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Sturm–Liouville systems for the survival probability in first-passage time problems

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We derive a Sturm–Liouville system of equations for the exact calculation of the survival probability in first-passage time problems. This system is the one associated with the Wiener–Hopf integral equation obtained from the theory of random walks. The derived approach is an alternative to the existing literature and we tested it against direct calculations from both discrete- and continuous-time random walks in a manageable, but meaningful, example. Within this framework, the Sparre Andersen theorem results to be a boundary condition for the system.

1. Introduction

The probability for a random walker started at $a > 0$ to remain in the initial half-axis after n steps is called survival probability [1]. If we denote it by $\phi_n(a)$, for $z, \xi \in \mathbb{R}_0^+$, it holds [1,2]

$$\int_0^\infty k(z-\xi)\, \phi_n(z)\, dz = \phi_{n+1}(\xi)\,, \quad n \geq 0\,, \tag{1.1}$$

with initial condition $\phi_0(\xi) = 1$, for all $\xi \in \mathbb{R}_0^+$, where $k(x)$, with $x \in \mathbb{R}$, is the distribution of the jumps. A general solution to (1.1) is known in literature with the name of Pollaczek–Spitzer formula [1]. This name refers to a formula, see [2, theorem 5, formula (4.6)] and [1, formula (12)], derived by F. Spitzer in 1957 [2, theorem 3, formula (3.1)] on the basis of an auxiliary formula obtained by F. Pollaczek in 1952 [3, formula (8)] but through a different method. Here we derive the Sturm–Liouville system associated with (1.1) for the calculation of the survival probability $\phi_{n+1}(\xi)$ as an alternative and easier approach with respect to the Pollaczek–Spitzer formula. Thus, first we show that $\phi_{n+1}(\xi)$ is indeed the solution of a differential equation and later we check this solution against direct calculations in a manageable, but meaningful, case within both the framework of discrete-time random walk and also that of continuous-time random walk (CTRW).

When $a = 0$, the survival probability $\phi_n(a)$ is indeed determined by the celebrated Sparre Andersen theorem derived on the basis of combinatorial arguments [4]. Within the provided setting, the Sparre Andersen theorem (1.2) actually is a boundary condition of the Sturm–Liouville system of equations in order to have a unique and non-decreasing solution $\phi_{n+1}(\xi)$. Sparre Andersen theorem is a fundamental result in the study of first-passage time problems for Markovian symmetric random walks with discrete time-step and starting at the origin. Originally established in 1954 [4], the theorem states

$$\phi_n(0) = 2^{-2n} \binom{2n}{n} \sim \frac{1}{\sqrt{n\pi}}, \quad n \to \infty, \tag{1.2}$$

where $n \in \mathbb{N}_0$ is the epoch of a one-dimensional random walk $S_n = X_1 + \cdots + X_n$ started at the origin $S_0 = 0$ with jumps i.i.d. random variables in $\mathbb{R}$ and $\phi_n(0)$ is the probability for the first ladder epoch $\{\mathcal{T} = n\} = \{S_1 \geq 0, \ldots, S_{n-1} \geq 0, S_n < 0\}$ to be larger than n, i.e., $\phi_n(0) = \mathbb{P}(\mathcal{T} > n)$. Thus, $\mathcal{T}$ is the epoch of the first entry of the walker into the strictly negative half-axis. See also F. Spitzer [5] and W. Feller [6, Section XII.7]. By adding a constant $a > 0$ to all terms, we obtain a random walk starting at a.

In opposition to the present approach, the derivation of the Sparre Andersen theorem (1.2) from (1.1) in the limit $\xi \to 0$ is possible by taking into account the boundedness of the survival probability at infinity as a consequence of its physical interpretation, see, e.g., [7–9]. By the way, the authors did not calculate with that approach also the survival probability [7–9]. Sometimes, the discussion of the Pollaczek–Spitzer formula suggests that it is possible to get the survival probability from it together with the Sparre Andersen theorem, see, e.g., [1,10], but this is misleading. Unfortunately, as reported by F. Spitzer in his proof [2, theorem 5], the so-called Pollaczek–Spitzer formula [2, formula (4.6)] requires two conditions. The first condition is provided by the following lemma [2, lemma 8]:

The non-decreasing limit function

$$\phi(\xi) = \lim_{n\to\infty} \frac{\phi_n(\xi)}{\phi_n(0)}, \quad \xi > 0, \tag{1.3}$$

satisfies the equation

$$\phi(\xi) = \int_0^\infty k(z - \xi)\phi(z)\, dz, \quad \xi > 0,$$

where point $\xi = 0$ is excluded because it is assumed to be known [2, formula (4.4)]. Later, through the same limit (1.3), it requires, as second condition, also the estimation of the following limit [2, formula (4.3)]:

$$\phi(\xi) = \lim_{n\to\infty} \sqrt{n\pi}\, \phi_n(\xi),$$

that uses formula (1.2) as an external information for quantifying $\phi_n(0)$, read - settled for this text -: *"This case is rather simple, because it is then known (see [4] or [5]) that* (1.2)*"* [2, formula (4.4) and above lines]. This means that claiming the derivation of the Sparre Andersen theorem (1.2) from the Pollaczek–Spitzer formula is indeed misleading, see, for example, reference [1, from formula (12) to (13) without steps], because Pollaczek–Spitzer formula in the limit $\xi \to 0$ provides (1.2) since it has been already assumed for its derivation. However, a further derivation of the Pollaczek–Spitzer formula is available in literature that allows for obtaining the survival probability together with the Sparre Andersen theorem by assuming the boundedness of the survival probability [11–13]. This is analogue to the assumption on the boundary conditions required in the approach here proposed.

The rest of the paper is organised as follows. In section 2, we derive the Sturm–Liouville system associated with the Wiener–Hopf equation (1.1). In section 3, we discuss our result through a meaningful example from the theory of the discrete-time random walk and in section 4 through the same example from the theory of the CTRW. Finally, in section 5 the paper is summarised and conclusions are reported.

2. The associated Sturm–Liouville system

We are now interested in solving (1.1) with a method that is different from the Pollaczek–Spitzer formula. First we observe that $z, \xi \in \mathbb{R}_0^+$ while $x = (z - \xi) \in \mathbb{R}$ and $\int_{\mathbb{R}} k(x) dx = 1$. Moreover, a noteworthy property of (1.2) is that it is independent of the distribution of the jumps if a symmetric kernel $k(x) = k(|x|)$ is considered. Therefore, in spite of the fact that the kernel $k(z,\xi) = k(z-\xi) = k(|z-\xi|)$ is assumed to be everywhere continuous in z, its derivative with respect to z has a jump discontinuity at $z = \xi$, i.e.,

$$k'_z(z,\xi) = \frac{dk(|z-\xi|)}{dz} = \mathrm{sgn}(z-\xi)\, \frac{dk(|x|)}{d|x|} \,, \tag{2.1}$$

and we define

$$-\frac{1}{p(\xi)} = k'_z(\xi^+,\xi) - k'_z(\xi^-,\xi) \,. \tag{2.2}$$

There are some general classes of boundary value problems for differential equations that can be reduced to integral equations [14]. More specifically, here we consider the reduction of the Sturm–Liouville systems to Fredholm integral equations of the second kind. Thus, by following F.G. Tricomi [15, Chapter 3.13], we introduce an operator $\mathcal{L}$ such that

$$\mathcal{L}[k(z,\xi)] = \frac{d}{dz}\left[p(z)\frac{dk}{dz}\right] + q(z)k(z,\xi) = 0 \,, \quad z \neq \xi \,, \tag{2.3}$$

and

$$\mathcal{L}[\phi_{n+1}(z)] = \frac{d}{dz}\left[p(z)\frac{d\phi_{n+1}}{dz}\right] + q(z)\phi_{n+1}(z) = -\phi_n(z) \,, \tag{2.4}$$

for an appropriate function $q(z)$ and for a source term $\phi_n(z)$, together with the corresponding boundary conditions at $z = 0$ and $z = \infty$ of $k(z,\xi)$, $\phi_{n+1}(z)$ and their first derivative. Since the distribution $k(x)$ of the jumps is assumed to be known, function $q(z)$ is given by the choice of $k(x)$, and $\phi_n(z)$ is indeed iteratively determined from the initial condition $\phi_0(z) = 1$. Then, from the subtraction $k(z,\xi)\mathcal{L}[\phi_{n+1}(z)] - \phi_{n+1}(z)\mathcal{L}[k(z,\xi)] = -k(z,\xi)\phi_n(z)$ we remove $q(z)$ and we have the Sturm–Liouville system

$$\frac{d}{dz}\left[p(z)\frac{d\phi_{n+1}}{dz}\right] = \frac{\phi_{n+1}(z)}{k(z,\xi)}\,\frac{d}{dz}\left[p(z)\frac{dk}{dz}\right] - \phi_n(z) \,. \tag{2.5}$$

We now show that (2.5) is the Sturm–Liouville system associated with the Wiener–Hopf integral equation (1.1) for proper boundary conditions.

