Classical Dynamics from Self-Consistency Equations in Quantum Mechanics – Extended Version

J.-B. Bru W. de Siqueira Pedra

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Abstract

During the last three decades, P. Bóna has developed a non-linear generalization of quantum mechanics, which is based on symplectic structures for normal states. One important application of such a generalization is a general setting which is very convenient to study the emergence of macroscopic classical dynamics from microscopic quantum processes. We propose here a new mathematical approach to Bona's non-linear quantum mechanics. It is based on C_0-semigroup theory and has a domain of applicability which is much brother than Bona's original one. It highlights the central role of self-consistency. This leads to a mathematical framework in which the classical and quantum worlds are naturally entangled. In this new mathematical approach, we build a Poisson bracket for the polynomial functions on the hermitian weak* continuous functionals on any C^*-algebra. This is reminiscent of a well-known construction for finite-dimensional Lie algebras. We then restrict this Poisson bracket to states of this C^*-algebra, by taking quotients with respect to Poisson ideals. This leads to densely defined symmetric derivations on the commutative C^*-algebras of real-valued functions on the set of states. Up to a closure, these are proven to generate C_0-groups of contractions. As a matter of fact, in general commutative C^*-algebras, even the closableness of unbounded symmetric derivations is a non-trivial issue. Some new mathematical concepts are introduced, which are possibly interesting by themselves: the convex weak* Gâteaux derivative, state-dependent C^*-dynamical systems and the weak*-Hausdorff hypertopology, a new hypertopology used to prove, among other things, that convex weak*-compact sets generically have weak*-dense extreme boundary in infinite dimension. Our recent results on macroscopic dynamical properties of lattice-fermion and quantum-spin systems with long-range, or mean-field, interactions corroborate the relevance of the general approach we present here. Note that this paper is an extended version of the corresponding published paper.

Keywords: C_0-semigroups, Poisson algebras, quantum mechanics, classical mechanics, self-consistency, hypertopology.

AMS Subject Classification: 81Q65, 47D06, 17B63, 81R15

Contents

1 Introduction

An indian, son of a dangerous witch,... *said to his wife: "It is my wish that you return with me to my mother's lodge – my home." His wife, knowing well who he was and who his mother was, readily consented to accompany him; by so doing she was faithfully carrying out the policy which her blind brother had advised her to pursue toward him. On their way homeward, while the husband was leading the trail, they came to a point where the path divided into two divergent ways which, however, after forming an oblong loop, reunited, forming once more only a single path. Here the woman was surprised to see her husband's body divide into two forms, one following the one path and the other the other trail. She was indeed greatly puzzled by this phenomenon, for she was at a loss to know which of the figures to follow as her husband. Fortunately, she finally resolved to follow the one leading to the right. After following this path for some distance, the wife saw that the two trails reunited and also that the two figures of her husband coalesced into one. It is said that this circumstance gave rise to the name of this strange man, which was Degiyanē'gĕñ'; that is to say, "They are two trails running parallel."*

Part of a Seneca[1] legend [1]

Recently, it was proven [2] that the Gross-Pitaevskii and Hartree hierarchies, which are infinite systems of coupled PDEs mathematically describing Bose gases with mean-field interactions, are equivalent to Liouville's equations for functions on a suitable phase space. This result is reminiscent of Hepp and Lieb's seminal paper [3] from year 1973, making explicit, for the first time, the existence of Poisson brackets in some space of functions, related to the classical effective dynamics for a permutation-invariant quantum-spin system with mean-field interactions. This research line was further developed by many other authors, at least until the nineties. For more details, see [4, Section 1]. We focus here on Bóna's impressive series of papers on the subject, starting in 1975 with [5]. In the middle of the eighties, his works [6, 7] lead him to consider a non-linear generalization of quantum mechanics. Based on his decisive progresses [8–11] on permutation-invariant quantum-spin systems with mean-field interactions, Bóna presents a full-fledged abstract theory in 1991 [12], which is improved later in a mature textbook published in 2000 (and revised in 2012) [13]. This theory does not seem to be incorporated by the physics and mathematics communities, yet.

Following [13, Section 1.1-a], Bóna's original motivation was to "*understand connections between quantum and classical mechanics more satisfactorily than via the limit* $\hbar \rightarrow 0$." This last limit refers to the semi-classical analysis, a well-developed research field in mathematics. In physics, it refers to Weyl quantization or, more generally, the quantization of classical systems with $\hbar$ as a deformation parameter. See, e.g., [14, Chapter 13]. This is the common understanding[2] of the relation between quantum and classical mechanics, which is seen as a limiting case of quantum mechanics, even if there exist physical features (such as the spin of quantum particles) which do not have a clear classical counterpart. Nonetheless, classical mechanics does not only appear in the limit $\hbar \rightarrow 0$, as explained for instance in [15, 16]. Bóna's major conceptual contribution is to highlight the possible emergence of classical mechanics without the disappearance of the quantum world, offering a general mathematical framework which is appropriate to study macroscopic coherence in large quantum systems.

Note that Bóna's view point is different from recent approaches of theoretical physics like [17–22] (see also references therein) which propose a general formalism to get a consistent description of interactions between classical and quantum systems, having in mind chemical reactions, decoherence or the quantum measurement theory. In these approaches [16–22], neither Bóna's papers nor Hepp and Lieb's results are mentioned, even if theoretical physicists are of course aware of the emergence of classical dynamics in presence of mean-field interactions. See, e.g., [16] where the mean-field (classical) theory corresponds to the leading term of a "large N" expansion while the quantum part of the theory (quantum fluctuations) is related to the next-to-leading order term. The approaches [17–22] (see also references therein) refer to quantum-classical hybrid theories for which the classical space exists by definition, in a *ad hoc* way, because of measuring instruments for instance. By contrast, the classical dynamics in Bóna's view point emerges as an *intrinsic* property of macroscopic quantum systems, like in [23]. This is also similar to [24], which is however a much more elementary example[3] referring to the Ehrenfest dynamics.

In the present paper we revisit Bóna's conceptual lines, but propose a new method to mathematically implement them, with a broader domain of applicability than Bóna's original version [13] (see also [15, 25] and references therein). In contrast with all previous approaches, including those of

[1]The Seneca was an important tribe of the Iroquois, the so-called Five Nations of New York. There is still a Seneca nation nowadays in the United States.

[2]At least in many textbooks on quantum mechanics. See for instance [14, Section 12.4.2, end of the 4th paragraph of page 178].

[3]It corresponds to a quantum systems with two species of particles in an extreme mass ratio limit: the particles of one species become infinitely more massive than the particles in the other one. In this limit, the species of massive particles, like nuclei, becomes classical while the other one, like electrons, stays quantum mechanical.

theoretical physics (see, e.g., [16–23]), in ours the classical and quantum worlds are *entangled*, with *backreaction*[4] (that is, feedbacks), as expected. Differently from Bóna's technical setting, ours has advantage of highlighting inherent *self-consistency* aspects, which are absolutely not exploited in [13], as well as in quantum-classical hybrid theories of physics (e.g., [17–24]).

The relevance of the abstract setting we propose here is corroborated by recent results [4,26] on the *macroscopic* (i.e., infinite-volume) dynamical properties of lattice-fermion and quantum-spin systems with long-range, or mean-field, interactions. Note that a simple illustration of them is available in [27, 28]. In fact, the outcomes of [4, 26] refer to objects that are far more general than quasi-free states or permutation invariant models and required the development of an appropriate mathematical framework to accommodate the macroscopic long-range dynamics, which turns out to be generally equivalent to an intricate combination of classical and short-range quantum dynamics. [4, 26] are therefore a strong motivation for a change of perspective, which is thus presented in a *systematic* way in the present paper. Several key ingredients of [4, 26] refer to abstract constructions discussed this paper, like the Poisson structures elaborated here. In other words, [4, 26] represent important applications, to the quantum many-body problem, of the general setting presented here.

To set up our approach, we use the algebraic formalism for quantum and classical mechanics [14, Chapter 12]. The most basic element of our mathematical framework is a generic non-commutative unital C^*-algebra $\mathcal{X}$, which will be called here the "primordial" algebra. For instance, $\mathcal{X}$ is the so-called CAR C^*-algebra for fermion systems or the spin C^*-algebras in the case of quantum spins. Then, the classical objects associated with $\mathcal{X}$ are defined as follows:

- *State and phase spaces* (Sections 2.1-2.3). The state space is the convex weak*-compact set E of *all* states on $\mathcal{X}$. We define the phase space as being the (weak*) closure[5] of the subset $\mathcal{E}(E) \subseteq E$ of all extreme points of E. Interestingly, in the case that the C^*-algebra $\mathcal{X}$ is antiliminal and simple (e.g., the CAR algebra associated with any separable infinite-dimensional one-particle Hilbert space, the spin algebra of any infinite countable lattice, etc.), the phase and state spaces coincide. More generally, by using a new (weak*) hypertopology, we show that this surprising property of the state space is *not* accidental, but *generic* in infinite-dimensional separable Banach spaces. Note that our definitions of phase and state spaces differ from Bóna's ones: he does not really distinguish both spaces and considers instead the set of all density matrices associated with a *fixed* Hilbert space [13, Section 2.1, see also 2.1-c]. In particular, Bóna's definition of the phase/state space is representation-dependent, in contrast with our approach. In fact, in [13, Sections 2.1c, footnote], Bóna proposes as a mathematically and physically interesting problem to "*formulate analogies of* [his] *constructions on the space of all positive normalized functionals on* $\mathcal{B}(\mathcal{H})$. *This leads to technical complications*." In Section 3.2 we propose a solution to this problem for any C^*-algebra $\mathcal{X}$.

- *Classical algebra* (Section 2.4). The classical (i.e., commutative) unital C^*-algebra[6] in our

[4] We do not mean here the so-called *quantum backreaction*, commonly used in physics, which refers to the backreaction effect of quantum fluctuations on the classical degrees of freedom. Note further that the phase spaces we consider are, generally, much more complex than those related to the position and momentum of simple classical particles.

[5] More properly, the phase space should be taken as being the set $\mathcal{E}(E)$ of extreme states on $\mathcal{X}$ itself. Note, however, that what is relevant in the algebraic approach is the algebra of continuous functions on the given topological space, and not the space itself. The algebras of continuous functions on the closure of $\mathcal{E}(E)$ is, of course, $*$-isomorphic to a C^*-subalgebra of continuous functions on $\mathcal{E}(E)$ and the closure of $\mathcal{E}(E)$ is taken to get a compact phase space, only.

[6] Analogously to the above distinction between phase and state spaces, more properly, the algebra related to the "classical world" should rather be the one of continuous functions on the phase space, but we expose in Section 2.5 the conceptual limitations of the use of this algebra in quantum physics. Moreover, in the case the C^*-algebra $\mathcal{X}$ is antiliminal and simple, both classical algebras are $*$-isomorphic to each other. In fact, the phase space turns out to be always conserved by the classical flows (in the state space) and we show that the classical dynamics studied in the present paper can always be pushed forward, by restriction of functions, from $\mathfrak{C}$ to the algebra of weak* continuous functions on the phase space.

approach is the algebra $\mathfrak{C} \doteq C(E;\mathbb{C})$ of continuous and complex-valued functions on the state space E.

- *Poisson structures* (Sections 3.4-3.5). By generalizing the well-known construction of a Poisson bracket for the polynomial functions on the dual space of finite dimensional Lie algebras [29, Section 7.1], we define a Poisson bracket for the polynomial functions on the hermitian continuous functionals (like the states) on any C^*-algebra $\mathcal{X}$. Then, the Poisson bracket is localized on the state or phase space associated with this algebra by taking quotients with respect to conveniently chosen Poisson ideals. This leads to a Poisson bracket for polynomial functions of the classical C^*-algebra $\mathfrak{C}$.

In our setting, we introduce state-dependent C^*-dynamical systems associated with the primordial algebra $\mathcal{X}$, as follows:

- *Secondary quantum algebra* (Section 5.1). Similar to quantum-classical hybrid theories of theoretical physics like in [17–22] we introduce an extended quantum algebra as being the[7] tensor product $\mathfrak{C} \otimes \mathcal{X}$ of the commutative C^*-algebra $\mathfrak{C}$ with the primordial one $\mathcal{X}$. This tensor product is nothing else (up to some $*$-isomorphism) than the unital C^*-algebra $\mathfrak{X} \doteq C(E;\mathcal{X})$, named here the *secondary* algebra associated with the *primordial* one, $\mathcal{X}$. There are natural inclusions $\mathcal{X} \subseteq \mathfrak{X}$ and $\mathfrak{C} \subseteq \mathfrak{X}$ by identifying elements of $\mathcal{X}$ with constant functions and elements of $\mathfrak{C}$ with functions whose values are scalar multiples of the unit of the primordial algebra $\mathcal{X}$. Note that in Bóna's approach, self-adjoint elements of $\mathfrak{X}$ refer to what he calls "*non-linear observables*" [13, Section 1.2.3].

- *State-dependent quantum dynamics* (Section 5.1). As in $\mathcal{X}$, a (possibly non-autonomous) quantum dynamics on $\mathfrak{X}$ is, by definition, a strongly continuous two-parameter family $\mathfrak{T} \equiv (\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{X}$ satisfying the reverse cocycle property:

$$\forall s,r,t \in \mathbb{R} : \qquad \mathfrak{T}_{t,s} = \mathfrak{T}_{r,s} \circ \mathfrak{T}_{t,r} \ .$$

  If $\mathfrak{T}$ preserves the classical algebra $\mathfrak{C} \subseteq \mathfrak{X}$, then we name the pair $(\mathfrak{X},\mathfrak{T})$ state-dependent, or secondary, C^*-dynamical system associated with the primordial algebra $\mathcal{X}$.

In this setting, the classical (i.e., commutative) and quantum (i.e., non-commutative) objects are strongly related to each other as follows:

- Any state-dependent C^*-dynamical system $(\mathfrak{X},\mathfrak{T})$ associated with $\mathcal{X}$, in the above sense, yields a classical dynamics on $\mathfrak{C}$, as explained in Section 5.2. This classical dynamics then induces a *Feller evolution system* [30], which in turn implies the existence of corresponding Markov transition kernels on E (which can be canonically identified with the Gelfand spectrum of the commutative unital C^*-algebra $\mathfrak{C}$). The full dynamics for (quantum) states on the primordial algebra $\mathcal{X}$ can then be recovered from the Markov transition kernels. A Feller evolution with similar properties also exists for the phase space (i.e., the closure of $\mathcal{E}(E)$).

- More interestingly, we remark in Section 4.2 that any *classical* differentiable Hamiltonian from $\mathfrak{C}$ is associated with a *state-dependent* quantum dynamics on the primordial C^*-algebra $\mathcal{X}$, in a natural way. This observation is then used to derive, mathematically, in Sections 4.3-4.4, classical dynamics associated with the Poisson structure of the (polynomial subalgebra of the) classical algebra $\mathfrak{C}$. These define again Feller evolution systems which turn out to be related to a self-consistency problem (Theorem 4.1). By Lemma 5.4, this yields, in turn, state-dependent quantum dynamics on the secondary (quantum) C^*-algebra $\mathfrak{X}$ of continuous ($\mathcal{X}$-valued) functions on states, associated with the primordial (quantum) algebra $\mathcal{X}$.

[7] Because commutative C^*-algebras are nuclear, the norm making the completion of the algebraic tensor product $\mathfrak{C}\otimes\mathcal{X}$ into a C^*-algebra is unique.

On the one hand, the classical world is embedded in the quantum world, as mathematically expressed by the above defined inclusion $\mathfrak{C} \subseteq \mathfrak{X}$. On the other hand, our approach entangles the quantum and classical worlds through self-consistency. As a result, *non-autonomous* and *non-linear* dynamics can emerge. Seeing both entangled worlds, quantum and classical, as "two sides of the same coin" looks like an oxymoron, but there is no contradiction there since everything refers to a *primordial* quantum world mathematically encoded in the structure of the non-commutative (unital) C^*-algebra $\mathcal{X}$. In fact, the quantum algebra $\mathcal{X}$ is the *arche*[8] of the theory. For instance, the state space E is the imprint left by $\mathcal{X}$ in the classical world, whose observables are the self-adjoint elements of the *commutative* C^*-algebra $\mathfrak{C} \doteq C(E;\mathbb{C})$, i.e., the continuous complex-valued functions on E. If $\mathcal{X}$ were a commutative algebra, note that the corresponding Poisson bracket and, hence, the associated classical dynamics would be trivial.

Note that the abstract setting proposed in this paper is not really useful to portray quantum dynamics of finite systems. In fact, in this case, the time evolution is *not* state-dependent. Nevertheless, as discussed above, such a mathematical framework turn out to be natural for the study of macroscopic dynamics of lattice-fermion or quantum-spin systems with long-range, or mean-field, interactions, because, in this case, the Heisenberg dynamics turns out to be effectively state dependent, in the thermodynamic limit. See again [4, 26], which uses self-consistency equations in a essential way, similar to Theorem 4.1. Moreover, since quantum many-body systems in the continuum are also expected to have, in general, a *state-dependent* Heisenberg dynamics in the thermodynamic limit (see, e.g., [32, Section 6.3]), the approach presented here is very likely relevant for future studies in this context. We thus consider important to have a systematic approach that can be used beyond specific applications, like [4, 26].

Our approach is not too far, in its spirit, to the one developed in [13], although it differs in its mathematical formulation. In comparison with [13], our formulation is more general in the case of an infinite-dimensional underlying C^*-algebra, which generally has several inequivalent irreducible representations, as a consequence of the Rosenberg theorem [33]: Whereas [13] has to use a representation of the underlying C^*-algebra to be able to define Poisson brackets in the associated classical algebra, we provide a definition for such brackets with no reference to representations. This is explained in more detail in Section 3.1. Notice at this point that, in condensed matter physics, the non-uniqueness of irreducible representations is intimately related to the existence of various inequivalent thermodynamically stable phases of the same material.

Last but not least, we observe that a large set of symmetric derivations can be defined on all polynomial elements of $\mathfrak{C}$ by using the Poisson bracket. See Section 3.6. These (unbounded) derivations are not a priori *closed* operators, but this property is necessary to generate (classical) dynamics, in its Hamiltonian formulation, via *strongly continuous semigroups*. In contrast with our approach, Bóna avoids this problem by using Hamiltonian flows in symplectic leaves of the corresponding Poisson manifold and by "gluing" together the flows within the leaves by showing continuity properties [13, Section 2.1-d].

The closabledness of a symmetric derivation is usually proven from its dissipativity [34, Definition 1.4.6, Proposition 1.4.7], which results from [34, Theorem 1.4.9] and the assumption that the square root of each positive element of the domain of the derivation also belongs to the same domain. We cannot expect this property to be satisfied for symmetric derivations acting on a dense domain of $\mathfrak{C}$. As a matter of fact, the closableness of unbounded symmetric derivations in commutative C^*-algebras like $\mathfrak{C}$ is, in general, a non-trivial issue. This property might not even be true since there exists norm-densely defined derivations of C^*-algebras that are *not* closable [35]. For instance, in [36, p. 306], it is even claimed that "Herman has constructed an extension of the usual differentiation on $C(0, 1)$ which

[8]Following Aristotle's use of the presocratic philosophical term "arche" (ἀρχὴ), here it means "the element or principle of a thing which, although undemonstrable and intangible in itself, provides the conditions of the possibility of that thing". See [31, p. 143].

is a non-closable derivation of $C(0,1)$." A complete classification of all closed symmetric derivations of functions on a compact subset of a *one-dimensional* space was obtained around 1990. However, quoting [34, Section 1.6.4, p. 27], "for more than 2 dimensions only *sporadic* results in this direction are known." See, e.g., [34, Section 1.6.4], [37], [38,39], and later [36, p. 306]. Since then, no progress has been made on this classification problem, at least to our knowledge.

In Section 4 (Theorem 4.5), via the analysis of certain self-consistency problems together with the one-parameter semigroups theory [40], we naturally obtain infinitely many closed symmetric derivations with dense domain in $\mathfrak{C}$. As it turns out, this method is very natural and efficient for the state space E, that is, a weak*-compact convex subset of the dual $\mathcal{X}^*$ of the unital (not necessarily separable) C^*-algebra $\mathcal{X}$, which is, in general, infinite-dimensional. In particular, E is generally *not* a subset of a finite-dimensional space. This construction of closed derivations of a commutative C^*-algebra via self-consistency problems is non-conventional and may motivate further studies. For more information, see Section 4.

Main results and structure of the paper. Recall that E is the state space of a non-commutative unital C^*-algebra $\mathcal{X}$. Our main results are the following:

- The *weak*-Hausdorff hypertopology* (Definitions 2.3 and 6.1) is a new notion, proposed here in order to characterize generic convex weak*-compact sets, by extending [41, 42] to weak* topological structures. We show in particular that convex weak*-compact sets of the dual space $\mathcal{X}^*$ of a (real or complex) Banach space $\mathcal{X}$ have generically weak*-dense set of extreme points in infinite dimension, in the sense of this new (hyper)topology. This refers to Theorems 2.4 and 2.5. These results has been extended in [43] for the dual space $\mathcal{X}^*$, endowed with its weak*-topology, of any infinite-dimensional, separable topological vector space $\mathcal{X}$.

- Corollary 3.6 defines, in a natural way, a Poisson bracket $\{\cdot,\cdot\}$ on polynomial functions of $C(E;\mathbb{C})$, while Corollary 3.7 shows that the restriction of $\{\cdot,\cdot\}$ to the phase space $\overline{\mathcal{E}(E)}$ also lead to a Poisson bracket on polynomial functions of $C(\overline{\mathcal{E}(E)};\mathbb{C})$. These Poisson brackets were previously used, for instance, in [4, 26].

- The *convex weak* Gateaux derivative* (Definition 3.8) is used to give an explicit expression for the Poisson bracket for functions on the state space E. This refers to Proposition 3.11, which is is an important result because it allows us to perform more explicit computations, both in this paper and in [4, 26].

- Theorem 4.1 shows the well-posedness of self-consistency equations, allowing us to define, for an appropriate continuous family $h \equiv (h(t))_{t\in\mathbb{R}} \subseteq C^1(E;\mathbb{R})$, a classical flow
$$\rho \mapsto \boldsymbol{\varpi}^h(s,t)(\rho), \qquad s,t\in\mathbb{R},$$
in the state space E and thus, a (generally non-autonomous classical) dynamics $(V_{t,s}^h)_{s,t\in\mathbb{R}}$ on $\mathfrak{C} \doteq C(E;\mathbb{C})$:
$$V_{t,s}^h(f) \doteq f\circ\boldsymbol{\varpi}^h(s,t), \qquad f\in\mathfrak{C},\ s,t\in\mathbb{R}.$$
Physically, the functions $h(t)$, $t\in\mathbb{R}$, are time-dependent classical energies. In Corollary 4.3, we show that the classical flow conserves both the set $\mathcal{E}(E)$ of extreme states and its weak* closure $\overline{\mathcal{E}(E)}$, which is the phase space.

- Proposition 4.4 proves that $(V_{t,s}^h)_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{C}$ satisfying the reverse cocycle property, i.e., the classical dynamics is a Feller evolution system [30].

- Theorem 4.5 shows that, given an appropriate function $h \in C^1(E;\mathbb{R})$, the Poissonian symmetric derivations
$$f \mapsto \{h, f\} \in \mathfrak{C}$$
defined for any polynomial functions f in $\mathfrak{C}$ is closable and is directly related to the generator of the C_0-group $(V_{t,0}^h)_{t\in\mathbb{R}}$ for the constant energy function h.

- Theorem 4.6 shows the non-autonomous evolution equations
$$\partial_t V_{t,s}^h(f) = V_{t,s}^h(\{h(t), f\}) \qquad \text{and} \qquad \partial_s V_{t,s}^h(f) = -\left\{h(s), V_{t,s}^h(f)\right\}$$
for any appropriate $h \equiv (h(t))_{t\in\mathbb{R}} \subseteq C^1(E;\mathbb{R})$, times $s, t \in \mathbb{R}$ and polynomial function f of $\mathfrak{C}$. In the autonomous case, i.e., when $h \in C^1(E;\mathbb{R})$, one gets Liouville's equation (Corollary 4.7), i.e.,
$$\partial_t V_{t,0}^h(f) = V_{t,0}^h(\{h, f\}) = \left\{h, V_{t,0}^h(f)\right\} .$$

- In Section 5.2, we show how the above classical dynamics defines a state-dependent quantum dynamics with *fixed-point* algebra including[9] the classical algebra $\mathfrak{C}$. This lead us to define a *state-dependent* C^*-dynamical system (Definition 5.3). See Lemma 5.4 and discussions afterwards. Such a quantum dynamics is relevant in the study of macroscopic dynamics of lattice-fermion systems or quantum-spin systems with long-range, or mean-field, interactions performed in [4, 26].

The paper is organized as follows: We first introduce, in Section 2, classical systems associated with arbitrary unital C^*-algebras. The Poisson structures for these systems are built in Section 3. Section 3 also gathers all the necessary definitions to describe, in Section 4, classical dynamics generated by a Poisson bracket, as is usual classical mechanics. Section 5 then explains the final setting of the theory. In Section 5.3 we discuss the role of symmetries as well as the notion of "reduction" of the classical dynamics. This is important in applications to simplify the self-consistency equations. Section 6 gives all arguments to deduce Theorems 2.4-2.5 by defining and studying the weak*-Hausdorff hypertopology. The proof of the most important result, that is, Theorem 4.1, is performed in Section 7, which also collects additional results used in Section 4.4. Finally, Section 8 is an appendix on liminal, postliminal and antiliminal C^*-algebras. Though these are standard notions in C^*-algebra theory, they may not be known by non-experts, but have major consequences on the structure of the set of states, which can be highly non-trivial and are relevant in our discussions.

Notation 1.1
(i) *A norm on a generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$ and the identity map of $\mathcal{X}$ by $\mathbf{1}_{\mathcal{X}}$. The space of all bounded linear operators on $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is denoted by $\mathcal{B}(\mathcal{X})$. The unit element of any algebra $\mathcal{X}$ is denoted by $\mathfrak{1}$, provided it exists. The scalar product of any Hilbert space $\mathcal{H}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$.*
(ii) *For all topological spaces $\mathcal{X}$ and $\mathcal{Y}$, $C(\mathcal{X};\mathcal{Y})$ denotes, as usual, the space of continuous maps from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X}$ is a locally compact topological space and $\mathcal{Y}$ is a Banach space, then $C_b(\mathcal{X};\mathcal{Y})$ denotes the Banach space of bounded continuous maps from $\mathcal{X}$ to $\mathcal{Y}$ along with the topology of uniform convergence. For any $p, n \in \mathbb{N}$, in the special case $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}$, $C_b^p(\mathbb{R}^n;\mathbb{R})$ denotes the Banach space of bounded continuous, real-valued, functions on $\mathbb{R}^n$ along with the topology of uniform convergence for the functions and all its m-th derivatives, where $m \in \{1, \ldots, p\}$.*
(iii) *We adopt the term "automorphism" in the sense of category theory and its precise meaning thus depends on the structure of the corresponding domain: An automorphism of a $*$-algebra is a bijective $*$-homomorphism from this algebra to itself, whereas an automorphism of a topological*

[9] This is a very important property, excluding the definition given by Equation (68).

space is a self-homeomorphism, that is, a homeomorphism of the space to itself. In fact, in the category of topological spaces the morphisms are precisely the continuous maps and the morphisms of the category of $*$-algebras are the $*$-homomorphisms. Recall that in category theory invertible morphisms are called isomorphisms and isomorphisms whose domain and codomain coincide are called automorphisms.

(iv) *In the sequel, a primordial C^*-algebra $\mathcal{X}$ is fixed and its state space is denoted by E. Then, various sets of functions on E are defined. The most important are $\mathfrak{C} \doteq C(E;\mathbb{C})$, $\mathfrak{X} \doteq C(E;\mathcal{X})$ and $\mathfrak{Y} \equiv \mathfrak{Y}(\mathcal{Y}) \doteq C^1(E;\mathcal{Y})$, $\mathcal{Y}$ being a Banach space, like $\mathbb{R}$ or $\mathbb{C}$. These spaces appear many times and we always use a shorter notation than the usual ones, like $C(E;\mathbb{C})$, the letter of the codomain within the Fraktur alphabet. More generally, any capital letter in Fraktur alphabet always refers to a space of functions on E. To denote real subspaces, we add the superscript $\mathbb{R}$ like in $\mathfrak{C}^{\mathbb{R}} \doteq C(E;\mathbb{R})$ or in $\mathcal{X}^{\mathbb{R}}$, which is the real Banach space of all self-adjoint elements of $\mathcal{X}$.*

2 Classical View on Quantum Systems

2.1 State Space of C^*-Algebras

Perhaps the philosophically most relevant feature of modern science is the emergence of abstract symbolic structures as the hard core of objectivity behind – as Eddington puts it – the colorful tale of the subjective storyteller mind.

Weyl, 1949 [44, Appendix B, p. 237]

Fix once and for all a C^*-algebra

$$\mathcal{X} \equiv (\mathcal{X}, +, \cdot_{\mathbb{C}}, \times, ^*, \|\cdot\|_{\mathcal{X}}) ,$$

that is, a (complex) Banach algebra endowed with an antilinear involution $A \mapsto A^*$ such that

$$(AB)^* = B^*A^* \qquad \text{and} \qquad \|A^*A\|_{\mathcal{X}} = \|A\|_{\mathcal{X}}^2 , \qquad A, B \in \mathcal{X} .$$

Here, $AB \equiv A \times B$. We always assume that $\mathcal{X}$ is unital, i.e., the product of $\mathcal{X}$ has a unit $\mathfrak{1} \in \mathcal{X}$. The (real) Banach subspace of all self-adjoint elements of $\mathcal{X}$ is denoted by

$$\mathcal{X}^{\mathbb{R}} \doteq \{A \in \mathcal{X} : A = A^*\} \equiv \left(\mathcal{X}^{\mathbb{R}}, +, \cdot_{\mathbb{R}}, \|\cdot\|_{\mathcal{X}}\right) . \tag{1}$$

The C^*-algebra $\mathcal{X}$ is named the *primordial* C^*-algebra. Note that it is not necessarily separable.

By [45, Theorem 3.10], the dual space $\mathcal{X}^*$ of $\mathcal{X}$ endowed with its weak* topology (i.e., the $\sigma(\mathcal{X}^*, \mathcal{X})$-topology of $\mathcal{X}^*$) is a locally convex space (in the sense of [45, Section 1.6]) whose dual is $\mathcal{X}$. Recall that $\mathcal{X}^*$ is a Banach space when it is endowed with the usual norm for linear functionals on a normed space. A subset of $\mathcal{X}^*$ which is pivotal in the algebraic formulation of quantum mechanics is the *state space* of $\mathcal{X}$, defined as follows:

Definition 2.1 (State space)
Let $\mathcal{X}$ be a unital C^*-algebra. The state space is the convex and weak*-closed set

$$E \doteq \bigcap_{A \in \mathcal{X}} \{\rho \in \mathcal{X}^* : \rho(A^*A) \geq 0,\ \rho(\mathfrak{1}) = 1\}$$

of all positive normalized linear functionals $\rho \in \mathcal{X}^*$.

Equivalently, $\rho \in \mathcal{X}^*$ is a state iff $\rho(\mathfrak{1}) = 1$ and $\|\rho\|_{\mathcal{X}^*} = 1$. Note that any state is hermitian: for all $\rho \in E$ and $A \in \mathcal{X}$, $\rho(A^*) = \overline{\rho(A)}$. From the Banach-Alaoglu theorem [45, Theorem 3.15], E is a weak*-compact subset of the unit ball of $\mathcal{X}^*$. Therefore, the Krein-Milman theorem [45, Theorem 3.23] tells us that E is the weak* closure of the convex hull of the (nonempty) set $\mathcal{E}(E)$ of its extreme points[10]:

$$E = \overline{\mathrm{co}\mathcal{E}\left(E\right)} \; . \tag{2}$$

The set $\mathcal{E}(E)$ is also called the extreme boundary of E. If $\mathcal{X}$ is separable then the weak* topology is metrizable on any weak*-compact subset of $\mathcal{X}^*$, by [45, Theorem 3.16]. In particular, the state space E of Definition 2.1 is metrizable, in this case, and by the Choquet theorem [46, p. 14], for any $\rho \in E$, there is a probability measure μ_ρ with support in $\mathcal{E}(E)$ such that, for any affine weak*-continuous complex-valued function g on E,

$$g\left(\rho\right) = \int_{\mathcal{E}(E)} g\left(\hat{\rho}\right) \mathrm{d}\mu_\rho\left(\hat{\rho}\right) \; . \tag{3}$$

The measure μ_ρ is unique for all $\rho \in E$, i.e., E is a Choquet simplex [36, Theorem 4.1.15], iff the C^*-algebra $\mathcal{X}$ is commutative, by [36, Example 4.2.6].

If E is not metrizable, meaning that $\mathcal{X}$ is not separable, note that such a probability measure μ_ρ is only pseudo–supported by $\mathcal{E}(E)$, i.e., $\mu_\rho(\mathcal{B}) = 1$ for all Baire sets $\mathcal{B} \supseteq \mathcal{E}(E)$. This refers to the Choquet-Bishop-de Leeuw theorem [46, p. 17]. Recall that the Baire sets are the elements of the σ-algebra generated by the compact G_δ sets. If $\mathcal{E}(E)$ is a Baire set then E must be metrizable [47]. The weak* closure $\overline{\mathcal{E}(E)}$ may even not be a G_δ set, or more generally a Baire set, when E is not metrizable. In fact, in the non-metrizable case, $\mathcal{E}(E)$ can have very surprising properties like being a *zero-measure* Borel set for μ_ρ (cf. [48]).

We use the state space E in the next section to define a classical algebra, the space $C(E;\mathbb{C})$ of complex-valued weak*-continuous functions on E. Note that our (quantum) state space E is different from the one considered in [13, Section 2.1, see also 2.1-c]. In Bóna's paper, the state space is defined to be the set of density matrices associated with a fixed Hilbert space. In relation to our approach, it corresponds to take, instead of *all* states of $\mathcal{X}$, only those which are π-normal, for some *fixed* represetation π of the C^*-algebra $\mathcal{X}$. Recall that the state $\rho \in E$ is called "π-normal" if the state $\rho \circ \pi$ on $\pi(\mathcal{X})$ has a (unique) normal extention to the von Neumann algebra $\pi(\mathcal{X})'' \supseteq \pi(\mathcal{X})$. By contrast, our definition of the (quantum) state space is *not* representation-dependent.

2.2 Phase Space of C^*-Algebras

Before the pioneer works of Jacobi and Boltzmann, then of Gibbs and Poincaré, the motion of a point-like particle was seen as a trajectory within the three-dimensional space. However, in classical mechanics, fixing only the position at a fixed time does not completely determine the trajectory, which only becomes unique after fixing the momentum. This leads to the term *phase*:

If we regard a phase as represented by a point in space of $2n$ dimensions, the changes which take place in the course of time in our ensemble of systems will be represented by a current in such space.

Gibbs, 1902 [49, p. 11, footnote]

This view point required the idea of "high dimensional" spaces, which widespread only in the first decade of the 20th century. This space refers to the illustrious concept of *phase space*, which seems to first appear in print in 1911 [50].

The historical origins of the notion of phase space can be found in [51], which makes explicit the "*tangle of independent discovery and misattributions that persist today*", even if this concept is

[10] I.e., the points which cannot be written as – non–trivial – convex combinations of other elements of E.

seen as "*one of the most powerful inventions of modern science*". For instance, the terminology of phase space is widely used in classical mechanics, and also in [13, Section 2.1], but its use is regularly confusing in many textbooks, which often view the state and phase spaces as the same thing.

The precise definition of phase space is an important, albeit non-trivial, issue in the understanding of a physical system because it is usually supposed to describe all its observable properties together with a deterministic motion, once the initial coordinates of the system is fixed in this phase space. In particular, it has to be sufficiently large to support a deterministic, or causal, motion.

In classical physics, the phase space is a locally compact Hausdorff space[11] K, like $\mathbb{R}^6$. In the algebraic formulation of classical mechanics [52, Chapter 3], one starts with a commutative C^*-algebra. By the Gelfand theorem (see, for instance, [36, Theorem 2.1.11A] or [52, Theorem 3.1]), such an algebra is $*$-isomorphic to the algebra $C_0(K;\mathbb{C})$ of all continuous functions $f : K \to \mathbb{C}$ vanishing at infinity, where K is a unique (up to a homeomorphism) locally compact Hausdorff space. In this case, K is, by definition, the phase space of the physical system. The phase space K is compact iff the commutative C^*-algebra is unital.

For *non*-commutative unital C^*-algebras, the definition of the associated phase space is less straightforward. To motivate the definition adopted here (Definition 2.2) for this space, we exhibit the relation between the phase space K and the state space E of Definition 2.1 for a commutative unital C^*-algebra seen as an algebra[12]

$$C\left(K;\mathbb{C}\right) \equiv \left(C\left(K;\mathbb{C}\right), +, \cdot_{\mathbb{C}}, \times, \overline{(\cdot)}, \|\cdot\|_{C(K;\mathbb{C})}\right)$$

of continuous complex-valued functions on the compact Hausdorff space K. Extreme points of E are the so-called characters of this C^*-algebra:

$$\mathcal{E}\left(E\right) = \left\{\mathfrak{c}\left(x\right) \in E : x \in K\right\} \ ,$$

where $\mathfrak{c}$ is the continuous and injective map from K to E defined by

$$\left[\mathfrak{c}\left(x\right)\right]\left(f\right) \doteq f\left(x\right) \ , \qquad f \in C\left(K;\mathbb{C}\right), \ x \in K \ . \tag{4}$$

Recall that the characters of a given C^*-algebra are, by definition, the unital $*$-homomorphisms from this algebras to $\mathbb{C}$ (i.e., the multiplicative hermitian functionals on the algebra). See [36, Proposition 2.3.27]. In this special case, $\mathcal{E}(E)$ is weak*-compact, like K, and the map $\mathfrak{c}$ is a homeomorphism. In particular, the map $f \mapsto \hat{f}$ from $C\left(K;\mathbb{C}\right)$ to $C(\mathcal{E}(E);\mathbb{C})$ defined by

$$\hat{f}\left(\mathfrak{c}\left(x\right)\right) = \left[\mathfrak{c}\left(x\right)\right]\left(f\right) \ , \qquad f \in C\left(K;\mathbb{C}\right), \ x \in K \ , \tag{5}$$

is a $*$-isomorphism of the commutative unital C^*-algebras $C\left(K;\mathbb{C}\right)$ and $C(\mathcal{E}(E);\mathbb{C})$. (See again [36, Theorem 2.1.11A] or [52, Theorem 3.1].) Therefore, as is usual, the phase space of any commutative unital C^*-algebra $\mathcal{X}$ can be identified with the weak*-compact set $\mathcal{E}(E)$ of extreme states of this algebra. The set of all characters of the commutative C^*-algebra $\mathcal{X}$ is called its (Gelfand) *spectrum* and its generalization to arbitrary C^*-algebras is not straightforward: Remark, for instance, that the algebra of $N \times N$ complex matrices, $N \geq 3$, has no characters, in the above sense, at all, by the celebrated Bell-Kochen-Specker theorem [52, Theorem 6.5]. The problem of properly defining a notion of spectrum for a general C^*-algebra is adressed, for instance, in [53, Chapters 3 & 4] in the context of decompositions of general representations of such an algebra in terms of its *irreducible* representations.

