Zindler-type hypersurfaces in $\mathbb{R}^4$ ⋆

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Abstract

In this paper the definition of Zindler-type hypersurfaces is introduced in $\mathbb{R}^4$ as a generalization of planar Zindler curves. After recalling some properties of planar Zindler curves, it is shown that Zindler hypersurfaces satisfy similar properties. Techniques from quaternions and symplectic geometry are used. Moreover, each Zindler hypersurface is fibrated by space Zindler curves that correspond, in the convex case, to some space curves of constant width lying on the associated hypersurface of constant width and with the same symplectic area.

Keywords: Zindler hypersurface, Zindler curve, Hypersurface of constant width, Curve of constant width, Hedgehog
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1. Introduction

It has been a century ago when Zindler published his article [40] where he described a special kind of planar curves today known by his name. Zindler curves are those such that all chords that divide the curve perimeter (or area) in a half, have the same length. These curves are also the boundaries of figures of constant density that float in water in equilibrium in any position [5] and serve as solutions to other famous problems, such as the ambiguous tire-track problem or the motion of an electron in a parabolic magnetic field (see e.g. [4], [34] and [2]).

There are known generalizations of these curves in the literature. For instance, some works studied Zindler curves in non-Euclidean geometries, such as

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in isotropic geometry [33] or together with some spherical motions [29]. Other works on Zindler curves in normed planes are also available [22, 23].

Zindler curves are very related to curves of constant width. In fact, Zindler curves can be generated by rotating double-normals of a closed plane curve of constant width a right angle about their midpoint (see e.g. [21], [12] or [31]). A visualization of this construction is presented in Figure 1.

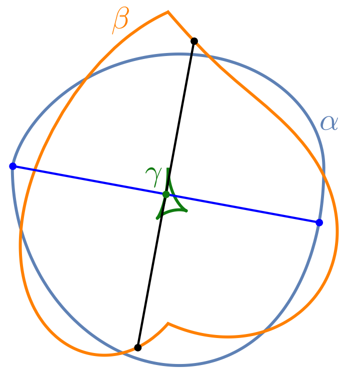

Figure 1: A Zindler curve β constructed by rotating double-normals of a constant width curve α. The curve described by the midpoint of the chords is the middle hedgehog γ.

The same idea led to other generalizations of Zindler curves. On the one hand, Hoschek extended Zindler curves to $\mathbb{R}^3$ using double-normals of a closed transnormal space curve of constant width in [13] and [14]. The resulting curve has analogous properties as the planar one. Wegner generalized the result to $\mathbb{R}^n$ in [38]. On the other hand, Wunderlich constructed Zindler curves without using spatial curves of constant width in [39] based on the family of tangent lines of the midpoint curve. From this, Pottmann generalized these results in [28], as the midpoints of the constant length chords lied on the striction curve of the ruled surface that is generated by these directions.

The generalization of constant width curves to hypersurfaces of constant width has been widely studied (see e.g. [21] and its references therein). The corresponding generalization to hedgehog hypersurfaces of constant width in $\mathbb{R}^n$ has been developed as well, see e.g. the works [16, 17, 19] by the first author, including the notion of a projective hedgehog as those hypersurfaces of constant width 0. Nevertheless, as far as the authors know, nobody has provided a generalization of Zindler curves as surfaces in $\mathbb{R}^3$ or, more generally, as hypersurfaces in $\mathbb{R}^n$.

The aim of this paper is to present the definition of a Zindler-type hypersurface in $\mathbb{R}^4$, which constitutes a generalization of planar Zindler curves and that satisfies analogous properties.

First of all, we recall some facts about planar Zindler curves in Section 2 and we describe a different construction of these curves from the evolute of a projective hedgehog (Proposition 5). Later, in Section 3 we introduce some concepts of symplectic geometry and the geometry of $\mathbb{R}^4$ as a quaternion algebra with a Kähler vector space structure. After that, in Section 4 our Zindler hypersurfaces are defined (Definition 1) and some of their properties are stated in Theorem 1.

It is known that any orthogonal projection of a hypersurface of constant width is a curve of constant width [21]. In particular, we deduce the analogous property for Zindler hypersurfaces: any orthogonal projection of a Zindler hypersurface is a planar Zindler curve. Another interesting property that comes from the same theorem is that Zindler hypersurfaces are fibrated by some spatial curves which are Zindler curves in the sense of Pottmann [28].

Convex hypersurfaces of constant width have associated Zindler hypersurfaces by definition. In Theorem 2 we show that the image of the Hopf circles on these hypersurfaces yields pairs of associated curves, a space curve of constant width (in the sense of Fujiwara [10]) and a space Zindler curve (in the sense of Pottmann), through a unit pure quaternion. In addition, the symplectic areas of these pairs of associated curves are the same (Theorem 3).

Finally, the property presented in Proposition 5 relating the evolute of the projective hedgehog and the associated Zindler curve is generalized to Zindler hypersurfaces in Theorem 4.

We want to remark that although the generalization to Zindler hypersurfaces in $\mathbb{R}^4$ introduced in this paper is quite natural, it does not seem to be the case for a generalization to $\mathbb{R}^3$, which remains an open problem.

2. Some properties of planar Zindler curves

The objective of this section is to describe some properties of planar Zindler curves, some of which will be generalized later on to Zindler hypersurfaces.

