OFFSETS AND FRONT TIRE TRACKS TO PROJECTIVE HEDGEHOGS

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ABSTRACT. There are some properties on curves of constant width and Zindler curves and their relationship with offsets and front tire-track curves which are already known. In this work, a generalization of all these concepts and results to hedgehogs is presented. Since hedgehogs usually have cusp singularities, the common parametric definition of offsets and front track curves induces some discontinuity issues. A constructive approach to a continuous parametric definition is given based on parameterizations by a support function. Finally, it is seen that there is a one-to-one correspondence between hedgehogs of constant width and generalized Zindler curves.

1. INTRODUCTION

There are many famous shapes in plane geometry that are generated by the motion of a constant length chord around a given curve. Two important examples of such curves are offset curves and front tire-track curves.

Offset (or parallel) curves are those which are at a constant distance from another in an orthogonal direction, while front wheel track curves are those which are found at a constant distance from another curve (the rear wheel track curve) in a tangential direction (see Figure 1). The movement of a bicycle, represented as an oriented segment in the plane, can be described by the trajectories of these rear and front wheel tire-track curves.

Offset curves have a wide range of applications, from Computer Aided Geometric Design (CAGD) and numerical-control machining to robot path-planning (see [6] or [7] and the references therein). The study of the front and rear track curves, besides its obvious engineering applicability, has a strong relationship with other famous problems in Physics such as the floating body problem or the motion of an electron in a parabolic magnetic field (see e.g. [3]).
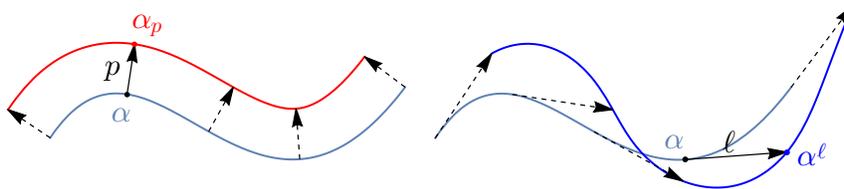
FIGURE 1. On the left, an offset α_p to a curve α at a distance p. On the right, the front wheel track curve α^ℓ to a rear wheel track curve α for a bicycle of length ℓ.

In this work we will consider hedgehogs, which are curves defined by the envelope of their inverse Gauss map. Hedgehogs constitute a generalization of convex curves in the plane (see e.g. [12], [14] and [15] for an introduction to hedgehogs).

If the initial curve exhibits some isolated singularities, as it happens with generic non-convex hedgehogs, then the usual parametric representation of offsets and front track curves is not appropriate because they lead to discontinuous curves. The common procedure to construct a continuous offset from the discontinuous one goes through adding some circular arcs centered at the singular points. Here we will also describe another approach to get continuous offsets that, in addition, will extend naturally some properties of offsets to convex curves. The same will be done for front track curves.

The generation of offsets and front track curves has a lot to do with two famous shapes in convex geometry, which are curves of constant width and Zindler curves (see [16] for an introduction to these curves and their applications).

The aim of this paper is to present a detailed, accessible and self-contained introduction, addressed to a broad scope, to all these concepts and their relationship in the general case of hedgehogs with no convexity constraints. Most of the results presented here might be known or "expected", but as far as the author knows, there is no single reference where this approach has been fully developed. In any case, this work can be thought of as a survey to get introduced into the topic.

In the first part of the paper, we present the parametric representation of a hedgehog by a support function and the corresponding generalization of constant with curves to hedgehogs of constant width (Definition 1). Hedgehogs of constant width satisfy the orthogonality property we have in the convex case (Proposition 1). Analogously, the corresponding generalization of Zindler curves is also presented (Definition 6).

Non-convex hedgehogs are likely to have cusps. As commented above, in these cases the parametric representation of offsets and front tracks is not appropriate. To solve this problem, a continuous definition of offset curves

and front track curves is presented based on a continuous vector field along the initial curve described by a support function.

It is known that there is a "duality" between constant width curves and Zindler curves. This duality can be written in terms of a projective hedgehog, which is a Wigner caustic, and the generation of offsets and front track curves. We show that the same duality is also valid for the corresponding extended notions to hedgehogs and that it induces a one-to-one correspondence between hedgehogs of constant width and generalized Zindler curves (Theorem 1).

We will assume along the paper that the involved curves are piecewise-differentiable with a sufficiently high order of differentiability without making explicit mention to it.

