170}$$

Now, each $-H_0^{-1}A^\dagger$ maps $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$ to itself and strictly increases the sector number. So the N–sector of $e^{-H_0^{-1}A^\dagger}\Psi$ may only depend on at most $N+1$ terms of the series (170). All terms are elements of $\dot{\mathcal{S}}_{\mathscr{F}}^\infty \subset \overline{\mathscr{F}}$. Hence, $e^{-H_0^{-1}A^\dagger}$ maps $\dot{\mathcal{S}}_{\mathscr{F}}^\infty$ into itself, as claimed. Clearly, the factor $e^{-\mathbf{\Lambda}/2}$ is an element of eRen, so $T\Psi \in \overline{\mathscr{F}}$.

□

Appendices

A Proof of Lemma 5.1

Proof. We show that $\mathcal{D}_{WS}$ is dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$. Since $(S_-\otimes I) : L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y \to \mathscr{F}$ is surjective and bounded with norm 1, we then immediately get density of $(S_-\otimes I)[\mathcal{D}_{WS}]$ within $\mathscr{F}$.

If $A_1^\dagger(\varphi)$ created a boson without giving a recoil to a fermion, we would be done: In this case, $W_{\mathscr{F},1}(\varphi)\Psi_m$ would be of the form

$$\Psi_{mx}\otimes W_y(\varphi)\Omega_y. \tag{171}$$

Now, $\Psi_{mx}\in\mathcal{S}(\mathcal{Q}_x)$, which is dense in $L^2(\mathcal{Q}_x)$. Further, we have that $\operatorname{span}\{W_y(\varphi)\Omega_y \mid \varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty\}$ is dense in $\mathscr{F}_y$, so

$$\operatorname{span}\big\{\Psi_{mx}\otimes W_y(\varphi)\Omega_y \;\big|\; \varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty,\ \Psi_{mx}\in\mathcal{S}(\mathcal{Q}_x)\big\}, \tag{172}$$

is dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$.
However, $W_{\mathscr{F},1}(\varphi)\Psi_m$ is not of the form (171), since $W_{\mathscr{F},1}(\varphi)$ shifts the momentum $\boldsymbol{p}_1$ by $\sum_{\ell=1}^N \boldsymbol{k}_\ell$, as in (36) (the first fermion gets a recoil). The same recoil occurs when applying $A_1^\dagger(v)$. In other words, creation and dressing "entangle" the fermion with the created boson by giving the fermion a recoil. In order to solve this problem, we introduce a **disentangling operator** $D_1 : L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y \to L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$, which removes all recoils:

$$(D_1\Psi)(\boldsymbol{P},\boldsymbol{K}) := \Psi\left(\boldsymbol{P}-\sum_{\ell=1}^N e_1\boldsymbol{k}_\ell,\boldsymbol{K}\right). \tag{173}$$

Clearly, D_1 is unitary. Now,

$$\begin{aligned}
(W_{\mathscr{F},1}(\varphi)\Psi_m)(\boldsymbol{P},\boldsymbol{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\left(\prod_{\ell=1}^N\varphi(\boldsymbol{k}_\ell)\right)\Psi_{mx}\left(\boldsymbol{P}+\sum_{\ell=1}^N e_1\boldsymbol{k}_\ell\right)\\
\Rightarrow (D_1W_{\mathscr{F},1}(\varphi)\Psi_m)(\boldsymbol{P},\boldsymbol{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\left(\prod_{\ell=1}^N\varphi(\boldsymbol{k}_\ell)\right)\Psi_{mx}(\boldsymbol{P})\\
\Leftrightarrow D_1W_{\mathscr{F},1}(\varphi)\Psi_m &= \Psi_{mx}\otimes\sum_{N=0}^\infty\frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\underbrace{\varphi\otimes\ldots\otimes\varphi}_{N\text{ times}}\\
&= \Psi_{mx}\otimes W_y(\varphi)\Omega_y.
\end{aligned} \tag{174}$$

So by density of (172), we have that

$$\operatorname{span}\big\{D_1W_{\mathscr{F},1}(\varphi)\Psi_m \;\big|\; \varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty,\ \Psi_m\in\mathcal{C}_{WS}\big\},$$

is dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$. And since D_1 is unitary, its preimage

$$\mathcal{D}_{WS}=\operatorname{span}\big\{W_{\mathscr{F},1}(\varphi)\Psi_m \;\big|\; \varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty,\ \Psi_m\in\mathcal{C}_{WS}\big\},$$

is dense in $L^2(\mathcal{Q}_x)\otimes\mathscr{F}_y$, as well. So $(S_-\otimes I)[\mathcal{D}_{WS}]$ is dense in $\mathscr{F}$.