We start by considering the integral

$$\begin{aligned} I &= \int_0^\infty k(z,\xi)\mathcal{L}[\phi_{n+1}(z)]\, dz \\ &= I_0 + \int_0^\infty k(z,\xi)q(z)\phi_{n+1}(z)\, dz \,, \end{aligned} \tag{2.6}$$

where the second line follows from the LHS of (2.4) with

$$I_0 = \int_0^\infty k(z,\xi)\frac{d}{dz}\left[p(z)\frac{d\phi_{n+1}}{dz}\right] dz \,. \tag{2.7}$$

From the RHS of (2.4) we also have that

$$I = -\int_0^\infty k(z,\xi)\phi_n(z)\, dz \,. \tag{2.8}$$

Integrating by parts I_0 (2.7), it holds

$$I_0 = k\,p\,\frac{d\phi_{n+1}}{dz}\bigg|_0^\infty - \int_0^\infty k'_z\,p\,\frac{d\phi_{n+1}}{dz}\,dz\,, \tag{2.9}$$

thus, by integrating again by parts the integral in the RHS of (2.9) and reminding (2.3), integral (2.6) turns into

$$I = k\,p\,\frac{d\phi_{n+1}}{dz}\bigg|_0^\infty - \int_0^\infty \frac{d}{dz}\left[k'_z\,p\,\phi_{n+1}\right]\,dz\,. \tag{2.10}$$

Because of jump discontinuity of k'_z (2.2), the integral term on the RHS of (2.10) gives

$$\int_0^\infty = \int_0^{\xi^-} + \int_{\xi^+}^\infty = \frac{dk}{dz}\,p\,\phi_{n+1}\bigg|_0^\infty + \phi_{n+1}(\xi)\,, \tag{2.11}$$

and, finally, by comparing (2.10) and (2.8) we recover (1.1) provided that

$$\left[p(z)\left(k(z,\xi)\frac{d\phi_{n+1}}{dz} - \phi_{n+1}(z)\frac{dk}{dz}\right)\right]_0^\infty = 0\,. \tag{2.12}$$

Hence, the kernel $k(z,\xi)$ and the survival probability $\phi_{n+1}(z)$ which are related by the Wiener–Hopf integral equation (1.1) are indeed the solutions of equations (2.3) and (2.4), or (2.5), when boundary conditions (2.12) are met. We highlight that in (2.12), the kernel $k(z,\xi)$ is supposed to be known from the process and $p(z)$ is determined through (2.2). Thus, boundary conditions (2.12) define indeed the boundary values of $\phi_{n+1}(z)$ and its first derivative. In the general case when boundary conditions (2.12) are different from 0, this procedure reduces differential equations to Fredholm integral equations of the second kind. In our case (2.12), this procedure reduces differential equation (2.5) to a homogeneous Fredhdolm integral equation of second kind, that is the Wiener–Hopf integral equation (1.1).

3. An example from the theory of discrete-time random walks

Wiener–Hopf integral equation (1.1) provides the survival probability in the case of a discrete-time random walk. In this section, first we directly solve (1.1) for the generating function by means of classical methods in complex analysis for the meaningful and manageable example with the exponential kernel $k(x) = \mathrm{e}^{-|x|}/2$, see, e.g., [16–19], and later we solve the Sturm–Liouville system derived in section 2 for the same special case.

We introduce the generating function

$$\overline{\phi}(\xi,u) = \sum_{n=0}^\infty u^n\phi_n(\xi) = 1 + \sum_{n=1}^\infty u^n\phi_n(\xi)\,, \tag{3.1}$$

which solves the equation

$$\overline{\phi}(\xi,u) = 1 + u\int_0^\infty k(z-\xi)\overline{\phi}(z,u)\,dz\,. \tag{3.2}$$

Equation (3.2) is a Fredholm integral equation of the second kind with respect to z and it can be solved by methods developed for Wiener–Hopf integral equations [20]. Thus, by following Wiener–Hopf technique, we extend the interval of ξ to $\mathbb{R}$ and we refer to $\overline{\phi}_+(\xi,u)$ and $\overline{\phi}_-(\xi,u)$ for *positive* and *negative* values of ξ, respectively, such that $\overline{\phi} = \overline{\phi}_+ + \overline{\phi}_-$. We are interested to find $\overline{\phi}_+(\xi,u)$ that corresponds to the physical solution of (3.2) while $\overline{\phi}_-(\xi,u)$ is an auxiliary function.