2. Hedgehogs of constant width and offsets

A closed curve $\alpha : [0, 2\pi] \to \mathbb{R}^2$ is said to be *parameterized by a support function h* if it can be written as

$$\alpha(t) = h(t) \left(\cos t,\, \sin t\right) + h'(t) \left(-\sin t,\, \cos t\right),$$

where h is a 2π-periodic function. The curve α is called a hedgehog [14], which is a representation of α as the envelope of its family of supporting lines. These lines are defined by a normal vector $(\cos t,\, \sin t)$ and the support function $h(t)$ is the signed distance from the origin to the corresponding supporting line (see Figure 2).
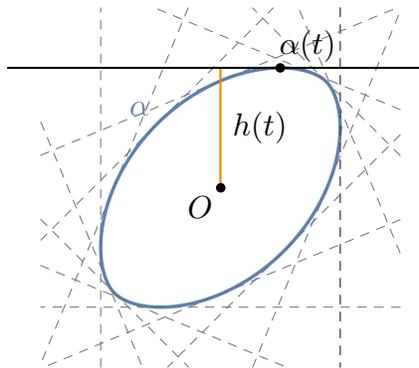

Figure 2. A convex curve α parameterized by a support function h. It is the envelope of its family of supporting lines.

It is well known that every regular convex curve can be written in terms of a support function and, therefore, is a hedgehog. In fact, since

(1) $$\alpha'(t) = \left(h(t) + h''(t)\right)\left(-\sin t,\, \cos t\right),$$

the condition of being α a convex (and regular) curve is simply that $h(t) + h''(t)$ has no zero.

Notice that if α is not convex, from Equation (1) we have that the tangent vector of α will be change from $(-\sin t, \, \cos t)$ to $(\sin t, \, -\cos t)$, or vice versa, when crossing each singular point of α. Of course, the corresponding normal vector of α will change too and, therefore, it will be equal either to the normal vector $(\cos t, \, \sin t)$ that describes the family of supporting lines or its opposite. See in Figure 3 two examples of non-convex hedgehogs.
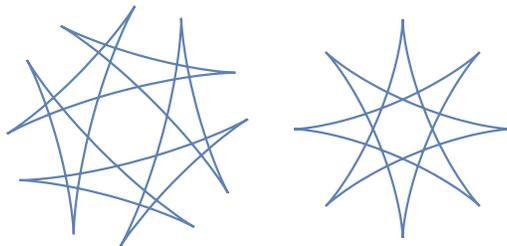

FIGURE 3. Two examples of non-convex hedgehogs. Hedgehogs have one and only one normal vector in each direction.

A very interesting particular case of convex curves are those of constant width [24]. These are curves in which the distance (width) between any pair of parallel supporting lines to the curve is constant. This constant distance, say d, is given by

$$h(t) + h(t + \pi) = d$$

from the definition of a support function.

If the curve α is regular, recall that the supporting lines to α are also tangent lines.

2.1. **Hedgehogs of constant width.** When one speaks about a constant width curve, it is implicitly assumed that the curve must be convex. In fact, it is a direct consequence from the definition of width in terms of pairs of supporting lines. However, there are some non-convex curves that, as Kelly pointed out in [9], have "some kind of constant width". It is the case for example of the curve of Figure 4.

Here, a constant length chord can travel orthogonally along the closed curve until a full revolution. The chord length does not represent the width of the curve in its proper sense but it represents some constant "width" which is maintained along the curve. Notice that this "width" is given as the distance between pairs of parallel tangent lines (rather than supporting lines) at the chord endpoints. For a hedgehog α parameterized by a support function, these points are precisely $\alpha(t)$ and $\alpha(t + \pi)$.
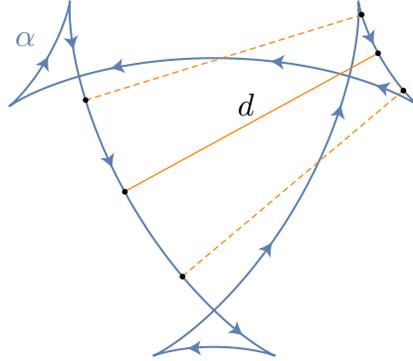
FIGURE 4. A non-convex hedgehog α with a kind of constant width d.

The constant distance property above can also be written in terms of the support function and it gives rise to the definition of a hedgehog of constant width [13].

Definition 1. A *hedgehog* parameterized by a support function h is *of constant width d* if

$$h(t) + h(t + \pi) = d.$$

Recall that a hedgehog α is called *projective* if $h(t) + h(t + \pi) = 0$. This is, projective hedgehogs are those of zero constant width.

