A \textit{b}-symplectic slice theorem

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Funding information
Spanish State Research Agency, Grant/Award Numbers: CEX2020-001084-M, PID2019-103849GB-I00/AEI/10.13039/501100011033; Severo Ochoa Program, Grant/Award Number: SEV-2017-0718; Basque Government, Grant/Award Number: 2018-2021; Foundation Sciences Mathématiques de Paris; French National Research Agency, Grant/Award Number: ANR-10-LABX-0098; ICREA, Catalan Institution for Research and Advanced Studies, Grant/Award Number: ICREA2016; ICREA, Catalan Institution for Research and Advanced Studies, Grant/Award Number: ICREA2021

Abstract
In this article, motivated by the study of symplectic structures on manifolds with boundary and the systematic study of \textit{b}-symplectic manifolds started in Guillemin, Miranda, and Pires Adv. Math. 264 (2014), 864–896, we prove a slice theorem for Lie group actions on \textit{b}-symplectic manifolds.

MSC 2020
53D17, 53D20 (primary)

1 | INTRODUCTION

The linearization of Lie group actions for compact groups in a neighbourhood of a fixed point is due to Bochner [3]. It gives a precise description of the local normal form for a Lie group action.
in the neighbourhood of a fixed point. The next level of difficulty in describing group actions is in the neighbourhood of an invariant submanifold for the action: It was not until the work of Palais in the 1960s that such a portrayal was achieved [29, 30]. The description of the normal form in this case is semilocal (in a neighbourhood of an orbit) and it is done in terms of the action of the group along the orbit and on the linearization of its (orthogonal) complement, which Palais denominated ‘slice’.

The existence of slices re-conducts the computation of the orbits for the action in terms of its normal space on which it acts linearly. When additional geometrical structures are added into the picture, the existence of normal forms gains interest as it can often be adapted to the new ingredient (the geometric structure). This is the case of symplectic structures where Lie group actions are naturally related to the investigation of Hamiltonian symmetries.

In particular, symplectic slice theorems give equivariant normal forms around orbits of symplectic group actions and become particularly handy when computing the orbits of fundamental vector fields of Hamiltonian actions. For example, the Marle–Guillemin–Sternberg normal form [18, 22] or its generalizations have been used extensively to study the local structure of symplectic manifolds with symmetries.

The purpose of this article is to extend these results to the singular set-up, more precisely to \( b \)-symplectic manifolds. In this singular framework, the motivation to find equivariant normal forms comes from the study of symmetric manifolds with boundary. These singular symplectic structures have been intensely studied since their introduction in [12]. A study of their geometry in the presence of symmetries was initiated in [13] (see also [10]) which gave global results on the structure of \( b \)-symplectic manifolds with toric symmetries and also semilocal models.

In [19], two of the authors in this paper described integrable systems as \( b \)-cotangent lifts of rotations on a Liouville torus to its cotangent bundle. These can be understood as semilocal models for free actions of tori (associated to integrable systems). Motivated by these models in the integrable case, we plan to give linearized models for general actions of Lie groups in the language of the symplectic slice theorem. Many examples motivating the study of more general symmetries comes from the study of non-commutative integrable systems on \( (b) \)-symplectic manifolds as considered in [20] and [4].

These cotangent lifts appear naturally on the study of geodesic flows: A \( P \)-manifold is a Riemannian manifold \( M \) with all the geodesics closed. Two-dimensional examples of \( P \)-manifolds are Zoll and Tannery surfaces (see [2, Chapter 4]). In this case, the geodesics admit a common period (see [2, Lemma 7.11]) inducing an \( S^1 \)-action on \( M \). In the same way that the standard cotangent lift induces a Hamiltonian action on \( T^* M \), we can use the twisted \( b \)-cotangent lift to obtain a \( b \)-Hamiltonian \( S^1 \)-action on \( T^* M \). In this case, the action is given as a twisted \( b \)-cotangent lift which is a ‘linear model’ of the \( b \)-Poisson structure parametrized by an additional invariant: a constant \( c \). This constant is the modular period of the structure.

The \( b \)-symplectic slice theorem gives a normal form for a \( b \)-symplectic form via the symplectic slice theorem, which we recall here (for details on the constructions, see [17] for the Hamiltonian case or [28] for the more general symplectic case):

**Theorem 1.** Let \((M, \omega)\) be a symplectic manifold and let \( H \) be a Lie group acting properly and by symplectomorphisms on \( M \). Let \( z \in M \). Denote the isotropy group of \( z \) by \( H_z \) and the orbit of \( m \) by \( \mathcal{O}_z^H \). Let \( V_z \) be the symplectic normal space

\[
V_z := \left( T_z \mathcal{O}_z^H \right)^{\omega} / \left( \left( T_z \mathcal{O}_z^H \right)^{\omega} \cap T_z \mathcal{O}_z^H \right).
\]
Let \( h \) be the Lie algebra of \( H \) and consider the following subalgebra of \( h \),

\[
\mathfrak{k} := \left\{ \eta \in h \mid \eta_M(z) \in (T_z \mathfrak{g}_z^H)^\omega \right\},
\]

where \( \eta_M \) is the generating vector field of \( \eta \). Let \( i \) be the Lie algebra of \( H_z \) and note that \( i \subset \mathfrak{k} \). Denote by \( m \) an \( \text{Ad}_{H_z} \)-invariant complement of \( i \) in \( \mathfrak{k} \). Then the twisted product

\[
Y_z^H := H \times_{H_z} (m^* \times V_z)
\]

is a symplectic \( H \)-space and can be chosen such that there is an \( H \)-invariant neighbourhood \( U \) of \( z \) in \( M \), an \( H \)-invariant neighbourhood \( U' \) of \([e,0,0]\) in \( Y_z^H \) and an equivariant symplectomorphism \( \phi : U \to U' \) satisfying \( \phi(z) = [e,0,0] \). Equipping the bundle \( Y_z^H \) with coordinates \([k,\eta,v]\) for \( k \in H, \eta \in m^* \) and \( v \in V_z \), \( H \) acts on \( Y_z^H \) as \( h \cdot [k,\eta,v] = [h \cdot k, \eta, v] \).

**Remark 2.** The symplectic form on the quotient bundle is called the MGS-symplectic form\(^\dagger\) and it is denoted by \( \omega_{\text{MGS}} \). This symplectic form constructed as explained above admits an explicit expression in terms of the Lie algebra decomposition associated to the isotropy group (confer \([28]\)). A particularly important class of symplectic actions are Hamiltonian actions when the action comes associated with a moment map \( \mu : M \to \mathfrak{g}^* \). In the Hamiltonian case, it is possible to describe the moment map for the group action as a splitting. Namely, the moment map \( \mu : M \to \mathfrak{g}^* \) may be written as \( \mu([g,\gamma,v]) = \text{Ad}_{g}^*(\mu(p) + \gamma + \phi(v)) \), where \( \phi : V \to \mathfrak{h}_z^* \) is the moment map for the slice representation.

In order to prove a \( b \)-symplectic analogue of Theorem 1, we show that \( b \)-symplectic manifolds equipped with \( b \)-symplectic actions transverse to the symplectic foliation possess a finite cover which is a product. The slice theorem then reduces to a ‘product slice theorem’ modulo the action of a finite group. This linearized model has an additional invariant compared to the symplectic one: the modular period of the component of the critical hypersurface where the orbit lies. We will prove:

**Theorem 3.** Let \((M, \omega, G)\) be a \( b \)-symplectic manifold together with an effective \( b \)-symplectic action by a compact connected Lie group \( G \). Let \( Z \) be the critical set of the \( b \)-symplectic form. Assume that \( Z \) is compact and connected and that there is one symplectic leaf of \( Z \) which is compact. Assume further that the orbits of \( G \) are transverse to the symplectic foliation of \( Z \). Let \( \mathcal{L} \) be a symplectic leaf of \( Z \). Then:

(i) \( G \) is necessarily of the form \( G = (S^1 \times H)/\mathbb{Z}_k \) where \( H \) is a compact, connected Lie group;
(ii) the action of \( G \) lifts to an action of the product group \( \tilde{G} = S^1 \times H \) on a finite cover \( \tilde{M} \) of a collar neighbourhood of \( Z, \tilde{M} := (-\varepsilon,\varepsilon) \times \tilde{Z}, \tilde{Z} \cong S^1 \times \mathcal{L}, \) where \( S^1 \) acts on \( \tilde{Z} \) by translations on the \( S^1 \)-factor and \( H \) by symplectomorphisms on the symplectic leaf \( \mathcal{L} \);
(iii) let \( \tilde{z} \in \tilde{Z} \). Denote by \( \mathcal{O}_{\tilde{z}}^{\tilde{G}} \) the orbit of \( \tilde{z} \) in \( \tilde{Z} \) under the action of \( \tilde{G} \) and by \( Y_{\tilde{z}}^H \) the bundle of Theorem 1 associated to the action of \( \tilde{H} \) on \( \mathcal{L} \). Then there is an equivariant \( b \)-symplectomorphism from a neighbourhood of the orbit \( \mathcal{O}_{\tilde{z}}^{\tilde{G}} \cong S^1 \times \mathcal{O}_{\tilde{z}}^H \) to the zero section of the bundle \( \tilde{E} = T^*S^1 \times Y_{\tilde{z}}^H \) where \( \tilde{G}/H_{\tilde{z}} \) is embedded as the zero section and the \( b \)-symplectic form

\(^\dagger\) MGS for Marle \([22]\) and Guillemin–Sternberg \([17]\).
on \( \tilde{E} \) is given by

\[
\tilde{\omega}_0 = \omega_{c'} + \omega_{MGS}.
\]

Here \( \omega_{c'} \) is the standard \( b \)-symplectic form of modular period \( c' = kc \) on the manifold \( T^*S^1 \), see Equation (5), \( c \) is the modular period of \( Z \) and \( \omega_{MGS} \) is the MGS normal form as given by Theorem 1;

(iv) let \( \mathcal{O}_z \) be the orbit of \( z \in Z \) under the action of \( G \). There is an equivariant \( b \)-symplectomorphism from a neighbourhood of \( \mathcal{O}_z \) to a neighbourhood of the zero section of the bundle \( E = (T^*S^1 \times Y^H_z)/\mathbb{Z}_l \) where \( \mathbb{Z}_l \) is a finite cyclic group acting by the cotangent lifted action on \( T^*S^1 \).

