Wiener–Hopf Integral Equations in Mean First-passage Time Problems for Continuous-time Random Walks

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Abstract

We study the mean first-passage time (MFPT) for asymmetric continuous time random walks in continuous space characterised by finite mean waiting times and jump amplitudes with both finite average and finite variance. We derive an inhomogeneous Wiener-Hopf integral equation that allows the exact estimation of the MFPT, which depends on the whole distribution of the jump amplitudes, but on the average of the waiting times only. Thus, our findings hold for general non-Markovian processes, since Markovianity emerges solely with an exponential distribution of the waiting times. Through the paradigmatic case study of a general class of asymmetric distributions of the jump-amplitudes that is exponential towards the boundary and arbitrary in the opposite direction, we show, that only the average of the jump amplitudes in the opposite direction of the boundary contributes to the MFPT. Moreover, we determine a length-scale, which depends only of the distribution of jumps in the direction of the boundary, such that for initial positions close to the boundary the MFPT depends on the specific whole distribution of jump amplitudes, in opposition to the appearing universality for initial positions far away from the boundary.

Keywords: mean first-passage time, continuous-time random walk, Wiener–Hopf integral equation

1. Finite-mean first-passage time

We consider a one-dimensional random walk in continuous space $x \in \mathbb{R}$ and continuous time $t \geq 0$. In particular, we study a continuous time random walk (CTRW) characterised by a distribution $q(\xi)$ of jump amplitudes and by a waiting times distribution between subsequent jumps $\psi(\tau)$, which normalise according to $\int_{-\infty}^{\infty} q(\xi)\, d\xi = 1$ and $\int_0^{\infty} \psi(\tau)\, d\tau = 1$, respectively. If the distribution of the waiting-times is exponential then the process is Markovian [1]. We assume that drawings of the jump-sizes and of the waiting-times are statistically independent, and both are independent and identically distributed (iid) random variables.

Let X_t be the walker's position at time t, and let the notation $X_{t-\tau}|y$ denoting the actual position of the random walker at time t under the condition that the walker was previously in $y \in \mathbb{R}$ at $\tau \in [0,t]$, which is the duration of the first random waiting time. Using the idea of the *first renewal picture* [2, 3], the walker may rest until time t in the starting point $x_0 \in \mathbb{R}$ with probability $\Psi(t) = 1 - \int_0^t \psi(\tau)\, d\tau$ or, it may be at time t in $X_{t-\tau}$ starting from the new initial-like position $x_0 + \xi$, $\xi \in \mathbb{R}$, with the complementary probability $\int_0^t \psi(\tau)\, d\tau$, in which ξ declares a random jump amplitude that may occurs at the new initial datum $\tau \in [0,t]$. Thus, in formulae, for the walkers position $X_t|x_0$ we may write

$$X_t|x_0 = \begin{cases} x_0; & \text{with probability}: \Psi(t) = 1 - \displaystyle\int_0^t \psi(\tau)\, d\tau\,, \\ X_{t-\tau}|(x_0+\xi); & \text{with probability}: \displaystyle\int_0^t \psi(\tau)\, d\tau\,. \end{cases} \tag{1}$$

The conditional probability density function (PDF) $p(x,t;x_0)$ of the statistically homogeneous process (1) may be described by the implicit integral equation

$$p(x,t;x_0) = \Psi(t)\delta(x-x_0) + \int_0^t \psi(\tau)\int_{-\infty}^{\infty} q(\xi - x_0)p(x,t-\tau;\xi)\,d\xi d\tau\,, \tag{2}$$

that in Fourier-Laplace space domain becomes the well-known Montroll-Weiss equation [4]

$$\widehat{\widetilde{p}}(\kappa,s;x_0) = \frac{\widetilde{\Psi}(s)\exp(ikx_0)}{1-\widetilde{\psi}(s)\,\hat{q}(\kappa)}\,. \tag{3}$$