Now we can show what we observed in the example of Figure 4: the chord which measures the constant "width" in a hedgehog is orthogonal to the pair of tangent lines. This generalizes the well-known analogous property for convex constant width curves (see e.g. [24]).

Proposition 1. *In a non-projective hedgehog of constant width parameterized by a support function, the chord connecting the contact points of any pair of parallel tangent lines is orthogonal to these lines and hence has length the constant width of the hedgehog.*

Proof. Let α be a hedgehog of constant width d parameterized by a support function h. It is easy to see that

$$\alpha(t + \pi) - \alpha(t) = \big(h(t) + h(t + \pi)\big)(-\cos t, -\sin t)$$

(2)
$$+ \big(h'(t) + h'(t + \pi)\big)(\sin t, -\cos t).$$

Since α is of constant width d, we have

$$h(t) + h(t + \pi) = d.$$

Therefore, the expression (2) turns into

$$\alpha(t + \pi) - \alpha(t) = -d(\cos t, \sin t),$$

which is orthogonal to a vector in the direction $(-\sin t, \cos t)$ defined by the parallel tangent lines.   □

Notice that at the points $\alpha(t)$ and $\alpha(t + \pi)$ the normal (and tangent) vectors to α are not necessarily opposite as it is shown in Figure 5 (this is, the pairs of points are not necessarily antipodal as it happens with convex curves). However, notice that the normal vector that defines the set of tangent lines, namely $(\cos t, \sin t)$, does satisfy such a property.
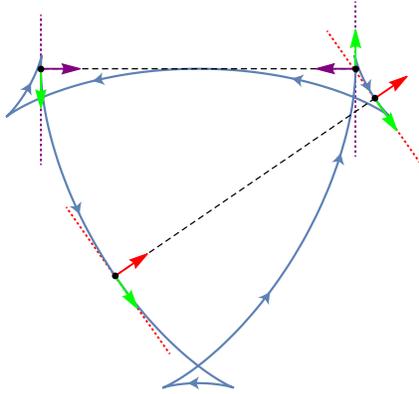

FIGURE 5. The normal vectors to a hedgehog of constant width at pairs of points with parallel tangent lines are not necessarily opposite.

2.2. **Offset curves.** An offset to a curve α at a distance p is defined as the envelope of a family of circles of radius p centered at each point of the curve. This construction actually yields a two-sided (or double) offset, which usually consists of two connected components (or curves). If α is differentiable, each of these two connected components, usually called *inner* and *outer offset* can be parameterized as

$$(3) \qquad \alpha_{\pm p}(t) = \alpha(t) \pm p\,\mathbf{n}(t),$$

where $\mathbf{n}$ is the normal vector of α and the sign $\pm$ determines the offset side which is parameterized.

But things are not always so easy. The appearance of isolated singularities in the initial curve α can be a problem when computing offset curves. This produces a discontinuous parametric offset at some distance p, which is usually not desired. A common way to overcame this problem is by introducing circular arcs of radius p centered at each singularity (see Figure 6). The resulting curve is called *untrimmed offset* (see [6] for more details).

Generic singularities of hedgehogs are cusps [10]. As we have pointed out above, when crossing each cusp of a hedgehog α, since the normal vector change to its opposite direction, to each side of α a discontinuous offset
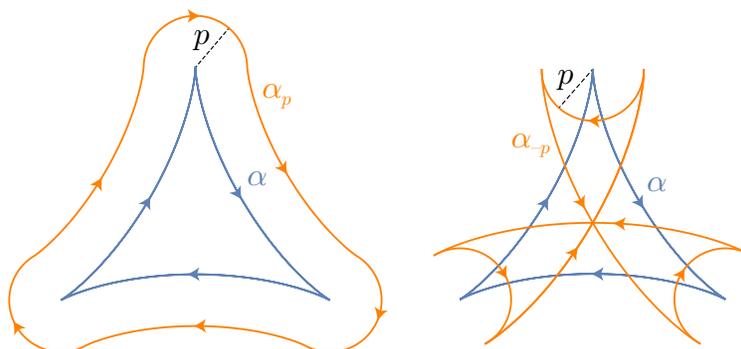
FIGURE 6. An untrimmed offset α_p at a distance p to a curve α with cusps. At each singularity, a circular arc is introduced to produce a continuous offset.

curve is produced. However, notice that the shape determined by the two branches (double offset) consists of to two continuous figures (see Figure 7).
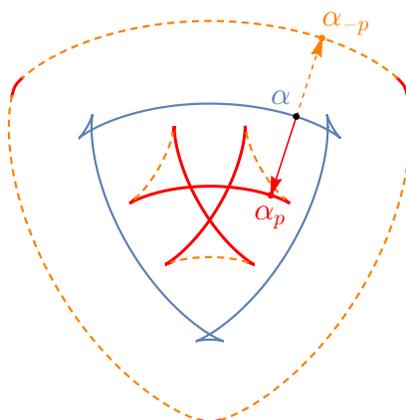

FIGURE 7. Double parametric offset to α (one of its sides dash plotted).