Now, with regard to the definition of the phase space as the set $\mathcal{E}(E) \neq E$ of extreme states, we want to emphasize that, for a *non-commutative* unital C^*-algebra $\mathcal{X}$, this set does *not* have to be

[11] I.e., a topological space whose open sets separate points (→Hausdorff) and whose points always have a compact neighborhood (→locally compact).

[12] $C\left(K;\mathbb{C}\right)$ is separable iff K is metrizable. See [54, Problem (d) p. 245].

weak*-closed (in E), and so weak*-compact. See, e.g., Lemma 8.5. As explained above, a classical physical system refers to the algebra of (complex-valued) continuous functions decaying at infinity on a locally compact Hausdorff space. Such an algebra is canonically $*$-isomorphic, via the restriction of functions, to a C^*-algebra of functions defined on any dense set of this Hausdorff space. Therefore, a natural definition of the (classical) phase space associated with a general quantum system, ensuring its compactness, is the weak* closure $\overline{\mathcal{E}(E)}$, instead of the set $\mathcal{E}(E)$ of extreme states itself:

Definition 2.2 (Phase space)
Let $\mathcal{X}$ be a unital C^*-algebra. The associated phase space is the weak* closure $\overline{\mathcal{E}(E)}$ of the extreme boundary of the state space E of Definition 2.1.

The phase space is, by definition, only a weak*-closed *subset* of the state space. However, in mathematical physics, the unital C^*-algebra associated with an infinitely extended (quantum) system is usually an approximately finite-dimensional (AF) C^*-algebra, i.e., it is generated by an increasing family of *finite-dimensional* C^*-subalgebras. They are all *antiliminal* (Definition 8.3) and *simple* (Definition 8.6). See Section 8 for more details. In this case, by Lemma 8.5, $\mathcal{E}(E)$ is weak*-dense in E, i.e.,

$$E = \overline{\mathcal{E}(E)} \ . \tag{6}$$

In other words, in general, the phase space of Definition 2.2 is the same as the state space of Definition 2.1 for infinitely extended quantum systems. The set E of states has therefore a *fairly complicated* geometrical structure. Compare, indeed, Equation (6) with (2). Provided the C^*-algebra $\mathcal{X}$ is separable, note that, surprisingly, (2) and (6) do not prevent E from having a unique center[13] [55].

2.3 Generic Weak*-Compact Convex Sets in Infinite Dimension

Accidens vero est quod adest et abest praeter subiecti corruptionem.[14]

An accident in the Middle Ages

The existence of convex sets with dense extreme boundary is well-known in infinite-dimensional vector spaces. For instance, the unit ball of any infinite-dimensional Hilbert space has a dense extreme boundary in the weak topology. In fact, a convex compact set with dense extreme boundary *is not an accident* in infinite-dimensional spaces, like Hilbert spaces or in the dual space of an antiliminal unital C^*-algebras (cf. (6) and Lemma 8.5).

In 1959, Klee shows [41] that, for convex norm-compact sets within a Banach space, the property of having a dense set of extreme points is *generic* in infinite dimension. More precisely, by [41, Proposition 2.1, Theorem 2.2], the set of all such convex compact subsets of an infinite-dimensional separable[15] Banach space $\mathcal{Y}$ is generic[16] in the complete metric space of compact convex subsets of $\mathcal{Y}$, endowed with the well-known Hausdorff metric topology [58, Definition 3.2.1]. Klee's result is refined in 1998 by Fonf and Lindenstraus [42, Section 4] for bounded norm-closed (but not necessarily

[13] I.e., a sort of maximally mixed point.

[14] Fr.: *L'accident est ce qui arrive et s'en va sans provoquer la perte du sujet.* See [56, V. L'accident]. It means that an accident is what is present or absent in a subject without affecting its essence. This comes from the *Isagoge* (ΕΙΣΑΓΩΓΗ, originally in greek) [56] written in the IIIe century by the Syrian Porphyry (of Tyr) as an introduction to *Aristotle's Categories*. The *Isagoge* was a pivotal textbook in medieval philosophy and more generaly on early logic during more than a millennium. Its reception by medieval (scholastic) philosophers has, in particular, initiated and fueled the celebrated *problem of universals* [57] from the XIIe to the XIVe century.

[15] [41, Proposition 2.1, Theorem 2.2] seem to lead to the asserted property for all (possibly non-separable) Banach spaces, as claimed in [41, 42, 59]. However, [41, Theorem 1.5], which assumes the separability of the Banach space, is clearly invoked to prove the corresponding density stated in [41, Theorem 2.2]. We do not know how to remove the separability condition.

[16] That is, the complement of a meagre set, i.e., a nowhere dense set.

norm-compact) convex subsets of $\mathcal{Y}$ having so-called empty quasi-interior (as a necessary condition). In this case, [42, Theorem 4.3] shows that such sets can be approximated in the Hausdorff metric topology by closed convex sets with a norm-dense set of strongly exposed points[17]. See, e.g., [59, Section 7] for a recent review on this subject.

In this section we demonstrate the same genericity in the dual space $\mathcal{X}^*$ of an infinite-dimensional, separable unital C^*-algebra $\mathcal{X}$, endowed with its weak*-topology. Of course, if one uses the usual norm topology on $\mathcal{X}^*$ for continuous linear functionals, then one can directly apply previous results [41, 42] to the separable Banach space $\mathcal{X}^*$. This is not anymore possible if one considers the weak*-topology. In particular, [42, Theorem 4.3] cannot be used because, in general, weak*-compact sets do not have an empty interior, in the sense of the norm topology. However, generic properties of convex weak*-compact sets, like the state space E of Definition 2.1, are relevent in the present paper. We thus prove, in this situation, results similar to [41, 42] in order to better understand the disconcerting structure of the state and phase spaces, respectively E and $\overline{\mathcal{E}(E)}$ defined above.

In order to talk about generic properties of convex weak*-compact sets, we first need to define an appropriate topological space of subsets of $\mathcal{X}^*$. It is naturally based on the set

$$\mathbf{CK}\left(\mathcal{X}^*\right) \doteq \left\{K \subseteq \mathcal{X}^* : K \neq \emptyset \text{ is convex and weak}^*\text{-compact}\right\} . \tag{7}$$

By Equation (83) and Lemma 6.5, note that

$$\mathbf{CK}\left(\mathcal{X}^*\right) = \left\{K \subseteq \mathcal{X}^* : K \neq \emptyset \text{ is convex, weak}^*\text{-closed and } \sup_{\sigma \in K} \|\sigma\|_{\mathcal{X}^*} < \infty\right\} . \tag{8}$$

This is a set of weak*-closed sets in a locally convex Hausdorff space $\mathcal{X}^*$. See, e.g., [60, Theorem 10.8].

We now make $\mathbf{CK}(\mathcal{X}^*)$ into a topological (hyper)space by defining a hypertopology on it. Recall that topologies for sets of closed subsets of topological spaces have been studied since the beginning of the last century and when such topologies, restricted to singletons, coincide with the original topology of the underlying space, we talk about hypertopologies and hyperspaces of closed sets. There exist several standard hypertopologies on the set of nonempty closed convex subsets of a topological space like, for instance, the slice topology [58, Section 2.4], the scalar and the linear topologies [58, Section 4.3]. Because of [58, Theorem 2.4.5], note that the slice topology is unappropriate here since it is not related to the weak*-topology of $\mathcal{X}^*$, but rather to its norm topology. In fact, we do not use any of those standard hypertopologies, but another natural topology on $\mathbf{CK}(\mathcal{X}^*)$ given by a family of pseudometrics[18] inspired by the Hausdorff metric topology for closed subsets of $\mathbb{C}$:

Definition 2.3 (Weak*-Hausdorff hypertopology for convex sets)
The weak*-Hausdorff hypertopology on $\mathbf{CK}(\mathcal{X}^*)$ is the topology induced by the family of Hausdorff pseudometrics $d_H^{(A)}$ defined, for all $A \in \mathcal{X}$, by

$$d_H^{(A)}(K, \tilde{K}) \doteq \max\left\{\max_{\sigma \in K} \min_{\tilde{\sigma} \in \tilde{K}} \left|(\sigma - \tilde{\sigma})(A)\right|, \max_{\tilde{\sigma} \in \tilde{K}} \min_{\sigma \in K} \left|(\sigma - \tilde{\sigma})(A)\right|\right\}, \qquad K, \tilde{K} \in \mathbf{CK}\left(\mathcal{X}^*\right) . \tag{9}$$

Compare (9) with the definition of the Hausdorff distance, given by (81). Definition 2.3 is a restriction of the weak*-Hausdorff hypertopology of Definition 6.1. In this topology, an arbitrary net $(K_j)_{j \in J}$ converges to K_∞ iff, for all $A \in \mathcal{X}$,

$$\lim_J d_H^{(A)}(K_j, K_\infty) = 0 . \tag{10}$$

[17] $x \in K$ is a strongly exposed point of a convex set $K \subseteq \mathcal{Y}$ when there is $f \in \mathcal{Y}^*$ satisfying $f(x) = 1$ and such that the diameter of $\{y \in K : f(y) \geq 1 - \varepsilon\}$ tends to 0 as $\varepsilon \to 0^+$. (Strongly) exposed points are extreme elements of K.

[18] Recall that a pseudometric d satisfies all properties of a metric but the identity of indiscernibles. In fact, $d(x, x) = 0$ but possibly $d(x, y) = 0$ for $x \neq y$.

This condition defines a unique topology in $\mathbf{CK}(\mathcal{X}^*)$, by [54, Chapter 2, Theorem 9]. In fact, because this topology is generated by a family of pseudometrics, it is a uniform topology, see, e.g., [54, Chapter 6].

It is completely obvious from the definition that any net $(\sigma_j)_{j\in J}$ in $\mathcal{X}^*$ converges to $\sigma \in \mathcal{X}^*$ in the weak* topology iff the net $(\{\sigma_j\})_{j\in J}$ converges in $\mathbf{CK}(\mathcal{X}^*)$ to $\{\sigma\}$ in the weak*-Hausdorff (hyper)topology. In other words, the embedding of $\mathcal{X}^*$ into $\mathbf{CK}(\mathcal{X}^*)$ is a bicontinuous bijection on its image. This justifies the use of the name weak*-Hausdorff *hyper*topology. We are not aware whether this particular hypertopology has already been considered in the past. We thus give in Section 6 its complete study along with interesting connections to other fields of mathematics and results that are more general than those stated in Section 2.3.

Endowed with the weak*-Hausdorff hypertopology, $\mathbf{CK}(\mathcal{X}^*)$ is a *Hausdorff* hyperspace. See Corollary 6.10. Observe also that the limit of weak*-Hausdorff convergent nets within $\mathbf{CK}(\mathcal{X}^*)$ is directly related to lower and upper limits à la Painlevé [61, § 29], as explained in Section 6.3. See, in particular, Equations (98) and (99). When $\mathcal{X}$ is a separable Banach space, Corollary 6.18 tells us that any weak*-Hausdorff convergent net $(K_j)_{j\in J} \subseteq \mathbf{CK}(\mathcal{X}^*)$ converges to its Kuratowski-Painlevé limit K_∞, which is thus the set of all weak* accumulation points of nets $(\sigma_j)_{j\in J}$ with $\sigma_j \in K_j$.

Recall that, by the Krein-Milman theorem [45, Theorem 3.23], any nonempty convex weak*-compact set $K \in \mathbf{CK}(\mathcal{X}^*)$ is the weak*-closure of the convex hull of the (nonempty) set $\mathcal{E}(K)$ of its extreme points:

$$K = \overline{\mathrm{co}\mathcal{E}\,(K)}\ .$$

The property $K = \overline{\mathcal{E}\,(K)}$ (with respect to the weak* topology) looks very peculiar. Nonetheless, as a matter of fact, typical elements of $\mathbf{CK}(\mathcal{X}^*)$ have this property:

Theorem 2.4 (Generic convex weak*-compact sets)
Let $\mathcal{X}$ be an infinite-dimensional separable Banach space. Then, the set $\mathcal{D}$ of all nonempty convex weak-compact sets K with a weak*-dense set $\mathcal{E}(K)$ of extreme points is a weak*-Hausdorff-dense G_δ subset of $\mathbf{CK}(\mathcal{X}^*)$.*

Proof. Combine Proposition 6.19 with Theorem 6.20. Note that the proof of Theorem 6.20 is crafted by following original Poulsen's intuitive construction [62], like in the proof of [42, Theorem 4.3]. The Hahn-Banach separation theorem [45, Theorem 3.4 (b)] plays a crucial role in this context. ■
As a consequence, $\mathcal{D}$ is generic in the hyperspace $\mathbf{CK}(\mathcal{X}^*)$, that is, the complement of a meagre set, i.e., a nowhere dense set. In other words, $\mathcal{D}$ is of second category in $\mathbf{CK}(\mathcal{X}^*)$.

The weak*-Hausdorff hypertopology on $\mathbf{CK}(\mathcal{X}^*)$ is finner than the scalar topology [58, Section 4.3] restricted to weak*-closed sets. The linear topology on the set of nonempty closed convex subsets is the supremum of the scalar and Wijsman topologies. Since the Wijsman topology [58, Definition 2.1.1] requires a metric space, one has to use the norm on $\mathcal{X}^*$ and the linear topology is not comparable with the weak*-Hausdorff hypertopology. If one uses the metric (108) generated the weak* topology on balls of $\mathcal{X}^*$ for a separable Banach space $\mathcal{X}$, then the Wijsman and linear topologies for norm-closed balls of $\mathcal{X}^*$ are coarser than the weak*-Hausdorff hypertopology, by Theorem 6.17. As a matter of fact, the Hausdorff metric topology is very fine, as compared to various standard hypertopologies (apart from the Vietoris[19] hypertopology). Consequently, the weak*-Hausdorff hypertopology can be seen as a very fine, weak*-type, topology on $\mathbf{CK}(\mathcal{X}^*)$. It shows that the density of the subset of all convex weak* compact sets with weak*-dense set of extreme points stated in Theorem 2.4 is a very *strong* property. Moreover, the genericity of such sets even holds true *inside* the state space E of any separable unital C^*-algebra:

Theorem 2.5 (Generic weak*-compact convex subset of the state space)
Let $\mathcal{X}$ be a infinite-dimensional, separable and unital C^*-algebra and E the state space (Definition

[19] Vietoris and Hausdorff metric topologies are not comparable.

2.1). Denote by $\mathbf{CK}(E)$ the set of all nonempty convex weak-compact subsets of E and by $\mathcal{D}(E)$ the set of all $K \in \mathbf{CK}(E)$ with a weak*-dense set $\mathcal{E}(K)$ of extreme points. Then, endowed with the weak*-Hausdorff hypertopology, $\mathbf{CK}(E)$ is a compact and completely metrizable hyperspace with $\mathcal{D}(E)$ being a dense G_δ subset.*

Proof. Since any state $\rho \in E$ has norm equal to $\|\rho\|_{\mathcal{X}^*} = 1$, we deduce from Theorem 6.17 that $\mathbf{CK}(E)$ belongs to the weak*-Hausdorff-compact and completely metrizable hyperspace $\mathbf{CK}_1(\mathcal{X}^*)$, defined by (106). By Corollary 6.18 and because E is a weak*-closed set, $\mathbf{CK}(E)$ is weak*-Hausdorff-closed, and thus a compact and completely metrizable hyperspace. It remains to prove that $\mathcal{D}(E)$ is a dense G_δ subset of $\mathbf{CK}(E)$.

The fact that $\mathcal{D}(E)$ is a G_δ subset of $\mathbf{CK}(E)$ can directly be deduced from the proof of Proposition 6.19 by repacing $\mathcal{F}_{D,m}$ with

$$\mathcal{F}_m(E) \doteq \{K \in \mathbf{CK}(E) : \exists \omega \in K,\ B(\omega, 1/m) \cap \mathcal{E}(K) = \emptyset\} \subseteq \mathbf{CK}(E) .$$

To prove the weak*-Hausdorff-density of $\mathcal{D}(E) \subseteq \mathbf{CK}(E)$, it suffices to reproduce the proof of Theorem 6.20, by adding one essential ingredient: the decomposition of any continuous linear functional into non-negative components proven in [63] for real Banach spaces. By noting that (i) $\mathcal{X}^{\mathbb{R}}$ (1) is a real Banach space, (ii) all states are hermitian functionals over $\mathcal{X}$, (iii) $(\mathcal{X}^{\mathbb{R}})^*$ is canonically identify with the real space of hermitian elements of $\mathcal{X}^*$, and (iv) any $\sigma \in \mathcal{X}^*$ is decomposed as $\sigma = \mathrm{Re}\{\sigma\} + i\mathrm{Im}\{\sigma\}$ with $\mathrm{Re}\{\sigma\}, \mathrm{Im}\{\sigma\} \in (\mathcal{X}^{\mathbb{R}})^*$, we deduce from [63] that any $\sigma \in \mathcal{X}^*$ can be decomposed as

$$\sigma = c_1\rho_1 - c_2\rho_2 + i(c_3\rho_3 - c_4\rho_4) , \qquad c_1, c_2, c_3, c_4 \in \mathbb{R}_0^+,\ \rho_1, \rho_2, \rho_3, \rho_4 \in E . \tag{11}$$

At *Step 1* of the proof of Theorem 6.20, because of (11), we observe that there is a non-zero positive functional

$$\sigma_1 \in (\mathcal{X}^* \backslash \mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon}\}) .$$

So, we proceed by using σ_1 as a (non-zero) positive functional with norm $\|\sigma_1\|_{\mathcal{X}^*} \leq 1$ and the state

$$\omega_{n_\varepsilon+1} \doteq (1 - \lambda_1\|\sigma_1\|_{\mathcal{X}^*})\,\varpi_1 + \lambda_1\sigma_1 \in E ,$$

instead of (121). One then iterates the arguments, as explained in the proof of Theorem 6.20, using always a (non-zero) positive functional σ_n with norm $\|\sigma_n\|_{\mathcal{X}^*} \leq 1$ and

$$\omega_{n_\varepsilon+n} \doteq (1 - \lambda_n\|\sigma_n\|_{\mathcal{X}^*})\,\varpi_n + \lambda_n\sigma_n \in E ,$$

instead of (128), as already explained. In doing so, we ensure that the convex weak*-compact set K_∞ of Equation (132) belongs to $\mathcal{D}(E) \subseteq \mathbf{CK}(E)$. ■

Note that Theorem 2.5 does *not* directly follow from Theorem 2.4 because the complement of $\mathbf{CK}(E)$ is open and dense in $\mathbf{CK}(\mathcal{X}^*)$.

Important examples of (antiliminal and simple) C^*-algebras with state space $E \in \mathcal{D}(E) \subseteq \mathcal{D} \subseteq \mathbf{CK}(\mathcal{X}^*)$, i.e., satisfying (6), are the (even subalgebra of the) CAR C^*-algebras for (non-relativistic) fermions on the lattice. Quantum-spin systems, i.e., infinite tensor products of copies of some elementary finite dimensional matrix algebra, referring to a spin variable, are also important examples. They are, for instance, widely used in quantum information theory as well as in condensed matter physics. In all these physical situations, the corresponding (non-commutative) C^*-algebra $\mathcal{X}$ is separable and E is thus a metrizable weak*-compact convex set. It is *not* a simplex [36, Example 4.2.6], but

$$E = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{P}_n} \tag{12}$$

is the weak*-closure of the union of a strictly increasing sequence $(\mathfrak{P}_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(E)$ of Poulsen simplices[20] [62]. Equation (12) is a consequence of well-known results (see, e.g., [60, 64]) and we give its complete proof in [4]. In other words, by Proposition 6.14, E is the *weak*-Hausdorff limit* of the increasing sequence $(\mathfrak{P}_n)_{n\in\mathbb{N}}$ within the set $\mathcal{D}(E)$ of all $K \in \mathbf{CK}(E)$ with weak*-dense set of extreme points.

Note that the Poulsen simplex $\mathfrak{P}$ is *not only* a metrizable simplex with dense extreme boundary $\mathcal{E}(\mathfrak{P})$. It has also the following remarkable properties:

- It is *unique*, up to an affine homeomorphism. Indeed, any two compact metrizable simplexes with dense extreme boundary are mapped into each other by an affine homeomorphism, by [65, Theorem 2.3].

- It is *universal* in the sense that every compact metrizable simplex is affinely homeomorphic to a (closed) face[21] of $\mathfrak{P}$, by [65, Theorem 2.5]. As a consequence, by [36, Example 4.2.6], the state space of *any* classical system with separable phase space can be seen as a face of $\mathfrak{P}$. Moreover, by [66], *every* Polish space[22] is homeomorphic to the extreme boundary of a face of $\mathfrak{P}$.

- It is *homogeneous* in the sense that any two proper closed isomorphic[23] faces of $\mathfrak{P}$ are mapped into each other by an affine automorphism of $\mathfrak{P}$. See [65, Theorem 2.3].

Together with Equation (12) this demonstrates, for infinite-dimensional quantum systems, the amazing structural richness of the state space E, while making mathematically clear the possible identification of the phase space $\overline{\mathcal{E}(E)}$ as the state space E.

In fact, because of Theorems 2.4-2.5, if the "primordial" (non-commutative) algebra $\mathcal{X}$ has *infinite dimension*, then, as is done without much attention in many textbooks, one should expect that the state and phase spaces, as we define them in the present paper, are identical, even if this feature has to be mathematically proven in each case (like for antiliminal and simple $\mathcal{X}$). For instance, if $\mathcal{X}$ is an infinite-dimensional, commutative and unital C^*-algebra, then the state and phase spaces, respectively E and $\overline{\mathcal{E}(E)}$, are cleary different from each other, even if E can always be approximated in the weak*-Hausdorff hypertopology by a convex weak*-compact set $K \subseteq E$ with weak*-dense extreme boundary, by Theorem 2.5.

2.4 Classical C^*-Algebra of Continuous Functions on the State Space

The space $C(E;\mathbb{C})$ of complex-valued weak*-continuous functions on the state space E of Definition 2.1, endowed with the point-wise operations and complex conjugation, is a unital *commutative* C^*-algebra denoted by

$$\mathfrak{C} \doteq \left(C\left(E;\mathbb{C}\right), +, \cdot_{\mathbb{C}}, \times, \overline{(\cdot)}, \|\cdot\|_{\mathfrak{C}} \right) , \tag{13}$$

where

$$\|f\|_{\mathfrak{C}} \doteq \max_{\rho\in E} |f(\rho)| , \qquad f \in \mathfrak{C} . \tag{14}$$

The (real) Banach subspace of all real-valued functions from $\mathfrak{C}$ is denoted by $\mathfrak{C}^{\mathbb{R}} \subsetneq \mathfrak{C}$. If $\mathcal{X}$ is separable then $\mathfrak{C}$ is also separable, E being in this case metrizable. See, e.g., [54, Problem (d) p. 245].

[20] It is the (unique up to a homeomorphism) metrizable simplex with dense extreme boundary.

[21] A face F of a convex set K is defined to be a subset of K with the property that, if $\rho = \lambda_1\rho_1 + \cdots + \lambda_n\rho_n \in F$ with $\rho_1, \ldots, \rho_n \in K$, $\lambda_1, \ldots, \lambda_n \in (0,1)$ and $\lambda_1 + \cdots + \lambda_n = 1$, then $\rho_1, \ldots, \rho_n \in F$.

[22] I.e., a separable topological space that is homeomorphic to a complete metric space.

[23] I.e, there is an affine homeomorphism between both faces.

Similar to the mapping defined by Equation (5) for commutative C^*-algebras, elements of the unital C^*-algebra $\mathcal{X}$ canonically define continuous affine functions $\hat{A} \in \mathfrak{C}$ by

$$\hat{A}(\rho) \doteq \rho(A) , \qquad \rho \in E, \ A \in \mathcal{X} . \tag{15}$$

This is the well-known *Gelfand transform*. Note that $A \neq B$ yields $\hat{A} \neq \hat{B}$, as states separates elements of $\mathcal{X}$. Since $\mathcal{X}$ is a (unital) C^*-algebra,

$$\|A\|_{\mathcal{X}} = \max_{\rho \in E} |\rho(A)| , \qquad A \in \mathcal{X}^{\mathbb{R}} , \tag{16}$$

and hence, the map $A \mapsto \hat{A}$ defines a linear isometry from the Banach space $\mathcal{X}^{\mathbb{R}}$ of all self-adjoint elements (cf. Equation (1)) to the space $\mathfrak{C}^{\mathbb{R}}$ of all real-valued functions on E.

For any self-adjoint[24] subspace $\mathcal{B} \subseteq \mathcal{X}$, we define the $*$-subalgebras

$$\mathfrak{C}_{\mathcal{B}} \equiv \mathfrak{C}_{\mathcal{B}}(E) \doteq \mathbb{C}[\{\hat{A} : A \in \mathcal{B}\}] \subseteq \mathfrak{C} \quad \text{and} \quad \mathfrak{C}_{\mathcal{B}}^{\mathbb{R}} \equiv \mathfrak{C}_{\mathcal{B}}^{\mathbb{R}}(E) \doteq \mathbb{R}[\{\hat{A} : A \in \mathcal{B} \cap \mathcal{X}^{\mathbb{R}}\}] \subseteq \mathfrak{C}^{\mathbb{R}} , \tag{17}$$

where $\mathbb{K}[\mathcal{Y}] \subseteq \mathfrak{C}$ denotes the $\mathbb{K}$-algebra generated by $\mathcal{Y}$, i.e., the subspace of polynomials in the elements of $\mathcal{Y}$, with coefficients in the field $\mathbb{K}$ $(= \mathbb{R}, \mathbb{C})$. The unit $\hat{\mathfrak{1}} \in \mathfrak{C}$, being the constant map $\hat{\mathfrak{1}}(\rho) = 1$ for $\rho \in E$ (cf. Definition 2.1), belongs, by definition, to $\mathfrak{C}_{\mathcal{B}}$ and $\mathfrak{C}_{\mathcal{B}}^{\mathbb{R}} \subseteq \mathfrak{C}_{\mathcal{B}}$. If $\mathcal{B}$ is dense in $\mathcal{X}$ then $\mathfrak{C}_{\mathcal{B}}$ separates states. Therefore, by the Stone-Weierstrass theorem [67, Chap. V, §8], for any dense self-adjoint subset $\mathcal{B} \subseteq \mathcal{X}$, $\mathfrak{C}_{\mathcal{B}}$ is dense in $\mathfrak{C}$, i.e., $\mathfrak{C} = \overline{\mathfrak{C}_{\mathcal{B}}}$.

2.5 Classical C^*-Algebra of Continuous Functions on the Phase Space

If the weak*-compact set $\overline{\mathcal{E}(E)}$ is supposed to play the role of a phase space (cf. Definition 2.2), then a classical dynamics should be defined on the space $C(\overline{\mathcal{E}(E)}; \mathbb{C})$ of complex-valued weak*-continuous functions on $\overline{\mathcal{E}(E)}$. Endowed with the usual point-wise operations and complex conjugation, it is again a unital *commutative* C^*-algebra. Of course, there is a natural $*$-homomorphism $\mathfrak{C} \to C(\overline{\mathcal{E}(E)}; \mathbb{C})$, by restriction on $\overline{\mathcal{E}(E)}$ of functions from $\mathfrak{C}$. Recall that $C(\overline{\mathcal{E}(E)}; \mathbb{C})$ is canonically $*$-isomorphic, via the restriction on $\mathcal{E}(E)$ of functions, to a C^*-subalgebra of $C(\mathcal{E}(E); \mathbb{C})$. In Corollary 4.3 and Equation (73), we show that the classical dynamics constructed in the present paper *can be pushed forward*, through the restriction map, from $\mathfrak{C}$ to either $C(\overline{\mathcal{E}(E)}; \mathbb{C})$ or $C(\mathcal{E}(E); \mathbb{C})$. The generator of the dynamics on $C(\overline{\mathcal{E}(E)}; \mathbb{C})$ can be expressed on polymonials via the Poisson bracket of Corollary 3.7, by Proposition 3.11.

In standard classical mechanics, in the case of compact phase spaces, even if the C^*-algebra $\mathfrak{C}$ is always well-defined, note that $\mathfrak{C}$ is usually never used, but rather $C(\overline{\mathcal{E}(E)}; \mathbb{C})$, and a classical system is always supposed to be in some extreme state. In fact, the same physical object cannot be at the same time on two distinct points of the phase space, according to the spatio-temporal identity of classical mechanics [68]. This refers to Leibniz's Principle of Identity of Indiscernibles[25]. This is related to the fact that any extreme classical state is dispersion-free, see [52, Eq. (6.3), V being the state]. In the classical situation, the space $\mathfrak{C}$ is therefore *not* fundamental: In this case, by the Riesz–Markov theorem, the state space is the same as the set of probability measures on the phase space $\overline{\mathcal{E}(E)}$ and a mixed, or non-extreme, state $\rho \in E \backslash \mathcal{E}(E)$ of a classical system is only used to reflect the lack of knowledge on the physical object along with a probabilistic interpretation. Compare with (3).

For quantum systems, this property is not as evident as it is for classical ones, as conceptually discussed for instance in [68]. The spatio-temporal identity of classical mechanics is questionable in

[24] This means that $A \in \mathcal{B}$ implies $A^* \in \mathcal{B}$.

[25] Leibniz's Principle of Identity of Indiscernibles [68, p. 1]: "*Two objects which are indistinguishable, in the sense of possessing all properties in common, cannot, in fact, be two objects at all. In effect, the Principle provides a guarantee that individual objects will always be distinguishable.*"

quantum mechanics. This is correlated with the celebrated EPR paradox of Einstein, Podolsky and Rosen. See also Einstein's conceptual opposition to quantum mechanics:

If one asks what, irrespective of quantum mechanics, is characteristic of the world of ideas of physics, one is first of all struck by the following: the concepts of physics relate to a real outside world... it is further characteristic of these physical objects that they are thought of as a range in a space-time continuum. An essential aspect of this arrangement of things in physics is that they lay clamed, at a certain time, to an existence independent of one another, provided these objects "are situated in different parts of space".

Einstein, 1948 [69]

The non-locality of quantum mechanics was in fact Einstein's main criticism on this theory [70], more than its weakly deterministic features.

The non-locality of quantum mechanics has been experimentally verified, for instance via Bell's inequalities, and it is not the subject of the present paper to discuss further related topics, like the existence of hidden variables in quantum physics. The point in this brief discussion is that there is no clear reason to restrict ourselves to the phase space $\overline{\mathcal{E}(E)}$ and not also consider the whole state space E, as, in contrast to classical physics, extreme states are not anymore dispersion-free for quantum systems. See, e.g., [52, Proposition 2.10]; cf. also the Bell-Kochen-Specker theorem [52, Theorem 6.5]. As a matter of fact, important phenomena, like the breakdown of the $U(1)$-gauge symmetry in the BCS theory of superconductivity, are related with non-extreme states. See, as an example, [71, Theorem 6.5]. What's more, the phase space and the state space turn out to be *identical* for important classes of (infinitely extended) quantum systems in condensed matter physics, as already explained. See Equation (6).

3 Poisson Structures in Quantum Mechanics

If $\mathfrak{g}$ is a finite dimensional Lie algebra, there is a standard contruction of a Poisson bracket for the polynomial functions on its dual space $\mathfrak{g}^*$. See, for instance, [29, Section 7.1]. Observe that the (real) space $\mathcal{X}^{\mathbb{R}}$ of all self-adjoint elements of an arbitrary C^*-algebra $\mathcal{X}$ forms a Lie algebra by endowing it with the Lie bracket $i[\cdot,\cdot]$, i.e., the skew-symmetric biderivation on $\mathcal{X}^{\mathbb{R}}$ defined by the commutator

$$i\,[A,B] \doteq i\,(AB-BA) \in \mathcal{X}^{\mathbb{R}}\ , \qquad A,B\in\mathcal{X}^{\mathbb{R}}\ . \tag{18}$$

One of the aims of our paper is to extend such a construction of a Poisson bracket to polynomial functions on the dual space of $\mathcal{X}^{\mathbb{R}}$, which is possibly infinite-dimensional. Before doing that, we first briefly present Bóna's setting [13, Sections 2.1b, 2.1c], which motivated the present work.

3.1 Bóna's Poisson Structures

Bóna [13, Sections 2.1b, 2.1c] proposes a Poisson structure for polynomial functions on the *predual* (instead of the dual) of a C^*-algebra. Recall that, if $\mathcal{X}$ is the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$, then its predual $\mathcal{X}_*$ can be identified with the Banach space $\mathcal{L}^1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$, with the (trace) norm

$$\|A\|_1 \doteq \mathrm{Tr}_{\mathcal{H}}\sqrt{A^*A}\ , \qquad A\in\mathcal{L}^1(\mathcal{H})\ .$$

More precisely, for all $A\in\mathcal{B}(\mathcal{H})$ $(=\mathcal{X})$, the linear map $\hat{A}$ defined by

$$\sigma \mapsto \mathrm{Tr}_{\mathcal{H}}(\sigma A)$$

from $\mathcal{L}^1(\mathcal{H})$ to $\mathbb{C}$ is continuous and, conversely, any linear continuous functional $\hat{A} : \mathcal{L}^1(\mathcal{H}) \to \mathbb{C}$ is of this form for a unique $A \in \mathcal{B}(\mathcal{H})$. From this, one concludes that the dual of the real Banach space $\mathcal{L}^1_{\mathbb{R}}(\mathcal{H})$ of self-adjoint trace-class operators on $\mathcal{H}$ is the real Banach space $\mathcal{B}(\mathcal{H})^{\mathbb{R}}$ of self-adjoint bounded operators on the Hilbert space $\mathcal{H}$. Thus,

$$\mathcal{B}(\mathcal{H})^{\mathbb{R}} \equiv (\mathcal{L}^1_{\mathbb{R}}(\mathcal{H}))^* \subseteq C(\mathcal{L}^1_{\mathbb{R}}(\mathcal{H}); \mathbb{R}). \tag{19}$$

Let

$$\mathfrak{C}^{\mathbb{R}}_{\mathcal{B}(\mathcal{H})^{\mathbb{R}}} \doteq \mathbb{R}[\mathcal{B}(\mathcal{H})^{\mathbb{R}}] \subseteq C(\mathcal{L}^1_{\mathbb{R}}(\mathcal{H}); \mathbb{R})$$

be the subalgebra of polynomials in the elements of $\mathcal{B}(\mathcal{H})^{\mathbb{R}}$ with real coefficients. The elements of this subalgebra are called "polynomial" functions on $\mathcal{L}^1_{\mathbb{R}}(\mathcal{H})$, the *predual* of the Lie algebra $(\mathcal{B}(\mathcal{H})^{\mathbb{R}}, i[\cdot,\cdot])$. In [13, Sections 2.1c], Bóna proves the existence of a unique Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{C}^{\mathbb{R}}_{\mathcal{B}(\mathcal{H})^{\mathbb{R}}}$, i.e., of a skew-symmetric biderivation satisfying the Jacobi identity on polynomial functions, such that

$$\{\hat{A}, \hat{B}\}(\sigma) = \mathrm{Tr}_{\mathcal{H}}(i[A,B]\sigma) = \widehat{i[A,B]}(\sigma), \qquad A, B \in \mathcal{B}(\mathcal{H})^{\mathbb{R}},\ \sigma \in \mathcal{L}^1_{\mathbb{R}}(\mathcal{H}) .$$

It turns out that the Poisson manifold $(\mathcal{L}^1_{\mathbb{R}}(\mathcal{H}), \{\cdot,\cdot\})$ has a non-trivial symplectic foliation: For any $\sigma \in \mathcal{L}^1_{\mathbb{R}}(\mathcal{H})$, we define its unitary orbit by

$$\mathrm{O}(\sigma) \doteq \{\mathrm{U}\sigma\mathrm{U}^* : \mathrm{U} \text{ a unitary operator on } \mathcal{H}\} \subseteq \mathcal{L}^1_{\mathbb{R}}(\mathcal{H}) . \tag{20}$$

If $\sigma \in \mathcal{L}^1_{\mathbb{R}}(\mathcal{H})$ has finite-dimensional range (i.e., $\dim \mathrm{ran}(\sigma) < \infty$), then $\mathrm{O}(\sigma)$ is a symplectic leaf of the Poisson manifold $(\mathcal{L}^1_{\mathbb{R}}(\mathcal{H}), \{\cdot,\cdot\})$. In particular, the restriction on such a leaf of the Poisson bracket of two functions f, g only depends on the restriction of f, g on the same leaf. Meanwhile, Bóna observes in [13, Lemma 2.1.7] that the union

$$\bigcup \left\{\mathrm{O}(\sigma) : \sigma \in \mathcal{L}^1_{\mathbb{R}}(\mathcal{H}),\ \sigma \geq 0,\ \mathrm{Tr}_{\mathcal{H}}(\sigma) = 1,\ \dim \mathrm{ran}(\sigma) < \infty\right\}$$

is dense in the set $\mathcal{S}_*$ of all normalized positive elements (i.e., density matrices) of $\mathcal{L}^1_{\mathbb{R}}(\mathcal{H})$. Using this observation, Bóna defines the Poisson bracket for polynomial functions defined on $\mathcal{S}_* \subseteq \mathcal{L}^1_{\mathbb{R}}(\mathcal{H})$, but he proposes [13, Sections 2.1c, footnote] as a mathematically and physically interesting problem to "*formulate analogies of* [his] *constructions on the space of all positive normalized functionals on* $\mathcal{B}(\mathcal{H})$. *This leads to technical complications*." In Sections 3.2 and 3.3 we give such a construction for the dual space of any C^*-algebra $\mathcal{X}$ (and not only for the special case $\mathcal{X} = \mathcal{B}(\mathcal{H})$). Sections 3.4-3.5 contribute an alternative, more explicit, construction of the same Poisson structure.

Remark 3.1
The construction given in the recent paper [72] for a Hamiltonian flow associated with Schrödinger's dynamics of one quantum particle corresponds to Bóna's symplectic leaf $\mathrm{O}(\sigma)$ *of density matrices* σ *of dimension one, i.e.,* $\dim \mathrm{ran}(\sigma) = 1$. *However, the author of [72] does not seem to be aware of Bóna's works.*

3.2 Poisson Algebra of Polynomial Functions on the Continuous Self-Adjoint Functionals on a C^*-Algebra

Recall that $(\mathcal{X}^{\mathbb{R}}, i[\cdot,\cdot])$ is a (possibly infinite-dimensional) Lie algebra. See (18). It is easy to check that the continuous (real) linear functionals $\mathcal{X}^{\mathbb{R}} \to \mathbb{R}$ are in one-to-one correspondance to the hermitian continuous (complex) linear functionals $\mathcal{X} \to \mathbb{C}$, simply by restriction to $\mathcal{X}^{\mathbb{R}} \subseteq \mathcal{X}$. Recall that a (complex) linear functional $\sigma : \mathcal{X} \to \mathbb{C}$ is, by definition, hermitian when

$$\sigma(A^*) = \overline{\sigma(A)} , \qquad A \in \mathcal{X} .$$

We denote by $\mathcal{X}_{\mathbb{R}}^{*}$ the (real) space of all hermitian elements of the (topological) dual space $\mathcal{X}^{*}$ and use the identification

$$\mathcal{X}_{\mathbb{R}}^{*} \equiv (\mathcal{X}^{\mathbb{R}})^{*} ,$$

as already done in the proof of Theorem 2.5. The space $\mathcal{X}_{\mathbb{R}}^{*}$ with $\mathcal{X} = \mathcal{B}(\mathcal{H})$ plays in our setting an analogous role as $\mathcal{L}_{\mathbb{R}}^{1}(\mathcal{H})$ in Bóna's approach [13, Sections 2.1b, 2.1c]. See Section 3.1.

