Lieb–Robinson Bounds for Multi–Commutators and Applications to Response Theory

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Abstract

We generalize to multi–commutators the usual Lieb–Robinson bounds for commutators. In the spirit of constructive QFT, this is done so as to allow the use of combinatorics of minimally connected graphs (tree expansions) in order to estimate time–dependent multi–commutators for interacting fermions. Lieb–Robinson bounds for multi–commutators are effective mathematical tools to handle analytic aspects of the dynamics of quantum particles with interactions which are non–vanishing in the whole space and possibly time–dependent. To illustrate this, we prove that the bounds for multi–commutators of order three yield existence of fundamental solutions for the corresponding non–autonomous initial value problems for observables of interacting fermions on lattices. We further show how bounds for multi–commutators of order higher than two can be used to study linear and non–linear responses of interacting fermions to external perturbations. The results discussed here are also valid for quantum spins on lattices, with obvious modifications. However, we only discuss the fermionic case in detail, in view of applications to microscopic quantum theory of electrical conduction discussed here and because this case is technically more involved.

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Contents

1 Introduction

Lieb–Robinson bounds are upper–bounds on time–dependent commutators and were originally used to estimate propagation velocities of information in quantum spin systems. They have first been derived in 1972 by Lieb and Robinson [LR]. Nowadays, they are widely used in quantum information and condensed matter physics. Phenomenological consequences of Lieb–Robinson bounds have been experimentally observed in recent years, see [Ch].

For the reader's convenience and completeness, we start by deriving such bounds for fermions on the lattice with (possibly non–autonomous) interactions. As explained in [NS] in the context of quantum spin systems, Lieb–Robinson bounds are only expected to hold true for systems with short–range interactions.

We thus define Banach spaces $\mathcal{W}$ of short–range interactions and prove Lieb–Robinson bounds for the corresponding fermion systems. The spaces $\mathcal{W}$ include density–density interactions resulting from the second quantization of two–body interactions defined via a real–valued and integrable interaction kernel $v(r) : [0, \infty) \rightarrow \mathbb{R}$. Considering fermions with spin $1/2$, our setting includes, for instance, the celebrated Hubbard model (and any other system with finite–range interactions) or models with Yukawa–type potentials. Two–body interactions decaying polynomially fast in space with sufficiently large degree are also allowed, but the Coulomb potential is excluded because it is not summable at large distances. The method of proof we use to get Lieb–Robinson bounds for non–autonomous C^*–dynamical systems related to lattice fermions is, up to simple adaptations, the one used in [NS] for (autonomous) quantum spin systems. Compare Theorem 3.1, Lemma 3.2, Theorem 4.1 and Corollary 4.2 with [NS, Theorems 2.3. and 3.1.]. See also [BMNS] where (usual) Lieb–Robinson bounds for non–autonomous quantum spin systems have already been derived [BMNS, Theorems 4.6].

Once the Lieb–Robinson bounds for commutators are established, we combine them with results of the theory of strongly continuous semigroups to derive properties of the infinite volume dynamics. These allow us to extend Lieb–Robinson bounds to time–dependent *multi*–commutators, see Theorems 3.8–3.9 and 4.4. The new bounds on multi–commutators make possible rigorous studies of dynamical properties that are relevant for response theory of interacting fermion systems. For instance, they yield tree–decay bounds in the sense of [BPH1, Section 4] if interactions decay sufficiently fast in space (typically some polynomial decay with large enough degree is needed). In fact, by using the Lieb–Robinson bounds for multi–commutators, we extend in [BP2, BP3] our results [BPH1, BPH2, BPH3, BPH4] on free fermions to interacting particles with short–range interactions. This is an important application of such new bounds: The rigorous microscopic derivation of Ohm and Joule's laws for *interacting* fermions, in the AC–regime. See Section 5 and [BP1] for a historical perspective on this subject.

Via Theorems 5.1 and 5.5, we show, for example, how Lieb–Robinson bounds for multi–commutators can be applied to derive decay properties of the so–called *AC–conductivity measure* at high frequencies. This result is new and is obtained in Section 5. Cf. [BP2, BP3]. Lieb–Robinson bounds for multi–commutators have, moreover, further applications which go beyond the use on linear response theory presented in Section 5. For instance, as explained in Sections 3.3 and 4.3, they also make possible the study of *non–linear* corrections to linear responses to external perturbations.

The new bounds can also be applied to *non–autonomous systems*. Indeed, the existence of a fundamental solution for the non–autonomous initial value problem related to infinite systems of fermions with time–dependent interactions is usually a non–trivial problem because the corresponding generators are time–dependent unbounded operators. The time–dependency cannot, in general, be isolated into a bounded perturbation around some unbounded time–constant generator and usual perturbation theory cannot be applied. In many important cases, the time–dependent part of the generator is not even relatively bounded with respect to (w.r.t.) the constant part. In fact, no unified theory of non–autonomous evolution equations that gives a complete characterization of the existence of fundamental solutions in terms of properties of generators, analogously to the Hille–Yosida generation theorems for the autonomous case, is available. See, e.g., [K3, C, S, P, BB] and references therein. Note that the existence of a fundamental solution implies the well–posedness of the initial value problem related to states or observables of interacting lattice fermions, provided the corresponding evolution equation has a unique solution for any initial condition.

The Lieb–Robinson bounds on multi–commutators we derive here yield the existence of fundamental solutions as well as other general results on non–autonomous initial value problems related to fermion systems on lattices with interactions which are non–vanishing in the whole space and time–dependent. This is done in a rather constructive way, by considering the large volume limit of finite volume dynamics, without using standard sufficient conditions for existence of fundamental solutions of non–autonomous linear evolution equations. If interactions decay exponentially fast in space, then we moreover show, also by using Lieb–Robinson bounds on multi–commutators, that the *non–autonomous* dynamics is smooth w.r.t. its generator on the dense set of local observables. See Theorem 4.6. Note that the generator of the (non–autonomous) dynamics generally has, in our case, a time–dependent domain, and the existence of a dense set of smooth vectors is a priori not at all clear.

Observe that the evolution equations for lattice fermions are not of parabolic type, in the precise sense formulated in [AT], because the corresponding generators do not generate analytic semigroups. They seem to be rather related to Kato's hyperbolic case [K1, K2, K3]. Indeed, by structural reasons – more precisely, the fact that the generators are derivations on a C^*–algebra – the time–dependent generator defines a stable family of operators in the sense of Kato. Moreover, this family always possesses a common core. In some specific situations one can directly show that the completion of this core w.r.t. a conveniently chosen norm defines a so–called admissible Banach space $\mathcal{Y}$ of the generator at any time, which satisfies further technical conditions leading to Kato's hyperbolic conditions [K1, K2, K3]. See also [P, Sect. 5.3.] and [BB, Sect. VII.1]. Nevertheless, the existence of such a Banach space $\mathcal{Y}$ is a priori unclear in the general case treated here (Theorem 4.5).

Our central results are Theorems 3.8–3.9 and 4.4. Other important assertions are Corollary 3.10 and Theorems 4.5–4.6, 4.8–4.9, 5.1, 5.5. This paper is organized as follows:

- Section 2 defines our setting. In particular, Banach spaces of short–range interactions are introduced.

- Section 3 is devoted to Lieb–Robinson bounds, which are generalized to multi–commutators. We also give a proof of the existence of the infinite–volume dynamics as well as some applications of such bounds. The tree–decay bounds on time–dependent multi–commutators (Corollary 3.10) are proven here. However, only the autonomous dynamics is considered in this section.

- Section 4 extends results of Section 3 to the non–autonomous case. We prove, in particular, the existence of a fundamental solution for the non–autonomous initial value problems related to infinite interacting systems of fermions on lattices with time–dependent interactions (Theorem 4.5). This implies well–posedness of the corresponding initial value problems for states and observables, provided their solutions are unique for any initial condition. Applications in (possibly non–linear) response theory (Theorems 4.8–4.9) are discussed as well.

- Finally, Section 5 explains how Lieb–Robinson bounds for multi–commutators can be applied to study (quantum) charged transport properties within the AC–regime. This analysis yields, in particular, the asymptotics at high frequencies of the so–called AC–conductivity measure. See Theorems 5.1 and 5.5.

Notation 1.1
(i) *We denote by D any positive and finite generic constant. These constants do not need to be the same from one statement to another.*
(ii) *A norm on the generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$ and the identity map of $\mathcal{X}$ by $\mathbf{1}_{\mathcal{X}}$. The space of all bounded linear operators on $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is denoted*

by $\mathcal{B}(\mathcal{X})$.
(iii) *If O is an operator, $\|\cdot\|_O$ stands for the graph norm on its domain.*
(iv) *By a slight abuse of notation, we denote in the sequel elements $X_i \in Y$ depending on the index $i \in I$ by expressions of the form $\{X_i\}_{i\in I} \subset Y$ (instead of $(X_i)_{i\in I} \subset I \times Y$).*

2 Algebraic Setting for Interacting Fermions on the Lattice

2.1 The Lattice CAR C^*–Algebra

We consider fermions on a lattice $\mathfrak{L}$. For convenience, the latter is taken to be a cubic one, i.e., $\mathfrak{L} \doteq \mathbb{Z}^d$ for $d \in \mathbb{N}$. This special choice is not essential in our proofs. In fact, we could take instead of $\mathbb{Z}^d$ any countable metric space $\mathfrak{L}$ as soon as it is regular (see Section 2.2), as defined in [NS, Section 3.1].

Let $\mathcal{P}_f(\mathfrak{L}) \subset 2^{\mathfrak{L}}$ be the set of all *finite* subsets of $\mathfrak{L}$. For any $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $\mathcal{U}_\Lambda$ is the finite dimensional C^*–algebra generated by $\mathbf{1}$ and generators $\{a_{x,\mathrm{s}}\}_{x\in\Lambda,\mathrm{s}\in\mathrm{S}}$ satisfying the canonical anti–commutation relations, S being some *finite* set of spins. To simplify notation, we omit the spin dependence of $a_{x,\mathrm{s}} \equiv a_x$, which is irrelevant in our proofs (up to trivial modifications). In fact, without loss of generality (w.l.o.g.), we will only consider spinless fermions, i.e., the case $\mathrm{S} = \{0\}$.

Let

$$\Lambda_L \doteq \{(x_1, \ldots, x_d) \in \mathfrak{L} : |x_1|, \ldots, |x_d| \leq L\} \in \mathcal{P}_f(\mathfrak{L}) \tag{1}$$

for all $L \in \mathbb{R}_0^+$ and observe that $\{\mathcal{U}_{\Lambda_L}\}_{L\in\mathbb{R}_0^+}$ is an increasing net of C^*–algebras. Hence, the set

$$\mathcal{U}_0 \doteq \bigcup_{L\in\mathbb{R}_0^+} \mathcal{U}_{\Lambda_L} \tag{2}$$

of local elements is a normed $*$–algebra with $\|A\|_{\mathcal{U}_0} = \|A\|_{\mathcal{U}_{\Lambda_L}}$ for all $A \in \mathcal{U}_{\Lambda_L}$ and $L \in \mathbb{R}_0^+$. The CAR C^*–algebra $\mathcal{U}$ of the infinite system is by definition the completion of the normed $*$–algebra $\mathcal{U}_0$. It is separable, by finite dimensionality of $\mathcal{U}_\Lambda$ for $\Lambda \in \mathcal{P}_f(\mathfrak{L})$. In other words, $\mathcal{U}$ is the inductive limit of the finite dimensional C^*–algebras $\{\mathcal{U}_\Lambda\}_{\Lambda\in\mathcal{P}_f(\mathfrak{L})}$.

For any fixed $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, the condition

$$\sigma_\theta(a_x) = \mathrm{e}^{-i\theta} a_x \tag{3}$$

defines a unique automorphism σ_θ of the C^*–algebra $\mathcal{U}$. A special role is played by σ_π. Elements $B_1, B_2 \in \mathcal{U}$ satisfying $\sigma_\pi(B_1) = B_1$ and $\sigma_\pi(B_2) = -B_2$ are respectively called *even* and *odd*, while elements $B \in \mathcal{U}$ satisfying $\sigma_\theta(B) = B$ for any $\theta \in [0, 2\pi)$ are called *gauge invariant*. The set

$$\mathcal{U}^+ \doteq \{B \in \mathcal{U} \ : \ B = \sigma_\pi(B)\} \subset \mathcal{U} \tag{4}$$

of all even elements and the set

$$\mathcal{U}^\circ \doteq \bigcap_{\theta \in \mathbb{R}/(2\pi\mathbb{Z})} \{B \in \mathcal{U} \ : \ B = \sigma_\theta(B)\} \subset \mathcal{U}^+ \tag{5}$$

of all gauge invariant elements are $*$–algebras. By continuity of σ_θ, it follows that $\mathcal{U}^+$ and $\mathcal{U}^\circ$ are closed and hence C^*–algebras. $\mathcal{U}^\circ$ is known as the fermion observable algebra.

2.2 Banach Spaces of Short–Range Interactions

An *interaction* is a family $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathfrak{L})}$ of even and self–adjoint local elements $\Phi_\Lambda = \Phi_\Lambda^* \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$ with $\Phi_\emptyset = 0$. Obviously, the set of all interactions can be endowed with a real vector space structure:

$$(\alpha_1 \Phi + \alpha_2 \Psi)_\Lambda \doteq \alpha_1 \Phi_\Lambda + \alpha_2 \Psi_\Lambda$$

for any interactions Φ, Ψ, and any real numbers α_1, α_2. We define Banach spaces of short–range interactions by introducing specific norms for interactions, taking into account space decay.

To this end, following [NOS, Eqs. (1.3)–(1.4)], we consider positive–valued and non–increasing decay functions $\mathbf{F} : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfying the following properties:

- *Summability on $\mathfrak{L}$.*

$$\|\mathbf{F}\|_{1,\mathfrak{L}} \doteq \sup_{y \in \mathfrak{L}} \sum_{x \in \mathfrak{L}} \mathbf{F}\left(|x - y|\right) = \sum_{x \in \mathfrak{L}} \mathbf{F}\left(|x|\right) < \infty \, . \tag{6}$$

- *Bounded convolution constant.*

$$\mathbf{D} \doteq \sup_{x,y \in \mathfrak{L}} \sum_{z \in \mathfrak{L}} \frac{\mathbf{F}\left(|x - z|\right) \mathbf{F}\left(|z - y|\right)}{\mathbf{F}\left(|x - y|\right)} < \infty \, . \tag{7}$$

In the case $\mathfrak{L}$ would be a general countable set with infinite cardinality and some metric d, the existence of such a function $\mathbf{F}$ satisfying (6)–(7) with $\mathrm{d}(\cdot,\cdot)$ instead of $|\cdot - \cdot|$ refers to the so–called *regular* property of $\mathfrak{L}$. For any $d \in \mathbb{N}$, $\mathfrak{L} \doteq \mathbb{Z}^d$ is in this sense regular with the metric $\mathrm{d}(\cdot,\cdot) = |\cdot - \cdot|$. Indeed, a typical example of such a $\mathbf{F}$ for $\mathfrak{L} = \mathbb{Z}^d$, $d \in \mathbb{N}$, and the metric induced by $|\cdot|$ is the function

$$\mathbf{F}(r) \doteq (1+r)^{-(d+\epsilon)} , \qquad r \in \mathbb{R}_0^+ , \tag{8}$$

which has convolution constant $\mathbf{D} \leq 2^{d+1+\epsilon} \|\mathbf{F}\|_{1,\mathfrak{L}}$ for $\epsilon \in \mathbb{R}^+$. See [NOS, Eq. (1.6)] or [Si, Example 3.1]. Note that the exponential function $\mathbf{F}(r) = \mathrm{e}^{-\varsigma r}$, $\varsigma \in \mathbb{R}^+$, satisfies (6) but not (7). Nevertheless, for every function $\mathbf{F}$ with bounded convolution constant (7) and any strictly positive parameter $\varsigma \in \mathbb{R}^+$, the function

$$\tilde{\mathbf{F}}(r) = \mathrm{e}^{-\varsigma r}\mathbf{F}(r) , \qquad r \in \mathbb{R}_0^+ ,$$

clearly satisfies Assumption (7) with a convolution constant that is no bigger than the one of $\mathbf{F}$. In fact, as observed in [Si, Section 3.1], the multiplication of such a function $\mathbf{F}$ with a non–increasing weight $f : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfying $f(r+s) \geq f(r) f(s)$ (logarithmically superadditive function) does not increase the convolution constant $\mathbf{D}$. In all the paper, (6)–(7) are assumed to be satisfied.

The function $\mathbf{F}$ encodes the short–range property of interactions. Indeed, an interaction Φ is said to be *short–range* if

$$\|\Phi\|_{\mathcal{W}} \doteq \sup_{x,y\in\mathfrak{L}} \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L}),\, \Lambda\supset\{x,y\}} \frac{\|\Phi_\Lambda\|_{\mathcal{U}}}{\mathbf{F}(|x-y|)} < \infty . \tag{9}$$

Since the map $\Phi \mapsto \|\Phi\|_{\mathcal{W}}$ defines a norm on interactions, the space of short–range interactions w.r.t. to the decay function $\mathbf{F}$ is the real separable Banach space $\mathcal{W} \equiv (\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ of all interactions Φ with $\|\Phi\|_{\mathcal{W}} < \infty$. Note that a short–range interaction $\Phi \in \mathcal{W}$ is not necessarily weak away from the origin of $\mathfrak{L}$: Generally, the element $\Phi_{x+\Lambda}$, $x \in \mathfrak{L}$, does not vanish when $|x| \to \infty$. It turns out that all the $\Phi \in \mathcal{W}$ define, in a natural way, infinite–volume quantum dynamics, i.e., they define C^*–dynamical systems on $\mathcal{U}$. For more details, see Section 3.1, in particular Theorem 3.6.

3 Lieb–Robinson Bounds for Multi–Commutators

Lieb–Robinson bounds for multi–commutators are studied here for fermion systems, only. In the case of quantum spin systems, $\mathcal{U}$ has to be replaced by the infinite tensor product of copies of some finite dimensional C^*–algebra attached to each site $x \in \mathfrak{L}$. All results of this section also hold in this situation. We concentrate our attention on fermion algebras in view of applications to microscopic foundations of the theory of electrical conduction [BP1, BP2]. Moreover, the fermionic case is, technically speaking, more involved, because of the non–commutativity of elements of the CAR algebra $\mathcal{U}$ sitting on different lattice sites.

3.1 Existence of Dynamics and Lieb–Robinson Bounds

Recall that an interaction is a family $\Psi = \{\Psi_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathfrak{L})}$ of even and self–adjoint local elements $\Psi_\Lambda = \Psi_\Lambda^* \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$ with $\Psi_\emptyset \doteq 0$. In Section 2.2, we define a Banach space $\mathcal{W}$ of short–range interactions by using a convenient norm $\| \cdot \|_\mathcal{W}$ for interactions, see (9). $\Psi \in \mathcal{W}$ ensures the existence of an infinite–volume derivation δ associated with Ψ by taking the thermodynamic limit of commutators involving Ψ_Λ, $\Lambda \in \mathcal{P}_f(\mathfrak{L})$. Every generator of a C^*–dynamical system is a derivation, but the converse does not generally hold. We show here that δ is the generator of a C^*–dynamical system in $\mathcal{U}$ when $\Psi \in \mathcal{W}$.