This gives rise to another point of view, which is that each connected component can be regarded as an "actual" inner and outer offset curve to α. Instead of adding any circular arc to the parametric inner and outer offset given by (3) to produce a continuous offset, we can consider the connected components of the double offset.

Each connected component of the double offset may be constructed by pieces of both, inner and outer parametric offsets. But is there any way to generate each connected component of the double offset as a non-piecewise parametric curve? The answer is affirmative, thanks to the parameterization by a support function. Here, the normal vector field given by $(\cos t, \sin t)$

is continuous and, therefore, does not change its direction when crossing each cusp. Instead, a curve generated by this normal vector field generates alternatively pieces of both sides of the double offset in a continuous manner. This produces one of the connected components of the double offset. See in Figure 8 some examples.
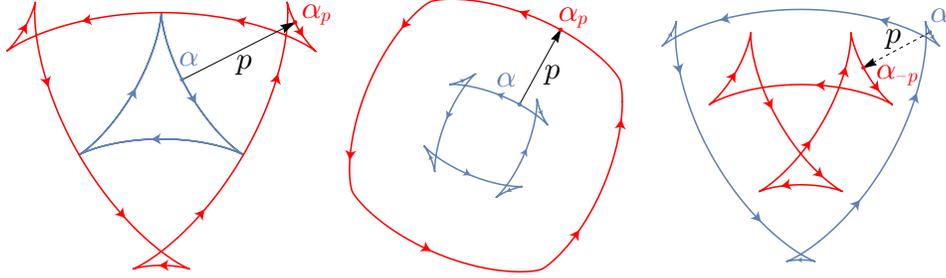

FIGURE 8. Some examples of continuous parametric offsets defining a connected component. Two outer offsets in the first two figures and an inner offset on the third.

The explicit parameterization of each connected component of the double offset to a curve is given in the next definition. From now on, when we speak about *an offset to a curve* we will refer to one of these two connected components as a parametric curve (and not to the classical parametric definition of an offset presented above).

Definition 2. Let α be a hedgehog parameterized by a support function h. An *offset to α at a distance p* is the parametric curve defined by

$$\alpha_{\pm p}(t) = \big(h(t) \pm p\big)(\cos t,\, \sin t) + h'(t)(-\sin t,\, \cos t),$$

where the choice of the sign $\pm$ determines the connected component of the double offset which is under consideration. These two components will also be referred as the *outer* and *inner offset* to α, respectively.

Now we can show that the offset to a constant width hedgehog is again a constant width hedgehog. The analogous property is well-known for convex curves, but notice that even in such a case, the inner offset to a convex curve is not necessarily convex, so that this property shall be understood in terms of hedgehogs.

Proposition 2. *Any offset at a distance p to a hedgehog of constant width d is also of constant width, being this width equal to $d \pm 2\,p$, where the sign $\pm$ is positive for outer offsets and negative for inner offsets.*

Proof. Let α be a hedgehog of constant width d parameterized by a support function h. We have that

$$h(t) + h(t + \pi) = d.$$

The offset curve to α at a distance p can be easily parameterized by a support function $h(t) \pm p$ as in Definition 2:

$$\alpha_{\pm p}(t) = \big(h(t) \pm p\big)(\cos t,\, \sin t) + h'(t)\,(-\sin t,\, \cos t).$$

Thus, the width of offset curve $\alpha_{\pm p}$ is

$$\big(h(t) \pm p\big) + \big(h(t + \pi) \pm p\big) = d \pm 2\,p,$$

which is constant. $\qquad\square$

As a trivial consequence of Proposition 2 we get the following result.

Proposition 3. *Any offset to a projective hedgehog at a distance d is a hedgehog of constant width $2\,d$.*

Now we can introduce the concept of a *middle hedgehog*. It is usually defined as the curve formed by the midpoints of all affine diameters of a convex body [18] (see also the survey [19] on affine diameters). A particular case considering tangent lines rather than supporting lines is the Wigner caustic [4]. The extended definition to hedgehogs can be written as follows.