An important step in the proof is the analysis of the group action along the critical set, \( Z \) which is naturally endowed with a cosymplectic structure. To achieve the proof, we first analyse the consequences of a cosymplectic manifold having a group action transverse to the symplectic foliation. In particular, we prove that given a cosymplectic action of a group \( G \), then there are two distinct cases.

1. \( G \) is a group isomorphic to the product of Lie groups \( G = S^1 \times H \) or
2. \( G = (S^1 \times H)/\Gamma \) where where \( \Gamma \subset S^1 \times H \) is of the form \( \Gamma = \mathbb{Z}_l \times \mathbb{Z}_k \) and \( \mathbb{Z}_k \) is a non-trivial cyclic subgroup of \( H \).

We will prove that there is a finite covering \( \tilde{Z} \) of the cosymplectic manifold \( Z \) which is trivial (in the sense that \( \tilde{Z} \cong S^1 \times \mathcal{L} \)) and this finite covering comes equipped with an \( S^1 \times H \)-action which projects to the action of \( G \) on \( Z \). Examples of cosymplectic manifolds with cosymplectic symmetries include in particular co-Kähler manifolds as discussed in [1], which inspired some techniques used here.

We remark that the aim here is to show the rigidity of \( b \)-symplectic group actions, for which the group action and symplectic form are completely determined in a neighbourhood of an orbit by the isotropy group and its representation on the symplectic normal space. Therefore, Theorem 3 does not reference the traditional moment map sometimes given as part of the symplectic slice theorem. Notwithstanding, the combination of our \( b \)-symplectic slice theorem with the normal form stated in Remark 2 yields a normal form for \( b \)-Hamiltonian actions on \( b \)-symplectic manifolds.

2 PRELIMINARIES

2.1 Introduction to \( b \)-symplectic geometry

We briefly recall the basics of \( b \)-symplectic geometry, see [12] for details.

A \( b \)-manifold is a pair \((M, Z)\) of an oriented manifold \( M \) and an oriented hypersurface \( Z \subset M \).

The hypersurface \( Z \) is called the critical hypersurface.

A \( b \)-vector field on a \( b \)-manifold \((M, Z)\) is a vector field which is tangent to \( Z \) at every point \( p \in Z \).

If \( a \) is a local defining function for the hypersurface \( Z \) on some open set \( U \subset M \) and \((a, z_2, \ldots, z_n)\) is a chart on \( U \), then the set of \( b \)-vector fields on \( U \) is a free \( C^\infty(U) \)-module with basis

\[
\left( a \cdot \frac{\partial}{\partial a}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n} \right).
\]
The corresponding vector bundle, which exists by the Serre–Swan theorem [31], is the \( b \)-tangent bundle:

**Definition 4.** The \( b \)-tangent bundle, \( bTM \), is the vector bundle whose sections are \( b \)-vector fields.

The classical exterior derivative \( d \) on the complex of (smooth) \( k \)-forms extends to the complex of \( b \)-forms in a natural way. Any \( b \)-form of degree \( k \) can locally be written

\[
\omega = \alpha \wedge \frac{da}{a} + \beta,
\]

where \( \alpha \in \Omega^{k-1}, \beta \in \Omega^k \), \( a \) is a local defining function of \( Z \) and \( \frac{da}{a} \) is the \( b \)-one-form dual to \( a \frac{\partial}{\partial a} \) in a frame of the form (2). The exterior derivative of \( \omega \) is then given by

\[
d\omega := d\alpha \wedge \frac{da}{a} + d\beta.
\]

**Definition 5.** A \( b \)-symplectic form is a \( b \)-form of degree 2 which is closed and non-degenerate as a \( b \)-form.

If \( Z \) is the critical hypersurface of a \( b \)-symplectic form, it can be shown that \( Z \) has a codimension-one foliation by symplectic leaves (see [11]). The hypersurface \( Z \) is then cosymplectic as studied in [21].

We recall the following notions for symplectic codimension one foliations given in [11]:

**Definition 6.** Let \( \mathcal{F} \) be a codimension one symplectic foliation of a manifold \( Z \). A form \( \alpha \in \Omega^1(Z) \) is a defining one-form of \( \mathcal{F} \) if it is nowhere vanishing and \( \iota_L^* \alpha = 0 \) for all leaves \( L \), where \( \iota_L \) is the inclusion \( L \hookrightarrow Z \), that is, the kernel of \( \alpha \) at any point \( p \in Z \) is the tangent space of the leaf through \( p \).

A form \( \omega \in \Omega^2(Z) \) is a defining two-form of \( \mathcal{F} \) if \( \iota_L^* \omega \) is the given symplectic form on each leaf of the foliation.

If \( Z \) is the critical hypersurface of a \( b \)-symplectic manifold, then the defining one- and two-form of the induced symplectic foliation can be chosen to be closed [11]. Conversely, a manifold \( Z \) with a codimension one symplectic foliation that admits closed defining one- and two-form \( \alpha \), respectively, \( \beta \) can be extended to a \( b \)-symplectic manifold \( M = Z \times \mathbb{R} \) with \( b \)-symplectic form

\[
\omega = \pi_Z^* \alpha \wedge \frac{da}{a} + \pi_{\mathbb{R}}^* \beta,
\]

where \( \pi_Z : Z \times \mathbb{R} \to Z \) is the canonical projection and \( a \) the coordinate on \( \mathbb{R} \).

\( b \)-Symplectic manifolds can also be viewed dually as a particular class of Poisson manifolds. As such they have a modular vector field:

**Definition 7.** Let \((M, \Pi)\) be a Poisson manifold equipped with a volume form \( \Omega \) and for each \( f \in C^\infty(M) \) denote by \( X_f \) the Hamiltonian vector field associated to \( f \). Then the modular vector
field of \((M, \Pi)\) is the following derivation on \(C^\infty(M)\):

\[
v_{\text{mod}} : C^\infty(M) \to \mathbb{R} : f \mapsto \frac{\mathcal{L}_{X_f} \Omega}{\Omega}.
\]

It can be shown that the modular vector field is a Poisson vector field and that the modular vector fields associated to different volume forms only differ by a Hamiltonian vector field. The following proposition gives a characterization of the modular vector field for \(b\)-symplectic manifolds:

**Proposition 8** [12, Proposition 25]. *The modular vector field of a \(b\)-symplectic manifold \((M, Z)\) is tangent to \(Z\) and transverse to the symplectic leaves inside \(Z\), independently of the volume form considered on \(M\).*

Having chosen a modular vector field \(v_{\text{mod}}\), we can choose defining one and two-forms of the symplectic foliation on \(Z\) uniquely by imposing

\[
\alpha(v_{\text{mod}}) = 1 \text{ and } \iota_{v_{\text{mod}}} \omega = 0.
\]

We will call defining one- and two-forms fulfilling this condition the defining one- and two-forms of the foliation. They are automatically closed [11].

**Remark 9.** Remark that the modular vector field of \(Z\) viewed as a Poisson manifold does not equal the modular vector field of the \(b\)-symplectic manifold \(M\) restricted to \(Z\). In fact, as shown in [11], \(Z\) is unimodular when viewed as a Poisson manifold and so the modular vector field vanishes (up to the addition of a Hamiltonian vector field).

Finally, we note that the flow of the modular vector field can be used to define the mapping torus structure of \(Z\) and define the **modular period** of the \(b\)-symplectic form as follows (cf. [11]):

**Definition 10.** Let \((M, Z)\) be a \(b\)-symplectic manifold and suppose that \(Z\) is compact and connected and that its symplectic foliation has a compact leaf \(\mathcal{L}\). Then \(Z\) is a mapping torus. More precisely, taking any modular vector field \(v_{\text{mod}}\), there exists a number \(c > 0\) such that

\[
Z \cong \left[0, c\right] \times \mathcal{L} \quad (0, x) \sim (c, \phi(x)),
\]

where the time \(t\)-flow of \(v_{\text{mod}}\) corresponds to translation by \(t\) in the first coordinate. In particular, \(\phi\) is the time \(c\)-flow of \(v_{\text{mod}}\). The number \(c > 0\) is called the **modular period** of \(Z\) and does not depend on the choice of modular vector field \(v_{\text{mod}}\).

