

# Implementing Bogoliubov Transformations Beyond the Shale–Stinespring Condition

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## Abstract

We provide two extensions of a dense subspace of Fock space, such that Bogoliubov transformations become implementable on them, even though they violate the Shale–Stinespring condition, so they are not implementable on Fock space. Both the bosonic and fermionic case are covered. Conditions for implementability in the extended sense are stated and proved. From these, we derive conditions for a quadratic Hamiltonian to be diagonalizable by a Bogoliubov transformation that is implementable in the extended sense. Three examples illustrate situations, in which an implementation in the extended sense is possible although the Shale–Stinespring condition fails to hold.

Key words: Second quantization, Bogoliubov transformation, Renormalization, Many–Body Quantum Mechanics, Quantum Field theory, Infinite tensor products.

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# 1 Introduction

Bogoliubov transformations are a powerful tool in many–body quantum mechanics, as they allow for simplifying Hamiltonians  $H$  and hence analyzing the dynamics of a system. Roughly speaking, a Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$  is realized by replacing all creation and annihilation operators  $a^\dagger(f), a(f)$  inside a product  $H$  of such operators by

$$b^\dagger(f) = a^\dagger(uf) + a(v\bar{f}), \quad b(f) = a^\dagger(v\bar{f}) + a(uf),$$

with  $f$  being an element of the one–particle Hilbert space  $\mathfrak{h}$ , where  $u, v$  are linear operators on  $\mathfrak{h}$  and where  $\bar{f}$  is the complex conjugate of  $f$ . This replacement can be used to eliminate pair creation and annihilation terms from quadratic Hamiltonians  $H$ , i.e., to “diagonalize  $H$ ” [1, 2, 3, 4]. Related transformations even allow for eliminating inconvenient terms of higher order from non–quadratic operator products  $H$  [5, 6].

In particular, it is desirable to find a unitary operator  $\mathbb{U}_\mathcal{V}$  on Fock space  $\mathcal{F}$ , such that  $\mathbb{U}_\mathcal{V}$  establishes the replacement  $a^\# \mapsto b^\#$  via

$$\mathbb{U}_\mathcal{V} a^\dagger(f) \mathbb{U}_\mathcal{V}^* = b^\dagger(f), \quad \mathbb{U}_\mathcal{V} a(f) \mathbb{U}_\mathcal{V}^* = b(f). \quad (1)$$

In that case we say that  $\mathbb{U}_\mathcal{V}$  implements the transformation  $\mathcal{V}$  and we call  $\mathcal{V}$  “implementable” (in the regular sense). It is well–known [7, 8] that  $\mathcal{V}$  is implementable, if and only if the **Shale–Stinespring condition** holds, which asserts that  $\text{tr}(v^*v) < \infty$ .

However, there are situations in which it is desirable to modify a Hamiltonian  $H$  or to describe its dynamics by a Bogoliubov transformation  $\mathcal{V}$  that is not implementable. Such situations occur, for instance, in relativistic models [9, 10], and within many–body systems of infinite size [11, 12, 13].

In this work, we therefore propose two frameworks with extensions of a dense subspace of Fock space  $\mathcal{D}_\mathcal{F} \subseteq \mathcal{F}$  on which  $\mathcal{V}$  is to be implemented.

- The **extended state space** (ESS) framework introduced in [14], which builds around vector spaces  $\mathcal{D}_\mathcal{F} \subset \overline{\mathcal{F}} \subset \overline{\mathcal{F}}_{\text{ex}}$ . Here,  $\overline{\mathcal{F}}$  contains elements of the form  $\Psi = e^\mathfrak{r} \Psi_m$ , where  $\mathfrak{r}$  is a (rigorously defined) and possibly infinite renormalization constant and  $\Psi_m$  is a function on the configuration space  $\mathcal{Q}$  that may be outside  $L^2(\mathcal{Q})$ . The space  $\overline{\mathcal{F}}_{\text{ex}}$  additionally contains elements of the form  $e^\mathfrak{r} \mathfrak{r} \Psi_m$ . Our construction differs from that in [14], as it is based on sequence spaces instead of smooth polynomially bounded functions with poles.
- The **infinite tensor product** (ITP) framework that builds on von Neumann’s ITP space [15]. Here, a Hilbert space  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^\otimes \mathcal{H}_k$  is defined, which extends the Fock space  $\mathcal{F}$ .

Both extensions are in the same spirit as Hilbert space riggings, which are discussed in detail within [16]. The construction of  $\overline{\mathcal{F}}$  may also remind about non–standard analysis,

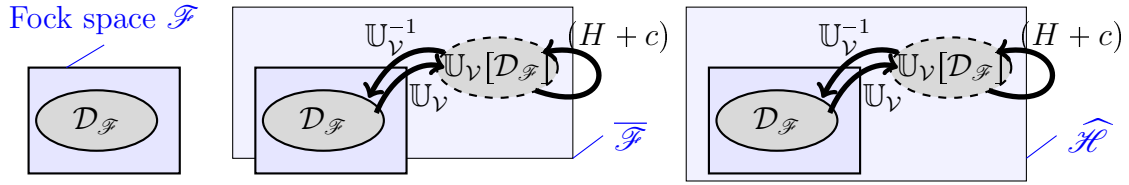


Figure 1: Renormalization of  $H$  using Fock space extensions. Note that  $\widehat{\mathcal{H}}$  extends  $\mathcal{F}$ , while  $\overline{\mathcal{F}}$  only extends a dense subspace of it. Color online.

where an extension  ${}^*\mathbb{R}$  of the real line  $\mathbb{R}$  is constructed [17]. However, our construction is not related to it, as we discuss in Remark 1. Our implementation procedure differs from that on Fock space, so let us quickly explain it: We take a formal expression  $H$  that consists of a product of  $a^\dagger$ - and  $a$ -operators. The aim is to define an operator (see Figure 1)

$$\tilde{H} = \mathbb{U}_V^{-1}(H + c)\mathbb{U}_V, \quad (2)$$

on a dense set  $\mathcal{D}_{\mathcal{F}} \subseteq \overline{\mathcal{F}}$ , where  $\tilde{H}$  is the version of  $H$  with  $a^\dagger$  replaced by  $b^\dagger$  and normal ordering applied. The “renormalization constant”  $c$  stems from normal ordering and may be formally infinite. We define a linear operator  $\mathbb{U}_V : \mathcal{D}_{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  or  $\mathbb{U}_V : \mathcal{D}_{\mathcal{F}} \rightarrow \widehat{\mathcal{H}}$  in such a way that  $(H + c)$  maps the space  $\mathbb{U}_V[\mathcal{D}_{\mathcal{F}}] \subset \overline{\mathcal{F}}$  or  $\mathbb{U}_V[\mathcal{D}_{\mathcal{F}}] \subset \widehat{\mathcal{H}}$  into itself. So  $\tilde{H}$  is the well-defined “renormalized version” of  $H$ .

In particular,  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$  are vector spaces without a scalar product or even a topology. Their only purpose is to allow for formal calculations involving infinities, which in the end produce an operator  $\tilde{H}$ . This  $\tilde{H}$  generates dynamics on a Hilbert space  $\mathcal{F}$ , which provides a suitable physical interpretation. Vectors in  $\overline{\mathcal{F}}$ ,  $\overline{\mathcal{F}}_{\text{ex}}$  can rather be seen as “virtual particle states” that contain no immediate physical meaning.

Our **main result** is now that in the extended sense, specified in Definition 5.1,  $\mathcal{V}$  can indeed be implemented:

- on  $\overline{\mathcal{F}}$  in the bosonic (Theorem 5.2) and the fermionic case (Theorem 5.4),
- on  $\widehat{\mathcal{H}}$  in the bosonic (Theorem 5.1) and the fermionic case (Theorem 5.3),

if the spectrum of the operator  $v^*v$  is countable. In Theorem 5.4, we need the additional assumption of  $\mathcal{V}$  inducing a full particle–hole transformation on at most finitely many modes. The reason is that  $\overline{\mathcal{F}}$  can only describe a finite number of particles.

It would be highly desirable to establish a similar result in the case of a generic  $v^*v$ . The main difficulty is the establishment of a suitable space  $\mathcal{E}(X)$  (14) of smooth functions on a non–open set  $X$ . This space  $\mathcal{E}(X)$  is needed to construct the extended state spaces  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$ . We conjecture that a statement similar to Theorems 5.2 and 5.4 can be established for generic  $v^*v$ . By contrast, the definition of ITP spaces  $\widehat{\mathcal{H}}$  directly

carries over to the case of generic  $v^*v$  and we obtain  $\prod_{x \in X}^{\otimes} \mathcal{H}_x$  with  $X$  being a possibly uncountable set related to  $\sigma(v^*v)$ . As a well-defined  $a^\dagger(f), a(f)$ , we would then only be able to treat sums  $a^\sharp = \sum_x f(x)a_x^\sharp$  over countably many  $x \in X$ . So  $f$  is equal to 0 apart from finitely many points and the set of such  $f$  is not dense in  $\mathfrak{h}$ . Thus, the generalizations of Theorems 5.1 and 5.3 would be limited to a very restrictive set of form factors  $f \in \mathfrak{h}$ , possibly only containing  $f = 0$ .

One may object that the construction of  $\mathbb{U}_\mathcal{V}$  creates an unnecessary effort for obtaining  $\tilde{H}$ , since  $\tilde{H}$  can directly be computed from  $H$  under a replacement  $a^\sharp \mapsto b^\sharp$ . While for Bogoliubov transformations, the latter way is indeed more efficient, such a replacement may not exist for other operator transformations. These operator transformations appear in Quantum Field Theory (QFT) and many-body systems, where a formal Hamiltonian  $H$  may not be well-defined in the infinite-volume limit [18, 19, 20]. In fact, the Fock space extension framework is intended to allow for a definition of more general operator transformations  $W$ , such that for a given formal  $H$ , the “renormalized Hamiltonian”

$$\tilde{H} = W^{-1}(H + c)W, \quad (3)$$

is well-defined on  $\mathcal{D}_\mathcal{F} \subseteq \mathcal{F}$  and allows for a self-adjoint extension, which generates quantum dynamics on  $\mathcal{F}$ . Here,  $c$  is a general counterterm and not necessarily just a constant. Our consideration of  $W = \mathbb{U}_\mathcal{V}$  can rather be seen as a proof of concept for this type of renormalization. In contrast to non-perturbative cutoff renormalization [18, 19, 20, 21, 22, 23], this framework does not require arbitrary cutoffs that necessarily break Lorentz invariance. Another advantage is that the Hamiltonian  $\tilde{H}$  can directly be written down without involving limit processes. Further, the amount “by how much  $H$  fails to map  $W[\mathcal{D}_\mathcal{F}]$  into itself” may provide useful heuristics for the choice of counterterms in  $c$ . In general, the ESS framework offers a variety of mathematical tools that appear on the way of constructing  $\overline{\mathcal{F}}$  and allow for a rigorous treatment of infinite quantities that appear in formal intermediate calculations. These tools do not appear in the ITP framework. We hope that in the future, further formal transformations  $W$ , which do not exist on Fock space, can be defined in either of both frameworks.

We remark that there have been attempts to apply the ITP framework in the context of Quantum Electrodynamics (QED) to scattering theory, where formal Weyl transformations occur, that cannot be defined on Fock space [24, 25, 26]. The ESS framework has also recently been successfully applied to Weyl transformations [14].

The rest of this paper is structured as follows: In Section 2, we give the basic definitions of the second quantization language, the ESS framework, and the ITP framework. Section 3 is a recap of known material about implementing Bogoliubov transformations on Fock space which, however, is necessary to establish the notation for an implementation in the extended sense and to understand which formulas are generalized therein. The extension beyond the Shale–Stinespring condition starts in Section 4. Here we de-

fine the  $\mathcal{V}$ -dependent Fock space extensions  $\widehat{\mathcal{H}}$ ,  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$ , and prove that creation and annihilation operators are well-defined on them (Lemmas 4.4 and 4.5).

Section 5 then concerns the implementability of  $\mathcal{V}$  in the extended sense, where a precise definition is given what this implementability means. We prove that the implementer  $\mathbb{U}_{\mathcal{V}}$  is well-defined (Lemma 5.1), set up conditions for  $\mathbb{U}_{\mathcal{V}}$  to implement  $\mathcal{V}$  (Lemma 5.2) and establish these conditions within Theorems 5.1–5.4 for different cases.

These implementers  $\mathbb{U}_{\mathcal{V}}$  can then be used to diagonalize quadratic Hamiltonians, i.e., to remove the  $a^\dagger a^\dagger$ - and  $aa$ -terms. In Section 6, we give a precise definition of what a diagonalization in the extended sense is, and provide conditions for when it can be performed, which is in Propositions 6.1 (bosonic case) and 6.2 (fermionic case).

Section 7 then offers three examples for a diagonalization in the extended sense. Proposition 7.1 concerns a bosonic field coupled to an external classical field by a Wick square, Proposition 7.2 considers the fermionic BCS model and Proposition 7.3 treats a toy model for fermions in a strong electromagnetic field that involves pair creation and annihilation.

## 2 Basic Definitions

### 2.1 Fock Space Notions

We start with some elementary notions of second quantization. Generally, in this framework, a particle is assigned a single coordinate  $\mathbf{x} \in X \subseteq \mathbb{R}^d$  with a measure  $\mu$  given on  $X$ . We will mainly be focusing on the cases  $X = \mathbb{R}^d$  and  $X = \mathbb{N}$ . A configuration of  $N \in \mathbb{N}_0$  particles is given by the tuple  $q = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ . For  $N = 0$ , there exists only one empty tuple, denoted  $q = \emptyset$ . The set of all tuples for all  $N \in \mathbb{N}_0$  is called ordered configuration space

$$\mathcal{Q}(X) := \bigsqcup_{N=0}^{\infty} \mathcal{Q}(X)^{(N)} := \bigsqcup_{N=0}^{\infty} X^N, \quad (4)$$

which allows for a standard topology and a measure  $\mu_N$  on each sector  $\mathcal{Q}(X)^{(N)}$ , hence yielding a topology and a measure  $\mu_{\mathcal{Q}}$  on  $\mathcal{Q}(X)$ . The ordered Fock space is then the  $L^2$ -Hilbert space with respect to that measure:

$$\mathcal{F}(X) := L^2(\mathcal{Q}(X), \mathbb{C}). \quad (5)$$

The corresponding scalar product will be denoted by  $\langle \Phi, \Psi \rangle := \int_{\mathcal{Q}(X)} \overline{\Phi(q)} \Psi(q) dq$  with  $\Phi(\cdot), \Psi(\cdot)$  being two representative functions of the vectors  $\Phi, \Psi \in \mathcal{F}(X)$  and the overline denoting complex conjugation. For  $\Psi \in C_0(\mathcal{Q}(X))$ , a unique continuous representative function exists. This includes the cases of smooth functions  $\Psi \in C^\infty(\mathcal{Q}(X))$  and smooth functions with compact support  $\Psi \in C_c^\infty(\mathcal{Q}(X))$  (called test functions), which are both dense in  $\mathcal{F}$ .

For describing bosonic/fermionic particle exchange symmetries, we introduce the symmetrization operators  $S_+, S_- : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ . They are defined by

$$(S_{\pm}\Psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) := \frac{1}{N!} \sum_{\sigma \in S_N} (\pm 1)^{(1-\text{sgn}(\sigma))/2} \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}), \quad (6)$$

with permutation group  $S_N$ . Here,  $(1 - \text{sgn}(\sigma))/2$  is 0 if the permutation is even, and 1 if it is odd. The bosonic (+) and fermionic (-) Fock space is given by:

$$\mathcal{F}_{\pm}(X) := S_{\pm}[\mathcal{F}(X)]. \quad (7)$$

In analogy to the  $(N)$ -particle sectors of configuration space  $\mathcal{Q}(X)^{(N)}$ , the corresponding  $(N)$ -particle sectors of Fock space are denoted by

$$\mathcal{F}(X)^{(N)} := L^2(\mathcal{Q}(X)^{(N)}, \mathbb{C}), \quad (8)$$

and  $\mathcal{F}_{\pm}^{(N)}$  are the symmetrized/antisymmetrized versions.

Often in the literature, the Fock space is equivalently constructed by starting from the one-particle sector  $\mathfrak{h} := \mathcal{F}(X)^{(1)} = \mathcal{F}_{\pm}(X)^{(1)}$ . The  $(N)$ -particle sector is then defined by the Hilbert space tensor product

$$\mathcal{F}(X)^{(N)} := \underbrace{\mathfrak{h} \otimes \dots \otimes \mathfrak{h}}_{N \text{ times}}, \quad (9)$$

where an algebraic tensor product space is taken, and then closed in the Hilbert space topology. The Fock space is defined here as:

$$\mathcal{F}(X) := \bigoplus_{N=0}^{\infty} \mathcal{F}(X)^{(N)}. \quad (10)$$

Note that by a change of basis, any separable Hilbert space  $L^2(X)$  can be isometrically identified with  $\ell^2 = L^2(\mathbb{N})$ . The same basis choice leads to an identification of the Fock space  $\mathcal{F}(X)$  with  $\mathcal{F}(\mathbb{N})$ . In the following, we will just write  $\mathcal{Q}$  and  $\mathcal{F}$  for configuration and Fock space and only make the  $(X)$  explicit if it is needed.

Creation and annihilation operators  $a^{\dagger}, a$  of some function  $f \in \mathfrak{h}$  can be defined on  $\mathcal{F}$  by using  $q \setminus \mathbf{X} \in X^{N-1}$  for denoting the removal of one particle from a configuration  $q \in X^N$ :

$$\begin{aligned} (a_{\pm}^{\dagger}(f)\Psi)(q) &= \sum_{j=1}^N \frac{(\pm 1)^j}{\sqrt{N}} f(\mathbf{x}_j) \Psi(q \setminus \mathbf{x}_j) \\ (a_{\pm}(f)\Psi)(q) &= \sqrt{N+1} \int \overline{f(\mathbf{x})} \Psi(q, \mathbf{x}) d\mu(\mathbf{x}). \end{aligned} \quad (11)$$

It is well-known that the fermionic operators  $a_-, a_-^\dagger$  are bounded and hence defined on all  $\Psi \in \mathcal{F}_-$ , while the bosonic  $a_+, a_+^\dagger$  are unbounded, but can still be defined on a dense subspace of  $\mathcal{F}_+$ . Also, the definitions naturally extend to  $\mathcal{F}$  or a subspace of it. Further, (11) implies the canonical commutation/anticommutation relations (CCR/CAR):

$$[a_\pm(f), a_\pm^\dagger(g)]_\pm = \langle f, g \rangle_{\mathfrak{h}}, \quad [a_\pm(f), a_\pm(g)]_\pm = 0 = [a_\pm^\dagger(f), a_\pm^\dagger(g)]_\pm, \quad (12)$$

with commutator  $[A, B]_+ = [A, B] = AB - BA$  and anticommutator  $[A, B]_- = \{A, B\} = AB + BA$ . In the following, we will often drop the indices  $\pm$  if there is no risk of confusion.

It is also customary to just consider the  $a(f), a^\dagger(f)$  not as operators, but as formal expressions within a  $*$ -algebra

$$\mathcal{A} = \mathcal{A}_\pm \quad \text{generated by} \quad \{a_\pm(f), a_\pm^\dagger(f) \mid f \in \mathfrak{h}\}. \quad (13)$$

The involution is given by  $a(f)^* = a^\dagger(f)$  and the multiplication in  $\mathcal{A}$  is such that the CCR/CAR hold. In particular,  $\mathcal{A}_-$  is a  $C^*$ -algebra by boundedness of operators.

## 2.2 Extended State Space

In this subsection, we define extensions  $\overline{\mathcal{F}}, \overline{\mathcal{F}}_{\text{ex}}$  of a dense subspace of Fock space  $\mathcal{F} = \mathcal{F}(\mathbb{N})$ , using some mathematical structures similar to those in [14, Sect. 3].

With a fixed basis, a single-particle wave function can be expressed by a sequence  $\phi = (\phi_j)_{j \in \mathbb{N}}, \phi \in \ell^2 = L^2(\mathbb{N})$ .  $N$ -particle wave functions are elements of  $L^2(\mathbb{N}^N)$  and the Fock space is  $\bigoplus_{N \in \mathbb{N}_0} L^2(\mathbb{N}^N)$ . We extend these three spaces to

- The space of **generalized one-particle wave functions**

$$\mathcal{E}(\mathbb{N}) = \mathbb{C}^{\mathbb{N}} := \{\phi : \mathbb{N} \rightarrow \mathbb{C}\}, \quad (14)$$

which is just the space of complex sequences  $\phi = (\phi_j)_{j \in \mathbb{N}}$ . It replaces the smooth function space  $\mathcal{S}_1^\infty$  from [14], see also Remark 2. And indeed, one may consider  $\mathcal{E}(\mathbb{N}) = C^\infty(\mathbb{N})$  as a “space of smooth functions” on the natural numbers, where smoothness is trivially satisfied.

- The space of **generalized  $N$ -particle wave functions**

$$\mathcal{E}^{(N)}(\mathbb{N}) := \{\Psi : \mathbb{N}^N \rightarrow \mathbb{C}\}. \quad (15)$$



- The space of **generalized Fock space functions**

$$\mathcal{E}_{\mathcal{F}}(\mathbb{N}) := \bigoplus_{N=0}^{\infty} \mathcal{E}^{(N)}(\mathbb{N}) = \{\Psi : \mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}\}, \quad (16)$$

which replaces  $\mathcal{S}_{\mathcal{F}}^{\infty}$  from [14].

$\mathcal{E}_{\mathcal{F}}$  is *not* yet the final extended state space. Formal state vectors occurring in QFT include products of functions  $\Psi_m \in \mathcal{E}_{\mathcal{F}}$  and exponentials of divergent sums  $e^{\mathbf{r}}$  with “ $\mathbf{r} = \pm\infty$ ”, see [14]. Hence, we need to construct structures accommodating such infinite quantities.

First, we introduce a **space of renormalization factors**  $\text{Ren}_1(\mathbb{N})$ , which contains all formal (and possible divergent) sums over sequences  $\mathbf{r} = (r_j)_{j \in \mathbb{N}} \in \mathcal{E}(\mathbb{N})$ . It replaces the space of formal (and possibly divergent) integrals  $\text{Ren}_1$  in [14]

- $\text{Ren}_1(\mathbb{N}) := \mathcal{E} / \sim_{\text{Ren}_1}$ . Here, for  $\mathbf{r}_1, \mathbf{r}_2 \in \mathcal{E}(\mathbb{N})$ , we define  $\mathbf{r}_1 \sim_{\text{Ren}_1} \mathbf{r}_2$  iff  $(\mathbf{r}_1 - \mathbf{r}_2) \in \ell^1 = L^1(\mathbb{N})$  and  $\sum_{j \in \mathbb{N}} (r_{1,j} - r_{2,j}) = 0$ .

We denote elements of  $\text{Ren}_1$  by  $\mathbf{r} = [\mathbf{r}]$  and identify  $\mathbf{r} = \sum_{j \in \mathbb{N}} r_j \in \mathbb{C}$ , if  $\mathbf{r} \in \ell^1$ . All  $\text{Ren}_1$ -elements not identifiable with a  $\mathbb{C}$ -number can be thought of as “controlled infinitely large numbers”.

Multiplication of  $\text{Ren}_1$ -elements is allowed by introducing the free vector spaces  $\text{Pol}_P(\mathbb{N})$ ,  $P \in \mathbb{N}$ , called space of **renormalization polynomials of degree  $P$** , which are generated by all commutative products  $\mathbf{r}_1 \cdot \dots \cdot \mathbf{r}_p$ ,  $p \leq P$ ,  $\mathbf{r}_j \in \text{Ren}_1(\mathbb{N})$ . Again, we identify equivalent terms by modding out an equivalence relation:

- $\text{Ren}_P(\mathbb{N}) := \text{Pol}_P(\mathbb{N}) / \sim_{\text{Ren}_P}$  with  $\sim_{\text{Ren}_P}$  generated by both  $\mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_p \sim_{\text{Ren}_P} c_1 \mathbf{r}_2 \dots \mathbf{r}_p$  for  $\mathbf{r}_1 = \sum_{j \in \mathbb{N}} r_{1,j} = c_1 \in \mathbb{C}$  and  $(c_1 c_2) \mathbf{r}_1 \dots \mathbf{r}_p \sim_{\text{Ren}_P} c_1 (c_2 \mathbf{r}_1) \dots \mathbf{r}_p$ .

The space of **renormalization polynomials** is then given by

$$\text{Ren}(\mathbb{N}) := \bigcup_{P \in \mathbb{N}} \text{Ren}_P(\mathbb{N}). \quad (17)$$

Wave function renormalization factors usually take the form  $e^{\mathbf{r}}$  with  $\mathbf{r} \in \text{Ren}_1$ , as well as linear combinations of such terms. We will interpret them as elements of a suitably defined field  $\text{eRen}$ . For this, consider the group algebra  $\mathbb{C}[\text{Ren}_1]$ , with elements denoted by  $c_1 e^{\mathbf{r}_1} + \dots + c_M e^{\mathbf{r}_M}$ ,  $c_j \in \mathbb{C}$ ,  $\mathbf{r}_j \in \text{Ren}_1$ . The group elements are correspondingly denoted by exponentials  $e^{\mathbf{r}_j}$  with group multiplication  $e^{\mathbf{r}_1} e^{\mathbf{r}_2} = e^{\mathbf{r}_1 + \mathbf{r}_2}$ . Some elements of  $\mathbb{C}[\text{Ren}_1]$  are intuitively equal to zero. We set them equivalent to zero by modding out an ideal  $\mathcal{I} \subset \mathbb{C}[\text{Ren}_1]$  generated by all elements  $e^c e^{\mathbf{r}} - e^{c+\mathbf{r}}$  with  $c \in \mathbb{C}$ ,  $\mathbf{r} \in \text{Ren}_1$ .

It is proven in [14] for a different but closely related definition of  $\text{Ren}_1$ , that the quotient ring  $\mathbb{C}[\text{Ren}_1]/\mathcal{I}$  has no proper zero divisors. So in that case, the following quotient field exists:

- $\text{eRen}(\mathbb{N}) := \{\mathbf{c} = a_1/a_2 \mid a_1, a_2 \in \mathbb{C}[\text{Ren}_1(\mathbb{N})]/\mathcal{I}\}$  is the **field of wave function renormalizations**

The proof only made use of the fact that  $\text{Ren}_1$  is a  $\mathbb{C}$ -vector space decaying into equivalence classes, which are defined by  $\mathbf{r}_1 \sim \mathbf{r}_2$ , if  $\mathbf{r}_1 - \mathbf{r}_2$  is identifiable with a  $\mathbb{C}$ -number. This remains true for our vector space  $\text{Ren}_1(\mathbb{N})$ , so a field  $\text{eRen}(\mathbb{N})$  can be constructed analogously.

The formal state vectors appearing in QFT are of the form  $\Psi = \sum_m \mathbf{c}_m \Psi_m$  with  $\mathbf{c}_j \in \text{eRen}$  and  $\Psi_m \in \mathcal{E}_{\mathcal{F}}$ . Those can be described as elements of the following vector space:

- $\overline{\mathcal{F}}(\mathbb{N}) := \overline{\mathcal{F}}_0(\mathbb{N})/\sim_{\mathbb{F}}$  is the **first extended state space**, where  $\overline{\mathcal{F}}_0(\mathbb{N})$  is the free  $\text{eRen}$ -vector space over  $\mathcal{E}_{\mathcal{F}}$  (all finite  $\text{eRen}$ -linear combinations) and  $\sim_{\mathbb{F}}$  is generated by  $(c\mathbf{c})\Psi_m \sim_{\mathbb{F}} c(c\Psi_m)$  for  $c \in \mathbb{C}$ .

For intermediate calculations, we will need an even larger vector space  $\overline{\mathcal{F}}_{\text{ex}}$ , which allows for multiplication of  $\Psi$  by elements of  $\text{Ren}$ . We define  $\text{Ren}^{\mathcal{Q}}(\mathbb{N})$  to consist of all functions  $\mathcal{Q}(\mathbb{N}) \rightarrow \text{Ren}$ , which replaces the similarly-defined  $\text{Ren}^{\mathcal{Q}}$  from [14].

- $\overline{\mathcal{F}}_{\text{ex}}(\mathbb{N}) := \overline{\mathcal{F}}_{\text{ex},0}(\mathbb{N})/\sim_{\text{Fex}}$  is the **second extended state space**, where  $\overline{\mathcal{F}}_{\text{ex},0}(\mathbb{N})$  is the set of all countable  $\text{eRen}$ -linear combinations  $\Psi = \sum_{m \in \mathbb{N}} \mathbf{c}_m \Psi_m$  with  $\mathbf{c}_m \in \text{eRen}(\mathbb{N})$ ,  $\Psi_m \in \text{Ren}^{\mathcal{Q}}(\mathbb{N})$  and where  $\sim_{\text{Fex}}$  is generated by  $(c\mathbf{c})\Psi_m \sim_{\text{Fex}} c(c\Psi_m)$  for  $c \in \mathbb{C}$ .

We may embed the complex numbers into  $\text{Ren}$  by identifying  $c \in \mathbb{C}$  with  $ce^0 \in \text{eRen}$ . Hence, the space  $\overline{\mathcal{F}}$  can be embedded into  $\overline{\mathcal{F}}_{\text{ex}}$ .

### Remarks.

1. This construction is similar to the non-standard analysis construction of the extended real line  ${}^*\mathbb{R}$  [17]. The extension  ${}^*\mathbb{R}$  of  $\mathbb{R}$  is also defined as a space of equivalence classes of real-valued sequences, just as the extension  $\text{Ren}_1(\mathbb{N})$  of  $\mathbb{C}$ . However, at a closer look, both constructions are quite different:
  - Two sequences are equivalent with respect to  ${}^*\mathbb{R}$ , if they agree on a sufficiently large set, and not if their difference is a null series.

- $\mathbb{R}$  is embedded into  ${}^*\mathbb{R}$  by identifying  $r \in \mathbb{R}$  with the constant sequence  $(r_j)_{j \in \mathbb{N}}, r_j = r$ , while  $\mathbb{C}$  is embedded into  $\text{Ren}_1(\mathbb{N})$  by identifying  $r \in \mathbb{C}$  with any sequence  $(r_j)_{j \in \mathbb{N}}$  such that  $\sum_j r_j = r$ .
  - Moreover,  ${}^*\mathbb{R}$  is a field, while  $\text{Ren}_1(\mathbb{N})$  is not. By contrast,  $\text{eRen}(\mathbb{N})$  is a field, but it is not defined as a set of equivalence classes of real- or complex-valued sequences.
2. Within the definitions of the above spaces, we may also have replaced  $(\mathbb{N})$  by  $(X)$ , with  $X \subseteq \mathbb{R}^d$  measurable. This is especially useful, if  $X$  is the spectral set of some operator. In that case, it becomes slightly ambiguous how to define a smooth function on the possibly “rugged” and non-open set  $X$ . One option would be to define  $\mathcal{E}(X)$  as the set of all functions  $X \rightarrow \mathbb{C}$ , where at each  $x \in X$ , all total derivatives exist<sup>1</sup>. Another option would be to consider smooth functions in  $\mathcal{E}(\mathbb{R}^d)$  and define each  $\phi \in \mathcal{E}(X)$  as an equivalence class of functions in  $\mathcal{E}(\mathbb{R}^d)$  agreeing on all  $x \in X$ .

The extended  $N$ -particle space  $\mathcal{E}^{(N)}(X)$  then allows for several reasonable definitions: either as the space of smooth functions  $X^N \rightarrow \mathbb{C}$  or as a topological tensor product space  $\mathcal{E}(X)^{\otimes N}$ . It would be useful to define  $\mathcal{E}(X)$  such that both definitions agree, as it is the case for  $\mathcal{E}(X)$  with open  $X \subseteq \mathbb{R}^d$ , see [34, Thm. 51.6].

The definition of  $\mathcal{E}_{\mathcal{F}}(X)$  follows naturally from  $\mathcal{E}^{(N)}(X)$ .  $\text{Ren}_1(X)$  becomes the space of formal integrals  $\mathfrak{r} = \int_X r(x) dx$  instead of sums and allows for an analogous definition of  $\text{Pol}(X)$ ,  $\text{eRen}(X)$ ,  $\mathcal{F}(X)$  and  $\mathcal{F}_{\text{ex}}(X)$ .

## 2.3 Infinite Tensor Products

In this subsection, we give a quick introduction to ITPs, as introduced by von Neumann [15], see also [16]. Some useful lemmas and remarks concerning this construction are given in Appendix A. For a more thorough discussion, we refer the reader to von Neumann’s original work.

We consider a (possibly uncountable) index set  $I$ , and for each  $k \in I$  a Hilbert space  $\mathcal{H}_k$  with scalar product  $\langle \cdot, \cdot \rangle_k$  and induced norm  $\| \cdot \|_k$ . The aim is to construct a vector space, which is generated formally by ITPs

$$\Psi = \prod_{k \in I}^{\otimes} \Psi_k, \tag{18}$$

or equivalently, by families  $(\Psi) = (\Psi_k)_{k \in I}$ ,  $\Psi_k \in \mathcal{H}_k$ . If  $I$  is countable, then  $(\Psi) =$

---

<sup>1</sup>We say that a total derivative of  $\phi : X \rightarrow \mathbb{C}$  at a non-interior point  $x \in X$  exists, if there exists a linear function approximating  $\phi(y)$  up to precision  $o(x - y)$  at all  $y \in B_\delta(x) \cap X$  for some  $\delta > 0$ .

$(\Psi_1, \Psi_2, \dots)$  defines a sequence. Each family gets assigned the formal expression

$$\|(\Psi)\| := \prod_{k \in I} \|\Psi_k\|_k, \quad (19)$$

which we will later use for defining a norm. In order to answer the question, whether (19) defines a complex number, one introduces the notions of convergence within a (possibly uncountable) sum:

- For  $z_k \in \mathbb{C}$ ,  $k \in I$ , we call  $\sum_{k \in I} z_k$  or  $\prod_{k \in I} z_k$  convergent to  $a \in \mathbb{C}$ , if for all  $\delta > 0$ , there exists some finite set  $I_\delta$ , such that for all finite sets  $J \subseteq I$  with  $I_\delta \subseteq J$ , we have

$$\left| a - \sum_{k \in J} z_k \right| \leq \delta \quad \text{or} \quad \left| a - \prod_{k \in J} z_k \right| \leq \delta, \quad \text{respectively.} \quad (20)$$

A simple consequence of this definition is that  $\sum_{k \in I} z_k$  can only converge if  $z_k \neq 0$  occurs for only countably many  $k \in I$ . So the question of convergence reduces to that of sequence convergence. Further, it is shown in [15] that  $\prod_{k \in I} z_k < \infty$  if and only if we have  $z_k = 0$  for at least one  $k \in I$  or if  $\sum_{k \in I} |z_k - 1| < \infty$ . The heuristic reason is that  $\prod_{k \in I} z_k = \exp(\sum_{k \in I} \ln z_k)$  and  $\ln z_k$  can be linearly approximated near 1 as  $\ln z_k = 1 - z_k + \mathcal{O}((1 - z_k)^2)$ .

In case  $\prod_{k \in I} |z_k|$  converges to a nonzero number, then  $\prod_{k \in I} z_k$  converges if and only if no infinite phase variation occurs. That is, if  $\arg(z_k) \in (-\pi, \pi]$  is the phase of the complex number  $z_k$ , then it is required that

$$\sum_{k \in I} |\arg(z_k)| < \infty. \quad (21)$$

In order to establish a notion of convergence, even when (21) is violated, one defines that:

- $\prod_{k \in I} z_k$  is quasi-convergent, if and only if  $\prod_{k \in I} |z_k|$  converges.

A family  $(\Psi) = (\Psi_k)_{k \in I}$  is now called a

- **$C$ -sequence** ( $(\Psi) \in \text{Cseq}$ ) iff  $\prod_{k \in I} \|\Psi_k\|_k < \infty$ ,
- **$C_0$ -sequence** iff  $\sum_{k \in I} \|\Psi_k\|_k - 1 < \infty \Leftrightarrow \sum_{k \in I} \|\Psi_k\|_k^2 - 1 < \infty$ .