The key ingredient in this analysis are the so–called *Lieb–Robinson bounds*. Indeed, they lead, among other things, to the existence of the infinite–volume dynamics for interacting particles. By using this, we define a C^*–dynamical system in $\mathcal{U}$ for any short–range interaction $\Psi \in \mathcal{W}$. These bounds are, moreover, a pivotal ingredient to study transport properties of interacting fermion systems later on. Thus, for the reader's convenience, below we review this topic in detail.

We start by defining the finite–volume dynamics as follows: Take any short–range interaction $\Psi \doteq \{\Psi_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} \in \mathcal{W}$ and any *potential* $\mathbf{V}$. By potential, we mean here a collection $\mathbf{V} \doteq \{\mathbf{V}_{\{x\}}\}_{x \in \mathfrak{L}}$ of even (cf. (4)) and self–adjoint elements such that $\mathbf{V}_{\{x\}} = \mathbf{V}_{\{x\}}^* \in \mathcal{U}^+ \cap \mathcal{U}_{\{x\}}$ for all $x \in \mathfrak{L}$. Indeed, strictly speaking, such a potential is nothing but a special case of interaction in the sense of Section 2.2. Such potentials are sometimes also called *on–site interactions*. The interaction representing $\mathbf{V}$ can possibly be outside $\mathcal{W}$ because we allow $\mathbf{V}$ to be unbounded, i.e., the case

$$\sup_{x \in \mathfrak{L}} \left\| \mathbf{V}_{\{x\}} \right\|_\mathcal{U} = \infty \tag{10}$$

is included in the discussion below. To such objects we associate the (internal) energy observable

$$H_L \doteq \sum_{\Lambda \subseteq \Lambda_L} \Psi_\Lambda + \sum_{x \in \Lambda_L} \mathbf{V}_{\{x\}} \ , \qquad L \in \mathbb{R}_0^+ \ , \tag{11}$$

of the cubic box Λ_L defined by (1). The finite–volume dynamics then corresponds to the continuous group $\{\tau_t^{(L)}\}_{t\in\mathbb{R}}$ of $*$–automorphisms of $\mathcal{U}$ defined by

$$\tau_t^{(L)}(B) = \mathrm{e}^{itH_L} B \mathrm{e}^{-itH_L} \ , \qquad B \in \mathcal{U} \ , \tag{12}$$

for any $L \in \mathbb{R}_0^+$, $\Psi \in \mathcal{W}$ and potential $\mathbf{V}$. Obviously, its generator is the bounded linear operator $\delta^{(L)}$ defined on $\mathcal{U}$ by

$$\delta^{(L)}(B) \doteq i \sum_{\Lambda \subseteq \Lambda_L} [\Psi_\Lambda, B] + i \sum_{x \in \Lambda_L} \left[\mathbf{V}_{\{x\}}, B\right] \ , \qquad B \in \mathcal{U} \ . \tag{13}$$

It is a symmetric derivation on $\mathcal{U}$ because, for all $B_1, B_2 \in \mathcal{U}$,

$$\delta^{(L)}(B_1^*) = \delta^{(L)}(B_1)^* \quad \text{and} \quad \delta^{(L)}(B_1 B_2) = \delta^{(L)}(B_1) B_2 + B_1 \delta^{(L)}(B_2) \ .$$

It is convenient to introduce at this point the notation

$$\mathcal{S}_\Lambda(\tilde{\Lambda}) \doteq \left\{ \mathcal{Z} \subset \Lambda : \mathcal{Z} \cap \tilde{\Lambda} \neq 0 \text{ and } \mathcal{Z} \cap \tilde{\Lambda}^c \neq 0 \right\} \tag{14}$$

for any set $\tilde{\Lambda} \subset \Lambda \subset \mathfrak{L}$ with complement $\tilde{\Lambda}^c \doteq \mathfrak{L} \backslash \tilde{\Lambda}$, as well as

$$\partial_\Psi \Lambda \doteq \{x \in \Lambda : \exists \mathcal{Z} \in \mathcal{S}_{\mathfrak{L}}(\Lambda) \text{ with } x \in \mathcal{Z} \text{ and } \Psi_{\mathcal{Z}} \neq 0\}$$

for any interaction $\Psi \doteq \{\Psi_{\mathcal{Z}}\}_{\mathcal{Z} \in \mathcal{P}_f(\mathfrak{L})}$ and any finite subset $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ of $\mathfrak{L}$. We are now in position to prove Lieb–Robinson bounds for finite–volume fermion systems with short–range interactions and in presence of potentials:

Theorem 3.1 (Lieb–Robinson bounds)
Let $\Psi \in \mathcal{W}$ and $\mathbf{V}$ be any potential. Then, for any $t \in \mathbb{R}$, $L \in \mathbb{R}_0^+$, and elements $B_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$, $B_2 \in \mathcal{U}_{\Lambda^{(2)}}$ with $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f(\mathfrak{L})$ and $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$,

$$\begin{aligned} \left\| [\tau_t^{(L)}(B_1), B_2] \right\|_{\mathcal{U}} \ &\leq \ 2\mathbf{D}^{-1} \|B_1\|_{\mathcal{U}} \|B_2\|_{\mathcal{U}} \left(\mathrm{e}^{2\mathbf{D}|t| \|\Psi\|_{\mathcal{W}}} - 1\right) \\ &\quad \times \sum_{x \in \partial_\Psi \Lambda^{(1)}} \sum_{y \in \Lambda^{(2)}} \mathbf{F}(|x-y|) \ . \end{aligned} \tag{15}$$

The constant $\mathbf{D} \in \mathbb{R}^+$ is defined by (7).

Proof: The arguments are essentially the same as those proving [NS, Theorem 2.3.] for quantum spin systems. Here, we consider fermion systems and we give the detailed proof for completeness and to prepare its extension to time–dependent interactions and potentials, in Theorem 4.1 (i). We fix $L \in \mathbb{R}_0^+$, $B_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$ and $B_2 \in \mathcal{U}_{\Lambda^{(2)}}$ with disjoint sets $\Lambda^{(1)}, \Lambda^{(2)} \subsetneq \Lambda_L$. [Note that $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$ yields $L \geq 1$.]

Let

$$C_{B_2}(\Lambda;t) \doteq \sup_{B \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda, B \neq 0} \frac{\left\| [\tau_t^{(L)}(B), B_2] \right\|_{\mathcal{U}}}{\|B\|_{\mathcal{U}}}, \qquad t \in \mathbb{R}, \ \Lambda \in \mathcal{P}_f(\mathfrak{L}).$$

At time $t = 0$, we observe that

$$|C_{B_2}(\Lambda;0)| \leq 2 \|B_2\|_{\mathcal{U}} \mathbf{1}\left[\Lambda \cap \Lambda^{(2)} \neq \emptyset\right],$$

while, for any $t \in \mathbb{R}$,

$$C_{B_2}(\Lambda;t) = \sup_{B \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda, B \neq 0} \frac{\left\| [\tau_t^{(L)} \circ \tau_{-t}^{(\Lambda)}(B), B_2] \right\|_{\mathcal{U}}}{\|B\|_{\mathcal{U}}}.$$

Here, $\{\tau_t^{(\Lambda)}\}_{t \in \mathbb{R}}$ is the continuous group of $*$–automorphisms defined like $\{\tau_t^{(L)}\}_{t \in \mathbb{R}}$ by replacing the box Λ_L with the (finite) set $\Lambda \in \mathcal{P}_f(\mathfrak{L})$.

Consider the function

$$f(t) \doteq \left[\tau_t^{(L)} \circ \tau_{-t}^{(\Lambda^{(1)})}(B_1), B_2\right], \qquad t \in \mathbb{R}. \tag{16}$$

Then, using $B_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$ and $\Lambda^{(1)} \subset \Lambda_L$, we deduce from (13) and explicit computations that

$$\begin{aligned} \partial_t f(t) &= i \sum_{\mathcal{Z} \in \mathcal{S}_{\Lambda_L}(\Lambda^{(1)})} \left[\tau_t^{(L)}(\Psi_{\mathcal{Z}}), f(t)\right] \\ &\quad - i \sum_{\mathcal{Z} \in \mathcal{S}_{\Lambda_L}(\Lambda^{(1)})} \left[\tau_t^{(L)} \circ \tau_{-t}^{(\Lambda^{(1)})}(B_1), \left[\tau_t^{(L)}(\Psi_{\mathcal{Z}}), B_2\right]\right]. \end{aligned} \tag{17}$$

Let $\mathfrak{g}_t(B)$ be the solution of

$$\forall t \geq 0: \qquad \partial_t \mathfrak{g}_t(B) = i \sum_{\mathcal{Z} \in \mathcal{S}_{\Lambda_L}(\Lambda^{(1)})} [\tau_t^{(L)}(\Psi_{\mathcal{Z}}), \mathfrak{g}_t(B)], \qquad \mathfrak{g}_0(B) = B \in \mathcal{U}.$$

Since $\|\mathfrak{g}_t(B)\|_{\mathcal{U}} = \|B\|_{\mathcal{U}}$ for any $B \in \mathcal{U}$, it follows from (17), by variation of constants, that

$$\|f(t)\|_{\mathcal{U}} \leq \|f(0)\|_{\mathcal{U}} + 2\|B_1\|_{\mathcal{U}} \sum_{\mathcal{Z}\in\mathcal{S}_{\Lambda_L}(\Lambda^{(1)})} \int_0^{|t|} \left\| \left[\tau_{\pm s}^{(L)}(\Psi_{\mathcal{Z}}), B_2 \right] \right\|_{\mathcal{U}} \mathrm{d}s \, . \quad (18)$$

[The sign of s in $\pm s$ depends whether t is positive or negative.] Hence, as $\Lambda^{(1)}, \Lambda^{(2)}$ are disjoint, for any $t \in \mathbb{R}$,

$$C_{B_2}\left(\Lambda^{(1)}; t\right) \leq 2 \sum_{\mathcal{Z}\in\mathcal{S}_{\Lambda_L}(\Lambda^{(1)})} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \int_0^{|t|} C_{B_2}(\mathcal{Z}; \pm s)\, \mathrm{d}s \, . \quad (19)$$

By estimating $C_{B_2}(\mathcal{Z}; s)$ in a similar manner and iterating this procedure, we show that, for every $L \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and all $B_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$, $B_2 \in \mathcal{U}_{\Lambda^{(2)}}$ with disjoint $\Lambda^{(1)}, \Lambda^{(2)} \subset \Lambda_L$,

$$C_{B_2}\left(\Lambda^{(1)}; t\right) \leq 2\|B_2\|_{\mathcal{U}} \sum_{k\in\mathbb{N}} \frac{|2t|^k}{k!} u_k \, , \quad (20)$$

where, for any $k \in \mathbb{N}$,

$$u_k \doteq \sum_{\mathcal{Z}_1\in\mathcal{S}_{\Lambda_L}(\Lambda^{(1)})} \sum_{\mathcal{Z}_2\in\mathcal{S}_{\Lambda_L}(\mathcal{Z}_1)} \cdots \sum_{\mathcal{Z}_k\in\mathcal{S}_{\Lambda_L}(\mathcal{Z}_{k-1})} \mathbf{1}\left[\mathcal{Z}_k \cap \Lambda^{(2)} \neq \emptyset\right] \prod_{j=1}^{k} \left\|\Psi_{\mathcal{Z}_j}\right\|_{\mathcal{U}} \, .$$

The above series is absolutely and uniformly convergent for $L \in \mathbb{R}_0^+$ (with fixed $\Lambda^{(1)}, \Lambda^{(2)} \subsetneq \Lambda_L$). Indeed, from straightforward estimates,

$$u_k \leq \mathbf{D}^{k-1} \|\Psi\|_{\mathcal{W}}^k \sum_{x\in\partial_\Psi\Lambda^{(1)}} \sum_{y\in\Lambda^{(2)}} \mathbf{F}(|x-y|) \, , \quad (21)$$

by Equations (7) and (9).

Note that (20)–(21) yield (15), provided $\Lambda^{(1)}, \Lambda^{(2)} \subsetneq \Lambda_L$. This last condition can easily be removed by taking, at any fixed $L \in \mathbb{R}_0^+$, an interaction $\tilde{\Psi}^{(L)} \in \mathcal{W}$ defined by $\tilde{\Psi}_{\mathcal{Z}}^{(L)} \doteq \Psi_{\mathcal{Z}}$ for any $\mathcal{Z} \subseteq \Lambda_L$, while $\tilde{\Psi}_{\mathcal{Z}}^{(L)} \doteq 0$ when $\mathcal{Z} \nsubseteq \Lambda_L$. Indeed, for all $L \in \mathbb{R}_0^+$, we obviously have $\|\tilde{\Psi}^{(L)}\|_{\mathcal{W}} \leq \|\Psi\|_{\mathcal{W}}$. Furthermore, for all $L, \tilde{L} \in \mathbb{R}_0^+$ with $\tilde{L} > L$, $\tilde{\tau}_t^{(\tilde{L})} = \tau_t^{(L)}$, where $\{\tilde{\tau}_t^{(\tilde{L})}\}_{t\in\mathbb{R}}$ is the (finite–volume) group of $*$–automorphisms of $\mathcal{U}$ defined by (12) with $L = \tilde{L}$ and $\Psi = \tilde{\Psi}^{(L)}$. Therefore, it suffices to apply (20)–(21) to the interaction $\tilde{\Psi}^{(L)}$ for sufficiently large $\tilde{L} \in \mathbb{R}_0^+$ in order to get the assertion without the condition $\Lambda^{(1)}, \Lambda^{(2)} \subsetneq \Lambda_L$. ■

As explained in [NS, Theorem 3.1] for quantum spin systems, Lieb–Robinson bounds lead to the existence of the infinite–volume dynamics:

Lemma 3.2 (Infinite–volume dynamics)
Let $\Psi \in \mathcal{W}$ and $\mathbf{V}$ be any potential. Then, for any $t \in \mathbb{R}$, $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $B \in \mathcal{U}_\Lambda$ and $L_1, L_2 \in \mathbb{R}_0^+$ with $\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$,

$$\left\| \tau_t^{(L_2)}(B) - \tau_t^{(L_1)}(B) \right\|_{\mathcal{U}} \leq 2 \|B\|_{\mathcal{U}} \|\Psi\|_{\mathcal{W}} |t| \, \mathrm{e}^{4\mathbf{D}|t| \|\Psi\|_{\mathcal{W}}} \times \sum_{y \in \Lambda_{L_2} \backslash \Lambda_{L_1}} \sum_{x \in \Lambda} \mathbf{F}(|x-y|) \ .$$

Proof: Again, the arguments are those proving [NS, Theorem 3.1.] for quantum spin systems. We give them for completeness, having also in mind the extension of the lemma to time–dependent interactions and potentials, in Theorem 4.1 (ii). We fix in all the proof $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ and $B \in \mathcal{U}_\Lambda$.

For any $L \in \mathbb{R}_0^+$ and $s, t \in \mathbb{R}$, define the unitary element

$$\mathbf{U}_L(t,s) \doteq \mathrm{e}^{it\mathbf{V}_{\Lambda_L}} \mathrm{e}^{-i(t-s)H_L} \mathrm{e}^{-is\mathbf{V}_{\Lambda_L}} \in \mathcal{U}_{\Lambda_L} \tag{22}$$

with

$$\mathbf{V}_{\mathcal{Z}} \doteq \sum_{x \in \mathcal{Z}} \mathbf{V}_{\{x\}} \in \mathcal{U}^+ \cap \mathcal{U}_{\mathcal{Z}} \ , \qquad \mathcal{Z} \in \mathcal{P}_f(\mathfrak{L}) \ .$$

Clearly, $\mathbf{U}_L(t,t) = \mathbf{1}_{\mathcal{U}}$ for all $t \in \mathbb{R}$ while

$$\partial_t \mathbf{U}_L(t,s) = -iG_L(t)\,\mathbf{U}_L(t,s) \quad \text{and} \quad \partial_s \mathbf{U}_L(t,s) = i\mathbf{U}_L(t,s)\,G_L(s)$$

with

$$G_L(t) \doteq \sum_{\mathcal{Z} \subseteq \Lambda_L} \mathrm{e}^{it\mathbf{V}_{\Lambda_L}} \Psi_{\mathcal{Z}} \mathrm{e}^{-it\mathbf{V}_{\Lambda_L}} \ .$$

Let

$$\tilde{\tau}_t^{(L)}(B) \doteq \mathbf{U}_L(0,t)\,B\,\mathbf{U}_L(t,0) \ , \qquad B \in \mathcal{U}_\Lambda \ .$$

For any $t \in \mathbb{R}$ and $L \in \mathbb{R}_0^+$ such that $\Lambda \subset \Lambda_L$,

$$\tau_t^{(L)}(B) = \tilde{\tau}_t^{(L)}\left(\mathrm{e}^{it\mathbf{V}_{\Lambda_L}} B \mathrm{e}^{-it\mathbf{V}_{\Lambda_L}}\right) = \tilde{\tau}_t^{(L)}\left(\mathrm{e}^{it\mathbf{V}_{\Lambda}} B \mathrm{e}^{-it\mathbf{V}_{\Lambda}}\right)$$

and it suffices to study the net $\{\tilde{\tau}_t^{(L)}(B)\}_{L\in\mathbb{R}_0^+}$ in $\mathcal{U}$. The equality above is related to the so–called "interaction picture" (w.r.t. potentials) of the time–evolution defined by the $*$–automorphism $\tau_t^{(L)}$.