We assume a CTRW with finite mean waiting times, i.e., $0 < \langle\tau\rangle = \int_0^\infty \tau\,\psi(\tau)\,d\tau < \infty$, and finite average and finite variance of the jump sizes $-\infty < \langle\xi\rangle = \int_{-\infty}^{\infty} \xi\,q(\xi)\,d\xi < \infty, 0 < \langle\xi^2\rangle = \int_{-\infty}^{\infty} \xi^2\,q(\xi)\,d\xi < \infty$. The Montroll-Weiss equation (3) may be re-arranged [5] such that in the diffusive limit $t, |x| \to \infty$ of the CTRW, for which applies $\widetilde{\psi}(s) \simeq 1 - s\langle\tau\rangle + \mathscr{O}(s^2)$, respectively $\widehat{q}(\kappa) \simeq 1 + i\kappa\langle\xi\rangle - \kappa^2\langle\xi^2\rangle/2 + \mathscr{O}\left(k^3\right)$, we may obtain

$$s\widehat{\widetilde{p}}(\kappa,s;x_0) - \exp(ikx_0) = \left[i\kappa v - D\kappa^2\right]\widehat{\widetilde{p}}(\kappa,s;x_0), \tag{4}$$

with the parameters $v = \langle\xi\rangle/\langle\tau\rangle$ and $D = \langle\xi^2\rangle/2\,\langle\tau\rangle$. Formula (4) proves the Central Limit Theorem insofar as its backward transformation into physically space becomes the advection-diffusion equation

$$\frac{\partial p(x,t;x_0)}{\partial t} = -v\frac{\partial p(x,t;x_0)}{\partial x} + D\frac{\partial^2 p(x,t;x_0)}{\partial x^2}\,, \quad v \in \mathbb{R}, \quad D \geq 0\,. \tag{5}$$

Plugging into an absorbing boundary at $x = 0$ by assuming that $x_0 > 0$ we may obtain the MFPT of an advective-diffusive system (5) from the literature [6], by remembering that $\mathrm{sgn}\{v\} = \mathrm{sgn}\{\langle\xi\rangle\}$

$$T(x_0) = -\frac{x_0}{v} > 0\,, \quad \text{with} \quad \langle\xi\rangle < 0\,. \tag{6}$$

2. An inhomogeneous Wiener–Hopf integral equation for the MFPT

The finite MFPT (6) derived in Section 1 holds in the limits $t, |x| \to \infty$, or analogously $s, \kappa \to 0$, in which the behaviour is governed by an advective-diffusive dynamic formula (5).

To avoid the asymptotic limit we have a look on the survival probability that may be estimated by

$$\Lambda(x_0,t) = \int_0^\infty p_{\text{abs}}(x,t;x_0)\,dx\,, \quad t \geq 0\,, \quad x_0 > 0\,, \tag{7}$$

where $p_{\text{abs}}(x,t;x_0)$ is the conditional PDF with an absorbing boundary at $x = 0$ that can be calculated by

$$p_{\text{abs}}(x,t;x_0) = \Psi(t)\delta(x-x_0) + \int_0^t \psi(\tau)\int_0^\infty q(\xi - x_0)p_{\text{abs}}(x,t-\tau;\xi)\,d\xi d\tau\,. \tag{8}$$

In equation (1), the variable ξ represents the size of the first jump, while in (2) and (8) ξ denotes the position immediately after the first jump events. The absorbing boundary was implemented in equation (8) through the spatial integration ξ that does not cover the full space (in contrast to formula (2)) but only the positive semi-infinite space above the absorbing boundary. Through the manipulation of the lower bound of the spatial integration the trajectories that pass the boundary at the first jump event were absorbed. Due to the implicitly of the integral equation, the random path is propagated in such a way that an occurrence of the particle in the negative area after any jump event is accompanied by its absorption.

From the formulas (7) and (8) we obtain

$$\Lambda(x_0,t) = \Psi(t) + \int_0^t \psi(\tau)\int_0^\infty q(\xi - x_0)\Lambda(\xi,t-\tau)\,d\xi d\tau\,, \tag{9}$$

and later, through the limit $s \to 0$ of the Laplace transform of (9) the MFPT $T(x_0)$ becomes

$$T(x_0) = \langle\tau\rangle + \int_0^\infty q(\xi - x_0)T(\xi)\,d\xi\,, \quad x_0 > 0\,, \tag{10}$$

that is an inhomogeneous Wiener–Hopf integral equation whose kernel is a PDF [7]. Formula (10) states that the exact MFPT depends only on the average of the waiting times, but on the whole distribution of the jump amplitudes, although in the asymptotic limit, formula (6), the MFPT depends on the mean of both, waiting-time and jump sizes, only.