Similar to (15), for any $A \in \mathcal{X}$, we define the weak*-continuous (complex) linear functional $\hat{A} : \mathcal{X}^{*} \rightarrow \mathbb{C}$ by

$$\hat{A}(\sigma) \doteq \sigma(A) , \qquad \sigma \in \mathcal{X}^{*} . \tag{21}$$

(Note that we use the same notation as in (15), for the canonical identification of $A \in \mathcal{X}$ with a linear functional on $\mathcal{X}^{*}$.) Any element of $\mathcal{X}^{**}$ is of this form, keeping in mind that the dual space $\mathcal{X}^{*}$ of $\mathcal{X}$ is here endowed with its weak* topology, see discussions before Definition 2.1. Note also that any weak*-continuous (real) linear functional on $\mathcal{X}_{\mathbb{R}}^{*}$ uniquely extends to a weak*-continuous (complex) linear hermitian functional on $\mathcal{X}^{*}$. In this case, by hermiticity, the corresponding $A \in \mathcal{X}$ belongs to $\mathcal{X}^{\mathbb{R}}$. Conversely, any $A \in \mathcal{X}^{\mathbb{R}}$ defines a weak*-continuous (real) linear functional $\hat{A} : \mathcal{X}_{\mathbb{R}}^{*} \rightarrow \mathbb{C}$, by restriction of (21) to $\mathcal{X}_{\mathbb{R}}^{*}$. Therefore, we identify the real Banach space $\mathcal{X}^{\mathbb{R}}$ of self-adjoint elements of the C^*-algebra $\mathcal{X}$ with the space of all weak*-continuous (real) linear functionals $\mathcal{X}_{\mathbb{R}}^{*} \rightarrow \mathbb{R}$, i.e.,

$$\mathcal{X}^{\mathbb{R}} \equiv (\mathcal{X}_{\mathbb{R}}^{*})^{*} . \tag{22}$$

In this view point, $\mathcal{X}^{\mathbb{R}} \subseteq C(\mathcal{X}_{\mathbb{R}}^{*};\mathbb{R})$. Let

$$\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}} \equiv \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*}) \doteq \mathbb{R}[\mathcal{X}^{\mathbb{R}}] \subseteq C(\mathcal{X}_{\mathbb{R}}^{*};\mathbb{R})$$

be the subalgebra of polynomials in the elements of $\mathcal{X}^{\mathbb{R}}$, with real coefficients. (Compare with (17) for $\mathcal{B} = \mathcal{X}^{\mathbb{R}}$.) The elements of this subalgebra are again called "polynomial" functions on $\mathcal{X}_{\mathbb{R}}^{*}$, the *dual* of the Lie algebra $(\mathcal{X}_{\mathbb{R}}, i[\cdot,\cdot])$.

Note that such polynomials are Gateaux differentiable and, for any $f \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}$ and any $\sigma \in \mathcal{X}_{\mathbb{R}}^{*}$, the Gateaux derivative $\mathrm{d}^G f(\sigma)$ is linear and weak* continuous, i.e., $\mathrm{d}^G f(\sigma) \in \mathcal{X}^{\mathbb{R}}$ (see (22)). In particular, for any $A \in \mathcal{X}$, by (21),

$$\mathrm{d}^G \hat{A}(\sigma) = A , \qquad \sigma \in \mathcal{X}_{\mathbb{R}}^{*} . \tag{23}$$

Thus, we can define a skew-symmetric biderivation $\{\cdot,\cdot\}_0$ on $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}$ as follows:

Definition 3.2 (Poisson bracket)
The skew-symmetric biderivation $\{\cdot,\cdot\}_0$ on $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}$ is defined by

$$\{f,g\}_0(\sigma) \doteq \sigma\left(i\left[\mathrm{d}^G f(\sigma), \mathrm{d}^G g(\sigma)\right]\right) , \qquad f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}} .$$

This skew-symmetric biderivation satisfies the Jacobi identity:

Proposition 3.3 (Usual properties of Poisson brackets)
$\{\cdot,\cdot\}_0$ is a Poisson bracket, i.e., it is a skew-symmetric biderivation satisfying the Jacobi identity

$$\{f,\{g,h\}_0\}_0 + \{h,\{f,g\}_0\}_0 + \{g,\{h,f\}_0\}_0 = 0 , \qquad f,g,h \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}} \doteq \mathbb{R}[\mathcal{X}^{\mathbb{R}}] .$$

Proof. $\{\cdot,\cdot\}_0$ is clearly skew-symmetric, by (18) and Definition 3.2. Note additionally that, for any $f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}$,

$$\mathrm{d}^G(f+g) = \mathrm{d}^G f + \mathrm{d}^G g \quad \text{and} \quad \mathrm{d}^G(fg) = f\mathrm{d}^G g + g\mathrm{d}^G f , \tag{24}$$

where the products in the last equality are meant point-wise. As a consequence, $\{\cdot,\cdot\}_0$ is bilinear and satisfies Leibniz's rule with respect to both arguments, by (18). In other words, $\{\cdot,\cdot\}_0$ is a skew-symmetric biderivation. Finally, by bilinearity, it suffices to prove the Jacobi identity for f,g,h being monomials in the elements of $\mathcal{X}^{\mathbb{R}}$. If the sum of the degree of the three monomials is 0, 1, or 2, then the Jacobi identity follows trivially. If the sum is exactly 3 then the Jacobi identity follows from the corresponding one for the commutators (18). (If one of the three monomials has zero degree then all terms in the Jacobi identity trivially vanish.) If the sum is bigger than 3 then at least one of the monomial has degree bigger than 1. Assume, without loss of generality, that this monomial is f. Then $f = f_1 f_2$ where the monomials f_1 and f_2 have degree at least 1, and, explicit computations using Leibniz's rule and the skew-symmetry yield

$$\begin{aligned}\{f,\{g,h\}_0\}_0+\{h,\{f,g\}_0\}_0+\{g,\{h,f\}_0\}_0 &= f_1\left(\{f_2,\{g,h\}_0\}_0+\{h,\{f_2,g\}_0\}_0+\{g,\{h,f_2\}_0\}_0\right)\\ &\quad + f_2\left(\{f_1,\{g,h\}_0\}_0+\{h,\{f_1,g\}_0\}_0+\{g,\{h,f_1\}_0\}_0\right) \ .\end{aligned}$$

Since f_1 and f_2 have in this case degree stricly smaller than the degree of f, the Jacobi identity follows by induction. ∎

Corollary 3.4 (Poisson algebra)
The subspace $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}$ of polynomials in the elements of $\mathcal{X}^{\mathbb{R}} \subseteq C(\mathcal{X}_{\mathbb{R}}^{*};\mathbb{R})$ with real coefficients, endowed with $\{\cdot,\cdot\}_0$ and the pointwise multiplications of $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}$, is a Poisson algebra, in the sense of [29, Definition 1.1].

3.3 Poisson Ideals Associated with State and Phase Spaces

Let $F \subseteq \mathcal{X}_{\mathbb{R}}^{*}$ be any nonempty subset of $\mathcal{X}_{\mathbb{R}}^{*}$ and define the algebra

$$\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(F) \doteq \left\{ f|_F : f \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*}) \right\} \tag{25}$$

of polynomials on F. If the restriction to F of the Poisson bracket $\{f,g\}_0$ of two polynomials $f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*})$ (Definition 3.2) only depends on the corresponding restrictions of f,g, then

$$\{f|_F, g|_F\} \doteq \{f,g\}_0|_F, \qquad f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*}) \ ,$$

is a well-defined Poisson bracket on $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(F)$. Equivalently, this means that the subalgebra

$$\mathfrak{I}_F \doteq \left\{ f \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}} : f(F) = \{0\} \right\}$$

of polynomials that vanish on $F \subseteq \mathcal{X}_{\mathbb{R}}^{*}$ is a Poisson ideal of the Poisson algebra $(\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}, \{\cdot,\cdot\}_0)$. Recall that a subalgebra $\mathfrak{I}$ of a Poisson algebra $(\mathcal{P}, \{\cdot,\cdot\})$ is called a Poisson ideal whenever, for all $f \in \mathfrak{I}$ and $g \in \mathcal{P}$, $fg \in \mathfrak{I}$ and $\{f,g\} \in \mathfrak{I}$. See, e.g., [29, Section 2.2.1]. As a consequence of this fact, the Poisson algebras

$$\left(\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(F), \{\cdot,\cdot\}\right) \qquad \text{and} \qquad \left(\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*}), \{\cdot,\cdot\}_0\right)/\mathfrak{I}_F$$

are isomorphic. See [29, Section 2.2.1] for the definition of the quotient of a Poisson algebra by one of its Poisson ideals. See also [29, Proposition 2.8].

For any state $\rho \in E$, we apply these observations to the follium E_ρ of states, defined by

$$E_\rho \doteq \left\{ \langle \varphi, \pi_\rho(\cdot)\varphi \rangle_{\mathcal{H}_\rho} : \varphi \in \mathcal{H}_\rho,\ \|\varphi\|_{\mathcal{H}_\rho} = 1 \right\} \subseteq E \subseteq \mathcal{X}_{\mathbb{R}}^{*} \ ,$$

where the triplet $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ is the GNS representation [36, Section 2.3.3] of ρ.

Proposition 3.5 (Folia of states and Poisson ideals)
For any $\rho \in E$ and any $f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*})$, the restriction $\{f,g\}_0|_{E_\rho}$ only depends on the corresponding restriction of f,g. In particular,

$$\{f|_{E_\rho}, g|_{E_\rho}\} \doteq \{f,g\}_0|_{E_\rho}, \qquad f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*}) \ ,$$

is a well-defined Poisson bracket on $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(E_\rho)$.

Proof. For any state $\rho \in E$ with GNS representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$, we define the unit sphere

$$\mathrm{S}_\rho \doteq \left\{ \varphi \in \mathcal{H}_\rho : \|\varphi\|_{\mathcal{H}_\rho} = 1 \right\} .$$

For any $f \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}} (\mathcal{X}_{\mathbb{R}}^*)$, we define the continuous function $f_\rho \in C(\mathrm{S}_\rho; \mathbb{R})$ by

$$f_\rho (\varphi) \doteq f(\langle \varphi, \pi_\rho (\cdot) \varphi \rangle_{\mathcal{H}_\rho}) , \qquad \varphi \in \mathrm{S}_\rho .$$

Let

$$\mathfrak{C}^{(\rho)} \doteq \left\{ f_\rho : f \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}} (\mathcal{X}_{\mathbb{R}}^*) \right\} \subseteq C(\mathrm{S}_\rho; \mathbb{R}) .$$

Then, we prove the existence of a skew-symmetric biderivation $\{\cdot, \cdot\}^{(\rho)}$ on $\mathfrak{C}^{(\rho)}$ satisfying

$$\{\hat{A}_\rho, \hat{B}_\rho\}^{(\rho)} = (\{\hat{A}, \hat{B}\}_0)_\rho , \qquad A, B \in \mathcal{X}^{\mathbb{R}} . \tag{26}$$

This last equality yields

$$\{f_\rho, g_\rho\}^{(\rho)} = \left(\{f, g\}_0\right)_\rho , \qquad f, g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}} \left(\mathcal{X}_{\mathbb{R}}^*\right) ,$$

by linearity and Leibniz's rule. In particular, for any $\varphi \in \mathrm{S}_\rho$,

$$\{f, g\}_0(\langle \varphi, \pi_\rho (\cdot) \varphi \rangle_{\mathcal{H}_\rho}) = \{f_\rho, g_\rho\}^{(\rho)} .$$

As f_ρ, g_ρ only depend on the restrictions $f|_{E_\rho}$ and $g|_{E_\rho}$, respectively, the assertion follows.

Now, in order to prove the existence of a skew-symmetric biderivation $\{\cdot, \cdot\}^{(\rho)}$ satisfying (26), let $\mathcal{L}_{\mathbb{R}}^1(\mathcal{H}_\rho)$ be the real Banach space of all self-adjoint trace-class operators on $\mathcal{H}_\rho$. For any $f \in \mathfrak{C}^{(\rho)}$ and $\varphi \in \mathrm{S}_\rho$, we denote by $\mathrm{d}_\rho^G f (\varphi)$ the Gateaux derivative at $A = 0$ of the map

$$A \mapsto f\left(\mathrm{e}^{iA} \varphi\right)$$

from $\mathcal{L}_{\mathbb{R}}^1(\mathcal{H}_\rho)$ to $\mathbb{R}$. For any $f \in \mathfrak{C}^{(\rho)}$, this Gateaux derivative is linear and continuous, i.e., $\mathrm{d}_\rho^G f (\varphi) \in \mathcal{B}(\mathcal{H}_\rho)^{\mathbb{R}}$. See, e.g., (19). Therefore, we can define a skew-symmetric biderivation $\{\cdot, \cdot\}^{(\rho)}$ on $\mathfrak{C}_\rho$ by

$$\{f, g\}^{(\rho)} (\varphi) = \left\langle \varphi, i \left[\mathrm{d}_\rho^G f (\varphi), \mathrm{d}_\rho^G g (\varphi)\right] \varphi \right\rangle_{\mathcal{H}_\rho} , \qquad f, g \in \mathfrak{C}^{(\rho)} .$$

For any $A \in \mathcal{X}^{\mathbb{R}}$ and $\varphi \in \mathrm{S}_\rho$, observe that

$$\mathrm{d}_\rho^G \hat{A}_\rho (\varphi) (B) = i \langle \varphi, [\pi_\rho (A), B] \varphi \rangle_{\mathcal{H}_\rho} = i \mathrm{Tr}_{\mathcal{H}_\rho}([P_\varphi, \pi_\rho (A)] B) ,$$

where P_φ is the orthogonal projection whose range is $\mathbb{C}\varphi$. In other words,

$$\mathrm{d}_\rho^G \hat{A}_\rho (\varphi) = i [P_\varphi, \pi_\rho (A)] \in \mathcal{B}(\mathcal{H}_\rho) , \qquad \varphi \in \mathrm{S}_\rho,\ A \in \mathcal{X}^{\mathbb{R}} .$$

Since $\pi_\rho : \mathcal{X} \to \mathcal{B}(\mathcal{H}_\rho)$ is a $*$-homomorphism, by Equation (23) and Definition 3.2, it follows that, for any $A, B \in \mathcal{X}^{\mathbb{R}}$,

$$\{\hat{A}_\rho, \hat{B}_\rho\}^{(\rho)} (\varphi) = i \langle \varphi, \pi_\rho ([A, B]) \varphi \rangle_{\mathcal{H}_\rho} = (\{\hat{A}, \hat{B}\}_0)_\rho (\varphi) , \qquad \varphi \in \mathrm{S}_\rho ,$$

i.e., Equation (26) holds true. ∎

The folia E_ρ, $\rho \in E$, play here an analogous role as the symplectic leaves $\mathrm{O}(\sigma)$ (20) of the Poisson manifold $(\mathcal{L}_{\mathbb{R}}^1(\mathcal{H}), \{\cdot, \cdot\})$ in Bóna's approach [13, Sections 2.1b, 2.1c]. See Section 3.1.

Corollary 3.6 (State space and Poisson ideals)
For any $f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*})$, the restriction $\{f,g\}_0|_E$ only depends on the corresponding restriction of f,g. In particular,

$$\{f|_E, g|_E\} \doteq \{f,g\}_0|_E, \qquad f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*}) \ ,$$

is a well-defined Poisson bracket on $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(E) \subseteq \mathfrak{C}^{\mathbb{R}} \doteq C(E;\mathbb{R})$.

Proof. The assertion is a direct consequence of Proposition 3.5 together with the obvious equality

$$E = \bigcup \{E_\rho : \rho \in E\} \ .$$

Equivalently, use that

$$\mathfrak{I}_E = \bigcap \left\{\mathfrak{I}_{E_\rho} : \rho \in E\right\} \ ,$$

is a Poisson ideal of the Poisson algebra $(\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}, \{\cdot,\cdot\}_0)$. ∎

In Sections 3.4-3.5, we also present an explicit construction of the Poisson bracket of Corollary 3.6, because it is technically more convenient for the subsequent sections.

Finally, recall that the phase space is the weak* closure $\overline{\mathcal{E}(E)}$ of the set $\mathcal{E}(E)$ of extreme points of the state space E, see Definition 2.2. Similar to Corollary 3.6, we prove from Proposition 3.5 the existence of a Poisson bracket for polynomials acting on the phase space:

Corollary 3.7 (Phase space and Poisson ideals)
For any $f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*})$, the restriction $\{f,g\}_0|_{\overline{\mathcal{E}(E)}}$ only depends on the corresponding restriction of f,g. In particular,

$$\{f|_{\overline{\mathcal{E}(E)}}, g|_{\overline{\mathcal{E}(E)}}\} \doteq \{f,g\}_0|_{\overline{\mathcal{E}(E)}}, \qquad f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*}) \ ,$$

is a well-defined Poisson bracket on $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\overline{\mathcal{E}(E)}) \subseteq C(\overline{\mathcal{E}(E)};\mathbb{R})$.

Proof. For any extreme (or pure) state $\rho \in \mathcal{E}(E)$, we infer from [73, Proposition 2.2.4] that the folium $E_\rho \subseteq \mathcal{E}(E)$ is a subset of extreme states and, hence,

$$\mathcal{E}(E) = \bigcup \{E_\rho : \rho \in \mathcal{E}(E)\} \ .$$

By Proposition 3.5 and continuity of polynomials, it follows that

$$\mathfrak{I}_{\overline{\mathcal{E}(E)}} = \mathfrak{I}_{\mathcal{E}(E)} = \bigcap \left\{\mathfrak{I}_{E_\rho} : \rho \in \mathcal{E}(E)\right\}$$

is again a Poisson ideal of the Poisson algebra $(\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}, \{\cdot,\cdot\}_0)$. ∎

3.4 Convex Weak* Gateaux Derivative

In order to construct the Poisson bracket $\{\cdot,\cdot\}$ of Corollary 3.6 more explicitly, as well as to analyze its properties as generator of (generally non-autonomous) classical dynamics, we introduce the notion of *convex* Gateaux derivative on the space $C(E;\mathcal{Y})$ of weak*-continuous functions on the convex and weak*-compact set E of states with values in an arbitrary Banach space

$$\mathcal{Y} \equiv \left(\mathcal{Y}, +, \cdot_{\mathbb{K}}, \|\cdot\|_{\mathcal{Y}}\right) \ , \qquad \mathbb{K} = \mathbb{R}, \mathbb{C} \ .$$

As far as only the construction of the Poisson bracket $\{\cdot,\cdot\}$ of Corollary 3.6 is concerned, the relevant example is $\mathcal{Y} = \mathbb{R} = \mathbb{K}$.

We first define the Banach space

$$\mathcal{A}(E;\mathcal{Y}) \doteq \{f \in C(E;\mathcal{Y}) : \forall \lambda \in (0,1),\ \rho, \upsilon \in E, \quad f((1-\lambda)\rho + \lambda\upsilon) = (1-\lambda) f(\rho) + \lambda f(\upsilon)\}$$

of all affine weak*-continuous $\mathcal{Y}$-valued functions on E, endowed with the supremum norm

$$\|f\|_{\mathcal{A}(E;\mathcal{Y})} \doteq \max_{\rho \in E} \|f(\rho)\|_{\mathcal{Y}}\ , \qquad f \in \mathcal{A}(E;\mathcal{Y})\ . \tag{27}$$

Again, the norm is *not* used in the contruction of the Poisson bracket $\{\cdot,\cdot\}$ of Corollary 3.6, but only in Section 7.

The convex Gateaux derivative of a weak*-continuous $\mathcal{Y}$-valued function on E at a fixed state is an affine weak*-continuous $\mathcal{Y}$-valued function on E defined as follows:

Definition 3.8 (Convex weak*-continuous Gateaux derivative)
For any continuous function $f \in C(E;\mathcal{Y})$ and any state $\rho \in E$, we say that $\mathrm{d}f(\rho) : E \to \mathcal{Y}$ is the (unique) convex weak-continuous Gateaux derivative of f at $\rho \in E$ if $\mathrm{d}f(\rho) \in \mathcal{A}(E;\mathcal{Y})$ and*

$$\lim_{\lambda \to 0^+} \lambda^{-1}\left(f((1-\lambda)\rho + \lambda\upsilon) - f(\rho)\right) = [\mathrm{d}f(\rho)](\upsilon)\ , \qquad \rho, \upsilon \in E\ .$$

To our knowledge, the concept of convex weak*-continuous Gateaux derivative defined above is new.

A function $f \in C(E;\mathcal{Y})$ such that $\mathrm{d}f(\rho)$ exists for all $\rho \in E$ is called *differentiable* and we use the notation

$$\mathrm{d}f \equiv (\mathrm{d}f(\rho))_{\rho \in E} : E \to \mathcal{A}(E;\mathcal{Y})\ .$$

Explicit examples of spaces of such differentiable functions are given, for any $n \in \mathbb{N}$, by

$$\mathfrak{Y}_n \equiv \mathfrak{Y}(\mathcal{Y})_n \doteq \Big\{ f \in C(E,\mathcal{Y}) : \exists \{B_j\}_{j=1}^n \subseteq \mathcal{X}^{\mathbb{R}},\ g \in C^1(\mathbb{R}^n,\mathcal{Y}) \text{ such that } f(\rho) = g(\rho(B_1),\ldots,\rho(B_n)) \Big\}\ . \tag{28}$$

Functions of this kind are said to be cylindrical. In fact, for any $n \in \mathbb{N}$ and $f \in \mathfrak{Y}_n$,

$$[\mathrm{d}f(\rho)](\upsilon) = \sum_{j=1}^{n} (\upsilon(B_j) - \rho(B_j))\, \partial_{x_j} g(\rho(B_1),\ldots,\rho(B_n))\,, \qquad \rho, \upsilon \in E\ . \tag{29}$$

We define the subspace of continuously differentiable $\mathcal{Y}$-valued functions on the convex and weak*-compact set E by

$$\mathfrak{Y} \equiv \mathfrak{Y}(\mathcal{Y}) \doteq C^1(E;\mathcal{Y}) \doteq \{f \in C(E;\mathcal{Y}) : \mathrm{d}f \in C(E;\mathcal{A}(E;\mathcal{Y}))\}\ . \tag{30}$$

We endow this vector space with the norm

$$\|f\|_{\mathfrak{Y}} \doteq \max_{\rho \in E} \|f(\rho)\|_{\mathcal{Y}} + \max_{\rho \in E} \|\mathrm{d}f(\rho)\|_{\mathcal{A}(E;\mathcal{Y})}\ , \qquad f \in \mathfrak{Y}\,, \tag{31}$$

in order to obtain a Banach space, also denoted by $\mathfrak{Y}$. Note again that we use "$\max$" instead "$\sup$" in the definition of the norm, because of the continuity of f and $\mathrm{d}f$ together with the weak* compactness of E. Observe that

$$\partial_\lambda^+ f_{\rho,\upsilon}(\lambda) \doteq \lim_{\varepsilon \to 0^+} \varepsilon^{-1}\left(f_{\rho,\upsilon}(\lambda+\varepsilon) - f_{\rho,\upsilon}(\lambda)\right) = \frac{1}{(1-\lambda)} [\mathrm{d}f((1-\lambda)\rho + \lambda\upsilon)](\upsilon)$$

with $f_{\rho,\upsilon}$ being the $\mathcal{Y}$-valued function on the interval $[0,1)$ defined, at fixed states $\rho, \upsilon \in E$, by

$$f_{\rho,\upsilon}(\lambda) \doteq f((1-\lambda)\rho + \lambda\upsilon)\ , \qquad \lambda \in [0,1)\ .$$

Moreover, the operator ∂_λ^+ is closed on $C([a,b);\mathcal{Y})$ for any real parameters $a<b$, in the sense of the supremum norm. Thus, by well-known properties of the uniform convergence of continuous functions, the normed vector space $\mathfrak{Y}$ is complete.

Remark that the family $\{\mathfrak{Y}_n\}_{n\in\mathbb{N}}$ is increasing with respect to inclusion and

$$\mathfrak{Y}_\infty \doteq \bigcup_{n\in\mathbb{N}} \mathfrak{Y}_n \subseteq \mathfrak{Y}$$

is the space of all cylindrical functions of $\mathfrak{Y}$. Additionally, if $f\in\mathcal{A}(E;\mathcal{Y})$ then

$$\mathrm{d}f\left(\rho\right) = f - f\left(\rho\right)\ , \qquad \rho\in E\ , \tag{32}$$

which means in particular that affine weak*-continuous $\mathcal{Y}$-valued functions on E are continuously differentiable, i.e., $\mathcal{A}(E;\mathcal{Y})\subseteq\mathfrak{Y}$.

3.5 Explicit Construction of Poisson Brackets for Functions on the State Space

We use the convex weak* Gateaux derivative in order to give an explicit expression for the Poisson bracket $\{\cdot,\cdot\}$ of Corollary 3.6. To this end, we only need the special case $\mathcal{Y}=\mathbb{R}$ in Definition 3.8. We also exploit the following result:

Proposition 3.9 (Affine weak*–continuous real-valued functions over E)
For any unital C^*-algebra $\mathcal{X}$, $\mathcal{A}(E;\mathbb{R})=\{\hat{A}:A\in\mathcal{X}^{\mathbb{R}}\}$, where $A\mapsto\hat{A}$ is the linear isometry from $\mathcal{X}^{\mathbb{R}}$ to $\mathfrak{C}^{\mathbb{R}}$ defined by (15). In particular, by (25), $\mathfrak{C}^{\mathbb{R}}_{\mathcal{X}^{\mathbb{R}}}(E)=\mathbb{R}[\mathcal{A}(E;\mathbb{R})]\subseteq\mathfrak{C}^{\mathbb{R}}\doteq C(E;\mathbb{R})$.

Proof. This statement is asserted without proof or references in [36, p 339]. A proof is only shortly sketched in [74, p 161] and we thus give it here for completeness and reader's convenience. It is based on preliminary results of convex analysis together with general properties of C^*-algebras: Clearly, $\{\hat{A}:A\in\mathcal{X}^{\mathbb{R}}\}\subseteq\mathcal{A}(E;\mathbb{R})$. Conversely, fix $f\in\mathcal{A}(E;\mathbb{R})$. Since E is a weak*-compact subset of $\mathcal{X}_{\mathbb{R}}^*$, we deduce from [74, Corollary 6.3] the existence of an increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of affine weak*-continuous real-valued functions on $\mathcal{X}_{\mathbb{R}}^*$ that uniformly converges to f, as $n\to\infty$. Meanwhile, observe that any affine weak*-continuous real-valued functions g on $\mathcal{X}_{\mathbb{R}}^*$ is of the form

$$g\left(\sigma\right)=\sigma\left(A\right)+g\left(0\right)\ , \qquad \sigma\in\mathcal{X}_{\mathbb{R}}^*\ ,$$

for some self-adjoint element $A\in\mathcal{X}^{\mathbb{R}}$, because the weak*-continuous real-valued function $g-g(0)$ on $\mathcal{X}_{\mathbb{R}}^*$ is linear. We thus deduce the existence of a sequence $\{A_n\}_{n\in\mathbb{N}}\subseteq\mathcal{X}^{\mathbb{R}}$ such that

$$f_n\left(\sigma\right)=\sigma\left(A_n\right)+f_n\left(0\right)\ , \qquad \sigma\in\mathcal{X}_{\mathbb{R}}^*\ .$$

Since $\rho\left(\mathfrak{1}\right)=1$ for $\rho\in E$, by (16), the uniform convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f on E yields that $\{A_n+f_n\left(0\right)\mathfrak{1}\}_{n\in\mathbb{N}}\subseteq\mathcal{X}^{\mathbb{R}}$ is a Cauchy sequence, which thus converges to some $A\in\mathcal{X}^{\mathbb{R}}$, as $n\to\infty$. It follows that $f=\hat{A}$. ■

Recall that $A\neq B$ yields $\hat{A}\neq\hat{B}$ for any $A,B\in\mathcal{X}$. Therefore, by (16), (27), (30) and Proposition 3.9, for any continuously differentiable real-valued function $f\in\mathfrak{Y}\left(\mathbb{R}\right)\subsetneq\mathfrak{C}$ there is a unique $\mathrm{D}f\in C(E;\mathcal{X}^{\mathbb{R}})$ such that

$$\mathrm{d}f\left(\rho\right)=\widehat{\mathrm{D}f\left(\rho\right)}\ , \qquad \rho\in E\ . \tag{33}$$

For instance, one infers from (29) that, for any $n\in\mathbb{N}$ and $f\in\mathfrak{Y}\left(\mathbb{R}\right)_n$,

$$\mathrm{D}f\left(\rho\right)=\sum_{j=1}^{n}\left(A_j-\rho\left(A_j\right)\mathfrak{1}\right)\partial_{x_j}g\left(\rho\left(A_1\right),\ldots,\rho\left(A_n\right)\right)\ , \qquad \rho\in E\ . \tag{34}$$

By (16) and (27), note that

$$\|\mathrm{D}f(\rho)\|_{\mathcal{X}} = \|\mathrm{d}f(\rho)\|_{\mathcal{A}(E;\mathbb{R})}\ , \qquad \rho \in E\ . \tag{35}$$

Therefore, we can define a skew-symmetric biderivation on $\mathfrak{Y}(\mathbb{R})$ for continuously differentiable real-valued functions depending on the state space:

Definition 3.10 (Skew-symmetric biderivation on $\mathfrak{Y}(\mathbb{R})$)
We define the map $\{\cdot,\cdot\} : \mathfrak{Y}(\mathbb{R}) \times \mathfrak{Y}(\mathbb{R}) \to C(E;\mathbb{R})$ by

$$\{f,g\}(\rho) \doteq \rho\left(i\left[\mathrm{D}f(\rho), \mathrm{D}g(\rho)\right]\right)\ , \qquad f,g \in \mathfrak{Y}(\mathbb{R})\ .$$

This map $\{\cdot,\cdot\}$ is clearly skew-symmetric, by (18) and Definition 3.10. This skew-symmetric biderivation is precisely the one already constructed in Corollary 3.6 on polynomials:

Proposition 3.11 (Poisson bracket)
Restricted to $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(E)$, the skew-symmetric biderivation of Definition 3.10 coincides with the Poisson bracket defined by

$$\{f|_E, g|_E\} \doteq \{f,g\}_0|_E, \qquad f,g \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}}(\mathcal{X}_{\mathbb{R}}^{*})\ .$$

See Corollary 3.6.

Proof. By Equation (34),

$$\mathrm{D}\hat{A}(\rho) = A - \rho(A)\mathbf{1}\ , \qquad A \in \mathcal{X}^{\mathbb{R}}\ , \tag{36}$$

and therefore,

$$\{\hat{A},\hat{B}\}(\rho) = \rho(i[A,B])\ , \qquad \rho \in E,\ A,B \in \mathcal{X}^{\mathbb{R}}\ . \tag{37}$$

Hence, by Definition 3.2 and Equation (23),

$$\{\hat{A}|_E, \hat{B}|_E\} = \{\hat{A},\hat{B}\}_0|_E\ , \qquad A,B \in \mathcal{X}^{\mathbb{R}}\ . \tag{38}$$

Linearity and Leibniz's rule then lead to the assertion. ■

The Poisson bracket can easily be extended to a complex Poisson bracket, i.e., a Poisson bracket for complex-valued polynomials: Since the sum of two affine functions stays affine, by Proposition 3.9, observe that

$$\mathcal{A}(E;\mathbb{C}) = \{\hat{A} : A \in \mathcal{X}\}\ .$$

Moreover, by (16), (27) and (30), for any continuously differentiable complex-valued function $f \in \mathfrak{Y}(\mathbb{C}) \subsetneq \mathfrak{C}$ there is a unique $\mathrm{D}f \in C(E;\mathcal{X})$ satisfying (33). Then, the Poisson bracket $\{\cdot,\cdot\}$ of Definition 3.10 can be extended to all $f,g \in \mathfrak{Y}(\mathbb{C})$, as a skew-symmetric biderivation. In fact, since

$$\mathrm{D}(f+g) = \mathrm{D}f + \mathrm{D}g \quad \text{and} \quad \mathrm{D}(fg) = f\mathrm{D}g + g\mathrm{D}f\ ,$$

this skew-symmetric biderivation satisfies

$$\{f,g\} = \{\mathrm{Re}\{f\},\mathrm{Re}\{g\}\} - \{\mathrm{Im}\{f\},\mathrm{Im}\{g\}\} + i\left(\{\mathrm{Im}\{f\},\mathrm{Re}\{g\}\} + \{\mathrm{Re}\{f\},\mathrm{Im}\{g\}\}\right) \tag{39}$$

for all $f,g \in \mathfrak{Y}(\mathbb{C})$. Note here that $\mathrm{Re}\{f\},\mathrm{Im}\{f\} \in \mathfrak{Y}(\mathbb{R})$ for all $f \in \mathfrak{Y}(\mathbb{C})$. Restricted to $\mathfrak{C}_{\mathcal{X}} \equiv \mathfrak{C}_{\mathcal{X}}(E)$, it is again a (complex) Poisson bracket, since it satisfies the Jacobi identity, by Proposition 3.11 together with tedious computations.

Remark 3.12 (Commutative case)
If $\mathcal{X}$ is already a commutative unital C^*-algebra then the Poisson bracket is of course trivial, being the zero biderivation, and any classical dynamics generated by this Poisson bracket corresponds to the identity map. This is reminiscent of the KMS dynamics, which becomes trivial when the corresponding von Neumann algebra is commutative. (In this case, the modular operator is the identity operator.)

3.6 Poissonian Symmetric Derivations

A *derivation* $\mathfrak{d}$ (on $\mathfrak{C}$) is a linear map from a dense $*$-subalgebra $\mathrm{dom}(\mathfrak{d})$ (i.e., its domain) of $\mathfrak{C}$ to the unital commutative C^*-algebra $\mathfrak{C}$ (13) of complex-valued weak*-continuous functions on E such that

$$\mathfrak{d}\left(fg\right)=\mathfrak{d}\left(f\right)g+f\mathfrak{d}\left(g\right)\ ,\qquad f,g\in \mathrm{dom}\left(\mathfrak{d}\right)\ . \tag{40}$$

It is *symmetric*, or a $*$-derivation, when

$$\mathfrak{d}(\bar{f})=\overline{\mathfrak{d}\left(f\right)}\ ,\qquad f\in \mathrm{dom}\left(\mathfrak{d}\right)\ . \tag{41}$$

For an exhaustive description of the theory of derivations, see [34, 36, 37] and references therein.

An important class of symmetric derivations can be defined by using the Poisson bracket $\{\cdot,\cdot\}$ of Definition 3.10:

Definition 3.13 (Poissonian symmetric derivations)
The Poissonian symmetric derivation associated with any continuously differentiable real-valued function $h\in\mathfrak{Y}\left(\mathbb{R}\right)$ *is the linear operator defined on its dense domain* $\mathrm{dom}(\mathfrak{d}^h)=\mathfrak{C}_{\mathcal{X}}\subseteq\mathfrak{C}$ *by*

$$\mathfrak{d}^h\left(f\right)\doteq\{h,f\}\ ,\qquad f\in\mathfrak{C}_{\mathcal{X}}\ .$$

Recall at this point that $\mathfrak{C}_{\mathcal{X}}\equiv\mathfrak{C}_{\mathcal{X}}(E)\subseteq\mathfrak{C}$ is the dense $*$-subalgebra of all polynomials in the elements of $\{\hat{A}:A\in\mathcal{X}\}$, with complex coefficients. See (17). Because of Definition 3.10 and Equations (30)-(31), (33), (35) and (39), $\mathfrak{d}^h$ is a symmetric derivation satisfying

$$\left\|\mathfrak{d}^h\left(f\right)\right\|_{\mathfrak{C}}\leq 4\left\|h\right\|_{\mathfrak{Y}(\mathbb{C})}\left\|f\right\|_{\mathfrak{Y}(\mathbb{C})}\ ,\qquad f\in\mathfrak{C}_{\mathcal{X}}\subseteq\mathfrak{Y}\left(\mathbb{C}\right)\ .$$

In particular, $\mathfrak{d}^h$ could be extended as a bounded symmetric derivation $\tilde{\mathfrak{d}}^h$ from $\mathfrak{Y}\left(\mathbb{C}\right)$ to $\mathfrak{C}$, i.e., as an element of $\mathcal{B}\left(\mathfrak{Y}\left(\mathbb{C}\right),\mathfrak{C}\right)$.

At first sight, the extension $\tilde{\mathfrak{d}}^h$ of $\mathfrak{d}^h$ to all continuously differentiable complex-valued functions of $\mathfrak{Y}\left(\mathbb{C}\right)$ seems to be natural, like for the usual differentiation on functions of the compact set $[0,1]$. On second thoughts, $\tilde{\mathfrak{d}}^h$ may *not* be closable, even if this property would be true for $\mathfrak{d}^h$. Of course, if $\tilde{\mathfrak{d}}^h$ is closable then $\mathfrak{d}^h$ is also closable.

For a large class of symmetric derivations, the closableness is proven from dissipativity [34, Definition 1.4.6, Proposition 1.4.7]. This property is in turn deduced from a theorem proven by Kishimoto [34, Theorem 1.4.9], which uses the assumption that the square root of each positive element of the domain of the derivation also belongs to the same domain. We cannot expect this last property to be satisfied for symmetric derivations like $\mathfrak{d}^h$ or $\tilde{\mathfrak{d}}^h$.

The closableness of unbounded symmetric derivations of C^*-algebras is, in general, a non-trivial issue, even in the commutative case like $\mathfrak{C}$. This property is not generally true: there exist norm-densely defined derivations of C^*-algebras that are *not* closable [35]. For instance, in [36, p. 306], it is even claimed that "*Herman has constructed an extension of the usual differentiation on* $C(0,1)$ *which is a nonclosable derivation of* $C(0,1)$." The general characterization of closed symmetric derivations depends heavily on the (Hausdorff) dimension of the locally compact set, here the weak*-compact set E. Around 1990, a characterization of all closed symmetric derivations were obtained by using spaces of functions acting on a compact subset of a one-dimensional space. However, "*for more than 2 dimensions only sporadic results are known*", as quoted in [34, Section 1.6.4, p. 27]. See, e.g., [34, Section 1.6.4], [37], [38, 39], and later [36, p. 306].

In our approach, the closableness of unbounded symmetric derivations like $\mathfrak{d}^h$ or $\tilde{\mathfrak{d}}^h$ is a necessary property to make sense of a classical dynamics, in its Hamiltonian formulation, via C_0-groups. In Section 4, we show that the symmetric derivation $\mathfrak{d}^h$ is closable, at least for all functions h in a dense subset of $\mathfrak{C}$, including $\mathfrak{C}_{\mathcal{X}}$. This is performed via a self-consistency problem together with the C_0-semigroup theory [40]. Our results are non-trivial since E is *not* a subset of a finite-dimensional space when $\mathcal{X}$ has infinite dimension. See proof of Theorem 2.5.