For notational convenience, we will consider rather \(t \in [0, 1]\). The modular vector field is then given by

\[
v_{\text{mod}} = \frac{1}{c} \frac{\partial}{\partial t}.
\]
and the defining one-form is given by

$$\alpha = c dt.$$  

The $b$-analogue of the Moser theorem for symplectic manifolds is proved in [12].

**Theorem 11** ($b$-Moser Theorem). Let $\omega_0$ and $\omega_1$ be two $b$-symplectic forms on $(M, Z)$. If they induce on $Z$ the same corank one Poisson structure and their modular vector fields differ on $Z$ by a Hamiltonian vector field, then there exist neighbourhoods $U_0, U_1$ of $Z$ in $M$ and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = id_Z$ and $\gamma^* \omega_1 = \omega_0$.

The condition that $\omega_0$ and $\omega_1$ induce the same Poisson structure on $Z$ and the same modular vector field (up to a Hamiltonian vector field) is equivalent to demanding that the induced symplectic foliations have the same defining one- and two-forms.

A consequence is the following semilocal model proved in [12]:

**Corollary 12** (Extension Theorem). Let $(M, Z)$ be a $b$-symplectic manifold where $Z$ is compact and connected. Then there is a neighbourhood of $Z$ in $M$ that is $b$-symplectomorphic to the collar neighbourhood $Z \times (-\epsilon, \epsilon)$ with $b$-symplectic form

$$\omega = \pi_Z^* \alpha \wedge \frac{da}{a} + \pi_Z^* \beta,$$

where $\alpha, \beta$ are the defining one-, respectively, two-forms on $Z$, $a$ is the coordinate on the interval $(-\epsilon, \epsilon)$ and $\pi_Z$ the projection of the collar to $Z$.

As noted in [13], by averaging the vector fields of the $b$-Moser theorem, given two $b$-symplectic forms invariant under a group action and $b$-symplectomorphic by the $b$-Moser theorem, we can choose the $b$-symplectomorphism to be equivariant with respect to the group action. In the special case where $M$ is two-dimensional, this yields the following semilocal normal form:

**Proposition 13.** Let $(M, Z)$ be a two-dimensional $b$-symplectic manifold with compact connected critical hypersurface $Z$ and modular period $c > 0$. Then $Z \cong S^1$ and there exists a neighbourhood of $Z$ which is $b$-symplectomorphic to $S^1 \times (-\epsilon, \epsilon)$ with $b$-symplectic form

$$\omega_c := c dt \wedge \frac{da}{a}.$$  

(5)

Here $(t, a)$ are the standard coordinates on $S^1 \times \mathbb{R}$.

It will be convenient to view the $b$-symplectic manifold $S^1 \times (-\epsilon, \epsilon)$ as a neighbourhood of the zero section of the cotangent bundle $T^* S^1 \cong S^1 \times \mathbb{R}$ with $b$-symplectic form given by the formula in Equation (5). We also remark for future purposes that $\omega_c$ is clearly invariant under the cotangent lift of the action of $S^1$ on itself by translations.

In this article, we characterize the normal form for actions which preserve a $b$-symplectic form ($b$-symplectic actions). Among the class of $b$-symplectic actions, the $b$-Hamiltonian class plays a central role. We end up this section of preliminaries with the definition of $b$-Hamiltonian action. We refer the interested reader to the articles [13–16].
Definition 14. The action of $G$ on a $b$-symplectic manifold $(M, Z, \omega)$ is called $b$-Hamiltonian if there exists a moment map $\mu \in {}^b C^\infty(M) \otimes \mathfrak{g}^*$ with

$$i(\nu_\xi)\omega = \langle d\mu, \xi \rangle,$$

where $\nu_\xi$ is the fundamental vector field associated to $\xi \in \mathfrak{g}$ and the set of $b$-functions is defined as $^b C^\infty(M) = \{a \log |t| + g, g \in C^\infty(M)\}$.

In other words, the action is $b$-Hamiltonian if it preserves the $b$-symplectic form and the form $i(\nu_\xi)\omega$ is exact.

2.2 Transversally equivariant fibrations

A bundle map $\pi : Z \to S^1$ is a transversally equivariant fibration if there is a smooth $S^1$-action on $Z$ such that the orbits of the action are transversal to the fibres of $\pi$ and $\pi(t \cdot x) - \pi(x)$ depends on $t \in S^1$ only. The following is a specialization of a theorem by Sadowski which was applied to the case of co-Kähler manifolds in [1].

Theorem 15. Let $Z \xrightarrow{\pi} S^1$ be a smooth bundle projection from a closed manifold $Z$ to the circle. The following are equivalent.

(1) $Z \xrightarrow{\pi} S^1$ is a mapping torus associated to a diffeomorphism of finite order.

(2) The bundle map $\pi$ is transversally equivariant with respect to an $S^1$-action on $Z$,
$$\rho : S^1 \times Z \to Z,$$

Let $\mathcal{L}$ be the fibre of $\pi$. If the above conditions are satisfied then $Z$ has a $\mathbb{Z}_k$-cover ($k \in \mathbb{N}$)

$$p : Z = S^1 \times \mathcal{L} \to Z$$
given by the action $(t, l) \mapsto \rho_t(l)$, where $\mathbb{Z}_k$ acts diagonally on $S^1 \times \mathcal{L}$ and by translations on $S^1$.

Explicitly, we can describe the $\mathbb{Z}_k$-action as follows: Consider the leaf-fixing subgroup of $S^1$,

$$\Gamma = \{s \in S^1 : \rho_s(\mathcal{L}) = \mathcal{L}\}. (6)$$

Identifying $S^1 \cong \mathbb{R} \mod 1$, the group $\Gamma$ is of the form $\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\}$ for some $k \in \mathbb{N}$ and hence we can identify it with $\mathbb{Z}_k$ in the natural way. Then for $m \in \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$, the action $\rho^m_k$ restricts to a leaf automorphism

$$\sigma_m : \mathcal{L} \to \mathcal{L}, \quad \sigma_m(l) = \rho^m_k(l). (7)$$

The mapping torus $Z$ is then the quotient of the cover $Z$ by the following action of $\mathbb{Z}_k$ on $Z$

$$\mu_m(t, l) = \left(t - \frac{m}{k}, \sigma_m(l)\right), \quad m \in \mathbb{Z}_k, (t, l) \in \tilde{Z} = S^1 \times \mathcal{L}. (8)$$
From the condition of transverse equivariance, it is clear that $\rho$ maps leaves to leaves. It induces an action on the base $S^1$ given by translations $t \mapsto t + ks$ and the equivariance condition reads

$$\pi(\rho_s(l)) = ks, \quad l \in \mathcal{L} := \pi^{-1}(\{0\}).$$

There is an associated $S^1$-action $\tilde{\rho}$ on the cover $\tilde{Z}$ given by

$$\tilde{\rho}_s(t, l) = (t + s, l), \quad s \in S^1, (t, l) \in S^1 \times \mathcal{L}. \quad (9)$$

The projection $\tilde{Z} \to Z$ is equivariant with respect to this action.

The existence of a finite trivializing cover of the critical hypersurface $Z$ will play a crucial role in the $b$-symplectic slice theorem.

### 3 A TRIVIALIZING COVER FOR THE CRITICAL HYPERSURFACE

Now we consider $(M, Z)$ a $b$-symplectic manifold. As we focus on a semi local result, we will assume $M \cong Z \times (-\epsilon, \epsilon)$ where the critical hypersurface $Z$ is compact and connected with $b$-symplectic form given by Equation (4). On a semilocal level, the last assumption is not an additional restriction, since as we have seen in the previous section, any $b$-symplectic manifold satisfying the previous conditions is of this form on a tubular neighbourhood of its critical hypersurface. We will further assume for the rest of the article that $Z$ has a compact leaf. Note that according to [11] this implies that $Z$ has a mapping torus structure (though not necessarily the same mapping torus structure given by Theorem 15).

**Definition 16.** A group action on a $b$-symplectic manifold is called *transverse* if it is transverse to the symplectic foliation of the critical hypersurface. If the action, restricted to the critical hypersurface, preserves the cosymplectic structure we will call the action *cosymplectic*. Finally, if the action preserves the $b$-symplectic form we will call the action *$b$-symplectic*.

Cosymplectic and $b$-symplectic actions are special cases of Poisson actions, when considering the manifolds with the associated Poisson structures.

As cosymplectic actions are leaf preserving, cosymplectic actions transverse to the symplectic foliation are automatically transversely equivariant where the relevant bundle map $\pi : Z \to S^1$ is a projection to the base of the mapping torus. Indeed, the $S^1$-action being leaf-preserving implies by definition that $\pi(t \cdot x) - \pi(x)$ depends only on $t \in S^1$. The next proposition then follows directly from Theorem 15:

**Proposition 17.** Let $Z$ be a cosymplectic manifold and suppose $Z$ has a transverse $S^1$-action preserving the cosymplectic structure. Then $Z$ has a finite cover $\tilde{Z} := S^1 \times \mathcal{L}$, $\mathcal{L}$ a leaf of the foliation, equipped with an $S^1$-action given by translation in the first coordinate for which the projection $p : S^1 \times \mathcal{L} \to Z$ is equivariant.

To get a cosymplectic structure on the cover, one simply lifts the associated defining one and two-forms.