□

B Proof of Lemma 5.2

Proof. For evaluating $[A^\dagger, W]$, we first consider the simple case where Ψ is replaced by a boson–only vector $\Psi_y\in\mathscr{F}_y$. A vector with one boson can be written as:

$$\phi=a^\dagger(\phi)\Omega_y, \tag{175}$$

with $\phi\in\mathfrak{h}=\mathscr{F}_y^{(1)}$. A coherent displacement can then be described using

$$W_y(\varphi)a^\dagger(\phi)=a^\dagger(\phi)W_y(\varphi)-[a^\dagger(\phi),W_y(\varphi)], \tag{176}$$

where $\varphi\in\mathfrak{h}\cap\dot{\mathcal{S}}_1^\infty$. The first expression is easily computed in momentum space:

$$\begin{aligned}(a^\dagger(\phi)\Psi_y)(\boldsymbol{K})&=\frac{1}{\sqrt{N}}\sum_{\ell=1}^N\phi(\boldsymbol{k}_\ell)\Psi_y(\boldsymbol{K}\setminus\boldsymbol{k}_\ell)\\ \Rightarrow\quad (a^\dagger(\phi)W_y(\varphi)\Omega_y)(\boldsymbol{K})&=\frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}}\sum_{\ell=1}^N\phi(\boldsymbol{k}_\ell)\left(\prod_{\ell'\neq\ell}\varphi(\boldsymbol{k}_{\ell'})\right).\end{aligned} \tag{177}$$

The commutator $[a^\dagger(\phi),W_y(\varphi)]$ is computed using

$$\begin{aligned}&(*)\qquad [a^\dagger(\varphi),a^\dagger(\phi)]=0\\ &(**)\qquad [a(\varphi),a^\dagger(\phi)]=\langle\varphi,\phi\rangle.\end{aligned} \tag{178}$$

We have

$$\begin{aligned}&[a^\dagger(\phi),W_y(\varphi)]=[a^\dagger(\phi),e^{a^\dagger(\varphi)-a(\varphi)}]=\sum_{k=0}^\infty\frac{1}{k!}[a^\dagger(\phi),(a^\dagger(\varphi)-a(\varphi))^k]\\ &\overset{(*)}{=}\sum_{k=0}^\infty\frac{(-1)^k}{k!}\big([a^\dagger(\phi),a(\varphi)](a^\dagger(\varphi)-a(\varphi))^{k-1}\\ &\quad+(a^\dagger(\varphi)-a(\varphi))[a^\dagger(\phi),a(\varphi)](a^\dagger(\varphi)-a(\varphi))^{k-2}\\ &\quad+\ldots+(a^\dagger(\varphi)-a(\varphi))^{k-1}[a^\dagger(\phi),a(\varphi)]\big)\\ &\overset{(**)}{=}-\sum_{k=1}^\infty\frac{(-1)^k}{k!}k\langle\varphi,\phi\rangle(a^\dagger(\varphi)-a(\varphi))^{k-1}=\sum_{k=0}^\infty\frac{(-1)^k}{k!}\langle\varphi,\phi\rangle(a^\dagger(\varphi)-a(\varphi))^k\\ &=\langle\varphi,\phi\rangle W_y(\varphi).\end{aligned} \tag{179}$$

It is easy to check that all above formulas hold as strong operator identities on the space of finite–boson states $\mathscr{F}_{\mathrm{fin},y}$ (defined in (99)). By (96), an application of k operators of the kind[4] $a^\sharp(\varphi) \in \{a(\varphi), a^\dagger(\varphi)\}$ to $\Psi_y \in \mathscr{F}_{\mathrm{fin},y}$ is bounded by

$$\Big\| \big(a(\varphi)^\sharp\big)^k \Psi_y \Big\| \leq \sqrt{\frac{(N_{\max}+k)!}{N_{\max}!}} \|\Psi_y\| \, \|\varphi\|^n. \tag{180}$$