4 Hamiltonian Flows for States from Self-Consistent Quantum Dynamics

Our approach to the construction of Hamiltonian flows and, in particular, closed derivations of a commutative C^*-algebra via self-consistency problems is *non-conventional*. However, it shares some similarity with the following simple example in the finite-dimensional case: Take $\mathcal{A}$ as being the commutative unital C^*-algebra of all continuous, bounded and complex-valued functions on $\mathbb{R}^{2N}$, $N \in \mathbb{N}$, and fix a smooth and compactly supported function $h : \mathbb{R}^{2N} \to \mathbb{R}$. From the Picard-Lindelöf iteration argument, the (Hamiltonian) vector field $J\nabla h$ (where ∇h is the gradient of h and J is the $2N$-dimensional symplectic matrix) generates a global smooth flow $\phi_t : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$, $t \in \mathbb{R}$. Let the one-parameter group $\{V_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{A}$ be defined by

$$[V_t(f)](x) = f \circ \phi_{-t}(x) \ , \qquad x \in \mathbb{R}^{2N},\ t \in \mathbb{R} \ .$$

Because of the compactness of the support of h, this one-parameter group is strongly continuous and the corresponding generator is a closed derivation in $\mathcal{A}$, denoted by δ_h. Moreover, it is straightforward to check that, in the dense set of smooth functions, this derivation acts as $\delta_h = \{h, \cdot\}$, where $\{\cdot, \cdot\}$ is the canonical Poisson bracket

$$\{f, g\}(x) \doteq \sum_{k,l=1}^{2N} J_{ij}[\partial_{x_i} f(x)][\partial_{x_j} g(x)] \ , \qquad x \in \mathbb{R}^{2N} \ ,$$

for smooth functions f, g on $\mathbb{R}^{2N}$. The analogy of the results presented in this section with this example is as follows: in our setting, the space E of all states on $\mathcal{X}$ replaces $\mathbb{R}^{2N}$ and the analogue of the global Hamiltonian flow $\{\phi_t\}_{t\in\mathbb{R}}$ is a one-parameter family of weak* automorphisms of E (or self-homeomorphisms of E). Note that Bóna uses such a construction only on symplectic leaves of the corresponding Poisson manifold and "glues" them together in order to construct the global flow [13, Section 2.1-d]. However, in strong contrast to this simple example, in our case, it is not clear at all how to construct the corresponding family of automorphisms from Hamiltonian vector fields. Instead, we construct it as the solution to a *self-consistency* problem. Similar to the above example, the closed derivations we obtain for the classical algebra $\mathfrak{C}$ are closed extensions of densely defined derivations of the form $f \mapsto \{h, f\}$, $f, h \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}$, where $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}} \subseteq \mathfrak{C}$ is the dense subalgebra of polynomials in the elements of $\mathcal{X}^{\mathbb{R}}$, defined by (17) for $\mathcal{B} = \mathcal{X}^{\mathbb{R}}$.

All this construction is performed in this section, supplemented with technical assertions proven in Section 7. We start by somehow tedious, albeit necessary, definitions and notation in Sections 4.1-4.3, the self-consistency equations being asserted in Theorem 4.1 and exploited afterwards.

4.1 Preliminary Definitions

Let $C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$ be the Banach space of bounded continuous maps from $\mathbb{R}$ to $\mathfrak{Y}(\mathbb{R})$ with the norm

$$\|h\|_{C_b(\mathbb{R};\mathfrak{Y}(\mathbb{R}))} \doteq \sup_{t\in\mathbb{R}} \|h(t)\|_{\mathfrak{Y}(\mathbb{R})} \ , \qquad h \in C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R})) \ . \tag{42}$$

We identify $\mathfrak{Y}(\mathbb{R})$ with the subalgebra of constant functions of $C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$, i.e.,

$$\mathfrak{Y}(\mathbb{R}) \subseteq C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R})) \ . \tag{43}$$

Let $C(E; E)$ be the set of weak*-continuous functions from the state space E to itself endowed with the topology of uniform convergence. In other words, a net $(f_j)_{j\in J} \subseteq C(E; E)$ converges to $f \in C(E; E)$ whenever

$$\lim_{j\in J} \max_{\rho\in E} |f_j(\rho)(A) - f(\rho)(A)| = 0 \ , \qquad \text{for all } A \in \mathcal{X} \ . \tag{44}$$

We denote by $\mathrm{Aut}\,(E) \subsetneq C\,(E;E)$ the subspace of all automorphisms of E, i.e., elements of $C\,(E;E)$ with weak*-continuous inverse. Equivalently, $\mathrm{Aut}\,(E)$ is the set of all bijective maps in $C\,(E;E)$, because E is a compact Hausdorff space. Recall that, here, the concept of an automorphism depends on the structure of the corresponding domain: elements of $\mathrm{Aut}\,(E)$ are self-homeomorphisms while a automorphism of a C^*-algebra is a $*$-automorphism of this C^*-algebra.

Any continuous function $h \in C_b\,(\mathbb{R};\mathfrak{Y}\,(\mathbb{R}))$ defines a non-autonomous, state-dependent, *quantum* dynamics on the C^*-algebra $\mathcal{X}$ via the family $\{\mathrm{D}h(t)\}_{t\in\mathbb{R}} \subseteq C(E;\mathcal{X}^{\mathbb{R}})$, satisfying (33) for each $t \in \mathbb{R}$. This quantum dynamics can in turn be used to define a (classical) dynamics on the commutative C^*-algebra $\mathfrak{C}$ of all continous complex-valued functions on E. This latter dynamics turns out to be the flow generated, as is usual in classical mechanics, by the Poisson bracket $\{h(t),\cdot\}$ of Definition 3.10 (see also Corollary 3.6 and Proposition 3.11). We start with the state-dependent quantum dynamics on the primordial C^*-algebra $\mathcal{X}$, in the next subsection.

4.2 Dynamics on the Primordial C^*-Algebra

Fix $h \in C_b\,(\mathbb{R};\mathfrak{Y}\,(\mathbb{R}))$, which plays the role of a time-dependent family of classical Hamiltonians. Then, for each state $\rho \in E$ and time $t \in \mathbb{R}$, we define the symmetric bounded derivation $X_t^{\rho} \in \mathcal{B}(\mathcal{X})$ by

$$X_t^{\rho}\,(A) \doteq i\,[\mathrm{D}h\,(t;\rho)\,, A] \doteq i\,(\mathrm{D}h\,(t;\rho)\,A - A\mathrm{D}h\,(t;\rho))\ , \qquad A \in \mathcal{X}\ , \tag{45}$$

where $[\cdot,\cdot]$ is the usual commutator defined by (18) and

$$\mathrm{D}h\,(t;\rho) \doteq [\mathrm{D}h\,(t)]\,(\rho) \in \mathcal{X}^{\mathbb{R}}\ , \qquad \rho \in E,\ t \in \mathbb{R}\ .$$

By Equations (31) and (35), note that

$$\sup_{\rho\in E} \|X_t^{\rho}\|_{\mathcal{B}(\mathcal{X})} \leq 2\,\|h\|_{C_b(\mathbb{R};\mathfrak{Y}(\mathbb{R}))} \tag{46}$$

and, for any state-valued continuous function $\xi \in C\,(\mathbb{R};E)$ and times $s,t \in \mathbb{R}$,

$$\|X_t^{\xi(t)} - X_s^{\xi(s)}\|_{\mathcal{B}(\mathcal{X})} \leq 2\,\|h\,(t) - h\,(s)\|_{\mathfrak{Y}(\mathbb{R})} + 2\,\|\mathrm{D}h\,(s;\xi\,(t)) - \mathrm{D}h\,(s;\xi\,(s))\|_{\mathcal{X}}\ ,$$

from (16) and (33).

Since $\mathrm{D}f \in C(E;\mathcal{X}^{\mathbb{R}})$ when $f \in \mathfrak{Y}\,(\mathbb{R})$, for any function $\xi \in C\,(\mathbb{R};E)$, $(X_t^{\xi(t)})_{t\in\mathbb{R}}$ is a norm-continuous family of bounded operators. Therefore, for any continuous functions $h \in C_b\,(\mathbb{R};\mathfrak{Y}\,(\mathbb{R}))$ and $\xi \in C\,(\mathbb{R};E)$, a norm-continuous two-parameter family $(T_{t,s}^{\xi})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{X}$ is uniquely defined in $\mathcal{B}(\mathcal{X})$ by the non-autonomous evolution equation

$$\forall s,t \in \mathbb{R}: \qquad \partial_t T_{t,s}^{\xi} = T_{t,s}^{\xi} \circ X_t^{\xi(t)}\ , \qquad T_{s,s}^{\xi} = \mathbf{1}_{\mathcal{X}}\ , \tag{47}$$

or, equivalently, by

$$\forall s,t \in \mathbb{R}: \qquad \partial_s T_{t,s}^{\xi} = -X_s^{\xi(s)} \circ T_{t,s}^{\xi}\ , \qquad T_{t,t}^{\xi} = \mathbf{1}_{\mathcal{X}}\ . \tag{48}$$

Note that $(T_{t,s}^{\xi})_{s,t\in\mathbb{R}}$ clearly satisfies the (reverse) cocycle property

$$\forall s,r,t \in \mathbb{R}: \qquad T_{t,s}^{\xi} = T_{r,s}^{\xi} \circ T_{t,r}^{\xi}\ . \tag{49}$$

The existence and uniqueness of a solution to these evolution equations follow from the usual theory of non-autonomous evolution equations for bounded norm-continuous generators, see, e.g., [75]. In this case, it is explicitly given by Dyson series. The fact that it defines a family of $*$-automorphisms of $\mathcal{X}$ results from the identity

$$\partial_t \left\{T_{t,s}^{\xi} T_{s,t}^{\xi}\right\} = 0\ , \qquad s,t \in \mathbb{R}\ ,$$

and the fact that the corresponding generators are symmetric derivations.

4.3 Self-Consistency Equations

Let $C(\mathbb{R}; C(E;E))$ be the set of continuous functions from $\mathbb{R}$ to $C(E;E)$. Any $\boldsymbol{\xi} \in C(\mathbb{R}; C(E;E))$ defines a function $\boldsymbol{\xi}(\cdot;\rho) \in C(\mathbb{R}; E)$ by

$$\boldsymbol{\xi}(t;\rho) \doteq [\boldsymbol{\xi}(t)](\rho) \ , \qquad \rho \in E, \ t \in \mathbb{R} \ . \tag{50}$$

Then, for any continuous functions $h \in C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$, $\boldsymbol{\xi} \in C(\mathbb{R}; C(E;E))$ and state $\rho \in E$, the norm-continuous two-parameter family $(T_{t,s}^{\boldsymbol{\xi}(\cdot;\rho)})_{s,t\in\mathbb{R}}$ of $\ast$-automorphisms of $\mathcal{X}$ defined above (Section 4.2) is used to define a family $(\phi_{t,s}^{(h,\boldsymbol{\xi})})_{s,t\in\mathbb{R}}$ of maps from the state space E to itself, as follows:

$$\phi_{t,s}^{(h,\boldsymbol{\xi})}(\rho) \doteq \rho \circ T_{t,s}^{\boldsymbol{\xi}(\cdot;\rho)} \ , \qquad \rho \in E, \ s,t \in \mathbb{R} \ . \tag{51}$$

By the reverse cocycle property (49) for $(T_{t,s}^{\xi})_{s,t\in\mathbb{R}}$, $(\phi_{t,s}^{(h,\boldsymbol{\xi})})_{s,t\in\mathbb{R}}$ has the (non-reverse) cocycle property, i.e.,

$$\phi_{t,s}^{(h,\boldsymbol{\xi})} = \phi_{t,r}^{(h,\boldsymbol{\xi})} \circ \phi_{r,s}^{(h,\boldsymbol{\xi})} \ , \qquad s,t,r \in \mathbb{R} \ . \tag{52}$$

By Lemma 7.1,

$$(\phi_{t,s}^{(h,\boldsymbol{\xi})}(\rho))_{s,t\in\mathbb{R}} \in C\left(\mathbb{R}^2; E\right) \ , \qquad \rho \in E \ .$$

As a consequence, the family $(\phi_{t,s}^{(h,\boldsymbol{\xi})})_{s,t\in\mathbb{R}}$ is a continuous flow on the state space E. Since $\{\mathrm{D}h(t)\}_{t\in\mathbb{R}} \subseteq C(E;\mathcal{X}^{\mathbb{R}})$, by Lemma 7.1 and Lebesgue's dominated convergence theorem, note additionally that $\{\phi_{t,s}^{(h,\boldsymbol{\xi})}\}_{s,t\in\mathbb{R}}$ is a family of automorphisms (self-homeomorphisms) of E, i.e.

$$\{\phi_{t,s}^{(h,\boldsymbol{\xi})}\}_{s,t\in\mathbb{R}} \subseteq \mathrm{Aut}(E) \, .$$

To understand the relevance of this flow with respect to classical dynamics, it is enlightening to consider the autonomous case for which h is the constant function $\hat{H}$ for some $H \in \mathcal{X}^{\mathbb{R}}$. See (15) for the definition of the function $\hat{H}$, the Gelfand transform of H. In this case, choose a state $\rho \in E$ and observe from (45), (47) and (51), together with Definition 3.10 and Equation (36), that

$$\partial_t \hat{A}_{t,s} = \{h, \hat{A}_{t,s}\} \qquad \text{with} \quad \hat{A}_{t,s} \doteq \hat{A} \circ \phi_{t,s}^{(\hat{H})} \in \mathfrak{C}$$

for any $A \in \mathcal{X}$ and $s,t \in \mathbb{R}$, noting that the flow $\phi_{t,s}^{(\hat{H})} \equiv \phi_{t,s}^{(h,\boldsymbol{\xi})}$, $s,t \in \mathbb{R}$, does not depend on $\boldsymbol{\xi} \in C(\mathbb{R}; C(E;E))$. Since $\phi_{t,s}^{(\hat{H})} = \phi_{t-s,0}^{(\hat{H})}$ for any $s,t \in \mathbb{R}$, the flow defined by $(\phi_{t,s}^{(\hat{H})})_{s,t\in\mathbb{R}}$ is associated with an autonomous classical dynamics, in the usual sense, on elementary elements $\{\hat{A} : A \in \mathcal{X}\}$.

In the general case of (non-autonomous) classical dynamics generated by time-dependent Poissonian symmetric derivations of the form $\{h(t), \cdot\}$, $t \in \mathbb{R}$, a convenient (and non-trivial) choice of the function $\boldsymbol{\xi}$ in Equation (51) has to be made. We determine it via a *self-consistency equation*. This is our first main result:

Theorem 4.1 (Self-consistency equations)
(a) *Let $\mathcal{X}$ be a unital C^*-algebra and $\mathfrak{B}$ a finite-dimensional real subspace of $\mathcal{X}^{\mathbb{R}}$.*
(b) *Take $h \in C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$ such that, for some constant $D_0 \in \mathbb{R}^+$,*

$$\sup_{t\in\mathbb{R}} \|\mathrm{D}h(t;\rho) - \mathrm{D}h(t;\tilde{\rho})\|_{\mathcal{X}} \leq D_0 \sup_{B\in\mathfrak{B}, \|B\|=1} |(\rho - \tilde{\rho})(B)| \ , \qquad \rho, \tilde{\rho} \in E \ .$$

Under Conditions (a)-(b), there is a unique function $\boldsymbol{\varpi}^h \in C(\mathbb{R}^2; \mathrm{Aut}(E))$ such that

$$\boldsymbol{\varpi}^h(s,t) = \phi_{t,s}^{(h,\boldsymbol{\varpi}^h(\alpha,\cdot))}|_{\alpha=s} \ , \qquad s,t \in \mathbb{R} \ , \tag{53}$$

where we recall that $\mathrm{Aut}(E) \subsetneq C(E;E)$ is the subspace of all automorphisms (or self-homeomorphisms) of E.

Proof. The theorem is a consequence of Lemmata 7.3 and 7.9. ■

Remark 4.2
(i) *Stronger results than Theorem 4.1 are proven in Section 7. See, in particular, Lemma 7.5.*
(ii) *If $\mathcal{X}$ is separable, recall that the state space E of Definition 2.1 is metrizable, which is a very useful property. In Theorem 4.1, however, the separability of $\mathcal{X}$ is not necessary at the cost of taking a finite dimensional space $\mathfrak{V}$ in Condition (b).*

Condition (b) of Theorem 4.1 is, for instance, satisfied for any cylindrical function h within the set

$$\mathfrak{Z} \doteq \Big\{ (f(t))_{t\in\mathbb{R}} \in \mathfrak{C} : f(t;\rho) = g(t;\rho(B_1),\ldots,\rho(B_n)) \text{ for } t \in \mathbb{R} \text{ and } \rho \in E \\ \text{with } n \in \mathbb{N},\ \{B_j\}_{j=1}^n \subseteq \mathcal{X}^{\mathbb{R}} \text{ and } g \in C_b\left(\mathbb{R}; C_b^3(\mathbb{R}^n,\mathbb{R})\right) \Big\} . \quad (54)$$

By (28), note that, for any $h \in \mathfrak{Z}$, there is $n \in \mathbb{N}$ such that $h(t) \in \mathfrak{Y}_n$ for all $t \in \mathbb{R}$. See also (34). Observe that $\mathfrak{Z} \subsetneq \mathfrak{C}$ is a dense subset since $\mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}^{\mathbb{R}} \subseteq \mathfrak{Z}$. In (54) we are quite generous by assuming that the function $g(t)$ belongs to $C_b^3(\mathbb{R}^n,\mathbb{R})$ for some $n \in \mathbb{N}$, but even $C_b^2(\mathbb{R}^n,\mathbb{R})$ would be sufficient to get Condition (b). We assume more regularity for $g(t)$, $t \in \mathbb{R}$, to be able to prove Theorem 4.6. Here, $C_b^p(\mathbb{R}^n;\mathbb{R})$, $p \in \mathbb{N}$, denotes the Banach space of bounded real-valued C^p-functions on $\mathbb{R}^n$, whose norm is the C^p-norm, i.e., the sum of the supremum norm of all derivatives of order from 0 to p.

As explained in Section 2.5, for quantum systems, we shall not restrict our study to the phase space $\overline{\mathcal{E}(E)}$ of Definition 2.2, but we generally consider the whole state space E of Definition 2.1. We show next that both the set $\mathcal{E}(E)$ of extreme points and its weak* closure $\overline{\mathcal{E}(E)}$ are conserved by the flow of Theorem 4.1, which is defined on the whole state space E:

Corollary 4.3 (Conservation of the phase space)
Under Conditions (a)-(b) of Theorem 4.1, for any $s,t \in \mathbb{R}$,

$$\boldsymbol{\varpi}^h(s,t)(\mathcal{E}(E)) \subseteq \mathcal{E}(E) \qquad \textit{and} \qquad \boldsymbol{\varpi}^h(s,t)(\overline{\mathcal{E}(E)}) \subseteq \overline{\mathcal{E}(E)} .$$

Proof. The proof is done by contradiction: Assume Conditions (a)-(b) of Theorem 4.1. Take $\rho \in \mathcal{E}(E)$ and assume the existence of $s,t \in \mathbb{R}$, $\lambda \in (0,1)$ and two distinct $\rho_1, \rho_2 \in E$ such that

$$\boldsymbol{\varpi}^h(s,t)(\rho) = \phi_{t,s}^{(h,\boldsymbol{\varpi}^h(s,\cdot))}(\rho) = (1-\lambda)\rho_1 + \lambda\rho_2 .$$

See Theorem 4.1. By (49) and (51), it follows that

$$\rho = (1-\lambda)\rho_1 \circ T_{s,t}^{\boldsymbol{\varpi}^h(s,\cdot)(\rho)} + \lambda\rho_2 \circ T_{s,t}^{\boldsymbol{\varpi}^h(s,\cdot)(\rho)} .$$

This is not possible whenever $\rho \in \mathcal{E}(E)$ because

$$\rho_1 \circ T_{s,t}^{\boldsymbol{\varpi}^h(s,\cdot)(\rho)} \qquad \text{and} \qquad \rho_2 \circ T_{s,t}^{\boldsymbol{\varpi}^h(s,\cdot)(\rho)}$$

are two distinct states. This proves that the image of an extreme state by $\boldsymbol{\varpi}^h(s,t)$ is always an extreme state. $\boldsymbol{\varpi}^h(s,t) \in \mathrm{Aut}(E)$ and thus preserves the phase space $\overline{\mathcal{E}(E)}$. ■

4.4 Classical Dynamics as Feller Evolution

The continuous family ϖ^h of Theorem 4.1 yields a family $(V_{t,s}^h)_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{C} \doteq C(E;\mathbb{C})$ defined by

$$V_{t,s}^h(f) \doteq f \circ \varpi^h(s,t) , \qquad f \in \mathfrak{C},\ s,t \in \mathbb{R} . \tag{55}$$

By Corollary 4.3, such a map can also be defined in the same way on $C(\overline{\mathcal{E}(E)};\mathbb{C})$ or $C(\mathcal{E}(E);\mathbb{C})$, where we recall that $\overline{\mathcal{E}(E)}$ is the phase space of Definition 2.2. In any case, it is a strongly continuous two-parameter family defining a classical dynamics:

Proposition 4.4 (Classical dynamics as Feller evolution system)
Under Conditions (a)-(b) of Theorem 4.1, $(V_{t,s}^h)_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{C}$ satisfying the reverse cocycle property:

$$\forall s,r,t \in \mathbb{R} : \qquad V_{t,s}^h = V_{r,s}^h \circ V_{t,r}^h . \tag{56}$$

If, additionally, $h \in \mathfrak{Y}(\mathbb{R})$ (cf. (43)), then $V_{t,s}^h = V_{t-s,0}^h$ for any $s,t \in \mathbb{R}$ and $(V_{t,0}^h)_{t\in\mathbb{R}}$ is a C_0-group of $*$-automorphisms of $\mathfrak{C}$.

Proof. The strong continuity of this family with respect to $s,t \in \mathbb{R}$ is a consequence of $\varpi^h \in C(\mathbb{R}^2; \mathrm{Aut}(E))$ and the fact that any continuous family of continuous functions on compacta is uniformly continuous. Recall that the topology of $\mathrm{Aut}(E)$ is the topology of uniform convergence of weak*-continuous functions from E to itself. (To prove continuity in such a strong sense, one could also use $V_{t,s}^h \in \mathcal{B}(\mathfrak{C})$ and the density of $\mathfrak{C}_{\mathcal{X}}$ in $\mathfrak{C}$.) Equation (56) follows from Corollary 7.7. Finally, if $h \in \mathfrak{Y}(\mathbb{R})$, while assuming Conditions (a)-(b) of Theorem 4.1, then the family $(T_{t,s}^{\xi})_{s,t\in\mathbb{R}}$ defined by (47)-(48) for any $\xi \in C(\mathbb{R}; E)$ satisfies $T_{t,s}^{\xi} = T_{t-s,0}^{\xi(\cdot+s)}$ for any $s,t \in \mathbb{R}$, where $\xi(\cdot + s) \in C(\mathbb{R}; E)$ is the function ξ translated by the real number s. As a consequence, at any fixed $s \in \mathbb{R}$ and $\rho \in E$, the function $\xi \in C(\mathbb{R}; E)$ defined by

$$\xi_s(t) \doteq \varpi^h(0, t-s; \rho) , \qquad t \in \mathbb{R} ,$$

is a solution to Equation (137). By Lemma 7.3, it follows that

$$\varpi^h(0, t-s) = \varpi^h(s,t) , \qquad s,t \in \mathbb{R} ,$$

i.e., $V_{t,s}^h = V_{t-s,0}^h$ for any $s,t \in \mathbb{R}$. By using (56) at $r = t - \alpha + s$ for any $\alpha \in \mathbb{R}$, one verifies that the one-parameter family $(V_{t,0}^h)_{t\in\mathbb{R}}$ satisfies the group property. ∎

Under Conditions (a)-(b) of Theorem 4.1, $(V_{t,s}^h)_{s,t\in\mathbb{R}}$ restricted on $\mathfrak{C}^{\mathbb{R}}$ is automatically a *Feller evolution system* in the following sense:

- As a $*$-automorphism of a C^*-algebra, $V_{t,s}^h$ is positivity preserving and $\|V_{t,s}^h\|_{\mathcal{B}(\mathfrak{C}^{\mathbb{R}})} = 1$;
- $(V_{t,s}^h)_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family satisfying (56).

Therefore, the classical dynamics defined on the real space $\mathfrak{C}^{\mathbb{R}}$ from $(V_{t,s}^h)_{s,t\in\mathbb{R}}$ can be associated in this case with Feller processes[26] in probability theory: By the Riesz-Markov representation theorem and the monotone convergence theorem, there is a unique two-parameter group $(p_{t,s}^h)_{s,t\in\mathbb{R}}$ of Markov transition kernels $p_{t,s}^h(\cdot,\cdot)$ on E such that

$$V_{t,s}^h f(\rho) = \int_E f(\hat{\rho})\, p_{t,s}^h(\rho, \mathrm{d}\hat{\rho}) , \qquad f \in \mathfrak{C}^{\mathbb{R}} .$$

[26] The positivity and norm-preserving property are reminiscent of Markov semigroups.

The right hand side of the above identity makes sense for bounded measurable functions from E to $\mathbb{R}$. In fact, one can naturally extend $(V^h_{t,s})_{s,t\in\mathbb{R}}$ to this more general class of functions on E. See (55).

Note that the notion of Feller evolution system, which is an extension of Feller semigroups to non-autonomous two-parameter families, has been (probably) introduced (only) in 2014 [30]. In contrast with [30], here the usual cocycle property is replaced by the reverse one and $C_\infty(\mathbb{R}^d)$ by $\mathfrak{C}^{\mathbb{R}}$, similar to [76, Section 8.1.15] or [77, Definition 1.6], because we do not have any differentiable structure on E. In fact, the term “Feller semigroup” can have different definitions[27] in the literature. See, e.g., [76, Section 8.1.15] and [77, Section 1.1].

For any constant function $h \in \mathfrak{Y}(\mathbb{R})$ satisfying Conditions (a)-(b) of Theorem 4.1, $(V^h_{t,0})_{t\in\mathbb{R}}$ is therefore a C_0-group of $*$-automorphisms of $\mathfrak{C}$ and we denote by $\daleth^h$ its (well-defined) generator. By [40, Chap. II, Sect. 3.11], it is a closed (linear) operator densely defined in $\mathfrak{C}$. Since $V^h_{t,0}$, $t \in \mathbb{R}$, are $*$-automorphisms, we infer from the Nelson theorem [34, Theorem 1.5.4], or the Lumer-Phillips theorem [36, Theorem 3.1.16], that $\pm\daleth^h$ are dissipative operators, i.e., $\daleth^h$ is conservative. The $*$-morphism property of $V^h_{t,0}$, $t \in \mathbb{R}$, is reflected by the fact that $\daleth^h$ has to be a symmetric derivation of $\mathfrak{C}$. This closed derivation is directly related with a Poissonian symmetric derivation:

Theorem 4.5 (Generators as Poissonian symmetric derivations)
Assume Conditions (a)-(b) of Theorem 4.1.
(i) *The Poissonian symmetric derivation $\mathfrak{d}^h$ of Definition 3.13 is closable. Its closure $\bar{\mathfrak{d}}^h$ is conservative and equals the generator $\daleth^h \supseteq \bar{\mathfrak{d}}^h$ on its domain.*
(ii) *If $\beth \supseteq \bar{\mathfrak{d}}^h$ is a conservative closed operator generating a C_0-group, then $\beth = \daleth^h$.*
(iii) *If $h \in \mathfrak{C}_{\mathcal{X}^{\mathbb{R}}}$ then $\bar{\mathfrak{d}}^h = \daleth^h$ is the generator of the C_0-group $(V^h_{t,0})_{t\in\mathbb{R}}$.*

Proof. Fix all assumptions of the theorem. Note first that one can compute $\daleth^h$ for any (elementary) functions of $\{\hat{A} : A \in \mathcal{X}\}$, see (15). In the light of the self-consistency equation given by Theorem 4.1, which is combined with (50)-(51) and (55), note that, for any $\rho \in E$, $s,t \in \mathbb{R}$ and $A \in \mathcal{X}$,

$$V^h_{t,s}(\hat{A})(\rho) = \rho \circ T^{\boldsymbol{\varpi}^h(s,\cdot;\rho)}_{t,s}(A) \ ,$$

which, by (47), in turn leads to the equality

$$\partial_t V^h_{t,s}(\hat{A})(\rho) = \boldsymbol{\varpi}^h(s,t;\rho) \circ X^{\boldsymbol{\varpi}^h(s,t;\rho)}_t(A) \ . \tag{57}$$

Using Definitions 3.10, 3.13, Equations (39), (45) and (55) as well as the fact that $(V^h_{t,0})_{t\in\mathbb{R}}$ is generated by $\daleth^h$, we deduce from the last equality that

$$\daleth^h(\hat{A}) = \mathfrak{d}^h(\hat{A}) \ , \qquad A \in \mathcal{X} \ .$$

Since both $\daleth^h$ and $\mathfrak{d}^h$ are symmetric derivations, it follows that

$$\daleth^h|_{\mathfrak{C}_{\mathcal{X}}} = \mathfrak{d}^h \ . \tag{58}$$

The operator $\mathfrak{d}^h$ is therefore (norm-) closable: For any sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{dom}(\mathfrak{d}^h) = \mathfrak{C}_{\mathcal{X}}$ converging to 0, if $(\mathfrak{d}^h(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence then it converges to 0, by (58) and the closedness of $\daleth^h$, as a generator of a C_0-group. Since $\daleth^h$ is conservative, we also infer from (58) that both the operator $\mathfrak{d}^h$ and its closure of $\mathfrak{d}^h$ are conservative. (See, e.g., [36, Proposition 3.1.15].) The generator $\daleth^h$ is a closed, not necessarily minimal, extension of $\mathfrak{d}^h$. This concludes the proof of Assertion (i). The second one (ii) thus follows from [36, Proposition 3.1.15].

[27] Feller semigroups have usually the same properties, but they can be defined on different classes of spaces in the litterature.

To prove Assertion (iii) we use (ii) and the Nelson theorem [34, Theorem 1.5.4]: Pick $h_1, h_2 \in \mathfrak{C}_{\mathcal{X}}$. Assume without loss of generality that h_1, h_2 are both not constant functions. Then, for any $\ell \in \{1,2\}$ there are $n_\ell \in \mathbb{N}$, $\{B_{\ell,j}\}_{j=1}^{n_\ell} \subseteq \mathcal{X}^{\mathbb{R}}$, and $g_\ell : \mathbb{R}^{n_\ell} \to \mathbb{R}$ being a polynomial of degree $m_\ell \in \mathbb{N}$ such that

$$h_\ell(\rho) = g_\ell\left(\rho\left(B_{\ell,1}\right), \ldots, \rho\left(B_{\ell,n_\ell}\right)\right) , \qquad \rho \in E .$$

Then, from Equation (34) and Definition 3.10, note that $\mathfrak{d}^{h_1}(h_2) \in \mathfrak{C}_{\mathcal{X}}$ with

$$\mathfrak{d}^{h_1}(h_2)(\rho) = \sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2} \rho\left(i\left[B_{1,j_1}, B_{2,j_2}\right]\right) \partial_{x_{j_1}} g_1\left(\rho\left(B_{1,1}\right), \ldots, \rho\left(B_{1,n_1}\right)\right) \times \partial_{x_{j_2}} g_2\left(\rho\left(B_{2,1}\right), \ldots, \rho\left(B_{2,n_2}\right)\right) \tag{59}$$

for any $\rho \in E$. Note that, for any $k \in \mathbb{N}$,

$$n_1^k \prod_{j=0}^{k-1}\left(j\left(n_1+1\right)+n_2\right) \leq n_1^k\left(k\left(n_1+1\right)+n_2\right)^k \leq k! \exp\left(n_1\left(k\left(n_1+1\right)+n_2\right)\right) , \tag{60}$$

because $x^n \leq n!\mathrm{e}^x$ for all $x \geq 0$ and $n \in \mathbb{N}$. Thus, using (59)-(60) together with Equations (14), (16) and straightforward estimates, one gets that

$$\left\|(\mathfrak{d}^{h_1})^k(h_2)\right\|_{\mathfrak{C}} \leq k!2^k\left(1+D_0\right)^k\left(1+D_1\right)^k\left(1+D_2\right)\exp\left(n_1\left(k\left(n_1+1\right)+n_2\right)\right) , \qquad k \in \mathbb{N} .$$

where

$$D_0 \doteq \max_{\ell\in\{1,2\}}\max_{j\in\{1,\ldots,n_\ell\}} \left\|B_{\ell,j}\right\|_{\mathcal{X}} , \quad D_\ell \doteq \max_{\underline{n}\in\mathbb{N}_0^{m_\ell}}\left\{\max_{\rho\in E}\left|\partial^{\underline{n}} g_\ell\left(\rho\left(B_{\ell,1}\right), \ldots, \rho\left(B_{\ell,n_\ell}\right)\right)\right|\right\}$$

for $\ell \in \{1,2\}$. It follows that

$$\sum_{k\in\mathbb{N}} \frac{t^k}{k!}\left\|(\mathfrak{d}^{h_1})^k(h_2)\right\|_{\mathfrak{C}} < \infty$$

for some positive time t satisfying

$$0 \leq t < \frac{\mathrm{e}^{-n_1(n_1+1)}}{2\left(1+D_0\right)\left(1+D_1\right)\left(1+D_2\right)} .$$

Therefore, by density of $\mathfrak{C}_{\mathcal{X}}$ in $\mathfrak{C}$, the conservative, densely defined, closed operator $\bar{\mathfrak{d}}^{h_1}$ has a dense set of analytic elements. By the Nelson theorem [34, Theorem 1.5.4], $\bar{\mathfrak{d}}^{h_1}$ is a conservative closed operator generating a C_0-group of $*$-automorphisms of $\mathfrak{C}$, whence Assertion (iii), following (ii). ■

Note that Equation (57) holds true for *any* $h \in C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$ satisfying Conditions (a)-(b) of Theorem 4.1. It follows that, for any $s, t \in \mathbb{R}$ and polynomial function $f \in \mathfrak{C}_{\mathcal{X}}$,

$$\partial_t V_{t,s}^h(f) = V_{t,s}^h\left(\{h(t), f\}\right) . \tag{61}$$

A similar expression for $\partial_s V_{t,s}^h$ like

$$\partial_s V_{t,s}^h(f) = -\left\{h(s), V_{t,s}^h(f)\right\} \tag{62}$$

is less obvious. First, we do not know, a priori, if $V_{t,s}^h$ maps elements from $\mathfrak{C}_{\mathcal{X}}$ to continuously differentiable complex-valued functions on E, i.e., if $V_{t,s}^h(\mathfrak{C}_{\mathcal{X}}) \subseteq \mathfrak{Y}(\mathbb{C})$. Secondly, even if $V_{t,s}^h(\mathfrak{C}_{\mathcal{X}}) \subseteq \mathfrak{Y}(\mathbb{R})$, one still has to prove that Equation (62) holds true. This is done in the next theorem:

Theorem 4.6 (Non-autonomous classical dynamics)
Take $h \in \mathfrak{Z}$. Then, for any $s,t \in \mathbb{R}$ and $f \in \mathfrak{C}_{\mathcal{X}}$, (61)-(62) hold true. See (54) for the definition of $\mathfrak{Z}$.

Proof. Note that any function $h \in \mathfrak{Z}$ satisfies Conditions (a)-(b) of Theorem 4.1. Equation (61) is already discussed before the theorem: it results from (57) for $h \in C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$ and properties of derivatives and symmetric derivations (linearity and Leibniz's rule, see, e.g., (40)). To prove (62), it suffices to invoke Corollary 7.12, which says that

$$\partial_s V^h_{t,s}(\hat{A}) = -\{h(s), V^h_{t,s}(\hat{A})\}$$

for any $s,t \in \mathbb{R}$ and $A \in \mathcal{X}$. Since $(V^h_{t,s})_{s,t\in\mathbb{R}}$ is a family of $*$-automorphisms of $\mathfrak{C}$, by using the (bi)linearity and Leibniz's rule satisfied by the derivatives and the bracket $\{\cdot,\cdot\}$, we deduce (62) for all polynomial functions $f \in \mathfrak{C}_{\mathcal{X}}$. ■

This theorem applied to the autonomous situation leads to the dynamical equation of classical mechanics (see, e.g., [78, Proposition 10.2.3]), i.e., (autonomous) *Liouville's equation*, which reads in our case as follows:

Corollary 4.7 (Autonomous Liouville's equation)
Take $h \in \mathfrak{Z}$ constant in time. Then, for any $t \in \mathbb{R}$ and $f \in \mathfrak{C}_{\mathcal{X}}$,

$$\partial_t V^h_{t,0}(f) = V^h_{t,0} \circ \urcorner^h(f) = V^h_{t,0}(\{h,f\}) = \left\{h, V^h_{t,0}(f)\right\} = \urcorner^h \circ V^h_{t,0}(f) \ . \tag{63}$$

Proof. Combine Theorem 4.6 with Theorem 4.5. ■

In the non-autonomous case, $(V^h_{t,s})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{C}$ solving the non-autonomous evolution equations

$$\forall s,t \in \mathbb{R}: \qquad \partial_t V^h_{t,s} = V^h_{t,s} \circ \urcorner^{h(t)} , \qquad V^h_{s,s} = \mathbf{1}_{\mathfrak{C}} ,$$

on $\mathfrak{C}_{\mathcal{X}}$, as explained before Theorem 4.6. See also Theorem 4.5. Theorem 4.6 suggests that, under stronger conditions, $(V^h_{t,s})_{s,t\in\mathbb{R}}$ is the solution to the non-autonomous evolution equations

$$\forall s,t \in \mathbb{R}: \qquad \partial_s V^h_{t,s} = -\urcorner^{h(s)} \circ V^h_{t,s} , \qquad V^h_{s,s} = \mathbf{1}_{\mathfrak{C}} , \tag{64}$$

on some dense subspace. To prove this, one could look for assumptions on h such that the family $(\urcorner^{h(t)})_{t\in\mathbb{R}}$ of closed dissipative operators satisfies sufficient conditions to generate an evolution family solving (64), as explained in [75, 79–82]. Then, $(V^h_{t,s})_{s,t\in\mathbb{R}}$ would be *the* solution to the non-autonomous evolution equation (64). This looks doable, but at the cost of many technical arguments. We thus refrain from doing such a study in this paper.