**Proposition 18.** In the setting of the previous proposition, the cosymplectic structure on $Z$ is given by the quotient of a cosymplectic structure on $Z = S^1 \times \mathcal{L}$ by the action of a finite cyclic group $\mathbb{Z}_k$. 
**Proof.** Let \( p : \tilde{Z} \to Z \) be the finite cover given by Proposition 17. Denote the one and two forms of the cosymplectic structure by \( \alpha \) and \( \beta \), respectively. Then \( \tilde{\beta} = p^* \beta \) and \( \tilde{\alpha} = p^* \alpha \) can easily be shown to define a cosymplectic structure on \( S^1 \times \mathcal{L} \) and by construction, the cosymplectic structure on the quotient agrees with the cosymplectic structure on \( Z \). \( \square \)

To extend this cover to a \( b \)-symplectic neighbourhood of \( Z \), we simply use the extension theorem (Corollary 12):

**Corollary 19.** Let \( M = Z \times (-\varepsilon, \varepsilon) \) come equipped with a transverse \( S^1 \)-action preserving the \( b \)-symplectic form \( \omega \). Then the \( b \)-symplectic structure on \( M \) is \( b \)-symplectomorphic in a neighbourhood of \( Z \) to the quotient of a \( b \)-symplectic structure on \( S^1 \times \mathcal{L} \times (-\varepsilon, \varepsilon) \) by the action of a finite cyclic group.

**Proof.** As before, let \( p : \tilde{Z} \to Z \) be the finite cover. Let \( \nu_{\text{mod}} \) be some choice of modular vector field and denote the defining one and two-forms of \( Z \) fulfilling the condition in Equation (3) by \( \alpha \) and \( \beta \), respectively. Denote by \( \tilde{\alpha}, \tilde{\beta} \) the correspond two forms defined in Proposition 18. By the extension theorem, we can assume that the \( b \)-symplectic form on \( M \) is

\[
\omega = \pi_{\tilde{Z}}^* \alpha \wedge \frac{da}{a} + \pi_{\tilde{Z}}^* \beta.
\]

Let \( \tilde{M} := \tilde{Z} \times (-\varepsilon, \varepsilon) \). Then we have a finite cover \( p_M : \tilde{M} \to M \) for \( M \) given by the product map of the cover \( p : \tilde{Z} \to Z \) and the identity on \( (-\varepsilon, \varepsilon) \). Let \( \pi_{\tilde{Z}} : \tilde{M} \to \tilde{Z} \) be the projection onto the first factor. Define for \( a \in (-\varepsilon, \varepsilon) \) the \( b \)-symplectic form on \( \tilde{M} \)

\[
\tilde{\omega} = \pi_{\tilde{Z}}^* \tilde{\alpha} \wedge \frac{da}{a} + \pi_{\tilde{Z}}^* \tilde{\beta}.
\]

Then by construction \( (p_M)^* \omega = \tilde{\omega} \). \( \square \)

**Remark 20.** Note that the modular period of the associated \( b \)-symplectic form on the \( \mathbb{Z}_k \) cover is \( k \) times the modular period of the \( b \)-symplectic form on the base. This can be seen as follows: let \( (t, l) \in \) be a coordinate system on \( \tilde{Z} = S^1 \times \mathcal{L} \). Then the projection \( p : \tilde{Z} \to Z \) acts on the \( S^1 \) factor as \( p(t) = kt \mod 1 \). Therefore, if the defining one-form on \( Z \) is \( \alpha = c dt \) the defining one-form on \( \tilde{Z} \) is given by \( \tilde{\alpha} = p^* (cdt) = cdkt = ck dt \) and the modular period of \( \tilde{Z} \) is \( ck \).

**Remark 21.** Similarly, any \( b \)-symplectic structure with defining one and two-forms \( \tilde{\alpha} \) and \( \tilde{\beta} \) equipped with a discrete \( b \)-symplectic group action gives a \( b \)-symplectic structure on the quotient. For such a group action, there are well-defined one and two-forms, \( \alpha \) and \( \beta \), on the base manifold defined by \( p^*(\alpha) = \tilde{\alpha} \) and \( p^*(\beta) = \tilde{\beta} \), where \( p \) is the projection to the quotient. Then \( \alpha \) and \( \beta \) automatically fulfill the conditions to define a cosymplectic structure on the image of the critical hypersurface. As the group action is discrete, the quotient of the symplectic structure on leaves is likewise symplectic.

### 4 | THE \( b \)-SYMPLECTIC SLICE THEOREM FOR AN \( S^1 \)-ACTION

First, we wish to simplify the expression of the \( b \)-symplectic form in the neighbourhood of an orbit. In the case that the leaf \( \mathcal{L} \) is simply connected, the \( b \)-symplectic form has a particularly simple expression.
Proposition 22. Let $M \cong Z \times (-\varepsilon, \varepsilon)$ be a $b$-symplectic manifold and suppose that $Z$ is a product, $Z \cong S^1 \times \mathcal{L}$, $\mathcal{L}$ a leaf of the symplectic foliation. Suppose furthermore that $\mathcal{L}$ is simply connected. Then for a suitable defining function $f$ of $Z$ the $b$-symplectic form is given by

$$\omega = cd t \wedge \frac{df}{f} + \pi^*_\mathcal{L}(\beta),$$

where $t$ is the standard coordinate on $S^1$, $\beta$ is the symplectic form on $\mathcal{L}$ and $\pi_\mathcal{L}$ is the projection $S^1 \times \mathcal{L} \to \mathcal{L}$.

Proof. A $b$-symplectic form on $S^1 \times \mathcal{L} \times (-\varepsilon, \varepsilon)$ equipped with coordinates $(t, l, a)$ can be written

$$\omega = cd t \wedge \frac{da}{a} + dt \wedge \eta + \pi^*_\mathcal{L}(\beta),$$

where $\beta$ is the symplectic form on $\mathcal{L}$.

When $\mathcal{L}$ is simply connected, $\eta = dh$ for some $h \in C^\infty(M)$. The function $f = ae^h$ is then a defining function for $Z$ and moreover

$$\frac{df}{f} = \frac{da}{a} + dh,$$

whence we have

$$\omega = cd t \wedge \frac{df}{f} + \pi^*_\mathcal{L}(\beta).$$

As in the symplectic slice theorem, the normal form of a $b$-symplectic form in the neighbourhood of an orbit is given by virtue of an equivariant Moser theorem. Equivariant $b$-Moser theorems for isotopic forms invariant under $S^1$-actions have been given in [13] and for more general groups in [25]. As we wish to compare $b$-symplectic forms in the neighbourhood of an orbit rather than on the whole of $Z$, we require an equivariant $b$-Moser theorem of a slightly different nature:

Proposition 23. Suppose that $\omega_1$ and $\omega_0$ are $b$-symplectic forms on $M$, invariant under an action of a group $G$ on $M$ which is transverse Poisson for $\omega_1$ and $\omega_0$. Denote by $O_z$ the orbit of some $z \in Z$ and suppose that $\omega_1$ and $\omega_0$ coincide at $z$. Then $\omega_1$ and $\omega_0$ are equivariantly $b$-symplectomorphic in some neighbourhood $U'$ of $O_z$.

Proof. As the defining one and two-forms associated to $\omega_1$ and $\omega_0$ are invariant under the $S^1$-action, it follows that on $O_z$ we have $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$. By the relative Poincaré lemma, in a contractible neighbourhood of $O_z$, we have that $\alpha_0 - \alpha_1 = dg$, an exact one-form on $U'$ and similarly $\beta_0 - \beta_1 = d\eta$, an exact two-form on $U'$. Whence $\omega_0 - \omega_1 = d(-g \frac{df}{f} + \eta)$. Then $\omega_1 = \omega_0 + (1-t)\omega_1$ is non-degenerate on $O_z$ and so on a neighbourhood of $O_z$. We use this to define a $b$-vector field $v_t$ by $v_t \omega_1 = g \frac{df}{f} - \eta$. As $v_t$ is zero on $O_z$, the time-one flow exists in a neighbourhood of $O_z$ and gives the required $b$-symplectomorphism. As both $b$-symplectic forms are invariant under the group action, we can choose the $b$-symplectomorphism to be equivariant.

Theorem 24. Let $M \cong (-\varepsilon, \varepsilon) \times Z$ be a $b$-symplectic manifold equipped with a $b$-symplectic form $\omega$ of modular period $c$ and a transverse $b$-symplectic $S^1$-action. Let $z \in L \subset Z$, let $O_z$ be its orbit under
the $S^1$-action, let $V := T_z \mathcal{L}$ and let $\mathcal{O}_z$ be the isotropy group of $z$. Then there exists an $S^1$-equivariant neighbourhood $(-\varepsilon, \varepsilon) \times U$ of $\mathcal{O}_z$ in $M$ and an $S^1$-equivariant mapping

$$\phi : (-\varepsilon, \varepsilon) \times U \to (T^*S^1 \times V)/\mathbb{Z}_l,$$

where $\mathcal{O}_z$ is embedded as the zero section of the bundle $S^1 \times \mathbb{R} \times V \cong (T^*S^1 \times V)/\mathbb{Z}_l$ and where the action of $\mathbb{Z}_l$ is given by the cotangent lifted action on $T^*S^1$ and by the isotropy representation on $V$.