We define the generalised Fourier transform with $\omega \in \mathbb{C}$ and we have the pairs

$$\widehat{g}_\pm(\omega) = \pm \int_0^{\pm\infty} \mathrm{e}^{i\omega x} g_\pm(x)\, dx\,, \tag{3.3}$$

$$g_\pm(x) = \frac{1}{2\pi} \int_{L_\pm} \mathrm{e}^{-i\omega x} \widehat{g}_\pm(\omega)\, d\omega\,, \tag{3.4}$$

where $L_\pm$ are proper integration paths in the complex plane. By inverting the Fourier transform, we explicitly obtain the generating function as follows [14]

$$\begin{aligned}
\overline{\phi}_+(\xi, u) &= \frac{1}{2\pi} \int_{L_+} \mathrm{e}^{-i\omega\xi}\, \widehat{\overline{\phi}}_+(\omega, u)\, d\omega\,, \\
&= -\mathrm{Res}\left\{ \frac{C\,\overline{\phi}_-(0,u)(i+\omega)\,\mathrm{e}^{-i\omega\xi}}{1+\omega^2-u} - \frac{(1+\omega^2)\,\mathrm{e}^{-i\omega\xi}}{\omega\,(1+\omega^2-u)} \right\}, \\
&= \frac{-C\overline{\phi}_-(0,u)}{2} \left\{ \frac{(\sqrt{1-u}-1)\exp(-\sqrt{1-u}\,\xi)}{\sqrt{1-u}} + \frac{(1+\sqrt{1-u})\exp(\sqrt{1-u}\,\xi)}{\sqrt{1-u}} \right\} + \\
&\quad + \frac{1}{1-u} + \frac{u\,\exp(-\sqrt{1-u}\,\xi)}{2(u-1)} + \frac{u\,\exp(\sqrt{1-u}\,\xi)}{2(u-1)}\,.
\end{aligned} \tag{3.5}$$

where C is a multiplicative constant that appears in the solution of the homogeneous case. Since the generating function is bounded because of its physical meaning, i.e.,

$$\lim_{\xi\to\infty} \overline{\phi}_+(\xi, u) = \overline{\phi}_\infty(u) < \infty\,, \tag{3.6}$$

then from (3.5) we have that the exponential term $\exp(\sqrt{1-u}\,\xi)$ must disappear, which gives

$$C\,\overline{\phi}_-(0,u) = \frac{\sqrt{1-u}-1}{\sqrt{1-u}}\,. \tag{3.7}$$

Finally, by plugging (3.7) in (3.5) we obtain the unique generating function of the survival probability for a discrete-time random walk in continuous space with exponential jump distribution, which is

$$\overline{\phi}_+(\xi, u) = \frac{\sqrt{1-u}-1}{1-u} \exp(-\sqrt{1-u}\,\xi) + \frac{1}{1-u}\,, \tag{3.8}$$

that is consistent with formula (32) in reference [18]. From formula (3.8) we also have the boundary conditions:

$$\overline{\phi}_+(0,u) = \frac{1}{\sqrt{1-u}}\,, \quad \overline{\phi}_+'(0,u) = \frac{1-\sqrt{1-u}}{\sqrt{1-u}}\,, \tag{3.9}$$

where $\overline{\phi}_+(0,u)$ is in agreement with the Sparre Anderson theorem, and

$$\overline{\phi}_\infty(u) = \frac{1}{1-u}\,. \tag{3.10}$$

We show now the effectiveness of the discussed approach based on the Sturm–Liouville systems in the same special case. If we consider the special case $k(x) = \mathrm{e}^{-|x|}/2$, see, e.g., [16–19], then from (2.1) we have that $k_z'(z,\xi) = -\,\mathrm{sgn}(z-\xi)\,\mathrm{e}^{-|z-\xi|}/2$ and from (2.2) that $p(z) = 1$, with $z \neq 0$. Therefore, by solving (2.3) with respect to $q(z)$ in all points except $z = \xi$ it results $q(z) = -1$. This means that the survival probability $\phi_{n+1}(z)$ from (2.4) is the solution of the equation

$$\frac{d^2\phi_{n+1}}{dz^2} - \phi_{n+1}(z) = -\phi_n(z)\,, \tag{3.11}$$

and the Sparre Andersen theorem (1.2) is a required boundary condition in analogy with the literature studies reported in Section 1. We introduce the generating function $\overline{\phi}(z,u)$, see definition in (3.1), then from (3.11) we have that it solves the equation

$$\frac{d^2\overline{\phi}(z,u)}{dz^2} - \overline{\phi}(z,u) + 1 = -u\,\overline{\phi}(z,u)\,. \tag{3.12}$$

By applying the Laplace transform with respect to z, where $\widetilde{\overline{\phi}}(s,u)$ stands for the Laplace transform of $\overline{\phi}(z,u)$ with $s \in \mathbb{C}$, formula (3.12) reads

$$s^2\,\widetilde{\overline{\phi}}(s,u) - s\overline{\phi}(0,u) - \overline{\phi}'(0,u) - \widetilde{\overline{\phi}}(s,u) + \frac{1}{s} = -u\,\widetilde{\overline{\phi}}(s,u)\,, \tag{3.13}$$

such that

$$\widetilde{\overline{\phi}}(s,u) = \frac{s^2\,\overline{\phi}(0,u) + s\,\overline{\phi}'(0,u) - 1}{s(s^2+u-1)}\,. \tag{3.14}$$