Fix $L_1, L_2 \in \mathbb{R}_0^+$ with $\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$. Note that, for any $t \in \mathbb{R}$,

$$\tilde{\tau}_t^{(L_2)}(B) - \tilde{\tau}_t^{(L_1)}(B) = \int_0^t \partial_s \{\mathbf{U}_{L_2}(0,s)\,\mathbf{U}_{L_1}(s,t)\,B\mathbf{U}_{L_1}(t,s)\,\mathbf{U}_{L_2}(s,0)\}\,\mathrm{d}s\ . \tag{23}$$

Straightforward computations yield

$$\begin{aligned} &\partial_s \{\mathbf{U}_{L_2}(0,s)\,\mathbf{U}_{L_1}(s,t)\,B\mathbf{U}_{L_1}(t,s)\,\mathbf{U}_{L_2}(s,0)\} \\ =\ & i\mathbf{U}_{L_2}(0,s)\left[G_{L_2}(s) - G_{L_1}(s)\,,\mathbf{U}_{L_1}(s,t)\,B\mathbf{U}_{L_1}(t,s)\right]\mathbf{U}_{L_2}(s,0) \\ =\ & i\mathbf{U}_{L_2}(0,s)\,\mathrm{e}^{is\mathbf{V}_{\Lambda_{L_1}}}\left[B_s, \tau_{t-s}^{(L_1)}(\tilde{B}_t)\right]\mathrm{e}^{-is\mathbf{V}_{\Lambda_{L_1}}}\mathbf{U}_{L_2}(s,0)\ , \end{aligned} \tag{24}$$

where, for any $s,t \in \mathbb{R}$, we define

$$B_s \doteq \mathrm{e}^{-is\mathbf{V}_{\Lambda_{L_1}}}\left(G_{L_2}(s) - G_{L_1}(s)\right)\mathrm{e}^{is\mathbf{V}_{\Lambda_{L_1}}} \quad \text{and} \quad \tilde{B}_t \doteq \mathrm{e}^{-it\mathbf{V}_\Lambda}B\mathrm{e}^{it\mathbf{V}_\Lambda}\ . \tag{25}$$

Thus, we infer from Equations (23)–(25) that

$$\left\|\tilde{\tau}_t^{(L_2)}(B) - \tilde{\tau}_t^{(L_1)}(B)\right\|_{\mathcal{U}} \leq \int_0^{|t|}\left\|\left[\tau_{\pm s-t}^{(L_1)}(B_{\pm s})\,,\tilde{B}_t\right]\right\|_{\mathcal{U}}\mathrm{d}s\ . \tag{26}$$

[The sign of s in $\pm s$ depends whether t is positive or negative.] Note that $\tilde{B}_t \in \mathcal{U}_\Lambda$ and

$$B_s = \sum_{\mathcal{Z}\subseteq\Lambda_{L_2},\ \mathcal{Z}\cap(\Lambda_{L_2}\backslash\Lambda_{L_1})\neq\emptyset} \mathrm{e}^{is\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}}}\Psi_{\mathcal{Z}}\mathrm{e}^{-is\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}}} \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda_{L_2}}$$

where, for any $\mathcal{Z} \subseteq \Lambda_{L_2}$,

$$\mathrm{e}^{is\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}}}\Psi_{\mathcal{Z}}\mathrm{e}^{-is\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}}} \in \mathcal{U}_{\mathcal{Z}}\ .$$

Now, we apply the Lieb–Robinson bounds given by Theorem 3.1 to deduce that, for any $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $s,t \in \mathbb{R}$, $B \in \mathcal{U}_\Lambda$ and $L_1, L_2 \in \mathbb{R}_0^+$ with $\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$,

$$\begin{aligned}\frac{\left\| \left[\tau_{s-t}^{(L_1)}\left(B_s\right), \tilde{B}_t \right] \right\|_{\mathcal{U}}}{2\left\|B\right\|_{\mathcal{U}}} &\leq \mathbf{D}^{-1}\left(\mathrm{e}^{2\mathbf{D}|s-t|\|\Psi\|_{\mathcal{W}}}-1\right) \\ &\quad\times \sum_{\substack{\mathcal{Z}\subseteq\Lambda_{L_2}, \\ \mathcal{Z}\cap(\Lambda_{L_2}\backslash\Lambda_{L_1})\neq\emptyset,\ \mathcal{Z}\cap\Lambda=\emptyset}} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \sum_{z\in\partial_\Psi\mathcal{Z}}\sum_{x\in\Lambda}\mathbf{F}\left(|x-z|\right) \\ &\quad+ \sum_{\substack{\mathcal{Z}\subseteq\Lambda_{L_2}, \\ \mathcal{Z}\cap(\Lambda_{L_2}\backslash\Lambda_{L_1})\neq\emptyset,\ \mathcal{Z}\cap\Lambda\neq\emptyset}} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}}\ . \end{aligned} \tag{27}$$

Direct estimates using (7) and (9) show that

$$\begin{aligned} &\sum_{\mathcal{Z}\subseteq\Lambda_{L_2},\ \mathcal{Z}\cap(\Lambda_{L_2}\backslash\Lambda_{L_1})\neq\emptyset} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \sum_{z\in\partial_\Psi\mathcal{Z}}\sum_{x\in\Lambda}\mathbf{F}\left(|x-z|\right) \\ \leq\ & \sum_{y\in\Lambda_{L_2}\backslash\Lambda_{L_1}} \sum_{\mathcal{Z}\subseteq\Lambda_{L_2},\ \mathcal{Z}\supset\{y\}} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \sum_{z\in\mathcal{Z}}\sum_{x\in\Lambda}\mathbf{F}\left(|x-z|\right) \\ \leq\ & \sum_{y\in\Lambda_{L_2}\backslash\Lambda_{L_1}} \sum_{z\in\Lambda_{L_2}} \sum_{\mathcal{Z}\subseteq\Lambda_{L_2},\ \mathcal{Z}\supset\{y,z\}} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \sum_{x\in\Lambda}\mathbf{F}\left(|x-z|\right) \\ \leq\ & \|\Psi\|_{\mathcal{W}} \sum_{y\in\Lambda_{L_2}\backslash\Lambda_{L_1}} \sum_{x\in\Lambda}\sum_{z\in\Lambda_{L_2}} \mathbf{F}\left(|y-z|\right)\mathbf{F}\left(|x-z|\right) \\ \leq\ & \mathbf{D}\|\Psi\|_{\mathcal{W}} \sum_{y\in\Lambda_{L_2}\backslash\Lambda_{L_1}} \sum_{x\in\Lambda}\mathbf{F}\left(|x-y|\right)\ , \end{aligned} \tag{28}$$

while, by using (9) only,

$$\begin{aligned} &\sum_{\mathcal{Z}\subseteq\Lambda_{L_2},\ \mathcal{Z}\cap(\Lambda_{L_2}\backslash\Lambda_{L_1})\neq\emptyset,\ \mathcal{Z}\cap\Lambda\neq\emptyset} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \\ \leq\ & \sum_{y\in\Lambda_{L_2}\backslash\Lambda_{L_1}} \sum_{x\in\Lambda} \sum_{\mathcal{Z}\subseteq\Lambda_{L_2},\ \mathcal{Z}\supset\{x,y\}} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \\ \leq\ & \|\Psi\|_{\mathcal{W}} \sum_{y\in\Lambda_{L_2}\backslash\Lambda_{L_1}} \sum_{x\in\Lambda}\mathbf{F}\left(|x-y|\right)\ . \end{aligned} \tag{29}$$

The lemma is then a direct consequence of (26)–(27) combined with the upper bounds (28)–(29). ■

The infinite–volume dynamics is obtained from Lemma 3.2 and the completeness of $\mathcal{U}$. Indeed, from the above lemma, for all $t \in \mathbb{R}$, $\tau_t^{(L)}$ converges strongly on $\mathcal{U}_0$ to τ_t, as $L \to \infty$. By density of $\mathcal{U}_0$ in the Banach space $\mathcal{U}$ and the fact that $\tau_t^{(L)}$ are isometries for all $L \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$, the limit τ_t, $t \in \mathbb{R}$, uniquely defines a $*$–automorphism, also denoted by τ_t, of the C^*–algebra $\mathcal{U}$. $\{\tau_t\}_{t\in\mathbb{R}}$ is clearly a group of $*$–automorphisms on $\mathcal{U}$. Again by the above lemma, for any element B in the dense subset $\mathcal{U}_0 \subset \mathcal{U}$, the convergence of $\tau_t^{(L)}(B)$, as $L \to \infty$, is uniform for t on compacta and $\{\tau_t\}_{t\in\mathbb{R}}$ thus defines a C_0–group on $\mathcal{U}$, that is, a strongly continuous group on $\mathcal{U}$.

We need in the sequel an explicit characterization of the infinitesimal generator of this C_0–group. Since the generator equals (13) at finite–volume, one expects that the infinitesimal generator equals on $\mathcal{U}_0$ the linear map δ from $\mathcal{U}_0$ to $\mathcal{U}$ defined by

$$\delta(B) \doteq i \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})} [\Psi_\Lambda, B] + i \sum_{x\in\mathfrak{L}} \left[\mathbf{V}_{\{x\}}, B\right] \ , \qquad B \in \mathcal{U}_0 \ , \tag{30}$$

for any $\Psi \in \mathcal{W}$ and potential $\mathbf{V}$. Indeed, for any $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ and local element $B \in \mathcal{U}_\Lambda$,

$$\begin{aligned} &\sum_{\mathcal{Z}\in\mathcal{P}_f(\mathfrak{L})} \|[\Psi_\mathcal{Z}, B]\|_\mathcal{U} + \sum_{x\in\mathfrak{L}} \left\| \left[\mathbf{V}_{\{x\}}, B\right] \right\|_\mathcal{U} \qquad (31)\\ \leq \ & 2\|B\|_\mathcal{U} \left(|\Lambda| \, \mathbf{F}(0) \, \|\Psi\|_\mathcal{W} + \sum_{x\in\Lambda} \left\|\mathbf{V}_{\{x\}}\right\|_\mathcal{U} \right) \end{aligned}$$

and the series (30) is absolutely convergent for all $B \in \mathcal{U}_0$. Moreover, by (13), we obviously have

$$\delta(B) = \lim_{L\to\infty} \delta^{(L)}(B) \ , \qquad B \in \mathcal{U}_0 \ . \tag{32}$$

To prove that the closure of the linear map $\delta : \mathcal{U}_0 \to \mathcal{U}$ is the generator of the C_0–group $\{\tau_t\}_{t\in\mathbb{R}}$ of $*$–automorphisms we use the second Trotter–Kato approximation theorem [EN, Chap. III, Sect. 4.9].

To this end, we first show that the (generally unbounded) operator δ on $\mathcal{U}$ with dense domain $\mathrm{Dom}(\delta) = \mathcal{U}_0$ is closable. Observe that both $\pm\delta$ are symmetric derivations and δ is thus conservative [BR, Definition 3.1.13.], by structure of the set $\mathcal{U}_0$ of local elements:

Lemma 3.3 (Conservative infinite–volume derivation)
Let $\Psi \in \mathcal{W}$ and $\mathbf{V}$ be any potential. Then, the derivation δ defined on $\mathcal{U}_0$ by (30) is a conservative symmetric derivation.

Proof: Let $B \in \mathcal{U}_0$ satisfying $B \geq 0$. By definition of $\mathcal{U}_0$, $B \in \mathcal{U}_\Lambda$ for some $\Lambda \in \mathcal{P}_f(\mathfrak{L})$. Since $\mathcal{U}_\Lambda$ is a unital C^*–algebra, there is $B^{1/2} \in \mathcal{U}_\Lambda \subset \mathcal{U}_0$ such that $B^{1/2} \geq 0$ and $(B^{1/2})^2 = B$. Therefore, the lemma follows from [BR, Proposition 3.2.22]. ∎

It follows that the symmetric derivation δ is (norm–) closable:

Lemma 3.4 (Closure of the infinite–volume derivation)
Let $\Psi \in \mathcal{W}$ and $\mathbf{V}$ be any potential. Then, the derivations $\pm\delta$ defined on $\mathcal{U}_0$ by (30) are closable and their closures, again denoted for simplicity by $\pm\delta$, are conservative.

Proof: $\pm\delta$ are densely defined dissipative operators on the Banach space $\mathcal{U}$. Therefore, the lemma is an obvious application of [BR, Proposition 3.1.15.]. ∎

In order to apply the second Trotter–Kato approximation theorem [EN, Chap. III, Sect. 4.9], we also prove that the range $\mathrm{Ran}\{(x\mathbf{1}_\mathcal{U}\mp\delta)\}$ of the closed operators $x\mathbf{1}_\mathcal{U} \mp \delta$ are dense in the Banach space $\mathcal{U}$ for $x > 0$. This is done in the following lemma:

Lemma 3.5 (Range of the infinite–volume derivation)
Let $\Psi \in \mathcal{W}$ and $\mathbf{V}$ be any potential. Then, for any $x \in \mathbb{R}^+$,

$$\mathcal{U}_0 \subseteq \mathrm{Ran}\{(x\mathbf{1}_\mathcal{U} \mp \delta)\} \subseteq \mathcal{U}$$

with $\mathbf{1}_\mathcal{U}$ being the identity on $\mathcal{U}$. In particular, $\mathrm{Ran}\{(x\mathbf{1}_\mathcal{U} \mp \delta)\}$ *is dense in $\mathcal{U}$.*

Proof: We only give the proof for the range of the operator $x\mathbf{1}_\mathcal{U}-\delta$, since the other case uses similar arguments.

Note that $\|\tau_t^{(L)}\|_{\mathcal{B}(\mathcal{U})} = 1$ for any $L \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$. Here, $\mathcal{B}(\mathcal{U})$ is the Banach space of bounded linear operators acting on $\mathcal{U}$. Thus, for any $L \in \mathbb{R}_0^+$, $x \in \mathbb{R}^+$, and $B \in \mathcal{U}$, the improper Riemann integral

$$\int_0^\infty \mathrm{e}^{-xs}\tau_s^{(L)}(B)\,\mathrm{d}s \doteq \lim_{t\to\infty}\int_0^t \mathrm{e}^{-xs}\tau_s^{(L)}(B)\,\mathrm{d}s$$

exists. By [EN, Chap. II, Sect. 1.10], it follows that, for any $L \in \mathbb{R}_0^+$ and $x \in \mathbb{R}^+$, the resolvent $(x\mathbf{1}_\mathcal{U} - \delta^{(L)})^{-1}$ of the generator $\delta^{(L)}$ of the group $\{\tau_t^{(L)}\}_{t\in\mathbb{R}}$ also exists and satisfies

$$(x\mathbf{1}_\mathcal{U}-\delta^{(L)})^{-1}(B) = \int_0^\infty \mathrm{e}^{-xs}\tau_s^{(L)}(B)\,\mathrm{d}s \tag{33}$$

for all $B \in \mathcal{U}$. Now, take $B \in \mathcal{U}_0$, $x \in \mathbb{R}^+$, and consider the element

$$B_L \doteq (x\mathbf{1}_{\mathcal{U}} - \delta^{(L)})^{-1}(B) \in \mathcal{U} \tag{34}$$

for some sufficiently large parameter $L \in \mathbb{R}_0^+$ such that $B \in \mathcal{U}_{\Lambda_L}$. Note that $\tau_s^{(L)}(\mathcal{U}_{\Lambda_L}) \subset \mathcal{U}_{\Lambda_L}$ and $B_L \in \mathcal{U}_{\Lambda_L} \subset \mathcal{U}_0$ because of (33). Then, we observe that

$$(x\mathbf{1}_{\mathcal{U}} - \delta)(B_L) = B + (\delta^{(L)} - \delta)(B_L) ,$$

where we recall that $L \in \mathbb{R}_0^+$, $x \in \mathbb{R}^+$, and $B \in \mathcal{U}_0$. Now, by the Lumer–Phillips theorem [BR, Theorem 3.1.16] (see also its proof), if there is $x \in \mathbb{R}^+$ such that

$$\lim_{L \to \infty} \left\| (\delta - \delta^{(L)})(B_L) \right\|_{\mathcal{U}} = 0 \tag{35}$$

for all $B \in \mathcal{U}_0$ then we obtain the assertion. Indeed, by using Lemma 3.2 together with $\|\tau_t^{(L)}\|_{\mathcal{B}(\mathcal{U})} = 1$ and (33), one verifies that $\{B_L\}_{L \in \mathbb{R}_0^+}$ is a Cauchy net, thus a convergent one in $\mathcal{U}$, while $x\mathbf{1}_{\mathcal{U}} - \delta$ is a closed operator, by Lemma 3.4.

To prove (35) we use Lieb–Robinson bounds (Theorem 3.1) as follows: Since $B_L \in \mathcal{U}_{\Lambda_L}$ for sufficiently large $L \in \mathbb{R}_0^+$, we can combine (13) and (30) with (33)–(34) to compute that

$$(\delta - \delta^{(L)})(B_L) = i \sum_{\mathcal{Z} \in \mathcal{P}_f(\mathfrak{L}),\ \mathcal{Z} \cap \Lambda_L^c \neq \emptyset} \int_0^\infty \mathrm{e}^{-xs} \left[\Psi_{\mathcal{Z}}, \tau_s^{(L)}(B) \right] \mathrm{d}s \tag{36}$$

for any $x \in \mathbb{R}^+$, sufficiently large $L \in \mathbb{R}_0^+$, and $B \in \mathcal{U}_0$. Here, $\Lambda_L^c \doteq \mathfrak{L} \backslash \Lambda_L$. It suffices to consider the case $B \neq 0$. Using now Theorem 3.1, similar to (27), one gets that, for all $s \in \mathbb{R}^+$ and any sufficiently large $L \in \mathbb{R}_0^+$ such that $B \in \mathcal{U}_\Lambda \subset \mathcal{U}_{\Lambda_L}$ with $\Lambda \in \mathcal{P}_f(\mathfrak{L})$,

$$\begin{aligned} &\sum_{\mathcal{Z} \in \mathcal{P}_f(\mathfrak{L}),\ \mathcal{Z} \cap \Lambda_L^c \neq \emptyset} \frac{\left\| \left[\Psi_{\mathcal{Z}}, \tau_s^{(L)}(B) \right] \right\|_{\mathcal{U}}}{2 \|B\|_{\mathcal{U}}} \\ \leq\ & \mathbf{D}^{-1} \left(\mathrm{e}^{2\mathbf{D}|s| \|\Psi\|_{\mathcal{W}}} - 1 \right) \sum_{\mathcal{Z} \in \mathcal{P}_f(\mathfrak{L}),\ \mathcal{Z} \cap \Lambda_L^c \neq \emptyset,\ \mathcal{Z} \cap \Lambda = \emptyset} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \sum_{x \in \partial_\Psi \mathcal{Z}} \sum_{y \in \Lambda} \mathbf{F}(|x - y|) \\ & + \sum_{\mathcal{Z} \in \mathcal{P}_f(\mathfrak{L}),\ \mathcal{Z} \cap \Lambda_L^c \neq \emptyset,\ \mathcal{Z} \cap \Lambda \neq \emptyset} \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} . \end{aligned} \tag{37}$$

Similar to Inequalities (28)–(29), we thus infer from (7) and (9) that

$$\sum_{\mathcal{Z}\in\mathcal{P}_f(\mathfrak{L}),\ \mathcal{Z}\cap\Lambda_L^c\neq\emptyset} \frac{\left\|\left[\Psi_{\mathcal{Z}},\tau_s^{(L)}(B)\right]\right\|_{\mathcal{U}}}{2\|B\|_{\mathcal{U}}} \leq \|\Psi\|_{\mathcal{W}}\,\mathrm{e}^{2\mathbf{D}|s|\|\Psi\|_{\mathcal{W}}}\sum_{y\in\Lambda_L^c}\sum_{x\in\Lambda}\mathbf{F}(|x-y|)\ , \tag{38}$$

while

$$\lim_{L\to\infty}\sum_{y\in\Lambda_L^c}\sum_{x\in\Lambda}\mathbf{F}(|x-y|)=0\ , \tag{39}$$

because of (6). Therefore, by (36)–(39), we deduce (35) for all $x > 2\mathbf{D}\|\Psi\|_{\mathcal{W}}$ and $B\in\mathcal{U}_0$. ∎

We now apply the second Trotter–Kato approximation theorem [EN, Chap. III, Sect. 4.9] to deduce that δ is the generator of the group $\{\tau_t\}_{t\in\mathbb{R}}$ of $*$–automorphisms and resume all the main results, so far, in the following theorem:

Theorem 3.6 (Infinite–volume dynamics and its generator)
Let $\Psi\in\mathcal{W}$, $\mathbf{V}$ be any potential, and $\mathbf{D}\in\mathbb{R}^+$ be defined by (7).
(i) *Infinite–volume dynamics. The continuous groups $\{\tau_t^{(L)}\}_{t\in\mathbb{R}}$, $L\in\mathbb{R}_0^+$, defined by (12) converge strongly to a C_0–group $\{\tau_t\}_{t\in\mathbb{R}}$ of $*$–automorphisms with generator δ.*
(ii) *Infinitesimal generator. δ is a conservative closed symmetric derivation which is equal on its core $\mathcal{U}_0$ to*

$$\delta(B)=i\sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})}[\Psi_\Lambda,B]+i\sum_{x\in\mathfrak{L}}\left[\mathbf{V}_{\{x\}},B\right]\ ,\qquad B\in\mathcal{U}_0\ .$$

(iii) *Rate of convergence. For any $\Lambda\in\mathcal{P}_f(\mathfrak{L})$, $B\in\mathcal{U}_\Lambda$ and $L\in\mathbb{R}_0^+$ such that $\Lambda\subset\Lambda_L$,*

$$\left\|\tau_t(B)-\tau_t^{(L)}(B)\right\|_{\mathcal{U}}\leq 2\|B\|_{\mathcal{U}}\|\Psi\|_{\mathcal{W}}|t|\,\mathrm{e}^{4\mathbf{D}|t|\|\Psi\|_{\mathcal{W}}}\sum_{y\in\mathfrak{L}\backslash\Lambda_L}\sum_{x\in\Lambda}\mathbf{F}(|x-y|)\ .$$

(iv) *Lieb–Robinson bounds. For any $t\in\mathbb{R}$ and $B_1\in\mathcal{U}^+\cap\mathcal{U}_{\Lambda^{(1)}}$, $B_2\in\mathcal{U}_{\Lambda^{(2)}}$ with disjoint sets $\Lambda^{(1)},\Lambda^{(2)}\in\mathcal{P}_f(\mathfrak{L})$,*

$$\|[\tau_t(B_1),B_2]\|_{\mathcal{U}}\leq 2\mathbf{D}^{-1}\|B_1\|_{\mathcal{U}}\|B_2\|_{\mathcal{U}}\left(\mathrm{e}^{2\mathbf{D}|t|\|\Psi\|_{\mathcal{W}}}-1\right)\times\sum_{x\in\partial_\Psi\Lambda^{(1)}}\sum_{y\in\Lambda^{(2)}}\mathbf{F}(|x-y|)\ .$$

Proof: By Lemma 3.4, the set $\mathcal{U}_0$ of local elements is a core of the dissipative derivation δ and one obtains (ii), see (30). Moreover, $\delta^{(L)}(B) \rightarrow \delta(B)$ for all $B \in \mathcal{U}_0$, see (32). Recall that $\delta^{(L)}$ is the generator of the group $\{\tau_t^{(L)}\}_{t\in\mathbb{R}}$ for any $L \in \mathbb{R}_0^+$. Therefore, since one also has Lemma 3.5, (i) is a direct consequence of [EN, Chap. III, Sect. 4.9]. The third statement (iii) thus follows from Lemma 3.2. (iv) is an obvious consequence of Theorem 3.1 and the first assertion (i). ■

3.2 Lieb–Robinson Bounds for Multi–Commutators

Recall that multi–commutators are defined by induction as follows:

$$[B_1, B_0]^{(2)} \doteq [B_1, B_0] \doteq B_1 B_0 - B_0 B_1 \ , \qquad B_0, B_1 \in \mathcal{U} \ , \tag{40}$$

and, for all integers $k \geq 2$,

$$[B_k, B_{k-1}, \ldots, B_0]^{(k+1)} \doteq [B_k, [B_{k-1}, \ldots, B_0]^{(k)}] \ , \quad B_0, \ldots, B_k \in \mathcal{U} \ . \tag{41}$$

The aim of this subsection is to extend Theorem 3.6 (iv) to multi–commutators. The arguments we use below to prove Lieb–Robinson bounds for multi–commutators are not a generalization of the proof of Theorem 3.1 or Theorem 3.6 (iv). Instead, we use a pivotal lemma deduced from Theorem 3.6 (iii), which in turn results from finite–volume Lieb–Robinson bounds of Theorem 3.1. This lemma expresses the C_0–group $\{\tau_t\}_{t\in\mathbb{R}}$ of Theorem 3.6 (i) as *telescoping* series.

To this end, it is convenient to introduce the family $\{\chi_x\}_{x\in\mathfrak{L}}$ of $*$–automorphisms of $\mathcal{U}$, which implements the action of the group of lattice translations on the CAR C^*–algebra $\mathcal{U}$. This family is uniquely defined by the conditions

$$\chi_x(a_y) = a_{y+x} \ , \qquad x, y \in \mathfrak{L} \ . \tag{42}$$

We also define, for any $n \in \mathbb{N}_0$, $x \in \mathfrak{L}$, $\Psi \in \mathcal{W}$ and potential $\mathbf{V}$, a *space translated* finite–volume dynamics which is the continuous group $\{\tau_t^{(n,x)}\}_{t\in\mathbb{R}}$ of $*$–automorphisms of $\mathcal{U}$ generated by the symmetric and bounded derivation

$$\delta^{(n,x)}(B) \doteq i \sum_{\Lambda \subseteq x+\Lambda_n} [\Psi_\Lambda, B] + i \sum_{y \in x+\Lambda_n} \left[\mathbf{V}_{\{y\}}, B\right] \ , \qquad B \in \mathcal{U} \ .$$

Note that the fermion system is generally *not* translation invariant and, in general,

$$\tau_t^{(n,x)} \circ \chi_x \neq \chi_x \circ \tau_t^{(n)} \ , \qquad x \in \mathfrak{L}, \ n \in \mathbb{N}_0 \ , \ t \in \mathbb{R} \ .$$

For $m \in \mathbb{N}_0$, $x \in \mathfrak{L}$, $B \in \mathcal{U}_{\Lambda_m}$ and $t \in \mathbb{R}$, we finally introduce the local elements

$$\mathfrak{B}_{B,t,x}(m) \equiv \mathfrak{B}_{B,t,x}^{(m)}(m) \doteq \tau_t^{(m,x)} \circ \chi_x(B) \in \mathcal{U}_{\Lambda_m+x} \tag{43}$$

and

$$\mathfrak{B}_{B,t,x}(n) \equiv \mathfrak{B}_{B,t,x}^{(m)}(n) \doteq (\tau_t^{(n,x)} - \tau_t^{(n-1,x)}) \circ \chi_x(B) \in \mathcal{U}_{\Lambda_n+x} , \qquad n \geq m+1 . \tag{44}$$

The family $\{\mathfrak{B}_{B,t,x}(n)\}_{n\geq m} \subset \mathcal{U}_0$ is used to define telescoping series:

Lemma 3.7 (Infinite–volume dynamics as telescoping series)
Let $\Psi \in \mathcal{W}$ and $\mathbf{V}$ be any potential. Then, for any $m \in \mathbb{N}_0$, $x \in \mathfrak{L}$, $B \in \mathcal{U}_{\Lambda_m}$ and $t \in \mathbb{R}$:

$$\sum_{n=m}^{\infty} \mathfrak{B}_{B,t,x}(n) = \tau_t \circ \chi_x(B) . \tag{45}$$

The above telescoping series is absolutely convergent in $\mathcal{U}$ with

$$\|\mathfrak{B}_{B,t,x}(n)\|_{\mathcal{U}} \leq 2\|B\|_{\mathcal{U}} \|\Psi\|_{\mathcal{W}} |t| \, \mathrm{e}^{4\mathbf{D}|t|\|\Psi\|_{\mathcal{W}}} \sum_{y\in\Lambda_n\backslash\Lambda_{n-1}} \sum_{z\in\Lambda_m} \mathbf{F}(|z-y|) \tag{46}$$

for any $n \geq m+1$, while $\|\mathfrak{B}_{B,t,x}(m)\|_{\mathcal{U}} = \|B\|_{\mathcal{U}}$.

Proof: Since, for any $N \in \mathbb{N}_0$ so that $N \geq m$,

$$\sum_{n=m}^{N} \mathfrak{B}_{B,t,x}(n) = \tau_t^{(N,x)} \circ \chi_x(B) , \tag{47}$$

it suffices to study the limit $N \to \infty$ of the group $\{\tau_t^{(N,x)}\}_{t\in\mathbb{R}}$ at any fixed $x \in \mathfrak{L}$. Similar to the proof of Theorem 3.6 (i), $\delta^{(N,x)}(B) \to \delta(B)$ for all $B \in \mathcal{U}_0$, as $N \to \infty$. By Lemma 3.5 and [EN, Chap. III, Sect. 4.9], the translated groups $\{\tau_t^{(N,x)}\}_{t\in\mathbb{R}}$, $N \in \mathbb{N}_0$, converge strongly to the C_0–group $\{\tau_t\}_{t\in\mathbb{R}}$ for any $x \in \mathfrak{L}$. In other words, we deduce Equation (45) from (47) in the limit $N \to \infty$. Moreover, one easily checks that Theorem 3.1 and thus Lemma 3.2 also hold for the (space translated) groups $\{\tau_t^{(n,x)}\}_{t\in\mathbb{R}}$, $n \in \mathbb{N}_0$, at any fixed $x \in \mathfrak{L}$. This yields Inequality (46) for $n > m$, while $\|\mathfrak{B}_{B,t,x}(m)\|_{\mathcal{U}} = \|B\|_{\mathcal{U}}$, because $\tau_t^{(m,x)}$ is a $*$–automorphism on $\mathcal{U}_{\Lambda_m}$. It follows that

$$\sum_{n=m+1}^{\infty} \|\mathfrak{B}_{B,t,x}(n)\|_{\mathcal{U}} \leq 2\|B\|_{\mathcal{U}} \|\Psi\|_{\mathcal{W}} |t| \, \mathrm{e}^{4\mathbf{D}|t|\|\Psi\|_{\mathcal{W}}} \sum_{z\in\Lambda_m} \sum_{n\in\mathbb{N}} \sum_{y\in\Lambda_n\backslash\Lambda_{n-1}} \mathbf{F}(|z-y|) .$$

Finally, by Assumption (6),

$$\sum_{z\in\Lambda_m}\sum_{n\in\mathbb{N}}\sum_{y\in\Lambda_n\backslash\Lambda_{n-1}}\mathbf{F}\left(|z-y|\right)\leq\sum_{z\in\Lambda_m}\sum_{y\in\mathfrak{L}}\mathbf{F}\left(|z-y|\right)=|\Lambda_m|\,\|\mathbf{F}\|_{1,\mathfrak{L}}<\infty\,.$$

■

To extend Lieb–Robinson bounds to multi–commutators we combine Lemma 3.7 with tree decompositions of sequences of clustering subsets of $\mathfrak{L}$ (cf. (56)): Let $\mathcal{T}_2$ be the set of all (non–oriented) trees with exactly two vertices. This set contains a unique tree $T=\{\{0,1\}\}$ which, in turn, contains the unique bond $\{0,1\}$, i.e., $\mathcal{T}_2\doteq\{\{\{0,1\}\}\}$. Then, for each integer $k\geq 2$, we recursively define a set $\mathcal{T}_{k+1}$ of trees with $k+1$ vertices by

$$\mathcal{T}_{k+1}\doteq\Big\{\{\{j,k\}\}\cup T\ :\ j=0,\ldots,k-1,\quad T\in\mathcal{T}_k\Big\}\,. \tag{48}$$

Therefore, for $k\in\mathbb{N}$ and any tree $T\in\mathcal{T}_{k+1}$, there is a map

$$\mathrm{P}_T:\{1,\ldots,k\}\rightarrow\{0,\ldots,k-1\} \tag{49}$$

such that $\mathrm{P}_T(j)<j$, $\mathrm{P}_T(1)=0$, and

$$T=\bigcup_{j=1}^{k}\{\{\mathrm{P}_T(j),j\}\}\,. \tag{50}$$

For any $k\in\mathbb{N}$, $T\in\mathcal{T}_{k+1}$, and every sequence $\{(n_j,x_j)\}_{j=0}^{k}$ in $\mathbb{N}_0\times\mathfrak{L}$ with length $k+1$, we define

$$\varkappa_T\left(\{(n_j,x_j)\}_{j=0}^{k}\right)\doteq\prod_{j=1}^{k}\mathbf{1}\left[(\Lambda_{n_j}+x_j)\cap(\Lambda_{n_{\mathrm{P}_T(j)}}+x_{\mathrm{P}_T(j)})\neq\emptyset\right]\in\{0,1\}\,, \tag{51}$$

while, for all $\ell\in\{1,\ldots,k\}$,

$$\mathcal{S}_{\ell,k}\doteq\{\pi\mid\pi:\{\ell,\ldots,k\}\rightarrow\{1,\ldots,k\}\ \text{such that}\ \pi\left(i\right)<\pi\left(j\right)\ \text{when}\ i<j\}\ . \tag{52}$$

Then, one gets the following bound on multi–commutators:

Theorem 3.8 (Lieb–Robinson bounds for multi–commutators – Part I)
Let $\Psi\in\mathcal{W}$ and $\mathbf{V}$ be any potential. Then, for any integer $k\in\mathbb{N}$, $\{m_j\}_{j=0}^{k}\subset\mathbb{N}_0$,

times $\{s_j\}_{j=1}^k \subset \mathbb{R}$*, lattice sites* $\{x_j\}_{j=0}^k \subset \mathfrak{L}$*, and local elements* $B_0 \in \mathcal{U}_0$*,* $\{B_j\}_{j=1}^k \subset \mathcal{U}_0 \cap \mathcal{U}^+$ *such that* $B_j \in \mathcal{U}_{\Lambda_{m_j}}$ *for* $j \in \{0, \ldots, k\}$*,*

$$\begin{aligned}
&\left\| \left[\tau_{s_k} \circ \chi_{x_k}(B_k), \ldots, \tau_{s_1} \circ \chi_{x_1}(B_1), \chi_{x_0}(B_0) \right]^{(k+1)} \right\|_{\mathcal{U}} \\
\leq \; & 2^k \prod_{j=0}^{k} \|B_j\|_{\mathcal{U}} \sum_{T \in \mathcal{T}_{k+1}} \left(\varkappa_T \left(\{(m_j, x_j)\}_{j=0}^{k} \right) + \Re_{T, \|\Psi\|_{\mathcal{W}}} \right)
\end{aligned}$$

with, for any $\alpha \in \mathbb{R}_0^+$*,*

$$\begin{aligned}
\Re_{T,\alpha} \doteq & \sum_{\ell=1}^{k} (2\alpha)^{k-\ell+1} \sum_{\pi \in \mathcal{S}_{\ell,k}} \left(\prod_{j \in \{\pi(\ell), \ldots, \pi(k)\}} |s_j| \, \mathrm{e}^{4\mathbf{D}\alpha |s_j|} \right) \\
& \sum_{n_{\pi(\ell)} = m_{\pi(\ell)}+1}^{\infty} \sum_{z_{\pi(\ell)} \in \Lambda_{m_{\pi(\ell)}}} \sum_{y_{\pi(\ell)} \in \Lambda_{n_{\pi(\ell)}} \backslash \Lambda_{n_{\pi(\ell)}-1}} \cdots \\
& \cdots \sum_{n_{\pi(k)} = m_{\pi(k)}+1}^{\infty} \sum_{z_{\pi(k)} \in \Lambda_{m_{\pi(k)}}} \sum_{y_{\pi(k)} \in \Lambda_{n_{\pi(k)}} \backslash \Lambda_{n_{\pi(k)}-1}} \\
& \varkappa_T \left(\{(n_j, x_j)\}_{j=0}^{k} \right) \prod_{j \in \{\pi(\ell), \ldots, \pi(k)\}} \mathbf{F}\left(|z_j - y_j| \right) .
\end{aligned} \tag{53}$$

In the right–hand side (r.h.s.) of (53), we set $n_j \doteq m_j$ *if*

$$j \in \{0, \ldots, k\} \backslash \{\pi(\ell), \ldots, \pi(k)\} .$$

The constant $\mathbf{D} \in \mathbb{R}^+$ *is defined by (7).*

Proof: Fix $k \in \mathbb{N}$, $\{m_j\}_{j=0}^k \subset \mathbb{N}_0$, $\{s_j\}_{j=1}^k \subset \mathbb{R}$, $\{x_j\}_{j=0}^k \subset \mathfrak{L}$ and elements $\{B_j\}_{j=0}^k \subset \mathcal{U}_0$ such that the conditions of the theorem are satisfied. From Lemma 3.7,

$$\begin{aligned}
& \left[\tau_{s_k} \circ \chi_{x_k}(B_k), \ldots, \tau_{s_1} \circ \chi_{x_1}(B_1), \chi_{x_0}(B_0) \right]^{(k+1)} \\
= \; & \sum_{n_1=m_1}^{\infty} \cdots \sum_{n_k=m_k}^{\infty} \left[\mathfrak{B}_{B_k,s_k,x_k}(n_k), \ldots, \mathfrak{B}_{B_1,s_1,x_1}(n_1), \chi_{x_0}(B_0) \right]^{(k+1)} .
\end{aligned} \tag{54}$$

Since $B_j \in \mathcal{U}_{\Lambda_{m_j}} \cap \mathcal{U}^+$ for $j \in \{1, \ldots, k\}$, we infer from (43)–(44) that

$$\begin{aligned} & \left[\mathfrak{B}_{B_k,s_k,x_k}(n_k), \ldots, \mathfrak{B}_{B_1,s_1,x_1}(n_1), \chi_{x_0}(B_0)\right]^{(k+1)} \\ = & \prod_{j=1}^{k} \mathbf{1}\left[\bigcup_{i=0}^{j-1} (\Lambda_{n_j} + x_j) \cap (\Lambda_{n_i} + x_i) \neq \emptyset\right] \\ & \left[\mathfrak{B}_{B_k,s_k,x_k}(n_k), \ldots, \mathfrak{B}_{B_1,s_1,x_1}(n_1), \chi_{x_0}(B_0)\right]^{(k+1)} \end{aligned} \quad (55)$$

for all integers $\{n_j\}_{j=0}^k \subset \mathbb{N}_0$ with $n_0 \doteq m_0$ and $n_j \geq m_j$ when $j \in \{1, \ldots, k\}$. The conditions inside characteristic functions in (55) refer to the fact that the sequence of sets $\{\Lambda_{n_j}\}_{j=0}^k$ has to be a cluster to have a non–zero multi–commutator. Note further that

$$\prod_{j=1}^{k} \mathbf{1}\left[\bigcup_{i=0}^{j-1} (\Lambda_{n_j} + x_j) \cap (\Lambda_{n_i} + x_i) \neq \emptyset\right] \leq \sum_{T \in \mathcal{T}_{k+1}} \varkappa_T\left(\{(n_j, x_j)\}_{j=0}^k\right) . \quad (56)$$

Using (54)–(56) one then shows that

$$\begin{aligned} & \left\|\left[\tau_{s_k} \circ \chi_{x_k}(B_k), \ldots, \tau_{s_1} \circ \chi_{x_1}(B_1), \chi_{x_0}(B_0)\right]^{(k+1)}\right\|_{\mathcal{U}} \\ \leq \; & 2^k \|B_0\|_{\mathcal{U}} \sum_{T \in \mathcal{T}_{k+1}} \sum_{n_1=m_1}^{\infty} \cdots \sum_{n_k=m_k}^{\infty} \varkappa_T\left(\{(n_j, x_j)\}_{j=0}^k\right) \\ & \times \prod_{j=1}^{k} \left\|\mathfrak{B}_{B_j,s_j,x_j}(n_j)\right\|_{\mathcal{U}} . \end{aligned} \quad (57)$$