Moreover, if we equip the bundle $T^*S^1 \times V$ with the $b$-symplectic form:

$$\bar{\omega}_0 = \omega_{c'} + \omega_V,$$

where $\omega_{c'}$ the $b$-symplectic normal form on $T^*S^1$ as given in Definition 5 with modular period $c' = kc$ and $\omega_V$ the linear symplectic form on $V$, and the quotient $(T^*S^1 \times V)/\mathbb{Z}_l$ with the quotient $b$-symplectic form $\bar{\omega}_0$ (see Remark 21), then the mapping becomes an equivariant $b$-symplectomorphism onto its image.

Proof. Let $z \in Z$ be a point in the critical set and $\mathcal{O}_z$ the orbit of $z$ under the $S^1$-action $\rho$. Denote by $\Gamma_z$ the isotropy group of $z$. Note that $\Gamma_z$ is automatically a subgroup of $\mathbb{Z}_k$ and so $\Gamma_z \cong \mathbb{Z}_l$ for some $l$. By the slice theorem, there exists a neighbourhood $U$ of $\mathcal{O}_z$ in $Z$ equivariantly diffeomorphic to a neighbourhood of the zero section of the vector bundle $S^1 \times \mathbb{Z}_l \mathbb{R} \times V$, where $S^1$ acts on the latter according to $s \cdot [t, v] = [t + s, v]$. By choosing the invariant Riemannian metric in the proof of the slice theorem in such a way that $T_z \mathcal{L}$ is orthogonal to $T_z \mathcal{O}_z$, the equivariant diffeomorphism can be expressed

$$S^1 \times \Gamma_z T_z \mathcal{L} \to U : [t, v] \mapsto \rho_t(\exp_z v).$$

Denote by $\psi$ the corresponding diffeomorphism on the neighbourhood $(-\varepsilon, \varepsilon) \times U$ of $\mathcal{O}_z$ in $M$:

$$\psi : (-\varepsilon, \varepsilon) \times U \to (-\varepsilon, \varepsilon) \times S^1 \times \Gamma_z T_z \mathcal{L}.$$

Restricting the defining one and two forms of $\omega$ to $U$, we have that $U$ is a cosymplectic manifold with a cosymplectic $S^1$-action. The symplectic leaves of $U$ are given by $L_U := U \cap \mathcal{L}$ and the leaf fixing subgroup as defined by Equation (6) is $\Gamma_z$. By Proposition 17, there is a trivial $\Gamma_z$-cover $\tilde{U} \cong S^1 \times L_U$ of $U$ pictured in Figure 1. Then $\omega|_{((-\varepsilon,\varepsilon)\times L_U)}$ is the quotient of a unique $b$-symplectic form $\tilde{\omega}$ on $(-\varepsilon, \varepsilon) \times \tilde{U}$ as given by Corollary 19. By Proposition 22, we may assume $\tilde{\omega}$ is of the form

$$\tilde{\omega} = ckdt \wedge \frac{da}{a} + \pi_{L_U}^* \beta,$$

where $a \in (-\varepsilon, \varepsilon)$ and $\beta$ is a symplectic two-form given on a leaf $L_U$. Consider the two form $\beta_z$ on $T_z \mathcal{L}$. On $(-\varepsilon, \varepsilon) \times S^1 \times T_z \mathcal{L}$, define the $b$-symplectic form

$$\bar{\omega}_0 = ckdt \wedge \frac{da}{a} + \beta_z.$$

Denote the quotient $b$-symplectic form on $[((-\varepsilon, \varepsilon) \times S^1 \times T_z \mathcal{L})]/\Gamma_z$ given in Remark 19 by $\omega_0$. Finally consider the $b$-symplectic form $\psi^* \omega_0$ on $(-\varepsilon, \varepsilon) \times U$. This is a $b$-symplectic structure,
invariant under the $S^1$-action agreeing with $\omega$ at $z$. By Theorem 23, there is an equivariant $b$-symplectomorphism $\varphi$ between neighbourhoods of $\mathcal{O}_z$ such that $\varphi^*(\psi^*\omega_0) = \omega$. Making $U'$ smaller if necessary and setting $\phi = \psi \circ \varphi$, we obtain the $b$-symplectomorphism given in the statement of the theorem. □

**Remark 25.** Note that the modular period of the form $\omega_0$ is $\frac{k}{l}c$ where $c$ is the modular period of the $b$-symplectic form. This is not necessarily the modular period of the original form $\omega$ cf. Remark 20.

**Example 26.** Consider the following symplectic mapping torus: take as a symplectic leaf a torus $\mathbb{T}^2$ with coordinates $(\varphi, \psi)$, $\varphi, \psi \in \mathbb{R}$ mod 1 equipped with the standard symplectic form and the holonomy map given by the diffeomorphism of $\mathbb{T}^2$ which descends from the diffeomorphism of $\mathbb{R}^2$ given by $\phi \in \text{GL}(2, \mathbb{Z})$:

$$
\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

The mapping on $\mathbb{R}^2$ corresponds to rotation by $\frac{\pi}{2}$ and so we have $\phi^4 = \text{Id}$. Denote the mapping torus $Z = ([0,1] \times \mathbb{T}^2)/(0,x) \sim (1,\phi(x))$.

Consider the following $b$-symplectic form on $(t, \varphi, \psi, s) \in Z \times S^1$:

$$
\omega = dt \wedge \frac{ds}{\sin(s)} + \beta,
$$

where $\beta$ is the standard symplectic form on $\mathbb{T}^2$. Consider the action of $S^1$ on $Z \times S^1$ given by translation in the $t$-coordinate. Then there is a neighbourhood of a regular orbit contained in $Z$ which is equivariantly diffeomorphic to a neighbourhood of the zero section $(t, 0)$ of $S^1 \times \mathbb{R}^3$ where $S^1$ acts by translations on the $S^1$ factor of $S^1 \times \mathbb{R}^3$. Moreover, there exist coordinates $(t, x, y, a)$ on $S^1 \times \mathbb{R}^3$ so that the equivariant diffeomorphism becomes a symplectomorphism where $S^1 \times \mathbb{R}^3$ is equipped with the $b$-symplectic form

$$
\omega = 4dt \wedge \frac{da}{a} + dx \wedge dy.
$$

(12)
On the critical set, there is also the exceptional orbit at $\phi = \psi = 0$. In a neighbourhood of the singular orbit, the $b$-symplectic form is the quotient of the $b$-symplectic structure \((\ref{b-symplectic})\) given above where the group action $\sigma_n \in GL(2, \mathbb{Z})$ on the vector space $(x, y)$ is given by

$$\sigma_n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n.$$  

**Example 27.** We can find examples from integrable systems having a naturally associated $S^1$-action model with non-trivial isotropy group.

Take $M = T^*S^1 \times \mathbb{R}^2$ endowed with coordinates $(p, t, x, y)$ and $b$-symplectic form $\omega = \frac{1}{p} dp \wedge dt + dx \wedge dy$. Consider the $b$-integrable system on $M$ given by $F = (\log(p), xy)$. This $b$-integrable system has hyperbolic singularities. Now let $\mathbb{Z}/2\mathbb{Z}$ act on $M$ in the following way: $(-1) \cdot (p, t, x, y) = (p, t, -x', -y')$ observe that this action leaves the hyperbola $xy = \text{cnst}$ invariant and switches its branches. The action clearly preserves the $b$-integrable system and induces a new integrable system on the quotient space $M/\sim$. Observe that the first component of the integrable system naturally induces an $S^1$-action given by the $b$-symplectic vector field associated to the singular Hamiltonian function $\log(p)$ (named as $b$-function, see [13] for a discussion). This circle action also descends to the quotient and the model for the circle action has non-trivial isotropy group of order two.

This twisted hyperbolic case in $b$-symplectic manifolds is a reminiscent of the twisted hyperbolic construction in the symplectic case in [7] and [26]$^\dagger$ and it is an invitation to study the invariants of a non-degenerate singularity of a $b$-symplectic manifold. This example can be extended to higher dimensions and the action of a $\mathbb{Z}/2\mathbb{Z}$ can be considered for every hyperbolic block added as long as the corank of the singularity is equal or bigger than 1. The situation can be visualized using the *curled torus*, the Figure 2 showing the structure of the set $p = 0, xy = 0$.

$^\dagger$ This example shows up in physical examples and corresponds to the 1:2 resonance (see, for instance, the example in page 32 of the monograph [8]).
5 | ACTIONS OF COMPACT LIE GROUPS ON COSYMPLECTIC MANIFOLDS

We treat the case of more general group actions on a $b$-symplectic manifold close to the critical set. First, we prove that only groups of a particular form can act on a $b$-symplectic manifold. For now, we will treat group actions on a mapping torus $Z$ and then extend the results to a neighbourhood of the critical set.

In the following, we assume that the group $G$ is compact and connected and acts on a mapping torus $Z$ via a transverse, effective and foliation preserving action $\rho$. For a more general treatment of the lifting of group actions, see [27].

**Proposition 28.** Suppose an element $h \in G$ fixes a leaf of the mapping torus, $\rho_h(L_0) = L_0$. Then $h$ fixes every leaf of the mapping torus.

This is an easy consequence of the fact that compact connected subgroups of $\text{Diffeo}^+(S^1)$ are conjugate to $\text{SO}(2)$, which is itself a consequence of $\text{Diffeo}^+(S^1)$ having a unique maximal compact subgroup, see [9] for the case of orientation preserving homeomorphisms which can be adapted mutatis mutandis for the smooth case.

