AN EXPLICIT CHARACTERIZATION OF ISOCHORDAL-VIEWED MULTIHEDGEHOGS WITH CIRCULAR ISOPTICS

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ABSTRACT. A curve α is called (ϕ, ℓ)-isochordal viewed if a straight segment of constant length ℓ can slide with its endpoints on α and such that their tangents to α at these endpoints make a constant angle ϕ. These tangents determine the so-called ϕ-isoptic curve of α. In this paper, an explicit characterization of all (ϕ, ℓ)-isochordal-viewed multihedgehogs with circular ϕ-isoptics is provided by their support functions, which are obtained as the solutions of a differential equation. This allows to construct any example of these curves in a very simple way from some free parameters. In addition, it is shown that a regular polygon of side length ℓ can slide smoothly along these multihedgehogs.

1. INTRODUCTION

In the recent years, the study of isoptic curves, support functions and related topics has attracted the interest of many researchers (see [3], [5], [15] or [6] among others). A curve α parameterized by a support function h can be written as

$$\alpha(t) = h(t)\,(\cos t,\, \sin t) + h'(t)\,(-\sin t,\, \cos t), \quad t \in I,$$

where $I = [0,\, 2\,m\,\pi]$, for some $m \in \mathbb{Z}$, and the function h is $2\,\pi\,m$-periodic. The curve α is called a m-hedgehog (sometimes called multihedgehog, see [7]) and it has m cooriented supporting lines to α with a given normal vector. Convex curves are examples of 1-hedgehogs (see in Figure 5 other examples). Any rosette (see e.g. those of Figure 2-right) is a multihedgehog [8].

Given $\phi \in \,]0, \pi[$, the ϕ-isoptic of α is defined as the locus of points through which a pair of tangents to α pass making an angle ϕ (see e.g. [1] and [2] for an introduction to isoptics and Figure 1 for a visualization).

Isoptics have many applications in engineering, mainly related to cam mechanisms and gears (see [12] and [13]). A very important particular case is when isoptics are circles. If the ϕ-isoptic of a closed curve α is circular, then α is called of constant ϕ-width [9]. A full characterization of this kind of curves in terms of their support function by a Fourier series has been given recently in [3].

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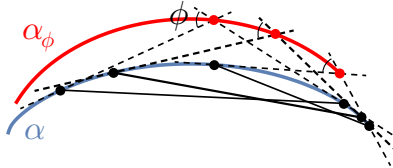
FIGURE 1. Definition of a ϕ-isoptic of α.

Given $\phi \in {]}0, \pi{[}$ and $\ell > 0$, a planar curve α is said to be (ϕ, ℓ)-isochordal viewed if a ϕ-isoptic of α is such that the chord joining the contact points with α of its supporting lines meeting at α_ϕ has constant length ℓ.

Examples of non-circular (ϕ, ℓ)-isochordal-viewed curves have been given before in [10] and [11]. In this paper we provide a full and explicit characterization of all (ϕ, ℓ)-isochordal-viewed multi-hedgehogs of constant ϕ-width (Theorem 2). Moreover, this characterization is constructive, in the sense that by setting two parameters (two natural numbers), we obtain a support function that describes the shape of a curve of this kind in a very easy way (Theorem 3). This is achieved by solving a differential equation that describes this kind of curves. The explicit expression given in the characterization also allows to derive easily some properties of these curves (see e.g. Proposition 2 on their singularities).

Another interesting fact on this kind of curves is that a regular polygon is allowed to move with its endpoints always lying on these curves. This was already proved in [11] for hedgehogs. Finally, in Section 4, we extend this result to multihedgehogs (Theorem 4) and we present it in a different way: given one of these curves, we show the existence of a regular polygon of n sides that can travel along such a curve. It is seen that except for two particular curves in which only a straight chord ($n = 2$) is allowed, the number n is greater than 2 in the general case.

2. PRELIMINARY RESULTS

In this section we are going to clarify some of definitions given above and to recall some known results that will be useful later.

First of all, notice that a ϕ-isoptic of α defined as above is not unique even for simple convex examples (see Figure 2-left). The uniqueness can be determined in this case by setting from which angle, either ϕ or $\pi - \phi$, the isoptic is "looking" at the curve. However, for general curves with cusps, the tangent vectors can flip when crossing these cusps and hence the angle between these tangents can change from ϕ to $\pi - \phi$. Therefore, we must understand isoptics as presented in the introduction in order to extend the concept to general non-convex curves (see the discussion in [10]). This is, isoptics will be determined by the angle between tangent lines rather than by the angle between tangent vectors.

Second, for general non-convex curves, there can be different points in the curve to construct a ϕ-isoptic given an angle $\phi \in {]}0, \pi{[}$ (see Figure 2-right). In this sense, the definition of ϕ-isoptic turns out to be even less specific than in the convex case.
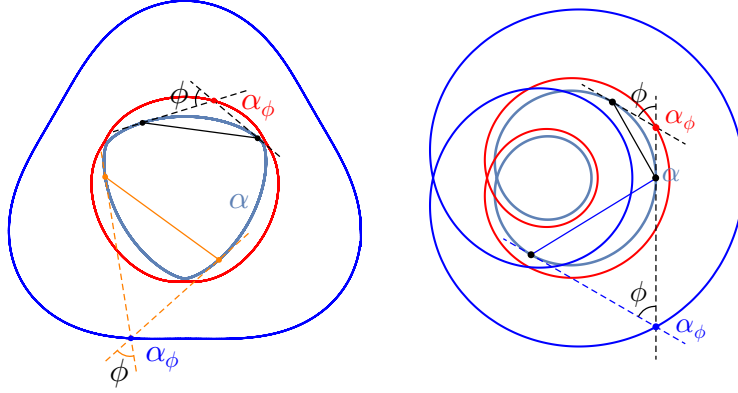
FIGURE 2. On the left, a convex curve α that have two different ϕ-isoptics α_ϕ. On the right, two different ϕ-isoptics α_ϕ to a self-intersecting curve α.