Each  $C_0$ -sequence is also a  $C$ -sequence. For all  $C$ -sequences, we have a well-defined value  $\|(\Psi)\| \in \mathbb{C}$  by (19) and each  $C$ -sequence, that is not a  $C_0$ -sequence, must automatically satisfy  $\|(\Psi)\| = 0$ .

Expression (19) only defines a seminorm, since there exist  $(\Psi) \neq 0$  with  $\prod_{k \in I} \|\Psi_k\|_k = 0$ . To make it a norm, one defines

- $\prod_{k \in I}^{\otimes} \mathcal{H}_k$  as the space of all conjugate-linear functionals on Cseq.

Following [15], we can embed  $\iota : \text{Cseq} \rightarrow \prod_{k \in I}^{\otimes} \mathcal{H}_k$  by identifying  $(\Phi) \in \text{Cseq}$  with the functional

$$\Phi = \iota((\Phi)) : (\Psi) \mapsto \prod_{k \in I} \langle \phi_k, \psi_k \rangle_k. \quad (22)$$

This identification essentially sets up an equivalence relation  $\sim_C$  on Cseq, where  $(\Phi) \sim_C (\Phi')$ , whenever  $\iota((\Phi)) = \iota((\Phi'))$ . In Proposition A.1, we show that equivalence is given if and only if  $(\Phi)$  and  $(\Phi')$  just differ by a family of complex factors  $(c_k)_{k \in I}$  with  $\prod_{k \in I} c_k = 1$ . The functionals in  $\iota[\text{Cseq}]$  are then the equivalence classes and the span of these functionals is denoted by [15]:

- $\prod_{k \in I}^{\otimes} \mathcal{H}_k := \text{span}(\iota[\text{Cseq}])$ .

In the following, we may drop the embedding map  $\iota$  and simply identify  $(\Phi)$  with  $\Phi$ , whenever the identification is obvious. An inner product  $\langle \cdot, \cdot \rangle$  can uniquely be defined on  $\prod_{k \in I}^{\otimes} \mathcal{H}_k$  via

$$\langle \Phi, \Psi \rangle = \prod_{k \in I} \langle \Phi_k, \Psi_k \rangle_k, \quad (23)$$

which makes  $\prod_{k \in I}^{\otimes} \mathcal{H}_k$  a pre-Hilbert space and induces a norm  $\|\Phi\|$  agreeing with (19) under identification  $\|\Phi\| = \|(\Phi)\|$ . This norm allows for completing  $\prod_{k \in I}^{\otimes} \mathcal{H}_k$  to a Hilbert space:

- The **infinite tensor product space**  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$  is defined as the space of all  $\Phi \in \prod_{k \in I}^{\otimes} \mathcal{H}_k$ , such that there exists a Cauchy sequence  $(\Phi^{(r)})_{r \in \mathbb{N}} \subset \prod_{k \in I}^{\otimes} \mathcal{H}_k$ , i.e.,

$$\lim_{r, s \rightarrow \infty} \|\Phi^{(r)} - \Phi^{(s)}\| = 0, \quad (24)$$

which converges to  $\Phi$  in the weak- $*$  topology on  $\prod_{k \in I}^{\otimes} \mathcal{H}_k$ . That means,

$$\lim_{r \rightarrow \infty} \Phi^{(r)}((\Psi)) = \Phi((\Psi)), \quad (25)$$

for all  $(\Psi) \in \text{Cseq}$ .

The Hilbert space  $\widehat{\mathcal{H}}$  is exactly the Fock space extension, we will use when talking about ITPs.

Checking that  $\langle \cdot, \cdot \rangle$  is indeed an inner product on  $\prod_{k \in I}^{\otimes} \mathcal{H}_k$ , and extends to  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$ , such that one obtains a Hilbert space, is a technical task accomplished in [15]. Note that replacing “ $C$ -sequence” by “ $C_0$ -sequence” in the construction results in the same space  $\widehat{\mathcal{H}}$  after completion, since all  $C$ -sequences that are not  $C_0$ -sequences get identified by  $\iota$  with the same functional 0.

In order to further analyze its structure, the space  $\widehat{\mathcal{H}}$  is divided into certain subspaces. For this, we divide the set of  $C_0$ -sequences into equivalence classes by defining:

- equivalence:  $(\Phi) \sim (\Psi) \quad :\Leftrightarrow \quad \sum_{k \in I} |\langle \Phi_k, \Psi_k \rangle - 1| < \infty$
- weak equivalence:  $(\Phi) \sim_w (\Psi) \quad :\Leftrightarrow \quad \sum_{k \in I} ||\langle \Phi_k, \Psi_k \rangle| - 1| < \infty$

The proof that  $\sim$  and  $\sim_w$  indeed divide the  $C_0$ -sequences into equivalence classes, called  $C$  and  $C_w$ , is given in [15]. The corresponding linear spaces of an equivalence class are defined by

- $\prod_{k \in I}^{\otimes C} \mathcal{H}_k := \overline{\text{span}\{\Psi \mid \exists(\Psi) \in C : \iota((\Psi)) = \Psi\}}^{\|\cdot\|}$  for equivalence
- $\prod_{k \in I}^{\otimes C_w} \mathcal{H}_k := \overline{\text{span}\{\Psi \mid \exists(\Psi) \in C_w : \iota((\Psi)) = \Psi\}}^{\|\cdot\|}$  for weak equivalence

Now, each  $C_0$ -sequence  $(\Psi)$  in an equivalence class  $[(\Omega)] = C$  (with  $\Omega$  being interpreted as the vacuum vector) can be written in **coordinates** as follows [15, Thm. V]: Choose an orthonormal basis  $(e_{k,n})_{n \in \mathbb{N}_0}$  for each  $\mathcal{H}_k$ , such that  $\Omega_k = e_{k,0}$  (we think of  $e_{k,0}$  as mode  $k$  being in the vacuum). Then,  $(\Psi) = (\Psi_k)_{k \in I}$  is uniquely specified by stating the coordinates  $c_{k,n} := \langle e_{k,n}, \Psi_k \rangle_k \in \mathbb{C}$ . In this coordinate representation, it is true that

- $\prod_{k \in I}^{\otimes C} \mathcal{H}_k$  is the closure of the space spanned by all normalized  $C_0$ -sequences, where  $c_{k,0} = 1$  for all but finitely many  $k \in I$ .

Or heuristically speaking, “almost all  $\Psi_k$  are in the vacuum”.

By [15, Thm. V], also a generic  $\Psi \in \prod_{k \in I}^{\otimes C} \mathcal{H}_k$  can be written as

$$\Psi = \sum_{n(\cdot) \in F} a(n(\cdot)) \prod_{k \in I}^{\otimes} e_{k,n(k)}, \quad (26)$$

with  $F$  being the countable set of all functions  $n : I \rightarrow \mathbb{N}_0$  with  $n(k) = 0$  for almost all  $k \in I$ , and  $a(n(\cdot)) \in \mathbb{C}$  being the coordinates of  $\Psi$  with  $\sum_{n(\cdot) \in F} |a(n(\cdot))|^2 < \infty$ .

### 3 Bogoliubov Transformations

In this section, we introduce our notation for Bogoliubov transformations and recap some important properties. In the literature, there exist several representations of Bogoliubov transformations (which are elements of a symplectic group) as linear operators on subspaces  $W_{1,j}$ , that are isomorphic to the one-operator subspace  $W_1$  of the algebra  $\mathcal{A}$ . We present three such choices for  $W_{1,j}$  in Appendix B.1, state the rules how to change between representations, and then finally fix the representation from Appendix B.4, which is used in Section 3.1 and thereafter. In Section 3.2, we recap the well-known implementation process in case when the Shale–Stinespring condition is valid. Standard references on the subject are [27] and [28].

### 3.1 Transformation on Operators

Consider the one-operator subspace  $W_1$  of  $\mathcal{A}$ , which is spanned by all  $a_{\pm}^{\dagger}(f), a_{\pm}(g)$ . By an algebraic Bogoliubov transformation, we mean any bijective map  $\mathcal{V}_A : W_1 \rightarrow W_1$ , which sends  $a_{\pm}^{\dagger}(f), a_{\pm}(g)$  to a new set of creation and annihilation operators  $b_{\pm}^{\dagger}(f), b_{\pm}(g)$ , such that  $b_{\pm}^{\dagger}(f)$  is the adjoint of  $b_{\pm}(f)$  and the CAR/CCR are conserved under  $\mathcal{V}$ . Further, the adjoint  $\mathcal{V}^*$ , defined in Appendix B, is also required to conserve the CAR/CCR.

The representation we use is the following: fix a basis  $(e_j)_{j \in \mathbb{N}} \subset \mathfrak{h}$ . Then, every  $f \in \mathfrak{h}$  can then be expressed by its coordinates  $f_j := \langle e_j, f \rangle$ . This way, we may identify  $f \in \mathfrak{h}$  with an equally denoted vector  $\mathbf{f} = (f_j)_{j \in \mathbb{N}} \in \ell^2$ . This way, we can identify  $a^{\dagger}(f) + a(\bar{g}) \in W_1$  with a vector  $(\mathbf{f}, \mathbf{g}) \in \ell^2 \oplus \ell^2$ , so on algebraic Bogoliubov transformations  $\mathcal{V}_A$  amounts to a linear operator  $\mathcal{V} = \mathcal{V}_3$  on  $\ell^2 \oplus \ell^2$ , which we just call ‘‘Bogoliubov transformation’’.

We now encode sums of creation and annihilation operators by vector pairs  $(\mathbf{f}, \mathbf{g}) \in \ell^2 \oplus \ell^2$ . This encoding is realized by the generalized creation/annihilation operators

$$\begin{aligned} A_{\pm}^{\dagger} : \ell^2 \oplus \ell^2 &\rightarrow \mathcal{A}_{\pm}, & (\mathbf{f}_1, \mathbf{f}_2) &\mapsto a_{\pm}^{\dagger}(\mathbf{f}_1) + a_{\pm}(\overline{\mathbf{f}_2}) = \sum_j (f_{1,j} a_{\pm}^{\dagger}(e_j) + f_{2,j} a_{\pm}(e_j)) \\ A_{\pm} : \ell^2 \oplus \ell^2 &\rightarrow \mathcal{A}_{\pm}, & (\mathbf{g}_1, \mathbf{g}_2) &\mapsto a_{\pm}(\mathbf{g}_1) + a_{\pm}^{\dagger}(\overline{\mathbf{g}_2}) = \sum_j (\overline{g_{1,j}} a_{\pm}(e_j) + \overline{g_{2,j}} a_{\pm}^{\dagger}(e_j)). \end{aligned} \tag{27}$$

A Bogoliubov transformation can then be encoded by a  $2 \times 2$  block matrix

$$\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}, \tag{28}$$

with operators  $u, v : \ell^2 \rightarrow \ell^2$ . The case of unbounded  $u, v$  is treated later in Section 4.

The Bogoliubov transformed operators are then given by

$$\begin{aligned} b_{\pm}^{\dagger}(\mathbf{f}) &= A_{\pm}^{\dagger}(\mathcal{V}(\mathbf{f}, 0)) = a_{\pm}^{\dagger}(u\mathbf{f}) + a_{\pm}(v\bar{\mathbf{f}}) \\ b_{\pm}(\mathbf{g}) &= A_{\pm}(\mathcal{V}(\mathbf{g}, 0)) = a_{\pm}^{\dagger}(v\bar{\mathbf{g}}) + a_{\pm}(u\mathbf{g}). \end{aligned} \tag{29}$$

In order for  $\mathcal{V}$  to be a Bogoliubov transformation, we require that both  $\mathcal{V}$  and  $\mathcal{V}^*$  conserve the CAR/CCR, so

$$[b_{\pm}(\mathbf{f}), b_{\pm}^{\dagger}(\mathbf{g})]_{\pm} = \langle \mathbf{f}, \mathbf{g} \rangle, \quad [b_{\pm}(\mathbf{f}), b_{\pm}(\mathbf{g})]_{\pm} = 0 = [b_{\pm}^{\dagger}(\mathbf{f}), b_{\pm}^{\dagger}(\mathbf{g})]_{\pm}, \tag{30}$$

and the same, if in (29)  $\mathcal{V}$  is replaced by  $\mathcal{V}^*$ . An explicit calculation (see Appendix B) shows that this conservation is equivalent to 4 conditions on  $u$  and  $v$ . To express

them, we define the transpose, complex conjugate and adjoint as  $(u^T)_{ij} = u_{ji}$ ,  $(\bar{u})_{ij} = \overline{u_{ij}}$ ,  $(u^*)_{ij} = \overline{u_{ji}}$  and the same for  $v_{ij}$ . The 4 Bogoliubov relations then read

$$\begin{aligned} u^*u \mp v^T\bar{v} &= 1 & u^*v \mp v^T\bar{u} &= 0 \\ uu^* \mp vv^* &= 1 & uv^T \mp vu^T &= 0. \end{aligned} \quad (31)$$

Since  $v^*v$  is self-adjoint, we have  $v^*v = \overline{v^*v} = v^T\bar{v}$ , so the first equation can equivalently be written as  $u^*u \mp v^*v = 1$ .

The generalized creation and annihilation operators also allow for particularly easy “generalized CAR/CCR”: Using the standard scalar product on  $\mathbf{F}, \mathbf{G} \in \ell^2 \oplus \ell^2$ :

$$\langle \mathbf{F}, \mathbf{G} \rangle = \left\langle \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} \right\rangle = \sum_j (\overline{f_{1,j}}g_{1,j} + \overline{f_{2,j}}g_{2,j}), \quad (32)$$

we obtain the generalized CAR/CCR:

$$\begin{aligned} [A_{\pm}(\mathbf{F}), A_{\pm}^{\dagger}(\mathbf{G})]_{\pm} &= \langle \mathbf{F}, \mathcal{S}_{\pm} \mathbf{G} \rangle, \\ [A_{\pm}(\mathbf{F}), A_{\pm}(\mathbf{G})]_{\pm} &= [A_{\pm}^{\dagger}(\mathbf{F}), A_{\pm}^{\dagger}(\mathbf{G})]_{\pm} = 0, \end{aligned} \quad (33)$$

where  $\mathcal{S}_- = id$  and  $\mathcal{S}_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

## 3.2 Implementation on Fock Space

In the following, we will drop the index  $\pm$  for bosonic/fermionic, as both cases are separately considered. A Bogoliubov transformation is called implementable (in the regular sense), if there exists a unitary operator  $\mathbb{U}_{\mathcal{V}} : \mathcal{F} \rightarrow \mathcal{F}$ , such that

$$\mathbb{U}_{\mathcal{V}} A^{\dagger}(\mathbf{F}) \mathbb{U}_{\mathcal{V}}^* = A^{\dagger}(\mathcal{V}\mathbf{F}). \quad (34)$$

It is well-known that a Bogoliubov transformation is implementable, if and only if the **Shale–Stinespring condition** holds. That is,  $\text{tr}(v^*v) < \infty$ , so  $v$  is Hilbert–Schmidt [7]. We recap some of the basic steps of the implementation process presented in [27], as some of those steps will have to be carried out in a slightly modified way for an implementation in the extended sense.

The main task within the implementation is to find the “Bogoliubov transformed vacuum”  $\Omega_{\mathcal{V}} \in \mathcal{F}$ , which is the vector annihilated by all operators  $b(\mathbf{f})$ :

$$b(\mathbf{f})\Omega_{\mathcal{V}} = (a^{\dagger}(u\mathbf{f}) + a(v\bar{\mathbf{f}}))\Omega_{\mathcal{V}} = 0 \quad \forall \mathbf{f} \in \ell^2. \quad (35)$$



If we can find such an  $\Omega_\nu$ , then it is an easy task to transform any product state vector  $a^\dagger(\mathbf{f}_1) \dots a^\dagger(\mathbf{f}_n)\Omega \in \mathcal{F}$  via:

$$\begin{aligned} \mathbb{U}_\nu a^\dagger(\mathbf{f}_1) \dots a^\dagger(\mathbf{f}_n)\Omega &= \underbrace{\mathbb{U}_\nu a^\dagger(\mathbf{f}_1)\mathbb{U}_\nu^*}_{b^\dagger(\mathbf{f}_1)} \mathbb{U}_\nu \dots \mathbb{U}_\nu^* \underbrace{\mathbb{U}_\nu a^\dagger(\mathbf{f}_n)\mathbb{U}_\nu^*}_{b^\dagger(\mathbf{f}_n)} \underbrace{\mathbb{U}_\nu \Omega}_{\Omega_\nu} \\ &= b^\dagger(\mathbf{f}_1) \dots b^\dagger(\mathbf{f}_n)\Omega_\nu. \end{aligned} \quad (36)$$

The span of these state vectors (also called algebraic tensor product) is dense within  $\mathcal{F}$ , so we can transform any  $\Psi \in \mathcal{F}$  by means of (36).

It remains to find the Bogoliubov transformed vacuum  $\Omega_\nu$ , in the following simply called *Bogoliubov vacuum*. For only one mode, i.e., with  $\ell^2$  replaced by  $\mathbb{C}$ , this would be an easier task, as  $u$  and  $v$  are just constants in that case. So we aim for “decomposing the transformation  $\mathcal{V}$  into modes”. More precisely, we seek vectors  $\mathbf{f}_j$ , such the Bogoliubov-transformed form factors  $u\mathbf{f}_j, v\overline{\mathbf{f}_j}$  are proportional to the same normalized vector  $\mathbf{g}_j \in \ell^2$ , i.e.

$$\mathcal{V} \begin{pmatrix} \mathbf{f}_j \\ 0 \end{pmatrix} = \begin{pmatrix} u\mathbf{f}_j \\ \overline{v\mathbf{f}_j} \end{pmatrix} = \begin{pmatrix} \mu_j \mathbf{g}_j \\ \nu_j \overline{\mathbf{g}_j} \end{pmatrix} \quad \Leftrightarrow \quad \mathbb{U}_\nu a^\dagger(\mathbf{f}_j) \mathbb{U}_\nu^* = \mu_j a^\dagger(\mathbf{g}_j) + \nu_j a(\mathbf{g}_j). \quad (37)$$

The constants  $\mu_j, \nu_j \in \mathbb{C}$  intuitively describe, how much of  $a^\dagger$  “stays”  $a^\dagger$  (this is  $\mu_j$ ) or “leaves”  $a^\dagger$  to become  $a$  (this is  $\nu_j$ ). If (37) holds, we only have to solve  $(\nu_j a^\dagger(\mathbf{g}_j) + \mu_j a(\mathbf{g}_j))\Omega_\nu$  for each mode  $\mathbf{g}_j$ , separately.

### 3.2.1 Bosonic Case

Here, (37) can indeed be fulfilled. Following [27], we define a suitable operator  $C$ , such that  $C\mathbf{f}_j = \lambda_j \mathbf{f}_j$ ,  $\lambda_j \in \mathbb{C}$ . We set  $\mathcal{V}(\mathbf{f}_j, 0) = (u\mathbf{f}_j, \overline{v\mathbf{f}_j}) =: (\tilde{\mathbf{g}}_j, J\tilde{\mathbf{h}}_j)$  for any  $\mathbf{f}_j \in \ell^2$  with  $J\tilde{\mathbf{h}}_j = J^*\tilde{\mathbf{h}}_j = \overline{\tilde{\mathbf{h}}_j}$  denoting complex conjugation. Then, we can translate (37) into  $\tilde{\mathbf{g}}_j$  and  $\tilde{\mathbf{h}}_j$  being parallel. Now,

$$\langle \tilde{\mathbf{g}}_j, \tilde{\mathbf{h}}_{j'} \rangle = \langle u\mathbf{f}_j, \overline{v\mathbf{f}_{j'}} \rangle = \langle \mathbf{f}_j, u^* v J \mathbf{f}_{j'} \rangle. \quad (38)$$

So the choice

$$C = u^* v J, \quad (39)$$

would lead to

$$\langle \tilde{\mathbf{g}}_j, \tilde{\mathbf{h}}_{j'} \rangle = \lambda_{j'} \langle \mathbf{f}_j, \mathbf{f}_{j'} \rangle = \lambda_j \delta_{jj'}, \quad (40)$$

if  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  was an eigenbasis of  $C$ . Now, by means of (31),  $C$  is Hermitian ( $C = u^* v J = v^T \overline{u} J = J v^* u = C^*$ ). Since by the Shale–Stinespring condition,  $v$  is Hilbert–Schmidt and  $u$  is bounded, the operator  $C$  is also Hilbert–Schmidt, so we can indeed find an orthonormal basis of eigenvectors  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  with  $C\mathbf{f}_j = \lambda_j \mathbf{f}_j$ . In order to obtain from

(40) that  $\tilde{\mathbf{g}}_j$  and  $\tilde{\mathbf{h}}_j$  are parallel, we still need to show that the  $\tilde{\mathbf{g}}_j$  provide an orthogonal basis of  $\ell^2$

One easily verifies  $[u^*u, C] = 0$ , so  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  can, without restrictions, be chosen to be a simultaneous eigenbasis of  $C$  and  $u^*u$ . So if  $\mu_j^2$  are the corresponding eigenvalues of  $u^*u$ , then

$$\langle \tilde{\mathbf{g}}_j, \tilde{\mathbf{g}}_{j'} \rangle = \langle \mathbf{f}_j, u^*u \mathbf{f}_{j'} \rangle = \mu_j^2 \langle \mathbf{f}_j, \mathbf{f}_{j'} \rangle = \mu_j^2 \delta_{jj'}, \quad (41)$$

meaning that the  $\tilde{\mathbf{g}}_j$  are orthogonal and by invertibility of  $u$  (as it was assumed to be bounded),  $(\tilde{\mathbf{g}}_j)_{j \in \mathbb{N}}$  is a basis. Hence, (37) is valid. By an appropriate choice of the complex phase of  $\mathbf{g}_j$ , we can further make  $\mu_j$  real and positive.

We even have that

$$(\mathbf{g}_j)_{j \in \mathbb{N}} := \left( \frac{1}{\mu_j} \tilde{\mathbf{g}}_j \right)_{j \in \mathbb{N}}, \quad (42)$$

is an orthonormal basis and (40) then implies that  $\tilde{\mathbf{h}}_j$  is indeed proportional to  $\mathbf{g}_j$ . So  $\tilde{\mathbf{h}}_j = \nu_j \mathbf{g}_j$  for some  $\nu_j \geq 0$  and with that choice of  $\mathbf{g}_j, \mu_j, \nu_j$ , indeed (37) holds true. Hence, for  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  being an eigenbasis of  $C$ , we have the following important formulas:

$$u \mathbf{f}_j = \mu_j \mathbf{g}_j \quad v \overline{\mathbf{f}_j} = v J \mathbf{f}_j = \nu_j \mathbf{g}_j. \quad (43)$$

By phase rotation, we can again enforce that  $\mu_j = |\mu_j|$ . Further,

$$\lambda_j \langle \mathbf{f}_i, \mathbf{f}_j \rangle = \langle \mathbf{f}_i, C \mathbf{f}_j \rangle = \langle \mathbf{f}_i, u^* v J \mathbf{f}_j \rangle = \nu_j \mu_j \langle \mathbf{g}_i, \mathbf{g}_j \rangle \quad \Rightarrow \quad \lambda_j = \mu_j \nu_j, \quad (44)$$

so  $\nu_j$  is real and by (31), we have

$$u^*u - v^*v = 1 \quad \Rightarrow \quad \mu_j^2 - \nu_j^2 = 1. \quad (45)$$

The Bogoliubov vacuum is determined by the condition  $(\mu_j a(\mathbf{g}_j) + \nu_j a^\dagger(\mathbf{g}_j)) \Omega_{\mathcal{V}} = 0$  for all modes  $\mathbf{g}_j$ ,  $j \in \mathbb{N}$ . This condition leads to one recursion relation per mode, which is formally solved by

$$\Omega_{\mathcal{V}} = \left( \prod_j \left( 1 - \frac{\nu_j^2}{\mu_j^2} \right)^{1/4} \right) \exp \left( - \sum_j \frac{\nu_j}{2\mu_j} (a^\dagger(\mathbf{g}_j))^2 \right) \Omega, \quad (46)$$

with the prefactor coming from normalization. The Shale–Stinespring condition now indicates when  $\Omega_{\mathcal{V}}$  lies in Fock space. Investigating the two–particle sector, we obtain

$$\Omega_{\mathcal{V}}^{(2)} = - \left( \prod_j \left( 1 - \frac{\nu_j^2}{\mu_j^2} \right)^{1/4} \right) \sum_j \frac{\nu_j}{2\mu_j} \mathbf{g}_j \otimes \mathbf{g}_j. \quad (47)$$

The prefactor in form of an infinite product is nonequal 0, if and only if  $\sum_j \frac{\nu_j^2}{\mu_j^2} < \infty$ . And by orthogonality of  $(\mathbf{g}_j)_{j \in \mathbb{N}}$ , the sum converges in  $L^2$ –norm under the very same

condition. This condition is equivalent to  $\sum_j \nu_j^2 < \infty$ :

Since  $\mu_j^2 - \nu_j^2 = 1$ , we have  $\mu_j \geq 1$ , so  $\sum_j \nu_j^2 < \infty$  implies  $\sum_j \frac{\nu_j^2}{\mu_j^2} < \infty$ .

Conversely,  $\sum_j \frac{\nu_j^2}{\mu_j^2} < \infty$  means that  $\frac{\nu_j^2}{\mu_j^2}$  is a null sequence, so it eventually drops below any  $\varepsilon > 0$ . Now, since  $\mu_j^2 \rightarrow 1$ , as  $\frac{\nu_j^2}{\mu_j^2} \rightarrow 0$ , we have  $\mu_j < 2$  for almost all  $j \in \mathbb{N}$  and hence  $\sum_j \nu_j^2 < 4 \sum_j \frac{\nu_j^2}{\mu_j^2} + \text{const.}$ , which is also convergent.

So the Shale–Stinespring condition indeed tells us, when exactly the formal expression (46) makes sense as a Fock space vector. Note that we exploited  $v$  being Hilbert Schmidt to arrive at (46) in the first place, namely when finding an orthonormal basis  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  for  $C = u^*vJ$ . So a proof of necessity of the Shale–Stinespring condition for implementability requires some further thoughts, see [27, Appendix F]

In case of implementability, the transformation is implemented by [5, (3.1)]:

$$\mathbb{U}_\nu = \exp \left( - \sum_j \frac{\xi_j}{2} ((a^\dagger(\mathbf{g}_j))^2 - (a(\mathbf{g}_j))^2) \right) \mathbb{U}_{\mathbf{g}\mathbf{f}} =: \prod_{j \in \mathbb{N}} \mathbb{U}_{j,\nu}, \quad (48)$$

$$\text{with} \quad \sinh \xi_j := \nu_j \quad \Rightarrow \quad \cosh \xi_j := \mu_j, \quad (49)$$

and where  $\mathbb{U}_{\mathbf{g}\mathbf{f}} : \mathcal{F} \rightarrow \mathcal{F}$  is the unitary transformation which changes the basis  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  to  $(\mathbf{g}_j)_{j \in \mathbb{N}}$  via

$$\mathbb{U}_{\mathbf{g}\mathbf{f}} : \mathbf{f}_{j_1} \otimes \dots \otimes \mathbf{f}_{j_n} \mapsto \mathbf{g}_{j_1} \otimes \dots \otimes \mathbf{g}_{j_n}, \quad \forall j_1, \dots, j_n \in \mathbb{N}. \quad (50)$$

Note that the presign of  $\pm \xi_j$  in the literature depends on whether the Bogoliubov transformation is defined as  $\mathbb{U}_\nu a^\# \mathbb{U}_\nu^*$  (as in our case) or  $\mathbb{U}_\nu^* a^\# \mathbb{U}_\nu$  with  $\# \in (\cdot, *)$ . For a more general discussion about the implementer  $\mathbb{U}_\nu$ , we refer the reader to [29, Thm. 16.47].

### 3.2.2 Fermionic Case

In the fermionic case, (37) cannot be fulfilled:  $C = u^*vJ$  is no longer Hermitian, so we cannot expect it to have an orthonormal eigenbasis. However,  $C^*C$  is Hermitian. By (31),  $C^*C = v^*v - (v^*v)^2$ , so  $C^*C$  is trace class and  $[u^*u, C] = 0$  implies  $[u^*u, C^*C] = 0$ . Thus, there exists a common orthonormal eigenbasis  $(\mathbf{f}_j)_{j \in J}$  of  $C^*C$  (eigenvalues  $\lambda_j^2$ ) and  $u^*u$  (eigenvalues  $\mu_j^2$ ). As  $u^*u + v^*v = u^*u + v^T \bar{v} = 1$ , the  $\mathbf{f}_j$  are also eigenvectors of  $v^*v$  (eigenvalues  $\nu_j^2 = 1 - \mu_j^2$ ). The index set  $J$  is assumed to be countable and specified below.

In particular, we can arrange the eigenvectors  $\mathbf{f}_j$  with  $\lambda_j \neq 0$  in pairs

$$C \mathbf{f}_{2i} = \lambda_{2i} \mathbf{f}_{2i-1}, \quad C \mathbf{f}_{2i-1} = -\lambda_{2i} \mathbf{f}_{2i}, \quad (51)$$

where  $i$  is an element of a countable index set  $I' \subseteq \mathbb{N}$  and the eigenvector indices are in  $J' := \{j \mid j = 2i \vee j = 2i - 1, i \in I'\}$ . The eigenvectors with  $\lambda_j = 0$  will be denoted by  $\mathbf{f}_j$  with index set  $j \in J'' \subseteq \mathbb{N}, J' \cap J'' = \emptyset$ . The set of all used indices is then  $J = J' \cup J'' \subseteq \mathbb{N}$  and  $(\mathbf{f}_j)_{j \in J}$  is an orthonormal basis of  $\ell^2$ .

Because of the pairing, the splitting into modes (37) with an orthonormal basis  $(\mathbf{g}_j)_{j \in \mathbb{N}}$  can no longer be achieved. Instead, we can obtain a splitting into modes  $j \in J''$  and pairs  $i \in I'$  using an orthonormal basis  $(\boldsymbol{\eta}_j)_{j \in J}$ . Again, we define  $(\tilde{\mathbf{g}}_j, \tilde{\mathbf{h}}_j) := (u\mathbf{f}_j, v\overline{\mathbf{f}}_j)$ .

The case  $j \in J''$  still allows for a split into single modes. It consists of 2 subcases: Since by (31),  $C^*C = u^*u(1 - u^*u)$ , we have that  $\lambda_j^2 = \mu_j^2(1 - \mu_j^2)$ , so  $\mu_j = 1$  or  $\mu_j = 0$ . In case  $\mu_j = 0$ , we have by  $u^*u + v^T\overline{v} = 1$ , that  $\nu_j = 1$ . We denote that case by  $j \in J''_1$  and get

$$\mathcal{V} \begin{pmatrix} \mathbf{f}_j \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{\eta}_j \end{pmatrix}, \quad j \in J''_1, \quad (52)$$

for a suitable choice of the phase  $\boldsymbol{\eta}_j = e^{i\varphi} \tilde{\mathbf{h}}_j = e^{i\varphi} v \overline{\mathbf{f}}_j$ . The case  $\mu_j = 1$  will be denoted by  $j \in J''_0 = J'' \setminus J''_1$ . Here,  $\nu_j = 0$ , so

$$\mathcal{V} \begin{pmatrix} \mathbf{f}_j \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\eta}_j \\ 0 \end{pmatrix}, \quad j \in J''_0, \quad (53)$$

for a suitable phase choice of  $\boldsymbol{\eta}_j = e^{i\varphi} \tilde{\mathbf{g}}_j = e^{i\varphi} u \mathbf{f}_j$ .

In case  $i \in I' \Rightarrow \lambda_{2i} \neq 0, \mu_{2i} \neq 0$ , we define the normalized vectors

$$\boldsymbol{\eta}_{2i} := \alpha_i^{-1} \tilde{\mathbf{g}}_{2i}, \quad \boldsymbol{\eta}_{2i-1} := \alpha_i^{-1} \tilde{\mathbf{g}}_{2i-1}, \quad (54)$$

where  $\alpha_i = \mu_{2i} = \mu_{2i-1}$  can be shown within a Bogoliubov pair  $i$ . Now,  $\tilde{\mathbf{h}}_j$  is not proportional to  $\boldsymbol{\eta}_j$ , but one can show that

$$\mathcal{V} \begin{pmatrix} \mathbf{f}_{2i} \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_i \boldsymbol{\eta}_{2i} \\ \beta_i \boldsymbol{\eta}_{2i-1} \end{pmatrix}, \quad \mathcal{V} \begin{pmatrix} \mathbf{f}_{2i-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_i \boldsymbol{\eta}_{2i-1} \\ -\beta_i \boldsymbol{\eta}_{2i} \end{pmatrix}, \quad i \in I', \quad (55)$$

where  $\beta_i \in \mathbb{R}$ ,  $\beta_i > 0$  is such that  $\alpha_i^2 + \beta_i^2 = 1$ . That means,  $\tilde{\mathbf{h}}_{2i}$  is proportional to  $\tilde{\mathbf{g}}_{2i-1}$ , which follows by an orthogonality argument, as in (41). The argument is based on the fact that  $(\boldsymbol{\eta}_j)_{j \in J}$  is an orthonormal basis. For proof details, see [27, (68)–(69)].

Relations (52), (53) and (55) now replace (37). Their implementation is a bit easier than in the bosonic case, since by the Pauli exclusion principle, all modes can be occupied by at most one fermion, i.e., the Fock space per mode is  $\cong \mathbb{C}^2$ .

For (52), (53), the condition  $A(\mathbf{f}_j, 0)\Omega_{\mathcal{V}} = 0$  is easily fulfilled. If  $j \in J_0''$ , then  $a(\boldsymbol{\eta}_j)\Omega_{\mathcal{V}} = 0$ , so the  $\boldsymbol{\eta}_j$ -mode is empty. If  $j \in J_1''$ , then  $a^\dagger(\boldsymbol{\eta}_j)\Omega_{\mathcal{V}} = 0$ , so the  $\boldsymbol{\eta}_j$ -mode is fully occupied.

For (55), the condition  $A(\mathbf{f}_i, 0)\Omega_{\mathcal{V}} = 0$  determines  $\Omega_{\mathcal{V}}$  on each two-mode subspace of  $\mathcal{F}$ . On the 4-dimensional subspace belonging to  $\boldsymbol{\eta}_{2i}, \boldsymbol{\eta}_{2i-1}$  for some  $i \in I'$ , we have a superposition of both modes being empty with amplitude  $\alpha_i$  and a ‘‘Cooper pair’’, where both modes are filled with amplitude  $-\beta_i$ :

$$\Omega_{\mathcal{V}} = \left( \prod_{j \in J_1''} a^\dagger(\boldsymbol{\eta}_j) \right) \left( \prod_{i \in I'} (\alpha_i - \beta_i a^\dagger(\boldsymbol{\eta}_{2i}) a^\dagger(\boldsymbol{\eta}_{2i-1})) \right) \Omega. \quad (56)$$

The corresponding implementer  $\mathbb{U}_{\mathcal{V}}$  is given by

$$\begin{aligned} \mathbb{U}_{\mathcal{V}} &= \left( \prod_{j \in J_1''} (a^\dagger(\boldsymbol{\eta}_j) + a(\boldsymbol{\eta}_j)) \right) \exp \left( - \sum_{i \in I'} \xi_i (a^\dagger(\boldsymbol{\eta}_{2i}) a^\dagger(\boldsymbol{\eta}_{2i-1}) - a(\boldsymbol{\eta}_{2i-1}) a(\boldsymbol{\eta}_{2i})) \right) \mathbb{U}_{\boldsymbol{\eta}\mathbf{f}} \\ \mathbb{U}_{\mathcal{V}} &=: \left( \prod_{j \in J''} \mathbb{U}_{j, \mathcal{V}} \right) \left( \prod_{i \in I'} \mathbb{U}_{2i, 2i-1, \mathcal{V}} \right), \end{aligned} \quad (57)$$

$$\text{with} \quad \sin \xi_i := \beta_i \quad \Rightarrow \quad \cos \xi_i := \alpha_i, \quad (58)$$

and where  $\mathbb{U}_{\boldsymbol{\eta}\mathbf{f}}$  is the unitary transformation changing the basis  $(\mathbf{f}_j)_{j \in J}$  to  $(\boldsymbol{\eta}_j)_{j \in J}$  via

$$\mathbb{U}_{\boldsymbol{\eta}\mathbf{f}} : \mathbf{f}_{j_1} \otimes \dots \otimes \mathbf{f}_{j_n} \mapsto \boldsymbol{\eta}_{j_1} \otimes \dots \otimes \boldsymbol{\eta}_{j_n}, \quad \forall j_1, \dots, j_n \in J. \quad (59)$$

A proof for (57) implementing  $\mathcal{V}$  can be found in Appendix C. For a general discussion of the implementer  $\mathbb{U}_{\mathcal{V}}$ , we refer the reader to [29, Thm. 16.47].