The middle hedgehog of any convex curve of constant width (i.e. the locus of midpoints of all the diameters) is a projective hedgehog. From it we can easily construct an associated Zindler curve. But notice that not every Zindler curve is generated from a middle hedgehog (see e.g. the example by Mampel in [15]). In this paper we will focus on Zindler curves which are associated with a convex curve of constant width and, thus, which can be generated from a projective hedgehog.

In general, the midpoint curve is the envelope of the halving chords. This is a consequence of the following property (see e.g. [7] and [8]).

Proposition 1. *Let z be a $\mathcal{C}^1$-regular parametric curve and let c be a vector defining its halving chords at each point following the parameterization z. Let m be the curve generated by the midpoints of the halving chords. The curve z is a Zindler curve if and only if c' is orthogonal to m'.*

In the generalization of Zindler curves to space curves in $\mathbb{R}^n$ proposed by Pottmann in [28], this condition is imposed in the definition.

Auerbach [1] was the first that set the "duality" between Zindler curves and curves of constant width. In particular, he proved that Zindler curves have associated curves of constant width with the same area. The reverse construction is true as well [15]: curves of constant width have associated Zindler curves and the area of these figures is invariant.

A simple proof of this area invariance can be given using Holditch's theorem (see e.g. [30] or [27] for an introduction to Holditch's theorem) as we show next. It can also be seen as a particular case of swept-out areas by bicycle tire-track curves (see e.g. [9]).

Proposition 2. *Pairs of associated curves, a convex curve of constant width and a Zindler curve, have the same area.*

Proof. Let x_h be a projective hedgehog parameterized by a support function h such that, given $r > 0$, x_{h+r} and $z_{h,r}$ are a convex curve of constant width and its associated Zindler curve, respectively, for chords of length $2\,r$. By Holditch's theorem, we have that $\mathcal{A}(x_{h+r}) - \mathcal{A}(x_h) = \pi\, r^2$ and $\mathcal{A}(z_{h,r}) - \mathcal{A}(x_h) = \pi\, r^2$. Therefore, $\mathcal{A}(x_{h+r}) = \mathcal{A}(z_{h,r})$. □

Proposition 3. *Zindler curves generated from a $\mathcal{C}^2$-projective hedgehog are regular.*

Proof. Let $x_h : [0, 2\pi] \to \mathbb{R}^2$ be a projective hedgehog parameterized by a $\mathcal{C}^2$-support function h:

$$x_h(t) = h(t)\, u(t) + h'(t)\, u'(t),$$

where $u(t) = (\cos t, \sin t)$. Given $r > 0$, the corresponding Zindler curve can be parameterized as

$$z_{h,r}(t) = x_h(t) + r\, u'(t) = h(t)\, u(t) + \big(h'(t) + r\big)\, u'(t).$$

Since

$$z'_{h,r}(t) = r\, u(t) + \big(h(t) + h''(t)\big)\, u'(t),$$

we have that

$$\big\| z'_{h,r}(t) \big\| = \sqrt{r^2 + \big(h(t) + h''(t)\big)^2} \neq 0$$

for all $t \in [0, 2\pi]$, so that $z_{h,r}$ is regular. □

Notice that the Zindler curve of Figure 1 has singularities. This is because the support function of its projective hedgehog is not twice differentiable at some points.

There is a well-known result about the angle that the halving chords of a Zindler curve make with the tangents at their endpoints. It can be stated as follows (see e.g. [39]).

Proposition 4. *The halving chords of a Zindler curve form the same angle with both tangent vectors to the curve at the corresponding endpoints.*

Proof. Let $u(t) = (\cos t, \sin t)$. The endpoints z_1 and z_2 of the halving chord of a Zindler curve can be described from its middle hedgehog x_h as follows:

$$\begin{aligned} z_1(t) &= x_h(t) - r\, u'(t), \\ z_2(t) &= x_h(t) + r\, u'(t). \end{aligned} \tag{1}$$

Thus, since

$$z_1'(t) = r\,u(t) + \big(h(t) + h''(t)\big)\,u'(t),$$
$$z_2'(t) = -r\,u(t) + \big(h(t) + h''(t)\big)\,u'(t), \tag{2}$$

we have $\langle z_2(t) - z_1(t),\, z_1'(t)\rangle = \langle z_2(t) - z_1(t),\, z_2'(t)\rangle$, from which the statement follows. □

The following proposition shows a nice geometrical property between the evolute of the middle hedgehog and the Zindler curve. We do not know if it is a new result or not, but at least we have not found it in the literature.

Proposition 5. *Let $\mathcal{H}_h$ be a plane projective hedgehog with a $\mathcal{C}^2$-support function h and let $\mathcal{E}$ be its evolute parameterized by $\varepsilon(t)$, which is a projective hedgehog. Let $\mathcal{Z}_{h,r}$ be the Zindler curve parameterized by $z_{h,r}$ that corresponds to a curve of constant width with a support function $h + r$. Then the vector $z_{h,r}(t) - \varepsilon(t)$ has the same length as $z'_{h,r}(t)$ and it is orthogonal to $\mathcal{Z}_{h,r}$ at $z_{h,r}(t)$*

Proof. Let $u(t) = (\cos t, \sin t)$. The evolute of $\mathcal{H}_h$ is

$$\varepsilon(t) = x_h(t) - R_h(t)\,u(t),$$

where $R_h(t) = h(t) + h''(t)$ is the radius of curvature of $\mathcal{H}_h$ at $x_h(t)$. Since

$$\begin{aligned}\varepsilon(t) &= h'(t)\,u'(t) + h''(t)\,u''(t)\\ &= p(t + \pi/2)\,u(t + \pi/2) + p'(t + \pi/2)\,u'(t + \pi/2),\end{aligned}$$

for $h'(t) = p(t + \pi/2)$, we have that $\mathcal{E}$ is a projective hedgehog with a support function $h'(t - \pi/2)$, see e.g. [18] or [20].