**Proof.** Let $\pi : Z \to S^1$ be the mapping torus projection. The action of the group $G$ on a symplectic mapping torus $Z$ induces an action of $G$ on the base $S^1$ in the obvious fashion

$$\tau : G \times S^1 \to S^1,$$

$$(g, \pi(x)) \mapsto \pi(\rho_g(x)) = := \tau_g(x).$$

As $G$ is compact and connected its image $\tau(G, \cdot)$ is a compact subgroup of $\text{Diffeo}^+(S^1)$, the group of orientation preserving diffeomorphisms of the circle. Whence $\tau(G, \cdot)$ is conjugate by some $w \in \text{Diffeo}^+(S^1)$ to $\text{SO}(2)$. Suppose $h \in G$ fixes a leaf $L_0$. This corresponds to a fixed point of the induced action $\tau_h$ on $S^1$, and so a fixed point for $w\tau_h w^{-1} \in \text{SO}(2)$. Whence $w\tau_h w^{-1} = \text{Id}_{S^1}$ and so $\tau_h = \text{Id}_{S^1}$. This corresponds to $h$ fixing all leaves of $Z$. □

It can be checked easily that this defines a subgroup of $G$. We call

$$H_0 = \{ h \in G \mid \rho_h(L_0) = L_0 \}$$

the leaf preserving subgroup of $G$.

**Proposition 29.** Let $G$ be a group acting in a transverse and foliation preserving manner on a symplectic mapping torus. Let $H_0$ be the leaf preserving subgroup of $G$. Then:

(i) $H_0$ is a normal subgroup of $G$;
(ii) $H_0$ is a closed Lie subgroup of $G$;
(iii) the codimension of $H_0$ in $G$ is one.

**Proof.**

(i) This follows immediately from the fact that for $h \in H_0$, $g \in G$, we have $\tau_{ghg^{-1}} = \tau_g \tau_h \tau_g^{-1} = \tau_g \tau^{-1} = \text{Id}_{S^1}$, hence $ghg^{-1} \in H_0$. 

(ii) Consider the projection

\[ \Phi : G \to SO(2) \]

\[ \Phi(g) = w \tau_g w^{-1} \]

corresponding to the map from \( G \) to \( SO(2) \) given in Proposition 28. It is clear that the level set \( \Phi^{-1}(\text{Id}) \) consists precisely of the elements of \( G \) which are leaf preserving. Hence \( \Phi^{-1}(\text{Id}) = H_0 \) is a closed subgroup of \( G \).

(iii) The codimension of \( H_0 \) is at most one since it is given as the level set \( \Phi^{-1}(\text{Id}) = H_0 \). As \( G \) induces an action transverse to the foliation of \( Z \), it follows that the codimension of \( H_0 \) is exactly one.

**Proposition 30.** The action of \( G \) on the mapping torus \( Z \) lifts to an action of a product group \( \tilde{G} = S^1 \times H \) on a finite trivializing cover of \( Z \) where \( H \) is compact and connected. Moreover, \( G \) is necessarily of the form \( G = (S^1 \times H)/\Gamma \) for a finite cyclic subgroup \( \Gamma \) (which might be trivial).

**Proof.** Let \( \mathfrak{h} \subset \mathfrak{g} \) be the Lie algebra of \( H_0 \), the leaf preserving subgroup of \( G \) and consider a complementary ideal \( \mathfrak{l} \) of \( \mathfrak{h} \) in \( \mathfrak{g} \). From the construction in the proof of Proposition 29, we can check that the subgroup \( K = \exp(\mathfrak{l}) \) is closed (and indeed isomorphic to \( SO(2) \cong S^1 \)). The action of \( K \) is transverse to the foliation and so by Proposition 17 there exists a finite trivializing cover \( \tilde{Z} \cong S^1 \times L \) of \( Z \), such that \( Z \) is the quotient of \( \tilde{Z} \) by the action of the leaf fixing subgroup \( \Gamma' \cong \mathbb{Z}_k \) of \( K \) on \( \tilde{Z} \) where \( \Gamma' \) acts as

\[ \mu_m(t, l) = (t - \frac{m}{k}, \sigma_m(l)), \quad m \in \mathbb{Z}_k, (t, l) \in S^1 \times L \]  

(15)

and \( \sigma \) is the leaf automorphism induced by the leaf-fixing elements of \( K \) on \( L \). Denote \( \exp(\mathfrak{h}) \subset G \) by \( H \). Denote by \( \tilde{G} \) the group \( K \times H \). Then we have an action \( \tilde{\rho} \) of \( \tilde{G} \) on \( \tilde{Z} \) given by

\[ \tilde{\rho} : \tilde{G} \times \tilde{Z} \to \tilde{Z}, \quad \tilde{\rho}_{(s, h)}(t, l) = (t + s, \rho_h(l)). \]

Suppose \( \sigma_m = \rho_h \) as an equality of maps on \( L \) for some \( h \in H, m \in \Gamma' \setminus \{0\} \). As \( H \) is connected, \( \sigma_1 = \rho_{h'} \) for some \( h' \in H \). Whence the action \( \mu \) of \( \Gamma' \) on \( \tilde{Z} \) is equivalent to the action \( \tilde{\rho} \) of \( \Gamma \subset \tilde{G} \) on \( \tilde{Z} \) where \( \Gamma \) is the group

\[ \Gamma = \left\{ (\frac{m}{k}, (h')^m) \middle| m = 0, \ldots, k - 1 \right\}, \]

that is, \( \mu_m = \tilde{\rho}_{(-\frac{m}{k}, (h')^m)} \) for all \( m \in \mathbb{Z}_k \). Letting \( p_\tilde{Z} \) and \( p_\tilde{G} \) denote the projections to \( \tilde{Z}/\Gamma \) and \( \tilde{G}/\Gamma \), respectively, we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{G} \times \tilde{Z} & \xrightarrow{\tilde{\rho}} & \tilde{Z} \\
\searrow \downarrow & & \downarrow \nearrow \\
\tilde{G}/\Gamma \times \tilde{Z} & \xrightarrow{\rho} & Z.
\end{array}
\]
By construction, the action of $\tilde{G}/\Gamma$ on $Z$ and the action of $G$ on $Z$ possess the same fundamental vector fields. Moreover, the action of both groups is effective. Necessarily, then, $\tilde{G}/\Gamma = G$.

Conversely, assume that $\sigma_1 \neq \rho_\iota$ for any $h \in H$.

Then $\exp(f) \cap \exp(h) = 0$ and so $G \cong K \times H$. The action $\bar{\rho}$ of $\tilde{G} \cong G$ on $\tilde{Z}$ projects to an action of $G$ on $Z$ where the projection $p_\iota$ is given by quotienting the group by the subgroup $\Gamma = \Gamma' \times \{e_H\}$ and the projection $p_Z$ is given by the action $\mu$ of $\Gamma \cong \Gamma'$ in (15).

Proposition 31. Let $G = S^1 \times H$ be a product group acting on a mapping torus $Z$ such that the $S^1$-factor acts transversely to the foliation. Let $z \in Z$ and denote by $G_z$ the isotropy group of $z$. Then $G_z \cong Z_l \times H_z$ where $H_z$ the isotropy group of $z$ under the $H$-action and $Z_l$ is a cyclic subgroup and $Z_l \times \{e_H\}$ acts as the identity on $\mathcal{O}^H_z$.

Proof. Let $\mathcal{L}_0$ be a leaf of $Z$. Denote by $\mathcal{O}^H_z \subset \mathcal{L}_0$ the orbit of $z$ under the action of $(0, H) \subset G$. Denote the subgroup $(S^1, e_H) \subset G$ by $K$. Let $\rho^K$ be the action of $K$ on $Z$. The leaf preserving subgroup of $K$ can be identified with $Z_k$, for $m \in Z_k$, that is, $m = \frac{m}{k} \in K$ is leaf-preserving, we either have $\rho^K(m(\mathcal{O}^H_z)) \cap \mathcal{O}^H_z = \emptyset$ or $\rho^K(m(\mathcal{O}^H_z)) \cap \mathcal{O}^H_z = \mathcal{O}^H_z$. Moreover, elements $m \in Z_k$ satisfying $\rho^K(m(\mathcal{O}^H_z)) \cap \mathcal{O}^H_z = \mathcal{O}^H_z$ form a subgroup $Z_l$ of $Z_k$.

If $\rho^K(z) \not\in \mathcal{O}^H_z$ for all $m \in Z_k$, then $Z_l = \{0\}$ and $G_z = \{0\} \times H_z$ where $H_z$ is the isotropy group of $z$ under the action of $(0, H)$. Alternatively suppose $Z_l \neq \{0\}$, so that $\rho^K(z) = h \cdot z$ for some $h \in H$. If $h \neq e_H$, we can find a new $K' \subset G$ which acts as the identity on $z$ as follows: let $\eta$ in $\mathfrak{h}$ be such that $K = \exp(t\eta)$ where $t \in [0, 1)$. Let $\nu \in \mathfrak{h}$ be such that $\exp(t \nu) = h^{-1}$. Consider the subgroup

$$K' = \{\exp(t(\eta + \nu)) | t \in [0, 1)\}.$$

Then the isotropy group of $z$ is of the form $Z_l \times H_z$ where $Z_l \cong \{\exp(n(\eta + \nu)) | n = 0, \ldots, l - 1\}$.