## 4 Bogoliubov Transformations: Extended

Now, consider the case where  $v$  is no longer Hilbert–Schmidt and (in the bosonic case) possibly not even bounded. So it is only defined on some domain  $\text{dom}(v) \subseteq \ell^2$ . Our aim in Section 4.1 is to show that the Bogoliubov relations (31) also hold in this case. So it makes sense to encode a Bogoliubov transformation in a block matrix  $\mathcal{V}$ .

In Section 4.2, we define an extended  $*$ -algebra  $\overline{\mathcal{A}}_e$  of creation and annihilation operator products, and give the precise definitions of  $\widehat{\mathcal{H}}$ ,  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$  with respect to a given Bogoliubov transformation  $\mathcal{V}$ . Lemmas 4.4 and 4.5 then establish that certain elements of  $\overline{\mathcal{A}}_e$  naturally define operators on suitable subspaces of  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ .

## 4.1 Extension of the Bogoliubov Relations

Throughout the following construction, we will assume that  $v^*v$  is densely defined and self-adjoint. In that case, the spectral theorem applies, and we can define the self-adjoint operators  $C^*C := v^*v(1 \pm v^*v)$  and  $|C| = \sqrt{C^*C}$  by spectral calculus. It will turn out convenient to work with spectral resolutions with respect to  $|C|$ . By the spectral theorem in the form of [30, Thm. 10.9], we may then decompose the Hilbert space  $\ell^2$  as a direct integral

$$\ell^2 = \int_{\sigma(|C|)}^{\oplus} \mathbb{C}^n d\mu_1(\lambda), \quad (60)$$

where  $\sigma(|C|) = \sigma$  is the spectrum of  $|C|$ ,  $\mu_1$  a suitable measure on it and  $n : \sigma \rightarrow \mathbb{N} \cup \{\infty\}$  [30, Def. 7.18] is a measurable dimension function (with  $\mathbb{C}^\infty \cong \ell^2$ ). Put differently, as visualized in Figure 2, we can find a spectral set  $X = \bigcup_{\lambda \in \sigma} \{\lambda\} \times Y_\lambda \subset \mathbb{R}^2$  with  $Y_\lambda \subseteq \mathbb{Z}$ ,  $|Y_\lambda| = n(\lambda)$  accounting for multiplicity and unitary maps<sup>2</sup>

$$U_{Xf} : \ell^2 \rightarrow L^2(X), \quad U_{fX} = U_{Xf}^{-1}, \quad (61)$$

such that

$$|C| = U_{fX} \lambda U_{Xf}, \quad (62)$$

with  $\lambda$  being the operator on  $L^2(X)$  that multiplies by  $\lambda(x)$ . In addition, we denote  $Y = \bigcup_{\lambda \in \sigma} Y_\lambda \subseteq \mathbb{Z}$  with  $|Y|$  being an upper bound for the multiplicity of any eigenvalue. Note that the  $\lambda$  here correspond to the  $\lambda_j$  in Section 3.2.

We also make use of the formulation [30, Thm. 10.4] of the spectral theorem, which provides us with a projection-valued measure  $P_{|C|}$ , such that

$$|C| = \int_X \lambda(x) dP_{|C|}(x) = \int_{\sigma \times Y} \lambda dP_{|C|}(\lambda, y). \quad (63)$$

For countable spectrum  $\sigma$ , the entire set  $X$  is countable, so we can index it by  $j \in \mathbb{N}$  and have a corresponding eigenbasis  $(\mathbf{f}_j)_{j \in \mathbb{N}} \subset \ell^2$ . This allows for resolving any  $\phi \in \ell^2$  as  $\phi = \sum_j \phi_j \mathbf{f}_j$ ,  $\phi_j \in \mathbb{C}$ . For the moment, we assume that the spectrum of  $|C|$  is arbitrary, but starting from Section 4.2, only countable spectra of  $|C|$  will be considered.

In the generic spectrum case, we can still choose  $Y \subseteq \mathbb{Z}$ , so  $X \subset \mathbb{R}^2$  consists of “lines” with distance 1.

The subsets of  $X$  with  $\lambda = 0$  will turn out to be critical, as the operators  $u$  or  $v$  amount to a multiplication by 0, there. We hence define the critical and regular spectral sets as

$$X_{\text{crit}} := \{x = (\lambda, y) \in X \mid \lambda = 0\}, \quad X_{\text{reg}} := X \setminus X_{\text{crit}}. \quad (64)$$

Our (dense) space of test functions on the spectral set is then given by:

$$\mathcal{D}_X := C_c^\infty(X_{\text{crit}}) \otimes C_c^\infty(X_{\text{reg}}). \quad (65)$$

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<sup>2</sup>See also the formulation [30, Thm. 10.10] of the spectral theorem.

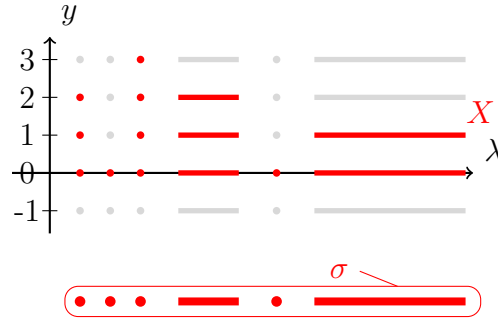


Figure 2: The spectral set  $X$  for a generic spectrum of  $|C|$ . Color online.

The corresponding test function space in  $\ell^2$  is

$$\mathcal{D}_{|C|} := U_{fX} \mathcal{D}_X. \quad (66)$$

For non-open  $X$ , we interpret definition (65) in the same way as the definition of  $\mathcal{E}(X)$  (14):

$$C_c^\infty(X) := C_c^\infty(\mathbb{R}^2) / \{\phi \mid \phi(x) = 0 \forall x \in X\}.$$

**Lemma 4.1** (Bogoliubov relations (31) survive the extension).

Suppose,  $u$  and  $v$  are defined on a common dense domain  $\mathcal{D} \subseteq \ell^2$ , such that  $v^*v$  is densely defined and self-adjoint, and such that the linear operator

$$\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}, \quad \mathcal{V} : \mathcal{D} \oplus \mathcal{D} \rightarrow \ell^2 \oplus \ell^2, \quad (67)$$

defines a Bogoliubov transformation. That means, both  $\mathcal{V}$  and  $\mathcal{V}^* = \begin{pmatrix} u^* & v^T \\ v^* & u^T \end{pmatrix}$  preserve the CAR/CCR, see (27), (30) or also Appendix B.4.

Then  $u, v, \bar{u}, \bar{v}, u^*, v^*, u^T$  and  $v^T$  are well-defined on all of  $\mathcal{D}_{|C|}$ , which was constructed above (66). Further, the Bogoliubov relations

$$\begin{aligned} (u^*u \mp v^*v) &= 1 & (u^*v \mp v^T\bar{u}) &= 0 \\ (uu^* \mp vv^*) &= 1 & (uv^T \mp vu^T) &= 0, \end{aligned} \quad (68)$$

hold as a weak operator identity on  $\mathcal{D}_{|C|}$ . Conversely, (68) implies conservation of the CAR/CCR under both  $\mathcal{V}$  and  $\mathcal{V}^*$ .

*Proof.* Well-definedness of  $u, v$  on  $\mathcal{D}_{|C|}$  follows from the polar decompositions

$$v = U_v |v|, \quad u = U_u |u|, \quad (69)$$

with unitary operators  $U_v, U_u : L^2(X) \rightarrow \ell^2$ . The operators  $|v| = \sqrt{v^*v}$  and  $|u| = \sqrt{u^*u} = \sqrt{1 \pm v^*v}$  on  $L^2(X)$  are bounded in the fermionic case (-), so  $u$  and  $v$  are

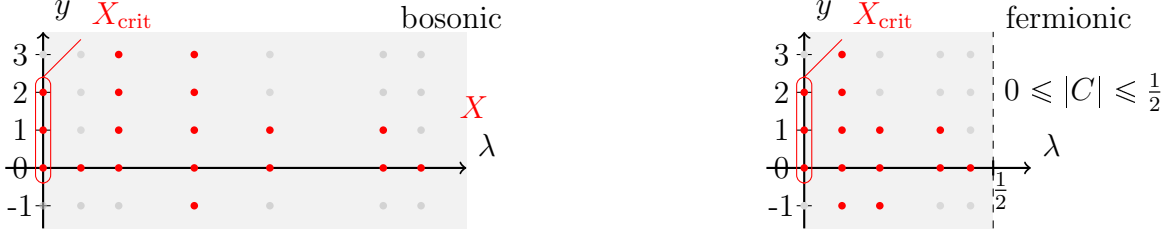


Figure 3: Left: Discrete spectrum of  $|C|$  in the bosonic case. Right: In the fermionic case,  $|C| \leq \frac{1}{2}$  holds. Color online.

defined on all of  $\ell^2$ . In the bosonic case (+), they are spectral multiplications by

$$\nu(\lambda) = \sqrt{-\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda^2}} \quad \text{and} \quad \mu(\lambda) = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda^2}}, \quad (70)$$

which are bounded on each bounded interval  $\lambda$ . Therefore,  $|v|$  and  $|u|$  map  $\mathcal{D}_X$  onto itself, and by definition (66) of  $\mathcal{D}_{|C|}$ , the operators  $v$  and  $u$  map  $\mathcal{D}_{|C|} \rightarrow \ell^2$ .

Well-definedness of  $\bar{u}$  and  $\bar{v}$  on  $\mathcal{D}_{|C|}$  follows by the same polar decomposition argument. And the domains of the adjoints  $u^*$ ,  $v^*$ ,  $u^T$  and  $v^T$  contain the domain of the respective original operators, so they all contain  $\mathcal{D}_{|C|}$ .

The CAR/CCR conservation under  $\mathcal{V}, \mathcal{V}^*$  can be equivalently translated into the two conditions  $\mathcal{V}^* \mathcal{S}_\pm \mathcal{V} = \mathcal{S}_\pm$  and  $\mathcal{V} \mathcal{S}_\pm \mathcal{V}^* = \mathcal{S}_\pm$  and a direct computation as in Appendix B.4 shows that they are formally equivalent to (68) in case of bounded  $u, v$ .

For general  $u$  and  $v$  defined on  $\mathcal{D}_{|C|}$ , the first two formulas are indeed weak operator identities on  $\mathcal{D}_{|C|}$ . The last two are also weak operator identities, since  $\text{dom}(v^*) \supseteq \text{dom}(v)$  and  $\text{dom}(u^*) \supseteq \text{dom}(u)$ . □

Next, we establish that  $v^*v$  having a countable spectrum implies a countable spectrum of  $|C|$ , as illustrated in Figures 3.

**Lemma 4.2.** *Let  $v^*v$  be self-adjoint. Then  $C = u^*vJ$  and  $C^*C$  are well-defined operators on  $\mathcal{D}_{|C|}$ . Here,  $J$  denotes complex conjugation (see Appendix B.1). The spectrum of  $C$  is contained in the real axis for bosons and the imaginary axis for fermions. Further, if  $v^*v$  has countable spectrum, then also  $C, C^*C$  and  $|C|$  have countable spectrum.*

*Proof.* By the Bogoliubov relations,  $C^*C = v^*v \pm (v^*v)^2$  holds, wherever it is defined. Now,  $\lambda \mapsto \lambda \pm \lambda^2$  is smooth apart from the critical points 0 (bosons) or 0 and 1 (fermions).



The condition  $\phi \in \mathcal{D}_{|C|}$  means that the corresponding spectral function  $\phi_X$  has compact support and is smooth apart from the critical points. This property is preserved by an application of  $C^*C$ , so  $C^*C : \mathcal{D}_{|C|} \rightarrow \mathcal{D}_{|C|}$  is well-defined. Hence, also  $|C| = \sqrt{C^*C}$  is well-defined, and by a polar decomposition also  $C = U_C|C|$  with  $U_C : \ell^2 \rightarrow \ell^2$  being a unitary operator.

In the bosonic case,  $C$  is symmetric by the Bogoliubov relations, so  $C^*C = C^2$  and further,  $\sigma(C)$  is a subset of the preimage of  $\sigma(C^2) \subseteq [0, \infty)$  under the complex map  $z \mapsto z^2$ . This preimage is contained within the real axis.

In the fermionic case, the Bogoliubov relations imply  $C^* = -C$ , so  $C^*C = -C^2$ . That means,  $\sigma(C)$  lies within the preimage of  $\sigma(C^2) \subseteq [0, \infty)$  under the map  $z \mapsto -z^2$ , which is contained within the imaginary axis.

Now, suppose  $\sigma(v^*v)$  is countable. Then,  $\sigma(|C|)$  is the image of  $\sigma(v^*v)$  under the map  $z \mapsto \sqrt{z(1 \pm z)}$ , which sends at most 2 arguments to the same value, so  $\sigma(|C|)$  is also countable. □

### Remarks.

3. For fermions,  $u^*u + v^*v = 1$  implies that  $v^*v$  is bounded, so if  $v^*v$  and also  $u^*u$  can be defined on all of  $\ell^2$ . Hence, also  $v$  and  $u$  are defined on all of  $\ell^2$ , so (68) holds as a strong operator identity on  $\ell^2$ .
4. For bosons, it is not obvious that (68) holds as a strong operator identity on a dense domain of  $\ell^2$ . In fact, it does not always hold as a strong operator identity on  $\mathcal{D}_{|C|}$ , as the following counter-example shows:

For the canonical basis  $(e_j)_{j \in \mathbb{N}}$  of  $\ell^2$ , let  $v$  be the multiplication by  $j$ , i.e.,  $ve_j = je_j$ . So  $v^*v$  and  $C^*C$  are the multiplications by  $j^2$  and  $j^2(1 + j^2)$ , respectively, and  $\mathcal{D}_{|C|}$  comprises all vectors  $\phi \in \ell^2$ , where there exists some  $N \in \mathbb{N}$ , such that  $f_j = 0 \ \forall j \geq N$ , i.e., there is a “maximum occupied basis vector”  $e_{N-1}$ . By  $u^*u - v^*v = 1$ , we also know that  $u^*u$  is a multiplication by  $(1 + j^2)$ , and by polar decomposition, we may write

$$u = U_u|u|, \tag{71}$$

with  $|u|$  being the multiplication by  $\sqrt{1 + j^2}$  and  $U_u$  a unitary operator. Similarly,  $u^* = |u|U_u^*$ . Now choose  $U_u$  such that

$$U_u^*e_1 = c \sum_j j^{-1/2-\varepsilon} e_j, \tag{72}$$

where  $\varepsilon > 0$  guarantees that the right-hand side is in  $\ell^2$ . Here,  $c > 0$  is a normalization constant depending on  $\varepsilon$ , chosen such that  $\|U_u^* e_1\| = 1$ . Then, for  $e_1 \in \mathcal{D}_{|C|}$ , consider the formal expression

$$u^* v e_1 = |u| U_u^* e_1 = \sum_j \sqrt{1 + j^2} j^{-1/2 - \varepsilon} e_j. \quad (73)$$

For  $\varepsilon \leq 1$ , this is obviously not in  $\ell^2$ , so  $u^* v$  is ill-defined on  $e_1 \in \mathcal{D}_{|C|}$  and the second formula in (68) does not hold as a strong operator identity on  $\mathcal{D}_{|C|}$ .

## 4.2 Extension of the Operator Algebra

In Section 2.1, we defined a  $*$ -algebra (bosonic) or  $C^*$ -algebra (fermionic)  $\mathcal{A}$  generated by  $a_{\pm}^{\dagger}(f), a_{\pm}(f), f \in \mathfrak{h}$ , so  $\mathcal{A}$  contains operator products that are densely defined on Fock space. On our two Fock space extensions, we will encounter formal expressions in creation and annihilation operators that belong to a larger algebra  $\overline{\mathcal{A}}_{\mathbf{e}}$ , which is defined with respect to a basis  $\mathbf{e} = (e_j)_{j \in \mathbb{N}} \subset \ell^2$ . We introduce the shorthand notations  $a_j := a(e_j), a_j^{\dagger} := a^{\dagger}(e_j)$ , and consider the set of finite operator products

$$\Pi_{\mathbf{e}} := \{a_{j_1}^{\#} \dots a_{j_m}^{\#} \mid j_l \in \mathbb{N}\}. \quad (74)$$

Then  $\overline{\mathcal{A}}_{\mathbf{e}}$  is defined as the set of all complex-valued maps

$$\overline{\mathcal{A}}_{\mathbf{e}} := \{H : \Pi_{\mathbf{e}} \rightarrow \mathbb{C}\}. \quad (75)$$

We formally write its elements as infinite sums

$$H = \sum_{m \in \mathbb{N}} \sum_{j_1, \dots, j_m \in \mathbb{N}} H_{j_1, \dots, j_m} a_{j_1}^{\#} \dots a_{j_m}^{\#}. \quad (76)$$

$\overline{\mathcal{A}}_{\mathbf{e}}$  is made a  $*$ -algebra by the involution

$$* : ca_j \mapsto \bar{c} a_j^{\dagger}, \quad ca_j^{\dagger} \mapsto \bar{c} a_j \quad \forall c \in \mathbb{C}. \quad (77)$$

It is easy to see that  $\overline{\mathcal{A}}_{\mathbf{e}}$  extends  $\mathcal{A}$ , as each element of  $\mathcal{A}$  is a finite sum of operator products  $a^{\#}(f_1) \dots a^{\#}(f_m)$ . Resolving each  $a^{\#}(f_j)$  with respect to the basis  $\mathbf{e}$ , we obtain a countable sum of the form (76), that contains each term  $a_{j_1}^{\#} \dots a_{j_m}^{\#}$  at most once.

Of particular interest will be elements of  $\overline{\mathcal{A}}_{\mathbf{e}}$  corresponding to finite sums. For  $a^{\#}(\phi) = \sum_{j \in \mathbb{N}} \phi_j a_j^{\#}, \phi \in \ell^2$ , this sum is finite if and only if the form factor  $\phi$  is an element of

$$\mathcal{D}_{\mathbf{e}} := \{\phi \in \ell^2 \mid \phi_j = 0 \text{ for all but finitely many } j \in \mathbb{N}\}. \quad (78)$$

If  $(e_j)_{j \in \mathbb{N}}$  is an orthonormal eigenbasis of  $|C|$ , then  $\mathcal{D}_{\mathbf{e}} = \mathcal{D}_{|C|}$ , since both domains are spanned by finite linear combinations of eigenvectors of  $C^*C$ .

Next, we will make the Fock space extensions precise, that will be used to implement a Bogoliubov transformation  $\mathcal{V}$  and we define products of  $a^\dagger(\phi), a(\phi)$  with  $\phi \in \mathcal{D}_e$  on suitable subspaces of them. Within these definitions, **it is assumed that  $v^*v$  has countable spectrum**, so the assumptions of Lemma 4.2 are valid and  $|C|$  also has countable spectrum.

We start with the ITP case. As argued around (63), the index set for the eigenvectors of  $|C|$  (i.e., modes)  $X \subseteq \sigma \times Y$  is countable and there exists an orthonormal eigenbasis  $(\mathbf{f}_j)_{j \in \mathbb{N}}$ . We may use it to construct the bases  $\mathbf{g} = (\mathbf{g}_j)_{j \in \mathbb{N}}$  (bosonic) and  $\boldsymbol{\eta} = (\boldsymbol{\eta}_j)_{j \in J}$  (fermionic), which will take the role of  $\mathbf{e}$  in  $\mathcal{D}_e$  and  $\overline{\mathcal{A}}_e$ . Note that the index set  $J \subseteq \mathbb{N}$  in the fermionic case is countable, so it can as well be re-indexed by  $\mathbb{N}$ .

For bosons, we follow the construction in Section 3.2.1, replacing the argument “ $C$  is Hilbert–Schmidt” below (40) by “ $|C|$  has countable spectrum”. This provides us with an orthonormal basis  $\mathbf{g} = (\mathbf{g}_j)_{j \in \mathbb{N}}$ . Further, it is used, that by Lemma 4.1, the Bogoliubov relations still hold as a weak operator identity.

Now, we consider the one–mode Fock space  $\mathcal{H}_k$  for mode  $k = j \in \mathbb{N}$  and take the ITP over all these modes:

**Definition 4.1.** The **bosonic infinite tensor product space** is given by

$$\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{F}(\{\mathbf{g}_k\}). \quad (79)$$

Note that the sequence  $(e_{k,n})_{n \in \mathbb{N}_0}$  of  $n$ –particle basis vectors is a canonical basis of each  $\mathcal{H}_k$ , and can be used to describe elements of  $\widehat{\mathcal{H}}$ .

For fermions, we obtain a similar orthonormal basis  $(\boldsymbol{\eta}_j)_{j \in J}$  with countable index set  $J \subseteq \mathbb{N}$ : Copying the construction in Section 3.2.2, while replacing “ $C^*C$  is trace class” by “ $|C|$  has countable spectrum” (which is true by Lemma 4.2), we have that  $(\boldsymbol{\eta}_j)_{j \in J}$  is orthonormal. The construction of the ITP space is, however, a bit more delicate in this case. The easy part are modes with a full particle–hole transformation, or no transformation at all ( $j \in J''$ ). Here we may just consider each Fock space  $\mathcal{F}(\{\boldsymbol{\eta}_j\}) \cong \mathbb{C}^2$  over the respective mode  $\boldsymbol{\eta}_j$  as a factor within the ITP. However, for Cooper pairs indexed by  $i \in I'$  (so  $j \in J'$ ), it will become necessary (see Remark 8) to introduce a separate Fock space  $\mathcal{F}(\{\boldsymbol{\eta}_{2i-1}\}) \otimes \mathcal{F}(\{\boldsymbol{\eta}_{2i}\}) \cong \mathbb{C}^4$  for each pair of modes. We index all  $j \in J''$  and  $i \in I'$  by a corresponding  $k(i)$  or  $k(j)$ , such that all  $k \in \mathbb{N}$  are used and take the tensor product over those  $k$ :

**Definition 4.2.** The **fermionic infinite tensor product space** is given by

$$\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k = \left( \prod_{j \in J''}^{\otimes} \mathcal{F}(\{\boldsymbol{\eta}_j\}) \right) \otimes \left( \prod_{i \in I'}^{\otimes} \mathcal{F}(\{\boldsymbol{\eta}_{2i-1}\}) \otimes \mathcal{F}(\{\boldsymbol{\eta}_{2i}\}) \right). \quad (80)$$

For a one-mode Fock space,  $\mathcal{H}_k := \mathcal{F}(\{\mathbf{g}_k\})$  or  $\mathcal{H}_{k(j)} := \mathcal{F}(\{\boldsymbol{\eta}_j\})$ , the sequence  $(e_{k,n})_{n \in \mathbb{N}_0}$  of  $n$ -particle state vectors (bosonic case) or the pair  $(e_{k,0}, e_{k,1})$  (fermionic case) forms a basis of each  $\mathcal{H}_k$ . For fermionic two-mode Fock spaces  $\mathcal{H}_{k(i)} := \mathcal{F}(\{\boldsymbol{\eta}_{2i-1}\}) \otimes \mathcal{F}(\{\boldsymbol{\eta}_{2i}\})$ , such a basis is given by the quadruple  $(e_{k,0,0}, e_{k,1,0}, e_{k,0,1}, e_{k,1,1})$ , where 0, 1 are the occupation numbers of the respective mode.

Our next challenge is to lift the one-mode creation and annihilation operators  $a_j^\dagger, a_j$  defined on the one- or two-mode Fock space  $\mathcal{H}_k$  to  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^\otimes \mathcal{H}_k$ .

**Lemma 4.3.** *Consider a (possibly unbounded) operator  $A_{j,j} : \mathcal{H}_j \supset \text{dom}(A_{j,j}) \rightarrow \mathcal{H}_j$ . Then, for  $\Psi_j^{(m)} \in \text{dom}(A_{j,j})$ ,*

$$A_j \Psi^{(m)} := \Psi_1^{(m)} \otimes \dots \otimes \Psi_{j-1}^{(m)} \otimes A_{j,j} \Psi_j^{(m)} \otimes \Psi_{j+1}^{(m)} \otimes \dots, \quad (81)$$

is independent of the choice of a  $C$ -sequence  $(\Psi^{(m)}) = (\Psi_k^{(m)})_{k \in \mathbb{N}}$  representing  $\Psi^{(m)}$ , and defines an operator  $A_j$  by linearity on

$$\Psi \in \text{dom}(A_j) := \left\{ \Psi = \sum_{m \in \mathcal{M}} d_m \Psi^{(m)} \in \widehat{\mathcal{H}} \left\| \left\| \sum_{m \in \mathcal{M}} d_m A_j \Psi^{(m)} \right\| < \infty \right. \right\}, \quad (82)$$

where  $\mathcal{M} \subseteq \mathbb{N}$ ,  $d_m \in \mathbb{C}$  and  $\Psi^{(m)}$  being such that  $A_j \Psi^{(m)}$  is well-defined by (81).

*Proof.* For a fixed choice of  $(\Psi^{(m)})$  representing  $\Psi^{(m)}$  such that  $\Psi_j^{(m)} \in \text{dom}(A_{j,j})$ , well-definedness of  $A_j \Psi^{(m)}$  is easy to see. By Lemma A.1, we can now represent  $\Psi = \sum_{m \in \mathcal{M}} d_m \Psi^{(m)}$ . And if  $\sum_{m \in \mathcal{M}} d_m A_j \Psi^{(m)}$  converges, then it is independent of the representation, since  $A_j$  is linear. So  $\text{dom}(A_j)$  and  $A_j \Psi$  are well-defined.

It remains to be proven that  $A_k \Psi^{(m)}$  (and hence  $A_k \Psi$ ) is independent of the choice of  $(\Psi^{(m)})$  representing  $\Psi^{(m)}$ . So, for  $m \in \mathcal{M}$ , consider a second representative  $C$ -sequence  $(\tilde{\Psi}^{(m)})$  with  $\tilde{\Psi}^{(m)} = \Psi^{(m)}$ . By Proposition A.1,  $\tilde{\Psi}_k^{(m)} = c_k \Psi_k^{(m)}$  for some  $c_k \in \mathbb{C}$  with  $\prod_{k \in \mathbb{N}} c_k = 1$ . By linearity,  $A_{j,j} \tilde{\Psi}_k^{(m)} = c_k A_{j,j} \Psi_k^{(m)}$ , so also  $A_j \Psi^{(m)}$  and  $A_j \tilde{\Psi}^{(m)}$  defined by (81) just differ by the sequence of complex factors  $(c_k)_{k \in \mathbb{N}}$  with  $\prod_{k \in \mathbb{N}} c_k = 1$ . Hence, according to Proposition A.1, they correspond to the same functional  $A_j \tilde{\Psi}^{(m)} = A_j \Psi^{(m)}$ .  $\square$

Bosonic creation and annihilation operators  $a_j, a_j^\dagger$  are usually not bounded. We need to carefully choose a non-dense domain in  $\mathcal{H}_k$  in order to make them bounded. Such a choice is made possible by restricting the allowed  $\Psi$  to the following space:

**Definition 4.3.** In the bosonic case, the space  $\mathcal{S}^\otimes$  with rapid decay in the particle number is defined as

$$\mathcal{S}^\otimes := \left\{ \Psi \in \bigcap_{\substack{n \in \mathbb{N} \\ k \in \mathbb{N}_0}} \text{dom}(N_k^n) \subseteq \widehat{\mathcal{H}} \mid \|N_k^n \Psi\| \leq c_{k,n} \|\Psi\| \forall k \in \mathbb{N}, n \in \mathbb{N}_0 \right\}, \quad (83)$$

where  $j = k$  (as we are in the bosonic case),  $c_{k,n} > 0$  are suitable constants for each  $n$  and  $k$ , and  $N_k$  is the number operator on  $\mathcal{H}_k$ , lifted to  $\widehat{\mathcal{H}}$ . The lift is possible by Lemma 4.3, which also yields a definition of  $\text{dom}(N_k^n)$ .

In the fermionic case, we simply set

$$\mathcal{S}^\otimes := \widehat{\mathcal{H}}, \quad (84)$$

as the maximum particle number per mode is 1, so we always have a rapid decay.

Within  $\mathcal{H}_k$  there may now exist one or two creation and annihilation operators  $a_j$  and  $a_j^\dagger$ , depending on whether  $\mathcal{H}_k$  describes one or two modes. We lift them to  $\widehat{\mathcal{H}}$  and formally define

$$a^\dagger(\phi) = \sum_j \phi_j a_j^\dagger, \quad a(\phi) = \sum_j \overline{\phi_j} a_j. \quad (85)$$

Now,  $a^\dagger(\phi)$  does only make sense for  $\phi \in \mathcal{D}_e \subseteq \ell^2$ , as will become clear in the proof of the following lemma:

**Lemma 4.4** (Products of  $a^\dagger, a$  are well-defined on the ITP space).

Consider the ITP space  $\widehat{\mathcal{H}} \supseteq \mathcal{S}^\otimes$  corresponding to the basis  $(e_j)_{j \in \mathbb{N}}$ , which is  $(g_j)_{j \in \mathbb{N}}$  (bosonic) or  $(\eta_j)_{j \in \mathbb{N}}$  (fermionic). If  $\phi \in \mathcal{D}_g$  or  $\mathcal{D}_\eta$  (defined in (78)), then

$$a^\dagger(\phi) : \mathcal{S}^\otimes \rightarrow \mathcal{S}^\otimes, \quad a(\phi) : \mathcal{S}^\otimes \rightarrow \mathcal{S}^\otimes, \quad (86)$$

as in (85), are well-defined linear operators.

Note that for  $(\eta_j)_{j \in \mathbb{N}}$ , we have re-indexed  $j \in J$  to  $j \in \mathbb{N}$ .

*Proof.* First, note that we can write

$$a^\dagger(\phi)\Psi = \sum_{j \in \mathbb{N}} \phi_j a_j^\dagger \Psi = \sum_{j: \phi_j \neq 0} \phi_j a_j^\dagger \Psi, \quad a(\phi)\Psi = \sum_{j: \phi_j \neq 0} \overline{\phi_j} a_j \Psi, \quad (87)$$

where the sum is finite by definition of  $\mathcal{D}_g$  and  $\mathcal{D}_\eta$ .

In the fermionic case,  $a_j, a_j^\dagger$  are bounded, so by [15, Lemma 5.1.1], they can be lifted to bounded operators on  $\mathcal{S}^\otimes = \widehat{\mathcal{H}}$ . Therefore, also the finite linear combination (87) is a bounded operator on  $\mathcal{S}^\otimes$ .

Within the bosonic case, where  $j = k$ , the first statement  $a^\dagger(\phi)\Psi \in \mathcal{S}^\otimes$  can be seen as follows: We start by verifying that  $a^\dagger(\phi)\Psi$  is well-defined. First, note that

$$\|a_k^\dagger\Psi\| = \|\sqrt{N_k + 1}\Psi\| \leq \|(N_k + 1)\Psi\| \leq (c_{k,1} + 1)\|\Psi\|, \quad (88)$$

where we used that the operator  $a_k^\dagger$  shifts all sectors up by one (keeping them orthogonal) and multiplies by  $\sqrt{N_k + 1}$ . So

$$\|a^\dagger(\phi)\Psi\| \leq \left( \sum_{k:\phi_k \neq 0} |\phi_k|(c_{k,1} + 1) \right) \|\Psi\| =: c_1\|\Psi\|, \quad (89)$$

where, by definition of  $\mathcal{D}_g$  and  $\mathcal{D}_\eta$ , the sum over  $k$  contains only finitely many nonzero terms, so we may call it  $c_1 > 0$ . Hence,  $a^\dagger(\phi)\Psi \in \widehat{\mathcal{H}}$ .

It remains to establish rapid decay. We observe that

$$\begin{aligned} \|N_k^n a_k^\dagger\Psi\| &= \|N_k^n \sqrt{N_k + 1}\Psi\| \leq \|N_k^{n+1}\Psi\| + \|N_k^n\Psi\| \leq (c_{k,n+1} + c_{k,n})\|\Psi\| \\ &\leq (c_{k,n+1} + c_{k,n})\|a_k^\dagger\Psi\|, \end{aligned} \quad (90)$$

so by summing over  $k$ , the rapid decay condition is again satisfied and  $a^\dagger(\phi)\Psi \in \mathcal{S}^\otimes$ .

For the second statement  $a(\phi)\Psi \in \mathcal{S}^\otimes$ , the same finite-sum argument can be used. By repeating all proof steps with  $\sqrt{N_k}$  instead of  $\sqrt{N_k + 1}$ , it can be seen that  $a(\phi)\Psi \in \widehat{\mathcal{H}}$ . However, verifying the rapid decay condition needs a bit more attention, since the inequality  $\|a_k^\dagger\Psi\| \leq \|\Psi\|$  in (90) does not generalize to  $a_k$ , see also Remark 5. However, for all  $k$  with  $a_k\Psi \neq 0$  and  $\phi_k \neq 0$ , there is a fixed ratio  $\frac{\|\Psi\|}{\|a_k\Psi\|} =: d_k > 0$ . Denote by  $d := \max_k d_k$  the maximum over these finitely many ratios for a fixed  $\Psi$ . Then,

$$\|N_k^n a_k\Psi\| \leq (c_{k,n+1} + c_{k,n})\|\Psi\| \leq d \cdot (c_{k,n+1} + c_{k,n})\|a_k\Psi\|. \quad (91)$$

For  $a_k\Psi = 0$ , the inequality is trivially satisfied. A finite sum over  $k$  establishes  $a(\phi)\Psi \in \mathcal{S}^\otimes$  and thus finishes the proof.  $\square$

The ESS for countable spectrum of  $v^*v$  is built using the complex sequence space

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_g := \mathcal{D}'_g \cong \mathcal{E}(\mathbb{N}) && \text{(bosonic)} \\ \mathcal{E} &= \mathcal{E}_\eta := \mathcal{D}'_\eta \cong \mathcal{E}(\mathbb{N}) && \text{(fermionic)}. \end{aligned} \quad (92)$$

This definition allows for a distribution pairing  $\langle \phi, \psi \rangle = \sum_j \overline{\phi_j} \psi_j$  for  $\phi \in \mathcal{D}, \psi \in \mathcal{E}$ . In particular, both  $\mathcal{D}$  and  $\ell^2$  can be embedded into  $\mathcal{E}$ .

The spaces  $\mathcal{E}^{(N)}, \mathcal{E}_{\mathcal{F}}, \text{Ren}_1, \text{Ren}, \text{eRen}, \overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$  are then defined as in Section 2.2.