In this paper, we will only consider curves which are m-hedgehogs. The supporting line of a m-hedgehog α makes m revolutions. In this case a ϕ-isoptic of α can be constructed from the points $\alpha(t)$ and $\alpha(t + \phi + c\,\pi)$, for any $c \in \mathbb{Z}$ such that $\phi + c\,\pi \in\,]0, m\,\pi[$. To simplify the notation, we want to avoid this kind of ambiguity on the definition of an isoptic to a m-hedgehog. This can be done by allowing an angle $\phi \in\,]0,\, m\,\pi[$ from the beginning such that $\alpha(t)$ and $\alpha(t + \phi)$ are the associated points to construct the associated ϕ-isoptic. Of course, the actual geometric angle of this ϕ-isoptic would be $\phi \pmod \pi$. It is important to avoid parallel supporting lines, so the constraint is that $\phi \pmod \pi \in\,]0, \pi[$. Thus, rigorously speaking, we will take $\phi \in \Phi_m$, where

$$\Phi_m = \,]0,\, m\,\pi[\, \setminus \big\{ k\,\pi : k \in \mathbb{Z} \big\} = \bigcup_{1 \le k \le m}](k-1)\,\pi,\, k\,\pi[.$$

So, the condition for a m-hedgehog α of being (ϕ, ℓ)-isochordal viewed, given $\phi \in \Phi_m$, is just that it satisfies the isochordal condition:

$$(1) \qquad\qquad \big\| \alpha(t + \phi) - \alpha(t) \big\| = \ell,$$
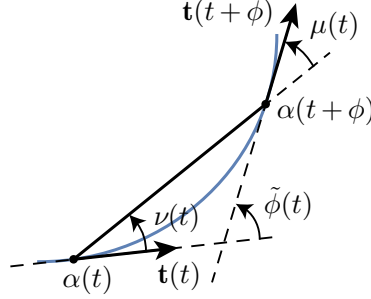
for all $t \in I$.

Recall now some definitions from [10]. Define $\nu(t)$ as the oriented angle function from $\mathbf{t}(t)$ to $\alpha(t + \phi) - \alpha(t)$ and $\mu(t)$ as the oriented angle function from $\alpha(t + \phi) - \alpha(t)$ to $\mathbf{t}(t + \phi)$. Assume that the starting angle at $\inf I$ is oriented in $]-\pi, \pi]$.

Define

$$\tilde{\phi}(t) := \nu(t) + \mu(t).$$

We have that $\tilde{\phi}$ is a piecewise-constant function (either ϕ or $\pi - \phi$ up to a multiple of π), which gives an oriented angle from $\mathbf{t}(t)$ to $\mathbf{t}(t + \phi)$. See Figure 3 for a visualization of these angles.

We can extend some of the results of [10] to m-hedgehogs easily with our current terminology (with the same proof). The generalized versions can be stated as follows. Assume that α is as sufficiently differentiable as needed.

FIGURE 3. Definition of the angles $\nu(t)$, $\mu(t)$ and $\tilde{\phi}(t)$.

Lemma 1. *Let $m \in \mathbb{N}$, $\phi \in \Phi_m$ and let $\alpha : I \to \mathbb{R}^2$ be a (ϕ, ℓ)-isochordal-viewed m-hedgehog. Then*

$$\left\|\alpha'(t)\right\| \cos \nu(t) = \left\|\alpha'(t + \phi)\right\| \cos \mu(t), \tag{2}$$

for all $t \in I$.

The expression (2) is found simply by differentiating the isochordal condition. This formula is known as a *projection rule* in the field of kinematics.

The following two results are not as obvious as previous lemma. We refer the reader to [10] for their proof.

Lemma 2. *Let $m \in \mathbb{N}$, $\phi \in \Phi_m$ and let $\alpha : I \to \mathbb{R}^2$ be a piecewise-$\mathcal{C}^2$ (ϕ, ℓ)-isochordal-viewed m-hedgehog parameterized by a support function $h \in \mathcal{C}^3$. Then*

$$\ell \left(\nu'(t) + 1\right) \cos \mu(t) = \left\|\alpha'(t)\right\| \sin \tilde{\phi}(t),$$

for all $t \in I$ such that α is regular at t.

Theorem 1. *Let α be a piecewise-$\mathcal{C}^2$ (ϕ, ℓ)-isochordal-viewed m-hedgehog parameterized by a support function $h \in \mathcal{C}^3$. The curve α is of constant ϕ-width if and only if $\nu'(t)$ is constant for all $t \in I$ such that $\tilde{\phi}$ is continuous at t.*

3. CHARACTERIZATION OF ISOCHORDAL-VIEWED MULTIHEDGEHOGS WITH CIRCLES AS ISOPTICS

The aim of this section is to translate the conditions of α being (ϕ, ℓ)-isochordal viewed and of constant ϕ-width into equations only dependent on the support function h of α and the constant angle ϕ. This will lead to a differential equation on h that will describe all the m-hedgehogs of this kind. Before stating the main theorem, we need some partial results.