6 A $b$-SYMPLECTIC SLICE THEOREM

Let $(M \cong (-\varepsilon, \varepsilon) \times Z, \omega)$ be a $b$-symplectic manifold together with an effective $b$-symplectic action by a compact connected Lie group $G$ acting transversely to the symplectic leaves inside the critical hypersurface $Z$. First we will construct the $b$-symplectic models which will give us a normal form for the $b$-symplectic form about an orbit of $G$. By Proposition 30, there are two distinct cases.

1. $G$ is a group isomorphic to the product of Lie groups $G = S^1 \times H$.
2. $G = (S^1 \times H)/\Gamma$ where $\Gamma \cong Z_l \times Z_k \subset S^1 \times H$ and $Z_k$ is a non-trivial cyclic subgroup of $H$.

Recall from Proposition 30 that there is a trivial finite cover $\tilde{Z} = S^1 \times \mathcal{L}$ of $Z$ equipped with an $(S^1 \times H)$-action which projects to the action of $G$ on $Z$.

Let $z \in \mathcal{L}_0$ be a point in a symplectic leaf of $Z$ and consider the orbit $\mathcal{O}^H_z$ of $z$ given by the group action of $H = \exp \mathfrak{h}$ on $\mathcal{L}_0$. Denote the isotropy group of $z$ by $H_z$. By the symplectic slice theorem (Theorem 1), there is an $H$-equivariant neighbourhood $U$ of $\mathcal{O}^H_z$ in $\mathcal{L}_0$ which is equivariantly symplectomorphic to a neighbourhood of the zero section of the vector bundle $Y^H_z = (H \times m^* \times V_z)/H_z$ with symplectic form $\omega_{MGS}$ as given by Theorem 1. Recall that $m$ is a Lie subalgebra of $\mathfrak{h}$, the vector space $V_z \subset T_z \mathcal{L}_0$ is the symplectic orthogonal $V_z = (T_m \mathcal{O}^H_z)_{\omega}/T_m \mathcal{O}^H_z$ and $H_z$ acts on $V_z$ by the isotropy representation.
**Definition 32** (b-Symplectic models). Consider the b-symplectic form on $\tilde{E} = T^*S^1 \times (H \times_{H_z} m^* \times V_z)$ given by

$$\tilde{\omega}_0 = \omega_{c'} + \omega_{MGS},$$  

(16)

where $\omega_{c'}$ is the b-symplectic form on $T^*S^1$ of modular period $c' = ck$ given by Definition 5, and $\omega_{MGS}$ is the symplectic form on $Y^H_z = H \times_{H_z} m^* \times V_z$ given by the symplectic slice theorem (Theorem 1). Consider the quotient b-symplectic structure on $E = \tilde{E} / \mathbb{Z}_l$ where $\frac{m}{l} \in \mathbb{Z}_l$ acts on $T^*S^1$ as the cotangent lift of $\mathbb{Z}_l$ acting by translations on $S^1$ and acts on the factor $H \times_{H_z} m^* \times V_z$ equipped with the coordinates $[k, \eta, \nu]$ of Theorem 1 either by:

1. $\frac{m}{l} \cdot [k, \eta, \nu] = [k, \sigma^m(\eta), \sigma^m(\nu)]$ for a linear symplectomorphism $\sigma$;
2. $\frac{m}{l} \cdot [k, \eta, \nu] = [h^m \cdot k, \eta, \nu]$ where $h$ is some element of $H$;

then $E$ has a unique b-symplectic structure such that the projection is a local b-symplectomorphism (see Remark 21). We call these normal forms b-symplectic models with symplectic slice $V_z$ and modular period $c^k_l$.

We will now show that a neighbourhood of an orbit of a $G$-action on a b-symplectic manifold with the properties stated in the beginning is b-symplectomorphic to a neighbourhood of the zero section of one of the models above, completing the proof of Theorem 3, which we recall here in a succinct way:

**Theorem 33.** Let $G$ be a compact Lie group acting on a b-symplectic manifold $(M, \omega)$ transverse to the symplectic foliation and assume there is one symplectic leaf of the critical set $Z$ which is compact. Suppose that the action of $G$ is b-symplectic, effective and Hamiltonian when restricted† to a symplectic leaf of $Z$. Let $O_z$ be an orbit of the group action contained in the critical set of $M$. Then there is a neighbourhood $V$ of $O_z$ in $M$ which is b-symplectomorphic to a neighbourhood of the zero section of a bundle given by the b-symplectic model $E$ in Definition 32.

**Proof.** By Proposition 30, there exists a finite cover $\tilde{Z} \cong S^1 \times \mathcal{L}$ which comes equipped with the action of a product group $\tilde{G} \cong S^1 \times H$ which covers the action of $G$ on $Z$. Let $z \in \mathcal{L}_0 \subset Z$ and let $\tilde{z} \in \tilde{Z}$ be a point projecting to $z$. Denote by $O^H_z$ the orbit of $z$ under the action of the subgroup $H$ and by $O^G_z$ the orbit of $\tilde{z}$ under the action of $\tilde{G} \cong S^1 \times H$ on the cover $\tilde{Z}$. Then an invariant open neighbourhood of $O^G_z$ in $\tilde{M} \cong (-\epsilon, \epsilon) \times \tilde{Z}$ is of the form $\tilde{V} = (-\epsilon, \epsilon) \times S^1 \times U'$ where $U'$ is an invariant open neighbourhood of $O^H_z$ in $\mathcal{L}_0$. Recall that $Z$ is the quotient of $\tilde{Z}$ by a cyclic subgroup $\Gamma$ of $\tilde{G}$. By Proposition 30, we may assume that $\Gamma$ is of the form

$$\Gamma = \left\{ \left( -\frac{m}{k}, h^m \right) \mid m = 0, \ldots, k - 1 \right\}.$$  

(17)

† The group action $\rho$ induces a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on $M$ via $d_{x} \rho_{x} : \mathfrak{g} \to T_x M$. In turn, by restricting the action to the critical set, we have a Lie algebra homomorphism $d_{x} \rho_{x}|_{Z} : \mathfrak{g} \to T_x Z \cong T_x S^1 \times T_x \mathcal{L}$ where $\mathcal{L}$ is a leaf of the foliation. The action is transverse to the leaf of the foliation if the restriction of this homomorphism to the first factor, $T_x S^1$, is surjective everywhere. The action is Hamiltonian when restricted to leaves if the vector fields given by the restricted homomorphism $d_{x} \rho_{x}|_{\mathcal{L}} : \mathfrak{g} \to T_x \mathcal{L}$ are Hamiltonian.
Let $\tilde{\omega}$ be the lift of $\omega$ to $\tilde{Z}$ as given by Proposition 17. By Theorem 22, we may assume that locally around $\tilde{z}$, $\tilde{\omega}$ is of the form

$$\tilde{\omega} = ck dt \wedge \frac{da}{a} + \beta,$$

where $\beta$ is the symplectic form on the leaf $L_0$. Denote by $H_Z$ the isotropy group of $z$ under the action of $H$. By the symplectic slice theorem, Theorem 1, a neighbourhood $U$ of $O^H_z$ with symplectic form $\beta$ is equivariantly symplectomorphic to a neighbourhood of the zero section of the bundle $Y^H_z = H \times_{H_Z} m^* \times V_z$ with symplectic form given by the symplectic slice theorem.

Consider the vector bundle $\tilde{E} = T^* S^1 \times (H \times_{H_Z} m^* \times V_z)$ with symplectic form given by $\tilde{\omega}_0$ in Equation (16), where $c$ is the modular period of $\omega$ and $k$ is the order of $\Gamma$. Let $\tilde{\psi}$ be the equivariant diffeomorphism form $\tilde{\mathcal{P}}$ to a neighbourhood of the zero section in $\tilde{E}$ obtained from the slice theorem on $L_0$. Then $\tilde{\psi}^* \tilde{\omega}_0$ is a $b$-symplectic form on a neighbourhood of $O^H_z$ and, since $(\tilde{\psi}^* \tilde{\omega}_0)_z = \tilde{\omega}_z$, by the equivariant relative Moser Theorem, Theorem 23, and after making $\tilde{\mathcal{P}}$ smaller if necessary, we can conclude that there is an equivariant $b$-symplectomorphism from $\tilde{\mathcal{P}}$ to a neighbourhood of the zero section of $\tilde{E}$ equipped with the $b$-symplectic form $\tilde{\omega}_0$.

Denote by $\Gamma_z$ the subgroup of $\Gamma$ defined by $\{ m \in \Gamma \mid \rho_m(O^H_z) \cap O^H_z = O^H_z \}$, where $\rho$ is the action of $\Gamma_z \subset \tilde{G}$ on $Z$ equivariant with respect to the projection $\tilde{Z} \to Z$, as in Proposition 30.