Let $J : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $J(a, b) = (-b, a)$. The Zindler curve $\mathcal{Z}_{h,r}$ can be parameterized in two ways, z_1 and z_2, as in (1). Using (2), we have

$$z_1(t) - \varepsilon(t) = \big(h(t) + h''(t)\big)\,u(t) - r\,u'(t) = -Jz_1'(t)$$
$$z_2(t) - \varepsilon(t) = \big(h(t) + h''(t)\big)\,u(t) + r\,u'(t) = -Jz_2'(t),$$

which are orthogonal to $\mathcal{Z}_{h,r}$ at $z_1(t)$ and $z_2(t)$, respectively, and have the same length as $z_1'(t)$ and $z_2'(t)$. □

Remark 1. Proposition 5 provides a method to construct Zindler curves geometrically from the evolute of a projective hedgehog (see Figure 2).

Let $\mathcal{H}_h$ be a projective $\mathcal{C}^2$-hedgehog. Consider the evolute $\mathcal{E}$ of $\mathcal{H}_h$ parameterized by $\varepsilon(t)$. For all t, take the circle centered at $\varepsilon(t)$ that cuts the support line of $\mathcal{H}_h$ at $x_h(t)$ in two points $z_1(t)$ and $z_2(t)$ such that $\big[z_1(t),\, z_2(t)\big]$ has length $2\,r$. We have that $z_1(t)$ and $z_2(t)$ are two parameterizations of a Zindler curve. Furthermore, the Frenet frame of z_i at $z_i(t)$ is given by

$$\left\{\frac{J\big(z_i(t) - \varepsilon(t)\big)}{\big\|z_i(t) - \varepsilon(t)\big\|},\, \frac{\varepsilon(t) - z_i(t)}{\big\|z_i(t) - \varepsilon(t)\big\|}\right\}.$$

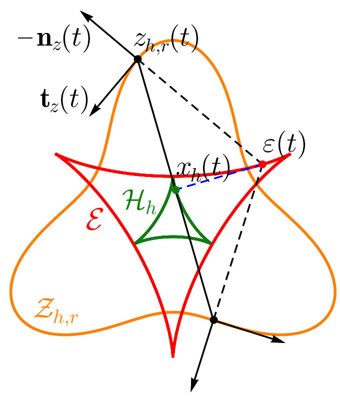

Figure 2: A projective hedgehog $\mathcal{H}_h$, its evolute $\mathcal{E}$ and the associated Zindler curve $\mathcal{Z}_{h,r}$ with tangent and normal vectors $\mathbf{t}_z$ and $\mathbf{n}_z$, respectively.

3. Geometric preliminaries in $\mathbb{R}^4$

In this section we will introduce the vector space in which we will work as well as some concepts on hedgehogs and symplectic geometry which are needed for our purpose.

Definition of a symplectic area

Let (V, J, ω) be a Kähler vector space, where J is a complex structure compatible with a symplectic form ω. The *symplectic area of a closed curve* $\gamma : \mathbb{S}^1 \to V$ is defined by

$$\mathcal{A}(\gamma) := \int_\gamma \alpha, \tag{3}$$

where α is the 1-form given by $(\alpha)_x(dx) = \frac{1}{2}\,\omega(x, dx)$, which is such that $d\alpha = \omega$. Notice that the integral (3) does not depend on the orientation of the curve γ (as if we change the orientation of γ, the 1-form α is changed into its opposite). Explicitly, if $\mathbb{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, the *symplectic area of* γ (sometimes called the *action of* γ) can be written as

$$\mathcal{A}(\gamma) = \frac{1}{2}\int_0^{2\pi} \omega\big(\gamma(t),\, \gamma'(t)\big)\, dt.$$

Geometrically, the symplectic area can be regarded as the sum of the algebraic areas of the projections of γ onto a set of orthogonal vector planes. The interested reader can find more details in [26], [25], [24] or [6].

Geometry of $\mathbb{R}^4$ as a Kähler vector space

Now, we are going to outfit a geometric structure to $\mathbb{R}^4$. The reader can see [20], [35] and [26] as main references for the background of this section.

We identify $\mathbb{R}^4$ with the quaternion algebra $\mathbb{H}$ and thus the unit sphere $\mathbb{S}^3$ with the set $\mathbb{S}^1_{\mathbb{H}}$ of unit quaternions. We denote by $\mathbb{S}^2$ the set $\mathbb{S}^1_{\mathbb{H}} \cap \mathrm{Im}\,(\mathbb{H})$ of pure unit quaternions. Furthermore, to any pure unit quaternion v we associate the linear complex structure $J_v : \mathbb{R}^4 \to \mathbb{R}^4$, $x \mapsto vx$. In other words, for any $v \in \mathbb{S}^2$, we choose to work in the Kähler vector space $\left(\mathbb{R}^4, J_v, \omega_v\right)$, where ω_v denotes the associated Kähler form (i.e. the alternating 2-form $\omega_v\,(X,Y) = \langle J_v X, Y\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean metric on $\mathbb{R}^4$). To any $v \in \mathbb{S}^2$, it corresponds a Hopf fibration and a Hopf flow on $\mathbb{S}^3 \cong \mathbb{S}^1_{\mathbb{H}}$ leaving the Hopf fibration invariant, namely the Hopf flow induced on $\mathbb{S}^3$ by the vector field $X_v\,(u) := J_v\,(u)$, that is the Hopf flow $\{(\phi_v)_\theta\}_{\theta \in \mathbb{S}^1}$ given by $(\phi_v)_\theta\,(u) := (\cos\theta)\,u + (\sin\theta)\,vu$, for any $u \in \mathbb{S}^3$.