Definition 3. The locus of the midpoints of chords connecting pairs of points with parallel tangents to a non-projective hedgehog α is called the *Wigner caustic of α*.

See in Figure 9 two examples of the Wigner caustic of a constant width hedgehog. Notice that they are projective hedgehogs. This is not by chance.
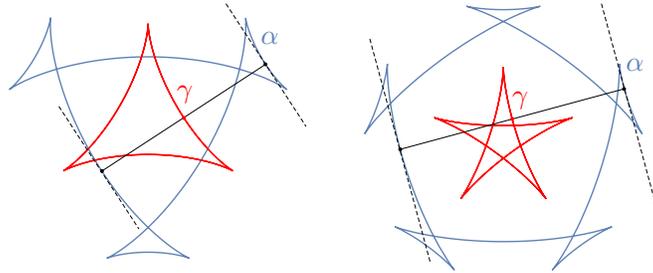

FIGURE 9. Examples of hedgehogs α of constant width and their Wigner caustic γ.

Proposition 4. *The Wigner caustic of a constant width hedgehog is a projective hedgehog.*

Proof. Notice that, by definition, the Wigner caustic of a hedgehog of constant width d is an inner offset at a distance $\frac{d}{2}$. By Proposition 2, we have that it is a hedgehog of zero constant width, which means that it is projective. $\qquad\square$

The number of cusps of a Wigner caustic of a convex curve is odd and not less than three (see [4] and [2]). In fact, any projective hedgehog α defined by a support function of class $\mathcal{C}^2$ has an odd number of cusps.

Thanks to the properties seen above, the construction of hedgehogs (and convex curves) of constant width can be done by offsetting a projective hedgehog γ, which can be parameterized by a support function h:

$$\gamma(t) = h(t)\,(\cos t,\,\sin t) + h'(t)\,(-\sin t,\,\cos t).$$

The offsets to γ at a distance d are then hedgehogs of constant width $2\,d$:

$$\alpha_d(t) = \big(h(t) \pm d\big)(\cos t,\,\sin t) + h'(t)\,(-\sin t,\,\cos t).$$

3. Generalized Zindler curves and front tire tracks

The motion of a bicycle is usually represented in the plane by an oriented segment that travels such that the trajectory of the front endpoint is on the tangent direction of the trajectory of the rear endpoint [20]. Given a rear tire track, there are two possible front tracks, depending if we move forward or backward. These can be parameterized as follows.

Definition 4. Given a regular curve α, a *front wheel tire-track curve to α for a bicycle of length ℓ* is a curve parameterized by

$$\alpha^{\pm\ell}(t) = \alpha(t) \pm \ell\,\mathbf{t}(t),$$

where $\mathbf{t}$ is the tangent vector of α and the sign $\pm$ is positive for the *forward front tire track* and negative for the *backward front tire track*. To shorten, we will say that $\alpha^{\pm\ell}$ is a *front track curve to α at a distance $\pm\ell$*. The shape generated by both curves is called the *double front tire track*.

A famous problem in this setting is: given the closed rear and front tracks of a bicycle, is it possible to determine the direction in which the bicycle went?

As we have seen, a rear track α determines two front tracks which, in general, produce different curves (see Figure 10).

When the same curve is generated by the forward and backward front track, say β, it is said that the pair of curves α and β is *ambiguous*. In these cases, one cannot determine the direction in which a bicycle went. This problem, called the "ambiguous tire track problem", has equivalent formulations in other physical contexts, such as the 2D floating bodies in equilibrium problem [3]. The interested reader can also see the works of Wegner, e.g. [21], [22] or [23].

Zindler curves [25] are those in which all chords that cut the curve perimeter into halves have the same length. These chords of constant length are called *halving chords*. Zindler curves serve as solutions to the 2D floating
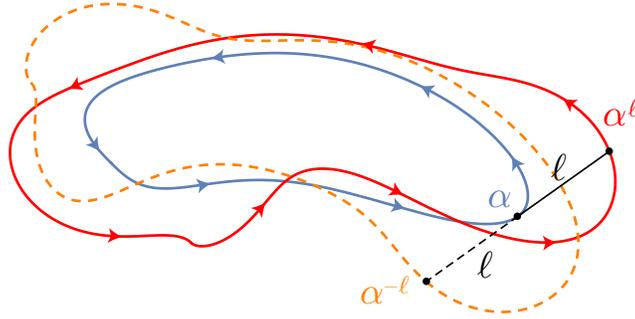
FIGURE 10. A rear track α determines two front tracks α^{ℓ} and $\alpha^{-\ell}$.
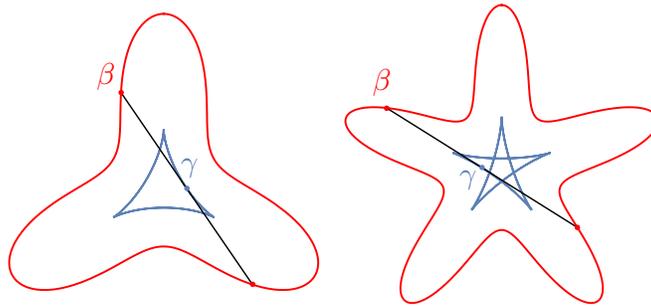