By taking boundary conditions (3.9), we have

$$\begin{aligned}
\widetilde{\overline{\phi}}(s,u) &= \frac{s^2 + s(1-\sqrt{1-u}) - \sqrt{1-u}}{s\sqrt{1-u}\,(s^2+u-1)} \\
&= \frac{s(s+1) - (s+1)\sqrt{1-u})}{s\sqrt{1-u}\,(s^2+u-1)} \\
&= \frac{s(s+1) - (s+1)\sqrt{1-u})}{s\sqrt{1-u}\,(s+\sqrt{1-u})(s-\sqrt{1-u})} \\
&= \frac{s+1}{s\sqrt{1-u}\,(s+\sqrt{1-u})}\,,
\end{aligned} \tag{3.15}$$

from which we recover formula (3.8) after the Laplace inversion.

4. An example from the theory of the CTRW

We discuss now the feasibility of the proposed approach based on the Sturm–Liouville system through the same example but from the theory of the CTRW. To this aim, we introduce the time variable $t \in \mathbb{R}^+$ and random waiting-times between two consecutive jumps $\tau_j \in \mathbb{R}_0^+$, with $j \in \mathbb{N}$, such that if the process starts at $t=0$ it holds $t = \sum_{j=1}^n \tau_j$. The waiting-times τ_j are assumed to be i.i.d. random variables distributed according to a density function $\psi(t)$. Moreover, we introduce also the probability that a given waiting interval between two consecutive jumps is greater than or equal to t, namely $\Psi(t) = 1 - \int_0^t \psi(\tau)d\tau$. In this continuous-time setting, we replace the notation of the survival probability in discrete epochs $\phi_n(\xi)$ with the new notation $\Lambda(\xi,t)$ and now (1.1) reads [19, formula (16)]

$$\widetilde{\Lambda}(\xi,s) = \widetilde{\Psi}(s) + \widetilde{\psi}(s)\int_0^\infty k(z-\xi)\widetilde{\Lambda}(z,s)\,dz\,, \tag{4.1}$$

where $\widetilde{\Lambda}(\xi,s)$ stands for the Laplace transform of $\Lambda(\xi,t)$ with $s \in \mathbb{C}$, and it holds $s\widetilde{\Psi}(s) = 1 - \widetilde{\psi}(s)$. After Laplace inversion and by setting $t = n\Delta t$ and $\psi(\tau) = \delta(\tau - \Delta t)$, then $\Psi(0) = 1$ and $\Psi(t \geq \Delta t) = 1 - \int_0^t \delta(\tau - \Delta t)d\tau = 0$, such that from (4.1) we recover (1.1) as expected.

Equation (4.1) is again a Fredholm integral equation of the second kind with respect to z and it can be solved by methods developed for Wiener–Hopf integral equations [14,20]. Thus, by following Wiener–Hopf technique, we extend the interval of ξ to $\mathbb{R}$ and we refer to $\widetilde{\Lambda}_+(\xi,s)$ and $\widetilde{\Lambda}_-(\xi,s)$ for *positive* and *negative* values of ξ, respectively, such that $\widetilde{\Lambda} = \widetilde{\Lambda}_+ + \widetilde{\Lambda}_-$. We are interested to find $\widetilde{\Lambda}_+(\xi,s)$ that corresponds to the physical solution of (4.1) while $\widetilde{\Lambda}_-(\xi,s)$ is an auxiliary function. Therefore, we study the following equation

$$\widetilde{\Lambda}(\xi,s) = \widetilde{\Psi}(s) + \widetilde{\psi}(s)\int_0^\infty k(z-\xi)\widetilde{\Lambda}_+(z,s)\,dz\,, \tag{4.2}$$

with $\widetilde{\Lambda}_-(\xi,s) = \widetilde{\psi}(s)\int_0^\infty k(z-\xi)\widetilde{\Lambda}_+(z,s)dz$.