Lemma 3. *Let $m \in \mathbb{N}$, $\phi \in \Phi_m$ and let α be a (ϕ, ℓ)-isochordal-viewed m-hedgehog.*

  (i) *If α is regular at $t_0 \in I$, then $\cos \mu(t_0) \neq 0$.*
  (ii) *If α is regular at $t_0 + \phi \in I$, then $\cos \nu(t_0) \neq 0$.*

Proof. Let us see (i). If $t_0 \in I$ is such that $\cos \mu(t_0) = 0$, then by Lemma 1 we would have $\|\alpha'(t_0)\| = 0$. This is because $\cos \nu(t_0) \neq 0$, otherwise the geometric angle given by ϕ would be equal to 0 or π. A similar discussion can be done for (ii). $\square$

Lemma 4. *Let $m \in \mathbb{N}$, $\phi \in \Phi_m$ and let α be a (ϕ, ℓ)-isochordal-viewed m-hedgehog parameterized by a support function h. Then:*

(3) $$\ell \, \|\alpha'(t)\| \cos \nu(t) = \big(h''(t) + h(t)\big) \big(\cos(\phi) \, h'(t + \phi) - h'(t) + \sin(\phi) \, h(t + \phi)\big).$$

and

(4) $$\ell \, \|\alpha'(t + \phi)\| \cos \mu(t) = \big(h(t + \phi) + h''(t + \phi)\big) \big(h'(t + \phi) - \cos(\phi) \, h'(t) + h(t) \sin(\phi)\big).$$

Proof. The equations of the statement can be found by a straightforward computation from the definition of the angle functions. Equation (3) follows by the definition of ν:

$$\ell \, \|\alpha'(t)\| \cos \nu(t) = \big\langle \alpha'(t), \, \alpha(t + \phi) - \alpha(t) \big\rangle.$$

Analogously, Equation (4) follows by the definition of μ:

$$\ell \, \|\alpha'(t + \phi)\| \cos \mu(t) = \big\langle \alpha'(t + \phi), \, \alpha(t + \phi) - \alpha(t) \big\rangle. \qquad \square$$

Proposition 1. *Let $m \in \mathbb{N}$, $\phi \in \Phi_m$ and let $\alpha : I \to \mathbb{R}^2$ be a (ϕ, ℓ)-isochordal-viewed m-hedgehog parameterized by a support function h. Then*

$$\big(h(t) + h''(t)\big) \big(\cos(\phi) \, h'(t + \phi) - h'(t) + \sin(\phi) \, h(t + \phi)\big)$$
$$= \big(h(t + \phi) + h''(t + \phi)\big) \big(h'(t + \phi) - \cos(\phi) \, h'(t) + h(t) \sin(\phi)\big),$$

for all $t \in I$.

Proof. The expression is easily found by substitution of (3) and (4) of Lemma 4 into the expression (2) of Lemma 1. $\square$

Notice that Proposition 1 provides a differential equation with deviated arguments that describes the support function h of any (ϕ, ℓ)-isochordal-viewed multihedgehog. If we add the hypothesis that the curve has a circular ϕ-isoptic, then we can get rid of the deviated arguments in the differential equation and the equation can be explicitly solved. This leads to the main result we state below: a complete characterization of all the m-hedgehogs which are (ϕ, ℓ)-isochordal-viewed and of constant ϕ-width.

Theorem 2. *Let $m \in \mathbb{N}$, $m \geq 1$. Let α_k be a piecewise-$\mathcal{C}^2$ m-hedgehog defined by a support function of the kind*

(5) $$h_k(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(k \, t) + c_4 \sin(k \, t),$$

where $k = \frac{a}{m}$, with $a \in \mathbb{N}$, $a \geq 0$. Then:

(i) *If $k = 0$, the curve α_k is a circle.*
(ii) *If $k = 1$ or $c_3 = c_4 = 0$, the curve α_k is a single point (degenerated curve).*

(iii) If $k \neq 0, 1$ and $c_3 \neq 0$ or $c_4 \neq 0$, the curve α_k is (ϕ, ℓ)-isochordal viewed and of constant ϕ-width for the values

$$\phi = \frac{2\pi c}{k \pm 1},$$

for any $c \in \mathbb{Z}$ such that $\phi \in \Phi_m$ and the corresponding values for ℓ satisfy

$$\ell = \sqrt{c_3^2 + c_4^2}\,|k \pm 1|\left|\sin\left(\frac{\pi c\,(k \mp 1)}{k \pm 1}\right)\right| > 0.$$

(iv) Any piecewise-$\mathcal{C}^2$-regular m-hedgehog which is (ϕ, ℓ)-isochordal viewed and of constant ϕ-width, for $\phi \in \Phi_m$, can be parameterized by a support function h_k as in (5) for some $k \in \mathbb{Q}$, $k \neq 1$.

Proof. Let J_ϕ be the set of points $t \in I$ such that α is regular at t and at $t + \phi$. Notice by Lemma 3 that $\cos\nu(t) \neq 0$ and $\cos\mu(t) \neq 0$ for all $t \in J_\phi$.

By Theorem 1, the curve α is of constant ϕ-width if and only if $\nu'(t) = 0$ for all $t \in J_\phi$. Remember that $\tilde{\phi}(t)$ is assumed to be differentiable, so that since it is piecewise-constant, its points of discontinuity correspond to the points where α is not regular either at t or at $t + \phi$.

From Lemma 2, being of constant ϕ-width is equivalent to saying that

$$(6) \qquad \frac{\|\alpha'(t)\|\sin\tilde{\phi}(t)}{\ell\cos\mu(t)} = b_0$$

is constant for all $t \in J_\phi$. Remember that $\sin\tilde{\phi}(t) = \varepsilon(t)\sin(\phi)$, where $\varepsilon(t)$ is a sign function.

By Lemma 1 we also have that it is also equivalent to saying that

$$(7) \qquad \frac{\|\alpha'(t+\phi)\|\sin\tilde{\phi}(t)}{\ell\cos\nu(t)} = b_0$$

is constant for all $t \in J_\phi$.

From Lemma 4, we have

$$(8) \qquad \ell\cos\nu(t) = s_1(t)\left(\cos(\phi)\,h'(t+\phi) - h'(t) + \sin(\phi)\,h(t+\phi)\right),$$

where

$$s_1(t) = \operatorname{sgn}\left(h(t) + h''(t)\right),$$

and

$$(9) \qquad \ell\cos\mu(t) = s_2(t)\left(h'(t+\phi) - \cos(\phi)\,h'(t) + h(t)\sin(\phi)\right),$$

where

$$s_2(t) = \operatorname{sgn}\left(h(t+\phi) + h''(t+\phi)\right).$$

These two equations hold for all $t \in J_\phi$.