Then $\Gamma_z$ is a cyclic subgroup $\Gamma_z \cong \mathbb{Z}_l$ of $\Gamma$ of the form

$$\Gamma_z = \left\{ \left(-\frac{m}{k}, h'(m)^m \right) \mid m = 0, \ldots, l - 1 \right\}$$

for some $h' \in H$. Denote by $p_{\tilde{\mathcal{P}}}, p_E$ the projections to the quotients of $\tilde{\mathcal{P}}$ and $E$ by the action of $\Gamma_z$, respectively. Define $\psi$ by the condition that the following diagram commutes:

$$\begin{array}{ccc}
\tilde{\mathcal{P}} & \xrightarrow{\tilde{\psi}} & \tilde{E} \\
p_{\tilde{\mathcal{P}}} \downarrow & & \downarrow p_E \\
\tilde{\mathcal{P}}/\Gamma_z & \xrightarrow{\psi} & E.
\end{array}$$

First consider the case where $G \cong S^1 \times H$. We may assume by Proposition 31 that the action of $\Gamma_z$ on $O^H_z$ and so on the base of the bundle $Y^H_z$ is trivial. Moreover, it preserves the slice $V_z$ and acts by linear symplectomorphisms and so $\psi$ is $b$-symplectomorphism to Model (1) of Definition 32.

For $h \neq e_H$ in the group $\Gamma$ in Equation (17) (that is, the case $G \cong (S^1 \times H)/\Gamma$ for $\Gamma$ non-trivial and $\Gamma_z \subset \Gamma$), the action of $\Gamma_z$ on $Y^H_z$ is given by the symplectic slice theorem, Theorem 1, and the equivariant normal form is given by Model (2) of Definition 32.

Some remarks are in order:

**Remark 34.** It would be possible to extend this slice theorem to proper group actions rather than compact group actions as done in [28].

**Remark 35.** This slice theorem puts a first step forwards towards understanding rigidity phenomena in the Poisson realm. In [24], a rigidity theorem for Hamiltonian actions on Poisson manifolds
is given. Our slice theorem yields a similar rigidity theorem also for \( b \)-symplectic actions. The general rigidity theorem in [24] uses sophisticated Nash–Moser techniques which are not necessary in the case of \( b \)-symplectic manifolds. In a more general context, this is connected to the problem of stability of symplectic leaves (the normal form obtained gives an equivariant version of this phenomenon). For the general problem of stability of symplectic leaves and rigidity and flexibility phenomena, refer to [6] and [5].

**Example 36.** Let \( G \) be a compact Lie group with non-trivial centre. Let \( \xi_1 \in \mathfrak{g} \) be a central element of the Lie algebra and \( \xi_2, \ldots, \xi_n \in \mathfrak{g} \) be such that \( \xi_1, \ldots, \xi_n \) form a basis of the Lie algebra. Denote by \( \eta_i \) the basis of the Lie algebra dual such that \( (\eta_i, \xi_j) = \delta_{ij} \). Denote the associated invariant vector fields \( L^g_\xi \xi_i \) by \( v_i \) and \( L^g_\eta \eta_i \) by \( m_i \), respectively. At each point \( g \in G \), these give a basis for the tangent and cotangent spaces at \( g \). Consider the singular two-form on \( T^*G \)

\[
\omega = \pi^* m_1 \wedge \frac{d(\lambda(v_1))}{\lambda(v_1)} + \sum_{i=2}^n \pi^* m_i \wedge d(\lambda(v_i)),
\]

where \( \pi \) the canonical projection \( T^*G \to G \). It can be checked directly that \( \omega \) is a \( b \)-symplectic form on \( T^*G \) invariant under the cotangent lifted action of \( G \) on \( T^*G \). By Proposition 30, \( G \) has a finite cover of the form \( S^1 \times H \). The \( b \)-symplectic model for the action of \( G \) on \( T^*G \) is given by \( \tilde{E} = (T^*S^1 \times T^*H)/\mathbb{Z}_k \) where \( \mathbb{Z}_k \) acts diagonally on \( T^*S^1 \times T^*H \) by the cotangent lift of translations on \( S^1 \) and the \( b \)-symplectic form on \( \tilde{E} \) is

\[
\tilde{\omega}_0 = \omega_c + \omega_H,
\]

where

- \( \omega_c \) is the standard \( b \)-symplectic form of modular period \( c \) on the manifold \( T^*S^1 \), as given in Definition 5;
- \( \omega_H \) is the canonical symplectic form on \( T^*H \).

**Example 37.** Consider the symplectic mapping torus

\[
Z = \frac{[0,1] \times \mathcal{L}}{(0, l) \sim (1, \phi(l))},
\]

where

- \( \mathcal{L} \cong S^2 \times S^2 \), where \( S^2 \) is the two sphere equipped with the standard symplectic form and \( \mathcal{L} \) is equipped with the product symplectic form;
- \( \phi : \mathcal{L} \to \mathcal{L} \) is the diffeomorphism of \( \mathcal{L} \) given by exchanging the \( S^2 \) factors of \( \mathcal{L} \), that is, \( \phi(x, y) = (y, x) \).

Consider the group \( G = S^1 \times SO(3) \) where \( SO(3) \) acts diagonally on the product \( \mathcal{L} \cong S^2 \times S^2 \) by rotation on each factor and \( S^1 \) acts by translations on the factor \([0,1]\) of the above mapping torus:

\[
(s, A) \cdot (t, x, y) = (t + 2s, A \cdot x, A \cdot y), \quad (s, A) \in S^1 \times SO(3), \quad (t, x, y) \in [0, 1] \times \mathcal{L}.
\]

Consider a point \( z = (0, x, y) \in Z \) and the corresponding orbit \( O_z \) in the \( b \)-symplectic manifold \( M = (-\varepsilon, \varepsilon) \times Z \). We distinguish three cases.
First, suppose $x \neq \pm y$. Then the action of $S^1 \times \text{SO}(3)$ on the orbit $\mathcal{O}_z$ is free. There is a neighbourhood $\mathcal{V}$ of $\mathcal{O}_z$ equivariantly $b$-symplectomorphic to the zero section of the bundle $E = T^*S^1 \times Y_{SO(3)}$, where $E$ is equipped with the $b$-symplectic form

$$\tilde{\omega}_0 = \omega_2 + \omega_{MGS}$$

and $\omega_2$ is the standard symplectic form of modular period 2 on $T^*S^1$ and $\omega_{MGS}$ is a symplectic form on $Y_{SO(3)}$ given by Theorem 1.

Now let $x = y$. Then $z$ has isotropy group $\mathbb{Z}_2 \times \text{SO}(2)$. The associated $b$-symplectic model is given by $E = T^*S^1 \times F$, where $F = \text{SO}(3) \times_{\text{SO}(2)} V$ is a bundle over the homogeneous space $\text{SO}(3)/\text{SO}(2) \cong S^2$, $V$ a two-dimensional vector space with Darboux symplectic form $\omega_V$. The $b$-symplectic form on $E$ is given by

$$\tilde{\omega}_0 = \omega_1 + 2\omega_{S^2} + \omega_V,$$

where $\omega_1$ is the standard symplectic form of modular period 1 on $T^*S^1$ and $\omega_{S^2}$ is the usual symplectic form on the sphere.

Finally, suppose $y = -x$. Let $\nu \in \mathfrak{k}$, $\mathfrak{k}$ the Lie algebra of $S^1$. Let $\exp(t\xi) \cong \text{SO}(2)$ be the 1-parameter subgroup of $\text{SO}(3)$ such that $g = \exp(\xi)$ acts on $S^2$ by $g(x) = -x$ and $d\rho_g = -Id$. Then the subgroup $(\exp(t\nu),\exp(t\xi)) \subset S^1 \times \text{SO}(3)$ acts as the identity on the orbit $\mathcal{O}_z \cong S^2$ and the $b$-symplectic model is given by the quotient bundle $E = T^*S^1 \times (\text{SO}(3) \times_{\text{SO}(2)} V)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $u, v \in V$ by $(u, v) \mapsto (-u, -v)$ and $E$ is equipped with the $b$-symplectic form

$$\tilde{\omega}_0 = \omega_1 + 2\omega_{S^2} + \omega_V.$$

ACKNOWLEDGEMENTS

We are grateful to the anonymous referee for their suggestions and comments that improved this article and increased its readability. We are thankful to Konstantinos Efthathiou for providing us the beautiful picture in this paper. Eva Miranda thanks her own institution, Universitat Politècnica de Catalunya BarcelonaTech, for covering the APC to publish this article in Open Access and making it available to the general public. Eva Miranda also wants to express her gratitude to Anna Agathopoulou and Muhusina Hashim for practical aspects concerning this publication. We are thankful to the Fondation Sciences Mathématiques de Paris for financing the stay of Roisin Braddell and Anna Kiesenhofer in Paris and to Observatoire de Paris for their hospitality in September 2017 to February 2018 during which this project was initiated.

Eva Miranda is supported by the Catalan institution for Research and Advanced Studies via an ICREA Academia Prize 2016 and an ICREA Academia Prize 2021 and by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (project CEX2020-001084-M) and through the project PID2019-103849GB-I00/AEI/10.13039/501100011033. Roisin Braddell is supported by the Severo Ochoa Program SEV-2017-0718, Basque Government BERC Program 2018-2021 and was supported by a predoctoral grant from UPC with the ICREA Academia project of Eva Miranda. We acknowledge support from the Foundation Sciences Mathématiques de Paris via the Chaire d’Excellence of Eva Miranda supported by a public grant overseen by the French National Research Agency (ANR) as part of the ‘Investissements d’Avenir’ program (reference: ANR-10-LABX-0098) to finance a research stay of Roisin Braddell and a research visit of Anna Kiesenhofer in Paris to start this project.
**JOURNAL INFORMATION**

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**REFERENCES**