FIGURE 11. A Zindler curve β and the locus γ of midpoints of the halving chords. The pair of curves β and γ is ambiguous.

body problem and they, together with the curve generated by the midpoint of the halving chords, are pairs of ambiguous tire-track curves (see Figure 11).

We are interested in front track curves to hedgehogs. If the rear track curve γ has cusps, then a similar problem to that of offsets happens: the tangent vector of γ reverses its direction when crossing each cusp, so that discontinuities happen. Each curve of the double front tire track do not correspond to a forward or backward front tire track, instead, it is constructed by pieces of both front tracks (see Figure 12).

Similarly as with offset curves, we can overcome this issue using parameterizations by a support function. This gives rise to the following definition of front tracks for hedgehogs.

Definition 5. Let α be a hedgehog parameterized by a support function h. A *front wheel tire-track curve to α for a bicycle of length ℓ* is the parametric curve defined by

$$\alpha^{\pm\ell}(t) = h(t)\left(\cos t,\, \sin t\right) + \left(h'(t) \pm \ell\right)\left(-\sin t,\, \cos t\right),$$
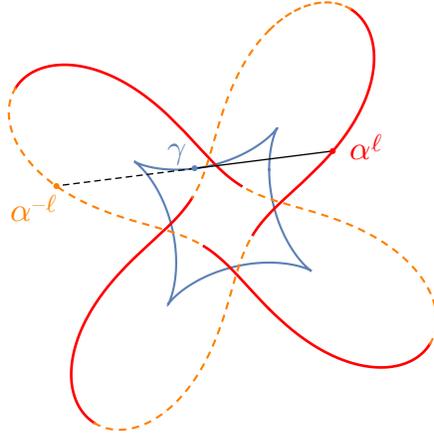
FIGURE 12. Double (forward and backward) front wheel tire track to a curve γ with cusps. The backward front track is dash-plotted.

where the choice of the sign $\pm$ determines the trajectory of the double front tire track which is under consideration. These two trajectories will be referred as the *forward* and *backward front tracks to α*. To shorten, we will say that $\alpha^{\pm\ell}$ is the front track curve to α at a (signed) distance $\pm\ell$. The shape generated by both curves will be called the *double front tire track*.

Notice that the parameterization of front tracks are based on a support function but they are not parameterized by it.

A common constraint for the definition of Zindler curves is that the halving chords cut the curve at precisely two points (and not more). That is, these chords of constant length must lie in the interior of the curve. Mampel in [11] considered generalized Zindler curves dropping this constraint and gave some examples of these curves that do not respect any convexity. The set of generalized Zindler curves we will consider here are described in the next definition. For simple curves, it corresponds to the definition given in the paper [5], where the authors refers to them as *closed simple curves of constant halving distance*.

Definition 6. A regular closed curve α is called a *generalized Zindler curve* if there is a continuous motion of a constant length chord with its endpoints along α such that the length of α is split into two halves by these endpoints.

See in Figure 13 some examples of generalized Zindler curves which are not Zindler curves, as the constant length chord intersects the curve at more than three points.
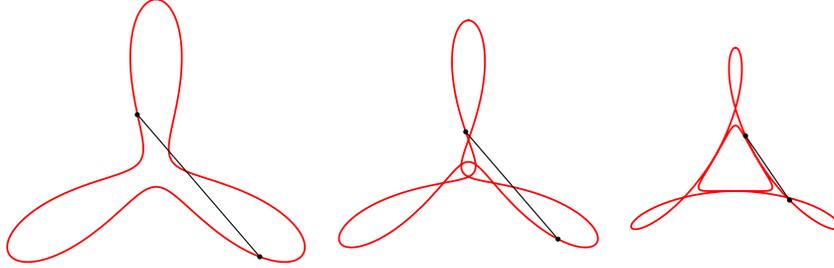
FIGURE 13. Examples of generalized Zindler curves which are not Zindler curves. A halving chord is displayed.