5 State-Dependent C^*-Dynamical Systems

5.1 Quantum C^*-Algebras of Continuous Functions on State Space

The space $C(E;\mathcal{X})$ of $\mathcal{X}$-valued weak*-continuous functions on the weak*-compact space E is a unital C^*-algebra with respect to the point-wise operations, denoted by

$$\mathfrak{X} \doteq (C(E;\mathcal{X}), +, \cdot_{\mathbb{C}}, \times, ^*, \|\cdot\|_{\mathfrak{X}}) \tag{65}$$

where

$$\|f\|_{\mathfrak{X}} \doteq \max_{\rho\in E} \|f(\rho)\|_{\mathcal{X}} \ , \qquad f \in \mathfrak{X} \ . \tag{66}$$

Clearly, $\mathfrak{X}$ is commutative iff $\mathcal{X}$ is commutative. The (real) Banach subspace of all $\mathcal{X}^{\mathbb{R}}$-valued functions from $\mathfrak{X}$ is denoted by $\mathfrak{X}^{\mathbb{R}} \subsetneq \mathfrak{X}$. $\mathfrak{X}$ is separable whenever $\mathcal{X}$ is separable, E being in this case metrizable.

We identify the primordial C^*-algebra $\mathcal{X}$, on which the quantum dynamics is usually defined, with the subalgebra of constant functions of $\mathfrak{X}$. Meanwhile, the classical dynamics appears in the space $\mathfrak{C} \doteq C(E;\mathbb{C})$ of complex-valued weak*-continuous functions on E. See (13)-(14). This unital *commutative* C^*-algebra is thus identified with the subalgebra of functions of $\mathfrak{X}$ whose values are multiples of the unit $\mathfrak{1} \in \mathcal{X}$. Compare (65)-(66) with (13)-(14). Hence, we have the inclusions

$$\mathcal{X} \subseteq \mathfrak{X} \qquad \text{and} \qquad \mathfrak{C} \subseteq \mathfrak{X} \ . \tag{67}$$

Both classical and quantum dynamics can then be extended to $\mathfrak{X}$. This is explained in the next subsection.

5.2 State-Dependent Quantum Dynamics

Since $\mathfrak{C} \subseteq \mathfrak{X}$, there is a natural extension to $\mathfrak{X}$ of the classical dynamics on $\mathfrak{C}$: The continuous family $\boldsymbol{\varpi}^h$ of Theorem 4.1 yields a family $(\mathfrak{V}_{t,s}^h)_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{X}$ defined by

$$\mathfrak{V}_{t,s}^h(f) \doteq f \circ \boldsymbol{\varpi}^h(s,t) \ , \qquad f \in \mathfrak{X}, \ s,t \in \mathbb{R} \ . \tag{68}$$

In particular, by (55), $\mathfrak{V}_{t,s}^h|_{\mathfrak{C}} = V_{t,s}^h$ for any $s,t \in \mathbb{R}$. However, it is *not* what we have in mind here: Emphasizing rather the inclusion $\mathcal{X} \subseteq \mathfrak{X}$, the classical algebra $\mathfrak{C}$ will become a subalgebra of the *fixed-point* algebra of the state-dependent dynamics we define below on $\mathfrak{X}$.

In Section 4.2, we explain how a fixed function $h \in C_b(\mathbb{R};\mathfrak{Y}(\mathbb{R}))$ is used to define (possibly non-autonomous) quantum dynamics $(T_{t,s}^{\xi})_{s,t\in\mathbb{R}}$ on the primordial C^*-algebra $\mathcal{X}$, for any $\xi \in C(\mathbb{R};E)$. This primal dynamics induces classical dynamics on the (classical) C^*-algebra $\mathfrak{C} \doteq C(E;\mathbb{C})$ of continuous functions on states, as discussed in Sections 4.3-4.4. By Theorem 4.1, it yields, in turn, a *state-dependent* quantum dynamics, referring in this case to a norm-continuous family

$$(\mathrm{T}_{t,s}^{\rho})_{(\rho,s,t)\in E\times\mathbb{R}^2} = (T_{t,s}^{\boldsymbol{\varpi}^h(s_0,\cdot;\rho)})_{(\rho,s,t)\in E\times\mathbb{R}^2}$$

of $*$-automorphisms of $\mathcal{X}$ for some *fixed* $s_0 \in \mathbb{R}$. This leads to a (state-dependent) dynamics on the (secondary) C^*-algebra $\mathfrak{X}$ of continuous functions on states.

As a matter of fact, any strongly continuous family $(\mathrm{T}^{\rho})_{\rho\in E}$ of linear contractions from $\mathcal{X}$ to itself can be viewed as a linear contraction $\mathfrak{T}$ from $\mathfrak{X}$ to itself defined by

$$[\mathfrak{T}(f)](\rho) \doteq \mathrm{T}^{\rho}(f(\rho)) \ , \qquad \rho \in E, \ f \in \mathfrak{X} \ . \tag{69}$$

Such contractions have the following properties:

Lemma 5.1 (State-dependent quantum dynamics)
Let $\mathcal{X}$ be a unital C^*-algebra. For any $s,t \in \mathbb{R}^2$, let $(\mathrm{T}_{t,s}^{\rho})_{\rho\in E}$ be any strongly continuous family of linear contractions from $\mathcal{X}$ to itself, and $\mathfrak{T}_{t,s}$ be defined by (69) with $\mathrm{T}^{\rho} = \mathrm{T}_{t,s}^{\rho}$.
(i) *If $\mathrm{T}_{t,s}^{\rho}$ is a $*$-automorphism of $\mathcal{X}$ at $s,t \in \mathbb{R}$ for any $\rho \in E$, then $\mathfrak{T}_{t,s}$ is a $*$-automorphism of $\mathfrak{X}$ and the classical subalgebra $\mathfrak{C} \subseteq \mathfrak{X}$ is contained in the fixed-point algebra of $\mathfrak{T}_{t,s}$, i.e.,*

$$\mathfrak{T}_{t,s}(f) = f \ , \qquad f \in \mathfrak{C} \ .$$

(ii) *If $(\mathrm{T}_{t,s}^{\rho})_{s,t\in\mathbb{R}}$ satisfies a reverse cocycle property for any $\rho \in E$, i.e.,*

$$\mathrm{T}_{t,s}^{\rho} = \mathrm{T}_{r,s}^{\rho} \circ \mathrm{T}_{t,r}^{\rho} \ , \qquad \rho \in E, \ s,t,r \in \mathbb{R} \ , \tag{70}$$

then $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$ has also this property.
(iii) *If $(\mathrm{T}_{t,s}^{\rho})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ is a strongly continuous family of contractions then so do $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$.*

Proof. Assertion (i)-(ii) directly follows from (69) and it remains to prove (iii). By contradiction, suppose that the family is not strongly continuous. Then, there is $f \in \mathfrak{X}$, times $s,t \in \mathbb{R}$, two zero nets $(\eta_j)_{j\in J}, (\varkappa_j)_{j\in J} \subseteq \mathbb{R}$, a net $(\rho_j)_{j\in J} \subseteq E$ of states and a positive constant $D > 0$ such that

$$\inf_{j\in J} \left\| \mathrm{T}_{t+\eta_j,s+\varkappa_j}^{\rho_j} \left(f \left(\rho_j \right) \right) - \mathrm{T}_{t,s}^{\rho_j} \left(f \left(\rho_j \right) \right) \right\|_{\mathcal{X}} \geq D > 0 \ .$$

By weak* compactness of E, we can assume without loss of generality that $(\rho_j)_{j\in J}$ converges to some $\rho \in E$. Because $(\mathrm{T}_{t,s}^{\rho})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ is a family of contractions, the above bound yields

$$\liminf_{j\in J} \left\| \mathrm{T}_{t+\eta_j,s+\varkappa_j}^{\rho_j} \left(f \left(\rho \right) \right) - \mathrm{T}_{t,s}^{\rho_j} \left(f \left(\rho \right) \right) \right\|_{\mathcal{X}} \geq D > 0 \ ,$$

which contradicts the strong continuity of this family. ■

If $(\mathrm{T}_{t,s}^{\rho})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ is a family of $*$-automorphisms of $\mathcal{X}$ then $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$ is a family of $*$-automorphisms of $\mathfrak{X}$ and, by Lemma 5.1 (i), the classical subalgebra $\mathfrak{C} \subseteq \mathfrak{X}$ is contained in the fixed-point algebra of the full quantum dynamics $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$, i.e., for all $f \in \mathfrak{C}$ and all $s,t \in \mathbb{R}$, $\mathfrak{T}_{t,s}(f) = f$. Any family $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{X}$ preserving each element of $\mathfrak{C}$ is of this form, at least when $\mathcal{X}$ is separable:

Lemma 5.2 (State-dependent quantum dynamics and fixed-point algebra)
Let $\mathcal{X}$ be a separable, unital C^*-algebra. The classical subalgebra $\mathfrak{C} \subseteq \mathfrak{X}$ is contained in the fixed-point algebra of a strongly continuous two-parameter family $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{X}$ iff there is a strongly continuous family $(\mathrm{T}_{t,s}^{\rho})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ of $*$-automorphisms of $\mathcal{X}$ satisfying (69).

Proof. In order to obtain the equivalence stated in the lemma, it only remains to prove that any strongly continuous family $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{X}$ whose fixed-point algebra contains $\mathfrak{C}$ comes from the strongly continuous family $(\mathrm{T}_{t,s}^{\rho})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ of $*$-homomorphisms defined by

$$\mathrm{T}_{t,s}^{\rho} \left(A \right) \doteq \left[\mathfrak{T}_{t,s} \left(A \right) \right] \left(\rho \right) \ , \qquad \rho \in E, \ A \in \mathcal{X} \subseteq \mathfrak{X}, \ s,t \in \mathbb{R} \ . \tag{71}$$

To this end, recall that, if $\mathcal{X}$ is separable then E is metrizable. So, take a distance $d(\cdot,\cdot)$ generating the weak* topology on E. For any $\rho \in E$ define the sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{C}$ of continuous functions by

$$g_n \left(\tilde{\rho} \right) = \frac{1}{1 + nd \left(\tilde{\rho}, \rho \right)} \ , \qquad \tilde{\rho} \in E, \ n \in \mathbb{N} \ .$$

Since, by assumption, $\mathfrak{T}_{t,s}$ is a $*$-automorphism of $\mathfrak{X}$ satisfying $\mathfrak{T}_{t,s}(g_n) = g_n$ for $s,t \in \mathbb{R}$ and $n \in \mathbb{N}$, we note that, for every fixed $\rho \in E$, $s,t \in \mathbb{R}$, $n \in \mathbb{N}$ and all functions $f \in \mathfrak{X}$,

$$\left[\mathfrak{T}_{t,s} \left(f \right) \right] \left(\rho \right) = \left[\mathfrak{T}_{t,s} \left(f g_n - f \left(\rho \right) g_n \right) \right] \left(\rho \right) + \mathrm{T}_{t,s}^{\rho} \left(f \left(\rho \right) \right) \ .$$

Because $\mathfrak{T}_{t,s}$ is a contraction (for it is a $*$-automorphism), by continuity of $f \in \mathfrak{X}$, it follows that

$$\lim_{n\to\infty} \left\| \mathfrak{T}_{t,s} \left(f g_n - f \left(\rho \right) g_n \right) \right\|_{\mathfrak{X}} = \lim_{n\to\infty} \left\| f g_n - f \left(\rho \right) g_n \right\|_{\mathfrak{X}} = 0 \ ,$$

and hence,

$$\left[\mathfrak{T}_{t,s} \left(f \right) \right] \left(\rho \right) = \mathrm{T}_{t,s}^{\rho} \left(f \left(\rho \right) \right) \ , \qquad \rho \in E, \ f \in \mathfrak{X}, \ s,t \in \mathbb{R} \ .$$

From the last equality we also conclude that $\mathrm{T}_{t,s}^{\rho}$ is a $*$-automorphism of $\mathcal{X}$ for all $(\rho,s,t) \in E \times \mathbb{R}^2$. ■

The above situation motivates the following notion of *state-dependent* C^*-dynamical system:

Definition 5.3 (State-dependent C^*-dynamical systems)
If $\mathfrak{T} \equiv (\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$ is a strongly continuous two-parameter family of $*$-automorphisms of $\mathfrak{X}$ preserving each element of $\mathfrak{C} \subseteq \mathfrak{X}$ and satisfying the reverse cocycle property

$$\mathfrak{T}_{t,s} = \mathfrak{T}_{r,s} \circ \mathfrak{T}_{t,r} \ , \qquad s,t,r \in \mathbb{R} \ ,$$

then we name the pair $(\mathfrak{X}, \mathfrak{T})$ "state-dependent C^*-dynamical system".

An example of such a C^*-dynamical system is given from Theorem 4.1 via the family $\mathfrak{T}^{h,s_0} \equiv (\mathfrak{T}^{h,s_0}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{X}$ defined by

$$\left[\mathfrak{T}^{h,s_0}_{t,s}(f)\right](\rho) \doteq T^{\varpi^h(s_0,\cdot;\rho)}_{t,s}(f(\rho)) \ , \qquad \rho \in E, \ f \in \mathfrak{X}, \ s,t \in \mathbb{R} \ ,$$

for any fixed $s_0 \in \mathbb{R}$ and every (time-depending classical Hamiltonian) $h \in C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$ satisfying all assumptions of Theorem 4.1. This is a state-dependent C^*-dynamical system:

Lemma 5.4 (From self-consistency equations to state-dependent quantum dynamics)
Assume Conditions (a)-(b) of Theorem 4.1. Then, for any $s_0 \in \mathbb{R}$, $(\mathfrak{X}, \mathfrak{T}^{h,s_0})$ is a state-dependent C^*-dynamical system.

Proof. Fix all parameters of the lemma. By Lemma 5.1 (i), $\mathfrak{T}^{h,s_0}_{t,s}$ is $*$-automorphism of $\mathfrak{X}$ and the classical subalgebra $\mathfrak{C} \subseteq \mathfrak{X}$ is contained in the fixed-point algebra of $\mathfrak{T}^{h,s_0}_{t,s}$ for any $s,t \in \mathbb{R}$. From Lemma 5.1 (ii), $\mathfrak{T}^{h,s_0}$ clearly satisfies the reverse cocycle property. Moreover, by Lemma 7.1 and Theorem 4.1, we can infer from Lemma 5.1 (iii) that $\mathfrak{T}^{h,s_0}$ is strongly continuous. ∎

Exactly like the classical dynamics defined in Section 4.4, state-dependent C^*-dynamical systems $(\mathfrak{X}, \mathfrak{T})$ induce Feller dynamics within the (classical) commutative C^*-algebra $\mathfrak{C}$:

- Recall that $\mathrm{Aut}(E)$ is the space of all automorphisms (or self-homeomorphisms) of the state space E, endowed with the topology of uniform convergence of weak*-continuous functions.
- From the family $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$, we define a continuous family $(\phi_{t,s})_{s,t\in\mathbb{R}} \subseteq \mathrm{Aut}(E)$ by
$$\phi_{t,s}(\rho) \doteq \rho \circ \mathrm{T}^{\rho}_{t,s} \ , \qquad \rho \in E, \ s,t \in \mathbb{R} \ , \tag{72}$$
where $(\mathrm{T}^{\rho}_{t,s})_{(\rho,s,t)\in E\times\mathbb{R}^2}$ is a strongly continuous family of $*$-automorphisms of $\mathcal{X}$ satisfying (69). See Lemma 5.2. Compare with Equation (51). Similar to Corollary 4.3,
$$\phi_{t,s}(\mathcal{E}(E)) \subseteq \mathcal{E}(E) \qquad \text{and} \qquad \phi_{t,s}(\overline{\mathcal{E}(E)}) \subseteq \overline{\mathcal{E}(E)} \ . \tag{73}$$
- This family in turn yields a strongly continuous two-parameter family $(V_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathfrak{C}$ defined by
$$V_{t,s} f \doteq f \circ \phi_{t,s} \ , \qquad f \in \mathfrak{C}, \ s,t \in \mathbb{R} \ . \tag{74}$$
Compare with Equation (55). Moreover, by (73), this map can also be defined in the same way on $C(\overline{\mathcal{E}(E)}; \mathbb{C})$, where we recall that $\overline{\mathcal{E}(E)}$ is the phase space of Definition 2.2.
- If (70) holds true, then $(V_{t,s})_{s,t\in\mathbb{R}}$ satisfies a reverse cocycle property, i.e., for $s,t,r \in \mathbb{R}$, $V_{t,s} = V_{t,r} \circ V_{r,s}$. This classical dynamics is a *Feller evolution system*, as defined in Section 4.4. Compare with Proposition 4.4.
- If, for any $\rho \in E$, the strongly continuous family $(\mathrm{T}^{\rho}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms defined by (69) satisfies in $\mathcal{B}(\mathcal{X})$ some non-autonomous evolution equation, then the family $(V_{t,s})_{s,t\in\mathbb{R}}$ would also satisfy some non-autonomous evolution equation, as discussed at the end of Section 4.4.

5.3 State-Dependent Symmetries and Classical Dynamics

Fix a state-dependent C^*-dynamical system $(\mathfrak{X}, \mathfrak{T})$. See Definition 5.3. A *state-dependent symmetry* of $(\mathfrak{X}, \mathfrak{T})$ is defined as follows:

Definition 5.5 (State-dependent symmetry)
A state-dependent symmetry $\mathfrak{G}$ of $(\mathfrak{X}, \mathfrak{T})$ is a $*$-automorphism of $\mathfrak{X}$ satisfying

$$\mathfrak{G} \circ \mathfrak{T}_{t,s} = \mathfrak{T}_{t,s} \circ \mathfrak{G} , \qquad s,t \in \mathbb{R} ,$$

and with fixed-point algebra containing $\mathfrak{C} \subseteq \mathfrak{X}$.

If $\mathfrak{G}$ is a state-dependent symmetry of $(\mathfrak{X}, \mathfrak{T})$, then, similar to Lemma 5.2, the equalities

$$G^{\rho}(A) \doteq [\mathfrak{G}(A)](\rho) , \qquad \rho \in E,\ A \in \mathcal{X} \subseteq \mathfrak{X} ,$$

define a strongly continuous family $(G^{\rho})_{\rho \in E}$ of $*$-automorphisms of $\mathcal{X}$. In this case, we define the weak*-compact space

$$E_{\mathfrak{G}} \doteq \{\rho \in E : \rho \circ G^{\rho} = \rho\} \tag{75}$$

of $\mathfrak{G}$-invariant states. By Equation (72) and Definition 5.5, together with (69) for $\mathfrak{T} = \mathfrak{T}_{t,s}$ and $\mathrm{T}^{\rho} = \mathrm{T}^{\rho}_{t,s}$, it follows that

$$\phi_{t,s}(E_{\mathfrak{G}}) \subseteq E_{\mathfrak{G}} \qquad \text{and} \qquad \phi_{t,s}(E \backslash E_{\mathfrak{G}}) \subseteq E \backslash E_{\mathfrak{G}} , \qquad s,t \in \mathbb{R} . \tag{76}$$

In particular, by Equation (74), for any function $f \in \mathfrak{C}$ and times $s,t \in \mathbb{R}$,

$$V_{t,s}(f|_{E_{\mathfrak{G}}}) \doteq (V_{t,s}f)|_{E_{\mathfrak{G}}} \qquad \text{and} \qquad V_{t,s}\left(f|_{E \backslash E_{\mathfrak{G}}}\right) \doteq (V_{t,s}f)|_{E \backslash E_{\mathfrak{G}}} \tag{77}$$

well define two two-parameter families of $*$-automorphisms respectively acting on

$$\{f|_{E_{\mathfrak{G}}} : f \in \mathfrak{C}\} \qquad \text{and} \qquad \left\{f|_{E \backslash E_{\mathfrak{G}}} : f \in \mathfrak{C}\right\} . \tag{78}$$

More generally, we can consider a faithful group homomorphism $g \mapsto \mathfrak{G}_g$ from a group G to the group of $*$-automorphisms of $\mathfrak{X}$. Then, a *state-dependent symmetry group* is defined as follows:

Definition 5.6 (State-dependent symmetry group)
A state-dependent symmetry group of $(\mathfrak{X}, \mathfrak{T})$ is a group $(\mathfrak{G}_g)_{g \in \mathrm{G}}$ of state-dependent symmetries of $(\mathfrak{X}, \mathfrak{T})$.

If $(\mathfrak{G}_g)_{g \in \mathrm{G}}$ is a state-dependent symmetry group of $(\mathfrak{X}, \mathfrak{T})$, then, defining the weak*-compact space

$$E_{\mathrm{G}} \doteq \left\{\rho \in E : \rho \circ G^{\rho}_g = \rho \quad \text{for all } g \in \mathrm{G}\right\} \tag{79}$$

of G-invariant states, we observe that

$$\phi_{t,s}(E_{\mathrm{G}}) \subseteq E_{\mathrm{G}} , \qquad \phi_{t,s}(E \backslash E_{\mathrm{G}}) \subseteq E \backslash E_{\mathrm{G}} , \qquad s,t \in \mathbb{R} , \tag{80}$$

(cf. (76)) and, exactly like in Equations (77)-(78), we infer from (74) the existence of two-parameter families of $*$-automorphisms respectively defined on

$$\{f|_{E_{\mathrm{G}}} : f \in \mathfrak{C}\} \qquad \text{and} \qquad \left\{f|_{E \backslash E_{\mathrm{G}}} : f \in \mathfrak{C}\right\} .$$

5.4 Reduction of Classical Dynamics via Invariant Subspaces

Any family $\mathcal{B} \subseteq \mathcal{X}$ defines an equivalence relation

$$\heartsuit_{\mathcal{B}} \doteq \left\{ (\rho_1, \rho_2) \in E^2 : \rho_1(A) = \rho_2(A) \text{ for all } A \in \mathcal{B} \right\}$$

on the set E of states. We say that the subset $E_{\mathcal{B}} \subseteq E$ represents E with respect to $\mathcal{B}$ whenever, for all $\rho_1 \in E$, there is $\rho_2 \in E_{\mathcal{B}}$ such that $(\rho_1, \rho_2) \in \heartsuit_{\mathcal{B}}$. In particular, one can identify continuous functions $f \in \overline{\mathfrak{C}_{\mathcal{B}}}$ with their restrictions to $E_{\mathcal{B}}$.

Fix now a state-dependent C^*-dynamical system $(\mathfrak{X}, \mathfrak{T})$. See Definition 5.3. For any self-adjoint subspace $\mathcal{B} \subseteq \mathcal{X}$, consider the following conditions:

Condition 5.7 (Reduction of dynamics)
(i) $(\mathcal{B} \cap \mathcal{X}^{\mathbb{R}}, i[\cdot,\cdot])$ *is a real Lie algebra,* $[\cdot,\cdot]$ *being the usual commutator in* $\mathcal{X}$*.*
(ii) $E_{\mathcal{B}}$ *is a weak***-compact space representing* E *with respect to* $\mathcal{B}$*.*
(iii) $\mathrm{T}^{\rho}_{t,s}(\mathcal{B}) \subseteq \mathcal{B}$ *for all* $\rho \in E$, $s,t \in \mathbb{R}$*, with* $\mathrm{T}^{\rho}_{t,s}$ *defined by (69).*
(iv) $\phi_{t,s}(E_{\mathcal{B}}) \subseteq E_{\mathcal{B}}$ *for all* $s,t \in \mathbb{R}$*, with* $(\phi_{t,s})_{s,t\in\mathbb{R}}$ *defined by (72).*

By (74), this condition yields that the polynomial algebra $\mathfrak{C}_{\mathcal{B}}$ (17) is preserved by the family $(V_{t,s})_{s,t\in\mathbb{R}}$, i.e.,

$$V_{t,s}(\mathfrak{C}_{\mathcal{B}}) \subseteq \mathfrak{C}_{\mathcal{B}} , \qquad s,t \in \mathbb{R} .$$

In this case, the state space of the classical dynamics coming from $(\mathfrak{T}_{t,s})_{s,t\in\mathbb{R}}$ can be restricted to the weak*-compact subset $E_{\mathcal{B}} \subseteq E$ with the corresponding Poisson algebra for observables being the subalgebra $\mathfrak{C}_{\mathcal{B}} \subseteq C(E_{\mathcal{B}}; \mathbb{C})$.

5.5 Other Constructions Involving Algebras of C^*-Valued Functions

We are not aware whether the C^*-algebra $\mathfrak{X}$ of $\mathcal{X}$-valued continuous functions on states has previously been systematically studied. However, other constructions of C^*-algebras of $\mathcal{X}$-valued, continuous or measurable, functions are well-known in the literature. For instance, in [83, Definition 1] C^*-algebras of $\mathcal{X}$-valued measurable functions on a locally compact group $\{\mathfrak{G}_g\}_{g\in\mathrm{G}}$ of $*$-automorphisms of $\mathcal{X}$ are introduced. This kind of construction goes under the name "covariance algebras". In contrast, note that the state space E has no natural group structure. Moreover, the product of covariances algebras are convolutions and not point-wise products as in $\mathfrak{X}$.

Covariance algebras are reminiscent of crossed products of C^*-algebras by groups acting on these algebras. Such products are relatively standard in the theory of operator algebras. For instance, they are fundamental in Haagerup's approach to noncommutative L_p-spaces [84].

6 The Weak*-Hausdorff Hypertopology

Deep in the human unconscious is a pervasive need for a logical universe that makes sense. But the real universe is always one step beyond logic.

"The Sayings of Muad'Dib" by the Princess Irulan[28]

The aim of this section is to provide all arguments to deduce Theorems 2.4-2.5. We adopt a broad perspective on the weak*-Hausdorff hypertopology because it does not seem to have been considered in the past. This leads, hopefully, to a good understanding of this hypertopology along with interesting connections to other fields of mathematics and more general results than those stated in Section 2.3.

[28] *Dune* by F. Herbert (1965).

This broader perspective also highlights the role played by the convexity of weak*-compact subsets in our arguments.

Recall that, when the restriction to singletons of a topology for sets of closed subsets of topological spaces coincide with the original topology of the underlying space, we talk about hypertopologies and hyperspaces of closed sets.

6.1 Immeasurable Hyperspaces

In all Section 6, $\mathcal{X}$ is not necessarily a C^*-algebra, but only a (real or complex) Banach space. Unless it is explicitly mentioned, for convenience, we always consider the complex case, as in all other sections. We study subsets of its dual $\mathcal{X}^*$, which, endowed with the weak*-topology, is a locally convex Hausdorff space. See, e.g., [60, Theorem 10.8]. As is usual in the theory of hyperspaces [58], we start with the set

$$\mathbf{F}(\mathcal{X}^*) \doteq \{F \subseteq \mathcal{X}^* : F \neq \emptyset \text{ is weak}^*\text{-closed}\}$$

of all nonempty weak*-closed subsets of $\mathcal{X}^*$. It is endowed below with some hypertopology.

Recall that there are various standard hypertopologies on general sets of nonempty closed subsets of a metric space $(\mathcal{Y}, d)$: the Fell, Vietoris, Wijsman, proximal or locally finite hypertopologies, to name a few well-known examples. See, e.g., [58]. The most well-studied and well-known hypertopology, named the Hausdorff metric topology [58, Definition 3.2.1], comes from the Hausdorff distance between two sets F_1, F_2, associated with the metric d on $\mathcal{Y}$:

$$d_H(F_1, F_2) \doteq \max\left\{\sup_{x_1\in F_1}\inf_{x_2\in F_2} d(x_1, x_2), \sup_{x_2\in F_2}\inf_{x_1\in F_1} d(x_1, x_2)\right\} \in \mathbb{R}_0^+ \cup \{\infty\} . \tag{81}$$

In this case, the corresponding hyperspace of nonempty closed subsets of $\mathcal{Y}$ is complete iff the metric space $(\mathcal{Y}, d)$ is complete. See, e.g., [58, Theorem 3.2.4]. The Hausdorff metric topology is the hypertopology used in [41, 42], the metric d being the one associated with the norm of a separable Banach space $\mathcal{Y}$, in order to prove the density of the set of convex compact subsets of $\mathcal{Y}$ with dense extreme boundary.

None of these well-known hypertopologies is used here for $\mathbf{F}(\mathcal{X}^*)$. Instead, we use a weak* version of the Hausdorff metric topology. This corresponds to the weak*-Hausdorff hypertopology of Definition 2.3, which is naturally extended to all weak*-closed sets of $\mathbf{F}(\mathcal{X}^*)$:

Definition 6.1 (Weak*-Hausdorff hypertopology)
The weak*-Hausdorff hypertopology on $\mathbf{F}(\mathcal{X}^*)$ is the topology induced (see (10)) by the family of Hausdorff pseudometrics $d_H^{(A)}$ defined, for all $A \in \mathcal{X}$, by

$$d_H^{(A)}(F, \tilde{F}) \doteq \max\left\{\sup_{\sigma\in F}\inf_{\tilde{\sigma}\in\tilde{F}} |(\sigma - \tilde{\sigma})(A)|, \sup_{\tilde{\sigma}\in\tilde{F}}\inf_{\sigma\in F} |(\sigma - \tilde{\sigma})(A)|\right\} \in \mathbb{R}_0^+\cup\{\infty\}, \quad F, \tilde{F} \in \mathbf{F}(\mathcal{X}^*) . \tag{82}$$

To our knowledge, this hypertopology has not been considered so far and we thus give here a detailed study of its main properties. Recall that it is an *hyper*topology because any net $(\sigma_j)_{j\in J}$ in $\mathcal{X}^*$ converges to $\sigma \in \mathcal{X}^*$ in the weak* topology iff the net $(\{\sigma_j\})_{j\in J}$ converges in $\mathbf{F}(\mathcal{X}^*)$ to $\{\sigma\}$ in the weak*-Hausdorff (hyper)topology.

Observe that (82) is always finite on the subspace

$$\mathbf{K}(\mathcal{X}^*) \doteq \left\{K \in \mathbf{F}(\mathcal{X}^*) : \sup_{\sigma\in K} \|\sigma\|_{\mathcal{X}^*} < \infty\right\} \subseteq \mathbf{F}(\mathcal{X}^*) \tag{83}$$

of all nonempty weak*-closed norm-bounded subsets of the dual space $\mathcal{X}^*$. Its complement, i.e., the set of all nonempty weak*-closed norm-unbounded subsets of $\mathcal{X}^*$, is denoted by

$$\mathbf{K}^c(\mathcal{X}^*) \doteq \mathbf{F}(\mathcal{X}^*) \backslash \mathbf{K}(\mathcal{X}^*) \ . \tag{84}$$

Both sets are weak*-Hausdorff closed since the weak*-Hausdorff hypertopology *immeasurably* separates norm-unbounded sets from norm-bounded ones:

Lemma 6.2 (Immeasurable separation of norm-unbounded sets from norm-bounded ones)
Let $\mathcal{X}$ be a Banach space. For any norm-unbounded weak-closed set $F \in \mathbf{K}^c(\mathcal{X}^*)$, there is $A \in \mathcal{X}$ such that*

$$d_H^{(A)}(F, K) = \infty \ , \qquad K \in \mathbf{K}(\mathcal{X}^*) \ . \tag{85}$$

Additionally, the union of any weak-Hausdorff convergent net $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ is norm-bounded.*

Proof. Take any norm-unbounded $F \in \mathbf{K}^c(\mathcal{X}^*)$. Then, there is a net $(\sigma_j)_{j\in J} \subseteq F$ such that

$$\lim_J \|\sigma_j\|_{\mathcal{X}^*} = \infty \ .$$

By the uniform boundedness principle (see, e.g., [45, Theorems 2.4 and 2.5]), there is $A \in \mathcal{X}$ such that

$$\lim_J |\sigma_j(A)| = \infty \ . \tag{86}$$

Now, pick any $K \in \mathbf{K}(\mathcal{X}^*)$. Then, by Definition 6.1 and the triangle inequality, for any $j \in J$,

$$d_H^{(A)}(F, K) \geq \inf_{\tilde{\sigma}\in K} |(\sigma_j - \tilde{\sigma})(A)| \geq |\sigma_j(A)| - \sup_{\tilde{\sigma}\in K} |\tilde{\sigma}(A)| \ .$$

Since K is, by definition, norm-bounded, by (86), the limit over j of the last inequality obviously yields (85).

Finally, any weak*-Hausdorff convergent net $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ has to converge in $\mathbf{K}(\mathcal{X}^*)$, by the first part of the lemma. Therefore, using an argument by contradiction and the uniform boundedness principle (see, e.g., [45, Theorems 2.4 and 2.5]) as above, one also checks that the union of any net $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ that weak*-Hausdorff converges must be norm-bounded. ∎

Because of Lemma 6.2, we say that the (nonempty) subhyperspaces $\mathbf{K}(\mathcal{X}^*)$ and $\mathbf{K}^c(\mathcal{X}^*)$ are weak*-Hausdorff-*immeasurable* with respect to each other.

Corollary 6.3 (Weak*-Hausdorff-clopen subhyperspaces)
Let $\mathcal{X}$ be a Banach space. Then, $\mathbf{K}(\mathcal{X}^*)$ is a weak-Hausdorff-closed subset of $\mathbf{F}(\mathcal{X}^*)$.*

Proof. The assertion is a consequence of Lemma 6.2. Note that a subset of a topological space is closed iff it contains the set of its accumulation points, by [54, Chapter 1, Theorem 5]. The accumulation points of a set are precisely the limits of nets whose elements are in this set, by [54, Chapter 2, Theorem 2]. ∎

Note that $\mathbf{K}(\mathcal{X}^*)$ is also a connected hyperspace:

Lemma 6.4 ($\mathbf{K}(\mathcal{X}^*)$ as connected subhyperspace)
Let $\mathcal{X}$ be a Banach space. Then, $\mathbf{K}(\mathcal{X}^*)$ is convex and path-connected.

Proof. Take any $K_0, K_1 \in \mathbf{K}(\mathcal{X}^*)$. Define the map f from $[0,1]$ to $\mathbf{K}(\mathcal{X}^*)$ by

$$f(\lambda) \doteq \{(1-\lambda)\sigma_0 + \lambda\sigma_1 : \sigma_0 \in K_0, \ \sigma_1 \in K_1\} \ , \qquad \lambda \in [0,1] \ .$$

(This already demonstrates that $\mathbf{K}(\mathcal{X}^*)$ is convex.) By Definition 6.1, for any $\lambda_1, \lambda_2 \in [0,1]$,

$$d_H^{(A)} (f(\lambda_1), f(\lambda_2)) \leq |\lambda_2 - \lambda_1| \max_{\sigma \in (K_0 - K_1)} |\sigma(A)| , \qquad A \in \mathcal{X} .$$

So, the map f is a continuous function from $[0,1]$ to $\mathbf{K}(\mathcal{X}^*)$ with $f(0) = K_0$ and $f(1) = K_1$. Therefore, $\mathbf{K}(\mathcal{X}^*)$ is path-connected. The image under a continuous map of a connected set is connected and, by [54, Chapter 1, Theorem 21], $\mathbf{K}(\mathcal{X}^*)$, being path-connected, is connected. ■

Note that one can prove that $\mathbf{K}(\mathcal{X}^*)$ is even a connected component[29] of $\mathbf{F}(\mathcal{X}^*)$. There are possibly many disconnected components, or even non-trivial weak*-Hausdorff-clopen subsets of $\mathbf{F}(\mathcal{X}^*)$, associated with different directions (characterized by some $A \in \mathcal{X}$) where the weak*-closed sets $F \in \mathbf{K}^c(\mathcal{X}^*)$ are unbounded. This would lead to a whole collection of weak*-Hausdorff-clopen sets, which could be used to form a Boolean algebra whose lattice operations are given by the union and intersection, as is usual in mathematical logics[30]. This is far from the scope of the article and we thus refrain from doing such a study here.

Meanwhile, note that the weak*-Hausdorff-closed set $\mathbf{K}(\mathcal{X}^*)$ of all nonempty weak*-closed norm-bounded subsets of $\mathcal{X}^*$ is nothing else than the set of all nonempty weak*-compact subsets:

Lemma 6.5 (Weak*-compactness vs. norm-boundedness)
Let $\mathcal{X}$ be a Banach space. Then,

$$\mathbf{K}(\mathcal{X}^*) = \{K \subseteq \mathcal{X}^* : K \neq \emptyset \text{ is weak}^*\text{-compact}\} .$$

Proof. The proof of the norm-boundedness of a weak*-compact set is completely standard (see, e.g., [58, Proposition 1.2.9]) and is given here only for completeness: Take any weak*-compact set $K \subseteq \mathcal{X}^*$ and use, for any $A \in \mathcal{X}$, the weak*-continuity of the map $\hat{A} : \sigma \mapsto \sigma(A)$ from $\mathcal{X}^*$ to $\mathbb{C}$ (cf. (15) and (21)) to show that $\sigma(K)$ is a bounded set, by weak* compactness of K. Then, the norm-boundedness of any weak*-compact set is a consequence of the uniform boundedness principle, see, e.g., [45, Theorems 2.4 and 2.5]. Since $\mathcal{X}^*$ is a Hausdorff space (see, e.g., [60, Theorem 10.8]), by [54, Chapter 5, Theorem 7], it follows that weak*-compact set are weak*-closed and norm-bounded subsets of $\mathcal{X}^*$. On the other hand, by the Banach-Alaoglu theorem [45, Theorem 3.15], weak*-closed and norm-bounded subsets of $\mathcal{X}^*$ are also weak*-compact and the assertion follows. ■

By Lemma 6.5, for any $K, \tilde{K} \in \mathbf{K}(\mathcal{X}^*)$, the suprema and infima in (82) become respectively maxima and minima. In this case, Definition 6.1 is the same as Definition 2.3, extended to all weak*-compact sets. Of course, by Lemma 6.5, $\mathbf{K}(\mathcal{X}^*)$ includes the hyperspace

$$\mathbf{CK}(\mathcal{X}^*) \doteq \{K \subseteq \mathcal{X}^* : K \neq \emptyset \text{ is convex and weak}^*\text{-compact}\} \subseteq \mathbf{K}(\mathcal{X}^*) \subseteq \mathbf{F}(\mathcal{X}^*) \qquad (87)$$

of all nonempty convex weak*-compact subsets of $\mathcal{X}^*$, already defined by (7) and used in Section 2.3.

6.2 Hausdorff Property and Closure Operator

One fundamental property one shall ask regarding the hyperspace $\mathbf{F}(\mathcal{X}^*)$ (or $\mathbf{K}(\mathcal{X}^*)$) is whether it is a Hausdorff space, with respect to the weak*-Hausdorff hypertopology, or not. The answer is *negative* for real Banach spaces of dimension greater than 1, as demonstrated in the next lemma:

Lemma 6.6 (Non-weak*-Hausdorff-separable points)
Let $\mathcal{X}$ be a real Banach space. Take any set $K \in \mathbf{CK}(\mathcal{X}^*)$ with weak-path-connected weak*-closed set $\mathcal{E}(K) \subseteq K$ of extreme points*[31]*. Then, $\mathcal{E}(K) \in \mathbf{K}(\mathcal{X}^*)$ and $d_H^{(A)}(K, \mathcal{E}(K)) = 0$ for any $A \in \mathcal{X}$.*

[29] That is, a maximal connected subset.
[30] See Stone's representation theorem for Boolean algebras.
[31] Cf. the Krein-Milman theorem [45, Theorem 3.23].