This inequality combined with (46) yields the assertion. ■

The above theorem extends Lieb–Robinson bounds to multi–commutators. Indeed, if $\mathbf{F}(r)$ decays fast enough as $r \to \infty$, then Theorem 3.8 and Lebesgue's dominated convergence theorem imply that, for any $j \in \{0, \ldots, k\}$,

$$\lim_{|x_j| \to \infty} \left\|\left[\tau_{s_k} \circ \chi_{x_k}(B_k), \ldots, \tau_{s_1} \circ \chi_{x_1}(B_1), \chi_{x_0}(B_0)\right]^{(k+1)}\right\|_{\mathcal{U}} = 0 . \quad (58)$$

The rate of convergence if this multi–commutator towards zero is, however, a priori unclear. Hence, to obtain bounds on the space decay of the above multi–commutator, more in the spirit of the original Lieb–Robinson bounds for commutators, we consider two situations w.r.t. the behavior of the function $\mathbf{F} : \mathbb{R}_0^+ \to \mathbb{R}^+$ at large arguments:

- *Polynomial decay.* There is a constant $\varsigma \in \mathbb{R}^+$ and, for all $m \in \mathbb{N}_0$, an absolutely summable sequence $\{\mathbf{u}_{n,m}\}_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$ such that, for all $n \in \mathbb{N}$ with $n > m$,

$$|\Lambda_n \backslash \Lambda_{n-1}| \sum_{z\in\Lambda_m} \max_{y\in\Lambda_n\backslash\Lambda_{n-1}} \mathbf{F}\left(|z-y|\right) \leq \frac{\mathbf{u}_{n,m}}{(1+n)^{\varsigma}} . \tag{59}$$

- *Exponential decay.* There is $\varsigma \in \mathbb{R}^+$ and, for $m \in \mathbb{N}_0$, a constant $\mathbf{C}_m \in \mathbb{R}^+$ such that, for all $n \in \mathbb{N}$ with $n > m$,

$$|\Lambda_n \backslash \Lambda_{n-1}| \sum_{z\in\Lambda_m} \max_{y\in\Lambda_n\backslash\Lambda_{n-1}} \mathbf{F}\left(|z-y|\right) \leq \mathbf{C}_m \mathrm{e}^{-2\varsigma n} . \tag{60}$$

For sufficiently large $\epsilon \in \mathbb{R}^+$, the function (8) clearly satisfies Condition (59), while (59)–(60) hold for the choice

$$\mathbf{F}\left(r\right) = \mathrm{e}^{-2\varsigma r}(1+r)^{-(d+\epsilon)} , \qquad r \in \mathbb{R}_0^+ , \tag{61}$$

with arbitrary $\varsigma, \epsilon \in \mathbb{R}^+$. Under one of these both very general assumptions, one can put the upper bound of Theorem 3.8 in a much more convenient form. In fact, one obtains an estimate on the norm of the multi–commutator (58) as a function of the distances between the points $\{x_0, \ldots, x_k\}$, like in the usual Lieb–Robinson bounds (i.e., the special case $k = 2$). To formulate such bounds, we need some preliminary definitions related to properties of trees.

For any $k \in \mathbb{N}$ and $T \in \mathcal{T}_{k+1}$, we define the sequence $\mathfrak{d}_T \equiv \{\mathfrak{d}_T(j)\}_{j=0}^k$ in $\{1, \ldots, k\}$ by

$$\mathfrak{d}_T(j) \doteq |\{b \in T \ : \ j \in b\}| , \qquad j \in \{0, \ldots, k\} ,$$

i.e., $\mathfrak{d}_T(j)$ is the *degree* of the j–th vertex of the tree T. For $k \in \mathbb{N}$ and $T \in \mathcal{T}_{k+1}$, observe that

$$\mathfrak{d}_T(0) + \cdots + \mathfrak{d}_T(k) = 2k . \tag{62}$$

We also introduce the following notation:

$$\mathfrak{d}_T! \doteq \mathfrak{d}_T(0)! \cdots \mathfrak{d}_T(k)!$$

for any tree $T \in \mathcal{T}_{k+1}$, $k \in \mathbb{N}$. The degree of any vertex of a tree is at least 1, by connectedness of such a graph, and (62) yields

$$\mathfrak{d}_T! \leq k! , \qquad k \in \mathbb{N} , \ T \in \mathcal{T}_{k+1} . \tag{63}$$

For any $k \in \mathbb{N}$, $T \in \mathcal{T}_{k+1}$, and any sequence $f : \mathbb{N}_0 \to \mathbb{R}^+$, note that

$$\prod_{j=0}^{k} \{f(j)\}^{\mathfrak{d}_T(j)} = \prod_{j=1}^{k} f(j) f(\mathrm{P}_T(j)) \ . \tag{64}$$

This property is elementary but pivotal to estimate the remainder $\mathfrak{R}_{T,\alpha}$, defined by (53), of Theorem 3.8.

Theorem 3.9 (Lieb–Robinson bounds for multi–commutators – Part II)
Let $\alpha \in \mathbb{R}_0^+$, $k \in \mathbb{N}$, $\{m_j\}_{j=0}^k \subset \mathbb{N}_0$, $\{s_j\}_{j=1}^k \subset \mathbb{R}$, $\{x_j\}_{j=0}^k \subset \mathfrak{L}$, and $T \in \mathcal{T}_{k+1}$.
(i) *Polynomial decay: Assume (59). Then,*

$$\begin{aligned}\mathfrak{R}_{T,\alpha} \leq \ & d^{\frac{\varsigma k}{2}} \sum_{\ell=1}^{k} (2\alpha)^{k-\ell+1} \sum_{\pi \in \mathcal{S}_{\ell,k}} \left(\prod_{j \in \{\pi(\ell),\ldots,\pi(k)\}} \left\| \mathbf{u}_{\cdot,m_j} \right\|_{\ell^1(\mathbb{N})} |s_j| \, \mathrm{e}^{4\mathbf{D}|s_j|\alpha} \right) \\ & \left(\prod_{j \in \{0,\ldots,k\} \backslash \{\pi(\ell),\ldots,\pi(k)\}} (1+m_j)^{\varsigma} \right) \prod_{\{j,l\} \in T} \frac{1}{(1+|x_j - x_l|)^{\varsigma (\max\{\mathfrak{d}_T(j),\mathfrak{d}_T(l)\})^{-1}}} \ .\end{aligned}$$

(ii) *Exponential decay: Assume (60). Then,*

$$\begin{aligned}\mathfrak{R}_{T,\alpha} \leq \ & \sum_{\ell=1}^{k} \left(\frac{2\alpha}{\mathrm{e}^{\varsigma} - 1} \right)^{k-\ell+1} \sum_{\pi \in \mathcal{S}_{\ell,k}} \left(\prod_{j \in \{\pi(\ell),\ldots,\pi(k)\}} \mathbf{C}_{m_j} |s_j| \, \mathrm{e}^{4\mathbf{D}|s_j|\alpha - \varsigma m_j} \right) \\ & \left(\prod_{j \in \{0,\ldots,k\} \backslash \{\pi(\ell),\ldots,\pi(k)\}} \mathrm{e}^{\varsigma m_j} \right) \prod_{\{j,l\} \in T} \exp\left(- \frac{\varsigma |x_j - x_l|}{\sqrt{d} \max\{\mathfrak{d}_T(j), \mathfrak{d}_T(l)\}} \right) \ .\end{aligned}$$

Proof: (i) Fix all parameters of the theorem. We infer from (53) and (59) that

$$\begin{aligned}\mathfrak{R}_{T,\alpha} \leq \ & \sum_{\ell=1}^{k} (2\alpha)^{k-\ell+1} \sum_{\pi \in \mathcal{S}_{\ell,k}} \left(\prod_{j \in \{\pi(\ell),\ldots,\pi(k)\}} |s_j| \, \mathrm{e}^{4\mathbf{D}|s_j|\alpha} \right) \sum_{n_{\pi(\ell)}=m_{\pi(\ell)}+1}^{\infty} \\ & \cdots \sum_{n_{\pi(k)}=m_{\pi(k)}+1}^{\infty} \varkappa_T \left(\{(n_j, x_j)\}_{j=0}^{k} \right) \prod_{j \in \{\pi(\ell),\ldots,\pi(k)\}} \frac{\mathbf{u}_{n_j,m_j}}{(1+n_j)^{\varsigma}} \ .\end{aligned}$$

Recall that $n_j \doteq m_j$ when $j \in \{0,\ldots,k\} \setminus \{\pi(\ell),\ldots,\pi(k)\}$. By Hölder's inequality, it follows that

$$\begin{aligned}\mathfrak{R}_{T,\alpha} \leq & \sum_{\ell=1}^{k} (2\alpha)^{k-\ell+1} \sum_{\pi \in \mathcal{S}_{\ell,k}} \left(\prod_{j\in\{\pi(\ell),\ldots,\pi(k)\}} \left\| \mathbf{u}_{\cdot,m_j} \right\|_{\ell^1(\mathbb{N})} |s_j| \, \mathrm{e}^{4\mathbf{D}|s_j|\alpha} \right) \\ & \times \max_{n_{\pi(\ell)},\ldots,n_{\pi(k)}\in\mathbb{N}} \left\{ \varkappa_T \left(\{(n_j,x_j)\}_{j=0}^{k} \right) \prod_{j\in\{\pi(\ell),\ldots,\pi(k)\}} \frac{1}{(1+n_j)^{\varsigma}} \right\} . \end{aligned} \tag{65}$$

Therefore, it suffices to bound the above maximum in an appropriate way. Using (64), note that

$$\begin{aligned}\prod_{j=0}^{k} \frac{1}{(1+n_j)^{\varsigma}} &= \prod_{j=0}^{k} \left(\frac{1}{(1+n_j)^{\frac{\varsigma}{\mathfrak{d}_T(j)}}} \right)^{\mathfrak{d}_T(j)} \\ &= \prod_{j=1}^{k} \frac{1}{(1+n_j)^{\frac{\varsigma}{\mathfrak{d}_T(j)}} \left(1+n_{\mathrm{P}_T(j)}\right)^{\frac{\varsigma}{\mathfrak{d}_T(\mathrm{P}_T(j))}}} \\ &\leq \prod_{j=1}^{k} \frac{1}{\left(1+n_j+n_{\mathrm{P}_T(j)}\right)^{\frac{\varsigma}{\mathfrak{m}_T(j)}}} , \end{aligned} \tag{66}$$

where, for $k \in \mathbb{N}$, any tree $T \in \mathcal{T}_{k+1}$, and $j \in \{1,\ldots,k\}$,

$$\mathfrak{m}_T(j) \doteq \max\{\mathfrak{d}_T(j), \mathfrak{d}_T(\mathrm{P}_T(j))\} .$$

Meanwhile, the condition

$$(\Lambda_{n_j} + x_j) \cap (\Lambda_{n_{\mathrm{P}_T(j)}} + x_{\mathrm{P}_T(j)}) \neq \emptyset$$

implies

$$\sqrt{d}(n_j + n_{\mathrm{P}_T(j)}) \geq |x_j - x_{\mathrm{P}_T(j)}| . \tag{67}$$

Therefore, we infer from (66)–(67) that

$$\begin{aligned} & \max_{n_{\pi(\ell)},\ldots,n_{\pi(k)}\in\mathbb{N}} \left\{ \varkappa_T \left(\{(n_j,x_j)\}_{j=0}^{k} \right) \prod_{j\in\{\pi(\ell),\ldots,\pi(k)\}} \frac{1}{(1+n_j)^{\varsigma}} \right\} \\ \leq & \left(\prod_{j\in\{0,\ldots,k\}\setminus\{\pi(\ell),\ldots,\pi(k)\}} (1+n_j)^{\varsigma} \right) \prod_{j=1}^{k} \frac{d^{\frac{\varsigma}{2}}}{(1+|x_j - x_{\mathrm{P}_T(j)}|)^{\frac{\varsigma}{\mathfrak{m}_T(j)}}} . \end{aligned}$$

Combined with (65), this last inequality yields Assertion (i).

(ii) The second assertion is proven exactly in the same way. We omit the details. ■

We defined in [BPH1, Section 4] the concept of *tree–decay bounds* for pairs (ρ, τ), where $\rho \in \mathcal{U}^*$ and $\tau \equiv \{\tau_t\}_{t\in\mathbb{R}}$ are respectively any state and any one–parameter group of $*$–automorphisms on the C^*–algebra $\mathcal{U}$. They are a useful tool to control multi–commutators of products of annihilation and creation operators. Such bounds are related to cluster or graph expansions in statistical physics. For more details see the preliminary discussions of [BPH1, Section 4]. As a straightforward corollary of Theorems 3.8–3.9 we give below an extension of the tree–decay bounds [BPH1, Section 4] to the case of interacting fermions on lattices:

Corollary 3.10 (Tree–decay bounds)
Let $\Psi \in \mathcal{W}$, $\mathbf{V}$ be any potential, $k \in \mathbb{N}$, $m_0 \in \mathbb{N}_0$, $t \in \mathbb{R}_0^+$, $\{s_j\}_{j=1}^k \subset [-t,t]$, $B_0 \subset \mathcal{U}_{\Lambda_{m_0}}$, and $\{x_j\}_{j=0}^k, \{z_j\}_{j=1}^k \subset \mathfrak{L}$ such that $|z_j| = 1$ for $j \in \{1,\ldots,k\}$.
(i) *Polynomial decay: Assume (59) for $m = 1$. Then,*

$$\begin{aligned}&\left\| \left[\tau_{s_k}(a^*_{x_k}a_{x_k+z_k}),\ldots,\tau_{s_1}(a^*_{x_1}a_{x_1+z_1}),\chi_{x_0}(B_0)\right]^{(k+1)}\right\|_{\mathcal{U}}\\ \leq\ & \|B_0\|_{\mathcal{U}}(1+m_0)^{\varsigma}\mathbf{K}_0^k\sum_{T\in\mathcal{T}_{k+1}}\prod_{\{j,l\}\in T}\frac{1}{(1+|x_j-x_l|)^{\varsigma(\max\{\mathfrak{d}_T(j),\mathfrak{d}_T(l)\})^{-1}}}\end{aligned}$$

with

$$\mathbf{K}_0 \doteq 2d^{\frac{\varsigma}{2}}\left(2^{\varsigma}+2\|\mathbf{u}_{\cdot,1}\|_{\ell^1(\mathbb{N})}\|\Psi\|_{\mathcal{W}}|t|\,\mathrm{e}^{4\mathbf{D}|t|\|\Psi\|_{\mathcal{W}}}\right).$$

(ii) *Exponential decay: Assume (60) for $m = 1$. Then,*

$$\begin{aligned}&\left\| \left[\tau_{s_k}(a^*_{x_k}a_{x_k+z_k}),\ldots,\tau_{s_1}(a^*_{x_1}a_{x_1+z_1}),\chi_{x_0}(B_0)\right]^{(k+1)}\right\|_{\mathcal{U}}\\ \leq\ & \|B_0\|_{\mathcal{U}}\mathrm{e}^{m_0\varsigma}\mathbf{K}_1^k\sum_{T\in\mathcal{T}_{k+1}}\prod_{\{j,l\}\in T}\exp\left(-\frac{\varsigma|x_j-x_l|}{\sqrt{d}\max\{\mathfrak{d}_T(j),\mathfrak{d}_T(l)\}}\right)\end{aligned}$$

with

$$\mathbf{K}_1 \doteq 2\left(\mathrm{e}^{\varsigma}+\frac{2\mathbf{C}_1\|\Psi\|_{\mathcal{W}}|t|\,\mathrm{e}^{4\mathbf{D}|t|\|\Psi\|_{\mathcal{W}}}}{\mathrm{e}^{2\varsigma}-\mathrm{e}^{\varsigma}}\right).$$

Proof: For all $k \in \mathbb{N}$, $T \in \mathcal{T}_{k+1}$, and any sequence $\{(m_j,x_j)\}_{j=0}^k$ in $\mathbb{N}_0 \times \mathfrak{L}$ of length $k+1$, the following upper bounds hold for $\varkappa_T$ (see (51)):

$$\varkappa_T\left(\{(m_j,x_j)\}_{j=0}^{k}\right) \leq d^{\frac{k\varsigma}{2}}\prod_{j=0}^{k}(1+m_j)^{\varsigma}\prod_{\{j,l\}\in T}\frac{1}{(1+|x_j-x_l|)^{\frac{\varsigma}{\max\{\mathfrak{d}_T(j),\mathfrak{d}_T(l)\}}}} \tag{68}$$

while

$$\varkappa_T\left(\{(m_j,x_j)\}_{j=0}^{k}\right) \leq \mathrm{e}^{(m_0+\cdots+m_k)\varsigma}\prod_{\{j,l\}\in T}\exp\left(-\frac{\varsigma\,|x_j-x_l|}{\sqrt{d}\max\{\mathfrak{d}_T(j),\mathfrak{d}_T(l)\}}\right). \tag{69}$$

Cf. proof of Theorem 3.9. Therefore, the corollary is a direct consequence of Theorems 3.8 and 3.9 together with the two previous inequalities. ■

Up to the powers $1/\max\{\mathfrak{d}_T(j),\mathfrak{d}_T(l)\}$, Corollary 3.10 gives for interacting systems upper bounds for multi–commutators like [BPH1, Eq. (4.14)] for the free case. We show in the next subsection how to use these bounds to obtain results similar to [BPH1, Theorem 3.4] on the dynamics perturbed by the presence of external electromagnetic fields.

Remark 3.11
All results of this subsection depend on Theorem 3.6 (iii), i.e., the rate of convergence, as $n\to\infty$, of the family $\{\tau^{(n,x)}\}_{n\in\mathbb{N}_0}$ of finite–volume groups introduced in the preliminary discussions before Lemma 3.7. It is the only information on the Fermi system we needed here.

Remark 3.12
The Lieb–Robinson bound for multi–commutators given by Theorems 3.8–3.9 at $k=1$ is not as good as the previous Lieb–Robinson bound of Theorem 3.6 (iv). Nevertheless, they are qualitatively equivalent in the following sense: For interactions with polynomial decay, the first bound also has polynomial decay, even if with lower degree than the second one. For interactions with exponential decay, both bounds are exponentially decaying, even if the first one has a worse prefactor and exponential rate than the second one.