Thus, from (8) and (9) the expression (6) can be written as

$$(10) \qquad \frac{\left(h(t) + h''(t)\right)\sin(\phi)}{h'(t+\phi) - \cos(\phi)\,h'(t) + h(t)\sin(\phi)} = b_0\,s(t),$$

where $s(t) = s_1(t)\,s_2(t)\,\varepsilon(t)$ is a sign function.

Now, notice that the jumps of the piecewise-constant function $\tilde{\phi}(t)$ occur when a cusp is crossed by $\alpha(t)$ or $\alpha(t + \phi)$. Therefore, if the sign of $h(t) + h''(t)$ or $h(t + \phi) + h''(t + \phi)$ changes, the sign $\varepsilon(t)$ also does. This imply that the sign function $s(t)$ must be constant equal to 1 or equal to -1. In any case, it can be added to the real constant b_0.

Hence, Equation (10) can be written as

$$(11) \qquad \frac{\big(h(t) + h''(t)\big) \sin(\phi)}{h'(t + \phi) - \cos(\phi)\, h'(t) + h(t) \sin(\phi)} = b,$$

for a real constant b. Differentiating this expression, we get

$$\frac{g(t)}{\big(h'(t + \phi) - \cos(\phi)\, h'(t) + h(t) \sin(\phi)\big)^2} = 0,$$

where

$$g(t) = \big(h'(t) + h^{(3)}(t)\big)\big(h'(t + \phi) - \cos(\phi)\, h'(t) + h(t) \sin(\phi)\big)$$
$$- \big(h''(t) + h(t)\big)\big(\sin(\phi)\, h'(t) + h''(t + \phi) - \cos(\phi)\, h''(t)\big).$$

Equivalently, such an equation holds if and only if $g(t) = 0$. This can be written as

$$(12) \qquad \frac{h'(t) + h^{(3)}(t)}{\sin(\phi)\, h'(t) + h''(t + \phi) - \cos(\phi)\, h''(t)} = \frac{h''(t) + h(t)}{h'(t + \phi) - \cos(\phi)\, h'(t) + h(t) \sin(\phi)}$$

for all $t \in J_\phi$ such that $h'(t) + h^{(3)}(t) \neq 0$. We know that the right-hand side of (12) is equal to $\frac{b}{\sin(\phi)}$. This leads to the equation

$$(13) \qquad \frac{\big(h'(t) + h^{(3)}(t)\big) \sin(\phi)}{\sin(\phi)\, h'(t) + h''(t + \phi) - \cos(\phi)\, h''(t)} = b.$$

The analogous discussion can be performed with the expression (7), which leads to the equation:

$$(14) \qquad \frac{\big(h(t + \phi) + h''(t + \phi)\big) \sin(\phi)}{\cos(\phi)\, h'(t + \phi) - h'(t) + \sin(\phi)\, h(t + \phi)} = b.$$

Now, given any real constant b we can solve the system of equations formed by (11), (13) and (14) for $h(t + \phi)$, $h'(t + \phi)$ and $h''(t + \phi)$. If $b \neq 0, 1$, the solution is

$$h(t + \phi) = \cos(\phi)\, h(t) + \frac{\sin(\phi)\big((b^2 - b + 1)h'(t) + h^{(3)}(t)\big)}{(b - 1)\, b},$$

$$h'(t + \phi) = \frac{-(b - 1) \sin(\phi)\, h(t) + b \cos(\phi)\, h'(t) + \sin(\phi)\, h''(t)}{b},$$

$$h''(t + \phi) = \frac{-(b - 1) \sin(\phi)\, h'(t) + b \cos(\phi)\, h''(t) + \sin(\phi)\, h^{(3)}(t)}{b}.$$

The expressions of $h'(t + \phi)$ and $h''(t + \phi)$ are clearly related by differentiation. The derivative of the first equation is

$$h'(t + \phi) = \cos(\phi)\, h'(t) + \frac{\sin(\phi)\big((b^2 - b + 1)\, h''(t) + h^{(4)}(t)\big)}{(b - 1)\, b},$$

which must be equal to the second solution of the system. Thus, their difference must be equal to zero. Doing that, a straightforward calculation produces the following equation:

$$\frac{\sin(\phi)\big((b^2 - 2b + 2)\, h''(t) + h^{(4)}(t) + (b - 1)^2\, h(t)\big)}{(b - 1)\, b} = 0.$$

Therefore, if $b \neq 0, 1$, we obtain a differential equation for the support function h:

$$(15) \qquad\qquad (b - 1)^2\, h(t) + (b^2 - 2b + 2)\, h''(t) + h^{(4)}(t) = 0.$$

The general real solution of this differential equation is

$$h(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 \cos\big(|b - 1|\, t\big) + c_4 \sin\big(|b - 1|\, t\big),$$

for any real constants c_1, c_2, c_3 and c_4.

Since b is an arbitrary real constant, $|b - 1|$ is an arbitrary non-negative constant. Hence, the desired family of curves is given in terms of a non-negative constant k and determined by a support function of the kind

$$(16) \qquad\qquad h_k(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 \cos\big(k\, t\big) + c_4 \sin\big(k\, t\big),$$

for any real constants c_1, c_2, c_3 and c_4.

The support function h_k must be $2\, m\, \pi$-periodic, for some integer m, i.e.

$$h_k(t + 2\, m\, \pi) - h_k(t) = 0.$$

We can compute explicitly this equation, which is

$$c_3 \big(\cos(2\, k\, \pi\, m + k\, t) - \cos(k\, t)\big) + c_4 \big(\sin(2\, k\, \pi\, m + k\, t) - \sin(k\, t)\big) = 0.$$

If $c_3 \neq 0$ or $c_4 \neq 0$, the only way to fulfill this equation is having an integer a such that

$$2\, k\, \pi\, m = 2\, \pi\, a.$$

This happens if and only if $k = \frac{a}{m}$, which means that k must be rational.