Proof. Let $\mathcal{X}$ be a real Banach space. Recall that any $A \in \mathcal{X}$ defines a weak*-continuous linear functional $\hat{A} : \mathcal{X}^* \to \mathbb{R}$ by

$$\hat{A}(\sigma) \doteq \sigma(A) , \qquad \sigma \in \mathcal{X}^* .$$

See (21). Observe next that

$$d_H^{(A)}(K, \mathcal{E}(K)) = \max \left\{ \max_{x_1 \in \hat{A}(K)} \min_{x_2 \in \hat{A}(\mathcal{E}(K))} |x_1 - x_2| , \max_{x_2 \in \hat{A}(\mathcal{E}(K))} \min_{x_1 \in \hat{A}(K)} |x_1 - x_2| \right\} . \qquad (88)$$

The right hand side is nothing else than the Hausdorff distance (81) between the sets $\hat{A}(K)$ and $\hat{A}(\mathcal{E}(K))$, where the metric used in $\mathcal{Y} = \mathbb{R}$ is the absolute-value distance. Now, clearly,

$$\hat{A}\left(\mathcal{E}\left(K\right)\right) \subseteq \hat{A}\left(K\right) \subseteq \left[\min \hat{A}\left(K\right), \max \hat{A}\left(K\right)\right] . \qquad (89)$$

By the Bauer maximum principle [60, Lemma 10.31] together with the affinity and weak*-continuity of $\hat{A}$,

$$\min \hat{A}\left(K\right) = \min \hat{A}\left(\mathcal{E}\left(K\right)\right) \qquad \text{and} \qquad \max \hat{A}\left(K\right) = \max \hat{A}\left(\mathcal{E}\left(K\right)\right) .$$

In particular, we can rewrite (89) as

$$\hat{A}\left(\mathcal{E}(K)\right) \subseteq \hat{A}\left(K\right) \subseteq \left[\min \hat{A}\left(\mathcal{E}\left(K\right)\right), \max \hat{A}\left(\mathcal{E}\left(K\right)\right)\right] . \qquad (90)$$

Since $\mathcal{E}(K)$ is, by assumption, path-connected in the weak* topology, there is a weak*-continuous path $\gamma : [0,1] \to \mathcal{E}(K)$ from a minimizer to a maximizer of $\hat{A}$ in $\mathcal{E}(K)$. By weak*-continuity of $\hat{A}$, it follows that

$$\left[\min \hat{A}\left(\mathcal{E}(K)\right), \max \hat{A}\left(\mathcal{E}(K)\right)\right] = \hat{A} \circ \gamma\left([0,1]\right) \subseteq \hat{A}\left(\mathcal{E}(K)\right)$$

and we infer from (90) that

$$\hat{A}\left(\mathcal{E}(K)\right) = \hat{A}\left(K\right) = \left[\min \hat{A}\left(K\right), \max \hat{A}\left(K\right)\right] = \left[\min \hat{A}\left(\mathcal{E}(K)\right), \max \hat{A}\left(\mathcal{E}(K)\right)\right] .$$

Together with (88), this last equality obviously leads to the assertion. Note that $\mathcal{E}(K) \in \mathbf{K}(\mathcal{X}^*)$ since it is, by assumption, a weak*-closed subset of the weak*-compact set K (Lemma 6.5). ■

Corollary 6.7 (Non-Hausdorff hyperspaces)
Let $\mathcal{X}$ be a real Banach space of dimension greater than 1. Then, $\mathbf{F}(\mathcal{X}^*)$ and $\mathbf{K}(\mathcal{X}^*)$ are non-Hausdorff topological spaces.

Proof. This corollary is a direct consequence of Lemma 6.6 by observing that the dual of a real Banach space of dimension greater than 1 contains a two-dimensional closed disc. ■

As a consequence, the Hausdorff property of the hyperspace $\mathbf{F}(\mathcal{X}^*)$ does not hold true, in general. A restriction to the sub-hyperspace $\mathbf{K}(\mathcal{X}^*)$ is also not sufficient to get the separation property. This fact, described in Lemma 6.6, also appears for other well-established hypertopologies, which cannot distinguish a set from its closed convex hull. The so-called scalar topology for closed sets is a good example of this phenomenon, as explained in [58, Section 4.3]. Actually, similar to the scalar topology, $\mathbf{CK}(\mathcal{X}^*)$ is a Hausdorff hyperspace. To get a better intuition of this fact, the following proposition is instructive:

Proposition 6.8 (Separation of the weak*-closed convex hull)
Let $\mathcal{X}$ be a Banach space and $K_1, K_2 \in \mathbf{K}(\mathcal{X}^*)$. If $d_H^{(A)}(K_1, K_2) = 0$ for all $A \in \mathcal{X}$, then $\overline{\mathrm{co}K_1} = \overline{\mathrm{co}K_2}$, where $\overline{\mathrm{co}F}$ denotes the weak*-closure of the convex hull of any set $F \in \mathbf{F}(\mathcal{X}^*)$.

Proof. Pick any weak*-compact sets K_1, K_2 satisfying $d_H^{(A)}(K_1, K_2) = 0$ for all $A \in \mathcal{X}$. Let $\sigma_1 \in K_1$. By Definition 6.1, it follows that

$$\min_{\sigma_2 \in K_2} |(\sigma_1 - \sigma_2)(A)| = 0 , \qquad A \in \mathcal{X} . \tag{91}$$

The dual space $\mathcal{X}^*$ of the Banach space $\mathcal{X}$ is a locally convex (Hausdorff) space in the weak* topology and its dual is $\mathcal{X}$. Note also that $\overline{\mathrm{co}K_2}$ is convex and weak*-compact, because it is a norm-bounded weak*-closed subset of $\mathcal{X}^*$, see Lemma 6.5. Since $\{\sigma_1\}$ is a convex weak*-closed set, if $\sigma_1 \notin \overline{\mathrm{co}K_2}$ then we infer from the Hahn-Banach separation theorem [45, Theorem 3.4 (b)] the existence of $A_0 \in \mathcal{X}$ and $x_1, x_2 \in \mathbb{R}$ such that

$$\max_{\sigma_2 \in \overline{\mathrm{co}K_2}} \mathrm{Re}\{\sigma_2(A_0)\} < x_1 < x_2 < \mathrm{Re}\{\sigma_1(A_0)\} , \tag{92}$$

which contradicts (91) for $A = A_0$. As a consequence, $\sigma_1 \in \overline{\mathrm{co}K_2}$ and hence, $K_1 \subseteq \overline{\mathrm{co}K_2}$. This in turn yields $\overline{\mathrm{co}K_1} \subseteq \overline{\mathrm{co}K_2}$. By switching the role of the weak*-compact sets, we thus deduce the assertion. ■

Proposition 6.8 motivates the introduction of the *weak*-closed convex hull operator*:

Definition 6.9 (The weak*-closed convex hull operator)
The weak-closed convex hull operator is the map* $\overline{\mathrm{co}}$ *from* $\mathbf{F}(\mathcal{X}^*)$ *to itself defined by*

$$\overline{\mathrm{co}}(F) \doteq \overline{\mathrm{co}F} , \qquad F \in \mathbf{F}(\mathcal{X}^*) ,$$

where $\overline{\mathrm{co}F}$ *denotes the weak*-closure of the convex hull of* F *or, equivalently, the intersection of all weak*-closed convex sets containing* F.

It is a *closure* (or hull) operator [85, Definition 5.1] since it satisfies the following properties:

- For any $F \in \mathbf{F}(\mathcal{X}^*)$, $F \subseteq \overline{\mathrm{co}}(F)$ (extensive);
- For any $F \in \mathbf{F}(\mathcal{X}^*)$, $\overline{\mathrm{co}}(\overline{\mathrm{co}}(F)) = \overline{\mathrm{co}}(F)$ (idempotent);
- For any $F_1, F_2 \in \mathbf{F}(\mathcal{X}^*)$ such that $F_1 \subseteq F_2$, $\overline{\mathrm{co}}(F_1) \subseteq \overline{\mathrm{co}}(F_2)$ (isotone).

Such a closure operator has probably been used in the past. It is a composition of (i) an *algebraic* (or finitary) closure operator [85, Definition 5.4] defined by $F \mapsto \mathrm{co}F$ with (ii) a *topological* (or Kuratowski) closure operator [54, Chapter 1, p.43] defined by $F \mapsto \overline{F}$ on $\mathbf{F}(\mathcal{X}^*)$.

As is usual, weak*-closed subsets $F \in \mathbf{F}(\mathcal{X}^*)$ satisfying $F = \overline{\mathrm{co}}(F)$ are by definition $\overline{\mathrm{co}}$*-closed* sets. In the light of Proposition 6.8, it is natural to propose the set $\overline{\mathrm{co}}(\mathbf{K}(\mathcal{X}^*))$ as the Hausdorff hyperspace to consider. This set is nothing else than the set of all nonempty convex weak*-compact sets defined by (7) or (87):

$$\overline{\mathrm{co}}(\mathbf{K}(\mathcal{X}^*)) = \mathbf{CK}(\mathcal{X}^*) . \tag{93}$$

We thus deduce the following assertion:

Corollary 6.10 ($\mathbf{CK}(\mathcal{X}^*)$ as an Hausdorff hyperspace)
Let $\mathcal{X}$ *be a Banach space. Then,* $\mathbf{CK}(\mathcal{X}^*)$ *is a Hausdorff hyperspace.*

Proof. This is a direct consequence of Proposition 6.8. ■

Note additionally that the restriction of the weak*-closed convex hull operator to $\mathbf{K}(\mathcal{X}^*)$ is a weak*-Hausdorff continuous map from the hyperspace $\mathbf{K}(\mathcal{X}^*)$ to $\mathbf{CK}(\mathcal{X}^*)$:

Proposition 6.11 (Weak*-Hausdorff continuity of the weak*-closed convex hull operator)
Let $\mathcal{X}$ be a Banach space. Then, $\overline{\mathrm{co}}$ preserves the set (84) of all nonempty weak*-closed norm-unbounded subsets of $\mathcal{X}^*$, i.e.,

$$\overline{\mathrm{co}}\left(\mathbf{K}^c\left(\mathcal{X}^*\right)\right) \subseteq \mathbf{K}^c\left(\mathcal{X}^*\right) \doteq \mathbf{F}\left(\mathcal{X}^*\right) \backslash \mathbf{K}\left(\mathcal{X}^*\right) \ , \tag{94}$$

and $\overline{\mathrm{co}}$ restricted to $\mathbf{K}(\mathcal{X}^*)$ is a weak*-Hausdorff continuous map onto $\mathbf{CK}(\mathcal{X}^*)$.

Proof. Let $\mathcal{X}$ be a Banach space. Equation (94) and surjectivity of $\overline{\mathrm{co}}$ seen as a map from $\mathbf{K}(\mathcal{X}^*)$ to $\mathbf{CK}(\mathcal{X}^*)$ are both obvious, by Definition 6.9 and (93). Now, take any weak*-Hausdorff convergent net $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ with limit $K_\infty \in \mathbf{K}(\mathcal{X}^*)$. Note that

$$\max_{\sigma\in\overline{\mathrm{co}}(K_\infty)} \min_{\tilde{\sigma}\in\overline{\mathrm{co}}(K_j)} \left|\left(\sigma-\tilde{\sigma}\right)(A)\right| = \sup_{\sigma\in\mathrm{co}K_\infty} \min_{\tilde{\sigma}\in\overline{\mathrm{co}}(K_j)} \left|\left(\sigma-\tilde{\sigma}\right)(A)\right| \ , \qquad A\in\mathcal{X} \ , \tag{95}$$

because, for any $A\in\mathcal{X}$, $j\in J$, $\sigma_1,\sigma_2\in\overline{\mathrm{co}}\left(K_\infty\right)$ and $\tilde{\sigma}\in\overline{\mathrm{co}}\left(K_j\right)$,

$$\left|\left|\left(\sigma_1-\tilde{\sigma}\right)(A)\right| - \left|\left(\sigma_2-\tilde{\sigma}\right)(A)\right|\right| \leq \left|\left(\sigma_1-\sigma_2\right)(A)\right| \ ,$$

which yields

$$\left|\min_{\tilde{\sigma}\in K_j}\left|\left(\sigma_1-\tilde{\sigma}\right)(A)\right| - \min_{\tilde{\sigma}\in K_j}\left|\left(\sigma_2-\tilde{\sigma}\right)(A)\right|\right| \leq \left|\left(\sigma_1-\sigma_2\right)(A)\right|$$

for any $A\in\mathcal{X}$, $j\in J$ and $\sigma_1,\sigma_2\in\overline{\mathrm{co}}\left(K_\infty\right)$. Fix $n\in\mathbb{N}$, $\sigma_1,\ldots,\sigma_n\in K_\infty$ and parameters $\lambda_1,\ldots,\lambda_n\in[0,1]$ such that

$$\sum_{k=1}^{n}\lambda_k = 1 \ .$$

For any $A\in\mathcal{X}$ and $k\in\{1,\ldots,n\}$, we define $\tilde{\sigma}_{k,j}\in K_j$ such that

$$\min_{\tilde{\sigma}\in K_j}\left|\left(\sigma_k-\tilde{\sigma}\right)(A)\right| = \left|\left(\sigma_k-\tilde{\sigma}_{k,j}\right)(A)\right| \ .$$

Then, for all $j\in J$,

$$\min_{\tilde{\sigma}\in\overline{\mathrm{co}}(K_j)}\left|\left(\sum_{k=1}^{n}\lambda_k\sigma_k-\tilde{\sigma}\right)(A)\right| \leq \sum_{k=1}^{n}\lambda_k\left|\left(\sigma_k-\tilde{\sigma}_{k,j}\right)(A)\right| \leq \max_{\sigma\in K_\infty}\min_{\tilde{\sigma}\in K_j}\left|\left(\sigma-\tilde{\sigma}\right)(A)\right| \ .$$

By using (95), we then deduce that, for all $j\in J$,

$$\max_{\sigma\in\overline{\mathrm{co}}(K_\infty)} \min_{\tilde{\sigma}\in\overline{\mathrm{co}}(K_j)} \left|\left(\sigma-\tilde{\sigma}\right)(A)\right| \leq \max_{\sigma\in K_\infty}\min_{\tilde{\sigma}\in K_j}\left|\left(\sigma-\tilde{\sigma}\right)(A)\right| \ , \qquad A\in\mathcal{X} \ . \tag{96}$$

By switching the role of K_∞ and K_j for every $j\in J$, we also arrive at the inequality

$$\max_{\tilde{\sigma}\in\overline{\mathrm{co}}(K_j)} \min_{\sigma\in\overline{\mathrm{co}}(K_\infty)} \left|\left(\sigma-\tilde{\sigma}\right)(A)\right| \leq \max_{\tilde{\sigma}\in K_j}\min_{\sigma\in K_\infty}\left|\left(\sigma-\tilde{\sigma}\right)(A)\right| \ , \qquad A\in\mathcal{X} \ . \tag{97}$$

Since $(K_j)_{j\in J}$ converges in the weak*-Hausdorff hypertopology to K_∞, Inequalities (96)-(97) combined with Definition 6.1 yield the weak*-Hausdorff convergence of $(\overline{\mathrm{co}}\left(K_j\right))_{j\in J}$ to $\overline{\mathrm{co}}\left(K_\infty\right)$. By [54, Chapter 3, Theorem 1], $\overline{\mathrm{co}}$ restricted to $\mathbf{K}(\mathcal{X}^*)$ is a weak*-Hausdorff continuous map onto $\mathbf{CK}(\mathcal{X}^*)$. ∎

Corollary 6.12 ($\mathrm{CK}(\mathcal{X}^*)$ as a connected, weak*-Hausdorff-closed set)
Let $\mathcal{X}$ be a Banach space. Then, $\mathbf{CK}(\mathcal{X}^*)$ is a convex, path-connected, weak*-Hausdorff-closed subset of $\mathbf{K}(\mathcal{X}^*)$.

Proof. By Corollary 6.10, $\mathbf{CK}(\mathcal{X}^*)$ endowed with the weak*-Hausdorff hypertopology is a Hausdorff space. Hence, by [54, Chapter 2, Theorem 3], each convergent net in this space converges in the weak*-Hausdorff hypertopology to at most one point, which, by Proposition 6.11, must be a convex weak*-compact set. Additionally, by Lemma 6.4, Proposition 6.11 and the fact that the image under a continuous map of a path-connected space is path-connected, $\mathbf{CK}(\mathcal{X}^*)$ is also path-connected. Convexity of $\mathbf{CK}(\mathcal{X}^*)$ is also obvious. ■

As is usual, the weak*-closed convex hull operator $\overline{\mathrm{co}}$ yields a notion of compactness, defined as follows: A set $K \in \mathbf{F}(\mathcal{X}^*)$ is $\overline{\mathrm{co}}$-*compact* iff it is $\overline{\mathrm{co}}$-closed and each family of $\overline{\mathrm{co}}$-closed subsets of K which has the finite intersection property has a non-empty intersection. Compare this definition with [54, Chapter 5, Theorem 1]. The set $\mathbf{CK}(\mathcal{X}^*)$ of all nonempty convex weak*-compact sets defined by (7) or (87) is precisely the set of $\overline{\mathrm{co}}$-compact sets:

Proposition 6.13 ($\mathbf{CK}(\mathcal{X}^*)$ as the space of $\overline{\mathrm{co}}$-compact sets)
Let $\mathcal{X}$ be a Banach space. Then,

$$\mathbf{CK}\left(\mathcal{X}^*\right) = \left\{K \in \mathbf{F}\left(\mathcal{X}^*\right) : K \textit{ is } \overline{\mathrm{co}}\textit{-compact}\right\} .$$

Proof. By [54, Chapter 5, Theorem 1], we clearly have

$$\mathbf{CK}\left(\mathcal{X}^*\right) \subseteq \left\{K \in \mathbf{F}\left(\mathcal{X}^*\right) : K \text{ is } \overline{\mathrm{co}}\text{-compact}\right\} .$$

Conversely, take any $\overline{\mathrm{co}}$-compact element $K \in \mathbf{F}(\mathcal{X}^*)$. If K is not norm-bounded, then we deduce from the uniform boundedness principle [45, Theorems 2.4 and 2.5] the existence of $A \in \mathcal{X}$ such that $\hat{A}(K) \subseteq \mathbb{C}$ is not bounded, where we recall that $\hat{A} : \mathcal{X}^* \to \mathbb{C}$ is the weak*-continuous (complex) linear functional defined by (21). Without loss of generality, assume that $\mathrm{Re}\{\hat{A}(K)\}$ is not bounded from above. Define for every $n \in \mathbb{N}$ the set

$$K_n \doteq \left\{\sigma \in K : \mathrm{Re}\{\hat{A}(\sigma)\} \geq n\right\} .$$

Clearly, by convexity of K, K_n is a convex weak*-closed subset of K and the family $(K_n)_{n\in\mathbb{N}}$ has the finite intersection property, but, by construction,

$$\bigcap_{n\in\mathbb{N}} K_n = \emptyset .$$

(The intersection of preimages is the preimage of the intersection.) This contradicts the fact that K is $\overline{\mathrm{co}}$-compact. Therefore, K is norm-bounded and, being $\overline{\mathrm{co}}$-compact, it is also weak*-closed and convex. Consequently, $K \in \mathbf{CK}\left(\mathcal{X}^*\right)$ (see, e.g., Equation 8). ■

The last proposition goes beyond the specific topic of the present article, and the proof of the weak*-Hausdorff density of convex weak*-compact sets with dense extreme boundary. This is however discussed here because, like (93), it is an elegant abstract characterization of $\mathbf{CK}(\mathcal{X}^*)$, only given in terms of a closure operator, namely the weak*-closed convex hull operator. It demonstrates connections with other mathematical fields, in particular with mathematical logics where fascinating applications of closure operators have been developed, already by Tarski himself during the 1930's.

6.3 Weak*-Hausdorff Hyperconvergence

In this subsection, we study weak*-Hausdorff convergent nets. Even if only the hyperspace $\mathbf{CK}(\mathcal{X}^*)$ of all nonempty convex weak*-compact sets is Hausdorff (Corollary 6.10), we study the convergence within the hyperspace $\mathbf{K}(\mathcal{X}^*)$ of all nonempty, weak*-closed and norm-bounded subsets of $\mathcal{X}^*$. Recall that $\mathbf{K}(\mathcal{X}^*)$ is a (path-) connected weak*-Hausdorff-closed subset of $\mathbf{F}(\mathcal{X}^*)$, by Corollary 6.3 and Lemma 6.4.

It is instructive to relate weak*-Hausdorff limits of nets to lower and upper limits of sets à la Painlevé [61, § 29]: The *lower limit* of any net $(K_j)_{j\in J}$ of subsets of $\mathcal{X}^*$ is defined by

$$\mathrm{Li}\,(K_j)_{j\in J} \doteq \{\sigma \in \mathcal{X}^* : \sigma \text{ is a weak}^* \text{ limit of a net } (\sigma_j)_{j\in J} \text{ with } \sigma_j \in K_j \text{ for all } j \in J\} \ , \tag{98}$$

while its *upper limit* equals

$$\mathrm{Ls}\,(K_j)_{j\in J} \doteq \{\sigma \in \mathcal{X}^* : \sigma \text{ is a weak}^* \text{ accumulation point of } (\sigma_j)_{j\in J} \text{ with } \sigma_j \in K_j \text{ for all } j \in J\} \ . \tag{99}$$

Clearly, $\mathrm{Li}(K_j)_{j\in J} \subseteq \mathrm{Ls}(K_j)_{j\in J}$. If $\mathrm{Li}(K_j)_{j\in J} = \mathrm{Ls}(K_j)_{j\in J}$ then $(K_j)_{j\in J}$ is said to be convergent to this set. See [61, § 29, I, III, VI], which however defines Li and Ls within metric spaces. This refers in the literature to the *Kuratowski* or *Kuratowski-Painlevé*[32] convergence, see e.g. [86, Appendix B] and [58, Section 5.2]. By [45, Theorem 1.22], if $\mathcal{X}$ is an infinite-dimensional space, then its dual $\mathcal{X}^*$, endowed with the weak* or norm topology, is not locally compact. In this case, the Kuratowski-Painlevé convergence is not topological [86, Theorem B.3.2]. See also [58, Chapter 5], in particular [58, Theorem 5.2.6 and following discussions] which relates the Kuratowski-Painlevé convergence to the so-called *Fell* topology.

We start by proving the weak*-Hausdorff convergence of monotonically increasing nets which are bounded from above within $\mathbf{K}(\mathcal{X}^*)$:

Proposition 6.14 (Weak*-Hausdorff hyperconvergence of increasing nets)
Let $\mathcal{X}$ be a Banach space. Any increasing net $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ such that

$$K \doteq \overline{\bigcup_{j\in J} K_j} \in \mathbf{K}\left(\mathcal{X}^*\right) \subsetneq \mathbf{F}\left(\mathcal{X}^*\right) \tag{100}$$

(with respect to the weak closure) converges in the weak*-Hausdorff hypertopology to the Kuratowski-Painlevé limit*

$$K = \mathrm{Li}\,(K_j)_{j\in J} = \mathrm{Ls}\,(K_j)_{j\in J} \ .$$

Proof. Let $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ be any increasing net, i.e., $K_{j_1} \subseteq K_{j_2}$ whenever $j_1 \prec j_2$, satisfying (100). Because $K \in \mathbf{K}(\mathcal{X}^*)$, it is norm-bounded. By the convergence of increasing bounded nets of real numbers, it follows that, for any $A \in \mathcal{X}$,

$$\lim_{J} \max_{\tilde{\sigma}\in K_j} \min_{\sigma\in K} \left|\left(\tilde{\sigma}-\sigma\right)(A)\right| = \sup_{j\in J} \max_{\tilde{\sigma}\in K_j} \min_{\sigma\in K} \left|\left(\tilde{\sigma}-\sigma\right)(A)\right| \leq \max_{\tilde{\sigma}\in K} \min_{\sigma\in K} \left|\left(\tilde{\sigma}-\sigma\right)(A)\right| = 0 \ .$$

Therefore, by Definition 6.1, if

$$\limsup_{J} \max_{\sigma\in K} \min_{\tilde{\sigma}\in K_j} \left|\left(\tilde{\sigma}-\sigma\right)(A)\right| = 0 \ , \qquad A \in \mathcal{X} \ , \tag{101}$$

then the increasing net $(K_j)_{j\in J}$ converges in $\mathbf{K}(\mathcal{X}^*)$ to K, which clearly equals the Kuratowski-Painlevé limit of the net. To prove (101), assume by contradiction the existence of $\varepsilon \in \mathbb{R}^+$ such that

$$\limsup_{J} \max_{\sigma\in K} \min_{\tilde{\sigma}\in K_j} \left|\left(\tilde{\sigma}-\sigma\right)(A)\right| \geq \varepsilon \in \mathbb{R}^+ \tag{102}$$

for some fixed $A \in \mathcal{X}$. For any $j \in J$, take $\sigma_j \in K$ such that

$$\max_{\sigma\in K} \min_{\tilde{\sigma}\in K_j} \left|\left(\tilde{\sigma}-\sigma\right)(A)\right| = \min_{\tilde{\sigma}\in K_j} \left|\left(\tilde{\sigma}-\sigma_j\right)(A)\right| \ . \tag{103}$$

[32] The idea of upper and lower limits is due to Painlevé, as acknowledged by Kuratowski himself in [61, § 29, Footnote 1, p. 335]. We thus use the name Kuratowski-Painlevé convergence.

By weak*-compactness of K (Lemma 6.5), there is a subnet $(\sigma_{j_l})_{l\in L}$ converging in the weak* topology to $\sigma_\infty \in K$. Via Equation (103) and the triangle inequality, we then get that, for any $l \in L$,

$$\max_{\sigma\in K}\min_{\tilde{\sigma}\in K_{j_l}} |(\tilde{\sigma}-\sigma)(A)| \leq |(\sigma_{j_l}-\sigma_\infty)(A)| + \min_{\tilde{\sigma}\in K_{j_l}} |(\tilde{\sigma}-\sigma_\infty)(A)| \ .$$

By (100) and the fact that $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ is an increasing net, it follows that

$$\lim_L \max_{\sigma\in K}\min_{\tilde{\sigma}\in K_{j_l}} |(\tilde{\sigma}-\sigma)(A)| = 0 \ . \tag{104}$$

By the convergence of decreasing bounded nets of real numbers, note that

$$\limsup_J \max_{\sigma\in K}\min_{\tilde{\sigma}\in K_j} |(\tilde{\sigma}-\sigma)(A)| = \liminf_J \max_{\sigma\in K}\min_{\tilde{\sigma}\in K_j} |(\tilde{\sigma}-\sigma)(A)|$$

and hence, (104) contradicts (102). As a consequence, Equation (101) holds true. ■

Nonmonotone weak*-Hausdorff convergent nets in $\mathbf{K}(\mathcal{X}^*)$ are not trivial to study, in general. In the next proposition we give preliminary, but completely general, results on limits of convergent nets.

Proposition 6.15 (Weak*-Hausdorff hypertopology vs. upper and lower limits)
Let $\mathcal{X}$ be a Banach space and $K_\infty \in \mathbf{K}(\mathcal{X}^*)$ any weak-Hausdorff limit of a convergent net $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$. Then,*

$$\mathrm{Li}\,(K_j)_{j\in J} \subseteq \overline{\mathrm{co}}\,(K_\infty) \qquad \textit{and} \qquad K_\infty \subseteq \overline{\mathrm{co}}\,(\mathrm{Ls}(K_j)_{j\in J}) \ ,$$

where we recall that $\overline{\mathrm{co}}$ is the weak-closed convex hull operator (Definition 6.9).*

Proof. Let $\mathcal{X}$ be a Banach space and $(K_j)_{j\in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ any net converging to K_∞. Assume without loss of generality that $\mathrm{Li}\,(K_j)_{j\in J}$ is nonempty. Let $\sigma_\infty \in \mathrm{Li}\,(K_j)_{j\in J}$, which is, by definition, the weak* limit of a net $(\sigma_j)_{j\in J}$ with $\sigma_j \in K_j$ for all $j \in J$. Then, for any $A \in \mathcal{X}$,

$$\min_{\sigma\in K_\infty} |(\sigma-\sigma_\infty)(A)| \leq |(\sigma_j-\sigma_\infty)(A)| + \min_{\sigma\in K_\infty} \{|(\sigma-\sigma_j)(A)|\} \ .$$

Taking this last inequality in the limit with respect to J and using Definition 6.1, we deduce that

$$\min_{\sigma\in K_\infty} |(\sigma-\sigma_\infty)(A)| = 0 \ , \qquad A \in \mathcal{X} \ . \tag{105}$$

If $\sigma_\infty \notin \overline{\mathrm{co}}\,(K_\infty)$ then, as it is done to prove (92), we infer from the Hahn-Banach separation theorem [45, Theorem 3.4 (b)] the existence of $A_0 \in \mathcal{X}$ and $x_1, x_2 \in \mathbb{R}$ such that

$$\max_{\sigma\in\overline{\mathrm{co}}(K_\infty)} \mathrm{Re}\,\{\sigma(A_0)\} < x_1 < x_2 < \mathrm{Re}\,\{\sigma_\infty(A_0)\} \ ,$$

which contradicts (105) for $A = A_0$. As a consequence, $\sigma_\infty \in \overline{\mathrm{co}}\,(K_\infty)$ and, hence, $\mathrm{Li}\,(K_j)_{j\in J} \subseteq \overline{\mathrm{co}}\,(K_\infty)$.

Conversely, let $\sigma_\infty \in K_\infty$. Since K_∞ is by definition the limit of $(K_j)_{j\in J}$ (see Definition 6.1), we deduce that

$$\lim_J \min_{\sigma\in K_j} |(\sigma-\sigma_\infty)(A)| = 0 \ , \qquad A \in \mathcal{X} \ .$$

By combining this equality with Lemma 6.2 and the Banach-Alaoglu theorem [45, Theorem 3.15], for any $A \in \mathcal{X}$, there is $\sigma_A \in \mathrm{Ls}\,(K_j)_{j\in J}$ such that

$$\sigma_A(A) = \sigma_\infty(A) \ .$$

Consequently, one infers from the Hahn-Banach separation theorem [45, Theorem 3.4 (b)] that σ_∞ belongs to the weak*-closed convex hull of the upper limit $\mathrm{Ls}\,(K_j)_{j\in J}$. ■

Applied to nonempty convex weak*-compact subsets of the dual space $\mathcal{X}^*$, Proposition 6.15 reads as follows:

Corollary 6.16 (Weak*-Hausdorff hypertopology and convexity vs. upper and lower limits)
Let $\mathcal{X}$ be a Banach space and $K_\infty \in \mathbf{CK}(\mathcal{X}^*)$ any weak*-Hausdorff limit of a convergent net $(K_j)_{j\in J} \subseteq \mathbf{CK}(\mathcal{X}^*)$. Then,

$$\overline{\mathrm{Li}(K_j)_{j\in J}} = \overline{\mathrm{co}}\left(\mathrm{Li}(K_j)_{j\in J}\right) \subseteq K_\infty \subseteq \overline{\mathrm{co}}\left(\mathrm{Ls}(K_j)_{j\in J}\right) \ .$$

Proof. The assertion is an obvious application of Proposition 6.15 to the subset $\mathbf{CK}(\mathcal{X}^*) \subseteq \mathbf{K}(\mathcal{X}^*)$ together with the idempotency of the weak*-closed convex hull operator $\overline{\mathrm{co}}$. Note that $\mathrm{Li}(K_j)_{j\in J}$ is a convex set. ■

6.4 Metrizable Hyperspaces

We are interested in investigating *metrizable* sub-hyperspaces of $\mathbf{F}(\mathcal{X}^*)$. Metrizable topological spaces are Hausdorff, so, in the light of Corollaries 6.7 and 6.10, we restrict our analysis on the Hausdorff hyperspace $\mathbf{CK}(\mathcal{X}^*)$ of all nonempty convex weak*-compact subsets of $\mathcal{X}^*$, already defined by Equation (7) or (87).

For a separable Banach space $\mathcal{X}$, we show how the well-known metrizability of the weak* topology on balls of $\mathcal{X}^*$ leads to the metrizability of the weak*-Hausdorff hypertopology on uniformly norm-bounded subsets of $\mathbf{CK}(\mathcal{X}^*)$: Let

$$\mathbf{CK}_D\left(\mathcal{X}^*\right) \doteq \left\{K \in \mathbf{CK}\left(\mathcal{X}^*\right) : K \subseteq \mathbf{B}\left(0, D\right)\right\} \tag{106}$$

where

$$\mathbf{B}\left(0, D\right) \doteq \left\{\sigma \in \mathcal{X}^* : \|\sigma\|_{\mathcal{X}^*} \leq D\right\} \subseteq \mathcal{X}^* \tag{107}$$

is the norm-closed ball of radius $D \in \mathbb{R}^+$ in $\mathcal{X}^*$. If $\mathcal{X}$ is separable then the weak* topology is metrizable on any ball $\mathbf{B}(0, D)$, $D \in \mathbb{R}^+$, by the Banach-Alaoglu theorem [45, Theorem 3.15] and [45, Theorem 3.16]. Take any countable dense set $(A_n)_{n\in\mathbb{N}}$ of the unit ball of $\mathcal{X}$ and define the metric

$$d\left(\sigma_1, \sigma_2\right) \doteq \sum_{n\in\mathbb{N}} 2^{-n} \left|\left(\sigma_1 - \sigma_2\right)\left(A_n\right)\right| \ , \qquad \sigma_1, \sigma_2 \in \mathcal{X}^* \ . \tag{108}$$

This metric is well-defined and induces the weak* topology on $\mathbf{B}(0, D)$. Denote by d_H the Hausdorff distance between two elements $K_1, K_2 \in \mathbf{CK}_D(\mathcal{X}^*)$, associated with the metric d, as defined by (81), that is[33],

$$d_H\left(K_1, K_2\right) \doteq \max\left\{\max_{\sigma_1\in K_1}\min_{\sigma_2\in K_2} d\left(\sigma_1, \sigma_2\right), \max_{\sigma_2\in K_2}\min_{\sigma_1\in K_1} d\left(\sigma_1, \sigma_2\right)\right\} \ . \tag{109}$$

This Hausdorff distance induces the weak*-Hausdorff hypertopology on $\mathbf{CK}_D(\mathcal{X}^*)$:

Theorem 6.17 (Complete metrizability of the weak*-Hausdorff hypertopology)
Let $\mathcal{X}$ be a separable Banach space and $D \in \mathbb{R}^+$. The family

$$\left\{\left\{K_2 \in \mathbf{CK}_D\left(\mathcal{X}^*\right) : d_H\left(K_1, K_2\right) < r\right\} : r \in \mathbb{R}^+,\ K_1 \in \mathbf{CK}_D\left(\mathcal{X}^*\right)\right\}$$

is a basis of the weak*-Hausdorff hypertopology of $\mathbf{CK}_D(\mathcal{X}^*)$. Additionally, $\mathbf{CK}_D(\mathcal{X}^*)$ is weak*-Hausdorff-compact and completely metrizable.

[33] Minima in (109) directly come from the compactness of sets and the continuity of d. The following maxima in (109) result from the compactness of sets and the fact that the minimum over a continuous map defines an upper semicontinuous function.

Proof. Recall that a topology is finer than a second one iff any convergent net of the first topology converges also in the second topology to the same limit. See, e.g., [54, Chapter 2, Theorems 4, 9]. We first show that the topology induced by the Hausdorff metric d_H is finer than the weak*-Hausdorff hypertopology of $\mathbf{CK}_D(\mathcal{X}^*)$ at fixed radius $D \in \mathbb{R}^+$: Take any net $(K_j)_{j\in J}$ converging in $\mathbf{CK}_D(\mathcal{X}^*)$ to K in the topology induced by the Hausdorff metric (109). Let $A \in \mathcal{X}$ and assume without loss of generality that $\|A\|_{\mathcal{X}} \leq 1$. By density of $(A_n)_{n\in\mathbb{N}}$ in the unit ball of $\mathcal{X}$, for any $\varepsilon \in \mathbb{R}^+$, there is $n \in \mathbb{N}$ such that, for all $j \in J$,

$$d_H^{(A)}(K, K_j) \leq \varepsilon + d_H^{(A_n)}(K, K_j) \leq \varepsilon + 2^n d_H(K, K_j) \ .$$

Thus, the net $(K_j)_{j\in J}$ converges to K also in the weak*-Hausdorff hypertopology.

Endowed with the Hausdorff metric topology, the space of closed subsets of a compact metric space is compact, by [58, Theorem 3.2.4]. In particular, by weak* compactness of norm-closed balls, $\mathbf{CK}_D(\mathcal{X}^*)$ endowed with the Hausdorff metric d_H is a compact hyperspace. By Corollary 6.12, $\mathbf{CK}_D(\mathcal{X}^*)$ is closed with respect to the weak*-Hausdorff hypertopology, and thus closed with respect to the topology induced by d_H, because this topology is coarser than the weak*-Hausdorff hypertopology, as proven above. Hence, $\mathbf{CK}_D(\mathcal{X}^*)$ is also compact with respect to the topology induced by d_H. Since the weak*-Hausdorff hypertopology is a Hausdorff topology (Corollary 6.10), as it is well-known [45, Section 3.8 (a)], both topologies must coincide: Take any subset $\mathcal{K} \subseteq \mathbf{CK}_D(\mathcal{X}^*)$ which is closed with respect to the Hausdorff metric d_H. By compactness of $(\mathbf{CK}_D(\mathcal{X}^*), d_H)$, $\mathcal{K}$ is compact with respect to the Hausdorff metric d_H (see, e.g., [54, Chapter 5, p. 140]) and, hence, also with respect to the weak*-Hausdorff hypertopology. Because any compact set in a Hausdorff space is closed [54, Chapter 5, Theorem 7], by Corollary 6.10, $\mathcal{K}$ is closed with respect to the weak*-Hausdorff hypertopology. ■

Note that Theorem 6.17 is similar to the assertion [58, End of p. 91]. It leads to a strong improvement of Proposition 6.14 and Corollary 6.16:

Corollary 6.18 (Weak*-Hausdorff hypertopology and Kuratowski-Painlevé convergence)
Let $\mathcal{X}$ be a separable Banach space. Then any weak-Hausdorff convergent net $(K_j)_{j\in J} \subseteq \mathbf{CK}(\mathcal{X}^*)$ converges to the Kuratowski-Painlevé limit*

$$K_\infty = \mathrm{Li}(K_j)_{j\in J} = \mathrm{Ls}(K_j)_{j\in J} \in \mathbf{CK}\left(\mathcal{X}^*\right) \ .$$

Proof. Recall that $\mathbf{CK}(\mathcal{X}^*) \subseteq \mathbf{K}(\mathcal{X}^*)$, see (87). By Lemma 6.2, the union of any weak*-Hausdorff convergent net in $\mathbf{CK}(\mathcal{X}^*)$ is norm-bounded and, as a consequence, we can restrict, without loss of generality, the study of weak*-Hausdorff hyperconvergent nets to the sub-hyperspace $\mathbf{CK}_D(\mathcal{X}^*)$ for some $D \in \mathbb{R}^+$. By Theorem 6.17 the weak*-Hausdorff hypertopology is induced by the Hausdorff distance d_H defined by (109). The assertion thus follows from [61, § 29, Section IX, Theorem 2]. ■

6.5 Generic Hypersets in Infinite Dimensions

By Corollary 6.10, recall that $\mathbf{CK}(\mathcal{X}^*)$ is a weak*-Hausdorff-closed subset of $\mathbf{K}\left(\mathcal{X}^*\right)$. Let

$$\mathcal{D} \doteq \left\{K \in \mathbf{CK}\left(\mathcal{X}^*\right) : K = \overline{\mathcal{E}\left(K\right)}\right\} \subseteq \mathbf{CK}\left(\mathcal{X}^*\right) \tag{110}$$

be the subset of all $K \in \mathbf{CK}(\mathcal{X}^*)$ with weak*-dense set $\mathcal{E}\left(K\right)$ of extreme points (cf. the Krein-Milman theorem [45, Theorem 3.23]).