3.3 Application to Perturbed Autonomous Dynamics

Let $\Psi\in\mathcal{W}$ and $\mathbf{V}$ be a potential. For any $l\in\mathbb{R}_0^+$, we consider a map $\eta\mapsto\mathbf{W}^{(l,\eta)}$ from $\mathbb{R}$ to the subspace of self–adjoint elements of $\mathcal{U}_{\Lambda_l}$. In the case that interests us, the following property holds:

$$\left\| \mathbf{W}^{(l,\eta)} \right\|_{\mathcal{U}} = \mathcal{O}(\eta \, |\Lambda_l|) \ . \tag{70}$$

More precisely, we consider elements $\mathbf{W}^{(l,\eta)}$ of the form

$$\mathbf{W}^{(l,\eta)} \doteq \sum_{x\in\Lambda_l} \sum_{z\in\mathfrak{L},|z|\leq 1} \mathbf{w}_{x,x+z}(\eta) a_x^* a_{x+z} \ , \qquad l \in \mathbb{R}_0^+ \ , \tag{71}$$

where $\{\mathbf{w}_{x,y}\}_{x,y\in\mathfrak{L}}$ are complex–valued functions of $\eta \in \mathbb{R}$ with

$$\overline{\mathbf{w}_{x,y}} = \mathbf{w}_{y,x} \qquad \text{and} \qquad \mathbf{w}_{x,y}(0) = 0 \tag{72}$$

for all $x, y \in \mathfrak{L}$.

Equation (71) has the form

$$\mathbf{W}^{(l,\eta)} = \sum_{x\in\Lambda_l} W_x(\eta) \tag{73}$$

where, for some fixed radius $R \in \mathbb{R}^+$ and any $x \in \mathfrak{L}$, $W_x(\eta)$ is a self–adjoint even element of $\mathcal{U}_{x+\Lambda_R}$ that depends on the real parameter η. All results below in this subsection hold for the more general case (73) as well, with obvious modifications. Indeed, we could even consider more general perturbations with $R = \infty$, see proofs of Inequality (131) and Theorem 4.6.

We refrain from treating cases more general than (71) to keep technical aspects as simple as possible. Observe that perturbations due to the presence of external electromagnetic fields are included in the class of perturbations defined by (71). In fact, as discussed in the introduction, our final aim is the microscopic quantum theory of electrical conduction [BP1, BP2, BP3]. Indeed, at fixed $l \in \mathbb{R}_0^+$, $\mathbf{W}^{(l,\eta)}$ defined by (71) is related to perturbations of dynamics caused by constant external electromagnetic fields that vanish outside the box Λ_l.

We assume that $\{\mathbf{w}_{x,y}\}_{x,y\in\mathfrak{L}}$ are uniformly bounded and Lipschitz continuous: There is a constant $K_1 \in \mathbb{R}^+$ such that, for all $\eta, \eta_0 \in \mathbb{R}$,

$$\sup_{x,y\in\mathfrak{L}} |\mathbf{w}_{x,y}(\eta) - \mathbf{w}_{x,y}(\eta_0)| \leq K_1 \, |\eta - \eta_0| \quad \text{and} \quad \sup_{x,y\in\mathfrak{L}} \sup_{\eta\in\mathbb{R}} |\mathbf{w}_{x,y}(\eta)| \leq K_1 \ . \tag{74}$$

These two uniformity conditions could hold for parameters η, η_0 on compact sets only, but we refrain again from considering this more general case, for simplicity.

The perturbed dynamics is defined via the symmetric derivation

$$\delta^{(l,\eta)} \doteq \delta + i \left[\mathbf{W}^{(l,\eta)}, \ \cdot \ \right] \ , \qquad l \in \mathbb{R}_0^+, \ \eta \in \mathbb{R} \ . \tag{75}$$

Recall that δ is the symmetric derivation of Theorem 3.6 which generates the C_0–group $\{\tau_t\}_{t\in\mathbb{R}}$ on $\mathcal{U}$. The second term in the r.h.s. of (75) is a bounded perturbation of δ. Hence, $\delta^{(l,\eta)}$ generates a C_0–group $\{\tilde{\tau}_t^{(l,\eta)}\}_{t\in\mathbb{R}}$ on $\mathcal{U}$, see [EN, Chap. III, Sect. 1.3]. By Lemma 3.4, the (generally unbounded) closed operator $\delta^{(l,\eta)}$ is a conservative symmetric derivation and $\tilde{\tau}_t^{(l,\eta)}$ is a $*$–automorphism of $\mathcal{U}$ for all $t\in\mathbb{R}$.

Let Φ be any interaction with energy observables

$$U_{\Lambda_L}^{\Phi} \doteq \sum_{\Lambda\subseteq\Lambda_L} \Phi_\Lambda , \qquad L\in\mathbb{R}_0^+ . \tag{76}$$

The main aim of this subsection is to study the energy increment

$$\mathbf{T}_{t,s}^{(l,\eta,L)} \doteq \tilde{\tau}_{t-s}^{(l,\eta)}(U_{\Lambda_L}^{\Phi}) - \tau_{t-s}(U_{\Lambda_L}^{\Phi}) , \qquad l,L\in\mathbb{R}_0^+,\ s,t,\eta\in\mathbb{R} , \tag{77}$$

in the limit $L\to\infty$ to obtain similar results as [BPH1, Theorem 3.4]. This can be done by using the (partial) Dyson–Phillips series:

$$\begin{aligned}
&\mathbf{T}_{t,s}^{(l,\eta,L)} - \mathbf{T}_{t,s}^{(l,\eta_0,L)} \qquad\qquad (78)\\
=\ &\sum_{k=1}^{m} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \left[\mathbf{X}_{s_k,s}^{(l,\eta_0,\eta)},\ldots,\mathbf{X}_{s_1,s}^{(l,\eta_0,\eta)},\tilde{\tau}_{t-s}^{(l,\eta_0)}(U_{\Lambda_L}^{\Phi})\right]^{(k+1)}\\
&+i^{m+1}\int_s^t \mathrm{d}s_1 \cdots \int_s^{s_m} \mathrm{d}s_{m+1}\\
&\qquad \tilde{\tau}_{s_{m+1}-s}^{(l,\eta)}\left(\left[\mathbf{W}^{(l,\eta)}-\mathbf{W}^{(l,\eta_0)},\mathbf{X}_{s_m,s_{m+1}}^{(l,\eta_0,\eta)},\ldots,\mathbf{X}_{s_1,s_{m+1}}^{(l,\eta_0,\eta)},\tilde{\tau}_{t-s_{m+1}}^{(l,\eta_0)}(U_{\Lambda_L}^{\Phi})\right]^{(m+2)}\right)
\end{aligned}$$

for any $m\in\mathbb{N}$, where

$$\mathbf{X}_{t,s}^{(l,\eta_0,\eta)} \doteq \tilde{\tau}_{t-s}^{(l,\eta_0)}(\mathbf{W}^{(l,\eta)}-\mathbf{W}^{(l,\eta_0)}) , \qquad l\in\mathbb{R}_0^+,\ s,t,\eta_0,\eta\in\mathbb{R} . \tag{79}$$

By (72), note that $\mathbf{T}_{t,s}^{(l,0,L)}=0$.

By (70), naive bounds on the r.h.s. of (78) predict that

$$\left[\mathbf{X}_{s_k,s}^{(l,\eta_0,\eta)},\ldots,\mathbf{X}_{s_1,s}^{(l,\eta_0,\eta)},\tilde{\tau}_{t-s}^{(l,\eta_0)}(U_{\Lambda_L}^{\Phi})\right]^{(k+1)} = \mathcal{O}(|\Lambda_l|^k\,|\Lambda_L|) .$$

To obtain more accurate estimates, we use the tree–decay bounds on multi–commutators of Corollary 3.10.

To this end, for any $x \in \mathfrak{L}$ and $m \in \mathbb{N}$, we define

$$\mathcal{D}(x,m) \doteq \{\Lambda \in \mathcal{P}_f(\mathfrak{L}) : x \in \Lambda,\ \Lambda \subseteq \Lambda_m + x,\ \Lambda \nsubseteq \Lambda_{m-1} + x\} \subset 2^{\mathfrak{L}} \ . \tag{80}$$

All elements of $\mathcal{D}(x,m)$ are finite subsets of the lattice $\mathfrak{L}$ that contain at least two sites which are separated by a distance greater or equal than m. Using, for any $x \in \mathfrak{L}$ and $m = 0$, the convention

$$\mathcal{D}(x,0) \doteq \{\{x\}\} \ , \tag{81}$$

we obviously have that

$$\mathcal{P}_f(\mathfrak{L}) = \bigcup_{x\in\mathfrak{L},\ m\in\mathbb{N}_0} \mathcal{D}(x,m) \ . \tag{82}$$

We now consider the following assumption on interactions Φ:

$$\sup_{x\in\mathfrak{L}} \sum_{m\in\mathbb{N}_0} \mathbf{v}_m \sum_{\Lambda\in\mathcal{D}(x,m)} \|\Phi_\Lambda\|_{\mathcal{U}} < \infty \tag{83}$$

for some (generally diverging) sequence $\{\mathbf{v}_m\}_{m\in\mathbb{N}_0} \subset \mathbb{R}_0^+$. For instance, if $\Phi \in \mathcal{W}$ and Condition (59) holds true, then one easily verifies (83) with $\mathbf{v}_m = (1+m)^\varsigma$. In the case (60) holds and $\Phi \in \mathcal{W}$, then (83) is also satisfied even with $\mathbf{v}_m = \mathrm{e}^{m\varsigma}$.

We are now in position to state the first main result of this section, which is an extension of [BPH1, Theorem 3.4 (i)] to interacting fermions:

Theorem 3.13 (Taylor's theorem for increments)
Let $l, \mathrm{T} \in \mathbb{R}_0^+$, $s,t \in [-\mathrm{T},\mathrm{T}]$, $\eta,\eta_0 \in \mathbb{R}$, $\Psi \in \mathcal{W}$, and $\mathbf{V}$ be any potential. Assume (59) with $\varsigma > d$, (72) and (74). Take an interaction Φ satisfying (83) with $\mathbf{v}_m = (1+m)^\varsigma$. Then:
(i) *The map $\eta \mapsto \mathbf{T}_{t,s}^{(l,\eta,L)}$ converges uniformly on $\mathbb{R}$, as $L \to \infty$, to a continuous function $\mathbf{T}_{t,s}^{(l,\eta)}$ of η and*

$$\mathbf{T}_{t,s}^{(l,\eta)} - \mathbf{T}_{t,s}^{(l,\eta_0)} = \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})} i \int_s^t \mathrm{d}s_1 \tilde{\tau}_{s_1-s}^{(l,\eta)} \left(\left[\mathbf{W}^{(l,\eta)} - \mathbf{W}^{(l,\eta_0)}, \tilde{\tau}_{t-s_1}^{(l,\eta_0)}(\Phi_\Lambda) \right] \right) \ .$$

(ii) *For any* $m \in \mathbb{N}$ *satisfying* $d(m+1) < \varsigma$,

$$\mathbf{T}_{t,s}^{(l,\eta)} - \mathbf{T}_{t,s}^{(l,\eta_0)} = \tag{84}$$
$$\sum_{k=1}^{m} \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \left[\mathbf{X}_{s_k,s}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t-s}^{(l,\eta_0)}(\Phi_\Lambda)\right]^{(k+1)}$$
$$+ \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} i^{m+1} \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_m} \mathrm{d}s_{m+1}$$
$$\tilde{\tau}_{s_{m+1}-s}^{(l,\eta)} \left(\left[\mathbf{W}^{(l,\eta)} - \mathbf{W}^{(l,\eta_0)}, \mathbf{X}_{s_m,s_{m+1}}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s_{m+1}}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t-s_{m+1}}^{(l,\eta_0)}(\Phi_\Lambda)\right]^{(m+2)} \right).$$

(iii) *All the above series in* Λ *absolutely converge: For any* $m \in \mathbb{N}$ *satisfying* $d(m+1) < \varsigma$, $k \in \{1, \ldots, m\}$, *and* $\{s_j\}_{j=1}^{m+1} \subset [-\mathrm{T}, \mathrm{T}]$,

$$\sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} \left\| \left[\mathbf{X}_{s_k,s}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t-s}^{(l,\eta_0)}(\Phi_\Lambda)\right]^{(k+1)} \right\|_{\mathcal{U}} \leq D \left|\Lambda_l\right| \left|\eta - \eta_0\right|^k$$

and

$$\sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} \left\| \tilde{\tau}_{s_{m+1}-s}^{(l,\eta)} \left(\left[\mathbf{W}^{(l,\eta)} - \mathbf{W}^{(l,\eta_0)}, \mathbf{X}_{s_m,s_{m+1}}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s_{m+1}}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t-s_{m+1}}^{(l,\eta_0)}(\Phi_\Lambda)\right]^{(m+2)} \right) \right\|_{\mathcal{U}}$$
$$\leq D \left|\Lambda_l\right| \left|\eta - \eta_0\right|^{m+1} ,$$

for some constant $D \in \mathbb{R}^+$ *depending only on* $m, d, \mathrm{T}, \Psi, K_1, \Phi, \mathbf{F}$. *The last assertion also holds for* $m = 0$.

Proof: We only prove (ii)–(iii), Assertion (i) being easier to prove by very similar arguments. For simplicity, we assume w.l.o.g. $\eta_0 = s = 0$ and $m \in \mathbb{N}$. Because of Equations (71), (78), (79) and (82), we first control the multi–commutator sum

$$\digamma_{k,L} \doteq \sum_{x_0 \in \mathfrak{L} \backslash \Lambda_L} \sum_{m_0 \in \mathbb{N}_0} \sum_{\Lambda \in \mathcal{D}(x_0,m_0)} \sum_{x_1 \in \Lambda_l} \sum_{z_1 \in \mathfrak{L}, |z_1| \leq 1} \cdots \sum_{x_k \in \Lambda_l} \sum_{z_k \in \mathfrak{L}, |z_k| \leq 1}$$
$$\left\| \xi_{x_1,z_1,\ldots,x_k,z_k} \left[\tau_{s_k}(a_{x_k}^* a_{x_k+z_k}), \ldots, \tau_{s_1}(a_{x_1}^* a_{x_1+z_1}), \tau_t(\Phi_\Lambda)\right]^{(k+1)} \right\|_{\mathcal{U}}$$

for any fixed $k \in \{1, \ldots, m\}$, $\mathrm{T} \in \mathbb{R}_0^+$, $\{s_j\}_{j=1}^{k} \subset [-\mathrm{T}, \mathrm{T}]$ and $L \in \mathbb{R}_0^+ \cup \{-1\}$, where we use the convention $\Lambda_{-1} \doteq \emptyset$ and

$$\xi_{x_1,z_1,\ldots,x_k,z_k} \doteq \prod_{j=1}^{k} \mathbf{w}_{x_j,x_j+z_j}(\eta) . \tag{85}$$

By (72)–(74), there is a constant $D \in \mathbb{R}^+$ (depending on K_1) such that

$$\sup_{x_1,z_1,\ldots,x_k,z_k \in \mathfrak{L}} \sup_{\eta \in \mathbb{R}} \left| \xi_{x_1,z_1,\ldots,x_k,z_k} \right| \leq D \text{ and } \sup_{x_1,z_1,\ldots,x_k,z_k \in \mathfrak{L}} \left| \xi_{x_1,z_1,\ldots,x_k,z_k} \right| \leq D |\eta|^k . \tag{86}$$

At fixed $k \in \{1, \ldots, m\}$ observe further that the condition $\varsigma > dk$ yields

$$\max_{x \in \mathfrak{L}} \sum_{y \in \mathfrak{L}} \frac{1}{(1 + |y - x|)^{\varsigma (\max\{\mathfrak{d}_T(j), \mathfrak{d}_T(l)\})^{-1}}} \leq \sum_{y \in \mathfrak{L}} \frac{1}{(1 + |y|)^{\frac{\varsigma}{k}}} < \infty \tag{87}$$

for any tree $T \in \mathcal{T}_{k+1}$ and all $j, l \in \{0, \ldots, k\}$. Using (83) with $\mathbf{v}_m = (1 + m)^{\varsigma}$, (86)–(87) and the equality

$$\begin{aligned} & \left\| \left[\tau_{s_k}(a^*_{x_k} a_{x_k+z_k}), \ldots, \tau_{s_1}(a^*_{x_1} a_{x_1+z_1}), \tau_t(\Phi_\Lambda) \right]^{(k+1)} \right\|_{\mathcal{U}} \\ = & \left\| \left[\tau_{s_k - t}(a^*_{x_k} a_{x_k+z_k}), \ldots, \tau_{s_1 - t}(a^*_{x_1} a_{x_1+z_1}), \Phi_\Lambda \right]^{(k+1)} \right\|_{\mathcal{U}} , \end{aligned} \tag{88}$$

we obtain from Corollary 3.10 that, for any $m \in \mathbb{N}$ and $k \in \{1, \ldots, m\}$ with $\varsigma > dk$, $\digamma_{k,-1} \leq D |\Lambda_l| |\eta|^k$ for some constant $D \in \mathbb{R}^+$ depending only on $m, d, \mathrm{T}, \Psi, K_1, \Phi, \mathbf{F}$.

Hence, by Lebesgue's dominated convergence theorem, for any $k \in \mathbb{N}$ satisfying $\varsigma > dk$, there is $R \in \mathbb{R}^+$ such that $\digamma_{k,L} < \varepsilon$ for any $L \geq R$. This ensures the convergence of the first k multi–commutators of (78) to the first k multi–commutators of (84) as well as the corresponding absolute summability. Cf. Assertions (ii)–(iii). The convergence is even uniform for $\eta \in \mathbb{R}$ because of the first assertion of (86).

Because $\tilde{\tau}_t^{(l,\eta)}$ is an isometry for any time $t \in \mathbb{R}$, the same arguments are used to control the multi–commutator

$$\tilde{\tau}^{(l,\eta)}_{s_{m+1} - s} \left(\left[\mathbf{W}^{(l,\eta)} - \mathbf{W}^{(l,\eta_0)}, \mathbf{X}^{(l,\eta_0,\eta)}_{s_m, s_{m+1}}, \ldots, \mathbf{X}^{(l,\eta_0,\eta)}_{s_1, s_{m+1}}, \tilde{\tau}^{(l,\eta_0)}_{t - s_{m+1}}(\Phi_\Lambda) \right]^{(m+2)} \right) \tag{89}$$

in (78). By (74), notice additionally that there is a constant $D \in \mathbb{R}^+$ and a family $\{\Psi^{(l,\eta)}\}_{l \in \mathbb{R}_0^+, \eta \in \mathbb{R}} \subset \mathcal{W}$ such that

$$\sup_{\eta \in \mathbb{R}} \sup_{l \in \mathbb{R}_0^+} \left\| \Psi^{(l,\eta)} \right\|_{\mathcal{W}} \leq D < \infty$$

and, for all $l \in \mathbb{R}_0^+$ and $\eta \in \mathbb{R}$, $\{\tilde{\tau}_t^{(l,\eta)}\}_{t \in \mathbb{R}}$ is the C_0–group of $*$–automorphisms on $\mathcal{U}$ associated to the interaction $\Psi^{(l,\eta)}$ and the potential $\mathbf{V}$. The norm $\|\cdot\|_{\mathcal{W}}$ in the last inequality, which defines the space $\mathcal{W}$ of interactions, is of course defined w.r.t. the same function $\mathbf{F}$ to which the conditions of the theorem are imposed. This property justifies the simplifying assumption $\eta_0 = 0$ at the beginning of the proof. This concludes the proof of Assertions (ii)–(iii).