Proposition 5. *Any front track curve from a projective hedgehog is a generalized Zindler curve.*

Proof. Let γ be a projective hedgehog parameterized by a support function h:
$$\gamma(t) = h(t)\,(\cos t,\, \sin t) + h'(t)\,(-\sin t,\, \cos t).$$
Let α be the front track curve to γ at a signed distance $\pm\ell$:
$$\alpha(t) = h(t)\,(\cos t,\, \sin t) + \big(h'(t) \pm \ell\big)\,(-\sin t,\, \cos t).$$
A straightforward computation shows that
$$\alpha(t+\pi) - \alpha(t) = -\big(h(t) + h(t+\pi)\big)\,(\cos t,\, \sin t)$$
$$- \big(h'(t) + h'(t+\pi) \pm 2\,\ell\big)\,(-\sin t,\, \cos t).$$
Since $h(t) + h(t+\pi) = 0$, we have that
$$\alpha(t+\pi) - \alpha(t) = \pm 2\,\ell\,(\sin t,\, -\cos t).$$
Therefore, the length of the chords joining the points $\alpha(t)$ and $\alpha(t+\pi)$ is constant:
$$\big\|\alpha(t+\pi) - \alpha(t)\big\| = 2|\ell|.$$

Moreover, since
$$\alpha'(t) = \mp\ell\,(\cos t,\, \sin t) + \big(h(t) + h''(t)\big)\,(-\sin t,\, \cos t)$$
and α is projective, we have
$$\big\|\alpha'(t)\big\| = \sqrt{\ell^2 + \big(h(t) + h''(t)\big)^2} = \big\|\alpha'(t+\pi)\big\|.$$
Thus, $\big\|\alpha'(t)\big\|$ is π-periodic and we deduce that
$$\int_{t_0}^{t_0+\pi} \big\|\alpha'(t)\big\|\,\mathrm{d}t = \int_0^\pi \big\|\alpha'(t)\big\|\,\mathrm{d}t = \frac{L}{2},$$
for any $t_0 \in [0, \pi]$, where L is the length of α. This means that the set of chords of constant length also cut the perimeter of α into halves. $\square$

A characterization for simple closed curves of constant halving distance can be given in terms of its midpoint curve (see [5] and [8]). Next, we state the analogous implication for generalized Zindler curves that we will use later. The proof is essentially the same as the given in the characterization mentioned above. We will reproduce it here to make the paper self-contained.

Lemma 1. *The midpoint curve of a generalized Zindler curve is the envelope of its halving chords.*

Proof. Let β be a piecewise-differentiable generalized Zindler curve. Suppose without loss of generality that $\beta : I \to \mathbb{R}^2$ is arc-length parameterized, so that $\left\| \beta(s) \right\| = 1$ for all $s \in I$. Denote by L the length of β. The halving chords are given by

$$b(s) = \beta(s) - \beta\left(s + \frac{L}{2}\right),$$

which are of constant length. Thus, $b'(s)$ is orthogonal to $b(s)$.

The midpoint curve m is

$$m(s) = \frac{1}{2}\left(\beta(s) + \beta\left(s + \frac{L}{2}\right)\right).$$

Hence,

$$\langle m'(s),\, b'(s)\rangle = \frac{1}{2}\left\langle \beta'(s) + \beta'\left(s + \frac{L}{2}\right),\, \beta'(s) - \beta'\left(s + \frac{L}{2}\right)\right\rangle$$

$$= \frac{1}{2}\left(\left\|\beta'(s)\right\|^2 - \left\|\beta'\left(s + \frac{L}{2}\right)\right\|^2\right) = 0.$$

This means that the tangent vector of the midpoint curve at $m(s)$ is on the direction of the halving chord $b(s)$. Since $m(s)$ is the midpoint of such a chord, we conclude that m is the envelope of the halving chords. $\square$

Mampel in [11] gave an example of a generalized Zindler curve whose midpoint curve is not a hedgehog, that is, there were more than one halving chord with the same normal direction. We will focus on generalized Zindler curves such that for each normal direction there is one and only one halving chord. These curves will be called *standard generalized Zindler curves*.