Let α_k be the curve defined by the support function h_k. By construction, α_k must be (ϕ, ℓ)-isochordal-viewed and of constant ϕ-width for some $\phi \in \Phi_m$. Let's see now for which values of ϕ the curve is (ϕ, ℓ)-isochordal viewed. We can easily compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\|\alpha_k(t + \phi) - \alpha_k(t)\big\|^2 = 2\, k\, (k^2 - 1)\, \big(\cos(k\, \phi) - \cos(\phi)\big)$$

$$(17) \qquad\qquad \Big(\big(c_3^2 - c_4^2\big) \sin\big(k\, (2\, t + \phi)\big) - 2\, c_3\, c_4 \cos\big(k\, (2\, t + \phi)\big)\Big).$$

This is equal to zero if and only if $k = 0$, $k = 1$, $c_3 = c_4 = 0$ or

$$\cos(k\, \phi) - \cos(\phi) = 0.$$

The solutions of this equation are

$$\phi = \frac{2\,\pi\,c}{k \pm 1},$$

for any $c \in \mathbb{Z}$. The corresponding values for ℓ can be simply computed as

$$\ell = \big\|\alpha_k(t + \phi) - \alpha_k(t)\big\|$$

to obtain the expression of the statement.

The condition of being α_k of constant ϕ-width is that

$$A(t) = \big\langle \alpha_k(t+\phi) - \alpha_k(t),\ J\alpha_k'(t) - J\alpha_k'(t+\phi) \big\rangle$$

is constant (see e.g. [10]). For our case, we can compute explicitly that

$$A'(t) = 2\,k\,(k^2 - 1)\,\big(\cos(k\,\phi) - \cos(\phi)\big)\,\big((c_3^2 - c_4^2)\sin\big(k\,(2\,t + \phi)\big) - 2\,c_3\,c_4\cos\big(k\,(2\,t + \phi)\big)\big).$$

which is the same expression as (17) obtained from being (ϕ, ℓ)-isochordal viewed, so that it holds for the same ϕ values seen above. Hence, for such values of ϕ, α_k is, indeed, also of constant ϕ-width.

Let's discuss now the particular cases that we left out above and that must be studied separately. The case $k = 1$ corresponds to $b = 0$. In this case, the system of equations takes the form

$$h(t) + h''(t) = 0,$$
$$h(t + \phi) + h''(t + \phi) = 0,$$
$$h'(t) + h^{(3)}(t) = 0.$$

The unique solution of these equations is

$$h(t) = c_1 \cos t + c_2 \sin t,$$

for any $c_1, c_2 \in \mathbb{R}$. The corresponding curve is degenerated to a single point:

$$\alpha_1(t) = (c_1,\ c_2).$$

Notice that this solution is included in the general solution (16) if $k = 1$ for appropriate values of the constants. The same curve is obtained if $c_3 = c_4 = 0$. But notice that it does not correspond to a (ϕ, ℓ)-isochordal-viewed curve of constant ϕ-width; in fact, the assumption $h'(t) + h^{(3)}(t) \neq 0$ to construct the system of equations above does not hold.

The case $k = 0$ corresponds to $b = 1$. In this case the system of equations (11), (13) and (14) with $b = 1$ is incompatible. Indeed, from (11) and (13) we can get expressions for $h'(t + \phi)$ and $h''(t + \phi)$. A substitution in (14) yields the equation $h'(t) + h^{(3)}(t) = 0$. But the solution of this differential equation cannot fulfill (13) for $b = 1$.

As said above, (13) is undefined if $h'(t) + h^{(3)}(t) = 0$. The solution of this differential equation is

$$h(t) = c_1 \cos(t) + c_2 \sin(t) + c_3,$$

which produces a circle. The circle is a trivial example of (ϕ, ℓ)-isochordal-viewed curve of constant ϕ-width, for all $\phi \in \,]0, \pi[$. This trivial solution has not been considered in our system of equations. However, the value $k = 0$ in the general solution (16) produces such a curve:

$$\alpha_0(t) = (c_1, \, c_2) + c_3 \, (\cos t, \sin t).$$

Therefore, the solution with $k = 0$ can be considered valid as well. $\square$

Convex curves are singularity-free 1-hedgehogs. The circle is the only example of a convex curve which is (ϕ, ℓ)-isochordal-viewed and of constant ϕ-width (see Theorem 2 of [4]). As a consequence of our Theorem 2, we can provide a different proof of such a result via support functions.

Corollary 1. *Let $\phi \in \,]0, \pi[$. The only convex (ϕ, ℓ)-isochordal-viewed curve of constant ϕ-width is the circle.*

Proof. First, notice that any convex curve can be parameterized by a support function h. By Theorem 2, the support function must be of the form (5). We have that

$$h_k(t) + h_k''(t) = \left(1 - k^2\right) \left(c_3 \cos(k\,t) + c_4 \sin(k\,t)\right).$$

The only way this expression not to be equal to zero for any $t \in \,]0, 2\,\pi[$ is that $k = 0$, which corresponds to the circle. $\square$

Next, we are going to simplify the statement of Theorem 2 by looking at the space of shapes. Recall that a *shape* is a closed curve modulo translations, rotations and scalings. For regular parameterized closed curves the space of shapes is introduced as a quotient space of the space of immersions $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^2)$ by the group of orientation-preserving diffeomorphisms on $\mathbb{S}^1$ and the group of translations, rotations and scalings (see e.g. [14]). Similarly can be done for piecewise-regular parameterized curves.