Creation and annihilation operators  $a^\dagger(\phi), a(\phi)$  are defined on configuration space functions  $\Psi_m : \mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}$  in similarity to (11). The only difference is that a configuration is no longer  $q \in \mathcal{Q}(\mathbb{R}^d)$ , but rather  $q \in \mathcal{Q}(\mathbb{N})$ , so  $q = \{j_1, \dots, j_N\}$ . Again, we have to distinguish the bosonic (+) and the fermionic (−) case. Formally,

$$\begin{aligned} (a_\pm^\dagger(\phi)\Psi_m)(q) &= \sum_{k=1}^N \frac{(\pm 1)^k}{\sqrt{N}} \phi_{j_k} \Psi_m(q \setminus j_k) \\ (a_\pm(\phi)\Psi_m)(q) &= \sqrt{N+1} \sum_j \overline{\phi_j} \Psi_m(q, j). \end{aligned} \tag{93}$$

The definition extends by linearity from  $\Psi_m \in \mathcal{E}_{\mathcal{F}}$  to any  $\Psi = \sum_m \mathbf{c}_m \Psi_m \in \overline{\mathcal{F}}$  with  $\mathbf{c}_m \in \text{eRen}, \Psi_m \in \mathcal{E}_{\mathcal{F}}$  and with the sum over  $m$  being finite. Note that the symmetrization operators  $S_\pm$  from (6) also naturally extend to ESS vectors, as they just permute entries within a configuration  $q$ . In the following, we will again drop the index  $\pm$ , meaning that statements about  $a^\dagger(\phi), a(\phi)$  are both about fermionic and the bosonic operators, if not stated otherwise.

The CAR/CCR are a direct consequence of definition (93) and hence still valid for the operator extensions.

**Lemma 4.5** (Products of  $a^\dagger, a$  are well-defined on the ESS).

*Consider the ESS  $\overline{\mathcal{F}}$  built over  $\mathcal{E}$ . Then, (93) uniquely defines operators  $a^\dagger(\phi), a(\phi)$  as follows: for  $\phi \in \mathcal{D}$ , we have*

$$a^\dagger(\phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}, \quad a(\phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}, \tag{94}$$

*and more generally, for  $\phi \in \mathcal{E}$ , we have*

$$a^\dagger(\phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}, \quad a(\phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}. \tag{95}$$

*Proof.* For  $\phi \in \mathcal{D}$ , it suffices to show that for  $\Psi_m \in \mathcal{E}_{\mathcal{F}}$ , we have  $a^\dagger(\phi)\Psi_m, a(\phi)\Psi_m \in \mathcal{E}_{\mathcal{F}}$ . Considering (93), it is easy to see that  $a^\dagger(\phi)\Psi_m : \mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}$  defines a function on configuration space, as it is just the tensor product of two functions. For  $(a(\phi)\Psi_m)(q)$ , since  $\phi \in \mathcal{D}$ , each sum over  $j$  in (93) has a finite number of nonzero terms and is hence finite. Therefore,  $(a(\phi)\Psi_m)(q)$  is finite and  $a(\phi)\Psi_m : \mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}$  is a well-defined function.

For  $\phi \in \mathcal{E}$ , it is again easy to see that  $a^\dagger(\phi)\Psi_m$  defines a function  $\mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}$  as a tensor product of two functions. In  $a(\phi)\Psi_m$ , the sum over  $j$  may now be infinite or even

divergent. However, it can be defined as a  $\text{Ren}_1$  renormalization constant: For each  $q \in \mathcal{Q}(\mathbb{N})$ , the function  $f(j) := \overline{\phi_j} \Psi_m(q, j)$  is  $f \in \mathcal{E}(\mathbb{N})$ , so

$$\sum_j \overline{\phi_j} \Psi_m(q, j) \in \text{Ren}_1(\mathbb{N}) \quad \Rightarrow \quad (a(\phi) \Psi_m) \in \text{Ren}^{\mathcal{Q}}(\mathbb{N}).$$

Hence, for  $\Psi = \sum_m \mathbf{c}_m \Psi_m \in \overline{\mathcal{F}}$ , we have

$$a(\phi) \Psi = \sum_m \mathbf{c}_m a(\phi) \Psi_m \in \overline{\mathcal{F}}_{\text{ex}}.$$

□

### Remarks.

5. The condition  $\phi \in \mathcal{D}_{\mathbf{f}}$  is indeed necessary, meaning we may not just allow any  $\phi \in \ell^2$  inside  $a^\sharp(\phi)$ , as the following counter-example shows: For the bosonic case ( $j = k$ ), consider  $\phi_k = \frac{1}{k}$ , so  $\phi \in \ell^2 \setminus \mathcal{D}_{\mathbf{f}}$ . For each mode  $k$ , consider the coherent state  $\Psi_k$  defined sector-wise by

$$\Psi_k^{(N_k)} = e^{-\frac{\alpha_k}{2}} \frac{\alpha_k^{\frac{N_k}{2}}}{\sqrt{N_k!}}, \quad (96)$$

where all  $\alpha_k \in \mathbb{R}$  are set equal to the same  $\alpha_k = \alpha > 0$  and where  $\|\Psi_k\|_k = 1$ . Then, define the ITP  $\Psi = \prod_{k \in \mathbb{N}}^{\otimes} \Psi_k$ . It is easy to see that  $\Psi$  satisfies the rapid decay condition (83), as for each  $\Psi_k$ ,  $\|\Psi_k^{(N_k)}\|_k$  decays exponentially in  $N_k$ . But still,  $(\alpha_k)_{k \in \mathbb{N}} \notin \ell^2$ , so we may think of  $\Psi$  as a ‘‘coherent state with a large displacement’’, living outside the Fock space. It follows from a well-known fact about coherent states that  $a_k \Psi = \alpha \Psi$ , so

$$\|a(\phi) \Psi\| = \left\| \sum_k \overline{\phi_k} \alpha \Psi \right\| = \alpha \sum_k \frac{1}{k} \|\Psi\| = \infty. \quad (97)$$

Hence,  $a(\phi)$  is ill-defined on  $\Psi$ .

The same happens with any coherent state product (96) and any  $\phi$ , where  $\sum_k \overline{\phi_k} \alpha_k = \infty$ . In particular, the space of allowed  $(\phi_k)_{k \in \mathbb{N}}$  is dual to the one of allowed  $(\alpha_k)_{k \in \mathbb{N}}$ .

6. The above-mentioned duality actually extends to the definition of  $\mathcal{D}$  and  $\mathcal{S}^{\otimes}$ : We may alter those definitions to allow for more form factors  $\phi \in \ell^2$  in  $a^\sharp(\phi)$ . The result is that fewer vectors  $\Psi \in \mathcal{S}^{\otimes}$  are allowed, if  $a^\sharp(\phi_1) \dots a^\sharp(\phi_N) \Psi$  shall still be well-defined. These alternative definitions are discussed in Appendix D.



7. It is possible to view the subspace  $\prod_{k \in \mathbb{N}}^{\otimes C} \mathcal{H}_k$  of the equivalence class  $C$  (see below (25)) as the *original Fock space* with respect to the vacuum  $\Omega = \prod_{k \in \mathbb{N}}^{\otimes} e_{k,0}$ :

Recall that each  $\Psi \in \prod_{k \in \mathbb{N}}^{\otimes C} \mathcal{H}_k$  can be written in coordinates as (26):

$$\Psi = \sum_{n(\cdot) \in F} a(n(\cdot)) \prod_{k \in \mathbb{N}}^{\otimes} e_{k,n(k)}, \quad (98)$$

with  $F$  containing all sequences  $(n(k))_{k \in \mathbb{N}}$ , such that  $n(k) = 0$  for almost all  $k$ . Hence, each  $\prod_{k \in \mathbb{N}}^{\otimes} e_{k,n(k)}$  is a tensor product state of finitely many particles. Since the Fock norm and the  $\widehat{\mathcal{H}}$ -norm coincide, the vector  $\prod_{k \in \mathbb{N}}^{\otimes} e_{k,n(k)}$  can be seen as a Fock space vector normalized to 1. The linear combination (98) with  $\sum_{n(\cdot)} |a(n(\cdot))|^2$  can hence also be interpreted a Fock space vector. Conversely, each Fock space vector can be written as a countable sequence (98), since the span of the above-mentioned tensor product states is dense in  $\mathcal{F}$ .

## 5 Implementation: Extended

Roughly speaking, implementability of  $\mathcal{V}$  on Fock space  $\mathcal{F}$  means that there exists a linear map  $\mathbb{U}_{\mathcal{V}} : \mathcal{F} \rightarrow \mathcal{F}$  that transforms  $a^{\sharp}$ - into  $b^{\sharp}$ -operators (see Section 3.2). In Section 5.1, we give a precise definition of how implementability of  $\mathcal{V}$  is to be interpreted on Fock space extensions  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ . Lemma 5.1 will then establish, that  $\mathbb{U}_{\mathcal{V}}$  is indeed well-defined and 5.2 will give suitable conditions for when it is an implementer in the extended sense.

Within Section 5.2, Theorems 5.1 and 5.2 then establish these suitable conditions for countable spectrum of  $v^*v$  in the bosonic case. In Section 5.3, Theorems 5.3 and 5.4 do the same for the fermionic case. Note that for implementability on  $\overline{\mathcal{F}}$  in Theorem 5.4, there is an additional requirement that only finitely many modes with full particle-hole transformations are allowed.

### 5.1 Definition of Extended Implementation

The implementer  $\mathbb{U}_{\mathcal{V}}$  is defined on a dense subspace of Fock space  $\mathcal{D}_{\mathcal{F}} \subset \mathcal{F}$ , that contains a finite number of particles from the space  $\mathcal{D}_{\mathbf{f}}$  (defined by (78) with  $\mathbf{e} = \mathbf{f}$ ):

$$\mathcal{D}_{\mathcal{F}} := \text{span}\{a^{\dagger}(\phi_1) \dots a^{\dagger}(\phi_N)\Omega, N \in \mathbb{N}_0, \phi_{\ell} \in \mathcal{D}_{\mathbf{f}}\}. \quad (99)$$

The operator  $\mathbb{U}_{\mathcal{V}}$  now maps from  $\mathcal{D}_{\mathcal{F}}$  into either an ITP space  $\widehat{\mathcal{H}}$  or an ESS  $\overline{\mathcal{F}}$ . All statements provided in this subsection hold regardless of the choice of this image space.

**Definition 5.1.** We say that a linear operator  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \widehat{\mathcal{H}}$  or  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \overline{\mathcal{F}}$  implements a Bogoliubov transformation  $\mathcal{V}$  in the extended sense, if for all  $\phi \in \mathcal{D}_\mathcal{f}$ ,  $\Psi \in \mathbb{U}_\mathcal{V}[\mathcal{D}_\mathcal{F}]$ , we have that

$$\mathbb{U}_\mathcal{V} a^\dagger(\phi) \mathbb{U}_\mathcal{V}^{-1} \Psi = b^\dagger(\phi) \Psi, \quad \mathbb{U}_\mathcal{V} a(\phi) \mathbb{U}_\mathcal{V}^{-1} \Psi = b(\phi) \Psi. \quad (100)$$

This requires, of course, that  $\mathbb{U}_\mathcal{V}^{-1}$  is well-defined. So before establishing (100), we have to show that  $\mathbb{U}_\mathcal{V}$  is invertible, in order to prove that  $\mathbb{U}_\mathcal{V}$  implements  $\mathcal{V}$  in the extended sense. This will be one main difficulty within the upcoming proofs.

The implementer  $\mathbb{U}_\mathcal{V}$  is defined as follows: First we define some new vacuum vector  $\Omega_\mathcal{V} = \mathbb{U}_\mathcal{V} \Omega$  within the respective Fock space extension, such that

$$b(\phi) \Omega_\mathcal{V} = 0. \quad (101)$$

Then we make  $\mathbb{U}_\mathcal{V}$  change  $a^\sharp$ - into  $b^\sharp$ -operators:

**Definition 5.2.** Given a Bogoliubov transformed vacuum state  $\Omega_\mathcal{V} \in \mathcal{S}^\otimes$  or  $\Omega_\mathcal{V} \in \overline{\mathcal{F}}$ , the **Bogoliubov implementer**  $\mathbb{U}_\mathcal{V}$  is formally defined on  $\mathcal{D}_\mathcal{F}$  by

$$\mathbb{U}_\mathcal{V} a^\dagger(\phi_1) \dots a^\dagger(\phi_n) \Omega := b^\dagger(\phi_1) \dots b^\dagger(\phi_n) \Omega_\mathcal{V}, \quad (102)$$

with  $\phi_\ell \in \mathcal{D}_\mathcal{f}$  and  $b^\dagger(\mathbf{f}_j) = (a^\dagger(u\mathbf{f}_j) + a(\overline{v\mathbf{f}_j}))$  for all basis vectors  $\mathbf{f}_j$  in  $\mathbf{f}$ .

**Lemma 5.1** ( $\mathbb{U}_\mathcal{V}$  is well-defined).

In the ITP case, if  $\Omega_\mathcal{V} \in \mathcal{S}^\otimes \subseteq \widehat{\mathcal{H}}$  (see (83)), then the formal implementer  $\mathbb{U}_\mathcal{V}$  (102) is a well-defined operator  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \mathcal{S}^\otimes$ .

For the ESS, if  $\Omega_\mathcal{V} \in \overline{\mathcal{F}}$ , then the formal implementer  $\mathbb{U}_\mathcal{V}$  (102) is a well-defined operator  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \overline{\mathcal{F}}$ .

*Proof.* In the ITP case, by (43),  $u\mathbf{f}_j$  and  $\overline{v\mathbf{f}_j}$  are both proportional to the same basis vector  $\mathbf{e}_j$  (bosonic:  $\mathbf{g}_j$ , fermionic:  $\boldsymbol{\eta}_j$ ). So the right-hand side of (102) is a finite linear combination of vectors  $a^\sharp(\mathbf{e}_{j_1}) \dots a^\sharp(\mathbf{e}_{j_n}) \Omega_\mathcal{V}$ . Now,  $\Omega_\mathcal{V} \in \mathcal{S}^\otimes$  and by Lemma 4.4, each application of  $a^\sharp(\mathbf{e}_j)$  leaves the vector in  $\mathcal{S}^\otimes$ . So the whole vector (102) is in  $\mathcal{S}^\otimes \subseteq \widehat{\mathcal{H}}$ .

For the ESS, the right-hand side of (102) is a finite linear combination of vectors of the kind  $a^\sharp(\mathbf{e}_{j_1}) \dots a^\sharp(\mathbf{e}_{j_n}) \Omega_\mathcal{V}$ , where  $\mathbf{e}_j \in \{u\mathbf{f}_j, \overline{v\mathbf{f}_j}\}$  is proportional to  $\mathbf{g}_j$  or  $\boldsymbol{\eta}_j$  and hence in  $\mathcal{E}$ . We have  $\Omega_\mathcal{V} \in \overline{\mathcal{F}}$ , and by Lemma 4.5, as  $\mathbf{f}_j \in \mathcal{D}_\mathcal{f}$ , each application of an  $a^\sharp(\mathbf{f}_j)$  maps again into  $\overline{\mathcal{F}}$ . So (102) is well-defined. □

We now provide conditions, for which  $\mathbb{U}_{\mathcal{V}}$  is indeed an implementer of  $\mathcal{V}$ .

**Lemma 5.2** (Conditions for an implementer  $\mathbb{U}_{\mathcal{V}}$ ). *Suppose that for a Bogoliubov transformation (i.e.,  $\mathcal{V}$  satisfying (68)) an  $\Omega_{\mathcal{V}}$  satisfying  $b(\phi)\Omega_{\mathcal{V}} = 0$  for all  $\phi \in \mathcal{D}_{\mathcal{F}} \subseteq \ell^2$  has been found, such that  $\mathbb{U}_{\mathcal{V}}$  in (102) on  $\mathcal{D}_{\mathcal{F}}$  is well-defined and has an inverse  $\mathbb{U}_{\mathcal{V}}^{-1}$  defined on  $\mathbb{U}_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}]$ .*

*Then,  $\mathbb{U}_{\mathcal{V}}$  implements  $\mathcal{V}$  in the sense of (100) on all  $\Psi \in \mathbb{U}_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}]$ .*

*Proof.* We write  $\Psi = \mathbb{U}_{\mathcal{V}}\Phi$  with  $\Phi \in \mathcal{D}_{\mathcal{F}}$ . By linearity, it suffices to prove the statement for  $\Phi = a^\dagger(\phi_1) \dots a^\dagger(\phi_n)\Omega$ , which implies by (102) that  $\Psi = b^\dagger(\phi_1) \dots b^\dagger(\phi_n)\Omega_{\mathcal{V}}$ . In that case, we have (with  $n \geq 0$ )

$$\begin{aligned} \mathbb{U}_{\mathcal{V}}a^\dagger(\phi)\mathbb{U}_{\mathcal{V}}^{-1}\Psi &= \mathbb{U}_{\mathcal{V}}a^\dagger(\phi)\mathbb{U}_{\mathcal{V}}^{-1}b^\dagger(\phi_1) \dots b^\dagger(\phi_n)\Omega_{\mathcal{V}} \\ &= \mathbb{U}_{\mathcal{V}}a^\dagger(\phi)a^\dagger(\phi_1) \dots a^\dagger(\phi_n)\Omega_{\mathcal{V}} \\ &= b^\dagger(\phi)b^\dagger(\phi_1) \dots b^\dagger(\phi_n)\Omega_{\mathcal{V}} = b^\dagger(\phi)\Psi, \end{aligned} \tag{103}$$

which is the first statement of (100). The second statement requires somewhat more attention, as our operator product also includes one annihilation operator. We make use of the CAR/CCR of  $a$ - and  $b$ -operators, using the combinatorial factor  $\varepsilon = (-1)$  for fermions and  $\varepsilon = 1$  for bosons. Here, the CAR/CCR are valid for  $a$ -operators by definition (93) and for  $b$ -operators, since by means of Lemma 4.1, the Bogoliubov relations survive the extension.

$$\begin{aligned} \mathbb{U}_{\mathcal{V}}a(\phi)\mathbb{U}_{\mathcal{V}}^{-1}\Psi &= \mathbb{U}_{\mathcal{V}}a(\phi)\mathbb{U}_{\mathcal{V}}^{-1}b^\dagger(\phi_1) \dots b^\dagger(\phi_n)\Omega_{\mathcal{V}} \\ &= \mathbb{U}_{\mathcal{V}}a(\phi)a^\dagger(\phi_1) \dots a^\dagger(\phi_n)\Omega \\ &= \sum_{\ell=1}^n \mathbb{U}_{\mathcal{V}}a^\dagger(\phi_1) \dots a^\dagger(\phi_{\ell-1})\varepsilon^{\ell+1}\langle\phi, \phi_\ell\rangle a^\dagger(\phi_{\ell+1}) \dots a^\dagger(\phi_n)\Omega \\ &= \sum_{\ell=1}^n b^\dagger(\phi_1) \dots b^\dagger(\phi_{\ell-1})\varepsilon^{\ell+1}\langle\phi, \phi_\ell\rangle b^\dagger(\phi_{\ell+1}) \dots b^\dagger(\phi_n)\Omega_{\mathcal{V}} \\ &= b(\phi)b^\dagger(\phi_1) \dots b^\dagger(\phi_n)\Omega_{\mathcal{V}} - \varepsilon^{n+1}b^\dagger(\phi_1) \dots b^\dagger(\phi_n)b(\phi)\Omega_{\mathcal{V}} \\ &\stackrel{(101)}{=} b(\phi)b^\dagger(\phi_1) \dots b^\dagger(\phi_n)\Omega_{\mathcal{V}} = b(\phi)\Psi, \end{aligned} \tag{104}$$

which is the second statement of (100), where we used the convention that the above sums are set to zero for  $N = 0$ .

□

## 5.2 Bosonic Case

We will now show that for a suitable choice of  $\Omega_{\mathcal{V}}$ , the operator  $\mathbb{U}_{\mathcal{V}}$  defined in (102) indeed implements the Bogoliubov transformation  $\mathcal{V}$ . The ITP case is treated in Theorem

5.1 and the ESS in Theorem 5.2.

**Theorem 5.1** (Implementation works, bosonic, ITPs). *Consider a bosonic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v^* & u \end{pmatrix}$  with  $v^*v$  having countable spectrum. Let  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k$  be the ITP space (Definition 4.1) with respect to the basis  $(\mathbf{g}_k)_{k \in \mathbb{N}} \subset \ell^2$ . Define the new vacuum vector*

$$\Omega_{\mathcal{V}} = \prod_{k \in \mathbb{N}}^{\otimes} \Omega_{k, \mathcal{V}} := \prod_{k \in \mathbb{N}}^{\otimes} \left( \left( \left( 1 - \frac{\nu_k^2}{\mu_k^2} \right)^{1/4} \right) \exp \left( -\frac{\nu_k}{2\mu_k} (a^\dagger(\mathbf{g}_k))^2 \right) \Omega_k \right). \quad (105)$$

Then,  $\mathcal{V}$  is implemented in the sense of (100) by  $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\mathcal{F}} \rightarrow \widehat{\mathcal{H}}$  (102).

In simple words, Theorem 5.1 says that for  $\phi \in \mathcal{D}_{\mathcal{F}} \subseteq \ell^2$  (78), the operators  $a(\phi), a^\dagger(\phi)$  defined on  $\mathcal{D}_{\mathcal{F}} \subset \mathcal{F}$  (99) are mapped to  $b(\phi), b^\dagger(\phi)$ , which still satisfy the CAR/CCR.

*Proof.* By Lemma 5.2,  $\mathbb{U}_{\mathcal{V}}$  implements  $\mathcal{V}$  (100), if we can show the following:

1. The new vacuum  $\Omega_{\mathcal{V}}$  is well-defined
2.  $\mathbb{U}_{\mathcal{V}}$  is well-defined on  $\mathcal{D}_{\mathcal{F}}$  (Lemma 5.1 will be used, here)
3.  $b(\phi)\Omega_{\mathcal{V}} = 0$
4.  $\mathbb{U}_{\mathcal{V}}^{-1}$  exists on  $\mathbb{U}_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}]$

1.) Well-definedness of  $\Omega_{\mathcal{V}}$ : Expression (105) is an ITP of one normalized factor per space  $\mathcal{H}_k$ . Hence, it is a  $C$ -sequence, which can be identified with  $\Omega_{\mathcal{V}} \in \widehat{\mathcal{H}}$ .

2.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}$ : follows from Lemma 5.1, if we can establish  $\Omega_{\mathcal{V}} \in \mathcal{S}^{\otimes}$ . By definition of  $\mathcal{S}^{\otimes}$ , we need to verify the rapid decay condition  $\|N_k^n \Omega_{\mathcal{V}}\| \leq c_{k,n} \|\Omega_{\mathcal{V}}\|$ . If it would hold, then  $\|N_k^n \Omega_{\mathcal{V}}\| < \infty$  and we would automatically obtain  $\Omega_{\mathcal{V}} \in \text{dom}(N_k^n)$ . Now, since all  $\Omega_{k, \mathcal{V}}$  are normalized, verifying rapid decay boils down to proving

$$\|N_k^n \Omega_{k, \mathcal{V}}\|_k^2 \leq c_{k,n}^2. \quad (106)$$

We may explicitly compute this expression:

$$\|N_k^n \Omega_{k, \mathcal{V}}\|_k^2 = (1 - 4t^2)^{1/2} \sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N)^{2n}, \quad (107)$$

with  $t = \left| \frac{\nu_k}{2\mu_k} \right| \in [0, 1/2)$ . Now, the function

$$N \mapsto \frac{t^{2N}(2N)!}{(N!)^2} (2N)^{2n} \leq (2t)^{2N} (2N)^{2n}, \quad (108)$$

is positive, bounded and decays exponentially at  $N \rightarrow \infty$  since  $0 \leq 2t < 1$ . So,

$$\sum_{N=0}^{\infty} \frac{t^{2N}(2N)!}{(N!)^2} (2N)^{2n} \leq \text{cons.} + \sum_{N=0}^{\infty} (2t)^{2N} (2N)^{2n} =: c_{k,n}^2 < \infty, \quad (109)$$

which establishes  $\Omega_{\mathcal{V}} \in \mathcal{S}^{\otimes}$  and hence the claim.

3.)  $b(\phi)$  annihilates  $\Omega_{\mathcal{V}}$ : This is straightforward to check. Since  $\phi \in \mathcal{D}_{\mathbf{f}}$ , the following sum over  $k$  is finite:

$$b(\phi)\Omega_{\mathcal{V}} = \sum_k \phi_k b(\mathbf{f}_k)\Omega_{\mathcal{V}}. \quad (110)$$

As in the case, where the Shale–Stinespring condition holds, each  $b(\mathbf{f}_k)$  annihilates the corresponding vacuum vector  $\Omega_{k,\mathcal{V}}$ , so the finite sum above is 0.

4.) Well–definedness of  $\mathbb{U}_{\mathcal{V}}^{-1}$ : The following set is a basis for  $\mathcal{D}_{\mathcal{F}}$ :

$$\{a^\dagger(\mathbf{f}_{k_1}) \dots a^\dagger(\mathbf{f}_{k_N})\Omega \mid N \in \mathbb{N}_0, k_\ell \in \mathbb{N}\}, \quad (111)$$

where  $\mathbf{f}_{k_\ell}$  are chosen out of the basis  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  (with  $j = k$ ). This can easily be seen, since every  $a^\dagger(\phi)$ ,  $\phi \in \mathcal{D}_{\mathbf{f}}$  can be decomposed by definition of  $\mathcal{D}_{\mathbf{f}}$  into a finite sum over operators proportional to  $a^\dagger(\mathbf{f}_k)$ . If we can show that the set

$$\{b^\dagger(\mathbf{f}_{k_1}) \dots b^\dagger(\mathbf{f}_{k_N})\Omega_{\mathcal{V}} \mid N \in \mathbb{N}_0, k_\ell \in \mathbb{N}\} \subset \widehat{\mathcal{H}}, \quad (112)$$

with  $b^\dagger(\mathbf{f}_k) = \mu_k a^\dagger(\mathbf{g}_k) + \nu_k a(\mathbf{g}_k)$ , is linearly independent, we are done, since then  $\ker(\mathbb{U}_{\mathcal{V}}) = \{0\}$ , so  $\mathbb{U}_{\mathcal{V}}$  is injective and hence invertible on its image.

Now, as the application of the operators  $b_k^\dagger := b^\dagger(\mathbf{f}_k)$  and  $\mathbb{U}_{\mathcal{V}}$  preserve the ITP structure, it suffices to show that on each mode  $k$ , the set

$$\{(b_k^\dagger)^N \Omega_{k,\mathcal{V}} \mid N \in \mathbb{N}_0\} \subset \mathcal{H}_k, \quad (113)$$

is linearly independent. Now, (113) is just the image of the set

$$\{(a^\dagger(\mathbf{f}_k))^N \Omega_k \mid N \in \mathbb{N}_0\} \subset \mathcal{F}(\{\mathbf{f}_k\}), \quad (114)$$

under a one–mode Bogoliubov transformation  $\mathbb{U}_{k,\mathcal{V}} : \mathcal{F}(\{\mathbf{f}_k\}) \rightarrow \mathcal{H}_k$  (defined as  $\mathbb{U}_{j,\mathcal{V}}$  in (48)), where  $\mathcal{F}(\{\mathbf{f}_k\})$  is the one–mode Fock space over  $\mathbf{f}_k$ . For a finite number  $m$  of modes, Bogoliubov transformations can always be implemented by unitary operators, as then the operator  $v : \mathbb{C}^m \rightarrow \mathbb{C}^m$  is always Hilbert–Schmidt. Now, (114) is an orthogonal set with no vector being 0, so its image (113) under the unitary  $\mathbb{U}_{k,\mathcal{V}}$  is also orthogonal with no vector being 0, and hence it is linearly independent. This finishes the proof.  $\square$

**Theorem 5.2** (Implementation works, bosonic, ESS). *Consider a bosonic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v^* & u \end{pmatrix}$  with  $v^*v$  having countable spectrum. Let  $\overline{\mathcal{F}}$  be the ESS over  $\mathcal{E}_{\mathbf{g}}$  (see (92)) with respect to the basis  $(\mathbf{g}_k)_{k \in \mathbb{N}} \subset \ell^2$ . Define the new vacuum vector*

$$\Omega_{\mathcal{V}} = \underbrace{\exp\left(\frac{1}{4} \sum_k \log\left(1 - \frac{\nu_k^2}{\mu_k^2}\right)\right)}_{=: e^{\mathfrak{r}}} \underbrace{\exp\left(-\sum_k \frac{\nu_k}{2\mu_k} (a^\dagger(\mathbf{g}_k))^2\right)}_{=: \Psi_{\mathcal{V}}} \Omega = e^{\mathfrak{r}} \Psi_{\mathcal{V}}. \quad (115)$$

Then,  $\mathcal{V}$  is implemented in the sense of (100) by  $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  (102).

*Proof.* By Lemma 5.2, it suffices to establish the four points in the proof of Theorem 5.1.

1.) Well-definedness of  $\Omega_{\mathcal{V}}$ : The factor  $\mathfrak{r} = \frac{1}{4} \sum_k \log\left(1 - \frac{\nu_k^2}{\mu_k^2}\right)$  can be interpreted as an element of  $\text{Ren}_1$ , as it is the sum over elements of a complex sequence. So the wave function renormalization is indeed  $e^{\mathfrak{r}} \in \text{eRen}$ . The second factor  $\Psi_{\mathcal{V}}$  is an infinite product of exponentials

$$\exp\left(-\frac{\nu_k}{2\mu_k} (a^\dagger(\mathbf{g}_k))^2\right) \Omega = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{(-\nu_k)^\ell}{2^\ell \mu_k^\ell} (a^\dagger(\mathbf{g}_k))^{2\ell} \Omega. \quad (116)$$

So the  $N$ -sector of  $\Psi_{\mathcal{V}}$  is 0 for odd  $N$ . For even  $N$ , the sector contains a sum over all choices of boson pair numbers  $(\ell_k)_{k \in \mathbb{N}}$ ,  $\ell_k \in \mathbb{N}_0$  such that  $\sum_k \ell_k = N/2$ . Each boson pair is described by a two-particle function supported on the diagonal, as shown in Figure 4.

Now, each point in the mode-configuration space  $q \in \mathcal{Q}(\mathbb{N})$  gets assigned at most one summand corresponding to one choice of a sequence  $(\ell_k)_{k \in \mathbb{N}}$ . This value  $\Psi_{\mathcal{V}}(q)$  can now be written as a convergent infinite product: On each mode  $\mathbf{g}_k$ , we fix the basis  $(e_{k,n})_{n \in \mathbb{N}_0} \subset \mathcal{F}(\{\mathbf{g}_k\})$ , as explained below (80), and write

$$\Psi_{k,\mathcal{V}}(n_k) := \langle e_{k,n_k}, \Psi_{k,\mathcal{V}} \rangle = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{(-\nu_k)^\ell}{2^\ell \mu_k^\ell} \langle e_{k,n_k}, (a^\dagger(\mathbf{g}_k))^{2\ell} \Omega_k \rangle, \quad (117)$$

where  $\Psi_{k,\mathcal{V}}$  is the sequence associated with the Bogoliubov vacuum of mode  $\mathbf{g}_k$ , meaning  $\Psi_{k,\mathcal{V}} = \exp\left(-\frac{\nu_k}{2\mu_k} (a^\dagger(\mathbf{g}_k))^2\right) \Omega_k \in \mathcal{F}(\{\mathbf{g}_k\})$ . Then, we have

$$\Psi_{\mathcal{V}}(q) = \prod_{k \in \mathbb{N}} \Psi_{k,\mathcal{V}}(n_k), \quad (118)$$

where  $n_k$  counts, how often mode  $k$  is contained in configuration  $q$ , see also Figure 4. As there are finitely many particles in  $q$ , we have  $n_k = 0$  for all but finitely many  $k$ . And as the vacuum is normalized to 1, i.e.,  $\Psi_{k,\mathcal{V}}(0) = 1$ , and all other  $\Psi_{k,\mathcal{V}}(n_k)$  are finite, the

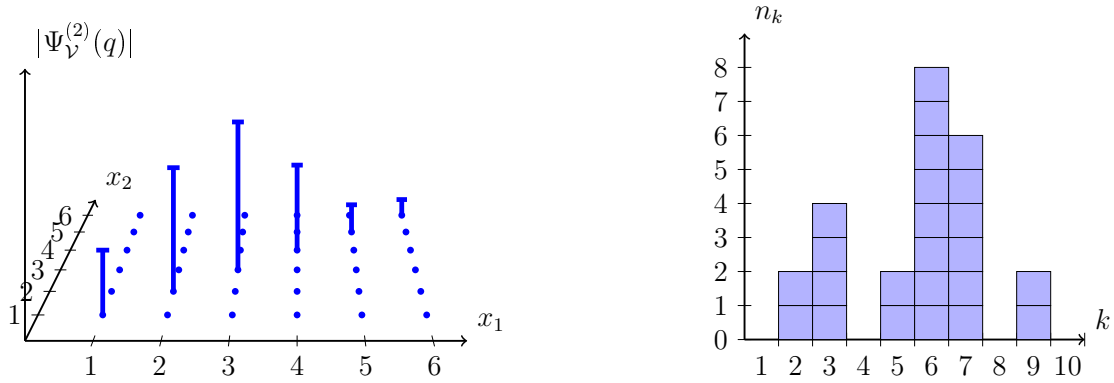


Figure 4: Left: The two-particle sector of  $\Psi_{\mathcal{V}}$ , visualized. Right: Particle numbers  $n_k$  for a typical configuration  $q$  with  $\Psi_{\mathcal{V}}(q) \neq 0$ . Color online.

infinite product (118) is finite, as well.

2.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}$ : This is an immediate consequence of Lemma 5.1.

3.)  $b(\phi)$  annihilates  $\Omega_{\mathcal{V}}$ : This is again checked mode-by-mode, as in proof step 3.) of Theorem 5.1.