This allows to estimate

$$\big\| a^\sharp(\phi) W_y(\varphi)\Psi \big\| \leq \sum_{k\in\mathbb{N}_0} \frac{1}{k!} \sqrt{\frac{(N_{\max}+k+1)!}{N_{\max}!}} \|\Psi_y\| \, \|2\varphi\|^n \, \|\phi\| < \infty, \tag{181}$$

and an analogous estimate shows that $W_y(\varphi)a^\sharp(\phi)$ is well–defined.

Putting together (176) and (179), we obtain the action of $W_y(\varphi)$ on single–boson states:

$$W_y(\varphi)a^\dagger(\phi)\Omega_y = (a^\dagger(\phi) - \langle\varphi,\phi\rangle) W_y(\varphi)\Omega_y. \tag{182}$$

Now, we turn to state vectors with many fermions and one boson, $A_1^\dagger(\phi)\Psi_m \in L^2(\mathcal{Q}_x) \otimes \mathscr{F}_y$. Further, we go over from dressings by $W_y(\varphi)$ to $W_{\mathscr{F},j}(\varphi)$, which is done replacing $a(\phi), a^\dagger(\phi)$ by $A_j(\phi), A_j^\dagger(\phi)$. Note that $A_j(\phi), A_j^\dagger(\phi)$ are no longer merely creating and annihilating bosons, but they also shift a fermion's momentum. Computations in (179) run through in almost the same manner. We have to replace $(**)$ by the CCR (86). If $j \neq j'$, we further use that $V_{jj'}(\varphi^*\phi)$ (which replaces $\langle\varphi,\phi\rangle$ in (179)) commutes with[5] $A_{j''}(\varphi)$ and $A_{j''}^\dagger(\varphi)$, so we can still pull it to the left.

The final result is

$$W_{\mathscr{F},j}(\varphi)A_{j'}^\dagger(\phi) = \begin{cases} (A_{j'}^\dagger(\phi) - \langle\varphi,\phi\rangle) W_{\mathscr{F},j}(\varphi) & \text{if } j = j' \\ (A_{j'}^\dagger(\phi) - V_{jj'}(\varphi^*\phi)) W_{\mathscr{F},j}(\varphi) & \text{if } j \neq j' \end{cases}. \tag{183}$$

This is one of the four identities claimed in (107). The other three identities follow by summation over j or j'.

The four identities in (108) are obtained analogously. In place of $(**)$, we use the CCR (86) together with $[A_j(\varphi), A_{j'}(\phi)] = 0$ and keep in mind that the factor $(-1)^k$ drops out in (179), which yields the desired result.

It is easy to check that all identities above hold as strong operator identities on the domain $\mathscr{F}_{\mathrm{fin}}$ defined in (106). The momentum space definitions of $A_j^\dagger(\varphi), A_j(\varphi)$ in (19) and (21) directly yield the well–known estimates

$$\big\| A_j^\dagger(\varphi)\Psi \big\| \leq \big\| (N+1)^{1/2}\Psi \big\| \, \|\varphi\|, \qquad \big\| A_j(\varphi)\Psi \big\| \leq \big\| N^{1/2}\Psi \big\| \, \|\varphi\|, \tag{184}$$

[4]Here, we allow $\sharp$ to be a different superscript in each factor. E.g., $(a^\sharp)^4$ would also represent $aa^\dagger aa^\dagger$.

[5]This is shown by a similar fiber decomposition argument, as used in the proof of Lemma 5.5.

which are analogous to (96) and allow for employing the same arguments as below (99). Thus, all expressions in (107) and (108) are well–defined on $\mathscr{F}_{\text{fin}}$.

□

Remarks.