We consider now, as a meaningful and manageable example, the exponential kernel $k(x) = \mathrm{e}^{-|x|}/2$, again, see, e.g., [16–19]. By applying the Fourier transform (3.3) to (4.2) we have

$$\widehat{\widetilde{\Lambda}}_+(\omega, s) = -\frac{\omega(1 - i\omega)\widetilde{\Lambda}_-(0, s) + i(1 + \omega^2)\widetilde{\Psi}(s)}{\omega(s\widetilde{\Psi}(s) + \omega^2)}\,, \tag{4.3}$$

where by exploiting the exponential kernel we have $\widehat{k}(\omega) = 1/(1 + \omega^2)$ and from the definition of $\widetilde{\Lambda}_-(\xi, s)$ it results $\widehat{\widetilde{\Lambda}}_-(\omega, s) = \widetilde{\Lambda}_-(0, s)/(1 + i\omega)$. We can invert (4.3) through (3.4) by using the residue theorem at the simple poles $\omega = 0$, $\omega = i\sqrt{s\widetilde{\Psi}(s)}$ and $\omega = -i\sqrt{s\widetilde{\Psi}(s)}$. Finally, we have

$$\begin{aligned}\widetilde{\Lambda}_+(\xi, s) = \frac{1}{s} &- \left[\frac{\sqrt{s\widetilde{\Psi}(s)}(\sqrt{s\widetilde{\Psi}(s)} + 1)}{2s\widetilde{\Psi}(s)}\widetilde{\Lambda}_-(0, s) + \frac{1 - s\widetilde{\Psi}(s)}{2s}\right] \mathrm{e}^{\xi\sqrt{s\widetilde{\Psi}(s)}} \\ &- \left[\frac{\sqrt{s\widetilde{\Psi}(s)}(\sqrt{s\widetilde{\Psi}(s)} - 1)}{2s\widetilde{\Psi}(s)}\widetilde{\Lambda}_-(0, s) + \frac{1 - s\widetilde{\Psi}(s)}{2s}\right] \mathrm{e}^{-\xi\sqrt{s\widetilde{\Psi}(s)}}\,.\end{aligned} \tag{4.4}$$

Hence, the survival probability $\widetilde{\Lambda}_+(\xi, s)$ cannot be determined form (4.4), and consequently from (4.1) and (4.2), because $\widetilde{\Lambda}_-(0, s)$ is unknown.

From the theory of CTRW we also have [21]

$$\widetilde{\Lambda}_+(\xi, s) = \widetilde{\Psi}(s) \sum_{n=0}^{\infty} \phi_n(\xi) \left[\widetilde{\psi}(s)\right]^n\,, \tag{4.5}$$

where now n counts the number of occurred jumps. If we set $\xi = 0$ in (4.5), then from (1.2) we can have $\phi_n(0)$. Since (1.2) holds for Markovian random walks, in the framework of CTRW we have that the process is Markovian when [22,23]

$$\psi(\tau) = \psi_{\mathrm{M}}(\tau) = \mathrm{e}^{-\tau}\,, \quad \widetilde{\psi}_{\mathrm{M}}(s) = \widetilde{\Psi}_{\mathrm{M}}(s) = \frac{1}{1 + s}\,, \tag{4.6}$$

and it holds

$$\begin{aligned}\widetilde{\Lambda}_+(0, s) &= \widetilde{\Psi}_{\mathrm{M}}(s) \sum_{n=0}^{\infty} \phi_n(0) \left[\widetilde{\psi}_{\mathrm{M}}(s)\right]^n \\ &= \frac{\widetilde{\Psi}_{\mathrm{M}}(s)}{\sqrt{s\widetilde{\Psi}_{\mathrm{M}}(s)}} = \frac{1}{\sqrt{s(1 + s)}}\,.\end{aligned} \tag{4.7}$$

In the long-time limit, i.e., $s \to 0$, we have $\widetilde{\Lambda}_+(0, s) \sim 1/\sqrt{s}$ that after Laplace inversion gives $\Lambda_+(0, t) \sim 1/\sqrt{t}$ for $t \to +\infty$ [24]. The above limit is the continuous-time counter-part of (1.2). This limit can be indeed obtained from (4.7) with any distribution $\psi(\tau)$ such that $\widetilde{\psi}(s) \sim 1 - s$ for $s \to 0$, that is a weak rearrangement of Markovianity in the sense of the existence of a finite time-scale that gives a finite mean for the waiting-times. Finally, by equalling (4.4) with $\xi = 0$ and (4.7) we can determine $\widetilde{\Lambda}_-(0, s)$ as follows

$$\widetilde{\Lambda}_-(0, s) = \widetilde{\Psi}_{\mathrm{M}}(s) \frac{\sqrt{s\widetilde{\Psi}_{\mathrm{M}}(s)} - 1}{\sqrt{s\widetilde{\Psi}_{\mathrm{M}}(s)}} = \frac{\sqrt{s} - \sqrt{1 + s}}{\sqrt{s}(1 + s)}\,. \tag{4.8}$$