4.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}^{-1}$ : As in proof step 4.) of Theorem 5.1, we have to check that  $\mathbb{U}_{\mathcal{V}}$  is injective. And as within that proof step, it suffices to prove that the set

$$\{b^\dagger(\mathbf{f}_{k_1}) \dots b^\dagger(\mathbf{f}_{k_N}) \Psi_{\mathcal{V}} \mid N \in \mathbb{N}_0, k_\ell \in \mathbb{N}\} \subset \overline{\mathcal{F}}, \quad (119)$$

is linearly independent. Here, the factor  $e^{\mathbf{t}}$  can be removed, as it is nonzero for  $\Omega_{\mathcal{V}}$ . Now, assume there was a (finite) nonzero linear combination

$$B = \sum_{m=1}^M b^\dagger(\mathbf{f}_{k_{m,1}}) \dots b^\dagger(\mathbf{f}_{k_{m,N_m}}), \quad \text{with } B\Psi_{\mathcal{V}} = 0. \quad (120)$$

Then, there exists a maximum mode number  $K \in \mathbb{N}$ , such that  $k_{m,\ell} \leq K$  for any creation operator  $b^\dagger(\mathbf{f}_{k_{m,\ell}})$  in  $B$ . We define the Fock space of all modes  $k \leq K$  by

$$\widehat{\mathcal{F}}_{\leq K} := \widehat{\mathcal{F}}(\{1, \dots, K\}), \quad (121)$$

where the Bogoliubov implementer restricted to this space

$$\mathbb{U}_{\leq K, \mathcal{V}} = \prod_{k=1}^K \mathbb{U}_{k, \mathcal{V}}, \quad (122)$$

is a unitary operator, as Bogoliubov transformations on finitely many modes are always implementable. Therefore, with  $\Omega_{\leq K} = \prod_{k \leq K}^{\otimes} \Omega_k$  and  $\Psi_{\leq K, \nu} = \prod_{k \leq K} \Psi_{k, \nu}$ , we have

$$\mathbb{U}_{\nu} a^{\dagger}(\mathbf{f}_{k_1}) \dots a^{\dagger}(\mathbf{f}_{k_N}) \Omega_{\leq K} = b^{\dagger}(\mathbf{f}_{k_1}) \dots b^{\dagger}(\mathbf{f}_{k_N}) \Psi_{\leq K, \nu} \in \mathcal{F}_{\leq K}. \quad (123)$$

We can also restrict the operator  $B$  to

$$\tilde{B} : \mathcal{F}_{\leq K} \rightarrow \mathcal{F}_{\leq K} \quad \tilde{B} = \sum_{m=1}^M b^{\dagger}(\mathbf{f}_{k_{m,1}}) \dots b^{\dagger}(\mathbf{f}_{k_{m,N_m}}). \quad (124)$$

Now, as the vectors  $a^{\dagger}(\mathbf{f}_{k_1}) \dots a^{\dagger}(\mathbf{f}_{k_N}) \Omega_{\leq K}$  with  $k_{\ell} \leq K$  and  $N \in \mathbb{N}_0$  are orthogonal without a zero vector and  $\mathbb{U}_{\nu}$  is unitary, also the set of  $b^{\dagger}(\mathbf{f}_{k_1}) \dots b^{\dagger}(\mathbf{f}_{k_N}) \Psi_{\leq K, \nu}$  is orthogonal without a zero vector, and hence linearly independent. So  $\Psi_{\tilde{B}} = \tilde{B} \Psi_{\leq K, \nu} \neq 0$ . For the associated function, that means,

$$\Psi_{\tilde{B}}(q_{\leq K}) \neq 0 \quad \text{for some } q_{\leq K} \in \mathcal{Q}(\{1, \dots, K\}). \quad (125)$$

Now, as  $B$  leaves all modes  $k_{m,\ell} > K$  invariant, we can write

$$\Psi_B(q) = \left( \prod_{k > K} \Psi_{k, \nu}(n_k) \right) \Psi_{\tilde{B}}(q_{\leq K}), \quad (126)$$

with  $(q_{\leq K})$  containing all modes of  $q$  with  $k \leq K$  and  $n_k$  being the number of times, mode  $k$  appears in configuration  $q$ . The left bracket is 1 for  $n_k \equiv 0$  and for the right term, there exists some  $q_{\leq K}$  with  $\Psi_{\tilde{B}}(q_{\leq K}) \neq 0$  (125). Hence, with  $q = q_{\leq K}$ , we have  $\Psi_B(q) \neq 0$ , which contradicts  $B\Omega_{\nu} = 0$  from (120). So we have linear independence of (119), leading to injectiveness of  $\mathbb{U}_{\nu}$  and finishing the proof.  $\square$

### 5.3 Fermionic Case

**Theorem 5.3** (Implementation works, fermionic, ITPs). *Consider a fermionic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v^* & u \end{pmatrix}$  with  $v^*v$  having countable spectrum. Let  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k$  be the ITP space (Definition 4.2). Define the new vacuum vector*

$$\begin{aligned} \Omega_{\mathcal{V}} &= \prod_{j \in J''}^{\otimes} \Omega_{j, \nu} \otimes \prod_{i \in I'}^{\otimes} \Omega_{2i, 2i-1, \nu} \\ &:= \left( \prod_{j \in J''_1}^{\otimes} a^{\dagger}(\boldsymbol{\eta}_j) \Omega_j \right) \otimes \left( \prod_{j \in J''_0}^{\otimes} \Omega_j \right) \otimes \left( \prod_{i \in I'}^{\otimes} (\alpha_i - \beta_i a^{\dagger}(\boldsymbol{\eta}_{2i}) a^{\dagger}(\boldsymbol{\eta}_{2i-1})) \Omega_{2i, 2i-1} \right), \end{aligned} \quad (127)$$

with  $\Omega_{2i, 2i-1}, \Omega_{2i, 2i-1, \nu} \in \widehat{\mathcal{H}}_{k(i)}$ . Then,  $\mathcal{V}$  is implemented in the sense of (100) by  $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\mathcal{F}} \rightarrow \widehat{\mathcal{H}}$  (102).



*Proof.* Again, by Lemma 5.2, it suffices to establish the four points in the proof of Theorem 5.1.

1.) Well-definedness of  $\Omega_{\mathcal{V}}$ : As in Theorem 5.1, we have a tensor product of infinitely many normalized vectors.

2.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}$ : follows from Lemma 5.1, as for fermions,  $\mathcal{S}^{\otimes} = \widehat{\mathcal{H}}$ .

3.)  $b(\phi)$  annihilates  $\Omega_{\mathcal{V}}$ : Follows by a similar argument, as in proof step 3.) of Theorem 5.1. We only have to check for  $i \in I'$ , that both  $b(\mathbf{f}_{2i})$  and  $b(\mathbf{f}_{2i-1})$  annihilate  $\Omega_{2i,2i-1,\mathcal{V}}$ , which is done exactly as in the case where the Shale–Stinespring condition holds.

4.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}^{-1}$ : We proceed as in proof step 4.) in Theorem 5.1. That means, we have  $\ker(\mathbb{U}_{\mathcal{V}}) = \{0\}$  and hence existence of an inverse, if we can prove that the set

$$\{b^\dagger(\mathbf{f}_{j_1}) \dots b^\dagger(\mathbf{f}_{j_N}) \Omega_{\mathcal{V}} \mid N \in \mathbb{N}_0, j_\ell \in J\} \subset \widehat{\mathcal{H}}, \quad (128)$$

with  $b^\dagger(\mathbf{f}_j) = a^\dagger(u\mathbf{f}_j) + a(v\overline{\mathbf{f}}_j)$  is linearly independent. This again boils down to proving a linear independence statement on each  $\mathcal{H}_k$ . The crucial difference now is, that each tensor product factor  $\mathcal{H}_k$  may be a Fock space over either one or two modes. We abbreviate  $b_j^\# := b^\#(\mathbf{f}_j)$  and  $a_j^\# := a^\#(\boldsymbol{\eta}_j)$ . For two-mode factors indexed by  $i \in I'$ , we need to prove linear independence of the set

$$\{(b_{2i}^\dagger)^{N_1} (b_{2i-1}^\dagger)^{N_2} \Omega_{k(i),\mathcal{V}} \mid N_1, N_2 \in \{0, 1\}\} \subset \mathcal{H}_{k(i)}. \quad (129)$$

As in proof step 4.) of Theorem 5.1, this follows by the fact that the finite-mode implementer  $\mathbb{U}_{2i,2i-1,\mathcal{V}}$  (see (57)) is unitary and maps the orthogonal, zero-free set

$$\{(a_{2i}^\dagger)^{N_1} (a_{2i-1}^\dagger)^{N_2} \Omega_{k(i)} \mid N_1, N_2 \in \{0, 1\}\} \subset \mathcal{F}(\{\mathbf{f}_{2i}\}) \otimes \mathcal{F}(\{\mathbf{f}_{2i-1}\}), \quad (130)$$

onto (129).

For one-mode factors indexed by  $j \in J''$  we need linear independence of

$$\{(b_j^\dagger)^N \Omega_{k(j),\mathcal{V}} \mid N \in \{0, 1\}\} \subset \mathcal{H}_{k(j)}. \quad (131)$$

This follows again by unitarity of  $\mathbb{U}_{j,\mathcal{V}}$ , as well as orthogonality and zero-freeness of the set

$$\{(a_j^\dagger)^N \Omega_{k(j)} \mid N \in \{0, 1\}\} \subset \mathcal{F}(\{\mathbf{f}_j\}), \quad (132)$$

which is mapped to (131). By linear independence of (129) and (131), we obtain linear independence of (128), which implies injectivity of  $\mathbb{U}_{\mathcal{V}}$  and finishes the proof.  $\square$

For the ESS, the situation is a bit more delicate: As in Theorem 5.2, we would again like to normalize the vacuum sector of each mode to  $\Psi_{j,\mathcal{V}}(0) = 1$  (compare (117)). However, for full particle–hole transformations  $j \in J_1''$ , we have  $\Psi_{j,\mathcal{V}}(0) = 0$ , so we cannot

normalize the vacuum. As the ESS can only cover a finite number of particles within a configuration  $q \in \mathcal{Q}(\mathbb{N})$ , we also need to restrict to a **finite number of particle–hole transformations**  $|J_1''| < \infty$ .

**Theorem 5.4** (Implementation works, fermionic, ESS). *Consider a fermionic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v^* & u \end{pmatrix}$  with  $v^*v$  having countable spectrum. Let  $\overline{\mathcal{F}}$  be the ESS over  $\mathcal{E}_\eta$  (see (92)) with respect to  $(\eta_j)_{j \in J}$ , and let  $|J_1''| < \infty$ , so **the number of modes with a full particle–hole transformation is finite**. Define the new vacuum vector*

$$\Omega_{\mathcal{V}} = \underbrace{\exp \left( \sum_{i \in I'} \log \alpha_i \right)}_{=: e^{\mathfrak{r}}} \underbrace{\left( \prod_{j \in J_1''}^{\otimes} a^\dagger(\eta_j) \right) \left( \prod_{i \in I'}^{\otimes} \left( 1 - \frac{\beta_i}{\alpha_i} a^\dagger(\eta_{2i}) a^\dagger(\eta_{2i-1}) \right) \right)}_{=: \Psi_{\mathcal{V}}} \Omega = e^{\mathfrak{r}} \Psi_{\mathcal{V}}. \quad (133)$$

Then,  $\mathcal{V}$  is implemented in the sense of (100) by  $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\overline{\mathcal{F}}} \rightarrow \overline{\mathcal{F}}$  (102).

*Proof.* Also in this case, by Lemma 5.2, it suffices to establish the four points from the proof of Theorem 5.1.

1.) Well–definedness of  $\Omega_{\mathcal{V}}$ : Since  $\alpha_i \in (0, 1)$ , we have  $\log \alpha_i \in \mathbb{R}$ . By setting  $\mathfrak{r}$  to be the equivalence class of the function  $r : J \rightarrow \mathbb{C}$ ,

$$r(j) = \begin{cases} \frac{1}{2} \log \alpha_j & \text{if } j \in J' \\ 0 & \text{if } j \in J'' \end{cases}, \quad (134)$$

we obtain  $\mathfrak{r} = \sum_{i \in I'} \log \alpha_i \in \text{Ren}_1$ , so  $e^{\mathfrak{r}} \in \text{eRen}$ .

The well–definedness argument for  $\Psi_{\mathcal{V}} \in \mathcal{E}_{\overline{\mathcal{F}}}$  is similar to that in proof step 1.) of Theorem 5.2. We can write

$$\Psi_{\mathcal{V}}(q) = \prod_{k \in \mathbb{N}} \Psi_{k, \mathcal{V}}(n_k), \quad (135)$$

where the occupation number  $n_k$  denotes, how often mode  $k$  appears in configuration  $q$ . We may allow for  $n_k \geq 2$ , but since we are in the fermionic case,  $\Psi_{k, \mathcal{V}}(n_k) = 0$  for  $n_k \notin \{0, 1\}$ . Further, for all  $i \in I'$ , the associated vector  $\Psi_{k(i), \mathcal{V}}$  describes two modes, so we have a pair of two occupation numbers  $(n_{2i}, n_{2i-1}) = n_{k(i)}$ , and

- for  $j \in J_0''$  :  $\Psi_{k(j), \mathcal{V}}(n_{k(j)}) = 1$  if  $n_{k(j)} = 0$  and else  $\Psi_{k(j), \mathcal{V}}(n_{k(j)}) = 0$
- for  $j \in J_1''$  :  $\Psi_{k(j), \mathcal{V}}(n_{k(j)}) = 1$  if  $n_{k(j)} = 1$  and else  $\Psi_{k(j), \mathcal{V}}(n_{k(j)}) = 0$
- for  $i \in I'$  :  $\Psi_{k(i), \mathcal{V}}(n_{k(i)}) = 1$  if  $n_{k(i)} = (0, 0)$

$$\Psi_{k(i), \mathcal{V}}(n_{k(i)}) = -\frac{\beta_i}{\alpha_i} \text{ if } n_{k(i)} = (1, 1) \quad \text{and else } \Psi_{k(i), \mathcal{V}}(n_{k(i)}) = 0. \quad (136)$$

So  $\Psi_{\mathcal{V}}(q)$  can only be nonzero if  $q$  contains none of the modes  $j \in J_0''$ , all of the (finitely many) modes  $j \in J_1''$  and a finite subset of the modes  $2i, 2i - 1$  for  $i \in I'$ . In that case,  $\Psi_{\mathcal{V}}(q)$  is a product of infinitely many times a factor 1 and finitely many times a finite factor, so all  $\Psi_{\mathcal{V}}(q)$  are well-defined and finite.

2.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}$ : This is an immediate consequence of Lemma 5.1.

3.)  $b(\phi)$  annihilates  $\Omega_{\mathcal{V}}$ : This is proven mode-by-mode in the same way, as within proof step 3.) of Theorem 5.3.

4.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}^{-1}$ : The argument is also analogous to that in proof step 4.) of Theorem 5.2. We reduce invertibility (which means injectivity) of  $\mathbb{U}_{\mathcal{V}}$  to linear independence of the set

$$\{b^\dagger(\mathbf{f}_{j_1}) \dots b^\dagger(\mathbf{f}_{j_N}) \Psi_{\mathcal{V}} \mid N \in \mathbb{N}_0, j_\ell \in J\} \subset \overline{\mathcal{F}}. \quad (137)$$

Again, we assume there was a (finite) nonzero linear combination

$$B = \sum_{m=1}^M b^\dagger(\mathbf{f}_{j_{m,1}}) \dots b^\dagger(\mathbf{f}_{j_{m,N_m}}), \quad \text{with } \Psi_B = B\Psi_{\mathcal{V}} = 0. \quad (138)$$

Then,  $B$  would possess a maximum occupied mode  $K$ , i.e., in  $B$ , there is no  $\mathbf{f}_{j_{m,N}}$  with  $k(j_{m,N}) > K$ . We denote the (finite-dimensional) fermionic Fock space and the associated Bogoliubov transformation by

$$\mathcal{F}_{\leq K} = \prod_{k \leq K}^{\otimes} \mathcal{H}_k, \quad \mathbb{U}_{\leq K, \mathcal{V}} = \left( \prod_{k(j) \leq K}^{\otimes} \mathbb{U}_{j, \mathcal{V}} \right) \otimes \left( \prod_{k(i) \leq K}^{\otimes} \mathbb{U}_{2i, 2i-1, \mathcal{V}} \right). \quad (139)$$

We restrict  $B$  to  $\tilde{B} : \mathcal{F}_{\leq K} \rightarrow \mathcal{F}_{\leq K}$ , as in (124), define  $\Psi_{\tilde{B}} = \tilde{B}\Psi_{\leq K, \mathcal{V}}$  and obtain

$$\Psi_B(q) = \left( \prod_{k > K} \Psi_{k, \mathcal{V}}(n_k) \right) \Psi_{\tilde{B}}(q_{\leq K}). \quad (140)$$

The first factor is nonzero for at least one choice of occupation numbers  $n_k$  per  $k > K$ , as  $\Psi_{\mathcal{V}} \neq 0$ . The factor  $\Psi_{\tilde{B}}$  is the image of a nonzero vector under the operator  $\mathbb{U}_{\leq K, \mathcal{V}}$ , which is unitary, as it acts on finitely many modes. So  $\Psi_{\tilde{B}}(q_{\leq K}) \neq 0$  for some  $q_{\leq K}$ , which yields  $\Psi_B \neq 0$  and hence the desired contradiction.  $\square$

8. It is crucial that the fermionic ITP space has been chosen as  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k$ , with two-mode spaces  $\mathcal{H}_k = \mathcal{F}(\{\boldsymbol{\eta}_{2i}\}) \otimes \mathcal{F}(\{\boldsymbol{\eta}_{2i-1}\})$  for Cooper pairs  $i \in I'$ . If we had just chosen a product of one-mode spaces  $\prod_{j \in J}^{\otimes} \mathcal{F}(\{\boldsymbol{\eta}_j\})$ , then  $\Omega_{\mathcal{V}}$  might not be in this space, depending on  $\mathcal{V}$ .

As a counter-example, consider a Bogoliubov transformation  $\mathcal{V}$  with countably infinitely many Cooper pairs  $i \in I'$ , such that  $\alpha_i = \beta_i = \frac{1}{\sqrt{2}}$ . Then, each Cooper pair is in the state

$$\Psi_i := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \in \mathbb{C}^4, \quad (141)$$

i.e., we have a “half particle–hole transformation”. The state (141) cannot be written as a tensor product of two vectors in  $\mathbb{C}^2$ . So when evaluating the formal ITP

$$\Omega_{\mathcal{V}} = \prod_{i \in I'}^{\otimes} \Psi_i, \quad (142)$$

we obtain a sum of  $C$ –sequences: For each pair  $i$ , one has to choose either  $|0\rangle \otimes |0\rangle$  or  $|1\rangle \otimes |1\rangle$  as a contribution to  $\Omega_{\mathcal{V}}$  and sum over all choices. But now, there are uncountably many such choices, as each corresponds to a binary number of infinitely many digits. And each one gives a contribution of norm  $\prod_{i \in I'} \frac{1}{\sqrt{2}} = 0$ . So  $\Omega_{\mathcal{V}} = 0$ , making  $\mathbb{U}_{\mathcal{V}}$  non-invertible.

## 6 Diagonalization: Extended

By Theorems 5.1–5.4, we know that  $\mathcal{V}$  is implementable by  $\mathbb{U}_{\mathcal{V}}$  in the extended sense under fairly mild assumptions (namely  $v^*v$  having countable spectrum). Now it would be interesting to know if this  $\mathbb{U}_{\mathcal{V}}$  can be used to diagonalize quadratic Hamiltonians  $H$ . This requires precise definitions of “quadratic Hamiltonian” and “diagonalized”, first, which we give in Section 6.1. The following Sections 6.2 and 6.3 provide diagonalizability criteria in Propositions 6.1 and 6.2, which are a simple consequence of combining results from [3] about when a diagonalizing  $\mathcal{V}$  exists with results from Theorems 5.1–5.4 about when  $\mathbb{U}_{\mathcal{V}}$  is implementable in the extended sense.

### 6.1 Definition of Extended Diagonalization

Recall that the extended operator algebra  $\overline{\mathcal{A}}_e$ , defined in (75) with respect to a basis  $e = (e_j)_{j \in \mathbb{N}}$ , consists of all maps  $H$  that assign to each finite operator product  $a_{j_1}^{\#} \dots a_{j_m}^{\#}$  a complex coefficient  $H_{j_1, \dots, j_m} \in \mathbb{C}$ . Each map  $H$  can be interpreted as a (possibly infinite) sum

$$H = \sum_m \sum_{j_1, \dots, j_m} H_{j_1, \dots, j_m} a_{j_1}^{\#} \dots a_{j_m}^{\#}. \quad (143)$$

A **formal quadratic Hamiltonian** is an element  $H \in \overline{\mathcal{A}}_e$ , where  $H_{j_1, \dots, j_m} \neq 0$  only appears for  $m = 2$  and  $H^* = H$ . We will impose a normal ordering on quadratic

Hamiltonians (see Remark 9), so they can be written as:

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} (2h_{jk} a_j^\dagger a_k \mp k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k), \quad (144)$$

where for  $\mp$ , we have to take  $+$  in the bosonic and  $-$  in the fermionic case. The term “formal” stresses that  $H$  is not necessarily an operator on Fock space.

Now, to each such  $H$  we can associate a block matrix (see Appendix E.1)

$$A_H = \begin{pmatrix} h & \mp k \\ \overline{k} & \mp \overline{h} \end{pmatrix}, \quad (145)$$

with  $h = (h_{jk})_{j,k \in \mathbb{N}}$ ,  $k = (k_{jk})_{j,k \in \mathbb{N}}$  being matrices of infinite size.

Consider a Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix}$ . The corresponding algebraic Bogoliubov transformation  $\mathcal{V}_{\overline{\mathcal{A}}}$  on  $\overline{\mathcal{A}}_e$  is defined similar to  $\mathcal{V}_{\mathcal{A}}$  on  $\mathcal{A}$  (see Appendix B.1):

$$\begin{aligned} \mathcal{V}_{\overline{\mathcal{A}}} : \overline{\mathcal{A}}_e \supseteq \text{dom}(\mathcal{V}_{\overline{\mathcal{A}}}) &\rightarrow \overline{\mathcal{A}}_e, \\ a_j^\dagger &\mapsto b_j^\dagger = \sum_k (u_{jk} a_k^\dagger + \overline{v_{jk}} a_k), \\ a_j &\mapsto b_j = \sum_k (v_{jk} a_k^\dagger + \overline{u_{jk}} a_k), \end{aligned} \quad (146)$$

and a **normal ordering** is performed after the transformation. Note that this expression might not be well-defined, as the sums over  $k$  might diverge. So we needed to restrict to a subspace  $\text{dom}(\mathcal{V}_{\overline{\mathcal{A}}}) \subseteq \overline{\mathcal{A}}_e$ . The transformed operator and its associated block matrices are

$$\tilde{H} = \mathcal{V}_{\overline{\mathcal{A}}}(H), \quad A_{\tilde{H}} = \mathcal{V}^* A_H \mathcal{V}, \quad (147)$$

where the latter can easily be seen by (260). A diagonalization is now given by an implementable Bogoliubov transformation  $\mathcal{V}$ , which eliminates all  $a^\dagger a^\dagger$ - and  $aa$ -terms from  $H$ :

**Definition 6.1.** A formal quadratic Hamiltonian  $H \in \overline{\mathcal{A}}_e$  is called **diagonalizable in the extended sense** if there exists a Bogoliubov transformation  $\mathcal{V}$ , such that

$$\mathcal{V}^* A_H \mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & \mp E \end{pmatrix}, \quad (148)$$

with  $E \geq 0$  being Hermitian,  $\mp$  being  $+$  in the bosonic and  $-$  in the fermionic case, and where  $\mathcal{V}$  is implementable in the extended sense (see Definition 5.1).

That means, the Hamiltonian associated with  $A_{\tilde{H}}$  is:

$$\tilde{H} = \sum_{j,k \in \mathbb{N}} E_{jk} a_j^\dagger a_k = d\Gamma(E). \quad (149)$$

The matrix  $E$  provides a well-defined positive semidefinite quadratic form on  $\mathcal{D}_e$ . So by Friedrichs' theorem, it has at least one self-adjoint extension on  $\text{dom}(E)$ , which we also denote by  $E$ . Following [31, Sect. VIII.10], the operator  $d\Gamma(E)$  is essentially self-adjoint on

$$\bigoplus_{n=0}^{\infty} \text{dom}(E)^{\otimes n} \subseteq \mathcal{F}, \quad (150)$$

so  $\tilde{H}$  defines quantum dynamics on  $\mathcal{F}$ .

9. *Normal ordering constant.* Our process of “diagonalizing” a Hamiltonian  $H$  actually consists of conjugating it with  $\mathbb{U}_{\mathcal{V}}$ , so  $a^\#$  is replaced by  $b^\#$ , *plus a subsequent normal ordering process.* This process is equivalent to adding a constant to the Hamiltonian, namely

$$c = \frac{1}{2} (\text{tr}(E) - \text{tr}(h)) = \frac{1}{2} \sum_j (E_{jj} - h_{jj}). \quad (151)$$

The sum might be divergent and hence not a complex number, but we may interpret it as an infinite renormalization constant  $c \in \text{Ren}_1(\mathbb{N})$  (see Section 2.2), namely the one associated with the equally denoted function  $c \in \mathcal{E}(\mathbb{N}), c(j) = \frac{1}{2}(E_{jj} - h_{jj})$ .

So the relation between the original Hamiltonian  $H$  and the diagonalized Hamiltonian  $\tilde{H}$  actually is

$$\tilde{H} = \mathbb{U}_{\mathcal{V}}^{-1}(H + c)\mathbb{U}_{\mathcal{V}},$$

which is in accordance with (2).

## 6.2 Bosonic Case

Conditions for the existence of a  $\mathcal{V}$ , such that  $\mathcal{V}^* A_H \mathcal{V}$  is block-diagonal, can be found in [3, Thms. 1 and 4]. We can use them to derive conditions for when a formal quadratic Hamiltonian  $H$  is diagonalizable in the extended sense:

**Proposition 6.1** (Extended diagonalizability, bosonic case). *Let a formal quadratic bosonic Hamiltonian  $H$  (144) be given such that for the associated block matrix  $A_H$  (145) we have  $h > 0$ , and that  $G = h^{-1/2} k h^{-1/2}$  is a bounded operator with  $\|G\| < 1$ .*

Following [3, Thm. 1], there exists a bosonic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$  such that

$$\mathcal{V}^* A_H \mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}. \quad (152)$$

Suppose further that  $v^*v$  has countable spectrum.

Then,  $H$  is diagonalizable in the extended sense, both on the ITP space  $\widehat{\mathcal{H}}$  and on the ESS  $\overline{\mathcal{F}}$ .

*Proof.* Consider Definition 6.1 for diagonalizability. The existence of  $\mathcal{V}$  as a block matrix associated with a bounded operator on  $\ell^2$  is a direct consequence of [3, Thm. 1], where the representation of  $\mathcal{V}$  and  $A_H$  on  $\mathfrak{h} \oplus \mathfrak{h}^*$  (see Appendix B.3) instead of our representation on  $\ell^2 \oplus \ell^2$  (see Appendix B.4) has been used. The same theorem also yields  $E > 0 \Rightarrow E \geq 0$ . Note that the matrix entries  $k_{jk}$  in our representation on  $\ell^2 \oplus \ell^2$  agree with the matrix elements in the representation on  $\mathfrak{h} \oplus \mathfrak{h}^*$ , which are  $\langle J e_j, k e_k \rangle = \langle e_j, J^* k e_k \rangle$ . So  $J^* k$  in [3] corresponds to  $k$  in our case.

If the spectrum of  $v^*v$  is countable, then implementability of  $\mathcal{V}$  in the extended sense follows from Theorems 5.1 (for  $\widehat{\mathcal{H}}$ ) and 5.2 (for  $\overline{\mathcal{F}}$ ). □

### 6.3 Fermionic Case

**Proposition 6.2** (Extended diagonalizability, fermionic case). *Let a formal quadratic fermionic Hamiltonian  $H$  (144) be given such that for the associated block matrix  $A_H$  (145),  $\dim \text{Ker}(A_H)$  is even or  $\infty$ . Following [3, Thm. 4], there exists some fermionic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$  such that*

$$\mathcal{V}^* A_H \mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}. \quad (153)$$

Suppose further that  $v^*v$  has countable spectrum.

Then,  $H$  is diagonalizable in the extended sense on the ITP space  $\widehat{\mathcal{H}}$ .

If 1 is not an eigenvalue of  $v^*v$ , or is an eigenvalue of finite multiplicity, then  $H$  is also diagonalizable in the extended sense on the ESS  $\overline{\mathcal{F}}$ .

*Proof.* Existence of a unitary  $\mathcal{V}$  and of  $E \geq 0$  follows from [3, Thm. 4]. By unitarity,  $\mathcal{V}^* \mathcal{V} = 1 = \mathcal{V} \mathcal{V}^*$ , so  $\mathcal{V}$  is a fermionic Bogoliubov transformation.

If  $\sigma(v^*v)$  is countable, then implementability on  $\widehat{\mathcal{H}}$  follows from Theorem 5.3.

If further, 1 is not an eigenvalue of  $v^*v$  with infinite multiplicity, then implementability on  $\overline{\mathcal{F}}$  follows from Theorem 5.4. □

## 7 Applications

### 7.1 Quadratic Bosonic Interaction

Our first example for a quadratic Hamiltonian whose diagonalization requires Bogoliubov transformations “beyond the Shale–Stinespring condition” is inspired by [13]. We consider a free massive bosonic scalar field, which is interacting by a Wick square  $:\phi(\mathbf{x})^2:$ ,  $\phi(\mathbf{x}) = a^\dagger(\mathbf{x}) + a(\mathbf{x})$ . We discretize the momentum by putting the system in a box  $\mathbf{x} \in [-\pi, \pi]^3$  with periodic boundary conditions. Further, the Wick square is weighted by a real-valued external field  $\kappa \in C_c^\infty([-\pi, \pi]^3)$ ,  $\kappa(\mathbf{x}) \in \mathbb{C}$ . The Hamiltonian then reads

$$\begin{aligned} H &= d\Gamma(\varepsilon_{\mathbf{p}}) + \frac{1}{2} \int \kappa(\mathbf{x}) : \phi(\mathbf{x})^2 : d\mathbf{x} \\ &= \frac{1}{2} \sum_{\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{Z}^3} (2\varepsilon_{\mathbf{p}_1} \delta(\mathbf{p}_1 - \mathbf{p}_2) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} + 2\hat{\kappa}(-\mathbf{p}_1 + \mathbf{p}_2) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} + \\ &\quad + \hat{\kappa}(-\mathbf{p}_1 - \mathbf{p}_2) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger + \hat{\kappa}(\mathbf{p}_1 + \mathbf{p}_2) a_{\mathbf{p}_1} a_{\mathbf{p}_2}), \end{aligned} \quad (154)$$

with  $\hat{\kappa}(\mathbf{p}) = \overline{\hat{\kappa}(-\mathbf{p})}$  denoting the Fourier transform of  $\kappa(\mathbf{x})$ . For simplicity, we assume that  $\kappa(\mathbf{x}) = \text{const.}$ , so we can write  $\hat{\kappa}(\mathbf{p}) = \kappa \delta(\mathbf{p})$ ,  $\kappa \in \mathbb{R}$ .

**Proposition 7.1.** *For interactions  $\kappa > -\frac{m}{2}$  but  $\kappa \neq 0$ , the Hamiltonian  $H$  is diagonalizable in the extended sense both on  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ . However, the transformation  $\mathcal{V}$  violates the Shale–Stinespring condition, so  $H$  is not diagonalizable on  $\mathcal{F}$ .*

*Proof.* We may directly compute  $\mathcal{V}$  and then apply Proposition 6.1. As demonstrated in Appendix E.1,  $H$  can be identified with the block matrix

$$A_H = \begin{pmatrix} h & k \\ \bar{k} & \bar{h} \end{pmatrix}, \quad (155)$$

with entries

$$h_{\mathbf{p}_1, \mathbf{p}_2} = (\varepsilon_{\mathbf{p}_1} + \kappa) \delta(\mathbf{p}_1 - \mathbf{p}_2), \quad k_{\mathbf{p}_1, \mathbf{p}_2} = \kappa \delta(\mathbf{p}_1 + \mathbf{p}_2). \quad (156)$$

The diagonalization of  $H$  now translates into diagonalizing  $A_H$  by some bosonic Bogoliubov transformation  $\mathcal{V}$ , such that

$$\mathcal{V}^* A_H \mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \quad E_{\mathbf{p}_1, \mathbf{p}_2} = E_{\mathbf{p}_1} \delta(\mathbf{p}_1 - \mathbf{p}_2). \quad (157)$$

This can be done by decomposing the matrices  $A_H$  into modes  $\mathbf{p}$  as

$$A_H = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} A_{H, \mathbf{p}}, \quad A_{H, \mathbf{p}} = \begin{pmatrix} h_{\mathbf{p}} & k_{\mathbf{p}} \\ k_{\mathbf{p}} & h_{\mathbf{p}} \end{pmatrix}, \quad (158)$$



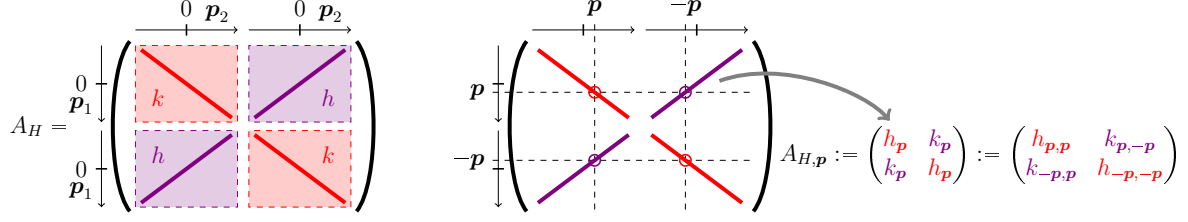


Figure 5: Decomposing the matrix  $A_H$  into modes. Color online.

(see Figure 5) with entries

$$h_{\mathbf{p}} = (\varepsilon_{\mathbf{p}} + \kappa), \quad k_{\mathbf{p}} = \kappa, \quad (159)$$

and diagonalizing all  $A_{H,\mathbf{p}} \in \mathbb{C}^{2 \times 2}$  separately via

$$\mathcal{V} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} \mathcal{V}_{\mathbf{p}}, \quad \mathcal{V}_{\mathbf{p}}^* A_{H,\mathbf{p}} \mathcal{V}_{\mathbf{p}} = \begin{pmatrix} E_{\mathbf{p}} & 0 \\ 0 & E_{\mathbf{p}} \end{pmatrix}. \quad (160)$$

Following [3, Subsec. 1.3], this is done for  $|k_{\mathbf{p}}| < h_{\mathbf{p}}$  by

$$\mathcal{V}_{\mathbf{p}} = \begin{pmatrix} u_{\mathbf{p}} & v_{\mathbf{p}} \\ v_{\mathbf{p}} & u_{\mathbf{p}} \end{pmatrix}, \quad u_{\mathbf{p}} = c_{\mathbf{p}}, \quad v_{\mathbf{p}} = c_{\mathbf{p}} \frac{-G_{\mathbf{p}}}{1 + \sqrt{1 - G_{\mathbf{p}}^2}}, \quad (161)$$

$$G_{\mathbf{p}} = k_{\mathbf{p}} h_{\mathbf{p}}^{-1}, \quad c_{\mathbf{p}} = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{1 - G_{\mathbf{p}}^2}}}, \quad (162)$$

with eigenvalues  $E_{\mathbf{p}} = \sqrt{h^2 - k^2}$ . The condition  $|k_{\mathbf{p}}| < h_{\mathbf{p}}$  amounts to

$$|k_{\mathbf{p}}| < h_{\mathbf{p}} \quad \Leftrightarrow \quad |\kappa| < \sqrt{|\mathbf{p}|^2 + m^2} + \kappa, \quad (163)$$

which is satisfied for all  $\mathbf{p} \in \mathbb{Z}^3$ , if and only if  $\kappa > -\frac{m}{2}$ .  $h_{\mathbf{p}} > 0$  also holds in that case and (160) defines a Bogoliubov transformation  $\mathcal{V}$  diagonalizing  $A_H$ .

Out of the conditions in Proposition 6.1,  $h > 0$  and  $\|h^{-1/2} k h^{-1/2}\| < 1$  follow from  $h_{\mathbf{p}}$  and  $|k_{\mathbf{p}}| < h_{\mathbf{p}}$  after taking a direct sum. For the spectrum of  $v^* v$ , since  $v = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} v_{\mathbf{p}}$  can be decomposed into modes, the same holds for  $v^* v$ , which has therefore countable spectrum. So Proposition 6.1 applies and  $H$  is diagonalizable in the extended sense on both  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ .

For the second claim, we have to show that

$$\text{tr}(v^* v) = \sum_{\mathbf{p} \in \mathbb{Z}^3} |v_{\mathbf{p}}|^2 = \infty. \quad (164)$$

If  $|\mathbf{p}|$  is large enough (say,  $|\mathbf{p}| > p_{\max} > 0$ ), we have the bounds

$$G_{\mathbf{p}} = \frac{\kappa}{\kappa + \sqrt{|\mathbf{p}|^2 + m^2}} \leq \frac{\kappa}{|\mathbf{p}|}, \quad G_{\mathbf{p}} \geq \frac{\kappa d}{|\mathbf{p}|}, \quad (165)$$

with  $d < 1$  arbitrary, so

$$|v_{\mathbf{p}}|^2 = \frac{1 + \sqrt{1 - G_{\mathbf{p}}^2}}{2\sqrt{1 - G_{\mathbf{p}}^2}} \frac{G_{\mathbf{p}}^2}{(1 + \sqrt{1 - G_{\mathbf{p}}^2})^2} \geq \frac{\kappa^2 d^2}{4|\mathbf{p}|^2}. \quad (166)$$

Now, we can write the sum in (164) as an integral over a weighted sum of indicator functions  $\chi_{Q(\mathbf{p})}(\cdot)$  with  $Q(\mathbf{p})$  being half-open cubes with side length 1 centered at  $\mathbf{p} = (p_1, p_2, p_3)$ :

$$Q(\mathbf{p}) = [p_1 - \frac{1}{2}, p_1 + \frac{1}{2}) \times [p_2 - \frac{1}{2}, p_2 + \frac{1}{2}) \times [p_3 - \frac{1}{2}, p_3 + \frac{1}{2}), \quad (167)$$

$$f(\mathbf{p}') = \sum_{\mathbf{p} \in \mathbb{Z}^3} |v_{\mathbf{p}}|^2 \chi_{Q(\mathbf{p})}(\mathbf{p}'), \quad \sum_{\mathbf{p} \in \mathbb{Z}^3} |v_{\mathbf{p}}|^2 = \int_{\mathbb{R}^3} f(\mathbf{p}') d\mathbf{p}'. \quad (168)$$

For  $|\mathbf{p}'| > p_{\max}$ , we have

$$f(\mathbf{p}') \geq \frac{\kappa^2 d^2}{4(|\mathbf{p}'| + \frac{\sqrt{3}}{2})^2}, \quad (169)$$

$$\Rightarrow \sum_{\mathbf{p} \in \mathbb{Z}^3} |v_{\mathbf{p}}|^2 \geq \int_{|\mathbf{p}'| > p_{\max}} f(\mathbf{p}') d\mathbf{p}' \geq \int_{p_{\max}}^{\infty} \frac{\kappa^2 d^2}{4(|\mathbf{p}'| + \frac{\sqrt{3}}{2})^2} 4\pi |\mathbf{p}'|^2 d|\mathbf{p}'| = \infty, \quad (170)$$

where the integral is linearly divergent, which establishes the claim that the Shale–Stinespring condition is violated. □

10. *Infinite volume and continuous  $\mathbf{p}$ .* The original Hamiltonian  $H$  in [13] is not restricted to a finite volume. That means,  $\mathbf{p} \in \mathbb{R}^3$  instead of  $\mathbf{p} \in \mathbb{Z}^3$  is considered. In that case, a decomposition into modes is also possible and even renders a  $\mathcal{V}$  diagonalizing  $A_H$ . However, the spectrum of  $v^*v$  is then no longer countable, so Proposition 7.1 no longer applies. This can be seen as  $k_{\mathbf{p}}$  is the same for all  $\mathbf{p}$ , but  $h_{\mathbf{p}}$  attains all values in  $[\kappa + m, \infty)$ , i.e., uncountably many of them. So  $v_{\mathbf{p}}$  also attains uncountably many values.

In order to treat this case, it would be necessary to extend Theorems 5.1 and 5.2 to generic  $v^*v$ . If this can be done, a proof of implementability for  $\mathcal{V}$  will be straightforward.

11. *Position-dependent  $\kappa(\mathbf{x})$ .* In contrast to [13], we also assumed an interaction strength  $\kappa(\mathbf{x})$  that is constant in the cube  $[-\pi, \pi]^3$ . Physically, it would be desirable to treat any  $\kappa \in C_c^\infty$ . However, in that case, we would no longer have a simple

decomposition into modes  $A_H = \bigoplus_{\mathbf{p} \in \mathbb{Z}} A_{H,\mathbf{p}}$ , so the diagonalization can no longer be done explicitly. Further, it might occur that  $v^*v$  has uncountable spectrum, so Proposition 7.1 may no longer apply.

However, if there was a version of Theorems 5.1 and 5.2 for generic  $v^*v$ , then we expect to obtain a diagonalizability result for small enough interactions (i.e.,  $k$  is a bounded operator with  $k < -m/2$ ).

12. *Wick square is not diagonalizable.* It may be tempting to set  $\varepsilon_{\mathbf{p}} = 0$  and to try a diagonalization of only the interaction Hamiltonian  $\frac{1}{2} \int \kappa(\mathbf{x}) : \phi(\mathbf{x}) : d\mathbf{x}$ . However, bosonic Wick squares are in general not diagonalizable by a Bogoliubov transformation, as “the interaction is too large”. For example, on one mode ( $\mathfrak{h} \cong \mathbb{C}$ ), the matrix associated with a Wick square  $H_W = 2a^\dagger a + a^\dagger a^\dagger + aa$  is

$$A_{H_W} = \begin{pmatrix} h & k \\ \bar{k} & \bar{h} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (171)$$

so  $\|h^{-1/2}kh^{-1/2}\| = 1$  and Proposition 6.1 does not apply. The same problem occurs when treating a direct sum of several independent modes. However, in the case of bosonic Wick products, an interpretation as a self-adjoint operator is possible by a suitable GNS construction [32].

## 7.2 BCS Model

Another example, where Bogoliubov transformations appear, which are not implementable on  $\mathcal{F}$ , is the Bardeen–Cooper–Schrieffer (BCS) model for explaining superconductivity [11, 12]. The “Hartree-like approximation” state [11, (2.16)] corresponds to a formal fermionic Bogoliubov vacuum state  $\Omega_V$  as in (127). A mathematical analysis by Haag [12] shows that in the infinite volume limit, the BCS Hamiltonian can indeed be diagonalized by a corresponding Bogoliubov transformation, which is not implementable on Fock space.

We consider a similar model of a fermionic gas inside a box with periodic boundary conditions  $\mathbf{x} \in [-\pi, \pi]^3$ . Hence, we have discretized, arbitrarily large momenta  $\mathbf{p} \in \mathbb{Z}^3$ , as well as two spins  $s \in \{\uparrow, \downarrow\}$ , leading to a one-particle Hilbert space  $\mathfrak{h} = L^2(\mathbb{Z}^3 \times \{\uparrow, \downarrow\})$ . The corresponding Fock space is  $\mathcal{F} = \mathcal{F}(\mathbb{Z}^3 \times \{\uparrow, \downarrow\})$ . We consider the following quadratic Hamiltonian (see [12]), which provides an approximate description for the fermionic gas that becomes exact in the infinite volume limit:

$$H' = H_0 + H_I' = \sum_{\mathbf{p} \in \mathbb{Z}^3} \left( \varepsilon_{\mathbf{p}} a_{\mathbf{p},\uparrow}^\dagger a_{\mathbf{p},\uparrow} + \varepsilon_{\mathbf{p}} a_{\mathbf{p},\downarrow}^\dagger a_{\mathbf{p},\downarrow} - \tilde{\Delta}_{\mathbf{p}} a_{\mathbf{p},\uparrow}^\dagger a_{\mathbf{p},\downarrow}^\dagger + \overline{\tilde{\Delta}_{\mathbf{p}}} a_{\mathbf{p},\uparrow} a_{\mathbf{p},\downarrow} \right), \quad (172)$$

with kinetic energy  $\varepsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \mu \in \mathbb{R}$  and interaction strength  $\tilde{\Delta}_{\mathbf{p}} \in \mathbb{C}$ , of which we assume  $\tilde{\Delta}_{\mathbf{p}} \neq 0$ . As a basis  $(e_j)_{j \in J}$  for identifying  $\mathfrak{h}$  with  $\ell^2$ , we choose

$$(e_{\mathbf{p},s})_{\substack{\mathbf{p} \in \mathbb{Z}^3 \\ s \in \{\uparrow, \downarrow\}}} \subset L^2(\mathbb{Z}^3 \times \{\uparrow, \downarrow\}), \quad e_{\mathbf{p},s}(\mathbf{p}', s') = \delta_{\mathbf{p}\mathbf{p}'} \delta_{ss'}, \quad (173)$$

with  $\delta$  being the Kronecker delta. The corresponding canonical basis of  $\ell^2$  is denoted  $(e_{\mathbf{p},s})_{\substack{\mathbf{p} \in \mathbb{Z}^3 \\ s \in \{\uparrow, \downarrow\}}}$  with a countable index set. In order to obtain momentum conservation, we

have to interpret  $a_{\mathbf{p},\downarrow}^\dagger, a_{\mathbf{p},\downarrow}$  as creating/annihilating a fermion of momentum  $-\mathbf{p}$ . The mode index  $\mathbf{p}$  is only used for an easier decomposition into modes.

**Proposition 7.2.** *The Hamiltonian  $H'$  (172) is diagonalizable in the extended sense both on  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ .*

*Proof.* We compute  $\mathcal{V}$  directly and apply Proposition 6.2. Following the identification in Appendix E.1, we can translate this Hamiltonian into a block matrix

$$A_{H'} = \begin{pmatrix} \varepsilon & -\tilde{\Delta} \\ \tilde{\Delta} & -\bar{\varepsilon} \end{pmatrix}, \quad (174)$$

with  $\varepsilon = \bar{\varepsilon}$  and  $\tilde{\Delta} = -\tilde{\Delta}^T$  being infinite-dimensional matrices.  $A_{H'}$  can be diagonalized by a Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$  mode-by-mode: We use the decompositions  $A_{H'} = \bigoplus_{\mathbf{p}} A_{H',\mathbf{p}}$  and  $\mathcal{V} = \bigoplus_{\mathbf{p}} \mathcal{V}_{\mathbf{p}}$  with

$$A_{H',\mathbf{p}} = \begin{pmatrix} \varepsilon_{\mathbf{p}} & 0 & 0 & -\tilde{\Delta}_{\mathbf{p}} \\ 0 & \varepsilon_{\mathbf{p}} & \tilde{\Delta}_{\mathbf{p}} & 0 \\ 0 & \tilde{\Delta}_{\mathbf{p}} & -\varepsilon_{\mathbf{p}} & 0 \\ -\tilde{\Delta}_{\mathbf{p}} & 0 & 0 & -\varepsilon_{\mathbf{p}} \end{pmatrix}, \quad \mathcal{V}_{\mathbf{p}} = \begin{pmatrix} u_{\mathbf{p}} & 0 & 0 & v_{\mathbf{p}} \\ 0 & u_{\mathbf{p}} & -v_{\mathbf{p}} & 0 \\ 0 & \bar{v}_{\mathbf{p}} & \bar{u}_{\mathbf{p}} & 0 \\ -\bar{v}_{\mathbf{p}} & 0 & 0 & \bar{u}_{\mathbf{p}} \end{pmatrix}, \quad (175)$$

with  $A_{H',\mathbf{p}}, \mathcal{V}_{\mathbf{p},s} \in \mathbb{C}^4 \otimes \mathbb{C}^4$ . The diagonalized matrix then reads

$$\mathcal{V}_{\mathbf{p}}^* A_{H',\mathbf{p}} \mathcal{V}_{\mathbf{p}} = \begin{pmatrix} E_{\mathbf{p}} & 0 & 0 & 0 \\ 0 & E_{\mathbf{p}} & 0 & 0 \\ 0 & 0 & -E_{\mathbf{p}} & 0 \\ 0 & 0 & 0 & -E_{\mathbf{p}} \end{pmatrix}, \quad E_{\mathbf{p}} = \sqrt{\varepsilon_{\mathbf{p}}^2 + |\tilde{\Delta}_{\mathbf{p}}|^2}, \quad (176)$$

and the diagonalization is established by

$$u_{\mathbf{p}} = \frac{\tilde{\Delta}_{\mathbf{p}}}{\sqrt{(E_{\mathbf{p}} - \varepsilon_{\mathbf{p}})^2 + |\tilde{\Delta}_{\mathbf{p}}|^2}}, \quad v_{\mathbf{p}} = \frac{E_{\mathbf{p}} - \varepsilon_{\mathbf{p}}}{\sqrt{(E_{\mathbf{p}} - \varepsilon_{\mathbf{p}})^2 + |\tilde{\Delta}_{\mathbf{p}}|^2}}. \quad (177)$$

From this form, it also follows that  $\dim \text{Ker}(A_H)$  is either even or  $\infty$ . So in order to apply Proposition 6.2, we only need to show that the spectrum of  $v^*v$  is countable. This

is the case, since  $v = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} v_{\mathbf{p}}$  decays into modes, so also  $v^*v = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} (v^*v)_{\mathbf{p}}$  decays into modes, where each  $(v^*v)_{\mathbf{p}}$  is a finite-dimensional matrix with finite spectrum. As the sum over  $\mathbf{p}$  is countable, also the spectrum of  $v^*v$  is countable, and by Proposition 6.2,  $H$  is diagonalizable on  $\widehat{\mathcal{H}}$ .

Now, since  $\tilde{\Delta}_{\mathbf{p}} \neq 0$ , we have only Cooper pairs labeled by  $\mathbf{p}$ , so in particular, there are no full particle-hole transformations. That means, 1 is not an eigenvalue of  $v^*v$  and by Proposition 6.2,  $H$  is also diagonalizable on  $\overline{\mathcal{F}}$ . □

Note that, for convenience, we can replace the pair index  $i \in I'$  by  $\mathbf{p} \in \mathbb{Z}^3$ . In particular,  $\tilde{\Delta}_{\mathbf{p}} > E_{\mathbf{p}} - \varepsilon_{\mathbf{p}}$  implies that  $u_{\mathbf{p}} > \frac{1}{\sqrt{2}}$ ,  $v_{\mathbf{p}} < \frac{1}{\sqrt{2}}$ , so we have “at most a half particle-hole transformation”.

### Remarks.

13. *Interaction-free case.* It is easy to see that, when relaxing the condition  $\tilde{\Delta}_{\mathbf{p}} \neq 0$ , then in each mode with  $\tilde{\Delta}_{\mathbf{p}} = 0$ , the matrix  $A_{H',\mathbf{p}}$  is already diagonal. So it is diagonalized by  $u_{\mathbf{p}} = 1, v_{\mathbf{p}} = 0$ , which does not generate further full particle-hole transformations. So the Bogoliubov transformation  $\mathcal{V}$  diagonalizing  $A_{H'}$  remains implementable on  $\prod_{\mathbf{p},s}^{\otimes} \mathcal{H}_{\mathbf{p},s}$  and  $\overline{\mathcal{F}}$  in this case.
14. *Infinite volume case.* The Hamiltonian  $H'$  is an approximation to  $H = H_0 + H_I$ , where  $H_I$  is an attractive quartic interaction between fermion pairs. As mentioned above, this approximation is only exact in the infinite volume limit. For infinite volumes, i.e.,  $\mathbf{p} \in \mathbb{R}^3$ , a decomposition into modes is also possible, so (175)–(177) still yield a Bogoliubov transformation  $\mathcal{V}$ , diagonalizing  $A_{H'}$ . However, as in the previous application, the spectrum of  $v^*v$  is generally uncountable in that case, so  $\mathcal{V}$  cannot be implemented by means of Theorems 5.3 and 5.4.

## 7.3 External Field QED

Similar cases, where a Bogoliubov transformation is not implementable, appear in the context of Dirac fields in external electromagnetic backgrounds. Again, we restrict to  $\mathbf{p} \in \mathbb{Z}^3$ . In Appendix E.2, we argue how to arrive at the simplified and time-dependent Hamiltonian:

$$H(t) = \sum_{\mathbf{p} \in \mathbb{Z}^3} (\varepsilon_{\mathbf{p},+}(t) a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \varepsilon_{\mathbf{p},-}(t) b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} - f_{\mathbf{p}}(t) a_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}^{\dagger} + f_{\mathbf{p}}(t) a_{\mathbf{p}} b_{\mathbf{p}}). \quad (178)$$

Here,  $b_{\mathbf{p}}^{\dagger}, b_{\mathbf{p}}$  are operators related to positrons (not to be confused with  $b^{\#} = \mathbb{U}_{\mathcal{V}} a^{\#} \mathbb{U}_{\mathcal{V}}^{-1}$ ) and  $a_{\mathbf{p}}^{\dagger}, a_{\mathbf{p}}$  are related to electrons. Further,  $\varepsilon_{\mathbf{p},\pm}(t) = \overline{\varepsilon_{\mathbf{p},\pm}(t)}$  is a kinetic term and

$f_{\mathbf{p}}(t) = \overline{f_{\mathbf{p}}(t)}$  is a time-dependent interaction. We assume both interactions to be continuous, bounded functions of time for each  $\mathbf{p}$ . As in the previous example,  $b_{\mathbf{p}}^\dagger, b_{\mathbf{p}}$  have to be interpreted as creating/annihilating a positron with momentum  $-\mathbf{p}$  for momentum conservation reasons. The index  $\mathbf{p}$  is again only chosen to simplify the decomposition into modes.

**Proposition 7.3.** *The Hamiltonian  $H(t)$  (178) is diagonalizable in the extended sense both on  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$  by a Bogoliubov transformation  $\mathcal{V}(t)$ .*

*Further, the Schrödinger dynamics  $\mathcal{U}(s, t) := \exp\left(-i \int_s^t \mathcal{V}(\tau) d\tau\right)$  exist for all  $s, t \in \mathbb{R}$  as a Bogoliubov transformation, which is implementable in the extended sense on  $\widehat{\mathcal{H}}$ .*

*Proof.* The matrix associated with  $H(t)$  is

$$A_H = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} A_{H, \mathbf{p}}, \quad A_{H, \mathbf{p}}(t) = \begin{pmatrix} \varepsilon_{\mathbf{p},+}(t) & 0 & 0 & -f_{\mathbf{p}}(t) \\ 0 & \varepsilon_{\mathbf{p},-}(t) & f_{\mathbf{p}}(t) & 0 \\ 0 & f_{\mathbf{p}}(t) & -\varepsilon_{\mathbf{p},+}(t) & 0 \\ -f_{\mathbf{p}}(t) & 0 & 0 & -\varepsilon_{\mathbf{p},-}(t) \end{pmatrix}, \quad (179)$$

which has the same structure as  $A_{H'}$  in Section 7.2, except for the existence of different kinetic terms. The diagonalization again boils down to diagonalizing the  $2 \times 2$  matrices

$$\tilde{A}_{H, \mathbf{p}, \pm}(t) = \begin{pmatrix} \varepsilon_{\mathbf{p}, \pm}(t) & \mp f_{\mathbf{p}}(t) \\ \mp f_{\mathbf{p}}(t) & -\varepsilon_{\mathbf{p}, \pm}(t) \end{pmatrix}, \quad (180)$$

with eigenvalues  $E_{\mathbf{p}, \pm}(t) = \sqrt{\varepsilon_{\mathbf{p}, \pm}(t)^2 + f_{\mathbf{p}}(t)^2}$ .

$A_{H, \mathbf{p}}(t)$  is diagonalized by a Bogoliubov transformation

$$\mathcal{V}(t) = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} \mathcal{V}_{\mathbf{p}}(t), \quad \mathcal{V}_{\mathbf{p}}(t) = \begin{pmatrix} u_{\mathbf{p},+}(t) & 0 & 0 & v_{\mathbf{p},+}(t) \\ 0 & u_{\mathbf{p},-}(t) & -v_{\mathbf{p},-}(t) & 0 \\ 0 & \bar{v}_{\mathbf{p},-}(t) & \bar{u}_{\mathbf{p},-}(t) & 0 \\ -\bar{v}_{\mathbf{p},+}(t) & 0 & 0 & \bar{u}_{\mathbf{p},+}(t) \end{pmatrix}, \quad (181)$$

with time-dependent coefficients

$$u_{\mathbf{p}, \pm}(t) = \frac{f_{\mathbf{p}}(t)}{\sqrt{(E_{\mathbf{p}, \pm}(t) - \varepsilon_{\mathbf{p}, \pm}(t))^2 + f_{\mathbf{p}}(t)^2}}, \quad v_{\mathbf{p}, \pm}(t) = \frac{E_{\mathbf{p}, \pm}(t) - \varepsilon_{\mathbf{p}, \pm}(t)}{\sqrt{(E_{\mathbf{p}, \pm}(t) - \varepsilon_{\mathbf{p}, \pm}(t))^2 + f_{\mathbf{p}}(t)^2}}. \quad (182)$$

It becomes again apparent from the form of  $\mathcal{V}^* A_H \mathcal{V}$ , that  $\dim \text{Ker}(A_H)$  is even or  $\infty$ , and by the same arguments as in the proof of Proposition 7.2, the operator  $v^* v$  has countable spectrum, which does not include 1. So by Proposition 6.2,  $H(t)$  is diagonalizable on both  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ .

Concerning the second claim, it is well-known [28, 33] and has also been argued in [10, II 2.4] that the Schrödinger dynamics generated by finite-dimensional matrices of the form  $A_{H,\mathbf{p}}$  are given by a Bogoliubov transformation

$$\mathcal{U}_{\mathbf{p}}(s, t) = \begin{pmatrix} U_{\mathbf{p}}(s, t) & V_{\mathbf{p}}(s, t) \\ \bar{V}_{\mathbf{p}}(s, t) & \bar{U}_{\mathbf{p}}(s, t) \end{pmatrix} := \exp \left( -i \int_s^t A_{H,\mathbf{p}}(\tau) d\tau \right). \quad (183)$$

The integral exists for finite times by the continuity and boundedness assumption on  $\varepsilon_{\mathbf{p},\pm}(t)$  and  $f_{\mathbf{p}}(t)$ . Now, the dynamics generated by  $A_H$  can easily be reconstructed from  $\mathcal{U}_{\mathbf{p}}(s, t)$  via:

$$\mathcal{U}(s, t) := \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} \mathcal{U}_{\mathbf{p}}(s, t), \quad \mathcal{U}(s, t) = \begin{pmatrix} U(s, t) & V(s, t) \\ \bar{V}(s, t) & \bar{U}(s, t) \end{pmatrix}. \quad (184)$$

The transformation  $\mathcal{U}_{\mathbf{p}}(s, t)$  acts on a finite number of modes (namely two) and are hence always implementable on the two-mode Fock space. As  $V = V(s, t)$  decays into countably many modes, so does  $V^*V$ . Therefore,  $V^*V$  has a countable spectrum and by Theorems 5.3 and 5.4,  $\mathcal{U}(s, t)$  is implementable on  $\widehat{\mathcal{H}}$ .  $\square$

Note that for a proof of implementability on  $\overline{\mathcal{F}}$ , we would need that there are at most finitely many particle-hole transformations, which cannot be guaranteed for all time intervals  $(s, t)$ .

### Remarks.

15. *Comparison with [10].* The toy model for external field QED in [10, II 2.4] is a similar one, where the matrix  $A_H$  (180) plays the role of the full Hamiltonian, i.e., the system is considered before second quantization. There,  $\mathbf{p} \in \mathbb{Z}^3$  is replaced by a mode index  $n \in \mathbb{N}^3$  and  $f_{\mathbf{p}}(t)$  decays like  $\frac{f(t)}{n}$ ,  $\varepsilon_{\mathbf{p},\pm}$  is chosen independent of time.
16. *Implementability on Fock space.* To check the Shale–Stinespring condition, note that since we can decompose  $A_{H,\mathbf{p}} = \tilde{A}_{H,\mathbf{p},+} \oplus \tilde{A}_{H,\mathbf{p},-}$ , it is also possible to decompose  $\mathcal{U}_{\mathbf{p}}(s, t) := \tilde{\mathcal{U}}_{\mathbf{p},+}(s, t) \oplus \tilde{\mathcal{U}}_{\mathbf{p},-}(s, t)$ . So  $\mathcal{U}_{\mathbf{p}}(s, t)$  takes the following form:

$$\mathcal{U}_{\mathbf{p}}(s, t) = \begin{pmatrix} U_{\mathbf{p},1}(s, t) & & & V_{\mathbf{p},1}(s, t) \\ & U_{\mathbf{p},2}(s, t) & V_{\mathbf{p},2}(s, t) & \\ & \bar{V}_{\mathbf{p},1}(s, t) & \bar{U}_{\mathbf{p},1}(s, t) & \\ \bar{V}_{\mathbf{p},2}(s, t) & & & \bar{U}_{\mathbf{p},2}(s, t) \end{pmatrix}. \quad (185)$$

Hence, the Shale–Stinespring condition for  $\mathcal{U}(s, t)$  amounts to

$$\sum_{\mathbf{p}} (|V_{\mathbf{p},1}|^2 + |V_{\mathbf{p},2}|^2) < \infty, \quad (186)$$

which does not necessarily hold: We may split the diagonalizing Bogoliubov transformation  $\mathcal{V}_{\mathbf{p}} = \mathcal{V}_{\mathbf{p},+} \oplus \mathcal{V}_{\mathbf{p},-}$  as in (181) and get

$$\begin{aligned} \mathcal{V}_{\mathbf{p},\pm}^* \widehat{\mathcal{U}}_{\mathbf{p},\pm}(s,t) \mathcal{V}_{\mathbf{p},\pm} &= \exp \left( -i \int_s^t \mathcal{V}_{\mathbf{p},\pm}^* \tilde{A}_{H,\mathbf{p},\pm}(\tau) \mathcal{V}_{\mathbf{p},\pm} d\tau \right) \\ &= \begin{pmatrix} e^{-i(t-s)E_{\mathbf{p},\pm}} & 0 \\ 0 & e^{i(t-s)E_{\mathbf{p},\pm}} \end{pmatrix}. \end{aligned} \quad (187)$$

Note that neither of the matrices  $\widehat{\mathcal{U}}_{\mathbf{p},\pm}(s,t), \mathcal{V}_{\mathbf{p},\pm}$  describes a Bogoliubov transformation. Reading off  $\mathcal{V}_{\mathbf{p},\pm}$  from (181), we conclude that

$$V_{\mathbf{p},\pm}(s,t) = \overline{u_{\mathbf{p},\pm}} v_{\mathbf{p},\pm} (\mp 2i \sin((t-s)E_{\mathbf{p},\pm})). \quad (188)$$

Although there is a rapid oscillation in  $\mathbf{p}$  for small times  $(t-s)$ , convergence of the sum (186) depends on the values of  $E_{\mathbf{p},\pm}$  and cannot be guaranteed in general.

# Appendices

## A Results on the Infinite Tensor Product Space

Our first proposition characterizes, when exactly two  $C$ -sequences correspond to the same functional  $\Psi \in \widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$ . This allows us to define operators on  $\widehat{\mathcal{H}}$  in Section 4.2. Recall that any  $C$ -sequence  $(\Psi) = (\Psi_k)_{k \in I}$  gives rise to a unique functional  $\Psi \in \widehat{\mathcal{H}}$  by the embedding  $\Psi = \iota((\Psi))$ .

**Proposition A.1.** *Whenever  $(\Psi), (\Psi') \in \text{Cseq}$  represent the same functional  $\Psi = \Psi'$ , then there exists a family of complex numbers  $(c_k)_{k \in I}$  such that*

$$\Psi_k = c_k \Psi'_k \quad \forall k \in I, \quad \text{and} \quad \prod_{k \in I} c_k = 1, \quad (189)$$

using the notion of convergence for an infinite product from Section 2.3.

*Conversely, if  $(\Psi), (\Psi') \in \text{Cseq}$  just differ by a family  $(c_k)_{k \in I}$  as in (189), then they represent the same functional  $\Psi = \Psi'$ .*

*Proof.* For the first statement, we must prove that  $\Psi_k$  and  $\Psi'_k$  are parallel for any  $k \in I$ . So let's fix a  $k$  and decompose  $\Psi_k = \Psi_k^{\parallel} + \Psi_k^{\perp}$  with  $\Psi_k^{\parallel} \parallel \Psi'_k$  and  $\Psi_k^{\perp} \perp \Psi'_k$ , and suppose that  $\Psi_k^{\perp} \neq 0$ . Now, choose some  $C$ -sequences  $(\Phi), (\Phi')$ , that agree on all  $k' \neq k$  and



with  $\Phi_k \parallel \Psi_k$  and  $\Phi'_k \parallel \Psi'_k$ , as well as  $\|\Phi_k\|_k = \|\Phi'_k\|_k = 1$ . Then,

$$\begin{aligned} \langle \Psi, \Phi' \rangle &= \langle \Psi_k, \Phi'_k \rangle_k \prod_{k \neq k'} \langle \Psi_{k'}, \Phi'_{k'} \rangle_{k'} = \|\Psi_k\|_k \prod_{k \neq k'} \langle \Psi_{k'}, \Phi'_{k'} \rangle_{k'} \\ &< \|\Psi_k\|_k \prod_{k \neq k'} \langle \Psi_{k'}, \Phi_{k'} \rangle_{k'} = \langle \Psi_k, \Phi_k \rangle_k \prod_{k \neq k'} \langle \Psi_{k'}, \Phi_{k'} \rangle_{k'} = \langle \Psi, \Phi \rangle. \end{aligned} \quad (190)$$

By the same arguments,  $\langle \Psi', \Phi \rangle < \langle \Psi', \Phi' \rangle$ . But since  $(\Psi), (\Psi')$  correspond to the same functional  $\Psi = \Psi'$ , we can freely exchange both expressions within the scalar product:

$$\langle \Psi, \Phi' \rangle = \langle \Psi', \Phi' \rangle > \langle \Psi', \Phi \rangle = \langle \Psi, \Phi \rangle. \quad (191)$$

This contradicts (190) and thus establishes  $\Psi_k = c_k \Psi'_k$ .

Convergence of  $\prod_{k \in I} c_k$  can be seen as follows: We have

$$\|\Psi\|^2 = \langle \Psi', \Psi \rangle = \prod_{k \in I} \langle \Psi'_k, c_k \Psi'_k \rangle_k = \prod_{k \in I} c_k \|\Psi'_k\|_k^2. \quad (192)$$

So if  $\prod_{k \in I} c_k$  was not convergent, i.e.,  $\sum_k |c_k - 1| = \infty$ , then for the product on the right-hand side, we would have

$$\begin{aligned} \sum_k |c_k \|\Psi'_k\|_k^2 - 1| &= \sum_k |c_k \|\Psi'_k\|_k^2 - \|\Psi'_k\|_k^2 + \|\Psi'_k\|_k^2 - 1| \\ &\geq \sum_k (|c_k \|\Psi'_k\|_k^2 - \|\Psi'_k\|_k^2| - |\|\Psi'_k\|_k^2 - 1|) \\ &= \underbrace{\sum_k \|\Psi'_k\|_k^2 |c_k - 1|}_{(*)} - \underbrace{\sum_k |\|\Psi'_k\|_k^2 - 1|}_{< \infty}. \end{aligned} \quad (193)$$

Now,  $\|\Psi'_k\|_k^2 > 1/2$  for all but finitely many  $k$ , so  $(*)$  and thus the first expression in (193) diverges. This is a contradiction to (192) being convergent. So  $\prod_{k \in I} c_k$  indeed yields a complex number.

But since  $\|\Psi\|^2 = \|\Psi'\|^2 = \prod_{k \in I} \|\Psi'_k\|_k^2$ , we immediately obtain  $\prod_{k \in I} c_k = 1$  from (192).

The converse statement can readily be seen by computing the action of the functionals  $\Psi, \Psi'$  on some  $\Phi \in \text{Cseq}$ :

$$\langle \Psi, \Phi \rangle = \prod_{k \in I} \langle \Psi_k, \Phi_k \rangle_k = \prod_{k \in I} \langle c_k \Psi'_k, \Phi_k \rangle_k = \left( \prod_{k \in I} \overline{c_k} \right) \prod_{k \in I} \langle \Psi'_k, \Phi_k \rangle_k = \langle \Psi', \Phi \rangle. \quad (194)$$

□

By [15, Lemma 4.1.1], all subspaces  $\prod_{k \in I}^{\otimes C} \mathcal{H}_k$  of  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$  are **mutually orthogonal**. This allows for a particularly simple decomposition:

**Lemma A.1.** For any  $\Psi \in \widehat{\mathcal{H}}$ , we can write

$$\Psi = \sum_{m \in \mathcal{M}} d_m \Psi^{(m)} = \sum_{m \in \mathcal{M}} d_m \prod_{k \in I}^{\otimes} \Psi_k^{(m)}, \quad (195)$$

with  $\mathcal{M}$  being a subset of  $\mathbb{N}$ ,  $\Psi^{(m)}$  defined by the mutually orthogonal  $C_0$ -sequences  $(\Psi^{(m)})$  with  $\|\Psi^{(m)}\| = 1$  and where  $\sum_m |d_m|^2 < \infty$  is a complex sequence.

Moreover, one can choose a fixed set  $Z = \{\Psi^{(a)}\}_{a \in A}$  defined by mutually orthogonal, normalized  $C_0$ -sequences  $(\Psi^{(a)})$ , such that for all  $\Psi \in \widehat{\mathcal{H}}$ , the form (195) can be achieved by taking only  $\Psi^{(m)} \in Z$ . The decomposition (195) is then unique up to the choice of the  $\Psi_k^{(m)}$  representing  $\Psi^{(m)}$ .

So  $Z$  is an orthonormal basis of  $\widehat{\mathcal{H}}$  that might be uncountable, but the elements  $\Psi \in \widehat{\mathcal{H}}$  are all countable linear combinations with coefficient sequences in  $\ell^2$ .

*Proof.* By definition, any  $\Psi \in \widehat{\mathcal{H}}$  can be approximated by a Cauchy sequence  $(\Psi^{(r)})_{r \in \mathbb{N}} \subseteq \prod_{k \in I}^{\otimes} \mathcal{H}_k$ ,  $\Psi^{(r)} \rightarrow \Psi$ . So each  $\Psi^{(r)}$  can be written a finite linear combination of  $C$ -sequences. All  $C$ -sequences, that are no  $C_0$ -sequences, must have norm 0, so we may drop them and simply write

$$\Psi^{(r)} = \sum_{\ell=1}^{L_r} \Psi_{\ell}^{(r)}, \quad (196)$$

with  $(\Psi_{\ell}^{(r)})$  being  $C_0$ -sequences. Now, the  $C_0$ -sequences decay into mutually orthogonal equivalence classes  $C$ , out of which countably many are occupied by any  $\Psi^{(r)}$ . So we have

$$\Psi^{(r)} = \sum_C \sum_{\ell: (\Psi_{\ell}^{(r)}) \in C} \Psi_{\ell}^{(r)} =: \sum_C \Psi_C^{(r)}. \quad (197)$$

By orthogonality of the subspaces  $\Psi \in \prod_{k \in I}^{\otimes C} \mathcal{H}_k$ , we have

$$\|\Psi^{(r)} - \Psi^{(s)}\|^2 = \sum_C \|\Psi_C^{(r)} - \Psi_C^{(s)}\|^2. \quad (198)$$

$(\Psi^{(r)})_{r \in \mathbb{N}}$  is a Cauchy sequence, so  $(\Psi_C^{(r)})_{r \in \mathbb{N}}$  is also a Cauchy sequence for all  $C$ . That means, the limit  $\Psi_C = \lim_{r \rightarrow \infty} \Psi_C^{(r)}$  exists and by orthogonality of the  $\Psi_C^{(r)}$  for each  $r$ ,

$$\lim_{r \rightarrow \infty} \sum_C \Psi_C^{(r)} = \sum_C \lim_{r \rightarrow \infty} \Psi_C^{(r)} \Leftrightarrow \Psi = \sum_C \Psi_C. \quad (199)$$

We may now write both  $\Psi_C$  and  $\Psi$  in coordinates:

$$\begin{aligned}\Psi_C &= \lim_{r \rightarrow \infty} \Psi_C^{(r)} = \lim_{r \rightarrow \infty} \sum_{\ell: (\Psi_\ell^{(r)}) \in C} \Psi_\ell^{(r)} = \sum_{n(\cdot) \in F} a_C(n(\cdot)) \prod_{k \in I}^{\otimes} e_{k, n(k)} \\ \Rightarrow \Psi &= \sum_C \lim_{r \rightarrow \infty} \Psi_C^{(r)} = \sum_C \sum_{n(\cdot) \in F} a_C(n(\cdot)) \prod_{k \in I}^{\otimes} e_{k, n(k)}.\end{aligned}\tag{200}$$

So  $\Psi$  can be written as a countable sum over mutually orthogonal, normalized  $C_0$ -sequences with coordinates  $a_C(n(\cdot))$ . We index the sequences and coordinates by  $(\Psi^{(m)})$  and  $d_m$ , which yields the desired form (195).

Square summability of the  $d_m$  can be seen by

$$\sum_m |d_m|^2 = \sum_C \sum_{n(\cdot) \in F} |a_C(n(\cdot))|^2.\tag{201}$$

Now, the set  $Z = \{\Psi^{(a)}\}_{a \in A}$  is exactly the union of all vectors  $\prod_{k \in I}^{\otimes} e_{k, n(k)}$  over all classes  $C$ , which is indeed a set of mutually orthogonal, normalized vectors.