3. Solution method applied to an asymmetric double-exponential jump distribution

In order to calculate the solution of (10), in the spirit of the Wiener–Hopf technique [8], we first generalise the MFPT to $T:\mathbb{R}\to\mathbb{R}$ and introduce $\rho(\xi)=q(-\xi)$, $\xi\in\mathbb{R}$, such that we obtain the inhomogeneous Wiener–Hopf equation in its actual form

$$T(x_0)=f(x_0)+\int_0^{\infty}\rho(x_0-\xi)T(\xi)\,d\xi\,,\quad x_0\in\mathbb{R}\,. \tag{11}$$

For our concern, we may set

$$T(x_0)=\begin{cases} T_+(x_0)\,; & \text{if} \quad x_0>0\,,\\ T_-(x_0)\,; & \text{if} \quad x_0\le 0\,,\end{cases}\,,\quad f(x_0)=\begin{cases}\langle\tau\rangle>0\,; & \text{if} \quad x_0>0\,,\\ 0\,; & \text{if} \quad x_0\le 0\,.\end{cases} \tag{12}$$

The spatial distinction provided by (12) allows us to re-write equation (11)

$$\begin{cases} T_+(x_0)=\langle\tau\rangle+\int_0^{\infty}\rho(x_0-\xi)T_+(\xi)\,d\xi\,, & x_0>0\,,\\ T_-(x_0)=\int_0^{\infty}\rho(x_0-\xi)T_+(\xi)\,d\xi\,, & x_0\le 0\,.\end{cases} \tag{13}$$

In Fourier-space that lead us to the following pairs

$$\widehat{T}_\pm(\kappa)=\pm\int_0^{\pm\infty}\exp(+i\kappa x_0)T_\pm(x_0)\,dx_0\,,\quad T_\pm(x_0)=\frac{1}{2\pi}\int_{L_\pm}\exp(-i\kappa x_0)\widehat{T}_\pm(\kappa)\,d\kappa\,,$$

where $L_\pm$ are proper integration paths in the complex plain.
The general solution of formula (13) is explicitly given in Fourier space and reads

$$\widehat{T}_+(\kappa)=-\frac{\widehat{T}_-(\kappa)}{[1-\widehat{\rho}(\kappa)]}-i\frac{\langle\tau\rangle}{\kappa[1-\widehat{\rho}(\kappa)]}\,. \tag{14}$$

By investigating the paradigmatic case of an asymmetric double-exponential jumps' distribution

$$\rho(\xi)=\begin{cases}(1-b)\exp(-\xi/\ell)/\ell\,; & \text{if} \quad \xi\ge 0\,,\\ ab\exp(a\xi/\ell)/\ell\,; & \text{if} \quad \xi<0\,,\end{cases}$$

with $0\le b\le 1$, $\ell\ge 0$, $a\ge 0$ and the average of the jump amplitudes

$$\langle\xi\rangle=\int_{-\infty}^{\infty}\xi\,q(\xi)\,d\xi=-\int_{-\infty}^{\infty}\xi\,\rho(\xi)d\xi=\left(\frac{1+a}{a}\right)\ell b-\ell\,, \tag{15}$$

the solution set may be obtained through formula (14)

$$\begin{aligned} T_+(x_0) &= T_-(0)\left\{\frac{\ell}{a\langle\xi\rangle}-\frac{(\ell-a\langle\xi\rangle)\exp(-a\langle\xi\rangle x_0/\ell^2)}{a\langle\xi\rangle}\right\}+\\ &+ \langle\tau\rangle\left\{-\frac{x_0}{\langle\xi\rangle}+\frac{\langle\xi\rangle\ell(1-a)+\ell^2}{a\langle\xi\rangle^2}-\frac{(\langle\xi\rangle+\ell)(\ell-a\langle\xi\rangle)\exp(-a\langle\xi\rangle x_0/\ell^2)}{a\langle\xi\rangle^2}\right\}\,. \end{aligned} \tag{16}$$