For every $(u,v) \in \mathbb{S}^3 \times \mathbb{S}^2$, let $\mathbb{S}^1_{u,v}$ be the oriented geodesic of $\mathbb{S}^3$ through u in the direction of $J_v(u)$. This oriented Hopf circle of $\mathbb{S}^3 \subset \left(\mathbb{R}^4, J_v\right)$ can be regarded as a unit circle of the vector plane $\mathbb{C}(u,v) := \mathbb{R}u + \mathbb{R}J_v\,(u)$ oriented by $(u, J_v(u))$. Conversely, any oriented vector plane ξ in $\mathbb{R}^4$ determines an oriented unit circle $\mathbb{S}^1_\xi = \mathbb{S}^3 \cap \xi$ and a pure unit quaternion v_ξ that is such that: for all $u \in \mathbb{S}^1_\xi$, $T_u\mathbb{S}^1_\xi$ is oriented by the unit vector $J_{v_\xi}(u)$.

Thus, the symplectic area of $\gamma : \mathbb{S}^1_{u,v} \to \mathbb{R}^4$ in the Kähler vector space $\left(\mathbb{R}^4, J_v, \omega_v\right)$ is the sum of the algebraic areas of its projections onto the planes $\mathbb{C}(u,v)$ and $\mathbb{C}(u,v)^\perp$. Notice that the orientation of $\mathbb{S}^1_{u,v}$ (or of the plane that contains this Hopf circle) is not relevant for the computation of the symplectic area.

Expression of the symplectic area of the image of a Hopf circle under hedgehogs

Let $\mathcal{H}_h$ be any C^2-hedgehog in $\mathbb{R}^4$ with a support function h. Let $s_{u,v}(h)$ be the symplectic area of the curve $x_h : \mathbb{S}^1_{u,v} \to \mathbb{R}^4$:

$$s_{u,v}\,(h) := \int_{x_h\left(\mathbb{S}^1_{u,v}\right)} \alpha_v,$$

where α_v is the 1-form given by $(\alpha_v)_x\,(dx) := \frac{1}{2}\omega_v(x, dx) = \frac{1}{2}\left\langle x, (-J_v)\,(dx)\right\rangle$. The first author proved the following proposition in [20]:

Proposition 6. *For all $h \in C^\infty\left(\mathbb{S}^3; \mathbb{R}\right)$ and $(u,v) \in \mathbb{S}^3 \times \mathbb{S}^2$,*

$$s_{u,v}\,(h) = \frac{1}{2}\int_0^{2\pi} \langle x_h(u_\theta), R_h(u_\theta, v)\,u_\theta\rangle\, d\theta,$$

where $u_\theta := (\cos\theta)\,u + (\sin\theta)\,J_v(u)$ and $R_h(u_\theta, v) := -v\,(T_{u_\theta}x_h)\,(J_v\,(u_\theta))\,\overline{u_\theta}$, with $\overline{u_\theta}$ being the quaternion conjugate of u_θ.

Decomposition of hedgehogs [20]

Let (v,w) be any couple of pure unit quaternions that are orthogonal when they are regarded as vectors of $\mathbb{R}^4$. The quadruple $(1, v, w, vw)$ is then a direct orthonormal basis of $\mathbb{H} \cong \mathbb{R}^4$. For any C^2-hedgehog $\mathcal{H}_h$ in $\mathbb{R}^4$ and, for any $u \in \mathbb{S}^3$, we have the following decompositions:

$$\begin{aligned} x_h(u) &= h(u)\,u + \nabla h(u) \\ &= h(u)\,u + \langle \nabla h(u), vu\rangle\, vu + \langle \nabla h(u), wu\rangle\, wu + \langle \nabla h(u), vwu\rangle\, vwu \\ &= h(u)\,u + \partial_v h(vu)\, vu + \partial_w h(wu)\, wu + \partial_{vw} h(vwu)\, vwu \\ &= \big(h(u) + \partial_v h(vu)\, v + \partial_w h(wu)\, w + \partial_{vw} h(vwu)\, vw\big)u. \end{aligned}$$

4. Zindler-type hypersurfaces in $\mathbb{R}^4$

Let $\mathcal{H}_h$ be a C^2-hedgehog in $\mathbb{R}^4$ that is projective. Recall that the condition for a C^2-hedgehog $\mathcal{H}_h$ to be projective is that $h(u) + h(-u) = 0$ for all $u \in \mathbb{S}^3$, and thus $x_h(-u) = x_h(u)$. For all $r > 0$, the C^2-hedgehog of $\mathbb{R}^4$ with support function $h+r$ is then of constant width $2r$ (in other words, the distance between the two support hyperplanes that are orthogonal to the line $\mathbb{R}u$ is equal to $2r$), and if r is large enough then $\mathcal{H}_{h+r}$ is necessarily convex and regular.