Recall that the so-called *exposed* points are particular examples of extreme ones: a point $\sigma_0 \in K$ in a convex subset $K \subseteq \mathcal{X}^*$ is *exposed* if there is $A \in \mathcal{X}$ such that the real part of the weak*-continuous functional $\hat{A} : \sigma \mapsto \sigma(A)$ from $\mathcal{X}^*$ to $\mathbb{C}$ (cf. (21)) takes its *unique* maximum on K at $\sigma_0 \in K$. Considering exposed points instead of general extreme points is technically convenient because of the weak*-density of the set of exposed points in the set of extreme points [87, Theorem 6.2] is an important ingredient to show that $\mathcal{D}$ is a G_δ subset of $\mathbf{CK}(\mathcal{X}^*)$:

Proposition 6.19 (**$\mathcal{D}$ as a G_δ set**)
Let $\mathcal{X}$ be a separable Banach space. Then $\mathcal{D}$ is a G_δ subset of $\mathbf{CK}(\mathcal{X}^*)$.

Proof. Let $\mathcal{X}$ be a separable Banach space. For any $D \in \mathbb{R}^+$, we can use the metric d defined by (108) and generating the weak* topology on the norm-closed ball $\mathbf{B}(0, D)$ of radius D, defined by (107). For any $D \in \mathbb{R}^+$, we denote by

$$B(\omega, r) \doteq \{\sigma \in \mathbf{B}(0, D) : d(\omega, \sigma) < r\} \tag{111}$$

the weak*-open ball of radius $r \in \mathbb{R}^+$ centered at $\omega \in \mathbf{B}(0, D)$. Then, for any $D \in \mathbb{R}^+$ and $m \in \mathbb{N}$, let $\mathcal{F}_{D,m}$ be the set of all nonempty convex weak*-compact subsets $K \subseteq \mathbf{B}(0, D)$ such that $B(\omega, 1/m) \cap \mathcal{E}(K) = \emptyset$ for some $\omega \in K$, i.e.,

$$\mathcal{F}_{D,m} \doteq \{K \in \mathbf{CK}_D(\mathcal{X}^*) : \exists \omega \in K,\ B(\omega, 1/m) \cap \mathcal{E}(K) = \emptyset\} \subseteq \mathbf{CK}_D(\mathcal{X}^*)\ . \tag{112}$$

Recall again that $\mathcal{E}(K)$ is the nonempty set of extreme points of K (cf. the Krein-Milman theorem [45, Theorem 3.23]). Now, by Equation (87), observe that the complement of $\mathcal{D}$ (110) in $\mathbf{CK}(\mathcal{X}^*)$ equals

$$\mathbf{CK}(\mathcal{X}^*) \backslash \mathcal{D} = \bigcup_{D,m \in \mathbb{N}} \mathcal{F}_{D,m}\ . \tag{113}$$

Therefore, $\mathcal{D}$ is a G_δ subset of $\mathbf{CK}(\mathcal{X}^*)$ if $\mathcal{F}_{D,m}$ is a weak*-Hausdorff-closed set for any $D, m \in \mathbb{N}$.

By Theorem 6.17, the weak*-Hausdorff hypertopology of $\mathbf{CK}_D(\mathcal{X}^*)$ is metrizable and $\mathbf{CK}_D(\mathcal{X}^*)$, being weak*-Hausdorff-compact, is a weak*-Hausdorff-closed subset of the Hausdorff hyperspace $\mathbf{CK}(\mathcal{X}^*)$ (see Corollary 6.10 and [54, Chapter 5, Theorem 7]). So, fix $D, m \in \mathbb{N}$ and take any sequence $(K_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_{D,m}$ converging with respect to the weak*-Hausdorff hypertopology to $K_\infty \in \mathbf{CK}_D(\mathcal{X}^*)$. For any $n \in \mathbb{N}$, there is $\omega_n \in K_n$ such that $B(\omega_n, 1/m) \cap \mathcal{E}(K_n) = \emptyset$. By metrizability and weak* compactness of the ball $\mathbf{B}(0, D)$ and Corollary 6.18, there is a subsequence $(\omega_{n_k})_{k \in \mathbb{N}}$ converging to some $\omega_\infty \in K_\infty$. Assume that, for some $\varepsilon \in (0, 1/m)$, there is $\sigma_\infty \in \mathcal{E}(K_\infty)$ such that

$$d(\omega_\infty, \sigma_\infty) \leq \frac{1}{m} - \varepsilon\ .$$

By the Mazur theorem (see, e.g., [87, Theorem 1.20]), the Straszewicz theorem extended to all weak Asplund spaces [87, Theorem 6.2] and the Milman theorem [60, Theorem 10.13], the set of exposed points of K_∞ is weak*-dense in $\mathcal{E}(K_\infty)$. As a consequence, we can assume without loss of generality that σ_∞ is an exposed point. In particular, there is $A \in \mathcal{X}$ such that

$$\max_{\sigma \in K_\infty} \mathrm{Re}\{\hat{A}(\sigma)\} = \hat{A}(\sigma_\infty)\ , \tag{114}$$

with σ_∞ being the *unique* maximizer in K_∞. Recall that $\hat{A}$ is the map $\sigma \mapsto \sigma(A)$ from $\mathcal{X}^*$ to $\mathbb{C}$ (cf. (21)). Consider now the sets

$$\mathcal{M}_n \doteq \left\{\tilde{\sigma} \in K_n : \max_{\sigma \in K_n} \mathrm{Re}\{\hat{A}(\sigma)\} = \hat{A}(\tilde{\sigma})\right\}\ , \qquad n \in \mathbb{N}\ .$$

By affinity and weak*-continuity of the function $\hat{A}$, together with the weak*-compactness of K_n, the set $\mathcal{M}_n$ is a convex weak*-compact subset of K_n for any $n \in \mathbb{N}$. In fact, $\mathcal{M}_n$ is a (weak*-closed) face[34] of K_n and thus, any extreme point of $\mathcal{M}_n$ belongs to $\mathcal{E}(K_n)$. So, pick any extreme point $\sigma_n \in \mathcal{E}(K_n)$ of $\mathcal{M}_n$ for each $n \in \mathbb{N}$. Since

$$\begin{aligned}
\max_{\sigma \in K_n} \mathrm{Re}\{\hat{A}(\sigma)\} - \max_{\tilde{\sigma} \in K_\infty} \mathrm{Re}\{\hat{A}(\tilde{\sigma})\} &= \max_{\sigma \in K_n} \min_{\tilde{\sigma} \in K_\infty} \mathrm{Re}\{\hat{A}(\sigma - \tilde{\sigma})\} \leq \max_{\sigma \in K_n} \min_{\tilde{\sigma} \in K_\infty} |(\sigma - \tilde{\sigma})(A)|\ , \\
\max_{\tilde{\sigma} \in K_\infty} \mathrm{Re}\{\hat{A}(\tilde{\sigma})\} - \max_{\sigma \in K_n} \mathrm{Re}\{\hat{A}(\sigma)\} &= \max_{\tilde{\sigma} \in K_\infty} \min_{\sigma \in K_n} \mathrm{Re}\{\hat{A}(\tilde{\sigma} - \sigma)\} \leq \max_{\tilde{\sigma} \in K_\infty} \min_{\sigma \in K_n} |(\sigma - \tilde{\sigma})(A)|\ ,
\end{aligned}$$

[34] It means that, if $\sigma \in \mathcal{M}_n$ is a finite convex combination of elements $\sigma_j \in K_n$ then all $\sigma_j \in \mathcal{M}_n$.

we deduce from Definition 2.3 and the weak*-Hausdorff convergence of $(K_n)_{n\in\mathbb{N}}$ to K_∞ that

$$\lim_{n\to\infty} \mathrm{Re}\{\hat{A}(\sigma_n)\} = \lim_{n\to\infty} \max_{\sigma\in K_n} \mathrm{Re}\{\hat{A}(\sigma)\} = \max_{\sigma\in K_\infty} \mathrm{Re}\{\hat{A}(\sigma)\} = \hat{A}\left(\sigma_\infty\right) .$$

Therefore, keeping in mind the convergence of the subsequence $(\omega_{n_k})_{k\in\mathbb{N}}$ towards $\omega_\infty \in K_\infty$, there is a subsequence $(\sigma_{n_{k(l)}})_{l\in\mathbb{N}}$ of $(\sigma_{n_k})_{k\in\mathbb{N}}$ (itself being a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$) converging to σ_∞, as it is the *unique* maximizer of (114) and $\hat{A}$ is weak*-continuous. Since, for any $l \in \mathbb{N}$,

$$\begin{aligned} d(\sigma_{n_{k(l)}}, \omega_{n_{k(l)}}) &\leq d(\sigma_\infty, \omega_\infty) + d(\omega_\infty, \omega_{n_{k(l)}}) + d(\sigma_{n_{k(l)}}, \sigma_\infty) \\ &\leq \frac{1}{m} - \varepsilon + d(\omega_\infty, \omega_{n_{k(l)}}) + d(\sigma_{n_{k(l)}}, \sigma_\infty) \end{aligned}$$

with $\varepsilon \in (0, 1/m)$ and $\sigma_n \in \mathcal{E}(K_n)$ for $n \in \mathbb{N}$, we thus arrive at a contradiction. Therefore, $K_\infty \in \mathcal{F}_{D,m}$. This means that $\mathcal{F}_{D,m}$ is a weak*-Hausdorff-closed set for any $D, m \in \mathbb{N}$ and hence, the countable union (113) is a F_σ set with complement being $\mathcal{D}$. The assertion follows, as the complement of an F_σ set is a G_δ set. ■

To show that $\mathcal{D}$ is weak*-Hausdorff dense in the hyperspace $\mathbf{CK}(\mathcal{X}^*)$, like in the proof of [42, Theorem 4.3] and in contrast with [41], we design elements of $\mathcal{D}$ that approximate $K \in \mathbf{CK}(\mathcal{X}^*)$ by using a procedure that is very similar to the construction of the Poulsen simplex [62]. Note however that Poulsen used the existence of orthonormal bases in infinite-dimensional Hilbert spaces[35]. Here, the Hahn-Banach separation theorem [45, Theorem 3.4 (b)] replaces the orthogonality property coming from the Hilbert space structure. In all previous results [41, 42] on the density of convex compact sets with dense extreme boundary, the norm topology is used, while the primordial topology is here the weak* topology. In this context, the metrizability of weak* and weak*-Hausdorff topologies on norm-closed balls is pivotal. See Theorem 6.17. We give now the precise assertion along with its proof:

Theorem 6.20 (Weak*-Hausdorf density of $\mathcal{D}$)
Let $\mathcal{X}$ be an infinite-dimensional separable Banach space. Then, $\mathcal{D}$ is a weak*-Hausdorff dense subset of $\mathbf{CK}(\mathcal{X}^*)$.

Proof. Let $\mathcal{X}$ be an infinite-dimensional separable Banach space and fix once and for all a convex weak*-compact subset $K \in \mathbf{CK}(\mathcal{X}^*)$. The construction of convex weak*-compact sets in $\mathcal{D}$ approximating K is done in several steps:

Step 0: By Lemma 6.5, K belongs to some norm-closed ball $\mathbf{B}(0, D)$ of radius $D \in \mathbb{R}^+$, in other words, $K \in \mathbf{CK}_D(\mathcal{X}^*)$, see (106)-(107). Therefore, we can use the metric d defined by (108) and generating the weak* topology on $\mathbf{B}(0, D)$. Then, for any fixed $\varepsilon \in \mathbb{R}^+$, there is a finite set $\{\omega_j\}_{j=1}^{n_\varepsilon} \subseteq K$, $n_\varepsilon \in \mathbb{N}$, such that

$$K \subseteq \bigcup_{j=1}^{n_\varepsilon} B\left(\omega_j, \varepsilon\right) , \tag{115}$$

where $B\left(\omega, r\right) \subseteq \mathbf{B}(0, D)$ denotes the weak*-open ball (111) of radius $r \in \mathbb{R}^+$ centered at $\omega \in \mathcal{X}^*$. We then define the convex weak*-compact set

$$K_0 \doteq \mathrm{co}\left\{\omega_1, \ldots, \omega_{n_\varepsilon}\right\} \subseteq \mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon}\} . \tag{116}$$

By (109) and (115), note that

$$d_H(K, K_0) \leq \varepsilon . \tag{117}$$

[35] In [62], Poulsen uses the Hilbert space $\ell^2(\mathbb{N})$ to construct his example of a convex compact set (in fact a simplex) with dense extreme boundary.

Step 1: Observe that the ball $\mathbf{B}(0, D)$ is weak*-separable, by its weak* compactness (the Banach-Alaoglu theorem [45, Theorem 3.15]) and metrizability (cf. separability of $\mathcal{X}$ and [45, Theorem 3.16]). Take any weak*-dense countable set $\{\varrho_{0,k}\}_{k\in\mathbb{N}}$ of K_0. By infinite dimensionality of $\mathcal{X}^*$, there is $\sigma_1 \in \mathcal{X}^*\backslash\mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon}\}$ with

$$\|\sigma_1\|_{\mathcal{X}^*} = D . \tag{118}$$

As in the proof of Proposition 6.8, recall that $\mathcal{X}^*$, endowed with the weak* topology, is a locally convex (Hausdorff) space with $\mathcal{X}$ as its dual. Since $\{\sigma_1\}$ is a convex weak*-compact set and $\mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon}\}$ is convex and weak*-closed [45, Theorem 1.42], we infer from the Hahn-Banach separation theorem [45, Theorem 3.4 (b)] the existence of $A_1 \in \mathcal{X}$ such that

$$\sup\left\{\mathrm{Re}\left\{\sigma\left(A_1\right)\right\} : \sigma \in \mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon}\}\right\} < \mathrm{Re}\left\{\sigma_1\left(A_1\right)\right\} .$$

Since $\mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon}\}$ is a linear space, observe that

$$\mathrm{Re}\left\{\sigma\left(A_1\right)\right\} = 0 , \qquad \sigma \in \mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon}\} . \tag{119}$$

Thus, by rescaling $A_1 \in \mathcal{X}$, we can assume without loss of generality that

$$\mathrm{Re}\left\{\sigma_1\left(A_1\right)\right\} = 1 . \tag{120}$$

Let

$$\omega_{n_\varepsilon+1} \doteq \left(1-\lambda_1\right)\varpi_1 + \lambda_1\sigma_1 , \qquad \text{with} \qquad \lambda_1 \doteq \min\left\{1, 2^{-2}D^{-1}\varepsilon\right\} ,\ \varpi_1 \doteq \varrho_{0,1} \in K_0 . \tag{121}$$

In contrast with the proof of [42, Theorem 4.3], we use a convex combination to automatically ensure that $\|\omega_{n_\varepsilon+1}\|_{\mathcal{X}^*} \leq D$, by convexity of the (norm-closed) ball $\mathbf{B}(0, D)$. The inequality $\lambda_1 \leq 2^{-2}D^{-1}\varepsilon$ yields

$$d\left(\omega_{n_\varepsilon+1}, \varpi_1\right) \leq \|\omega_{n_\varepsilon+1} - \varpi_1\|_{\mathcal{X}^*} \leq 2^{-1}\varepsilon . \tag{122}$$

Define the new convex weak*-compact set

$$K_1 \doteq \mathrm{co}\left\{\omega_1, \ldots, \omega_{n_\varepsilon+1}\right\} \subseteq \mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon+1}\} .$$

Observe that $\omega_{n_\varepsilon+1}$ is an exposed point of K_1, by (119) and (120). By (109), (116) and (122), note that $d_H(K_0, K_1) \leq 2^{-1}\varepsilon$, which, by the triangle inequality and (117), yields

$$d_H(K, K_1) \leq \left(1 + 2^{-1}\right)\varepsilon \tag{123}$$

for an arbitrary (but previously fixed) $\varepsilon \in \mathbb{R}^+$.

Step 2: Take any weak* dense countable set $\{\varrho_{1,k}\}_{k\in\mathbb{N}}$ of K_1. By infinite dimensionality of $\mathcal{X}^*$, there is $\sigma_2 \in \mathcal{X}^*\backslash\mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon+1}\}$ with

$$\|\sigma_2\|_{\mathcal{X}^*} = \min\left\{D, 2^{-1}\|A_1\|_{\mathcal{X}}^{-1}\lambda_1\right\} . \tag{124}$$

As before, we deduce from the Hahn-Banach separation theorem [45, Theorem 3.4 (b)] the existence of $A_2 \in \mathcal{X}$ such that

$$\mathrm{Re}\left\{\sigma_2\left(A_2\right)\right\} = 1 \qquad \text{and} \qquad \mathrm{Re}\left\{\sigma\left(A_2\right)\right\} = 0 , \qquad \sigma \in \mathrm{span}\{\omega_1, \ldots, \omega_{n_\varepsilon+1}\} . \tag{125}$$

Let

$$\omega_{n_\varepsilon+2} \doteq \left(1-\lambda_2\right)\varpi_2 + \lambda_2\sigma_2 , \qquad \text{with} \qquad \lambda_2 \doteq \min\left\{1, 2^{-3}D^{-1}\varepsilon\right\} ,\ \varpi_2 \doteq \varrho_{1,1} \in K_1 . \tag{126}$$

In this case, similar to Inequality (122),

$$d\left(\omega_{n_\varepsilon+2},\varpi_2\right)\leq\left\|\omega_{n_\varepsilon+2}-\varpi_2\right\|_{\mathcal{X}^*}\leq 2^{-2}\varepsilon\ . \tag{127}$$

Define the new convex weak*-compact set

$$K_2\doteq\mathrm{co}\left\{\omega_1,\ldots,\omega_{n_\varepsilon+2}\right\}\subseteq\mathrm{span}\{\omega_1,\ldots,\omega_{n_\varepsilon+2}\}\ .$$

By (125), $\omega_{n_\varepsilon+2}$ is an exposed point of K_2, but it is not obvious that the exposed point $\omega_{n_\varepsilon+1}$ of K_1 is still an exposed point of K_2, with respect to $A_1\in\mathcal{X}$. This property is a consequence of

$$\mathrm{Re}\left\{\omega_{n_\varepsilon+2}\left(A_1\right)\right\}=\left(1-\lambda_2\right)\mathrm{Re}\left\{\varpi_2\left(A_1\right)\right\}+\lambda_2\mathrm{Re}\left\{\sigma_2\left(A_1\right)\right\}<\mathrm{Re}\left\{\omega_{n_\varepsilon+1}\left(A_1\right)\right\}=\lambda_1\ ,$$

(see (119), (121) and (126)), which holds true because

$$\mathrm{Re}\left\{\sigma_2\left(A_1\right)\right\}\leq 2^{-1}\lambda_1<\lambda_1\ ,$$

by Equation (124). By (109), (123) and (127) together with the triangle inequality,

$$d_H(K,K_2)\leq\left(1+2^{-1}+2^{-2}\right)\varepsilon$$

for an arbitrary (but previously fixed) $\varepsilon\in\mathbb{R}^+$.

Step $n\to\infty$: We now iterate the above procedure, ensuring, at each step $n\geq 3$, that the addition of the element

$$\omega_{n_\varepsilon+n}\doteq\left(1-\lambda_n\right)\varpi_n+\lambda_n\sigma_n\ ,\qquad\text{with}\qquad\lambda_n\doteq\min\left\{1,2^{-(n+1)}D^{-1}\varepsilon\right\}\ , \tag{128}$$

in order to define the convex weak*-compact set

$$K_n\doteq\mathrm{co}\left\{\omega_1,\ldots,\omega_{n_\varepsilon+n}\right\}\subseteq\mathrm{span}\{\omega_1,\ldots,\omega_{n_\varepsilon+n}\}\ , \tag{129}$$

does not destroy the property of the elements $\omega_{n_\varepsilon+1},\ldots,\omega_{n_\varepsilon+n-1}$ being exposed. To this end, for any $n\geq 2$, we choose $\sigma_n\in\mathcal{X}^*\backslash\mathrm{span}\{\omega_1,\ldots,\omega_{n_\varepsilon+n-1}\}$ such that

$$\left\|\sigma_n\right\|_{\mathcal{X}^*}=\min\left\{D,2^{-1}\left\|A_1\right\|_{\mathcal{X}}^{-1}\lambda_1,\ldots,2^{-1}\left\|A_{n-1}\right\|_{\mathcal{X}}^{-1}\lambda_{n-1}\right\}\ . \tag{130}$$

Compare with (118) and (124). Here, for any $j\in\{1,\ldots,n-1\}$, $A_j\in\mathcal{X}$ satisfies

$$\mathrm{Re}\left\{\sigma_j\left(A_j\right)\right\}=1\qquad\text{and}\qquad\mathrm{Re}\left\{\sigma\left(A_j\right)\right\}=0\ ,\qquad\sigma\in\mathrm{span}\{\omega_1,\ldots,\omega_{n_\varepsilon+j-1}\}\ . \tag{131}$$

Compare with (119)-(120) and (125). We also have to conveniently choose $\varpi_n\in K_{n-1}$ in order to get the asserted weak* density. Like in the proof of [42, Theorem 4.3] the sequence $(\varpi_n)_{n\in\mathbb{N}}$ is chosen such that

$$\{\varpi_n\}_{n\in\mathbb{N}}=\left\{\varrho_{n,k}\right\}_{n\in\mathbb{N}_0,k\in\mathbb{N}}$$

and all the functionals $\varrho_{n,k}$ appear infinitely many times in the sequence $(\varpi_n)_{n\in\mathbb{N}}$. In this case, we obtain a weak*-dense set $\{\omega_n\}_{n\in\mathbb{N}}$ in the convex weak*-compact set

$$K_\infty\doteq\overline{\mathrm{co}\left\{\{\omega_n\}_{n\in\mathbb{N}}\right\}}\in\mathbf{CK}_D\left(\mathcal{X}^*\right)\ , \tag{132}$$

which, by construction, satisfies

$$d_H(K,K_\infty)\leq\sum_{n=0}^{\infty}2^{-n}\varepsilon=2\varepsilon$$

for an arbitrary (but previously fixed) $\varepsilon \in \mathbb{R}^+$.

Step $n = \infty$: It remains to verify that $\omega_{n_\varepsilon+j}$, $j \in \mathbb{N}$, are exposed points of K_∞, whence $K_\infty \in \mathcal{D}$. By (128) with $\varpi_n \in K_{n-1}$ (see (129)), for each natural number $n \geq j+1$, there are $\alpha_{n,j-1}^{(j)}, \ldots, \alpha_{n,n}^{(j)} \in [0,1]$ and $\rho_n^{(j)} \in \mathrm{co}\,\{\omega_1, \ldots, \omega_{n_\varepsilon+j-1}\}$ such that

$$\alpha_{n,j-1}^{(j)} + \alpha_{n,j}^{(j)} + \sum_{k=j+1}^{n} \alpha_{n,k}^{(j)} \lambda_k = 1 \quad \text{and} \quad \omega_{n_\varepsilon+n} = \alpha_{n,j-1}^{(j)} \rho_n^{(j)} + \alpha_{n,j}^{(j)} \omega_{n_\varepsilon+j} + \sum_{k=j+1}^{n} \alpha_{n,k}^{(j)} \lambda_k \sigma_k \ . \tag{133}$$

Additionally, define $\alpha_{n,k}^{(j)} \doteq 1$ for all natural numbers $k \geq n$ while $\alpha_{n,k}^{(j)} \doteq 0$ for $k \in \mathbb{N}_0$ such that $k \leq j-2$. Using (130), (131) and (133), at fixed $j \in \mathbb{N}$, we thus obtain that

$$\begin{aligned} \mathrm{Re}\,\{\omega_{n_\varepsilon+n}(A_j)\} &= \alpha_{n,j}^{(j)} \mathrm{Re}\,\{\omega_{n_\varepsilon+j}(A_j)\} + \sum_{k=j+1}^{n} \alpha_{n,k}^{(j)} \lambda_k \mathrm{Re}\,\{\sigma_k(A_j)\} \\ &\leq \lambda_j \left(1 - 2^{-1} \sum_{k=j+1}^{n} \alpha_{n,k}^{(j)} \lambda_k\right) \end{aligned} \tag{134}$$

for any $n \geq j+1$, while, for any natural number $n \leq j-1$,

$$\mathrm{Re}\,\{\omega_{n_\varepsilon+n}(A_j)\} = 0 \ ,$$

using (131). Fix $j \in \mathbb{N}$ and let $\omega_\infty \in K_\infty$ be a solution to the variational problem

$$\max_{\sigma \in K_\infty} \mathrm{Re}\,\{\sigma(A_j)\} = \mathrm{Re}\,\{\omega_\infty(A_j)\} \geq \mathrm{Re}\,\{\omega_{n_\varepsilon+j}(A_j)\} = \lambda_j \ . \tag{135}$$

(K_∞ is weak*-compact.) By weak*-density of $\{\omega_n\}_{n\in\mathbb{N}}$ in K_∞, there is a sequence $(\omega_{n_\varepsilon+n_l})_{l\in\mathbb{N}}$ converging to ω_∞ in the weak* topology. Since K_j is weak*-compact and $\alpha_{n,k}^{(j)} \in [0,1]$ for all $k \in \mathbb{N}_0$ and $n, j \in \mathbb{N}$, by using a standard argument with a so-called diagonal subsequence, we can choose the sequence $(n_l)_{l\in\mathbb{N}}$ such that $(\rho_{n_l}^{(j)})$ weak*-converges to $\rho_\infty^{(j)} \in K_{j-1}$, and $(\alpha_{n_l,k}^{(j)})_{l\in\mathbb{N}}$ has a limit for any fixed $k \in \mathbb{N}_0$ and $j \in \mathbb{N}$. Using (128), (134) and the inequality

$$\sum_{k=j+1}^{n_l} \alpha_{n_l,k}^{(j)} \lambda_k \leq D^{-1} \varepsilon \sum_{k=j+1}^{\infty} 2^{-(k+1)} = 2^{-(j+1)} D^{-1} \varepsilon$$

together with Lebesgue's dominated convergence theorem, we thus obtain that

$$\mathrm{Re}\,\{\omega_\infty(A_j)\} = \lim_{l\to\infty} \mathrm{Re}\,\{\omega_{n_\varepsilon+n_l}(A_j)\} \leq \lambda_j \left(1 - 2^{-1} \sum_{k=j+1}^{\infty} \lambda_k \lim_{l\to\infty} \alpha_{n_l,k}^{(j)}\right) \ .$$

Because of (135), it follows that

$$\lim_{l\to\infty} \alpha_{n_l,k}^{(j)} = 0 \ , \qquad k \in \{j+1, \ldots, \infty\} \ ,$$

leading to $\omega_\infty \in K_j$, by (128), (133) and Lebesgue's dominated convergence theorem. (Recall that K_j is defined by (129) for $n = j \in \mathbb{N}$.) Since $\omega_{n_\varepsilon+j}$ is by construction the unique maximizer of

$$\max_{\sigma \in K_j} \mathrm{Re}\,\{\sigma(A_j)\} = \mathrm{Re}\,\{\omega_{n_\varepsilon+j}(A_j)\}$$

and (135) holds true with $\omega_\infty \in K_j$, we deduce that $\omega_\infty = \omega_{n_\varepsilon+j}$, which is thus an exposed point of K_∞ for any $j \in \mathbb{N}$. ■

Our proof differs in several important aspects from the one of [42, Theorem 4.3], even if it has the same general structure, inspired by Poulsen's construction [62], as already mentioned. To be more precise, as compared to the proof of [42, Theorem 4.3], *Step 0* is new and is a direct consequence of the compactness and metrizability of K, a property not assumed in [42, Theorem 4.3]. *Step 1* to *Step* $n \rightarrow \infty$ are similar to what is done in [42], but with the essential difference that convex combinations are used to produce new (strongly) exposed points and the required bounds on $\{\lambda_n, \sigma_n\}_{n\in\mathbb{N}}$ are thus quite different. Compare Equations (128) and (130) with the bounds on $\upsilon_1, \upsilon_2, \upsilon_3$ given in [42, p. 27-29], at parameters $r_1(t), r_2(t), r_3(t) = 1$. In particular, [42, Lemma 4.2], which is essential to prove that the Poulsen-type construction leads to a dense set of (strongly) exposed points in [42, Theorem 4.3], is *never* used here. Instead, we use other direct estimates on convex combinations to deduce this property. This corresponds to *Step* $n = \infty$.

Note finally that [42, Theorem 4.3] shows the density of convex compact sets with dense set of *strongly* exposed points. A strongly exposed point σ_0 in some convex set $K \subseteq \mathcal{X}^*$ is an exposed point for some $A \in \mathcal{X}$ with the additional property that any minimizing net of the real part of $\hat{A}$ (cf. (21)) has to converge to σ_0 in the weak* topology[36]. Observe that the only weak* accumulation point of such a minimizing net is the exposed point σ_0, by weak* continuity of $\hat{A}$. If K is weak*-compact, this yields that any minimizing net converges to σ_0 in the weak* topology. In other words, any exposed point is *automatically* strongly exposed in all convex weak*-compact sets $K \in \mathbf{CK}(\mathcal{X}^*)$.

7 Technical Proofs

The aim of this section is to prove Theorems 4.1 and 4.6. In fact, we prove here stronger results than these theorems. The proof of Theorem 4.1 is done in six lemmata and two corollaries. The proof of Theorem 4.6 is a direct consequence of Corollary 7.12.

We start with a useful estimate on the norm-continuous two-parameter family $(T^{\xi}_{t,s})_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{X}$ defined by the non-autonomous evolution equations (47)-(48).

Lemma 7.1 (Continuity of quantum dynamics)
Let $\mathcal{X}$ be a unital C^*-algebra. For any $h \in C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$, $\xi_1, \xi_2 \in C(\mathbb{R}; E)$ and $s_1, s_2, t_1, t_2 \in \mathbb{R}$,

$$\left\| T^{\xi_2}_{t_2,s_2} - T^{\xi_1}_{t_1,s_1} \right\|_{\mathcal{B}(\mathcal{X})} \leq 2\left(|t_2 - t_1| + |s_2 - s_1|\right) \|h\|_{C_b(\mathbb{R};\mathfrak{Y}(\mathbb{R}))} + 2\int_{s_2}^{t_2} \left\|\mathrm{D}h\left(\alpha; \xi_1(\alpha)\right) - \mathrm{D}h\left(\alpha; \xi_2(\alpha)\right)\right\|_{\mathcal{X}} \mathrm{d}\alpha \, .$$

Proof. Fix $h \in C_b(\mathbb{R}; \mathfrak{Y}(\mathbb{R}))$, $\xi_1, \xi_2 \in C(\mathbb{R}; E)$ and $s_1, s_2, t_1, t_2 \in \mathbb{R}$. Via (49), observe that

$$T^{\xi_2}_{t_2,s_2} - T^{\xi_1}_{t_1,s_1} = T^{\xi_1}_{t_1,s_1} \circ (T^{\xi_1}_{t_2,t_1} - \mathbf{1}_{\mathcal{X}}) + (T^{\xi_1}_{s_1,s_2} - \mathbf{1}_{\mathcal{X}}) \circ T^{\xi_1}_{t_2,s_1} + T^{\xi_2}_{t_2,s_2} - T^{\xi_1}_{t_2,s_2} \, .$$

Using (47)-(48) together with (49), we thus obtain the equality

$$T^{\xi_2}_{t_2,s_2} - T^{\xi_1}_{t_1,s_1} = \int_{t_1}^{t_2} T^{\xi_1}_{\alpha,s_1} \circ X^{\xi_1(\alpha)}_{\alpha} \mathrm{d}\alpha + \int_{s_2}^{s_1} X^{\xi_1(\alpha)}_{\alpha} \circ T^{\xi_1}_{t_2,\alpha} \mathrm{d}\alpha + \int_{s_2}^{t_2} T^{\xi_2}_{\alpha,s_2} \circ \left(X^{\xi_2(\alpha)}_{\alpha} - X^{\xi_1(\alpha)}_{\alpha}\right) \circ T^{\xi_1}_{t_2,\alpha} \mathrm{d}\alpha \, . \qquad (136)$$

For any $\xi \in C(\mathbb{R}; E)$, $(T^{\xi}_{t,s})_{s,t\in\mathbb{R}}$ is a two-parameter family of $*$-automorphisms of $\mathcal{X}$ and the generator $X^{\xi(t)}_t$ defined by (45) has its operator norm bounded by (46). Therefore, the sum of the first two terms in the right hand side of (136) is bounded by

$$2\,|t_2 - t_1| \, \|h\|_{C_b(\mathbb{R};\mathfrak{Y}(\mathbb{R}))} + 2\,|s_2 - s_1| \, \|h\|_{C_b(\mathbb{R};\mathfrak{Y}(\mathbb{R}))} \ ,$$

[36]One should not mistake the notion of strongly exposed points discussed here for the notion of weak* strongly exposed points of [87, Definition 5.8] where a weak* strongly exposed point is a (weak*) exposed point with the additional property that any minimizing net of the real part of $\hat{A}$ has to converge to σ_0 *in the norm topology* of $\mathcal{X}^*$.

while the last term in (136) is bounded by

$$2\int_{s_2}^{t_2} \|\mathrm{D}h(\alpha;\xi_1(\alpha)) - \mathrm{D}h(\alpha;\xi_2(\alpha))\|_{\mathcal{X}}\, \mathrm{d}\alpha \ .$$

■

We start now more specifically with the proof of Theorem 4.1, by showing the existence and uniqueness of the solution to the self-consistency equation. To this end, we basically use the Banach fixed point theorem.

In contrast with Section 6, note that, below, the dual $\mathcal{X}^*$ of the unital C^*-algebra $\mathcal{X}$ is always equipped with the usual norm for linear functionals on a normed space. In particular, $\mathcal{X}^*$ is in this case a Banach space. The set E of states is a weak*-compact subset of $\mathcal{X}^*$ in the weak* topology, but not in the norm topology, unless $\mathcal{X}$ is finite-dimensional. This issue leads us to introduce Conditions (a)-(b) of Theorem 4.1, that is:

Condition 7.2

(a) *Let $\mathcal{X}$ be a unital C^*-algebra and $\mathfrak{B}$ a finite-dimensional real subspace of $\mathcal{X}^{\mathbb{R}}$.*

(b) *Take $h \in C_b(\mathbb{R};\mathfrak{Y}(\mathbb{R}))$ and a constant $D_0 \in \mathbb{R}^+$ such that, for all $t \in \mathbb{R}$,*

$$\|\mathrm{D}h(t;\rho) - \mathrm{D}h(t;\tilde{\rho})\|_{\mathcal{X}} \leq D_0 \sup_{B\in\mathfrak{B},\|B\|=1} |(\rho-\tilde{\rho})(B)| \ , \qquad \rho,\tilde{\rho}\in E \ .$$

We are now in a position to show the existence and uniqueness of the solution to the self-consistency equation:

Lemma 7.3 (Self-consistency equations)

Under Condition 7.2, for any $s \in \mathbb{R}$ and $\rho \in E$, there is a unique solution $\varpi_{\rho,s} \in C(\mathbb{R};E)$ to the following equation in $\xi \in C(\mathbb{R};E)$:

$$\forall t \in \mathbb{R}: \qquad \xi(t) = \rho\circ T_{t,s}^{\xi} \ . \tag{137}$$

Moreover, $\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t)$ for any $r,s,t \in \mathbb{R}$.

Proof. We prove the existence and uniqueness of a solution to (137) by using the Banach fixed point theorem, similar to the Picard-Lindelöf theory for ODEs, keeping in mind that E is endowed with the weak* topology: Pick a function $h \in C_b(\mathbb{R};\mathfrak{Y}(\mathbb{R}))$, an initial time $s \in \mathbb{R}$ and a state $\rho \in E$. For $\epsilon \in \mathbb{R}^+$, define the map $\mathfrak{F}$ from

$$\mathcal{C}_{\epsilon,s} \doteq C([s-\epsilon,s+\epsilon];\mathcal{X}^*)\cap C([s-\epsilon,s+\epsilon];E)$$

to itself by

$$\mathfrak{F}(\xi)(t) \doteq \rho\circ T_{t,s}^{\xi} \ , \qquad t\in[s-\epsilon,s+\epsilon] \ . \tag{138}$$

The continuity of $\mathfrak{F}(\xi)$ in the Banach space $C([s-\epsilon,s+\epsilon];\mathcal{X}^*)$ can directly be read from Lemma 7.1 and Condition 7.2 (b). The same also yields the contractivity of $\mathfrak{F}$ for sufficiently small $\epsilon\in\mathbb{R}^+$, uniformly with respect to $s\in\mathbb{R}$ and $\rho\in E$. Using the Banach fixed point theorem, there is a unique solution $\varpi_{\rho,s}$ to $\mathfrak{F}(\xi)=\xi$ in $\mathcal{C}_{\epsilon,s}$. By exactly the same arguments, observe that, for each $r\in[s-\epsilon,s+\epsilon]$, the following self-consistency equation

$$\forall t\in[r-\tilde{\epsilon},r+\tilde{\epsilon}]: \qquad \xi(t) = \varpi_{\rho,s}(r)\circ T_{t,r}^{\xi} \ , \tag{139}$$

has also a unique solution $\varpi_{\varpi_{\rho,s}(r),r}$ in $\mathcal{C}_{\tilde{\epsilon},r}$ for any $\tilde{\epsilon}\in(0,\epsilon]$. By the reverse cocycle property (49), at fixed $s\in\mathbb{R}$ and $\rho\in E$, $\varpi_{\rho,s}$ solves (139) for any $r\in(s-\epsilon,s+\epsilon)$ and $t\in[s-\tilde{\epsilon},s+\tilde{\epsilon}]$ with $\tilde{\epsilon}=\epsilon-|s-r|\in\mathbb{R}^+$, whence

$$\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t) \ , \qquad r\in(s-\epsilon,s+\epsilon),\ t\in[s-\tilde{\epsilon},s+\tilde{\epsilon}] \ . \tag{140}$$

Now, assume the existence and uniqueness of a solution $\varpi_{\rho,s}$ to $\mathfrak{F}(\xi) = \xi$ in $\mathcal{C}_{\epsilon_0,s}$ for some parameter $\epsilon_0 \in \mathbb{R}^+$. Take $r \in (s-\epsilon_0, s-\epsilon_0+\epsilon) \cup (s+\epsilon_0-\epsilon, s+\epsilon_0)$. By combining the existence and uniqueness of a solution $\varpi_{\varpi_{\rho,s}(r),r}$ to (139) in $\mathcal{C}_{\tilde{\epsilon},r}$ together with the reverse cocycle property (49), we deduce that

$$\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t) , \qquad t \in (s-\epsilon_0, s+\epsilon_0) ,$$

as well as the existence of a unique solution $\varpi_{\rho,s}$ to $\mathfrak{F}(\xi) = \xi$ in $\mathcal{C}_{\epsilon_0+\epsilon,s}$. As a consequence, one can infer from a contradiction argument the existence and uniqueness of a solution in $C(\mathbb{R};\mathcal{X}^*) \cap C(\mathbb{R};E)$ to (137). Moreover, this solution must satisfy the equality $\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t)$ for any $r,s,t \in \mathbb{R}$.