Assertion (i) is proven in the same way and we omit the details. Note only that the convergence of $\digamma_{1,L}$ as $L \to \infty$ is uniform for $\eta \in \mathbb{R}$ because of the first assertion of (86). The latter implies the continuity of the map $\eta \mapsto \mathbf{T}_{t,s}^{(l,\eta)}$ for $\eta \in \mathbb{R}$. ∎

A direct consequence of Theorem 3.13 is that $\mathbf{T}_{t,s}^{(l,\eta)} = \mathcal{O}(|\Lambda_l|)$. Note furthermore that Theorem 3.13 also holds when the cubic box Λ_l is replaced by *any* finite subset $\Lambda \in \mathcal{P}_f(\mathfrak{L})$. The assumptions of this theorem are fulfilled for any interactions $\Psi, \Phi \in \mathcal{W}$ with (8), provided the parameter $\epsilon \in \mathbb{R}^+$ is sufficiently large. Theorem 3.13 is thus a *significant* extension of [BPH1, Theorem 3.4 (i)] in the sense that very general inter–particle interactions and the full range of parameters $\eta \in \mathbb{R}$ are now allowed.

In the case of exponentially decaying interactions we can bound the derivatives $|\Lambda_l|^{-1}\partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta)}$ for all $m \in \mathbb{N}$, uniformly w.r.t. $l \in \mathbb{R}_0^+$. We thus extend [BPH1, Theorem 3.4 (ii)] for interactions Φ satisfying (83).

Under these conditions, we show below that the map $\eta \mapsto |\Lambda_l|^{-1}\mathbf{T}_{t,s}^{(l,\eta)}$ from $\mathbb{R}$ to $\mathcal{U}$ is bounded in the sense of Gevrey norms, uniformly w.r.t. $l \in \mathbb{R}_0^+$. Note that real analytic functions (cf. [BPH1, Theorem 3.4 (ii)]) are a special case of Gevrey functions.

Theorem 3.14 (Increments as Gevrey maps)
Let $l, \mathrm{T} \in \mathbb{R}_0^+$, $s,t \in [-\mathrm{T},\mathrm{T}]$, $\Psi \in \mathcal{W}$, and $\mathbf{V}$ be any potential. Assume (60) and take an interaction Φ satisfying (83) with $\mathbf{v}_m = \mathrm{e}^{m\varsigma}$. Assume further the real analyticity of the maps $\eta \mapsto \mathbf{w}_{x,y}(\eta)$, $x,y \in \mathfrak{L}$, from $\mathbb{R}$ to $\mathbb{C}$ as well as the existence of $r \in \mathbb{R}^+$ such that

$$K_2 \doteq \sup_{x,y\in\mathfrak{L}} \sup_{m\in\mathbb{N}} \sup_{\eta\in\mathbb{R}} \frac{r^m \partial_\eta^m \mathbf{w}_{x,y}(\eta)}{m!} < \infty\,. \tag{90}$$

(i) *Smoothness. As a function of $\eta \in \mathbb{R}$, $\mathbf{T}_{t,s}^{(l,\eta)} \in C^\infty(\mathbb{R};\mathcal{U})$ and for any $m \in \mathbb{N}$,*

$$\partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta)} = \sum_{k=1}^{m} \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k$$
$$\partial_\varepsilon^m \left[\mathbf{X}_{s_k,s}^{(l,\eta,\eta+\varepsilon)}, \ldots, \mathbf{X}_{s_1,s}^{(l,\eta,\eta+\varepsilon)}, \tilde{\tau}_{t,s}^{(l,\eta)}(\Phi_\Lambda)\right]^{(k+1)}\Bigg|_{\varepsilon=0}\,.$$

The above series in Λ are absolutely convergent.
(ii) *Uniform boundedness of the Gevrey norm of density of increments. There exist $\tilde{r} \equiv \tilde{r}_{d,\mathrm{T},\Psi,K_2,\mathbf{F}} \in \mathbb{R}^+$ and $D \equiv D_{\mathrm{T},\Psi,K_2,\Phi} \in \mathbb{R}^+$ such that, for all $l \in \mathbb{R}_0^+$, $\eta \in \mathbb{R}$ and $s,t \in [-\mathrm{T},\mathrm{T}]$,*

$$\sum_{m\in\mathbb{N}} \frac{\tilde{r}^m}{(m!)^d} \sup_{l\in\mathbb{R}_0^+} \left\| |\Lambda_l|^{-1} \partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta)} \right\|_{\mathcal{U}} \leq D \, .$$

Before giving the proof, note first that the assumptions of Theorem 3.14 are satisfied for any interactions $\Psi, \Phi \in \mathcal{W}$ with (61). Moreover, under conditions of Theorem 3.14, the family $\{|\Lambda_l|^{-1}\mathbf{T}_{t,s}^{(l,\eta)}\}_{l\in\mathbb{R}_0^+}$ of functions of the variable η at dimension $d = 1$ is uniformly bounded w.r.t. analytic norms. In particular, for $d = 1$ and any state $\varrho \in \mathcal{U}^*$, the limit of the increment density $|\Lambda_l|^{-1}\varrho(\mathbf{T}_{t,s}^{(l,\eta)})$, as $l \to \infty$ (possibly along subsequences), is either identically vanishing for all $\eta \in \mathbb{R}$, or is different from zero for η outside a discrete subset of $\mathbb{R}$. Note that, by contrast, general non–vanishing Gevrey functions can have arbitrarily small support. We discuss this with more details at the end of Section 4.3.

We now conclude this subsection by proving Theorem 3.14. To this end, we need the following estimate:

Proposition 3.15
There is a constant $D \in \mathbb{R}^+$ such that, for all $k \in \mathbb{N}$,

$$\sum_{T\in\mathcal{T}_{k+1}} \max_{j\in\{0,\ldots,k\}} \max_{x_j\in\mathfrak{L}} \sum_{x_0,\ldots,\not{x}_j,\ldots,x_k\in\mathfrak{L}} \prod_{\{p,l\}\in T} \mathrm{e}^{-\frac{\varsigma|x_p-x_l|}{\sqrt{d}\max\{\mathfrak{d}_T(p),\mathfrak{d}_T(l)\}}} \leq D^k (k!)^d \, .$$

The proof of this upper bound uses the fact that trees with vertices of large degree are "*rare*" in a way that summing up the numbers $(\mathfrak{d}_T!)^\alpha$ for $T \in \mathcal{T}_{k+1}$ and any $\alpha \in \mathbb{R}^+$ gives factors behaving, at worse, like $D^k(k!)^\alpha$. The arguments are standard results of finite mathematics. We prove them below for completeness, in two simple lemmata.

Let $k \in \mathbb{N}$. For any fixed sequence $\mathfrak{d} = (\mathfrak{d}(0),\ldots,\mathfrak{d}(k)) \in \mathbb{N}^{k+1}$ define the set $\mathcal{T}_{k+1}(\mathfrak{d}) \subset \mathcal{T}_{k+1}$ by

$$\mathcal{T}_{k+1}(\mathfrak{d}) \doteq \{T \in \mathcal{T}_{k+1} : \mathfrak{d}_T \equiv (\mathfrak{d}_T(0),\ldots,\mathfrak{d}_T(k)) = \mathfrak{d}\} \, .$$

In other words, $\mathcal{T}_{k+1}(\mathfrak{d})$ is the set of all trees of $\mathcal{T}_{k+1}$ with vertices having their degree fixed by the sequence $\mathfrak{d}$. The cardinality of this set is bounded as follows:

Lemma 3.16 (Number of trees with vertices of fixed degrees)
For all $k \in \mathbb{N}$ and $\mathfrak{d} \in \mathbb{N}^{k+1}$,

$$|\mathcal{T}_{k+1}(\mathfrak{d})| \leq \frac{(k-1)!}{(\mathfrak{d}(0)-1)! \cdots (\mathfrak{d}(k)-1)!} \ .$$

Proof: The bound can be proven, for instance, by using so–called "Prüfer codes". We give here a proof based on a simplified version of such codes, well adapted to the particular sets of trees $\mathcal{T}_{k+1}$. At fixed $k \in \mathbb{N}$, define the map $\mathfrak{C} : \mathcal{T}_{k+1} \to \{0, \ldots, k-1\}^{k-1}$ by

$$\mathfrak{C}(T) \doteq (\mathrm{P}_T(2), \ldots, \mathrm{P}_T(k)) \ .$$

See (48)–(50). This map is clearly injective and if $j \in \{0, \ldots, k\}$ is a vertex of degree $\mathfrak{d}_T(j)$, then it appears exactly $(\mathfrak{d}_T(j)-1)$ times in the sequence $\mathfrak{C}(T)$. Note that $\mathfrak{d}_T(k) = 1$ for all $T \in \mathcal{T}_{k+1}$. To finish the proof, fix $\mathfrak{d} = (\mathfrak{d}(0), \ldots, \mathfrak{d}(k)) \in \mathbb{N}^{k+1}$ and observe that if $\mathfrak{d}(0) + \cdots + \mathfrak{d}(k) = 2k$ then there are exactly

$$\frac{(k-1)!}{(\mathfrak{d}(0)-1)! \cdots (\mathfrak{d}(k)-1)!}$$

sequences in $\{0, \ldots, k-1\}^{k-1}$ with $j \in \{0, \ldots, k\}$ appearing exactly $(\mathfrak{d}(j)-1)$ times in such sequences. If $\mathfrak{d}(0) + \cdots + \mathfrak{d}(k) \neq 2k$ then such a sequence does not exist. ■

Lemma 3.17
For all $k \in \mathbb{N}$,

$$\sum_{\mathfrak{d}(0), \ldots, \mathfrak{d}(k) \in \mathbb{N}} \mathbf{1}[\mathfrak{d}(0) + \cdots + \mathfrak{d}(k) = 2k] \leq 4^k \ .$$

Proof: For $k \in \mathbb{N}$, the coefficient c_{2k} of the analytic function

$$z \mapsto \frac{z^{k+1}}{(1-z)^{k+1}} = \sum_{m=1}^{\infty} c_m z^m$$

on the complex disc $\{z \in \mathbb{C} : |z| < 1\}$ is exactly the finite sum

$$\sum_{\mathfrak{d}(0), \ldots, \mathfrak{d}(k) \in \mathbb{N}} \mathbf{1}[\mathfrak{d}(0) + \cdots + \mathfrak{d}(k) = 2k] \ .$$

In particular,

$$\sum_{\mathfrak{d}(0),\ldots,\mathfrak{d}(k)\in\mathbb{N}} \mathbf{1}[\mathfrak{d}(0)+\cdots+\mathfrak{d}(k)=2k] = \frac{1}{2\pi i}\oint_{|z|=1/2} \frac{1}{z^k(1-z)^{k+1}}\mathrm{d}z \ ,$$

which combined with the inequality

$$\left|\frac{1}{2\pi i}\oint_{|z|=1/2} \frac{1}{z^k(1-z)^{k+1}}\mathrm{d}z\right| \leq 4^k$$

yields the assertion. ■

By using the two above lemmata, we now prove Proposition 3.15:

Proof: Fix $\alpha \in \mathbb{R}^+$ and note first that, for all $d \in \mathbb{N}$,

$$\lim_{g\to\infty} \frac{1}{g^d}\sum_{x\in\mathfrak{L}} \mathrm{e}^{-\frac{\alpha|x|}{g\sqrt{d}}} = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\alpha|x|}{\sqrt{d}}}\mathrm{d}^d x < \infty \ .$$

Hence, for $d \in \mathbb{N}$, there is a constant $S_d \in \mathbb{R}^+$ such that

$$\sum_{x\in\mathfrak{L}} \mathrm{e}^{-\frac{\alpha|x|}{g\sqrt{d}}} \leq S_d g^d \ , \qquad g \in \mathbb{N} \ .$$

From this estimate and by using the Stirling–type bounds [R]

$$g^g\mathrm{e}^{-g}\mathrm{e}^{\frac{1}{12g+1}}\sqrt{2\pi g} \leq g! \leq g^g\mathrm{e}^{-g}\mathrm{e}^{\frac{1}{12g}}\sqrt{2\pi g} \ , \qquad g \in \mathbb{N} \ , \tag{91}$$

we obtain

$$\begin{aligned} &\max_{j\in\{0,\ldots,k\}}\max_{x_j\in\mathfrak{L}} \sum_{x_0,\ldots,\not{x}_j,\ldots,x_k\in\mathfrak{L}} \prod_{\{p,l\}\in T} \exp\left(-\frac{\varsigma\,|x_p-x_l|}{\sqrt{d}\max\{\mathfrak{d}_T(p),\mathfrak{d}_T(l)\}}\right) \\ \leq\ & S_d^k\prod_{j=0}^k \mathfrak{d}_T(j)^{\mathfrak{d}_T(j)d} \leq S_d^k\mathrm{e}^{\mathfrak{d}_T(j)d}(\mathfrak{d}_T!)^d \end{aligned} \tag{92}$$

for all $d, k \in \mathbb{N}$ and $T \in \mathcal{T}_{k+1}$. We infer from (63) that

$$\sum_{T\in\mathcal{T}_{k+1}} (\mathfrak{d}_T!)^d \leq (k!)^{d-1}\sum_{T\in\mathcal{T}_{k+1}} (\mathfrak{d}_T!) \ . \tag{93}$$

We use now Lemma 3.16 to get

$$\begin{aligned}\sum_{T\in\mathcal{T}_{k+1}}(\mathfrak{d}_T!) &= \sum_{\mathfrak{d}(0),\ldots,\mathfrak{d}(k)\in\mathbb{N}} \mathbf{1}[\mathfrak{d}(0)+\cdots+\mathfrak{d}(k)=2k] \sum_{T\in\mathcal{T}_{k+1}((\mathfrak{d}(0),\ldots,\mathfrak{d}(k)))}(\mathfrak{d}_T!)\\ &\leq k! \sum_{\mathfrak{d}(0),\ldots,\mathfrak{d}(k)\in\mathbb{N}} \mathbf{1}[\mathfrak{d}(0)+\cdots+\mathfrak{d}(k)=2k]\, \mathfrak{d}(0)\cdots\mathfrak{d}(k)\\ &\leq k! \sum_{\mathfrak{d}(0),\ldots,\mathfrak{d}(k)\in\mathbb{N}} \mathbf{1}[\mathfrak{d}(0)+\cdots+\mathfrak{d}(k)=2k]\, \mathrm{e}^{\mathfrak{d}(0)}\cdots\mathrm{e}^{\mathfrak{d}(k)}\ .\end{aligned}$$

We invoke (62) and Lemma 3.17 to arrive at

$$\sum_{T\in\mathcal{T}_{k+1}}(\mathfrak{d}_T!) \leq (k!)\mathrm{e}^{2k} \sum_{\mathfrak{d}(0),\ldots,\mathfrak{d}(k)\in\mathbb{N}} \mathbf{1}[\mathfrak{d}(0)+\cdots+\mathfrak{d}(k)=2k] \leq (k!)(4\mathrm{e}^2)^k\ . \tag{94}$$

Proposition 3.15 is then a consequence of (92), (93) and (94). ■

We are now in position to prove Theorem 3.14:

Proof: (i) Observe that

$$\partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta,L)} = \partial_\varepsilon^m \left(\mathbf{T}_{t,s}^{(l,\eta+\varepsilon,L)} - \mathbf{T}_{t,s}^{(l,\eta,L)}\right)\Big|_{\varepsilon=0}\ . \tag{95}$$

The difference $\mathbf{T}_{t,s}^{(l,\eta+\varepsilon,L)} - \mathbf{T}_{t,s}^{(l,\eta,L)}$ is explicitly given by a Dyson–Phillips series involving multi–commutators (40)–(41): Use (78) to produce an infinite series. As the function $\eta \mapsto \mathbf{W}^{(l,\eta)}$ is, by assumption, real analytic, it follows that

$$\begin{aligned}&\partial_\varepsilon^m \left(\mathbf{T}_{t,s}^{(l,\eta+\varepsilon,L)} - \mathbf{T}_{t,s}^{(l,\eta,L)}\right)\Big|_{\varepsilon=0} = \\ &\sum_{k=1}^m i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \partial_\varepsilon^m \left[\mathbf{X}_{s_k,s}^{(l,\eta,\eta+\varepsilon)},\ldots,\mathbf{X}_{s_1,s}^{(l,\eta,\eta+\varepsilon)},\tilde{\tau}_{t,s}^{(l,\eta)}(U_{\Lambda_L}^\Phi)\right]^{(k+1)}\Bigg|_{\varepsilon=0}\end{aligned} \tag{96}$$

for any $m \in \mathbb{N}$, $l \in \mathbb{R}_0^+$, and $s,t,\eta \in \mathbb{R}$. Set

$$\xi_{x_1,z_1,\ldots,x_k,z_k} \doteq \partial_\varepsilon^m \left\{\prod_{j=1}^k \left(\mathbf{w}_{x_j,x_j+z_j}(\eta+\varepsilon) - \mathbf{w}_{x_j,x_j+z_j}(\eta)\right)\right\}\Bigg|_{\varepsilon=0}\ .$$

By (90), these coefficients are uniformly bounded w.r.t. $x_1,z_1,\ldots,x_k,z_k$ and η:

$$\sup_{x_1,z_1,\ldots,x_k,z_k\in\mathfrak{L}}\ \sup_{\eta\in\mathbb{R}} |\xi_{x_1,z_1,\ldots,x_k,z_k}| \leq D^m m! \tag{97}$$

for some constant $D \in \mathbb{R}^+$ depending on K_2 but not on $m \geq k$. Bounding the above multi–commutators exactly as done for the proof of Theorem 3.13 and by taking the limit $L \to \infty$, we deduce from (95)–(96) that, for any $m \in \mathbb{N}$ and $s, t, \eta \in \mathbb{R}$,

$$\lim_{L\to\infty} \partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta,L)} = \sum_{k=1}^{m} \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \qquad (98)$$
$$\partial_\varepsilon^m \left[\mathbf{X}_{s_k,s}^{(l,\eta,\eta+\varepsilon)}, \ldots, \mathbf{X}_{s_1,s}^{(l,\eta,\eta+\varepsilon)}, \tilde{\tau}_{t,s}^{(l,\eta)}(\Phi_\Lambda) \right]^{(k+1)} \Bigg|_{\varepsilon=0} .$$

This limit is uniform for $\eta \in \mathbb{R}$ because of (97). As in Theorem 3.13 (ii), the above series in Λ are absolutely convergent. Moreover, the uniform convergence of $\partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta,L)}$, $m \in \mathbb{N}$, together with Theorem 3.13 (i) implies that the energy increment limit $\mathbf{T}_{t,s}^{(l,\eta)}$ is a smooth function of η with m–derivatives

$$\partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta)} = \lim_{L\to\infty} \partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta,L)}$$

for all $m \in \mathbb{N}$ and $s, t, \eta \in \mathbb{R}$. Because of (98), Assertion (i) thus follows.
(ii) is a direct consequence of (i), Corollary 3.10, and Proposition 3.15 together with (97) and

$$\int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \leq \frac{(2\mathrm{T})^k}{k!} .$$

■

4 Lieb–Robinson Bounds for Non–Autonomous Dynamics

Like in Section 3, we only consider fermion systems, but all results can easily be extended to quantum spin systems. For quantum spin systems, note that Lieb–Robinson bounds for non–autonomous dynamics have already been considered in [BMNS]. However, [BMNS] only proves Lieb–Robinson bounds for commutators, while the multi–commutator case was not considered, in contrast with results of this section. Observe also that some aspects of the non–autonomous case can be treated in a similar way to the autonomous case. However, several important arguments cannot be directly extended to the non–autonomous situation.