Theorem 3. *Let $m \in \mathbb{N}$, $m \geq 1$. The shape of any non-circular piecewise-$\mathcal{C}^2$-regular (ϕ, ℓ)-isochordal-viewed m-hedgehog of constant ϕ-width can be parameterized by a support function*

$$(18) \hspace{4cm} h_k(t) = \sin(k\,t),$$

where $k = \frac{a}{m}$, with $a \in \mathbb{N}$, $a \geq 1$ and $a \neq m$, for the values

$$\phi = \frac{2\,\pi\,c}{k \pm 1},$$

for any $c \in \mathbb{Z}$ such that $\phi \in \Phi_m$ and the corresponding values for ℓ satisfy

$$\ell = |k \pm 1| \left| \sin\left(\frac{\pi\,c\,(k \mp 1)}{k \pm 1} \right) \right| > 0.$$

Proof. By Theorem 2, the only thing we must show is that any curve $\alpha_{c_1, c_2, c_3, c_4, k}$ parameterized by a support function (5), namely,

$$h_{c_1, c_2, c_3, c_4, k}(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 \cos\!\left(k\,t\right) + c_4 \sin\!\left(k\,t\right),$$

where $c_3 \neq 0$ or $c_4 \neq 0$ and $k \neq 0$, is equal to $\alpha_{0,0,0,1,k}$ up to translations, rotations, scalings and reparameterizations.

First of all, notice that

$$\alpha_{c_1,c_2,c_3,c_4,k}(t) = (c_1, c_2) + c_3\, \alpha_{0,0,1,0,k}(t) + c_4\, \alpha_{0,0,0,1,k}(t).$$

Now, it is easy to check that

$$c_3\, \alpha_{0,0,1,0,k}(t) + c_4\, \alpha_{0,0,0,1,k}(t) = \sqrt{c_3^2 + c_4^2}\; R\big(-g(c_3, c_4, k)\big)\, \alpha_{0,0,0,1,k}\big(t - g(c_3, c_4, k)\big)$$

where

$$g(c_3, c_4, k) = \frac{\mathrm{sgn}(c_4)}{k}\, \arccos\left(\frac{c_3}{\sqrt{c_3^2 + c_4^2}}\right).$$

and R is the usual rotation matrix

$$R(v) = \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}.$$

Therefore, if $k \neq 0$, $\alpha_{c_1,c_2,c_3,c_4,k}$ is $\alpha_{0,0,0,1,k}$ up to a translation, a rotation, a scaling and a reparameterization. $\qquad\square$

Example 1. Theorem 3 provides explicit parameterizations by a support function of (ϕ, ℓ)-isochordal-viewed curves of constant ϕ-width easily. See some examples of these curves for different rational values of k in Figure 4.

$$k = \frac{1}{3};\; \phi = \frac{3}{2}\pi \qquad\qquad k = \frac{1}{7};\; \phi = \frac{7}{2}\pi \qquad\qquad k = \frac{4}{3};\; \phi = \frac{6}{7}\pi$$

$$k = \frac{3}{5};\; \phi = \frac{5}{4}\pi \qquad\qquad k = \frac{10}{7};\; \phi = \frac{14}{17}\pi \qquad\qquad k = \frac{3}{8};\; \phi = \frac{16}{11}\pi$$
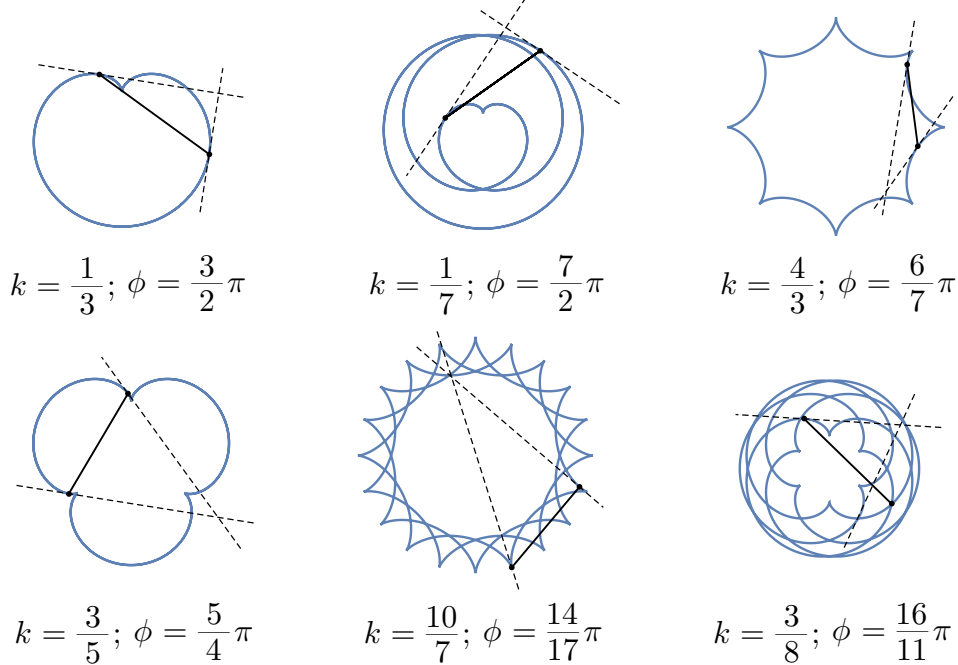
FIGURE 4. Some examples of (ϕ, ℓ)-isochordal-viewed m-hedgehogs for different values of $k = \frac{a}{m} \in \mathbb{Q}$.