We can now complete the calculation of $\widetilde{\Lambda}_+(\xi, s)$ in the Markovian case that results to be

$$\widetilde{\Lambda}_+(\xi, s) = \frac{1}{s}\left\{1 - \left[1 - \sqrt{s\widetilde{\Psi}_{\mathrm{M}}(s)}\right] \mathrm{e}^{-\xi\sqrt{s\widetilde{\Psi}_{\mathrm{M}}(s)}}\right\}\,, \tag{4.9}$$

with

$$\widetilde{\Lambda}_+(0, s) = \frac{\sqrt{1 - \widetilde{\psi}_{\mathrm{M}}(s)}}{s}\,, \quad \widetilde{\Lambda}'_+(0, s) = \frac{\sqrt{1 - \widetilde{\psi}_{\mathrm{M}}(s)} - 1 + \widetilde{\psi}_{\mathrm{M}}(s)}{s}\,, \tag{4.10}$$

and it is in agreement with a previous derivation on the basis of probabilistic arguments [17]. In fact, by introducing the quantity $\widetilde{\lambda}(\xi,s) = 1 - s\,\widetilde{\Lambda}_+(\xi,s)$ we have

$$\widetilde{\lambda}(\xi,s) = \left[1 - \sqrt{s\widetilde{\Psi}_{\rm M}(s)}\right] {\rm e}^{-\xi\sqrt{s\widetilde{\Psi}_{\rm M}(s)}}\,, \tag{4.11}$$

that is the formula obtained by S.G. Kou and H. Wang for the same simple case considered here [17, Theorem 3.1, formula (3.1)].

In analogy with the previous section, we show that the proposed Sturm–Liouville system can be used for calculating the survival probability also for CTRW models. In this case, instead to consider the generating function, we consider formula (4.5) that allows for stepping from the discrete-time to the continuous-time setting. By taking into account (2.4), we have in the most general case that $\widetilde{\Lambda}_+(z,s)$ satisfies the equation

$$\frac{d}{dz}\left\{p(z)\,\frac{d}{dz}\widetilde{\Lambda}_+(z,s)\right\} + q(z)\,\widetilde{\Lambda}_+(z,s) = -\widetilde{\psi}(s)\,\widetilde{\Lambda}_+(z,s) + q(z)\,\widetilde{\Psi}(s)\,. \tag{4.12}$$

Hence, in the special case $k(x) = {\rm e}^{-|x|}/2$, see, e.g., [16–19], it holds $p(z) = 1$ and $q(z) = -1$ and, together with the Markovian assumption (4.6), equation (4.12) reduces to

$$\frac{d^2\widetilde{\Lambda}_+}{dz^2} - \widetilde{\Lambda}_+(z,s) = -\widetilde{\psi}_{\rm M}\,\widetilde{\Lambda}_+(z,s) - \widetilde{\Psi}_{\rm M}(s)\,. \tag{4.13}$$

To conclude, by applying the Laplace transform to z into v it holds

$$\widetilde{\widetilde{\Lambda}}_+(v,s) = \frac{v\,\widetilde{\Lambda}'_+(0,s) + v^2\,\widetilde{\Lambda}_+(0,s) - \widetilde{\Psi}_{\rm M}(s)}{v\,\left(v^2 - 1 + \widetilde{\psi}_{\rm M}(s)\right)}\,, \tag{4.14}$$

and, after plugging boundary conditions (4.10), we have

$$\widetilde{\widetilde{\Lambda}}_+(v,s) = \frac{(v+1)\,\sqrt{1-\widetilde{\psi}_{\rm M}(s)}}{vs\,\left(v + \sqrt{1-\widetilde{\psi}_{\rm M}(s)}\right)}\,. \tag{4.15}$$

Finally, it is possible to check that, by the Laplace inversion in v, from (4.15) we can recover (4.9).

5. Conclusion

In this paper we derived the Sturm–Liouville system of equations that is associated with the Wiener–Hopf integral equation (1.1) for the calculation of the survival probability in first-passage time problems for random walks. By studying a simple, but meaningful, example we showed the feasibility of the proposed approach within the framework of the discrete-time random walk and that of the CTRW. This approach is an alternative to the Pollaczek–Spitzer formula.

Summarising, we showed that $k(z,\xi)$ and $\phi_{n+1}(z)$ are indeed related accordingly to (1.1) if and only if they solve (2.3) and (2.4), respectively, and boundary conditions (2.12) are met. Hence, from the derived Sturm–Liouville system (2.5) the Sparre Andersen theorem (1.2) cannot be obtained but it is indeed a necessary boundary condition. In other words, from the simple and concrete example here studied, it follows that when the survival probability is calculated by the proposed Sturm–Liouville system (2.5) an extra condition must be added, and this extra condition is actually the statement of the Sparre Andersen theorem (1.2). As we discussed in the Introduction, this is a limitation also in the use of the Pollaczek–Spitzer formula in its original derivation [2] and analogue to the assumption of the boundedness of the survival probability in a more cumbersome derivation [11–13].