Definition 1. Let $\mathcal{H}_h$ be a projective hedgehog in $\mathbb{R}^4$ and let $r > 0$ be such that $\mathcal{H}_{h+r}$ is a convex hypersurface of constant width. Given a pure unit quaternion $v \in \mathbb{S}^2$, the hypersurface $\mathcal{Z}^v_{h,r}$ of $\mathbb{R}^4$ that is parametrized by

$$\begin{aligned} z^v_{h,r} :\ & \mathbb{S}^3 \to \mathbb{R}^4 \\ & u \mapsto x_h\,(u) + rvu. \end{aligned}$$

will be called the *v-Zindler hypersurface associated with $\mathcal{H}_{h+r}$.*

Notice that the hypersurface $\mathcal{Z}^v_{h,r} = z^v_{h,r}\left(\mathbb{S}^3\right)$ is fibrated by the smooth curves that are the image of the Hopf circles $\mathbb{S}^1_{u,v}$ under $z^v_{h,r} : \mathbb{S}^3 \to \mathbb{R}^4$. The following result essentially states some properties that make $\mathcal{Z}^v_{h,r}$ a Zindler-type hypersurface.

Theorem 1. *Let $\mathcal{H}_h$ be a projective hedgehog in $\mathbb{R}^4$ and let $r > 0$ be such that $\mathcal{H}_{h+r}$ is a convex hypersurface of constant width $2\,r$. Let $\mathcal{Z}^v_{h,r}$ be the associated v-Zindler hypersurface, for some $v \in \mathbb{S}^2$. Then*

1. *Any chord $\left[z^v_{h,r}(-u), z^v_{h,r}(u)\right]$, with $u \in \mathbb{S}^3$, has the projective hedgehog $x_h(u)$ as its midpoint and has constant length $2\,r$.*
2. *The curve $\mathcal{Z}^{u,v}_{h,r} := z^v_{h,r}\left(\mathbb{S}^1_{u,v}\right)$ is regular and has perimeter halving chords*

   $$\left[z^v_{h,r}\,(-u_\theta)\,,\, z^v_{h,r}\,(u_\theta)\right],$$

   where $u_\theta := (\cos\theta)\,u + (\sin\theta)\,vu \in \mathbb{S}^1_{u,v}$ for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.
3. *The halving chords $\left[z^v_{h,r}\,(-u_\theta)\,,\, z^v_{h,r}\,(u_\theta)\right]$ form the same angle with both tangent vectors to the curve $\mathcal{Z}^{u,v}_{h,r}$ at the corresponding endpoints.*

4. *The orthogonal projection of the curve* $\mathcal{Z}_{h,r}^{u,v}$ *onto the vector plane* $\mathbb{C}(u,v) := \mathbb{R}u + \mathbb{R}J_v(u)$ *oriented by* $(u, J_v(u))$ *is a Zindler curve of* $\mathbb{C}(u,v)$ *that is parametrized by*

$$\begin{aligned} z: \quad \mathbb{R}/2\pi\mathbb{Z} &\to \mathbb{C}\,(u,v) \\ \theta &\mapsto x_{h|_{\mathbb{S}^1_{u,v}}}(\theta) + rvu_\theta, \end{aligned}$$

where $\mathcal{H}_{h|_{\mathbb{S}^1_{u,v}}}$ *is the projective hedgehog of* $\mathbb{C}(u,v)$ *whose support function is the restriction of* h *to* $\mathbb{S}^1_{u,v}$.

Proof. 1. By construction, the Zindler hypersurface satisfies

$$\frac{z_{h,r}(u) + z_{h,r}(-u)}{2} = x_h(u),$$

which is the projective hedgehog $\mathcal{H}_h$. Moreover, since $x_h(-u) = x_h(u)$ for all $u \in \mathbb{S}^3$, we have

$$\left\| z_{h,r}(u) - z_{h,r}(-u) \right\| = \left\| 2\,rvu \right\| = 2\,r.$$

2. First, notice that $u_{\theta+\pi} = -u_\theta \in \mathbb{S}^1_{u,v}$. We must show that the length of each part of $\mathcal{Z}_{h,r}^{u,v}$ connecting $z_{h,r}^v\,(-u_\theta)$ and $z_{h,r}^v\,(u_\theta)$ is one half the length of $\mathcal{Z}_{h,r}^{u,v}$. Indeed, the curve $\theta \mapsto z_{h,r}^v\,(\theta) := z_{h,r}^v\,(u_\theta)$ is such that for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$,

$$\left(z_{h,r}^v\right)'(\theta) = \left(T_{u_\theta} x_h\right)(vu_\theta) + rv\,(vu_\theta) = \left(T_{u_\theta} x_h\right)(vu_\theta) - ru_\theta,$$

where $T_{u_\theta} x_h$ is the tangent map of x_h at u_θ. Since $\mathcal{H}_h$ is projective,

$$\begin{aligned} (z_{h,r}^v)'(\theta+\pi) &= (T_{u_{\theta+\pi}} x_h)(vu_{\theta+\pi}) - ru_{\theta+\pi} \\ &= (T_{-u_\theta} x_h)(-vu_\theta) + ru_\theta = (T_{u_\theta} x_h)(vu_\theta) + ru_\theta. \end{aligned}$$

Thus,

$$\left\| \left(z_{h,r}^v\right)'(\theta+\pi) \right\| = \sqrt{\left\| \left(T_{u_\theta} x_h\right)(vu_\theta) \right\|^2 + r^2} = \left\| \left(z_{h,r}^v\right)'(\theta) \right\|.$$

This means that the norm $\left\| \left(z_{h,r}^v\right)'(\theta) \right\|$ is π-periodic and, therefore,

$$\int_t^{t+\pi} \left\| \left(z_{h,r}^v\right)'(\theta) \right\| dt = \int_0^\pi \left\| \left(z_{h,r}^v\right)'(\theta) \right\| dt = \frac{L}{2},$$

where L is the length of $\mathcal{Z}_{h,r}^{u,v}$. The curve $\mathcal{Z}_{h,r}^{u,v}$ is indeed regular because $r > 0$.