7. If we replace the form factors $\varphi(\boldsymbol{k})$ and $\phi(\boldsymbol{k})$ in $A^\dagger(\phi)$ by $\varphi(\boldsymbol{p},\boldsymbol{k})$ and $\phi(\boldsymbol{p},\boldsymbol{k})$ (as they appear in more realistic QFT models), then $[A_j^\dagger(\varphi), A_{j'}^\dagger(\phi)] = 0$ will no longer hold true. The operators $A_j^\dagger(\phi)$ change the momentum of the fermion which is emitting a boson. So when one fermion creates two bosons with different form factors, which depend on the fermion momentum, then it can make a difference, which boson is created first. In addition, $V_{j'j''}$ will no longer commute with A_j and $A_j^\dagger$. In this case, several multi–commutators of the form $[A_{j_1}^\sharp, [A_{j_2}^\sharp, \ldots, [A_{j_n}^\sharp, A_{j'}^\sharp]]]$ with $A_j^\sharp \in \{A_j, A_j^\dagger\}$ appear.

C Proof of Lemma 6.1

Proof. We evaluate the pullbacks of $A(v)$ and E_∞, keeping in mind that by Lemma 5.4, $\Psi = W(s)W_{\mathscr{F},1}(\varphi)\Psi_m = W(s)W_1(\varphi)\Psi_m$.

- $A(v)$: The commutator $[W(s)W_1(\varphi), A(v)]$ is evaluated using the extended commutation relations established in Lemma 5.7:

$$\begin{aligned} A(v)\Psi &= A(v)W(s)W_1(\varphi)\Psi_m \\ &= [A(v), W(s)W_1(\varphi)]\Psi_m + W(s)W_1(\varphi)\underbrace{A(v)\Psi_m}_{=0} \\ &= \big(M\langle v,s\rangle + \langle v,\varphi\rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s)\big)\Psi. \end{aligned} \tag{185}$$

- E_∞: The self–energy operator (140) now exactly removes the term $M\langle v,s\rangle$, so we have

$$(A(v) - E_\infty)\Psi = \big(\langle v,\varphi\rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s)\big)\Psi. \tag{186}$$

Now, by Definition 5.4, we may commute $\langle v,\varphi\rangle, V$ and $V_{\bullet 1}$ with $W(s)$ yielding

$$\begin{aligned} W^{-1}(s)\big(A(v) - E_\infty\big)W(s)W_1(\varphi)\Psi_m &= \big(\langle v,\varphi\rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s)\big)W_1(\varphi)\Psi_m, \\ &= (\text{res}_1(\varphi) + V)\, W_1(\varphi)\Psi_m \quad \in \overline{\mathscr{F}}_{\text{ex}}. \end{aligned} \tag{187}$$

Now, Lemma 5.4 allows replacing W_1 by $W_{\mathscr{F},1}$, which renders the desired result.

□

Remarks.

8. If the form factor v depended on the fermion momentum $\boldsymbol{p}$, then $s = -\frac{v}{\omega}$ would also depend on $\boldsymbol{p}$. In that case, $W^{-1}(s)VW(s) = V$ will no longer hold true in general, as the function $s(\boldsymbol{k}, \boldsymbol{p})$ in the dressing operator $W(s)$ then depends on $\boldsymbol{p}$ and fermion momenta are changed by V. If $(A(v) - E_\infty)W(s)W_{\mathscr{F},1}(\varphi)\Psi_m$ is then expressed in terms of dressed coherent states, multiple commutators appear.

D Proof of Lemma 6.2

Proof. Again, we may use Lemmas 5.3 and 5.4, to write

$$\Psi = W(s)W_{\mathscr{F},1}(\varphi)\Psi_m = W(s)W_1(\varphi)\Psi_m \in \overline{\mathscr{F}}.$$

Following Proposition 4.1 we further have $H_{0,y}\Psi, A^\dagger(v)\Psi \in \overline{\mathscr{F}}$. We first evaluate $(H_{0,y} + A^\dagger(v))W(s)W_1(\varphi)\Psi_m$ in momentum space and then use Definition 5.3 to pull $W(s)$ to the left.

- $H_{0,y}$: The important point is that the expression $(H_{0,y} + A^\dagger(v))W(s)$ cancels terms in $W(s)H_{0,y}$. We therefore investigate the commutator expression $[H_{0,y}, W(s)]W_1(\varphi)\Psi_m$ and compare it to $A^\dagger(v)W(s)W_1(\varphi)\Psi_m$.