Uniqueness of the decomposition follows by orthogonality of the  $\Psi^{(a)}$ . □

### Remarks.

17. Although the coefficient sequence  $a_C(n(\cdot))$  is an element of  $\ell^2$ , the space  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$  is generally not isomorphic to  $\ell^2$ , and hence not separable. The reason is that the set  $Z$  of eligible  $\Psi^{(m)}$ -vectors can be uncountable. For instance, already with a countable  $I$  and at least two basis vectors  $e_{k,0}, e_{k,1}$  for each  $k$ , the space  $\widehat{\mathcal{H}}$  contains the orthonormal set of vectors  $\prod_{k \in I}^{\otimes} e_{k, n}$  with  $n \in \{0, 1\}$ . This set corresponds to all families of binary digits, which is uncountable for  $I$  of infinite cardinality. For the same reason,  $Z$  is uncountable, as well as the number of equivalence classes  $C$ .

18. We may sum up all components with sequences  $(\Psi^{(m)})$  in the same equivalence class  $C$  as

$$\Psi_C := \sum_{m: (\Psi^{(m)}) \in C} d_m \Psi^{(m)} \quad \Rightarrow \quad \Psi = \sum_C \Psi_C,\tag{202}$$

where the sum runs only over countably many equivalence classes.

Further, by [15, Def. 6.1.1], two  $C_0$ -sequences  $(\Psi), (\Phi)$  are weakly equivalent, if and only if there exists a family  $(z_k)_{k \in I} \subseteq \mathbb{C}$  with  $(z_k \Psi_k)_{k \in I}$  being (strongly) equivalent to  $(\Phi_k)_{k \in I}$ . From that, we may conclude:

**Lemma A.2.** *Let  $C_w$  be the weak equivalence class of a  $C_0$ -sequence  $(\Phi) = (\Phi_k)_{k \in I}$ , choose an orthonormal basis  $(e_{k,n})_{n \in \mathbb{N}_0}$  for each  $\mathcal{H}_k$ , such that  $\Phi_k = c_{k,0}$  and define for any  $C_0$ -sequence  $(\Psi) = (\Psi_k)_{k \in I}$  the coordinates  $c_{k,n} := \langle e_{k,n}, \Psi_k \rangle_k$ . Then,  $\prod_{k \in I}^{\otimes C_w} \mathcal{H}_k$  is exactly the closure of the span of all normalized  $C_0$ -sequences, where  $|c_{k,0}| = 1$  for all but finitely many  $k \in I$ .*

Note that the last statement means  $c_{k,n} = 0$  for  $n \geq 1$  and for those  $k$ . In simple words, Lemma A.2 asserts that replacing  $C$  by  $C_w$  in the equivalence class is done via replacing  $c_{k,0} = 1$  by  $|c_{k,0}| = 1$ .

*Proof.* First, we prove that any normalized  $C_0$ -sequence  $(\Psi)$  with  $|c_{k,0}| = 1$  almost everywhere is weakly equivalent to  $(\Phi)$ : We can define a family of phase rotations  $|z_k| = 1$ , such that  $z_k c_{k,0} = 1$  for all  $k$  with  $|c_{k,0}| = 1$ . So  $(z_k \Psi_k)_{k \in I}$  has  $z_k c_{k,0} = 1$  almost everywhere and is hence a  $C_0$ -sequence strongly equivalent to  $(\Phi)$ . Therefore,  $(\Psi)$  is weakly equivalent to  $(\Phi)$ . So  $\Psi \in \prod_{k \in I}^{\otimes C_w} \mathcal{H}_k$  and the same holds for the span of these  $C_0$ -sequences, and their closure with respect to the Hilbert space topology on  $\widehat{\mathcal{H}}$ .

Conversely, any  $\Psi \in \prod_{k \in I}^{\otimes C_w} \mathcal{H}_k$  is within the closure of the span of normalized  $C_0$ -sequences with  $|c_{k,0}| = 1$  almost everywhere: By Lemma A.1, we may write

$$\Psi = \sum_{m \in \mathcal{M}} d_m \prod_{k \in I}^{\otimes} \Psi_k^{(m)},$$

where  $\sum_m |d_m|^2 < \infty$  and the  $(\Psi^{(m)}) = (\Psi_k^{(m)})_{k \in I}$  with  $\|\Psi^{(m)}\| = 1$  are orthogonal. Further, we may choose  $(\Psi^{(m)}) \sim_w (\Phi)$ , since all  $(\Psi^{(m)})$  were constructed to come from a (strong) equivalence class  $C$  contained within  $C_w$ . So there exist families  $(z_k^{(m)})_{k \in I}$ ,  $|z_k^{(m)}| = 1$ , such that  $(z_k^{(m)} \Psi_k^{(m)})_{k \in I} \sim (\Phi)$  for all  $m \in \mathcal{M}$ . By strong equivalence, we may approximate each  $(z_k^{(m)} \Psi_k^{(m)})$  up to arbitrary precision  $\varepsilon > 0$  by a linear combination of (normalized) families  $\Psi^{(m,\varepsilon)}$ , such that, when writing these families in coordinates, we have  $c_{k,0}^{(m,\varepsilon)} = 1$  almost everywhere in  $k \in I$ . Hence, the families  $((z_k^{(m)})^{-1} \Psi_k^{(m,\varepsilon)})_{k \in I}$  approximate  $\Psi^{(m)}$  up to precision  $\varepsilon$ . They satisfy  $|(z_k^{(m)})^{-1} c_{k,0}^{(m,\varepsilon)}| = 1$  almost everywhere, so  $\Psi^{(m)}$  can be approximated up to arbitrary precision by a linear combination of  $C_0$ -sequences with the above-mentioned property. And by (195) and convergence of  $|d_m|^2$ , also an arbitrary approximation of  $\Psi$  is possible by linear combinations of normalized  $C_0$ -sequences with  $|c_{k,0}| = 1$  almost everywhere. □

## B Representations of Bogoliubov Transformations

Here, we compare four different ways of representing a Bogoliubov transformation: One transformation  $\mathcal{V}_A$  directly acts on the  $*$ -algebra  $\mathcal{A}$  of creation and annihilation opera-

tors. The three others  $\mathcal{V}_1, \mathcal{V}_2$  and  $\mathcal{V}_3$ , are equivalent to  $\mathcal{V}_A$  and act on different spaces  $W_{1,1}, W_{1,2}, W_{1,3}$  isomorphic to the one-operator subspace  $W_1$  of  $\mathcal{A}$ . More precisely, we consider as  $W_1$  either  $\mathfrak{h} \oplus \mathfrak{h}$ ,  $\mathfrak{h} \oplus \mathfrak{h}^*$ , or  $\ell^2 \oplus \ell^2$ . Since  $\mathcal{A}$  is the algebra generated by the one-operator subspace  $W_1$ , it suffices to define  $\mathcal{V}_j$  on  $W_{1,j}$ , in order to obtain its action on operator products. The one used in this paper is  $\mathcal{V} = \mathcal{V}_3$ . We explicitly derive the Bogoliubov relations in all representations.

## B.1 Bogoliubov Transformations on Operators

By a (bounded) algebraic Bogoliubov transformation  $\mathcal{V}_A$ , we understand any transformation mapping creation and annihilation operators  $a_{\pm}^{\dagger}(f), a_{\pm}(g)$  with  $f, g \in \mathfrak{h}$  into operators  $b_{\pm}^{\dagger}(f), b_{\pm}(g)$  via:

$$\begin{aligned} \mathcal{V}_A : \mathcal{A} &\rightarrow \mathcal{A}, & a_{\pm}^{\dagger}(f) &\mapsto b_{\pm}^{\dagger}(f) \\ b_{\pm}^{\dagger}(f) &= a_{\pm}^{\dagger}(\hat{u}f) + a_{\pm}(\hat{v}f) \\ b_{\pm}(g) &= a_{\pm}^{\dagger}(\hat{v}g) + a_{\pm}(\hat{u}g), \end{aligned} \tag{203}$$

with a linear operator  $\hat{u}$  and an antilinear operator  $\hat{v}$ , both of which are defined on  $\mathfrak{h}$  (i.e., bounded), such that both  $\mathcal{V}_A$  and its adjoint  $\mathcal{V}_A^*$  conserve the CAR/CCR (30).

In order to express  $\mathcal{V}_A$  and the conservation conditions, we define for the linear  $\hat{u}$ :

- the transpose  $\hat{u}^T$  by  $\langle f, \hat{u}^T g \rangle = \langle g, \hat{u} f \rangle$
- the complex conjugate  $\bar{\hat{u}}$  by  $\langle f, \bar{\hat{u}} g \rangle = \overline{\langle \bar{f}, \hat{u} \bar{g} \rangle}$
- the adjoint by  $\hat{u}^* = \overline{\hat{u}^T} = \bar{\hat{u}}^T$

The antilinear  $\hat{v}$  can be written as  $\hat{v} = \hat{v}_{\ell} J$  with a linear operator  $\hat{v}_{\ell}$  and complex conjugation  $(Jf)(x) := \overline{f(x)}$ , see [37, 38]. We define the corresponding transpose by  $\hat{v}^T := \hat{v}_{\ell}^T J$ , the complex conjugate by  $\bar{\hat{v}} := \overline{\hat{v}_{\ell}} J$  and the adjoint by  $\hat{v}^* := \hat{v}_{\ell}^* J$ . So in particular,

$$\langle f, \hat{v}^T g \rangle = \langle f, \hat{v}_{\ell}^T J g \rangle = \langle \overline{\hat{v}_{\ell}} f, J g \rangle = \langle \overline{\hat{v}_{\ell}} J J f, J g \rangle = \langle \overline{\hat{v}} f, \bar{g} \rangle = \overline{\langle \hat{v} f, g \rangle}, \tag{204}$$

$$\Rightarrow \langle \hat{v} f, g \rangle = \overline{\langle f, \hat{v}^T g \rangle} = \langle \hat{v}^T g, f \rangle, \tag{205}$$

which is the ‘‘correct law for shifting antilinear operators from one side of the scalar product to the other’’ and replaces the familiar  $\langle f, \hat{v}_{\ell} g \rangle = \langle \hat{v}_{\ell}^* f, g \rangle$  from the linear case.

The adjoint transformation  $\mathcal{V}_A^*$  is given by replacing  $\hat{u} \rightarrow \hat{u}^*$  and  $\hat{v}_{\ell} \rightarrow \hat{v}_{\ell}^*$ , so  $\hat{v} \rightarrow \hat{v}^T$  in  $\mathcal{V}_A$  (203).

Now, we investigate conservation of the CAR/CCR. From  $[a(f), a^\dagger(g)]_\pm = \langle f, g \rangle$  being preserved, we get

$$\begin{aligned}
\langle f, g \rangle &= [b(f), b^\dagger(g)]_\pm \\
&= [a(\hat{u}f), a^\dagger(\hat{u}g)]_\pm + [a^\dagger(\hat{v}f), a(\hat{v}g)]_\pm \\
&= [a(\hat{u}f), a^\dagger(\hat{u}g)]_\pm \mp [a(\hat{v}g), a^\dagger(\hat{v}f)]_\pm \\
&= \langle \hat{u}f, \hat{u}g \rangle \mp \langle \hat{v}g, \hat{v}f \rangle \stackrel{(205)}{=} \langle \hat{u}^* \hat{u}f, g \rangle \mp \langle \hat{v}^T \hat{v}f, g \rangle \\
\Rightarrow 1 &= \hat{u}^* \hat{u} \mp \hat{v}^T \hat{v}.
\end{aligned} \tag{206}$$

From the conservation of  $[a(f), a(g)]_\pm = 0$  we obtain

$$\begin{aligned}
0 &= [b(f), b(g)]_\pm \\
&= [a(\hat{u}f), a^\dagger(\hat{v}g)]_\pm + [a^\dagger(\hat{v}f), a(\hat{u}g)]_\pm \\
&= [a(\hat{u}f), a^\dagger(\hat{v}g)]_\pm \mp [a(\hat{u}g), a^\dagger(\hat{v}f)]_\pm \\
&= \langle \hat{u}f, \hat{v}g \rangle \mp \langle \hat{u}g, \hat{v}f \rangle \stackrel{(205)}{=} \langle f, \hat{u}^* \hat{v}g \rangle \mp \langle f, \hat{v}^T \hat{u}g \rangle \\
\Rightarrow 0 &= \hat{u}^* \hat{v} \mp \hat{v}^T \hat{u}.
\end{aligned} \tag{207}$$

The two conditions (206) and (207) are required also for  $\mathcal{V}_A^*$ , where we replace  $\hat{u} \rightarrow \hat{u}^*$  and  $\hat{v} \rightarrow \hat{v}^T$ . This leads to 4 conditions in total:

$$\begin{aligned}
\hat{u}^* \hat{u} \mp \hat{v}^T \hat{v} &= 1 & \hat{u}^* \hat{v} \mp \hat{v}^T \hat{u} &= 0 \\
\hat{u} \hat{u}^* \mp \hat{v} \hat{v}^T &= 1 & \hat{v} \hat{u}^* \mp \hat{u} \hat{v}^T &= 0.
\end{aligned} \tag{208}$$

Note that the term ‘‘Bogoliubov transformation’’ or ‘‘canonical transformation’’ is also used in a wider sense, denoting any transformation leaving the CAR/CCR invariant. This includes transformations of the kind  $b^\sharp = a^\sharp + c$  with  $c \in \mathbb{C}$ , as in [39, (4.3)]. We will refer to the latter transformations as ‘‘Weyl transformations’’ and reserve the term ‘‘Bogoliubov transformation’’ for (203).

## B.2 Representation by $W_{1,1} = \mathfrak{h} \oplus \mathfrak{h}$

It is most natural to encode  $a_\pm^\dagger(f_1) + a_\pm(\overline{f_2})$  by a direct sum  $F = (f_1, f_2)$ . We do this by introducing generalized creation and annihilation operators

$$\begin{aligned}
A_{1\pm}^\dagger : \mathfrak{h} \oplus \mathfrak{h} &\rightarrow \mathcal{A}_\pm, & (f_1, f_2) &\mapsto a_\pm^\dagger(f_1) + a_\pm(\overline{f_2}) \\
A_{1\pm} : \mathfrak{h} \oplus \mathfrak{h} &\rightarrow \mathcal{A}_\pm, & (g_1, g_2) &\mapsto a_\pm(g_1) + a_\pm^\dagger(\overline{g_2}).
\end{aligned} \tag{209}$$

In a more abstract language, the representation is fixed by a bijective identification  $\iota_1 : \mathfrak{h} \oplus \mathfrak{h} \rightarrow W_1$  such that  $\iota_1(F) = A_{1\pm}^\dagger(F)$ . The operator  $A_{1\pm}(F)$  is then defined as the adjoint of  $\iota_1(F)$ .

In this representation, a Bogoliubov transformation acts by  $\mathcal{V}_1 : \mathfrak{h} \oplus \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h}$ ,  $\mathcal{V}_1 = \iota_1^{-1} \circ \mathcal{V}_A \circ \iota_1$ , where

$$\mathcal{V}_1 \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{u}f \\ \hat{v}f \end{pmatrix} \quad \text{and} \quad \mathcal{V}_1 \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} \hat{v}\bar{g} \\ \hat{u}\bar{g} \end{pmatrix}. \quad (210)$$

This transformation is obviously not linear on  $\mathfrak{h} \oplus \mathfrak{h}$ . Hence, we cannot encode subsequent Bogoliubov transformations by a matrix multiplication, which makes the representation inconvenient. The nonlinearity issues can be resolved within a different representation, by complex conjugation of the second one-particle Hilbert space.

### B.3 Representation by $W_{1,2} = \mathfrak{h} \oplus \mathfrak{h}^*$

We define the complex conjugation operator  $J : \mathfrak{h} \rightarrow \mathfrak{h}^*$  with adjoint  $J^* : \mathfrak{h}^* \rightarrow \mathfrak{h}$  and denote  $F = (f_1 \oplus Jf_2)$ . The generalized creation and annihilation operators are then given by

$$\begin{aligned} A_{2\pm}^\dagger : \mathfrak{h} \oplus \mathfrak{h}^* &\rightarrow \mathcal{A}_\pm, & (f_1 \oplus Jf_2) &\mapsto a_\pm^\dagger(f_1) + a_\pm(f_2) \\ A_{2\pm} : \mathfrak{h} \oplus \mathfrak{h}^* &\rightarrow \mathcal{A}_\pm, & (g_1 \oplus Jg_2) &\mapsto a_\pm(g_1) + a_\pm^\dagger(g_2). \end{aligned} \quad (211)$$

The corresponding identification is  $\iota_2 : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow W_1$  with  $\iota_2(F) = A_{2\pm}^\dagger(F)$ . In this representation, a Bogoliubov transformation is represented by  $\mathcal{V}_2 : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow \mathfrak{h} \oplus \mathfrak{h}^*$ ,  $\mathcal{V}_2 = \iota_2^{-1} \circ \mathcal{V}_A \circ \iota_2$ , where

$$\mathcal{V}_2 \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{u}f \\ J\hat{v}f \end{pmatrix} \quad \text{and} \quad \mathcal{V}_2 \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} \hat{v}g \\ J\hat{u}g \end{pmatrix}. \quad (212)$$

The operator  $\mathcal{V}_2$  is linear and can be written in block matrix form

$$\mathcal{V}_2 = \begin{pmatrix} U & J^*VJ^* \\ V & JUJ^* \end{pmatrix}, \quad (213)$$

with linear operators  $U : \mathfrak{h} \rightarrow \mathfrak{h}$ ,  $U = \hat{u}$  and  $V : \mathfrak{h} \rightarrow \mathfrak{h}^*$ ,  $V = J\hat{v}$ . So we can write successive transformations as a matrix product. This representation is used, for instance in [3, 27].

Conservation of the CAR/CCR now translates into

$$\mathcal{V}_2^* \mathcal{S}_\pm \mathcal{V}_2 = \mathcal{V}_2 \mathcal{S}_\pm \mathcal{V}_2^* = \mathcal{S}_\pm, \quad (214)$$

with  $\mathcal{S}_- = id$  and  $\mathcal{S}_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  being operators on  $\mathfrak{h} \oplus \mathfrak{h}^*$ .

In the *fermionic* case  $(-)$ , the first conservation condition reads:

$$\mathcal{V}_2^* \mathcal{V}_2 = \begin{pmatrix} U^*U + V^*V & U^*J^*VJ^* + V^*JUJ^* \\ JV^*JU + JU^*J^*V & JV^*VJ^* + JU^*UJ^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (215)$$

From the first line, we recover

$$U^*U + V^*V = 1 \quad U^*J^*V + V^*JU = 0, \quad (216)$$

and the second line yields exactly the same conditions.

The CAR/CCR conservation for the adjoint transformation reads

$$\mathcal{V}_2 \mathcal{V}_2^* = \begin{pmatrix} UU^* + J^*VV^*J & UV^* + J^*VU^*J^* \\ VU^* + JUV^*J & VV^* + JUU^*J^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (217)$$

Again, we recover from the first line

$$UU^* + J^*VV^*J = 1 \quad UV^* + J^*VU^*J^* = 0, \quad (218)$$

and the second line yields the same conditions.

The calculations for *bosons* (+) are the same, except for additional minus signs. From (216) and (218), we arrive at 4 conditions in total, which are:

$$\begin{aligned} U^*U \mp V^*V &= 1 & V^*JU \mp U^*J^*V &= 0 \\ UU^* \mp J^*VV^*J &= 1 & UV^* \mp J^*VU^*J^* &= 0. \end{aligned} \quad (219)$$

## B.4 Representation by $W_{1,3} = \ell^2 \oplus \ell^2$

It is also possible to write the four blocks in  $\mathcal{V}_j$  as infinite matrices with countably many complex-valued entries. This is the representation introduced in Section 3. Recall that we fixed a basis  $(e_j)_{j \in \mathbb{N}}$  of  $\mathfrak{h}$  and wrote  $f_j := \langle e_j, f \rangle$  for  $f \in \mathfrak{h}$ , so  $f$  could be identified with the sequence  $\mathbf{f} = (f_j)_{j \in \mathbb{N}}$ ,  $\mathbf{f} \in \ell^2$ . The generalized creation and annihilation operators are as in (27):

$$\begin{aligned} A_{3\pm}^\dagger : \ell^2 \oplus \ell^2 &\rightarrow \mathcal{A}_\pm, & (\mathbf{f}_1, \mathbf{f}_2) &\mapsto a_\pm^\dagger(\mathbf{f}_1) + a_\pm(\overline{\mathbf{f}_2}) = \sum_j (f_{1,j} a_\pm^\dagger(e_j) + f_{2,j} a_\pm(e_j)) \\ A_{3\pm} : \ell^2 \oplus \ell^2 &\rightarrow \mathcal{A}_\pm, & (\mathbf{g}_1, \mathbf{g}_2) &\mapsto a_\pm(\mathbf{g}_1) + a_\pm^\dagger(\overline{\mathbf{g}_2}) = \sum_j (\overline{g_{1,j}} a_\pm(e_j) + \overline{g_{2,j}} a_\pm^\dagger(e_j)). \end{aligned}$$

The identification is  $\iota_3 : \ell^2 \oplus \ell^2 \rightarrow W_1$  with  $\iota_3(\mathbf{F}) = A_{3\pm}^\dagger(\mathbf{F})$ . A Bogoliubov transformation can now indeed be expressed by a matrix  $\mathcal{V}_3 := \iota_3^{-1} \circ \mathcal{V}_A \circ \iota_3$  with  $\mathcal{V}_3 : \ell^2 \oplus \ell^2 \rightarrow \ell^2 \oplus \ell^2$ , which is explicitly given by

$$\mathcal{V}_3 = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix},$$

as in (28). Here,  $v$  and  $u$  are defined by its matrix elements  $v_{ij} = \langle e_i, \hat{v}e_j \rangle$  and  $u_{ij} = \langle e_i, \hat{u}e_j \rangle$ . In addition, transpose, complex conjugate and adjoint are given by  $(u^T)_{ij} = u_{ji}$ ,  $(\bar{u})_{ij} = \overline{u_{ij}}$ ,  $(u^*)_{ij} = \overline{u_{ji}}$  and the same for  $v_{ij}$



As in the previous representation, the CAR/CCR conservation amounts to

$$\mathcal{V}_3^* \mathcal{S}_\pm \mathcal{V}_3 = \mathcal{V}_3 \mathcal{S}_\pm \mathcal{V}_3^* = \mathcal{S}_\pm, \quad (220)$$

but this time,  $\mathcal{S}_- = id$  and  $\mathcal{S}_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  are operators on  $\ell^2 \oplus \ell^2$ .

For *fermions*  $(-)$ ,

$$\mathcal{V}_3^* \mathcal{V}_3 = \begin{pmatrix} u^*u + v^T \bar{v} & u^*v + v^T \bar{u} \\ v^*u + u^T \bar{v} & v^*v + u^T \bar{u} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (221)$$

Note that both the operators  $v^T \bar{v} = v^*v$  and  $u^T \bar{u} = u^*u$  are real. As for the previous representation, the second line is equivalent to the first one, which in turn yields two conditions:

$$u^*u + v^T \bar{v} = 1 \quad u^*v + v^T \bar{u} = 0. \quad (222)$$

The CAR/CCR conservation for the adjoint transformation is

$$\mathcal{V}_3 \mathcal{V}_3^* = \begin{pmatrix} uu^* + vv^* & uv^T + vu^T \\ \bar{v}u^* + \bar{u}v^* & \bar{v}v^T + \bar{u}u^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (223)$$

From which we recover the two conditions

$$uu^* + vv^* = 1 \quad uv^T + vu^T = 0. \quad (224)$$

So in total, with (222) and (224), we have 4 conditions.

For *bosons*, we get the same conditions with an additional minus signs. So in total, we have

$$\begin{aligned} u^*u \mp v^T \bar{v} &= 1 & u^*v \mp v^T \bar{u} &= 0 \\ uu^* \mp vv^* &= 1 & uv^T \mp vu^T &= 0. \end{aligned} \quad (225)$$

Again, the transformation can be written by a matrix multiplication, which makes this representation very convenient to handle. It is hence widely used, e.g., in [2, 13, 28, 33]. Note that depending on the resource, the definitions of  $v$  and  $\bar{v}$  might be interchanged.

## C Proof that Fermionic $\mathbb{U}_\mathcal{V}$ Implements $\mathcal{V}$

In this section, we prove that the fermionic Bogoliubov implementer  $\mathbb{U}_\mathcal{V}$  in (57) indeed implements the Bogoliubov transformation  $\mathcal{V}$ . The proof for the bosonic case (48) works similar. For shortness, we write  $a^\sharp(\boldsymbol{\eta}_j) = a_j^\sharp$ .

**Proposition C.1.** *Let  $\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$  be a fermionic Bogoliubov transformation. If  $\text{tr}(v^*v) < \infty$ , then the operator*

$$\mathbb{U}_\mathcal{V} = \left( \prod_{j \in J_1''} (a_j^\dagger + a_j) \right) \exp \left( - \sum_{i \in I'} \xi_i (a_{2i}^\dagger a_{2i-1}^\dagger - a_{2i-1} a_{2i}) \right) \mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}},$$

with  $J_1'', I'$  defined in Section 3.2.2, implements  $\mathcal{V}$ , i.e.,

$$\mathbb{U}_{\mathcal{V}} a(\phi) \mathbb{U}_{\mathcal{V}}^* = a(u\phi) + a^\dagger(v\bar{\phi}), \quad \mathbb{U}_{\mathcal{V}} a^\dagger(\phi) \mathbb{U}_{\mathcal{V}}^* = a^\dagger(u\phi) + a(v\bar{\phi}) \quad \forall \phi \in \ell^2. \quad (226)$$

*Proof.* Recall that for a fermionic Bogoliubov transformation,  $u$  and  $v$  are bounded, so the above expressions are well-defined.

We first decompose  $\phi$  according to the orthonormal basis  $(\mathbf{f}_j)_{j \in J}$  defined in Section 3.2.2 as an eigenbasis of the matrix  $C^*C$ . This basis is mapped under both  $u$  and  $v$  to an orthonormal basis  $(\boldsymbol{\eta}_j)_{j \in J}$ , also defined in Section 3.2.2. Since  $\mathbb{U}_{\boldsymbol{\eta}\mathbf{f}}$  just performs a unitary transformation, which replaces the  $\mathbf{f}_{j^-}$  by  $\boldsymbol{\eta}_j$ -vectors, we may also remove the  $\mathbb{U}_{\boldsymbol{\eta}\mathbf{f}}$  from  $\mathbb{U}_{\mathcal{V}}$ , and replace the  $\mathbf{f}_{j^-}$  with  $\boldsymbol{\eta}_j$ -vectors for all  $j \in J$ .

Recall that one may decompose  $J = J' \cup J_0'' \cup J_1''$ , where  $J'$  contains ‘‘Cooper pairs’’ of indices  $(2i, 2i-1)$  with  $i \in I'$ ,  $J_0''$  contains all ‘‘invariant modes’’ and  $J_1''$  all modes with a full ‘‘particle-hole transformation’’. Then, it holds that

$$\begin{aligned} v\bar{\boldsymbol{\eta}}_j &= 0, & u\boldsymbol{\eta}_j &= \boldsymbol{\eta}_j & \text{for } j \in J_0'' \\ v\bar{\boldsymbol{\eta}}_j &= \boldsymbol{\eta}_j, & u\boldsymbol{\eta}_j &= 0 & \text{for } j \in J_1'' \\ v\bar{\boldsymbol{\eta}}_{2i} &= \beta_i \boldsymbol{\eta}_{2i-1}, & u\boldsymbol{\eta}_{2i} &= \alpha_i \boldsymbol{\eta}_{2i} & \text{for } i \in I' \\ v\bar{\boldsymbol{\eta}}_{2i-1} &= -\beta_i \boldsymbol{\eta}_{2i}, & u\boldsymbol{\eta}_{2i-1} &= \alpha_i \boldsymbol{\eta}_{2i-1} & \text{for } i \in I', \end{aligned} \quad (227)$$

with  $\sin \xi_i := \beta_i$  and  $\cos \xi_i := \alpha_i$ .

(226) can now be checked to hold mode-by-mode.

For  $j \in J_0''$ , it is easy to see that  $\mathbb{U}_{\mathcal{V}}$  implements  $\mathcal{V}$ , as it acts as the identity.

For  $j \in J_1''$ , the implementation is also quickly verified using  $a_j a_j = a_j^\dagger a_j^\dagger = 0$ :

$$(a_j^\dagger + a_j) a_j (a_j^\dagger + a_j) = a_j^\dagger a_j a_j^\dagger \stackrel{(\text{CAR})}{=} a_j^\dagger - a_j^\dagger a_j^\dagger a_j = a_j^\dagger. \quad (228)$$

The statement  $(a_j^\dagger + a_j) a_j^\dagger (a_j^\dagger + a_j) = a_j$  follows by swapping the roles of  $a_j$  and  $a_j^\dagger$ .

For  $i \in I'$ , we make use of the *Lie-Schwinger formula* [40] (for a proof, see [41]):

$$e^A B a^{-A} = \sum_{n=0}^{\infty} \frac{\text{ad}^n(A) B}{n!}, \quad (229)$$

with  $\text{ad}^n(A) B := [A, [A, \dots [A, B] \dots]]$  being the  $n$ -fold commutator. In our case,

$$A = -\xi_i (a_{2i}^\dagger a_{2i-1}^\dagger - a_{2i-1} a_{2i}), \quad B = a_{2i}^\# \quad \text{or} \quad B = a_{2i-1}^\#. \quad (230)$$

The following formula will turn out useful for the calculations

$$[XY, Z] = XYZ + XZY - XZY - ZXY = X\{Y, Z\} - \{X, Z\}Y, \quad (231)$$

with  $\{.,.\} = [.,.]_-$  denoting the anticommutator. We start with  $B = a_{2i}$  and compute

$$\begin{aligned} \text{ad}^0(A)B &= B = a_{2i} \\ \text{ad}^1(A)B &= [A, B] = -\xi_i[a_{2i}^\dagger a_{2i-1}^\dagger, a_{2i}] + \xi_i[a_{2i-1} a_{2i}, a_{2i}] \\ &= \xi_i\{a_{2i}^\dagger, a_{2i}\}a_{2i-1}^\dagger = \xi_i a_{2i-1}^\dagger \\ \text{ad}^2(A)B &= [A, [A, B]] = -\xi_i^2[a_{2i}^\dagger a_{2i-1}^\dagger, a_{2i-1}^\dagger] + \xi_i^2[a_{2i-1} a_{2i}, a_{2i-1}^\dagger] \\ &= -\xi_i^2\{a_{2i-1}, a_{2i-1}^\dagger\}a_{2i} = -\xi_i^2 a_{2i}. \end{aligned} \quad (232)$$

Now,  $\text{ad}^n(A)B$ -terms of higher order repeatedly change between  $a_{2i}$  and  $a_{2i-1}^\dagger$ . More precisely,

$$\text{ad}^n(A)B = \begin{cases} (-1)^m \xi_i^{2m} a_{2i} & \text{for } n = 2m \\ (-1)^m \xi_i^{2m+1} a_{2i-1}^\dagger & \text{for } n = 2m + 1 \end{cases}. \quad (233)$$

So we can split the series (229) into an even and an odd part:

$$\mathbb{U}_\mathcal{V} a_{2i} \mathbb{U}_\mathcal{V}^* = \sum_{m=0}^{\infty} \left( \frac{(-1)^m \xi_i^{2m}}{2m!} a_{2i} + \frac{(-1)^m \xi_i^{2m+1}}{(2m+1)!} a_{2i-1}^\dagger \right) = \cos(\xi_i) a_{2i} + \sin(\xi_i) a_{2i-1}^\dagger, \quad (234)$$

which is the desired result.

For  $\mathbb{U}_\mathcal{V} a_{2i-1} \mathbb{U}_\mathcal{V}^*$ , we have to interchange  $a_{2i}^\dagger$  and  $a_{2i-1}^\dagger$ , which in (230) leads to a change of  $\xi_i$  to  $-\xi_i$ , so we obtain

$$\mathbb{U}_\mathcal{V} a_{2i-1} \mathbb{U}_\mathcal{V}^* = \cos(\xi_i) a_{2i-1} - \sin(\xi_i) a_{2i}^\dagger. \quad (235)$$

The computations for  $a_{2i}^\dagger$  and  $a_{2i-1}^\dagger$  go through analogously. The series in (232) now start with  $a_{2i-1}^\dagger$  instead of  $a_{2i}$  and  $a_{2i}^\dagger$  instead of  $a_{2i-1}$ , which leads to

$$\mathbb{U}_\mathcal{V} a_{2i-1}^\dagger \mathbb{U}_\mathcal{V}^* = \cos(\xi_i) a_{2i-1}^\dagger - \sin(\xi_i) a_{2i}, \quad \mathbb{U}_\mathcal{V} a_{2i}^\dagger \mathbb{U}_\mathcal{V}^* = \cos(\xi_i) a_{2i}^\dagger + \sin(\xi_i) a_{2i-1}, \quad (236)$$

as desired. □

We remark that the implementation of a bosonic  $\mathcal{V}$  by  $\mathbb{U}_\mathcal{V}$  (48) can be checked similarly.

## D Alternative Definition $a^\dagger(\phi), a(\phi)$ on Infinite Tensor Products

In Section 4.2, we defined some spaces  $\mathcal{D}_e \subseteq \ell^2$  (78) and  $\mathcal{S}^\otimes \subseteq \widehat{\mathcal{H}}$  (83), such that for  $\phi_1, \dots, \phi_N \in \mathcal{D}_e$  and  $\Psi \in \mathcal{S}^\otimes$ , arbitrary operator products

$$a^\sharp(\phi_1) \dots a^\sharp(\phi_N) \Psi \in \mathcal{S}^\otimes \subseteq \widehat{\mathcal{H}},$$

were well-defined (Lemma 4.4). However, the definition of  $\mathcal{D}_e$  resembling test functions is quite restrictive. It is possible to consider larger domains for the form factor  $\phi \in \mathcal{D}_e$ . The price one has to pay is a stricter rapid decay condition in  $\mathcal{S}^\otimes$ . We propose two alternatives: one with a uniform rapid decay in the particle number ( $\mathcal{S}_{\text{uni}}^\otimes$ , Lemma D.1) and a class of even smaller ones, based on Hölder sequence spaces ( $\mathcal{S}_q^\otimes$ , Lemma D.2).

### D.1 Uniform Rapid Decay

As ITP vectors with uniform rapid decay, we define

$$\mathcal{S}_{\text{uni}}^\otimes := \left\{ \Psi \in \widehat{\mathcal{H}} \mid \|N_k^n \Psi\| \leq c_n \|\Psi\| \ \forall k \right\}. \quad (237)$$

In particular,  $\mathcal{S}_{\text{uni}}^\otimes \subseteq \mathcal{S}^\otimes$ , as the sequence of decay constants  $c_{k,n} = c_n$  in the definition of  $\mathcal{S}^\otimes$  (83) is additionally required to be in  $\ell^\infty$ . Correspondingly, the form factors  $\phi \in \ell^2$  have to be additionally in the dual space  $\ell^1$ , i.e.,

$$\sum_{k \in \mathbb{N}} |\phi_k| = \|\phi\|_{\ell^1} < \infty. \quad (238)$$

**Lemma D.1** (Creation and annihilation operator products for  $\mathcal{S}_{\text{uni}}^\otimes$ ).

For  $\phi_1, \dots, \phi_n \in \ell^1$  and  $\Psi \in \mathcal{S}_{\text{uni}}^\otimes$ , any operator product application

$$a^\sharp(\phi_1) \dots a^\sharp(\phi_n) \Psi \in \widehat{\mathcal{H}}, \quad (239)$$

is well-defined.