3. We have that $z_{h,r}^v(u_\theta) - z_{h,r}^v(-u_\theta) = 2\,rvu_\theta$ is orthogonal to u_θ for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Therefore,

$$\left\langle z_{h,r}^v(u_\theta) - z_{h,r}^v(-u_\theta),\, (z_{h,r}^v)'(\theta) \right\rangle = \left\langle z_{h,r}^v(u_\theta) - z_{h,r}^v(-u_\theta),\, (z_{h,r}^v)'(\theta+\pi) \right\rangle.$$

4. By the decomposition of a hedgehog given in the previous section, we have that

$$x_f(u) = f(u)\,u + \partial_v f(vu)\,vu \in \mathbb{C}(u,v),$$

where $f = h|_{\mathbb{C}(u,v)}$. Therefore, the orthogonal projection of $\mathcal{Z}^{u,v}_{h,r}$ onto $\mathbb{C}(u,v)$ can be parameterized by

$$z(\theta) = x_{h|_{\mathbb{S}^1_{u,v}}}(\theta) + rvu_\theta,$$

for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. It is a planar Zindler curve by construction, as $x_{h|_{\mathbb{S}^1_{u,v}}}$ is a planar hedgehog parameterized by supporting lines which are parallel to vu_θ. Notice, in fact, that the length r is large enough to have a Zindler curve because it is associated with the curve defined by the same orthogonal projection of $\mathcal{H}_{h+r}$ onto $\mathbb{C}(u,v)$, which is known to be a planar convex curve of constant width. □

Remark 2. The curve $\mathcal{Z}^{u,v}_{h,r} := z^v_{h,r}\left(\mathbb{S}^1_{u,v}\right)$ from Theorem 1 is a spatial Zindler curve in the sense of Pottmann [28] because it is written as

$$z^v_{h,r}(\theta) = x_h(u_\theta) + rvu_\theta$$

where $m(\theta) = x_h(u_\theta)$ is the striction line of the ruled surface generated by the halving chords with directions $e(\theta) = vu_\theta$. This is because $m'(\theta)$ is orthogonal to $e'(\theta)$. In addition, we have $m(\theta+\pi) = m(\theta)$ and $e(\theta+\pi) = -e(\theta)$.

Therefore, we can say that the Zindler hypersurface $\mathcal{Z}^v_{h,r}$ is fibrated by space Zindler curves in the sense of Pottmann.

The following result aims to show that the space Zindler curve $\mathcal{Z}^{u,v}_{h,r}$ on the Zindler hypersurface $\mathcal{Z}^v_{h,r}$ has an associated space curve of constant width lying on the corresponding hypersurface of constant width $\mathcal{H}_{h+r}$.

There are several generalizations of constant width curves to space in the literature and which are not necessarily equivalent. Fujiwara [10] was the first one to make a definition in $\mathbb{R}^3$. He also considered a more restrictive class of curves there and later in [11]. After him, some other authors provided similar definitions based on his work (see e.g. [37], [3, p. 147] or [36]).

The original definition by Fujiwara extended to $\mathbb{R}^n$ reads as follows.

Definition 2 (Fujiwara)**.** Let C be a closed and regular curve in $\mathbb{R}^n$. Given a point $A \in C$, let $\delta_{A,P}$ be the shortest distance between the tangent line at A and that at another point $P \in C$. The *width of C with respect to A* is defined by

$$M_A = \max_{P\in C} \delta_{A,P}.$$

The curve C is said to be *of constant width* if M_A is constant for all $A \in C$.

It can be proved that a sufficient condition to have a space curve of constant width is the following (see also the characterizations provided in [32]):

Proposition 7. *If there exists a diffeomorphism $P \rightleftarrows P'$ between points of C such that PP' are maximal chords which are double-normal (i.e. orthogonal to the tangents to C at the endpoints P and P') and such that as a point M on C moves from P to P' according to an orientation, the point M' moves from P' to P with the same orientation, then the curve C is of constant width.*

We are going to use this proposition in the following result.

Theorem 2. *Let $\mathcal{H}_h$ be a projective hedgehog in $\mathbb{R}^4$ and let $r > 0$ be such that $\mathcal{H}_{h+r}$ is a convex and regular hypersurface of constant width $2r$. Let $\mathcal{Z}^v_{h,r}$ be its associated v-Zindler hypersurface, for some $v \in \mathbb{S}^2$. Then, for all $(u, v) \in \mathbb{S}^3 \times \mathbb{S}^2$, the curve $x_{h+r}(\mathbb{S}^1_{u,v})$ is a space curve of constant width that has the space Zindler curve $z^v_{h,r}(\mathbb{S}^1_{u,v})$ as its associated by a rotation of the constant length chords through the unit pure quaternion v.*

Proof. Let $\alpha : \mathbb{S}^1_{u,v} \to \mathbb{R}^4$ be defined by $\alpha(\theta) = x_h(u_\theta) + r\,u_\theta$, where $u_\theta = (\cos\theta)\,u + (\sin\theta)\,vu$ for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. First, notice that α is regular since x_{h+r} is regular.