  First, we note that the application of $H_{0,y}$ to a dressed state just changes one single photon dispersion relation. Therefore, it is equivalent to applying a creation operator to the dressed state:

$$\begin{aligned}
&(H_{0,y}W_1(\varphi)\Psi_m)(\boldsymbol{P}, \boldsymbol{K}) \\
=&\frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \left(\sum_{\ell'=1}^{N} \omega(\boldsymbol{k}_{\ell'})\right) \left(\prod_{\ell=1}^{N} \varphi(\boldsymbol{k}_\ell)\right) \Psi_{mx}(\boldsymbol{P}') \\
=&\frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \sum_{\ell=1}^{N} \omega(\boldsymbol{k}_\ell)\varphi(\boldsymbol{k}_\ell) \left(\prod_{\ell' \neq \ell}^{N} \varphi(\boldsymbol{k}_{\ell'})\right) \Psi_{mx}(\boldsymbol{P}') \\
=&(A_1^\dagger(\omega\varphi)W_1(\varphi)\Psi_m)(\boldsymbol{P}, \boldsymbol{K}),
\end{aligned} \tag{188}$$

  with $\boldsymbol{P}' = \big(\boldsymbol{p}_1 + \sum_{\ell'=1}^{N} \boldsymbol{k}_{\ell'}, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_M\big)$. Replacing the dressing $W_1(\varphi)$ by $W(s)W_1(\varphi)$ and applying the commutation relations from Definitions 5.3 and 5.4, we obtain

$$\begin{aligned}
&H_{0,y}W(s)W_1(\varphi)\Psi_m \\
=&(A^\dagger(\omega s) + A_1^\dagger(\omega\varphi))W(s)W_1(\varphi)\Psi_m \\
=&-A^\dagger(v)W(s)W_1(\varphi)\Psi_m \\
&+ W(s)W_1(\varphi)\big(A_1^\dagger(\omega\varphi) - \langle v, \varphi\rangle - V_{\bullet 1}(v^*\varphi) + \langle \varphi, \omega\varphi\rangle\big)\Psi_m.
\end{aligned} \tag{189}$$

Here, we used $s^*\omega = \omega s = -v$ and Definition 5.4, which allows us to pull the formal scalar products and the $V_{\bullet 1}$–term past the dressing operators.
By means of (188) and the commutation relations from Definitions 5.3 and 5.4, we also have

$$\begin{aligned} &W(s)H_{0,y}W_1(\varphi)\Psi_m \\ =&W(s)W_1(\varphi)A_1^\dagger(\omega\varphi)\Psi_m + W(s)W_1(\varphi)\langle\varphi,\omega\varphi\rangle\Psi_m. \end{aligned} \tag{190}$$

Combining (189) and (190), we finally obtain

$$\begin{aligned} &[H_{0,y}, W(s)]W_1(\varphi)\Psi_m \\ =&-A^\dagger(v)W(s)W_1(\varphi)\Psi_m - W(s)W_1(\varphi)\big(\langle v,\varphi\rangle + V_{\bullet 1}(v^*\varphi)\big)\Psi_m. \end{aligned} \tag{191}$$

- $A^\dagger(v)$: Here, we do not need to perform any calculations. The appearing term simply cancels the $-A^\dagger(v)W(s)W_1(\varphi)\Psi_m$ from (191).

Now, adding both terms and using Definition 5.4, (191) amounts to

$$\begin{aligned} &\big([H_{0,y}, W(s)] + A^\dagger(v)W(s)\big)W_1(\varphi)\Psi_m \\ =&W(s)W_1(\varphi)\big(-\langle v,\varphi\rangle - V_{\bullet 1}(v^*\varphi)\big)\Psi_m \\ =&W(s)\big(-\mathrm{res}_1(\varphi)\big)W_1(\varphi)\Psi_m. \end{aligned} \tag{192}$$

This is equivalent to

$$\big(H_{0,y} + A^\dagger(v)\big)W(s)W_1(\varphi)\Psi_m = W(s)\big(H_{0,y} - \mathrm{res}_1(\varphi)\big)W_1(\varphi)\Psi_m. \tag{193}$$

Lemma 5.4 allows again to replace W_1 by $W_{\mathscr{F},1}$, which yields the final result.

□

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