For integer values of k we obtain the already known examples of (ϕ, ℓ)-isochordal viewed 1-hedgehogs provided in [10] and [11] (see in Figure 5 some examples).
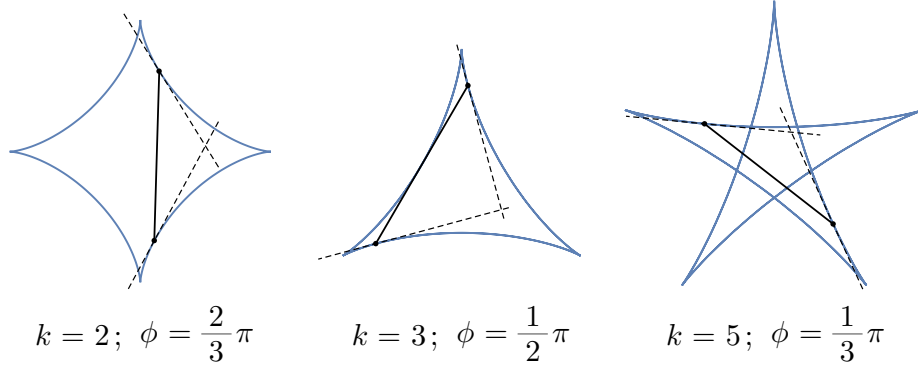

$$k = 2; \ \phi = \frac{2}{3}\pi \qquad k = 3; \ \phi = \frac{1}{2}\pi \qquad k = 5; \ \phi = \frac{1}{3}\pi$$

FIGURE 5. Some examples of (ϕ, ℓ)-isochordal-viewed 1-hedgehogs.

Once we know the explicit expression of all (ϕ, ℓ)-isochordal-viewed m-hedgehogs of constant ϕ-width, we can derive some of their properties by using the explicit expression of the support function.

Proposition 2. *Let $m \in \mathbb{N}$, $\phi \in \Phi_m$ and let α be a non-circular piecewise-$\mathcal{C}^2$-regular (ϕ, ℓ)-isochordal viewed m-hedgehog of constant ϕ-width. Then α has a singular point.*

Proof. By Theorem 3 the shape of the m-hedgehog α can be parameterized by a support function h_k of the kind (18) with $k \in \mathbb{Q}$, $k \neq 1$. Thus, we have
$$h_k(t) + h_k''(t) = (1 - k^2)\sin(k\,t).$$
This is equal to zero if and only if
$$t = \frac{\pi\,c}{k},$$
for some $c \in \mathbb{Z}$. Since $k = \frac{a}{m}$, for $a, m \in \mathbb{N}$, $a \neq m$, we can always set $c = a$ to have
$$t = \pi\,\frac{m}{a}\,c = \pi\,m \in \,]0, 2\,\pi\,m[.$$
Therefore, α is singular at least at this point. $\qquad \square$

Remark 1. Given $k = \frac{a}{m}$, the number of zeros of h_k in $]0, 2\,\pi\,m[$ is equal to $2\,a - 1$. This will correspond to the number of singular points (counted by multiplicity) of the m-hedgehog generated by the support function h_k.

Recall that a closed curve α has rotational symmetry of angle β about a point $\mathbf{x}$ if there is a rotation around $\mathbf{x}$ such that the trace of the rotated curve $\tilde{\alpha}$ coincides with the trace of α (see [4] for the definition in the case of convex curves).

Many of our curves have some rotational symmetry (see Figure 4). However, note that not all of them are rotationally symmetric (see the first curve of such a figure).

Remark 2. A very important question is still open, that is if there is any (ϕ, ℓ)-isochordal-viewed curve which is not of constant ϕ-width. This question is related, in the convex case, to the question: "is the circle the only isochordal-viewed convex curve?".

It was proved in [4] (and also in our Corollary 1) that the circle is the only convex curve which is (ϕ, ℓ)-isochordal viewed and of constant ϕ-width. But no relations between being (ϕ, ℓ)-isochordal viewed and of constant ϕ-width more than Theorem 1 have been established up to know, as far as the author knows.

4. Curves generated by regular polygons

Consider now the class of closed curves where a regular polygon can slide smoothly with its vertices always lying on such curves. These curves might be of interest in engineering, mainly for the development of constrained mechanisms constituted by fixed bars and rotation devices.

The set of closed curves where a regular polygon of side length ℓ can slide smoothly with its vertices always lying on such curves contains the set of (ϕ, ℓ)-isochordal-viewed hedgehogs of constant ϕ-width. This was proved in [11]. The methodology of the same paper can also be extended to m-hedgehogs. In the next result we do so, but we state it in a different manner, which is that in these curves we can always find a regular n-gon which can move along them.

First, recall that a m-hedgehog is called projective if its support function h satisfies
$$h(t) + h(t + \pi m) = 0,$$
for all $t \in I$.

Theorem 4. *Let $m \in \mathbb{N}$, $m \geq 1$, $\phi \in \Phi_m$ and let α be a piecewise-$\mathcal{C}^2$ (ϕ, ℓ)-isochordal-viewed m-hedgehog of constant ϕ-width. Then there exists $n \in \mathbb{N}$, $n \geq 2$, such that a regular polygon of n sides is allowed to move along α with its vertices always lying on α. If α is not the curve associated with $k = \frac{1}{3}$ or $k = 3$ given by (18), then $n \geq 3$.*

Proof. The case of the circle is trivial, since it satisfies the hypothesis for all $\phi \in \,]0, \pi[$ and any regular n-gon is allowed to rotate with its vertices along it.

Suppose now a non-circular m-hedgehog α. By Theorem 3, α can be parameterized by a support function $h_k(t) = \sin(k\,t)$ with $k = \frac{a}{m}$, where $a \in \mathbb{N}$, $a \geq 1$, $a \neq m$. The values of ϕ are of the kind
$$\phi = \frac{2\,\pi\,c}{k \pm 1} = \frac{2\,\pi\,m\,c}{a \pm m},$$
for any $c \in \mathbb{Z}$ such that $\phi \in \Phi_m$ and $\ell > 0$. Assume that a and m are relatively prime.