Finally, to prove uniqueness in $C(\mathbb{R};E)$, we observe from Lemma 7.1 and Condition 7.2 that any solution in $C(\mathbb{R};E)$ (i.e., continuous with respect to the weak* topology in E) to (137) is automatically in $C(\mathbb{R};\mathcal{X}^*)$ (i.e., continuous with respect to the norm topology in $\mathcal{X}^*$). ∎

Corollary 7.4 (Bijectivity of the solution to the self-consistency equation)
Under Condition 7.2, for any $s,t \in \mathbb{R}$, $\varpi_s(t) \equiv (\varpi_{\rho,s}(t))_{\rho\in E}$ is a bijective map from E to itself.

Proof. This is a straightforward consequence of Lemma 7.3, in particular the equality $\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t)$ for any $r,s,t \in \mathbb{R}$. ∎

Lemma 7.5 (Differentiability of the solution – I)
Under Condition 7.2, for $s \in \mathbb{R}$ and $\rho \in E$, $\varpi_{\rho,s} \in C^1(\mathbb{R};\mathcal{X}^*)$ with derivative given by

$$\partial_t \varpi_{\rho,s}(t) = \rho \circ T_{t,s}^{\varpi_{\rho,s}} \circ X_t^{\varpi_{\rho,s}(t)} , \qquad t \in \mathbb{R} .$$

Proof. This is a direct consequence of Equation (47) together with Lemma 7.3. ∎

Lemma 7.6 (Continuity with respect to the initial condition)
Under Condition 7.2, for any $s,t \in \mathbb{R}$, $\varpi_s(t) \equiv (\varpi_{\rho,s}(t))_{\rho\in E} \in C(E;E)$.

Proof. Take $s \in \mathbb{R}$ and two states $\rho_1, \rho_2 \in E$. Then, define the quantity

$$\mathbf{X}(\epsilon) \doteq \max_{t\in[s-\epsilon,s+\epsilon]} \max_{B\in\mathfrak{B},\|B\|=1} \left|\left(\varpi_{\rho_1,s}(t) - \varpi_{\rho_2,s}(t)\right)(B)\right| , \qquad \epsilon \in \mathbb{R}^+ .$$

Because $\varpi_{\rho,s}(t) = \rho \circ T_{t,s}^{\varpi_{\rho,s}}$ (Lemma 7.3) with $(T_{t,s}^{\xi})_{s,t\in\mathbb{R}}$ being a family of $*$-automorphisms of $\mathcal{X}$ for any $\xi \in C(\mathbb{R};E)$, this positive number is bounded by

$$\mathbf{X}(\epsilon) \leq \max_{B\in\mathfrak{B},\|B\|=1} \left|(\rho_1 - \rho_2) \circ T_{t,s}^{\varpi_{\rho_1,s}}(B)\right| + \mathbf{Y}(\epsilon) , \tag{141}$$

where

$$\mathbf{Y}(\epsilon) \doteq \max_{t\in[s-\epsilon,s+\epsilon]} \left\|T_{t,s}^{\varpi_{\rho_1,s}} - T_{t,s}^{\varpi_{\rho_2,s}}\right\|_{\mathcal{B}(\mathcal{X})} . \tag{142}$$

By Lemma 7.1, the last quantity is bounded by

$$\mathbf{Y}(\epsilon) \leq 2 \max_{t\in[s-\epsilon,s+\epsilon]} \left\{ \int_s^t \left\|\mathrm{D}h\left(\alpha;\varpi_{\rho_1,s}(\alpha)\right) - \mathrm{D}h\left(\alpha;\varpi_{\rho_2,s}(\alpha)\right)\right\|_{\mathcal{X}} \mathrm{d}\alpha \right\} , \tag{143}$$

which, together with Condition 7.2 (b), leads to

$$\mathbf{Y}(\epsilon) \leq 2D_0\epsilon\mathbf{X}(\epsilon) , \qquad \epsilon \in \mathbb{R}^+ . \tag{144}$$

By Inequality (141), it follows that

$$(1 - 2D_0\epsilon)\,\mathbf{X}(\epsilon) \leq \max_{B\in\mathfrak{B},\|B\|=1} \left|(\rho_1 - \rho_2) \circ T_{t,s}^{\varpi_{\rho_1,s}}(B)\right| , \qquad \epsilon \in \mathbb{R}^+ . \tag{145}$$

Now, we combine $\varpi_{\rho,s}(t) = \rho \circ T_{t,s}^{\varpi_{\rho,s}}$ with (142) and (144)-(145) to get the inequality

$$\begin{aligned}\left|\varpi_{\rho_1,s}(t)(A) - \varpi_{\rho_2,s}(t)(A)\right| &\leq \left|(\rho_1 - \rho_2) \circ T_{t,s}^{\varpi_{\rho_1,s}}(A)\right| \\ &\quad + \frac{2D_0\epsilon \|A\|_{\mathcal{X}}}{(1-2D_0\epsilon)} \max_{B\in\mathfrak{B},\|B\|=1} \left|(\rho_1 - \rho_2) \circ T_{t,s}^{\varpi_{\rho_1,s}}(B)\right|\end{aligned} \tag{146}$$

for any $s \in \mathbb{R}$, $\rho_1, \rho_2 \in E$, $A \in \mathcal{X}$, $\epsilon \in (0, D_0/2)$ and $t \in [s-\epsilon, s+\epsilon]$. By finite dimensionality of $\mathfrak{B}$ (Condition 7.2 (a)), the norm and weak* topologies of $\mathfrak{B}^*$ are the same and the weak* continuity property of $\varpi_s(t)$ follows from (146) for any times $s \in \mathbb{R}$ and $t \in [s-\epsilon, s+\epsilon]$, provided $\epsilon < D_0/2$. Using now the equality $\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t)$ for any $r, s, t \in \mathbb{R}$ (Lemma 7.3), we thus deduce the weak* continuity of $\varpi_s(t)$ for all times $s, t \in \mathbb{R}$. ∎

Corollary 7.7 (Solution to the self-consistency equation as homeomorphism family)
Under Condition 7.2, at any fixed times $s, t \in \mathbb{R}$, $\varpi_s(t) \equiv (\varpi_{\rho,s}(t))_{\rho\in E} \in \mathrm{Aut}(E)$, i.e., $\varpi_s(t)$ is a automorphism of the state space E. Moreover, it satisfies a cocycle property:

$$\forall s, r, t \in \mathbb{R}: \qquad \varpi_s(t) = \varpi_r(t) \circ \varpi_s(r) \ . \tag{147}$$

Proof. By Corollary 7.4 and Lemma 7.6, for any $s, t \in \mathbb{R}$, $\varpi_s(t)$ is a weak*-continuous bijective map from E to itself. Recall that E is the (Hausdorff) topological space of all states on $\mathcal{X}$ with the weak* topology. It is weak*-compact. Therefore, the inverse of $\varpi_s(t)$ is also weak*-continuous. Equation (147) is only another way to write the equality $\varpi_{\rho,s}(t) = \varpi_{\varpi_{\rho,s}(r),r}(t)$ of Lemma 7.3. ∎

Recall that the set $\mathrm{Aut}(E)$ of all automorphisms (or self-homeomorphisms) of E is endowed with the topology of uniform convergence of weak*-continuous functions from E to itself. See (44). Having this in mind, we obtain now the following lemma:

Lemma 7.8 (Well-posedness of the self-consistency equation)
Under Condition 7.2, for any $s \in \mathbb{R}$,

$$\boldsymbol{\varpi}_s \equiv (\varpi_s(t))_{t\in\mathbb{R}} \equiv ((\varpi_{\rho,s}(t))_{\rho\in E})_{t\in\mathbb{R}} \in C(\mathbb{R}; \mathrm{Aut}(E)) \ .$$

Proof. Take any net $(t_j)_{j\in J} \subseteq \mathbb{R}$ converging to some arbitrary time $t \in \mathbb{R}$. Assume that $\varpi_s(t_j)$ does not converge to $\varpi_s(t)$, in the topology of uniform convergence of weak*-continuous functions. In this case, by (44), there is a net $(\rho_j)_{j\in J} \subseteq E$ of states, $A \in \mathcal{X}$ and $\varepsilon \in \mathbb{R}^+$ such that

$$\liminf_{j\in J} \left|\left[\varpi_{\rho_j,s}(t_j) - \varpi_{\rho_j,s}(t)\right](A)\right| \geq \varepsilon > 0 \ . \tag{148}$$

By weak* compactness of E, there is a subnet $(\rho_{j_i})_{i\in I}$ weak*-converging to some $\rho \in E$. By Lemmata 7.3, 7.6 and Inequality (148), it follows that

$$\liminf_{i\in I} \left|\left[\rho_{j_i} \circ T_{t_{j_i},s}^{\varpi_{\rho_{j_i},s}} - \varpi_{\rho,s}(t)\right](A)\right| \geq \varepsilon > 0 \ .$$

Using (142) and (144)-(145) together with the reverse cocycle property (49) and the fact that $(T_{t,s}^{\xi})_{s,t\in\mathbb{R}}$ is a family of $*$-automorphisms of $\mathcal{X}$ for any $\xi \in C(\mathbb{R}; E)$, we thus deduce from the last inequality that

$$\liminf_{i\in I} \left|\left[\rho_{j_i} \circ T_{t_{j_i},s}^{\varpi_{\rho,s}} - \varpi_{\rho,s}(t)\right](A)\right| \geq \varepsilon > 0 \ . \tag{149}$$

This is a contradiction because $(T_{t,s}^{\varpi_{\rho,s}})_{s,t\in\mathbb{R}}$ is a norm-continuous two-parameter family. Hence, for any $A \in \mathcal{X}$,

$$\lim_{i\in I} \rho_{j_i} \circ T_{t_{j_i},s}^{\varpi_{\rho,s}}(A) = \rho \circ T_{t,s}^{\varpi_{\rho,s}}(A) = [\varpi_{\rho,s}(t)](A) \ .$$

∎

Lemma 7.9 (Joint continuity with respect to initial and final times)
Under Condition 7.2, the solution to the self-consistency equation is jointly continuous with respect to initial and final times:

$$\boldsymbol{\varpi} \equiv (\boldsymbol{\varpi}_s)_{s\in\mathbb{R}} \equiv (\varpi_s(t))_{s,t\in\mathbb{R}} \equiv ((\varpi_{\rho,s}(t))_{\rho\in E})_{s,t\in\mathbb{R}} \in C\left(\mathbb{R}^2; \mathrm{Aut}(E)\right) \ .$$

Proof. We use again the Banach fixed point theorem: Fix $s \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$. Similar to (138), we define the map $\mathfrak{F}$ from

$$C\left([s-\epsilon, s+\epsilon]^2; \mathcal{X}^*\right) \cap C\left([s-\epsilon, s+\epsilon]^2; E\right)$$

to itself by

$$\mathfrak{F}(\zeta)(r,t) \doteq \rho \circ T_{t,r}^{\zeta(r,\cdot)}, \qquad r,t \in [s-\epsilon, s+\epsilon] \ ,$$

where

$$\zeta(r,\cdot) \in C([s-\epsilon, s+\epsilon]; \mathcal{X}^*) \cap C([s-\epsilon, s+\epsilon]; E)$$

is the function defined, at fixed $r \in [s-\epsilon, s+\epsilon]$, by $\zeta(r,t)$ for any $t \in [s-\epsilon, s+\epsilon]$. By Lemma 7.1 and Condition 7.2 (b), $\mathfrak{F}$ is a contraction for sufficiently small times $\epsilon \in \mathbb{R}^+$ and we use similar arguments as in the proof of Lemma 7.3 to show the existence of a unique solution $\daleth$ to the following equation in $\zeta \in C([s-\epsilon, s+\epsilon]^2; E)$:

$$\forall r,t \in [s-\epsilon, s+\epsilon] : \qquad \zeta(r,t) = \rho \circ T_{t,r}^{\zeta(r,\cdot)} \ .$$

By uniqueness of the solution to (137) in $C(\mathbb{R}; E)$ at any fixed $s \in \mathbb{R}$, $\varpi_{\rho,r}(t) = \daleth(r,t)$ for any $r,t \in [s-\epsilon, s+\epsilon]$. By Corollary 7.7 and Lemma 7.8, it follows that

$$(\varpi_{\rho,s}(t))_{s,t\in\mathbb{R}} \in C\left(\mathbb{R}^2; E\right) \ , \qquad \rho \in E \ .$$

Finally, by similar compactness arguments as in the proof of Lemma 7.8, we deduce the assertion. ∎

Lemma 7.10 (Differentiability of the solution – II)
Fix $n \in \mathbb{N}$, $g \in C_b(\mathbb{R}; C_b^3(\mathbb{R}^n, \mathbb{R}))$, $\{B_j\}_{j=1}^n \subseteq \mathcal{X}^{\mathbb{R}}$ and

$$h(t;\rho) \doteq g(t; \rho(B_1), \ldots, \rho(B_n)) \ , \qquad t \in \mathbb{R}, \ \rho \in E \ . \tag{150}$$

Then, for any $s,t \in \mathbb{R}$ and $A \in \mathcal{X}$,

$$(\varpi_{\rho,s}(t)(A))_{\rho\in E} \equiv (\varpi_{\rho,s}(t,A))_{\rho\in E} \in C^1(E; \mathbb{C}) \tag{151}$$

and, for any $\upsilon \in E$,

$$[\mathrm{d}\varpi_{\rho,s}(t,A)](\upsilon) = \upsilon(\mathrm{D}\varpi_{\rho,s}(t,A)) = (\upsilon - \rho) \circ T_{t,s}^{\varpi_{\rho,s}}(A) + \maltese_A[\mathrm{d}\varpi_{\rho,s}(\cdot,\cdot)(\upsilon)]$$

where, for any continuous function $\xi : \mathbb{R} \times \mathcal{X} \to \mathbb{C}$,

$$\begin{aligned} \maltese_A[\xi] \ &\doteq \ \sum_{j,k=1}^n \int_s^t \mathrm{d}\alpha \, \xi(\alpha, B_k) \, \rho \circ T_{\alpha,s}^{\varpi_{\rho,s}}\left(i\left[B_j, T_{t,\alpha}^{\varpi_{\rho,s}}(A)\right]\right) \\ &\quad \times \partial_{x_k}\partial_{x_j} g(\alpha; \varpi_{\rho,s}(\alpha, B_1), \ldots, \varpi_{\rho,s}(\alpha, B_n)) \ . \end{aligned} \tag{152}$$

Moreover, for all $s \in \mathbb{R}$ and $\rho \in E$, the map $(t,A) \mapsto \mathrm{D}\varpi_{\rho,s}(t,A)$ from $\mathbb{R} \times \mathcal{X}$ to $\mathcal{X}$ is continuous.

Proof. Fix all parameters of the lemma. Observe first that Taylor's theorem applied to $\partial_{x_j} g(t)$ for each $t \in \mathbb{R}$ and $j \in \{1, \ldots, n\}$ yields that, for all $x, y \in \mathbb{R}^n$,

$$\partial_{x_j} g(t; y) = \partial_{x_j} g(t; x) + \sum_{k=1}^{n} (y_k - x_k) \left(\partial_{x_k} \partial_{x_j} g(t; x_1, \ldots, x_n) + r_k(t, x, y) \right) \tag{153}$$

where, for any $k \in \{1, \ldots, n\}$, $r_k(\cdot, \cdot, \cdot)$ is a continuous real-valued function on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\lim_{y \to x} r_k(t, x, y) = 0 , \tag{154}$$

uniformly for t and x in a compact set. Note additionally that the function h, as defined by (150), satisfies Condition 7.2.

For any $s, t \in \mathbb{R}$, $\rho, \upsilon \in E$, $\lambda \in (0, 1]$ and $A \in \mathcal{X}$, we infer from Lemma 7.3 that

$$\begin{aligned} \beth(\lambda, t, A; \upsilon) &\doteq \lambda^{-1} \left(\varpi_{(1-\lambda)\rho + \lambda\upsilon, s}(t, A) - \varpi_{\rho, s}(t, A) \right) \\ &= (\upsilon - \rho) \circ T_{t,s}^{\varpi_{\rho,s}}(A) + \lambda^{-1} ((1-\lambda)\rho + \lambda\upsilon) \circ \left(T_{t,s}^{\varpi_{(1-\lambda)\rho+\lambda\upsilon,s}} - T_{t,s}^{\varpi_{\rho,s}} \right)(A) . \end{aligned}$$

Through Equations (34), (45), (136), (150) and (153), we deduce that

$$\begin{aligned} \beth(\lambda, t, A; \upsilon) = (\upsilon - \rho) \circ T_{t,s}^{\varpi_{\rho,s}}(A) + \sum_{j,k=1}^{n} \int_s^t \mathrm{d}\alpha \, \beth(\lambda, \alpha, B_k; \upsilon) \\ \times ((1-\lambda)\rho + \lambda\upsilon) \circ T_{\alpha,s}^{\varpi_{(1-\lambda)\rho+\lambda\upsilon,s}} \left(i \left[B_j, T_{t,\alpha}^{\varpi_{\rho,s}}(A) \right] \right) \\ \times \left(\partial_{x_k} \partial_{x_j} g(\alpha; \mathbf{x}(0, \alpha)) + r_k(\alpha; \mathbf{x}(0, \alpha), \mathbf{x}(\lambda, \alpha)) \right) , \end{aligned} \tag{155}$$

where

$$\mathbf{x}(\lambda, \alpha) \doteq \left(\varpi_{(1-\lambda)\rho+\lambda\upsilon,s}(\alpha, B_1), \ldots, \varpi_{(1-\lambda)\rho+\lambda\upsilon,s}(\alpha, B_n) \right) \in \mathbb{R}^n .$$

From Equation (155), one sees that $\beth(\lambda, t, A; \upsilon)$ is given by a Dyson-type series which is absolutely summable, uniformly with respect to $\lambda \in (0, 1]$, because $(T_{t,s}^{\xi})_{s,t \in \mathbb{R}}$ is a family of $*$-automorphisms of $\mathcal{X}$ for any $\xi \in C(\mathbb{R}; E)$. By Lemmata 7.1 and 7.8 together with Condition 7.2 (b),

$$\lim_{\lambda \to 0^+} ((1-\lambda)\rho + \lambda\upsilon) \circ T_{\alpha,s}^{\varpi_{(1-\lambda)\rho+\lambda\upsilon,s}} \left(i \left[B_j, T_{t,\alpha}^{\varpi_{\rho,s}}(A) \right] \right) = \rho \circ T_{\alpha,s}^{\varpi_{\rho,s}} \left(i \left[B_j, T_{t,\alpha}^{\varpi_{\rho,s}}(A) \right] \right)$$

while

$$\lim_{\lambda \to 0^+} r_k(\alpha, \mathbf{x}(0, \alpha), \mathbf{x}(\lambda, \alpha)) = 0 ,$$

using (154). (Both limits are uniform for α in a compact set.) Hence, we deduce from Lebesgue's dominated convergence theorem that

$$\beth(0, t, A; \upsilon) \doteq \lim_{\lambda \to 0^+} \beth(\lambda, t, A; \upsilon) = \lim_{\lambda \to 0^+} \lambda^{-1} \left(\varpi_{(1-\lambda)\rho+\lambda\upsilon,s}(t, A) - \varpi_{\rho,s}(t, A) \right) \tag{156}$$

exists for all $s, t \in \mathbb{R}$, $\rho, \upsilon \in E$ and $A \in \mathcal{X}$, as given by a Dyson-type series. In particular, for any $\upsilon \in E$, the complex-valued function $(t, A) \mapsto \beth(0, t, A, \upsilon)$ on $\mathbb{R} \times \mathcal{X}$ is the unique solution in $\xi \in C(\mathbb{R} \times \mathcal{X}; \mathbb{C})$ to the equation

$$\xi(t, A) = (\upsilon - \rho) \circ T_{t,s}^{\varpi_{\rho,s}}(A) + \maltese_A[\xi] \tag{157}$$

with $\maltese_A$ defined by (152). Compare with (155) taken at $\lambda = 0$. Note that the integral equation

$$\begin{aligned} \mathfrak{D}(t, A) = T_{t,s}^{\varpi_{\rho,s}}(A) - \rho \circ T_{t,s}^{\varpi_{\rho,s}}(A) \mathbf{1} + \sum_{j,k=1}^{n} \int_s^t \mathrm{d}\alpha \, \mathfrak{D}(\alpha, B_k) \\ \times \rho \circ T_{\alpha,s}^{\varpi_{\rho,s}} \left(i \left[B_j, T_{t,\alpha}^{\varpi_{\rho,s}}(A) \right] \right) \partial_{x_k} \partial_{x_j} g(\alpha; \mathbf{x}(0, \alpha)) \end{aligned} \tag{158}$$

uniquely determines, by absolutely summable (in $\mathcal{X}$) Dyson-type series, a continuous map $(t, A) \mapsto \mathfrak{D}(t, A)$ from $\mathbb{R} \times \mathcal{X}$ to $\mathcal{X}$, which, by (157), satisfies

$$\upsilon\left(\mathfrak{D}(t, A)\right) = \beth(0, t, A; \upsilon) \doteq \lim_{\lambda \to 0^+} \lambda^{-1}\left(\varpi_{(1-\lambda)\rho + \lambda\upsilon, s}(t, A) - \varpi_{\rho, s}(t, A)\right) \tag{159}$$

for all $s, t \in \mathbb{R}$, $\rho, \upsilon \in E$ and $A \in \mathcal{X}$. By Definition 3.8, the assertion follows. ■

Lemma 7.11 (Differentiability of the solution – III)
Under the assumptions of Lemma 7.10, for any $t \in \mathbb{R}$, $\rho \in E$ and $A \in \mathcal{X}$,

$$(\varpi_{\rho,s}(t)(A))_{s\in\mathbb{R}} \equiv (\varpi_{\rho,s}(t, A))_{s\in\mathbb{R}} \in C^1(\mathbb{R}; \mathbb{C})$$

with derivative given, for any $A \in \mathcal{X}$, by

$$\partial_s \varpi_{\rho,s}(t, A) = -\rho \circ X_s^{\rho} \circ T_{t,s}^{\varpi_{\rho,s}}(A) + \maltese_A\left[\partial_s \varpi_{\rho,s}\right] . \tag{160}$$

Here, $\maltese_A$ is defined by (152) and $(t, A) \mapsto \partial_s \varpi_{\rho,s}(t, A)$ is a continuous function on $\mathbb{R} \times \mathcal{X}$.

Proof. By Lemma 7.3, for any $\rho \in E$, $s, t \in \mathbb{R}$, $A \in \mathcal{X}$ and $\varepsilon \in \mathbb{R}\backslash\{0\}$,

$$\begin{aligned} \tilde{\beth}(\varepsilon, t, A) &\doteq \varepsilon^{-1}\left(\varpi_{\rho,s+\varepsilon}(t, A) - \varpi_{\rho,s}(t, A)\right) \\ &= \varepsilon^{-1}\rho \circ \left(T_{t,s+\varepsilon}^{\varpi_{\rho,s}} - T_{t,s}^{\varpi_{\rho,s}}\right)(A) + \varepsilon^{-1}\rho \circ \left(T_{t,s+\varepsilon}^{\varpi_{\rho,s+\varepsilon}} - T_{t,s+\varepsilon}^{\varpi_{\rho,s}}\right)(A) . \end{aligned}$$

Similar to (155), via Equations (34), (45), (136), (150) and (153) we deduce that

$$\begin{aligned} \tilde{\beth}(\varepsilon, t, A) = \varepsilon^{-1}\rho \circ \left(T_{t,s+\varepsilon}^{\varpi_{\rho,s}} - T_{t,s}^{\varpi_{\rho,s}}\right)(A) + \sum_{j,k=1}^{n} \int_{s+\varepsilon}^{t} \mathrm{d}\alpha\, \tilde{\beth}(\varepsilon, \alpha, B_k) \\ \times \rho \circ T_{\alpha,s}^{\varpi_{\rho,s+\varepsilon}}\left(i\left[B_j, T_{t,\alpha}^{\varpi_{\rho,s}}(A)\right]\right)\left(\partial_{x_k}\partial_{x_j} g(\alpha; \mathbf{y}(0, \alpha)) + r_k(\alpha; \mathbf{y}(0, \alpha), \mathbf{y}(\varepsilon, \alpha))\right) \end{aligned} \tag{161}$$

with

$$\mathbf{y}(\varepsilon, \alpha) \doteq \left(\varpi_{\rho,s+\varepsilon}(\alpha, B_1), \ldots, \varpi_{\rho,s+\varepsilon}(\alpha, B_n)\right) \in \mathbb{R}^n .$$

Again, one sees from Equation (161) that $\tilde{\beth}(\varepsilon, t, A)$ is given by a Dyson-type series which is absolutely summable, uniformly with respect to ε in a bounded set. Recall that $(T_{t,s}^{\xi})_{s,t\in\mathbb{R}}$ is a norm-continuous two-parameter family of $*$-automorphisms of $\mathcal{X}$ satisfying in $\mathcal{B}(\mathcal{X})$ the non-autonomous evolution equation (48) for any fixed $\xi \in C(\mathbb{R}; E)$. Therefore, similar to (156), by Equation (154), Lemmata 7.1 and 7.9 together with Condition 7.2 (b) and Lebesgue's dominated convergence theorem, we deduce that

$$\partial_s \varpi_{\rho,s}(t, A) \doteq \lim_{\varepsilon \to 0} \tilde{\beth}(\varepsilon, t, A) = \lim_{\varepsilon \to 0} \varepsilon^{-1}\left(\varpi_{\rho,s+\varepsilon}(t, A) - \varpi_{\rho,s}(t, A)\right)$$

exists for all $s, t \in \mathbb{R}$, $\rho \in E$ and $A \in \mathcal{X}$, as given by a Dyson-type series. In particular, the complex-valued function $(t, A) \mapsto \partial_s \varpi_{\rho,s}(t, A)$ on $\mathbb{R} \times \mathcal{X}$ is the unique solution in $\xi \in C(\mathbb{R} \times \mathcal{X}; \mathbb{C})$ to the equation

$$\xi(t, A) = -\rho \circ X_s^{\rho} \circ T_{t,s}^{\varpi_{\rho,s}}(A) + \maltese_A[\xi] \tag{162}$$

with $\maltese_A$ defined by (152). Compare with (161) taken at $\varepsilon = 0$. ■

We conclude this section with the derivation of Liouville's equation for the time-evolution of (elementary) continuous and affine functions defined by (15), from which Theorem 4.6 is deduced.

Corollary 7.12 (Liouville's equation for affine functions)
Under the assumptions of Lemma 7.10,

$$\partial_s V_{t,s}^{h}(\hat{A}) = -\{h(s), V_{t,s}^{h}(\hat{A})\} , \qquad s, t \in \mathbb{R},\ A \in \mathcal{X},$$

with $\hat{A} \in \mathfrak{C}$ being the elementary continuous and affine function defined by (15). In particular, both side of the equation are well-defined functions in $\mathfrak{C}$.

Proof. Fix $s \in \mathbb{R}$ and $\rho \in E$. By (15) and (55), note that

$$V_{t,s}^{h}(\hat{A}) = \varpi_{\rho,s}(t)(A) \equiv \varpi_{\rho,s}(t,A) \ , \qquad t \in \mathbb{R},\ A \in \mathcal{X} \ .$$

By Lemma 7.10, the map

$$(t,A) \mapsto \mathrm{D}V_{t,s}^{h}(\hat{A})(\rho) = \mathrm{D}\varpi_{\rho,s}(t;A) = \mathfrak{D}(t,A)$$

from $\mathbb{R} \times \mathcal{X}$ to $\mathcal{X}$ is continuous. See also Definition 3.8 and Equation (33). Therefore, the map

$$(t,A) \mapsto -\{h(s), V_{t,s}^{h}(\hat{A})\}(\rho) \doteq -\rho\left(i\left[\mathrm{D}h(s;\rho), \mathrm{D}V_{t,s}^{h}(\hat{A})(\rho)\right]\right)$$

from $\mathbb{R} \times \mathcal{X}$ to $\mathbb{C}$ is a well-defined continuous function. See Definition 3.10 and (39). By (158), it solves Equation (162), like the well-defined continuous map

$$(t,A) \mapsto \partial_s V_{t,s}^{h}(\hat{A})(\rho) = \partial_s \varpi_{\rho,s}(t)(A) \equiv \partial_s \varpi_{\rho,s}(t,A)$$

from $\mathbb{R} \times \mathcal{X}$ to $\mathbb{C}$ (Lemma 7.11). By uniqueness of the solution to (162), the assertion follows. ■

8 Appendix: Liminal, Postliminal and Antiliminal C^*-Algebras

La même structure qui, si vous montez, comporte une distance, si vous descendez, n'en comporte pas.[37]

A. de Libera, 2015

As explained in [53, p. 99], the notion of *liminal* C^*-algebras was first introduced in 1951 by Kaplansky under the name of *CCR*-algebras. Remark that, in this context, *CCR* does not mean "Canonical Commutation Relations" but "Completely Continuous Representations", "completely continuous" standing for "compact". *CCR* usually means nowadays "Canonical Commutation Relations" and thus, like Dixmier in his textbook [53] on C^*-algebras, we rather prefer the terminology "liminal". This concept is strongly related to the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators acting on a Hilbert space $\mathcal{H}$ via the concept of C^*-algebra representations. See also [89] for a recent compendium on operator algebras.

Recall that a *representation* on the Hilbert space $\mathcal{H}$ of a C^*-algebra $\mathcal{X}$ is, by definition [36, Definition 2.3.2], a $*$-homomorphism $\boldsymbol{\pi}$ from $\mathcal{X}$ to the unital C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators acting on $\mathcal{H}$. Injective representations are called *faithful*. The representation of a C^*-algebra $\mathcal{X}$ is not unique: For any representation $\boldsymbol{\pi} : \mathcal{X} \to \mathcal{B}(\mathcal{H})$, we can construct another one by doubling the Hilbert space $\mathcal{H}$ and the map $\boldsymbol{\pi}$, via a direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of two copies $\mathcal{H}_1, \mathcal{H}_2$ of $\mathcal{H}$. Thus, recall also the notion of "minimal" representations of C^*-algebras: If $\boldsymbol{\pi} : \mathcal{X} \to \mathcal{B}(\mathcal{H})$ is a representation of a C^*-algebra $\mathcal{X}$ on the Hilbert space $\mathcal{H}$, we say that it is *irreducible*, whenever $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ which are invariant with respect to any operator of $\boldsymbol{\pi}(\mathcal{X}) \subseteq \mathcal{B}(\mathcal{H})$.

Every C^*-algebra which is isomorphic to the C^*-algebra $\mathcal{K}(\mathcal{H})$ of all compact operators acting on some Hilbert space $\mathcal{H}$ is said to be *elementary*. The concept of *liminal* C^*-algebras generalizes this notion (see [53, Definition 4.2.1] or [89, Section IV.1.3.1]):

[37] Engl.: *The same structure which, if you go up, contains a distance, if you go down, does not contain it.* See [88, p. 38]. This citation refers to the highly political and theological issues of hierarchies in Christianity, as discussed in the Late Middle Ages. Indeed, for some theologians of the XIIIe-XIVe centuries like Giles of Rome, the increasing hierarchy refers to the existence of an order, implying in particular a distance (cf. *"potentia dei ordinaria"*). From the top down, the relation can be direct, immediate, without distance (cf. "*potentia dei absoluta*"). In the mathematical context, from the bottom up, we have in mind the ordering of measures having same barycenter ρ in a compact convex space K to arrive, by "removing" the mass farther away from ρ, at a (maximal) measure only supported by extreme points, as proved and stated in the Choquet(-Bishop-de Leeuw) Theorem. See, e.g., (3). From the top down, we have in mind the disconcerting property that, for some K, extreme points are meanwhile dense, i.e., any $x \in K$ is arbitrarily close to an extreme point.

Definition 8.1 (Liminal C^*-algebras)
A C^*-algebra $\mathcal{X}$ is called liminal if, for every irreducible representation π of $\mathcal{X}$ and each $A \in \mathcal{X}$, $\pi(A)$ is compact.

All finite-dimensional C^*-algebras are of course liminal. All commutative C^*-algebras are also liminal. See [53, 4.2.1-4.2.2] or [89, Examples IV.1.3.3]. Note that the set of elements of a C^*-algebra $\mathcal{X}$ whose images under any irreducible representation are compact operators is the largest liminal closed two-sided ideal of $\mathcal{X}$, by [53, Proposition 4.2.6].

Later, Kaplansky and Glimm also introduced the term *GCR*[38] for a generalization of Definition 8.1, much later replaced by *postliminal* (see [53, Section 4.3.1] or [89, Section IV.1.3.1]). On the one hand, observe that the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators acting on a Hilbert space $\mathcal{H}$ is a closed two-sided ideal of the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators acting on $\mathcal{H}$. On the other hand, from a closed, self-adjoint two-sided ideal $\mathcal{I}$ of a C^*-algebra $\mathcal{X}$ and the quotient $\mathcal{X}/\mathcal{I}$, we can construct a C^*-algebra. Keeping this information in mind, the notion of postliminal C^*-algebras are defined as follows [53, Section 4.3.1]:

Definition 8.2 (Postliminal C^*-algebras)
A C^*-algebra $\mathcal{X}$ is postliminal if every non-zero quotient C^*-algebra of $\mathcal{X}$ possesses a non-zero liminal closed two-sided ideal.

All liminal C^*-algebras are postliminal, by [53, Proposition 4.2.4], but the converse is false.

Kaplansky and Glimm named important C^*-algebras that are not *GCR* (postliminal), *NGCR* C^*-algebras. Such algebras were later called *antiliminal* [53, Section 4.3.1] (see also [89, Section IV.1.3.1]):

Definition 8.3 (Antiliminal C^*-algebras)
A C^*-algebra $\mathcal{X}$ is antiliminal if the zero ideal is its only liminal closed two-sided ideal.

Remark that a quotient C^*-algebra of an antiliminal C^*-algebra is not antiliminal, in general.

If the image of $\mathcal{X}$ by an *irreducible* representation π would intersect the set of compact operators, then the set of compact operators would automatically be included in $\pi(\mathcal{X})$, by [53, Corollary 4.1.10]. In other words, the image of a C^*-algebra by an irreducible representation either contains the set of compact operators or does not intersect it. Antiliminal and separable C^*-algebras are related to the second situation [90, Theorem 1 (b)]:

Theorem 8.4 (Glimm)
Let $\mathcal{X}$ be a separable[39] *C^*-algebra. Then, the following conditions are equivalent:*
(i) *$\mathcal{X}$ is antiliminal.*
(ii) *$\mathcal{X}$ has a faithful type II representation.*
(iii) *$\mathcal{X}$ has a faithful type III representation.*
(iv) *$\mathcal{X}$ has a faithful representation which is a direct sum of a family of representations of $\mathcal{X}$ whose range does not contain the compact operators.*

Proof. By [53, Proposition 1.8.5], an antiliminal C^*-algebra does not possess any postliminal closed two-sided ideal, apart from the zero ideal. Therefore, the theorem is a direct consequence of [90, Theorem 1 (b)], keeping in mind that "completely continuous" in [90] is a synonym of "compact". ■

In other words, no irreducible representation of an antiliminal separable C^*-algebra $\mathcal{X}$ contains the compact operators on the representation space. Additionally, antiliminal C^*-algebras are directly

[38] The definition of *GCR* given in [89, Section IV.1.3.1] is different from the original one.
[39] In the non-separable situation, any of the assertions (ii)-(iv) yields (i), by [90, Theorem 1 (c)].

related with von Neumann algebras of type II and III, while postliminal C^*-algebras are directly associated with von Neumann algebras of type I, by [90, Theorem 1 (a), (c)].

Antiliminal (unital) C^*-algebras $\mathcal{X}$ have a set E of states with fairly complicated geometrical structure, similar to the Poulsen simplex [62]. Recall that E is a weak*-compact convex subset of $\mathcal{X}^*$ with (nonempty) set of extreme points denoted by $\mathcal{E}(E)$. See (2). Then, one has the following result [53, Lemma 11.2.4]:

Lemma 8.5 (Weak* density of the set of extremes states)
Let $\mathcal{X}$ be a antiliminal unital C^*-algebra. Assume that any two (different) non-zero closed two-sided ideals of $\mathcal{X}$ always have a non-zero intersection. Then $E = \overline{\mathcal{E}(E)}$, in the weak topology.*

C^*-algebras $\mathcal{X}$ relevant for mathematical physics often have a faithful type III representation. See, e.g., [91, Section 4] for UHF (uniformly hyperfinite) algebras. Note that every $*$-representation of a UHF algebra, like for instance a CAR algebra, is faithful. For key statements on representations of CAR C^*-algebras, see, e.g., [92, Theorem 2.4] or [93, Theorem 12.3.8] and references therein. Hence, by Theorem 8.4, many C^*-algebras $\mathcal{X}$ with physical applications are antiliminal. Note also that they are generally separable and *simple*:

Definition 8.6 (Simple C^*-algebras)
A C^*-algebra $\mathcal{X}$ is simple if the only closed two-sided ideals of $\mathcal{X}$ are the trivial sets $\{0\}$ and $\mathcal{X}$.

C^*-algebras $\mathcal{B}(\mathcal{H})$ of all (bounded) linear operators acting on some finite-dimensional Hilbert space $\mathcal{H}$ are of course simple. See, e.g., [53, Corollary 4.1.7]. However, finite-dimensional C^*-algebras are not generally simple, but semisimple only, as direct sums of simple algebras.

In mathematical physics, unital C^*-algebras of infinitely extended (quantum) systems are usually built from a family of local finite-dimensional C^*-subalgebras. It refers to approximately finite-dimensional (AF) C^*-algebras, originally introduced in 1972 by Bratteli [94]. See also the quasi-local algebras [36, Definition 2.6.3]. AF C^*-algebras used in physics are usually simple, by [36, Corollary 2.6.19], because they are generally constructed from simple local algebras (typically as some inductive limit, with respect to boxes Λ, of a family of C^*-algebras $\mathcal{B}(\mathcal{H}_\Lambda)$, with $\dim\mathcal{H}_\Lambda < \infty$, like, for instance, Cuntz, lattice CAR or quantum-spin C^*-algebras). Therefore, by Lemma 8.5, for infinitely extended quantum systems, like fermions on the lattice or quantum-spin systems, the corresponding set E of states has a *dense* subset of extreme points. This fact is well-known and already discussed in [94, p. 226]. See also [36, Example 4.1.31] for a direct proof in the context of the so-called UHF (uniformly hyperfinite) C^*-algebras [36, Examples 2.6.12].

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Jean-Bernard Bru
Departamento de Matemáticas
Facultad de Ciencia y Tecnología
Universidad del País Vasco
Apartado 644, 48080 Bilbao

BCAM - Basque Center for Applied Mathematics
Mazarredo, 14.
48009 Bilbao

IKERBASQUE, Basque Foundation for Science
48011, Bilbao

Walter de Siqueira Pedra
Departamento de Física Matemática
Instituto de Física,
Universidade de São Paulo
Rua do Matão 1371
CEP 05508-090 São Paulo, SP Brasil