The points $\alpha(\theta)$ and $\alpha(\theta + \pi)$ are unequivocally associated by means of a diffeomorphism $\theta \mapsto \theta + \pi$. In addition,

$$\alpha'(\theta) = \big(T_{u_\theta} x_h\big)(v\,u_\theta) + rvu_\theta,$$
$$\alpha'(\theta + \pi) = \big(T_{u_\theta} x_h\big)(v\,u_\theta) - rvu_\theta.$$

and

$$\alpha(\theta) - \alpha(\theta + \pi) = 2ru_\theta.$$

Thus, the chords with endpoints $\alpha(\theta)$ and $\alpha(\theta + \pi)$ are double-normal. Since $\mathcal{H}_h$ is projective, these chords are of constant length:

$$\big\|\alpha(\theta) - \alpha(\theta + \pi)\big\| = 2r.$$

As α lies on a hypersurface of constant width $2r$, these chords are also maximal. Therefore, we conclude by Proposition 7 that the curve α is of constant width.

Finally, notice that if the chords of constant length

$$\alpha(\theta) - \alpha(\theta + \pi) = 2ru_\theta$$

are rotated through the pure unit quaternion v, then we obtain the halving chords

$$z^v_{h,r}(u_\theta) - z^v_{h,r}(u_{\theta+\pi}) = 2rvu_\theta$$

of $\mathcal{Z}^{u,v}_{h,r} = z^v_{h,r}(\mathbb{S}^1_{u,v})$, so that α is associated with $z^v_{h,r}$ through the unit pure quaternion v. □

Remark 3. The space curve of constant width of Theorem 2 is not transnormal, as the vectors $\alpha'(\theta)$ and $\alpha'(\theta + \pi)$ are not parallel and, therefore, they do not share the same normal hyperplane at the corresponding points $\alpha(\theta)$ and $\alpha(\theta+\pi)$.

The previous results can be complemented with the following theorem, which states that the symplectic areas of these pairs of associated curves (a space curve of constant width and a space Zindler curve) are the same. This constitutes a generalization of Proposition 2.

Theorem 3. *Let $\mathcal{H}_h$ be a projective hedgehog in $\mathbb{R}^4$ and let $r > 0$ be such that $\mathcal{H}_{h+r}$ is a convex hypersurface of constant width $2r$. Let $\mathcal{Z}^v_{h,r}$ be its associated v-Zindler hypersurface, for some $v \in \mathbb{S}^2$. Then, for all $(u, v) \in \mathbb{S}^3 \times \mathbb{S}^2$, the symplectic area of $z^v_{h,r}(\mathbb{S}^1_{u,v})$ is equal to the symplectic area $s_{u,v}(h+r)$ of $x_{h+r}(\mathbb{S}^1_{u,v})$. More precisely, both symplectic areas are equal to $s_{u,v}(h) + \pi r^2$.*

Proof. By definition, the sympectic area of $z^v_{h,r}(\mathbb{S}^1_{u,v})$ in the Kähler vector space $\left(\mathbb{R}^4, J_v, \omega_v\right)$ is

$$\begin{aligned}\int_{z^v_{h,r}(\mathbb{S}^1_{u,v})} \alpha_v &= \frac{1}{2}\int_0^{2\pi} \Big\langle z^v_{h,r}(\theta), (-J_v)\Big(\big(z^v_{h,r}\big)'(\theta)\Big)\Big\rangle\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \Big\langle x_h(u_\theta) + rvu_\theta, (-J_v)\big((T_{u_\theta}x_h)(vu_\theta) - ru_\theta\big)\Big\rangle\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \Big\langle x_h(u_\theta) + rvu_\theta, (-J_v)\big((T_{u_\theta}x_h)(vu_\theta)\big) + rvu_\theta\Big\rangle\, d\theta.\end{aligned}$$

By bilinearity of $\langle\cdot,\cdot\rangle$ we then deduce

$$\begin{aligned}\int_{z^v_{h,r}(\mathbb{S}^1_{u,v})} \alpha_v = s_{u,v}(h) + \pi r^2 &+ \frac{r}{2}\int_0^{2\pi} \big\langle x_h(u_\theta), vu_\theta\big\rangle\, d\theta \\ &- \frac{r}{2}\int_0^{2\pi} \big\langle vu_\theta, v(T_{u_\theta}x_h)(vu_\theta)\big\rangle\, d\theta.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^{2\pi} \big\langle x_h(u_\theta), vu_\theta\big\rangle\, d\theta &= \int_0^{2\pi} \big\langle h(u_\theta)u_\theta + \nabla h(u_\theta), vu_\theta\big\rangle\, d\theta \\ &= \int_0^{2\pi} \big\langle \nabla h(u_\theta), vu_\theta\big\rangle\, d\theta = 0,\end{aligned}$$

because for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we have

$$\big\langle \nabla h(u_{\theta+\pi}), vu_{\theta+\pi}\big\rangle = -\big\langle \nabla h(u_\theta), vu_\theta\big\rangle,$$

by the fact that $\mathcal{H}_h$ is projective. Since $J_v : \mathbb{R}^4 \to \mathbb{R}^4$, $x \mapsto vx$ is an isometry we also have

$$\int_0^{2\pi} \big\langle vu_\theta, v(T_{u_\theta}x_h)(vu_\theta)\big\rangle\, d\theta = \int_0^{2\pi} \big\langle u_\theta, (T_{u_\theta}x_h)(vu_\theta)\big\rangle\, d\theta = 0.$$