The non-unique solution, formula (16), is only in agreement with the asymptotic limit for $x_0\to\infty$, equation (6), if $T_-(0)=-\langle\tau\rangle(\ell+\langle\xi\rangle)/\langle\xi\rangle$ such that the unique MFPT is finally given by

$$T_+(x_0)=-\frac{\langle\tau\rangle}{\langle\xi\rangle}(\ell+x_0)\,,\quad \text{for} \quad x_0>0\,,\quad \langle\xi\rangle<0\,,\quad \langle\tau\rangle>0\,. \tag{17}$$

The result (17) is indeed in agreement with the literature [9] and we may see that the asymptotic limit can be used as approximation if $x_0\gg\ell$: $T_+(x_0)\sim -x_0/v$ by remembering that $v=\langle\xi\rangle/\langle\tau\rangle$. Moreover, through formula (15) the condition that $\langle\xi\rangle<0$ can be used to define the range for the parameter setting $0\le b<a/(1+a)$ that guarantees a finite MFPT.

4. Indirect estimation of the jump-sizes distribution

The identification of the MFPT of a CTRW in continuous space with the inhomogeneous Wiener-Hopf integral equation [10] offers the possibility to determine the jump amplitudes PDF for a given MFPT by distinguishing between the jump directions governed by the probability $b \in [0,1]$

$$\rho(\xi) = \begin{cases} (1-b)\,\rho_>(\xi)\,; & \text{if} \quad \xi \geq 0\,, \\ b\,\rho_<(\xi)\,; & \text{if} \quad \xi < 0\,, \end{cases} \tag{18}$$

for which it yields $\int_{-\infty}^{0} \rho_<(\xi)\,d\xi = 1$ and $\int_0^\infty \rho_>(\xi)\,d\xi = 1$ without limitation. Moreover, we set that

$$\int_{-\infty}^{0} \xi\,\rho_<(\xi)d\xi = -\ell/a\,, \quad \text{and} \quad \int_0^\infty \xi\,\rho_>(\xi)d\xi = \ell\,, \quad \ell > 0\,, \quad a > 0, \tag{19}$$

to obtain the same average as provided by equation (15).
Plugging in (17), (18) and (19) into equation (13) leads us to the relation

$$-\frac{\langle\tau\rangle}{\langle\xi\rangle}(\ell + x_0) = \langle\tau\rangle - \frac{b\,\langle\tau\rangle}{\langle\xi\rangle}\left(\ell + x_0 + \frac{\ell}{a}\right) - \frac{(1-b)\langle\tau\rangle}{\langle\xi\rangle}\int_0^{x_0}(\ell+\xi)\,\rho_>(x_0-\xi)d\xi\,, \quad x_0 > 0.$$

In Laplace-space we may explicitly obtain $\tilde{\rho}_>(s)$ that may be transformed back into physically space

$$\tilde{\rho}_>(s) = \frac{1}{\ell s + 1} \quad \Leftrightarrow \quad \rho_>(\xi) = \frac{\exp(-\xi/\ell)}{\ell}\,, \quad \xi \geq 0\,, \quad \ell > 0. \tag{20}$$

We derived an exponential distribution of the jump amplitudes towards the absorbing boundary, but there is no restriction for the jumps away from the target besides the constrain (19). Thus, the PDF of jumps in the opposite direction of the boundary may be chosen arbitrarily and we obtain the same result (17).

5. Conclusion

We studied the problem of a finite MFPT of a CTRW in continuous space characterised by finite mean waiting times and by jump amplitudes with both finite average and finite variance. In particular, we obtain a inhomogeneous Wiener–Hopf integral equation that allows for an exact calculation of the MFPT by avoiding asymptotic limits. This formula results to depend on the whole PDF of the jump amplitudes and on the mean value of the waiting times only such that our findings holds for general non-Markovian processes. The derived Wiener–Hopf integral equation (10) has been used for the paradigmatic case study of an asymmetric double-exponential distribution of the jump amplitudes, whose MFPT turns out to be valid for a more general class of asymmetric distributions of the jump-sizes, namely exponential towards the boundary and arbitrary in the opposite direction. Especially, we show that when the jumps towards the boundary are exponentially distributed, the MFPT is indeed independent of the jump amplitudes PDF in the opposite direction. Furthermore, a length-scale emerges that depends only on the distribution of the jump amplitudes in the direction of the boundary, which defines the transition to the asymptotic behaviour (6). Thus, in opposition to the universal MFPT for starting points that are far-away from the boundary, for starting points that are close to the boundary this universality is lost.

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