*Proof.* By the modified rapid decay condition (237),

$$\|a_k^\dagger \Psi\| = \|\sqrt{N_k + 1} \Psi\| \leq \|(N_k + 1) \Psi\| \leq (c_1 + 1) \|\Psi\|, \quad (240)$$

where we used that on the one-mode Fock space  $\mathcal{H}_k$ , the operator  $a_k^\dagger$  shifts all sectors up by one (keeping them orthogonal) and multiplies by  $\sqrt{N_k + 1}$ . So

$$\|a^\dagger(\phi) \Psi\| \leq \sum_k |\phi_k| (c_1 + 1) \|\Psi\| = \|\phi\|_{\ell^1} (c_1 + 1) \|\Psi\|. \quad (241)$$

For an  $n$ -fold application of  $a^\dagger(\phi)$ , we obtain in similarity to (87):

$$a^\dagger(\phi_1) \dots a^\dagger(\phi_n) \Psi = \sum_{k_1, \dots, k_n} \phi_{1, k_1} \dots \phi_{n, k_n} a_{k_1}^\dagger \dots a_{k_n}^\dagger \Psi. \quad (242)$$

So creations are applied to at most  $n$  modes, with at most  $n$  applications, each. In the maximal application number case,

$$\|(a_k^\dagger)^n \Psi\| = \left\| \prod_{\ell=1}^n \sqrt{N_k + \ell} \Psi \right\| \leq \|(N_k + n)^n \Psi\|. \quad (243)$$

The operator  $(N_k + n)^n$  is a polynomial of degree  $n$  in  $N_k$ . So by the modified rapid decay condition (237) we can estimate

$$\|(N_k + n)^n \Psi\| \leq \tilde{c}_n \|\Psi\|, \quad (244)$$

with a constant  $\tilde{c}_n$  depending on  $c_1, \dots, c_n$  and  $n$ . A similar estimate holds true for all application numbers  $l \leq n$  with a respective constant  $\tilde{c}_l$ . Denoting the maximum of those constants by  $\tilde{c}_{n, \max}$ , and keeping in mind that at most  $n$  modes can receive a creation operator, we finally obtain

$$\|a_{k_1}^\dagger \dots a_{k_n}^\dagger \Psi\| \leq \|\tilde{c}_{n, \max}^n \Psi\|. \quad (245)$$

Hence, when summing over all  $k$ ,

$$\|a^\dagger(\phi_1) \dots a^\dagger(\phi_n) \Psi\| \leq \left\| \sum_{k_1, \dots, k_n} |\phi_{1, k_1}| \dots |\phi_{n, k_n}| \tilde{c}_{n, \max}^n \Psi \right\| = \tilde{c}_{n, \max}^n \prod_{\ell=1}^n \left( \sum_{k_\ell} |\phi_{k_\ell}| \right) \|\Psi\|. \quad (246)$$

Replacing any number of creation by annihilation operators will lower the number  $\prod_{\ell=1}^n \sqrt{N_k + \ell}$  in (243), so the right-hand side stays an upper bound and all estimates remain valid. Hence, also  $a^\#(\phi_1) \dots a^\#(\phi_n) \Psi \in \widehat{\mathcal{H}}$ , which establishes the claim.  $\square$

### Remarks.

19. As in Lemma 4.4,  $\Psi \in \mathcal{S}_{\text{uni}}^\otimes$  also result in  $a^\dagger(\phi) \Psi \in \mathcal{S}_{\text{uni}}^\otimes$ , but we may have  $a(\phi) \Psi \notin \mathcal{S}_{\text{uni}}^\otimes$ .

The first statement can be seen by

$$\begin{aligned} \|N_k^n a_k^\dagger \Psi\| &= \|(N_k + 1)^n \sqrt{N_k + 1} \Psi\| \leq \|(N_k + 1)^{n+1} \Psi\| \leq \text{const.} \|\Psi\| \\ &\leq \text{const.} \|a_k^\dagger \Psi\|, \end{aligned} \quad (247)$$

with the const. depending on  $c_1, \dots, c_{n+1}$ . So the uniform rapid decay condition is satisfied.

This argument does not work for  $a_k$ , since  $\|\Psi\| \leq \|a_k^\dagger \Psi\|$  does not generalize to annihilation operators. A simple counter-example, where  $a(\phi)\Psi \notin \mathcal{S}_{\text{uni}}^\otimes$  is the following: Define  $\Psi = \prod_{k \in \mathbb{N}}^\otimes \Psi_k$  such that

$$\Psi_k^{(k+1)} = e^{-k}, \quad \Psi_k^{(0)} = \sqrt{1 - e^{-2k}}, \quad (248)$$

and all other sectors are unoccupied ( $\Psi_k^{(N_k)} = 0$ ). Here,  $\Psi_k^{(n)} = \langle e_{k,n}, \Psi_k \rangle_k = c_{k,n}$  with respect to the basis  $(e_{k,n})_{n \in \mathbb{N}_0}$  of  $n$ -particle states in mode  $k$ . So  $\|\Psi_k\|_k = 1$  and  $\|\Psi\| = 1$ . Then, on  $\mathcal{H}_k$  we have

$$\|N_k^n \Psi_k\|_k = (k+1)^n \|\Psi_k^{(k+1)}\|_k = (k+1)^n e^{-k} = (k+1)^n e^{-k} \|\Psi_k\|_k. \quad (249)$$

Since the function  $x \mapsto (x+1)^n e^{-x}$  is bounded on  $[0, \infty)$ , we have a uniform rapid decay, e.g., with  $c_n = \sup_{x \geq 0} (x+1)^n e^{-x}$ . However, in  $a_k \Psi_k$ , only the  $k$ -sector is occupied with  $\|a_k \Psi_k\|_k = \sqrt{k} e^{-k}$ , so

$$\|N_k^n a_k \Psi_k\|_k = \sqrt{k} k^n e^{-k} = k^n \|a_k \Psi_k\|_k, \quad (250)$$

and  $k^n$  cannot be uniformly bounded in all  $k \in \mathbb{N}$ . Hence,  $a(\phi)\Psi$  does not meet the rapid decay condition if we choose  $\phi \in \ell^1$  such that  $\phi_k \neq 0$  holds for infinitely many  $k$ , for instance by  $\phi_k = \frac{1}{k^2}$ .

20. Theorem 5.1 *does not hold* in a similar form for  $\mathcal{S}_{\text{uni}}^\otimes$  instead of  $\mathcal{S}^\otimes$ . The Bogoliubov vacuum  $\Omega_\nu$  (105) is in  $\mathcal{S}_{\text{uni}}^\otimes$ , **if and only if**  $C = u^* v J$  is bounded. This can be seen by checking the rapid decay condition. We recall (107):

$$\|N_k^n \Omega_{k,\nu}\|_k^2 = (1 - 4t^2)^{1/2} \sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N)^{2n} =: f_n(t),$$

with  $t = \left| \frac{\nu_k}{2\mu_k} \right| \in [0, 1/2)$  depending continuously and monotonically on the eigenvalues  $\lambda_k$  of  $C$  and with  $t \rightarrow 1/2$  as  $\lambda_k \rightarrow \infty$ .

If  $C$  is bounded, then there is a largest  $\lambda_k$ , meaning a largest  $t$  exists, called  $t_{\text{max}}$ . The continuous functions  $f_n$  attain a maximum on  $t \in [0, t_{\text{max}}]$ , called  $c_n$ . So  $\|N_k^n \Omega_{k,\nu}\|_k^2 \leq c_n$ , which establishes the uniform rapid decay and hence  $\Omega_\nu \in \mathcal{S}_{\text{uni}}^\otimes$ .

However, for unbounded  $C$ , the sequence  $\|N_k \Omega_{k,\nu}\|_k$  diverges as  $\lambda_k \rightarrow \infty$ :

$$\begin{aligned}
(1 - 4t^2)^{-1/2} \|N_k \Omega_{k,\nu}\|_k^2 &= \sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N)^2 \geq \sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N) \\
&= t \frac{d}{dt} \left( \sum_{N=1}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} \right) + 1 \\
&= t \frac{d}{dt} (1 - 4t^2)^{-1/2} + 1 = 4t^2 (1 - 4t^2)^{-3/2} + 1 \\
\Leftrightarrow \|N_k \Omega_{k,\nu}\|_k^2 &\geq \underbrace{4t^2 (1 - 4t^2)^{-1}}_{\rightarrow \infty} + \underbrace{(1 - 4t^2)^{1/2}}_{\rightarrow 0} \rightarrow \infty \quad \text{as } t \rightarrow 1/2.
\end{aligned} \tag{251}$$

So there is no way to set up a uniform bound for  $\|N_k \Omega_{k,\nu}\|_k^2$  or  $\|N_k \Omega_{k,\nu}\|_k$ , meaning that the uniform rapid decay condition cannot be fulfilled and  $\Omega_\nu \notin \mathcal{S}_{\text{uni}}^\otimes$ .

## D.2 Hölder Condition in Decay Coefficients

We may also consider other conditions on  $\phi \in \mathcal{D}_e$ , for instance

$$\phi \in \ell^p \Leftrightarrow \sum_k |\phi_k|^p < \infty, \tag{252}$$

and define

$$\mathcal{D}_p := \ell^p, \quad 1 \leq p \leq 2. \tag{253}$$

The largest space on which an estimate of the kind (246) still goes through is the following:

$$\mathcal{S}_q^\otimes := \left\{ \Psi \in \widehat{\mathcal{H}} \left| \|(N_k + 1)^{n/2} \Psi\| \leq c_k^n \|\Psi\| \text{ with } \sum_k c_k^q < \infty \quad \forall n \in \mathbb{N} \right. \right\}, \tag{254}$$

where  $2 \leq q \leq \infty$  is the Hölder dual of  $p$ , i.e.,  $1/q + 1/p = 1$ . With that definition, it is easy to see that  $\mathcal{S}_2^\otimes \subseteq \dots \subseteq \mathcal{S}_q^\otimes \subseteq \dots \subseteq \mathcal{S}_\infty^\otimes$ .

**Lemma D.2** (Creation and annihilation operator products for  $\mathcal{S}_q^\otimes$ ).

For  $\phi_1, \dots, \phi_n \in \mathcal{D}_p$  and  $\Psi \in \mathcal{S}_q^\otimes$ , any operator product application

$$a^\sharp(\phi_1) \dots a^\sharp(\phi_n) \Psi \in \widehat{\mathcal{H}}, \tag{255}$$

is well-defined.

*Proof.* The arguments are almost the same as for Lemma D.1, where in definition (254), the estimation step  $\sqrt{N_k + 1} \leq N_k$  for  $N_k \geq 1$  is dropped. An additional simplification comes from the fact that within an  $n$ -fold application of  $a_k^\dagger$ , each application gets out a maximum factor of  $c_k$ . So we don't have to find a  $\tilde{c}_{n,\max}$  anymore, but can directly estimate each application of  $a_k^\dagger$  by  $c_k$ . Thus, we get

$$\|a^\dagger(\phi_1) \dots a^\dagger(\phi_n)\Psi\| \leq \sum_{k_1, \dots, k_n} |\phi_{1,k_1}| \dots |\phi_{n,k_n}| c_{k_1} \dots c_{k_n} \|\Psi\| = \prod_{\ell=1}^n \left( \sum_{k_\ell} |\phi_{\ell,k_\ell}| c_{k_\ell} \right) \|\Psi\|. \quad (256)$$

The bracket can now be guaranteed to be finite, since  $(\phi_{\ell,k_\ell})_{k_\ell \in \mathbb{N}} \in \ell^p$  and  $(c_{k_\ell})_{k_\ell \in \mathbb{N}} \in \ell^q$ . So expression (256) is finite. The same holds true after replacing any number of  $a^\dagger$  by  $a$ -operators, which establishes the claim.  $\square$

## E Concerning Applications

### E.1 Translation of Hamiltonians into Block Matrices

Consider the following formal quadratic bosonic Hamiltonian, that is only an element of  $\overline{\mathcal{A}}_e$ , (75) but not necessarily an operator generating dynamics on Fock space  $\mathcal{F} = \mathcal{F}(\mathbb{N})$ :

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} (2h_{jk} a_j^\dagger a_k + k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k), \quad (257)$$

where  $h_{jk}, k_{jk} \in \mathbb{C}$ , so  $h = (h_{jk})_{j,k \in \mathbb{N}}, k = (k_{jk})_{j,k \in \mathbb{N}}$  are matrices of infinite size. By symmetry of  $H$ , we have  $h^* = h \Leftrightarrow h_{jk} = \overline{h_{kj}}$  and we can arrange for  $k^T = k \Leftrightarrow k_{jk} = k_{kj}$ . Now recall from Appendix B.4, that each  $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2) \in \ell^2 \oplus \ell^2$  can be identified with a bosonic algebraic expression

$$A^\dagger(\mathbf{F}) = A_{3+}^\dagger(\mathbf{F}) = a^\dagger(\mathbf{f}_1) + a(\overline{\mathbf{f}_2}) = \sum_j (f_{1,j} a_+^\dagger(e_j) + f_{2,j} a_+(e_j)), \quad (258)$$

or with  $A(\mathbf{F}) = (A^\dagger(\mathbf{F}))^*$ . We may now identify  $H$  with the following block matrices [2, 3, 13, 28]:

$$A_H = \begin{pmatrix} h & k \\ \overline{k} & \overline{h} \end{pmatrix}, \quad B_H = \begin{pmatrix} h & -k \\ \overline{k} & -\overline{h} \end{pmatrix} = A_H \mathcal{S}, \quad (259)$$

where  $\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . With this identification,

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} \langle \mathbf{F}_j, A_H \mathbf{F}_k \rangle A^\dagger(\mathbf{F}_j) A(\mathbf{F}_k), \quad (260)$$

where  $\mathbf{F}_j = \begin{pmatrix} \mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix}$  for a canonical basis vector  $\mathbf{e}_j$  and where by a direct calculation

$$[A(\mathbf{F}), [H, A^\dagger(\mathbf{G})]] = \langle \mathbf{F}, (\mathcal{S} B_H) \mathbf{G} \rangle_{\ell^2 \oplus \ell^2}, \quad (261)$$



holds for all finite linear combinations

$$\mathbf{F}, \mathbf{G} \in \mathcal{D}_F, \quad \mathcal{D}_F := \text{span} \left\{ \begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{e}_\ell \end{pmatrix}, \mid \ell \in \mathbb{N} \right\}, \quad \mathcal{D}_F \subset \ell^2 \oplus \ell^2. \quad (262)$$

Infinite linear combinations may or may not be well-defined.

Often  $H$  is called “second quantization of  $A_H$ ” and  $A_H$  is called “first quantization of  $H$ ”, where the first one is unique up to a normal ordering constant  $c$ , which we set to 0.

Further, the Schrödinger dynamics generated by  $-B_H$  on  $\mathbf{F} \in \mathcal{D}_F$  via

$$i\partial_t \mathbf{F} = -B_H \mathbf{F}, \quad (263)$$

correspond to the respective Heisenberg dynamics generated by  $H$  on expressions  $A^\dagger(\mathbf{F})$  in the following sense:

**Lemma E.1.** *In the bosonic case, for all  $\mathbf{F} \in \mathcal{D}_F$  we have:*

$$A^\dagger(iB_H \mathbf{F}) = i[H, A^\dagger(\mathbf{F})]. \quad (264)$$

This lemma can also be found in a similar form in [13, (2.27)]. For completeness, we give a proof here:

*Proof.* Consider any canonical basis vector  $\mathbf{e}_\ell \in \ell^2$ . Then, for  $\mathbf{F} = \begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}$ ,

$$B_H \mathbf{F} = \begin{pmatrix} \sum_j h_{j\ell} \mathbf{e}_j \\ \sum_j \overline{k_{j\ell}} \mathbf{e}_j \end{pmatrix} \Rightarrow A^\dagger(iB_H \mathbf{F}) = i \sum_j (h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j). \quad (265)$$

On the other hand since  $k_{j\ell} = k_{\ell j}$ ,

$$i[H, A^\dagger(\mathbf{F})] = \frac{i}{2} \sum_j (2h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j + \overline{k_{\ell j}} a_j) = i \sum_j (h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j). \quad (266)$$

This verifies (264) for  $\mathbf{F} = \begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}$ .

A similar calculation can be carried out for  $\mathbf{F}' = \begin{pmatrix} 0 \\ \mathbf{e}_\ell \end{pmatrix}$ , using  $\overline{h_{j\ell}} = h_{\ell j}$ :

$$B_H \mathbf{F}' = \begin{pmatrix} -\sum_j k_{j\ell} \mathbf{e}_j \\ -\sum_j h_{j\ell} \mathbf{e}_j \end{pmatrix} \Rightarrow A^\dagger(iB_H \mathbf{F}') = -i \sum_j (k_{j\ell} a_j^\dagger + h_{\ell j} a_j), \quad (267)$$

$$i[H, A^\dagger(\mathbf{F}')] = \frac{i}{2} \sum_j (-2h_{\ell j} a_j - k_{\ell j} a_j^\dagger - k_{j\ell} a_j^\dagger) = -i \sum_j (h_{\ell j} a_j^\dagger + k_{j\ell} a_j). \quad (268)$$

So by taking finite linear combinations of  $\begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ \mathbf{e}_\ell \end{pmatrix}$ , (264) holds for all  $\mathbf{F} \in \mathcal{D}_F$ .  $\square$

A similar statement is valid for fermions. Here, the formal quadratic Hamiltonian is of the form

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} (2h_{jk} a_j^\dagger a_k - k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k), \quad (269)$$

with  $h^* = h$  and where we can arrange for  $k^T = -k$ . This time, we only associate one block matrix to  $H$ :

$$A_H = \begin{pmatrix} h & -k \\ \overline{k} & -\overline{h} \end{pmatrix}. \quad (270)$$

It serves for both translation conditions

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} \langle \mathbf{F}_j, A_H \mathbf{F}_k \rangle A^\dagger(\mathbf{F}_j) A(\mathbf{F}_k), \quad (271)$$

and [28, (2b.38)]:

$$\{(A^\dagger(\mathbf{F}))^*, [H, A^\dagger(\mathbf{G})]\} = \langle \mathbf{F}, A_H \mathbf{G} \rangle_{\ell^2 \oplus \ell^2}, \quad \mathbf{F}, \mathbf{G} \in \mathcal{D}_F. \quad (272)$$

Note that the CAR imply  $k_{jj} = 0$ . For  $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2)$ , the expression  $A_{3-}^\dagger(\mathbf{F})$  is again defined as

$$A_{3-}^\dagger(\mathbf{F}) = a^\dagger(\mathbf{f}_1) + a(\overline{\mathbf{f}_2}) = \sum_j (f_{1,j} a_+^\dagger(e_j) + f_{2,j} a_+(e_j)). \quad (273)$$

The Schrödinger dynamics of  $(-A_H)$  can now be translated into the Heisenberg dynamics of  $H$ :

**Lemma E.2.** *In the fermionic case, for all  $\mathbf{F} \in \mathcal{D}_F$ , we have:*

$$A^\dagger(iA_H \mathbf{F}) = i[H, A^\dagger(\mathbf{F})]. \quad (274)$$

*Proof.* The proof is similar to that of Lemma E.1. We consider again a canonical basis vector  $\mathbf{e}_\ell \in \ell^2$  and  $\mathbf{F} = \begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}$ ,  $\mathbf{F}' = \begin{pmatrix} 0 \\ \mathbf{e}_\ell \end{pmatrix}$ . Formulas (265) and (267) can be copied from the proof of Lemma E.1:

$$A^\dagger(iA_H \mathbf{F}) = i \sum_j (h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j), \quad A^\dagger(iA_H \mathbf{F}') = -i \sum_j (k_{j\ell} a_j^\dagger + h_{\ell j} a_j). \quad (275)$$

For computing the commutators, we make use of the CAR and  $k_{j\ell} = -k_{\ell j}$ :

$$\begin{aligned}
i[H, A^\dagger(\mathbf{F})] &= \frac{i}{2} \sum_{jk} [2h_{jk}a_j^\dagger a_k - k_{jk}a_j^\dagger a_k^\dagger + \overline{k_{jk}}a_j a_k, a_\ell^\dagger] \\
&= \frac{i}{2} \sum_{jk} (2h_{jk} \underbrace{(a_j^\dagger a_k a_\ell^\dagger + a_j^\dagger a_\ell^\dagger a_k - a_j^\dagger a_\ell^\dagger a_k - a_\ell^\dagger a_j^\dagger a_k)}_{=a_j^\dagger \delta_{k\ell}} + \overline{k_{jk}} \underbrace{(a_j a_k a_\ell^\dagger + a_j a_\ell^\dagger a_k - a_j a_\ell^\dagger a_k - a_\ell^\dagger a_j a_k)}_{=a_j \delta_{k\ell} - a_k \delta_{j\ell}}) \\
&= \frac{i}{2} \sum_j 2h_{j\ell} a_j^\dagger + \frac{i}{2} \sum_j \overline{k_{j\ell}} a_j - \frac{i}{2} \sum_k \overline{k_{\ell k}} a_k \\
&= i \sum_j (h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j),
\end{aligned} \tag{276}$$

$$\begin{aligned}
i[H, A^\dagger(\mathbf{F}')] &= \frac{i}{2} \sum_{jk} [2h_{jk}a_j^\dagger a_k - k_{jk}a_j^\dagger a_k^\dagger + \overline{k_{jk}}a_j a_k, a_\ell] \\
&= \frac{i}{2} \sum_{jk} (2h_{jk} \underbrace{(a_j^\dagger a_k a_\ell + a_j^\dagger a_\ell a_k - a_j^\dagger a_\ell a_k - a_\ell a_j^\dagger a_k)}_{=0} - k_{jk} \underbrace{(a_j^\dagger a_k^\dagger a_\ell + a_j^\dagger a_\ell a_k^\dagger - a_j^\dagger a_\ell a_k^\dagger - a_\ell a_j^\dagger a_k^\dagger)}_{=a_j^\dagger \delta_{k\ell} - a_k^\dagger \delta_{j\ell}}) \\
&= -\frac{i}{2} \sum_k 2h_{\ell k} a_k - \frac{i}{2} \sum_j k_{j\ell} a_j^\dagger + \frac{i}{2} \sum_k k_{\ell k} a_k^\dagger \\
&= -i \sum_j (h_{\ell j} a_j + k_{j\ell} a_j^\dagger).
\end{aligned} \tag{277}$$

Both results agree with (275). Taking finite linear combinations of  $\mathbf{F}$ ,  $\mathbf{F}'$  establishes the proof.  $\square$

Of course, Lemmas E.1 and E.2 remain valid, if the number of modes is finite, i.e.,  $\mathbf{F} \in \mathbb{C}^N \oplus \mathbb{C}^N$ .

Both lemmas may also remain valid for further  $\mathbf{F} \in \ell^2 \oplus \ell^2$ , provided the expression  $A_H \mathbf{F} \in \ell^2 \oplus \ell^2$  exists. In that case, also an infinite linear combination of modes is well-defined.

## E.2 Deriving Pair Creation in External Field QED

In Section 7.3, we considered a quadratic Hamiltonian which is intended to describe a simplified model for external field QED. Starting from the formal Hamiltonian of a Dirac field coupled to a time-dependent homogeneous classical electromagnetic field  $A_\mu(t) = (0, 0, 0, A_3(t)) \in \mathbb{C}^4$ , we may justify the quadratic Hamiltonian as follows: Consider a Dirac field with discrete momentum  $\mathbf{p} \in \mathbb{Z}^3$  and spin index  $s \in \{1, 2, 3, 4\}$ ,

where  $s \in \{1, 2\}$  denotes an electron and  $s \in \{3, 4\}$  a positron. The formal Hamiltonian now reads:

$$\begin{aligned} H(t) &= H_0 + H_I(t), & H_0 &= d\Gamma(E_{\mathbf{p},0}), & E_{\mathbf{p},0} &= \sqrt{|\mathbf{p}|^2 + m^2} \\ H_I(t) &= e \int \overline{\Psi(\mathbf{x})} \gamma^\mu \Psi(\mathbf{x}) A_\mu(t) d\mathbf{x}, \end{aligned} \quad (278)$$

where  $\gamma^\mu$  are the gamma matrices in Dirac representation with Einstein summation convention in the index  $\mu$  assumed and  $e$  is the coupling constant. The field operator-valued distributions  $\Psi(\mathbf{x})$  and  $\overline{\Psi(\mathbf{x})} = \Psi(\mathbf{x})^* \gamma_0$  are given by

$$\Psi(\mathbf{x}) = \sum_{\mathbf{p} \in \mathbb{Z}^3} \sum_{s \in \{1,2\}} (a_{\mathbf{p},s} \mathbf{u}_{\mathbf{p},s} e^{-i\mathbf{p}\mathbf{x}} + b_{\mathbf{p},s}^* \mathbf{v}_{\mathbf{p},s} e^{i\mathbf{p}\mathbf{x}}), \quad (279)$$

with positron operators  $b_{\mathbf{p},1}^\# = a_{\mathbf{p},3}^\#, b_{\mathbf{p},2}^\# = a_{\mathbf{p},4}^\#$ . For the normalized Dirac spinors, there exist many conventions, out of which we adopt the following:

$$\mathbf{u}_{\mathbf{p},s} = c \begin{pmatrix} \phi_s \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \phi_s \end{pmatrix}, \quad \mathbf{v}_{\mathbf{p},s} = c \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \chi_s \\ \chi_s \end{pmatrix}, \quad \phi_1 = \chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_2 = \chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (280)$$

with normalization constant  $c \in \mathbb{C}$ , such that  $\|\mathbf{u}_{\mathbf{p},s}\| = \|\mathbf{v}_{\mathbf{p},s}\| = 1$  and with  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  being the Pauli matrix vector.

Evaluating  $H(t)$  now leads to a quadratic operator that can be translated into a block matrix (see Appendix E.1):

$$\tilde{A}_H(t) = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} \tilde{A}_{H,\mathbf{p}}(t), \quad \tilde{A}_{H,\mathbf{p}}(t) = \begin{pmatrix} h(t) & -k(t) \\ k(t) & -h(t) \end{pmatrix} \in \mathbb{C}^{8 \times 8}. \quad (281)$$

A direct calculation verifies

$$\begin{aligned} eA_3(t) \mathbf{u}_{\mathbf{p},s}^* \gamma^0 \gamma^3 \mathbf{u}_{\mathbf{p},s'} &= eA_3(t) \mathbf{u}_{\mathbf{p},s}^* \begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix} \mathbf{u}_{\mathbf{p},s'} = (\Delta E_{\mathbf{p}}(t)) \delta_{ss'}, \\ \Delta E_{\mathbf{p}}(t) &= \frac{2c^2 e A_3(t)}{E+m} p_3, \end{aligned} \quad (282)$$

so the  $a^\dagger a$ - and  $b^\dagger b$ -terms within  $H_I(t)$  render an additional kinetic term in

$$h(t) = \begin{pmatrix} E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t) & & & \\ & E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t) & & \\ & & E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t) & \\ & & & E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t) \end{pmatrix}. \quad (283)$$

By charge conservation, no  $a^\dagger a^\dagger$ ,  $aa$ ,  $b^\dagger b^\dagger$ - or  $bb$ -terms will appear. Moreover, by symmetry  $u_{\mathbf{p}s} \gamma_0 \gamma_3 v_{-\mathbf{p}s'} = v_{-\mathbf{p}s'} \gamma_0 \gamma_3 u_{\mathbf{p}s}$ , and by  $a_{\mathbf{p},s}^\dagger b_{-\mathbf{p},s'}^\dagger = -b_{-\mathbf{p},s'}^\dagger a_{\mathbf{p},s}^\dagger$ , we have

$$k = \begin{pmatrix} 0 & 0 & f_{\mathbf{p},11} & f_{\mathbf{p},12} \\ 0 & 0 & f_{\mathbf{p},21} & f_{\mathbf{p},22} \\ -f_{\mathbf{p},11} & -f_{\mathbf{p},21} & 0 & 0 \\ -f_{\mathbf{p},12} & -f_{\mathbf{p},22} & 0 & 0 \end{pmatrix}, \quad (284)$$

with  $f_{\mathbf{p},ss'} = \overline{f_{\mathbf{p},ss'}}$ . We perform a physical simplification by setting  $f_{\mathbf{p},11} = f_{\mathbf{p},22} = 0$ , i.e., eliminating creation and annihilation of equal spins. As a result,  $A_{H,\mathbf{p}}$  decays into a direct sum of two  $\mathbb{C}^{4 \times 4}$ -matrices, one for  $s \in \{1, 4\}$  and one for  $s \in \{2, 3\}$ , of the form

$$A_{H,\mathbf{p}}(t) = \begin{pmatrix} E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t) & 0 & 0 & -f_{\mathbf{p}}(t) \\ 0 & E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t) & f_{\mathbf{p}}(t) & 0 \\ 0 & f_{\mathbf{p}}(t) & -E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t) & 0 \\ -f_{\mathbf{p}}(t) & 0 & 0 & -E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t) \end{pmatrix}, \quad (285)$$

with  $f_{\mathbf{p}} \in \{f_{\mathbf{p},12}, f_{\mathbf{p},21}\}$ . Setting  $\varepsilon_{\mathbf{p},+}(t) = E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t)$  and  $\varepsilon_{\mathbf{p},-}(t) = E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t)$ , we arrive at the form (179).

*Data Availability.* Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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## References

- [1] H. Araki: On the Diagonalization of a Bilinear Hamiltonian by a Bogoliubov Transformation *Publ. Res. Inst. Math. Sci.* **4.2**: 387–412 (1968)
- [2] V. Bach, J.B. Bru: Diagonalizing Quadratic Bosonic Operators by Non-Autonomous Flow Equation *Mem Am Math Soc.* **240**: 1138 (2016) <https://arxiv.org/abs/1608.05591v1>
- [3] P.T. Nam, M.Napiórkowski, J.P.Solovej: Diagonalization of bosonic quadratic Hamiltonians by Bogoliubov transformations *J. Funct. Anal.* **270.11**: 4340–4368 (2016) <https://arxiv.org/abs/1508.07321v2>
- [4] Y. Matsuzawa, I. Sasaki, K. Usami: Explicit diagonalization of pair interaction models *Anal. Math. Phys.* **11**: 48 (2021) <https://arxiv.org/abs/1910.13487>
- [5] B. Schlein: Bogoliubov excitation spectrum for Bose-Einstein condensates *Proceedings of the ICM 2018* 2669–2686 (2019) <https://arxiv.org/abs/1802.06662v1>

- [6] M. Falconi, E.L. Giacomelli, C. Hainzl, M. Porta: The Dilute Fermi Gas via Bogoliubov Theory *Ann. Henri Poincaré* **22**: 2283—2353 (2021) <https://arxiv.org/abs/2006.00491v1>
- [7] D. Shale: Linear Symmetries of Free Boson Fields *Trans. Amer. Math. Soc.* **103**: 149–167 (1962)
- [8] S.N.M. Ruijsenaars: On Bogoliubov transformations for systems of relativistic charged particles *J. Math. Phys.* **18**: 517 (1977)
- [9] C.G. Torre, M. Varadarajan: Functional Evolution of Free Quantum Fields *Classical and Quantum Gravity* **16**: 2651–2668 (1999) <https://arxiv.org/abs/hep-th/9811222>
- [10] P. Marecki: Quantum Electrodynamics on Background External Fields *PhD Thesis*, University of Hamburg (2003) <https://arxiv.org/pdf/hep-th/0312304.pdf>
- [11] J. Bardeen, L.N. Cooper, J.R. Schrieffer: Theory of Superconductivity *Phys. Rev.* **108**: 1175 (1957)
- [12] R. Haag: The Mathematical Structure of the Bardeen-Cooper-Schrieffer Model *Nuovo Cimento* **25.2**: 287–299 (1962)
- [13] J. Dereziński: Bosonic quadratic Hamiltonians *J. Math. Phys.* **58**: 121101 (2017) <https://arxiv.org/abs/1608.03289v2>
- [14] S. Lill: Extended State Space for describing renormalized Fock spaces in QFT *Preprint* (2020) <https://arxiv.org/abs/2012.12608>
- [15] J.v. Neumann: On infinite tensor products *Comp. Math.* **6**: 1—77 (1939)
- [16] Yu.M. Berezansky, Y.G. Kondratiev: Spectral Methods in Infinite-Dimensional Analysis *Springer* (1995)
- [17] S. Albeverio, J.E. Fenstad, R. Høegh-Krohn, T. Lindstrøm: Nonstandard Methods in Stochastic Analysis and Mathematical Physics *Academic Press* (1986)
- [18] E. Nelson: Interaction of Nonrelativistic Particles with a Quantized Scalar Field *J. Math. Phys.* **5**: 1190 (1964)
- [19] J. Glimm: Boson fields with the  $:\Phi^4$ : Interaction in three dimensions *Comm. Math. Phys.* **10**: 1–47 (1968)
- [20] J. Dereziński: Van Hove Hamiltonians – Exactly Solvable Models of the Infrared and Ultraviolet Problem *Ann. Henri Poincaré* **4**: 713—738 (2003)
- [21] J. Glimm, A. Jaffe: The  $\lambda(\phi)_2^4$  quantum field theory without cutoffs. I *Phys. Rev.* **176**: 1945 (1968)

- [22] J. Glimm, A. Jaffe: Self-Adjointness of the Yukawa<sub>2</sub>, Hamiltonian *Annals of Physics* **60**: 321–383 (1970)
- [23] L. Gross: The Relativistic Polaron without Cutoffs *Comm. Math. Phys.* **31**: 25–73 (1973)
- [24] V. Chung: Infrared Divergence in Quantum Electrodynamics *Phys. Rev.* **140**: B1110 (1965)
- [25] T.W.B. Kibble: Coherent Soft-Photon States and Infrared Divergences. I. Classical Currents *J. Math. Phys.* **9**: 315 (1968)
- [26] P.P. Kulish, L.D. Faddeev: Asymptotic conditions and infrared divergences in quantum electrodynamics *Theor. Math. Phys.* **4**: 745—757 (1970)
- [27] J.P. Solovej: Many Body Quantum Mechanics *lecture notes* <http://web.math.ku.dk/~solovej/MANYBODY/mbnotes-ptn-5-3-14.pdf> (2014), visited on Jul 1, 2021
- [28] V. Bach, E.H. Lieb, J.P. Solovej: Generalized Hartree-Fock Theory and the Hubbard Model *J. Stat. Phys.* **76**: 3–89 (1994) <https://arxiv.org/abs/cond-mat/9312044>
- [29] J. Dereziński, C. Gérard: Mathematics of Quantization and Quantum Fields *Cambridge University Press* (2013)
- [30] B.C. Hall: Quantum Theory for Mathematicians *Springer* (2013)
- [31] M. Reed and B. Simon: Methods of Modern Mathematical Physics I: Functional Analysis *Academic Press* (1980)
- [32] K. Sanders: Essential self-adjointness of Wick squares in quasi-free Hadamard representations on curved spacetimes *J. Math. Phys.* **53**: 042502 (2012) <https://arxiv.org/abs/1010.3978>
- [33] L. Bruneau, J. Dereziński: Bogoliubov Hamiltonians and one-parameter groups of Bogoliubov transformations *J. Math. Phys.* **48**: 022101 (2007) <https://arxiv.org/abs/math-ph/0511069v1>
- [34] F. Trèves: Topological Vector Spaces, Distributions and Kernels *Academic Press* (1967)
- [35] H.H. Schaefer, M.P. Wolff: Topological Vector Spaces *Springer* (1999)
- [36] A. Grothendieck: Résumé des résultats essentiels dans la théorie des produits tensoriels topologiques et des espaces nucléaires *Ann. de l'Institut Fourier* **4**: 73–112 (1952)

- [37] E.P. Wigner: Phenomenological Distinction between Unitary and Antiunitary Symmetry Operators *J. Math. Phys.* **1**: 414 (1960)
- [38] E.P. Wigner: Normal Form of Antiunitary Operators *J. Math. Phys.* **1**: 409 (1960)
- [39] F.A. Berezin: The Method of Second Quantization *Academic Press* (1992)
- [40] N. Datta, R. Fernandez, J. Fröhlich: Effective Hamiltonians and Phase Diagrams for Tight-Binding Models *J. Stat. Phys.* **96.3**: 545—611 (1999) <https://arxiv.org/abs/math-ph/9809007>
- [41] J. Fröhlich: Application of commutator theorems to the integration of representations of Lie algebras and commutation relations *Comm. Math. Phys.* **54.2**: 135—150 (1977)