Therefore,

$$\int_{z^v_{h,r}(\mathbb{S}^1_{u,v})} \alpha_v = s_{u,v}(h) + \pi r^2.$$

Now, by Proposition 6 we have

$$\begin{aligned}s_{u,v}(h+r) &= \frac{1}{2}\int_0^{2\pi} \big\langle x_{h+r}(u_\theta), R_{h+r}(u_\theta, v)u_\theta\big\rangle\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \big\langle x_h(u_\theta) + ru_\theta, R_h(u_\theta, v)u_\theta + ru_\theta\big\rangle\, d\theta\end{aligned}$$

and by bilinearity of $\langle\cdot,\cdot\rangle$ we deduce then

$$s_{u,v}(h+r) = s_{u,v}(h) + \pi r^2 + \frac{r}{2}\int_0^{2\pi} \big\langle x_h(u_\theta), u_\theta \big\rangle \, d\theta + \frac{r}{2}\int_0^{2\pi} \big\langle u_\theta, R_h(u_\theta, v) u_\theta \big\rangle \, d\theta.$$

The first integral is

$$\int_0^{2\pi} \big\langle x_h(u_\theta), u_\theta \big\rangle \, d\theta = \int_0^{2\pi} \big\langle h(u_\theta)u_\theta + \nabla h(u_\theta), u_\theta \big\rangle \, d\theta = \int_0^{2\pi} h(u_\theta)\, d\theta = 0,$$

because $\mathcal{H}_h$ is projective. Again, since J_v is an isometry, we also have

$$\int_0^{2\pi} \big\langle u_\theta, R_h(u_\theta, v) u_\theta \big\rangle \, d\theta = \int_0^{2\pi} \big\langle u_\theta, -v(T_{u_\theta} x_h)(v u_\theta)\big\rangle \, d\theta = \int_0^{2\pi} \big\langle v u_\theta, (T_{u_\theta} x_h)(v u_\theta)\big\rangle \, d\theta = 0,$$

by the fact that

$$\big\langle v u_{\theta+\pi}, (T_{u_{\theta+\pi}} x_h)(v u_{\theta+\pi})\big\rangle = \big\langle -v u_\theta, -(T_{u_\theta} x_h)(-v u_\theta)\big\rangle$$

for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Therefore,

$$s_{u,v}(h+r) = s_{u,v}(h) + \pi r^2. \qquad \square$$

The relation between the evolute of the projective hedgehog and the Zindler curve in the plane described in Proposition 5 can also be extended to $\mathbb{R}^4$.

Recall that the evolute of a projective hedgehog $\mathcal{H}_h$ in $(\mathbb{R}^4, J_v, \omega_v)$ with respect to the pure unit quaternion $v \in \mathbb{S}^2$ is a projective hedgehog that can be parameterized as

$$\varepsilon_v(u) = x_h(u) - R_h(u, v)\, u,$$

where $R_h(u,v) = -v\, T_u x_h\big(J_v(u)\big)\, \overline{u}$, for any $u \in \mathbb{S}^3$ (see [20]).

Theorem 4. *Let $\mathcal{H}_h$ be a projective hedgehog in $\mathbb{R}^4$ with a $\mathcal{C}^2$-support function h and let $\mathcal{E}_v$ be its evolute with respect to a pure unit quaternion $v \in \mathbb{S}^2$, which is a projective hedgehog. Given $r > 0$, let $\mathcal{Z}^v_{h,r}$ be the v-Zindler hypersurface associated with $\mathcal{H}_{h+r}$. Then, for all $(u,v) \in \mathbb{S}^3 \times \mathbb{S}^2$, the vector $z^v_{h,r}(u_\theta) - \varepsilon_v(u_\theta)$ has the same length as $(z^v_{h,r})'(u_\theta)$ and it is orthogonal to $\mathcal{Z}^v_{h,r}$ at $z^v_{h,r}(u_\theta)$, where $u_\theta = (\cos\theta)\, u + (\sin\theta)\, vu \in \mathbb{S}^1_{u,v}$.*

Proof. The v-Zindler hypersurface $\mathcal{Z}^v_{h,r}$ can be parameterized as $z^v_{h,r} : \mathbb{S}^1 \to \mathbb{R}^4$ by

$$z^v_{h,r}(\theta) = x_h(u_\theta) + r v u_\theta.$$

We have

$$\left(z_{h,r}^{v}\right)'(\theta) = (T_{u_\theta} x_h)(v u_\theta) - r u_\theta.$$

The image of a Hopf circle $\mathbb{S}^1_{u,v}$ through the parameterization of the evolute $\mathcal{E}_v$ can be written as $\varepsilon_v : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^4$ given by

$$\varepsilon_v(\theta) = x_h(u_\theta) + v\,(T_{u_\theta} x_h)(v u_\theta).$$

Therefore,

$$z_{h,r}^{v}(\theta) - \varepsilon_v(\theta) = r v u_\theta - v\,(T_{u_\theta} x_h)(v u_\theta) = -v\left(z_{h,r}^{v}\right)'(\theta),$$

which implies that

$$\left\| z_{h,r}^{v}(\theta) - \varepsilon_v(\theta) \right\| = \sqrt{r^2 + \left\|(T_{u_\theta} x_h)(v u_\theta)\right\|^2} = \left\| \left(z_{h,r}^{v}\right)'(\theta) \right\|$$

and

$$\left\langle z_{h,r}^{v}(\theta) - \varepsilon_v(\theta),\, \left(z_{h,r}^{v}\right)'(\theta) \right\rangle = 0. \qquad \square$$

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