Take (for $c = 1$) the angle
$$\phi = \frac{2\,\pi\,m}{a + m}.$$
We know that $\frac{m}{a+m} < 1$. Therefore $\phi < 2\,\pi$ so we deduce that $\phi \in \Phi_m$ (we have that $m \geq 1$ and $a \neq m$). Moreover, notice that the associated length
$$\ell = \frac{a + m}{m} \left| \sin\left(\frac{a - m}{a + m}\,\pi \right) \right|$$

cannot be equal to zero. This could only happen if $a + m$ divides $a - m$, which is not possible.

Let us prove now that α is projective if and only if a is odd. The curve α is projective if

$$h_k(t) + h_k(t + m\,\pi) = 0.$$

This equation explicitly written is

$$\sin\left(\frac{a\,t}{m} + a\,\pi\right) + \sin\left(\frac{a\,t}{m}\right) = 0.$$

This happens if and only if a is odd.

To get a closed polygon of side length ℓ, we look for the lowest integer n such that

$$\alpha(t + n\,\phi) = \alpha(t).$$

If α is not projective (a even), then m is odd (it cannot be even because a and m would have 2 as a common factor). In this case we take $n = a + m \geq 3$, which satisfies that

$$\alpha\left(t + (a + m)\,\frac{2\,m}{a + m}\,\pi\right) = \alpha(t + 2\,m\,\pi) = \alpha(t).$$

If α is projective (a odd), we have two cases. First, if m is also odd, we have that $a + m$ is even, so it can be divided by 2. In this case we take $n = \frac{a+m}{2}$, which satisfies

$$\alpha\left(t + \frac{a + m}{2}\,\frac{2\,m}{a + m}\,\pi\right) = \alpha(t + m\,\pi) = \alpha(t).$$

Notice that $a \neq m$, so that $n \geq 2$. The case $n = 2$ can only be achieved if $a = 1$ and $m = 3$ or if $a = 3$ and $m = 1$. Other cases lead to $n \geq 3$.

The second case is if m is even (so that $a + m$ is odd). In this case we take $n = a + m \geq 3$, which satisfies

$$\alpha\left(t + (a + m)\,\frac{2\,m}{a + m}\,\pi\right) = \alpha(t + 2\,m\,\pi) = \alpha(t).$$

The integer n is the lowest one since it cannot be divided by 2.

With this, we have proved that there is a polygon of n sides that can move along α. That it is equiangular comes from the fact that α is of constant ϕ-width (follow the second part of the proof of Proposition 2.1 of [11], which can be reproduced in this context as well). $\square$

See in Figure 6 some multihedgehogs featuring the regular polygon property we have just seen.

Theorem 4 ensures the existence of a regular polygon for each of these curves, but there could be more than one for the same figure. This is associated to the fact that the same curve can be (ϕ, ℓ)-isochordal-viewed and of constant ϕ-width for different values of ϕ (and ℓ). This can be observed for instance in Figure 7.

We must remark that from the same paper [11] some examples of curves exhibiting the regular polygon property which are not (ϕ, ℓ)-isochordal-viewed nor of constant ϕ-width can be found. These curves can be constructed as Holditch curves of (ϕ, ℓ)-isochordal-viewed curves of constant
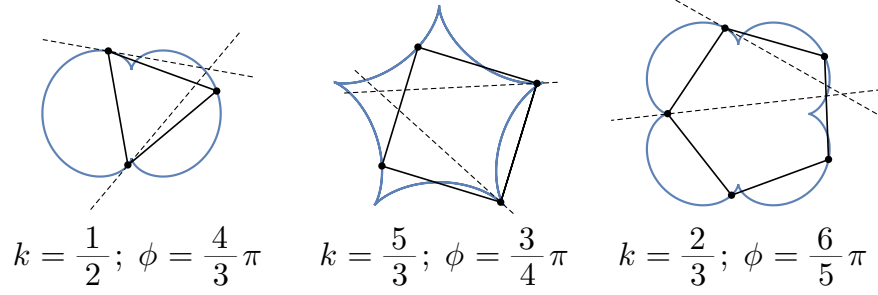
$$k = \frac{1}{2}\,;\ \phi = \frac{4}{3}\,\pi \qquad k = \frac{5}{3}\,;\ \phi = \frac{3}{4}\,\pi \qquad k = \frac{2}{3}\,;\ \phi = \frac{6}{5}\,\pi$$

FIGURE 6. Some regular polygons are allowed to travel along a (ϕ, ℓ)-isochordal-viewed multihedgehog of constant ϕ-width.
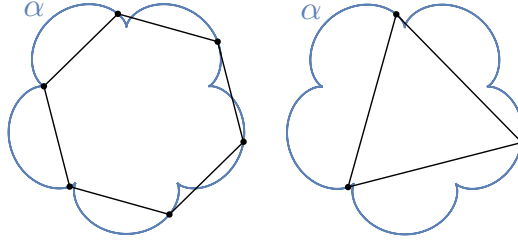
FIGURE 7. A (ϕ, ℓ)-isochordal-viewed curve α of constant ϕ-width found with $k = \frac{5}{7}$. A regular hexagon slides along α for $\phi = \frac{7\pi}{6}$ and a regular triangle for $\phi = \frac{7\pi}{3}$.

ϕ-width (see Figure 8). Therefore, the class of curves with the regular polygon property is, indeed, strictly larger than the set of (ϕ, ℓ)-isochordal-viewed multihedgehogs of constant ϕ-width.
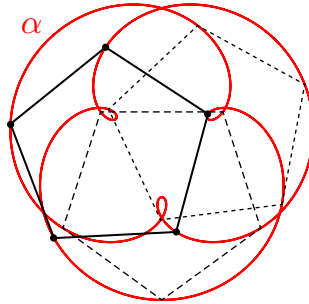

FIGURE 8. A curve α which is not (ϕ, ℓ)-isochordal-viewed of constant ϕ-width and such that a regular polygon can move along its perimeter.

Acknowledgements

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