Proposition 6. *The midpoint curve of a standard generalized Zindler curve is a projective hedgehog.*

Proof. By Lemma 1, the midpoint curve m of the halving chords of a standard generalized Zindler curve β is given by its envelope. Since for each direction there is one and only one halving chord, we have that the curve m

can be parameterized by a support function h as a hedgehog $m : [0, 2\pi] \rightarrow \mathbb{R}^2$ by

$$(4) \qquad m(t) = h(t)\,(\cos t,\, \sin t) + h'(t)\,(-\sin t,\, \cos t).$$

If the halving chords have length $2\,\ell$, then the curve β can be parameterized by

$$\beta(t) = h(t)\,(\cos t,\, \sin t) + \big(h'(t) + \ell\big)\,(-\sin t,\, \cos t)$$

and the pairs of halving points are $\beta(t)$ and $\beta(t + \pi)$.

Finally, we have that

$$m'(t) = \big(h(t) + h''(t)\big)\,(-\sin t,\, \cos t)$$

and

$$\beta(t + \pi) - \beta(t) = -\big(h(t) + h(t + \pi)\big)\,(\cos t,\, \sin t)$$
$$- \big(h'(t) + h'(t + \pi) + 2\,\ell\big)\,(-\sin t,\, \cos t)$$

are parallel if and only if $h(t) + h(t + \pi) = 0$. Therefore, the hedgehog m must be projective. $\qquad\square$

Now we can state the duality between hedgehogs of constant width and standard generalized Zindler curves.

Theorem 1. *There is a one-to-one correspondence between hedgehogs of constant width and standard generalized Zindler curves.*

Proof. Given a hedgehog of constant width d, we know that its midpoint curve m is a projective hedgehog by Proposition 4. The front track curve to m for a bicycle of length $\frac{d}{2}$ is a generalized Zindler curve for halving chords of length d by Proposition 5. It is a standard generalized Zindler curve by construction. The reverse procedure can also be done unequivocally. Given a standard generalized Zindler curve, its midpoint curve m is a projective hedgehog by Proposition 6. Now, the offset to m at a distance $\frac{d}{2}$ is a hedgehog of constant width d. $\qquad\square$

The "duality" between classical curves of constant width and Zindler curves has been stated before (see [8], [1], [16] or [17]). The geometric construction of this duality is given by simply rotating the constant length chords around their midpoint a right angle. The endpoints of these two chords (the original and the rotated one) describe the pairs of dual curves (see Figure 14).

An important detail, which is taken into account in Corollary 3.11 of [8], is that one of the directions of the duality in the classical setting might "fail": given a classical Zindler curve, the construction above (rotation by a right angle the halving chords around their midpoint) produces a curve of constant width **if it is convex**. Sometimes the resulting curve is not convex, so that the correspondence cannot be set in this manner (see Figure 15). With
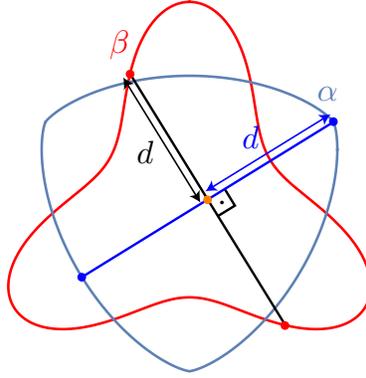
FIGURE 14. A curve α of constant width $2\,d$ and its dual
Zindler curve β for halving chords of constant length $2\,d$.

the extended notions of constant width and generalized Zindler curves to
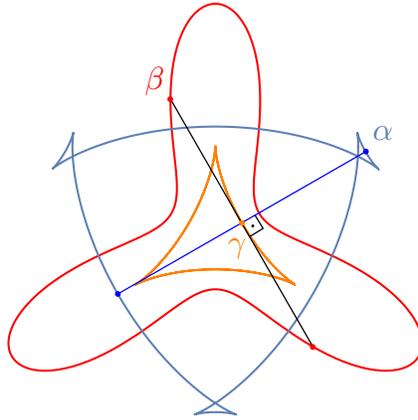hedgehogs, this duality can always be considered.

FIGURE 15. A Zindler curve β and its dual curve α, which is
not convex. The curve α is actually a hedgehog of constant
width generated as an offset from the projective hedgehog γ.

The same discussion done for offset curves above is applicable here. In
order to generate the full shape of a generalized Zindler curve from a projec-
tive hedgehog a double front tire track must be considered. Equivalently, we
can parameterize a standard generalized Zindler curve continuously thanks
to parameterizations by a support function. If γ is a projective hedgehog
parameterized by a support function h, then

$$\beta_{\pm d}(t) = h(t)\,(\cos t, \sin t) + \big(h'(t) \pm d\big)\,(-\sin t, \cos t)$$

is a generalized Zindler curve. Notice that $\beta_{\pm d}$ is not parameterized by a
support function.

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