We end by underlying that the proposed approach allows for deriving explicit formulae for the survival probability in the discrete-time setting and also in the continuous-time one, this last feasibility supports to extend its successful application even to non-Markovian CTRW models.

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References

1. Bray, A. J., Majumdar, S. N. & Schehr, G., 2013 Persistence and first-passage properties in nonequilibrium systems.
   Adv. Phys. **62**, 225–361.
2. Spitzer, F., 1957 The Wiener–Hopf equation whose kernel is a probability density.
   Duke Math. J. **24**, 327–343.
3. Pollaczek, F., 1952 Fonctions caractéristiques de certaines répartitions définies au moyen de la notion d'ordre. Application à la théorie des attentes.
   C. R. Acad. Sci. Paris **234**, 2334–2336.
4. Sparre Andersen, E., 1954 On the fluctuations af sums of random variables II.
   Math. Scand. **2**, 195–223.
5. Spitzer, F., 1956 A combinatorial lemma and its application to probability theory.
   Trans. Am. Math. Soc. **82**, 323–339.
6. Feller, W., 1971 *An Introduction to Probability Theory and its Applications*, volume 2.
   New York: Wiley, second edition.
7. Frisch, U. & Frisch, H., 1975 Non-LTE transfer. $\sqrt{\epsilon}$ revisited.
   Mon. Not. R. Astr. Soc. **173**, 167–182.
8. Frisch, U. & Frisch, H., 1994 Universality of escape from a half-space for symmetrical random walks.
   In *Lévy Flights and Related Topics in Physics: Proceedings of the International Workshop Held at Nice, France, 27–30 June 1994* (eds. M. F. Shlesinger, G. M. Zaslavsky & U. Frisch), Lecture Notes in Physics, pp. 262–268. Berlin, Heidelberg: Springer–Verlag.
9. Vergassola, M., Dubrulle, B., Frisch, U. & Noullez, A., 2014 Burgers' equation, Devil's staircases and the mass distribution for large-scale structures.
   Astron. Astrophys. **289**, 325–356.
10. Majumdar, S. N., 2010 Universal first-passage properties of discrete-time random walks and Lévy flights on a line: Statistics of the global maximum and records.
    Physica A **389**, 4299–4316.
11. Ivanov, V. V., 1994 Resolvent method: exact solutions of half-space transport problems by elementary means.
    Astron. Astrophys. **286**, 328–337.
12. Comtet, A. & Majumdar, S. N., 2005 Precise asymptotics for a random walker's maximum.
    J. Stat. Mech. **2005**, P06013.
13. Majumdar, S. N., Mounaix, P. & Schehr, G., 2014 On the gap and time interval between the first two maxima of long random walks.
    J. Stat. Mech. **2014**, P09013.
14. Kisil, A. V., Abrahams, I. D., Mishuris, G. & Rogosin, S. V., 2021 The Wiener–Hopf technique, its generalizations and applications: constructive and approximate methods.
    Proc. R. Soc. A **477**, 20210533.
15. Tricomi, F. G., 1985 *Integral Equations*.
    New York: Dover Publication, Inc.
16. Gutkowicz–Krusin, D., Procaccia, I. & Ross, J., 1978 Analytic results for asymmetric random walk with exponential transition probabilities.
    J. Stat. Phys. **19**, 525–541.
17. Kou, S. G. & Wang, H., 2003 First passage times of a jump diffusion process.
    Adv. Appl. Probab. **35**, 504–531.
18. Majumdar, S. N., Comtet, A. & Ziff, R. M., 2006 Unified solution of the expected maximum of a discrete time random walk and the discrete flux to a spherical trap.
    J. Stat. Phys. **122**, 833–856.
19. Dahlenburg, M. & Pagnini, G., 2022 Exact calculation of the mean first-passage time of continuous-time random walks by nonhomogeneous Wiener–Hopf integral equations.
    J. Phys. A: Math. Theor. **55**, 505003.
20. Estrada, R. & Kanwal, R. P., 2000 *Singular Integral Equations*.
    New York: Springer Science+Business Media, LLC.

21. Gorenflo, R., Mainardi, F. & Vivoli, A., 2007 Continuous-time random walk and parametric subordination in fractional diffusion.
Chaos Solitons Fract. **34**, 87–103.
22. Zwanzig, R., 1983 From classical dynamics to continuous time random walks.
J. Stat. Phys. **30**, 255–262.
23. Mainardi, F., Raberto, M., Gorenflo, R. & Scalas, E., 2000 Fractional calculus and continuous-time finance II: the waiting-time distribution.
Physica A **287**, 468–481.
24. Artuso, R., Cristadoro, G., Degli Esposti, M. & Knight, G., 2014 Sparre–andersen theorem with spatiotemporal correlations.
Phys. Rev. E **89**, 052111.