Here, we only address in detail the technical issues which are specific to the non–autonomous problem. See for instance Corollary 4.2 (iii), Lemma 4.3, Theorem 4.5, and Theorem 4.7.

4.1 Existence of Non–Autonomous Dynamics

We now consider time–dependent models. So, let $\Psi \doteq \{\Psi^{(t)}\}_{t\in\mathbb{R}}$ be a map from $\mathbb{R}$ to $\mathcal{W}$ such that

$$\|\Psi\|_{\infty} \doteq \sup_{t\in\mathbb{R}} \left\|\Psi^{(t)}\right\|_{\mathcal{W}} < \infty \ .$$

I.e., $\{\Psi^{(t)}\}_{t\in\mathbb{R}}$ is a *bounded* family in $\mathcal{W}$. We could easily extend the study of this section to families $\{\Psi^{(t)}\}_{t\in\mathbb{R}}$ which are only bounded for t on compacta. We refrain from considering this more general case, for simplicity. Take, furthermore, any collection $\{\mathbf{V}^{(t)}\}_{t\in\mathbb{R}}$ of potentials. Note that (10) is allowed for any $t\in\mathbb{R}$.

For all $x\in\mathfrak{L}$ and $\Lambda\in\mathcal{P}_f(\mathfrak{L})$, assume the continuity of the two maps $t\mapsto\Psi_{\Lambda}^{(t)}$, $t\mapsto\mathbf{V}_{\{x\}}^{(t)}$ from $\mathbb{R}$ to $\mathcal{U}$, i.e., $\Psi_{\Lambda},\mathbf{V}_{\{x\}}\in C(\mathbb{R};\mathcal{U})$. For any $L\in\mathbb{R}_0^+$, this yields the existence, uniqueness and an explicit expression, as a Dyson–Phillips series, of the solution $\{\tau_{t,s}^{(L)}\}_{s,t\in\mathbb{R}}$ of the (finite–volume) non–autonomous evolutions equations

$$\forall s,t\in\mathbb{R}: \qquad \partial_s\tau_{t,s}^{(L)} = -\delta_s^{(L)}\circ\tau_{t,s}^{(L)} \ , \qquad \tau_{t,t}^{(L)} = \mathbf{1}_{\mathcal{U}} \ , \tag{99}$$

and

$$\forall s,t\in\mathbb{R}: \qquad \partial_t\tau_{t,s}^{(L)} = \tau_{t,s}^{(L)}\circ\delta_t^{(L)} \ , \qquad \tau_{s,s}^{(L)} = \mathbf{1}_{\mathcal{U}} \ . \tag{100}$$

Here, for any $t\in\mathbb{R}$ and $L\in\mathbb{R}_0^+$, the bounded linear operator $\delta_t^{(L)}$ is defined on $\mathcal{U}$ by

$$\delta_t^{(L)}(B) \doteq i\sum_{\Lambda\subseteq\Lambda_L}\left[\Psi_{\Lambda}^{(t)},B\right] + i\sum_{x\in\Lambda_L}\left[\mathbf{V}_{\{x\}}^{(t)},B\right] \ , \qquad B\in\mathcal{U} \ .$$

Compare this definition with (13).

Similar to the autonomous case, for any $L\in\mathbb{R}_0^+$, $\{\tau_{t,s}^{(L)}\}_{s,t\in\mathbb{R}}$ is a continuous two–parameter family of bounded operators that satisfies the (reverse) cocycle property

$$\forall s,r,t\in\mathbb{R}: \qquad \tau_{t,s}^{(L)} = \tau_{r,s}^{(L)}\tau_{t,r}^{(L)} \ . \tag{101}$$

Its time–dependent generator $\delta_t^{(L)}$ is clearly a symmetric derivation and $\tau_{t,s}^{(L)}$ is thus a $*$–automorphism on $\mathcal{U}$ for all $L\in\mathbb{R}_0^+$ and $s,t\in\mathbb{R}$. Moreover, similar to the autonomous case (cf. Theorem 3.1 and Lemma 3.2), for all $L\in\mathbb{R}_0^+$ and $s,t\in\mathbb{R}$, $\tau_{t,s}^{(L)}$ satisfies Lieb–Robinson bounds and thus converges in the strong sense on $\mathcal{U}_0$, as $L\to\infty$:

Theorem 4.1 (Properties of non–autonomous finite–volume dynamics)
Let $\Psi \doteq \{\Psi^{(t)}\}_{t\in\mathbb{R}}$ *be a bounded family on* $\mathcal{W}$ *(i.e.,* $\|\Psi\|_{\infty} < \infty$ *) and* $\{\mathbf{V}^{(t)}\}_{t\in\mathbb{R}}$ *a collection of potentials. For any* $x \in \mathfrak{L}$ *and* $\Lambda \in \mathcal{P}_f(\mathfrak{L})$*, assume* $\Psi_{\Lambda}, \mathbf{V}_{\{x\}} \in C(\mathbb{R};\mathcal{U})$*. Fix* $s, t \in \mathbb{R}$*.*
(i) *Lieb–Robinson bounds. For any* $L \in \mathbb{R}_0^+$*,* $B_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$*, and* $B_2 \in \mathcal{U}_{\Lambda^{(2)}}$ *with* $\Lambda^{(1)}, \Lambda^{(2)} \subsetneq \Lambda_L$ *and* $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$*,*

$$\begin{aligned}&\left\|\left[\tau_{t,s}^{(L)}\left(B_1\right), B_2\right]\right\|_{\mathcal{U}} \\ \leq\ & 2\mathbf{D}^{-1}\left\|B_1\right\|_{\mathcal{U}}\left\|B_2\right\|_{\mathcal{U}}\left(\mathrm{e}^{2\mathbf{D}|t-s|\|\Psi\|_{\infty}}-1\right)\sum_{x\in\partial_{\Psi}\Lambda^{(1)}}\sum_{y\in\Lambda^{(2)}}\mathbf{F}\left(|x-y|\right) .\end{aligned}$$

(ii) *Convergence of the finite–volume dynamics. For any* $\Lambda \in \mathcal{P}_f(\mathfrak{L})$*,* $B \in \mathcal{U}_{\Lambda}$*, and* $L_1, L_2 \in \mathbb{R}_0^+$ *with* $\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$*,*

$$\begin{aligned}&\left\|\tau_{t,s}^{(L_2)}\left(B\right)-\tau_{t,s}^{(L_1)}\left(B\right)\right\|_{\mathcal{U}} \\ \leq\ & 2\left\|B\right\|_{\mathcal{U}}\left\|\Psi\right\|_{\infty}|t-s|\,\mathrm{e}^{4\mathbf{D}|t-s|\|\Psi\|_{\infty}}\sum_{y\in\Lambda_{L_2}\backslash\Lambda_{L_1}}\sum_{x\in\Lambda}\mathbf{F}\left(|x-y|\right) .\end{aligned}$$

Proof: (i) The arguments are a straightforward extension of those proving Theorem 3.1 to non–autonomous dynamics: Fix $L \in \mathbb{R}_0^+$, $B_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$ and $B_2 \in \mathcal{U}_{\Lambda^{(2)}}$ with disjoint sets $\Lambda^{(1)}, \Lambda^{(2)} \subsetneq \Lambda_L$. Similar to (16)–(17), we infer from (99)–(100) that the derivative w.r.t. to t of the function

$$f\left(s,t\right) \doteq \left[\tau_{t,s}^{(L)}\circ\tau_{s,t}^{(\Lambda^{(1)})}\left(B_1\right), B_2\right] , \qquad s,t\in\mathbb{R} ,$$

equals

$$\begin{aligned}\partial_t f\left(s,t\right) &= i\sum_{\mathcal{Z}\in\mathcal{S}_{\Lambda_L}(\Lambda^{(1)})}\left[\tau_{t,s}^{(L)}(\Psi_{\mathcal{Z}}^{(t)}), f\left(s,t\right)\right] \\ &\quad -i\sum_{\mathcal{Z}\in\mathcal{S}_{\Lambda_L}(\Lambda^{(1)})}\left[\tau_{t,s}^{(L)}\circ\tau_{s,t}^{(\Lambda^{(1)})}\left(B_1\right), \left[\tau_{t,s}^{(L)}(\Psi_{\mathcal{Z}}^{(t)}), B_2\right]\right] .\end{aligned} \tag{102}$$

Exactly like (18), it follows that

$$\left\|f\left(s,t\right)\right\|_{\mathcal{U}} \leq \left\|f\left(s,s\right)\right\|_{\mathcal{U}}+2\left\|B_1\right\|_{\mathcal{U}}\sum_{\mathcal{Z}\in\mathcal{S}_{\Lambda_L}(\Lambda^{(1)})}\int_{\min\{s,t\}}^{\max\{s,t\}}\left\|\left[\tau_{\alpha,s}^{(L)}(\Psi_{\mathcal{Z}}^{(\alpha)}), B_2\right]\right\|_{\mathcal{U}}\mathrm{d}\alpha$$

for any $s, t \in \mathbb{R}$. Therefore, by using estimates that are similar to (19)–(21), we deduce Assertion (i).
(ii) The arguments are extensions to the non–autonomous case of those proving Lemma 3.2: Since $\Psi_\Lambda, \mathbf{V}_{\{x\}} \in C(\mathbb{R};\mathcal{U})$ for any $x \in \mathfrak{L}$ and $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, the time–dependent energy observables

$$H_L^{(t)} \doteq \sum_{\Lambda \subseteq \Lambda_L} \Psi_\Lambda^{(t)} + \sum_{x \in \Lambda_L} \mathbf{V}_{\{x\}}^{(t)} , \qquad L \in \mathbb{R}_0^+ , \ \ t \in \mathbb{R} ,$$

and potentials

$$\mathbf{V}_{\mathcal{Z}}^{(t)} \doteq \sum_{x \in \mathcal{Z}} \mathbf{V}_{\{x\}}^{(t)} \in \mathcal{U}^+ \cap \mathcal{U}_{\mathcal{Z}} , \qquad \mathcal{Z} \in \mathcal{P}_f(\mathfrak{L}) , \ \ t \in \mathbb{R} ,$$

generate two solutions $\{\mathcal{V}_{s,t}(H_L)\}_{s,t \in \mathbb{R}}$ and $\{\mathcal{V}_{s,t}(\mathbf{V}_{\mathcal{Z}})\}_{s,t \in \mathbb{R}}$, respectively, of the non–autonomous evolution equations

$$\partial_t \left(\mathcal{V}_{s,t}(X) \right) = i \mathcal{V}_{s,t}(X) X^{(t)} \qquad \text{and} \qquad \partial_s \left(\mathcal{V}_{s,t}(X) \right) = -i X^{(s)} \mathcal{V}_{s,t}(X) \quad (103)$$

with $X^{(t)} = H_L^{(t)}$ or $\mathbf{V}_{\mathcal{Z}}^{(t)}$. These evolution families satisfy $\mathcal{V}_{t,t}(X) = \mathbf{1}_{\mathcal{U}}$ for $t \in \mathbb{R}$ as well as the (usual) cocycle (Chapman–Kolmogorov) property

$$\forall t, r, s \in \mathbb{R} : \qquad \mathcal{V}_{s,t}(X) = \mathcal{V}_{s,r}(X) \mathcal{V}_{r,t}(X) . \quad (104)$$

For any $L \in \mathbb{R}_0^+$ and $s, t, \alpha \in \mathbb{R}$, we then replace (22) in the proof of Lemma 3.2 with

$$\mathbf{U}_L(t, \alpha) \doteq \mathcal{V}_{s,t}(\mathbf{V}_{\Lambda_L}) \mathcal{V}_{t,\alpha}(H_L) \mathcal{V}_{\alpha,s}(\mathbf{V}_{\Lambda_L}) . \quad (105)$$

By (104), $\mathbf{U}_L(t, t) = \mathbf{1}_{\mathcal{U}}$ for all $t \in \mathbb{R}$ while

$$\partial_t \mathbf{U}_L(t, \alpha) = -i G_L(t) \mathbf{U}_L(t, \alpha) \quad \text{and} \quad \partial_\alpha \mathbf{U}_L(t, \alpha) = i \mathbf{U}_L(t, \alpha) G_L(\alpha) \quad (106)$$

with

$$G_L(t) \doteq \sum_{\mathcal{Z} \subseteq \Lambda_L} \mathcal{V}_{s,t}(\mathbf{V}_{\Lambda_L}) \, \Psi_{\mathcal{Z}} \, \mathcal{V}_{t,s}(\mathbf{V}_{\Lambda_L}) . \quad (107)$$

Using the notation

$$\tilde{\tau}_{t,s}^{(L)}(B) \doteq \mathbf{U}_L(s, t) B \mathbf{U}_L(t, s) , \qquad B \in \mathcal{U}_\Lambda , \quad (108)$$

for any $s, t \in \mathbb{R}$ and $L \in \mathbb{R}_0^+$ such that $\Lambda \subset \Lambda_L$, observe that

$$\tau_{t,s}^{(L)}(B) = \mathcal{V}_{s,t}(H_L) B \mathcal{V}_{t,s}(H_L) = \tilde{\tau}_{t,s}^{(L)} \left(\mathcal{V}_{s,t}(\mathbf{V}_\Lambda) B \mathcal{V}_{t,s}(\mathbf{V}_\Lambda) \right) . \quad (109)$$

Note that, for any $s, t \in \mathbb{R}$, $\Lambda, \mathcal{Z} \in \mathcal{P}_f(\mathfrak{L})$ and $B \in \mathcal{U}_\Lambda$,

$$\mathcal{V}_{s,t}(\mathbf{V}_\mathcal{Z}) B \mathcal{V}_{t,s}(\mathbf{V}_\mathcal{Z}) \in \mathcal{U}_\Lambda \quad \text{and} \quad \|\mathcal{V}_{s,t}(\mathbf{V}_\mathcal{Z}) B \mathcal{V}_{t,s}(\mathbf{V}_\mathcal{Z})\|_\mathcal{U} = \|B\|_\mathcal{U} \ . \tag{110}$$

Hence, it suffices to study the net $\{\tilde{\tau}_{t,s}^{(L)}(B)\}_{L \in \mathbb{R}_0^+}$ with $B \in \mathcal{U}_\Lambda$. Up to straightforward modifications taking into account the initial time $s \in \mathbb{R}$, the remaining part of the proof is now identical to the arguments starting from Equation (23) in the proof of Lemma 3.2. ∎

Corollary 4.2 (Infinite–volume dynamics)
Under the conditions of Theorem 4.1, finite–volume families $\{\tau_{t,s}^{(L)}\}_{s,t\in\mathbb{R}}$, $L \in \mathbb{R}_0^+$, *converge strongly and uniformly for* s, t *on compact sets to a strongly continuous two–parameter family* $\{\tau_{t,s}\}_{s,t\in\mathbb{R}}$ *of* $*$*–automorphisms on* $\mathcal{U}$ *satisfying the following properties:*
(i) *Reverse cocycle property.*

$$\forall s, r, t \in \mathbb{R} : \qquad \tau_{t,s} = \tau_{r,s}\tau_{t,r} \ .$$

(ii) *Lieb–Robinson bounds. For any* $s, t \in \mathbb{R}$, $B_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$, *and* $B_2 \in \mathcal{U}_{\Lambda^{(2)}}$ *with disjoint sets* $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f(\mathfrak{L})$,

$$\begin{aligned} & \|[\tau_{t,s}(B_1), B_2]\|_\mathcal{U} \\ \leq \ & 2\mathbf{D}^{-1} \|B_1\|_\mathcal{U} \|B_2\|_\mathcal{U} \left(\mathrm{e}^{2\mathbf{D}|t-s|\|\Psi\|_\infty} - 1\right) \sum_{x \in \partial_\Psi \Lambda^{(1)}} \sum_{y \in \Lambda^{(2)}} \mathbf{F}(|x-y|) \ . \end{aligned}$$

(iii) *Non–autonomous evolution equation. If* $\Psi \in C(\mathbb{R}; \mathcal{W})$ *then* $\{\tau_{t,s}\}_{s,t\in\mathbb{R}}$ *is the unique family of bounded operators on* $\mathcal{U}$ *satisfying, in the strong sense on the dense domain* $\mathcal{U}_0 \subset \mathcal{U}$,

$$\forall s, t \in \mathbb{R} : \qquad \partial_t \tau_{t,s} = \tau_{t,s} \circ \delta_t \ , \qquad \tau_{s,s} = \mathbf{1}_\mathcal{U} \ . \tag{111}$$

Here, δ_t, $t \in \mathbb{R}$, *are the conservative closed symmetric derivations, with common core* $\mathcal{U}_0$, *associated to the interactions* $\Psi^{(t)} \in \mathcal{W}$ *and the potentials* $\mathbf{V}^{(t)}$. *See Theorem 3.6.*

Proof: The existence of a strongly continuous two–parameter family $\{\tau_{s,t}\}_{s,t\in\mathbb{R}}$ of $*$–automorphisms satisfying Lieb–Robinson bounds (ii) is a direct consequence of Theorem 4.1 together with the density of $\mathcal{U}_0 \subset \mathcal{U}$ and completeness of $\mathcal{U}$. This limiting family also satisfies the reverse cocycle property (i) because of (101).

(iii) For any $B \in \mathcal{U}_0 \subset \mathrm{Dom}(\delta_t)$, the map $t \mapsto \tau_{t,s} \circ \delta_t(B)$ from $\mathbb{R}$ to $\mathcal{U}$ is continuous. Indeed, for any $B \in \mathcal{U}_0$ and $\alpha, t \in \mathbb{R}$,

$$\|\tau_{\alpha,s} \circ \delta_\alpha (B) - \tau_{t,s} \circ \delta_t (B)\|_{\mathcal{U}} \leq \|(\tau_{\alpha,s} - \tau_{t,s}) \circ \delta_t (B)\|_{\mathcal{U}} + \|\delta_\alpha (B) - \delta_t (B)\|_{\mathcal{U}} .$$

By applying (31) to the interaction $\Psi^{(t)} - \Psi^{(\alpha)}$ and the potential $\mathbf{V}^{(t)} - \mathbf{V}^{(\alpha)}$ together with the strong continuity of $\{\tau_{t,s}\}_{s,t\in\mathbb{R}}$, one sees that, in the limit $\alpha \to t$, the r.h.s of the above inequality vanishes when $B \in \mathcal{U}_0$ and $\Psi \in C(\mathbb{R}; \mathcal{W})$. Now, because of (100), for any $L \in \mathbb{R}_0^+$, $B \in \mathcal{U}_0$, and $s, t \in \mathbb{R}$,

$$\begin{aligned} \left\|\tau_{t,s} (B) - B - \int_s^t \tau_{\alpha,s} \circ \delta_\alpha (B) \mathrm{d}\alpha\right\|_{\mathcal{U}} \leq & \left\|\tau_{t,s} (B) - \tau_{t,s}^{(L)} (B)\right\|_{\mathcal{U}} \\ & + \int_s^t \left\|\left(\tau_{\alpha,s}^{(L)} - \tau_{\alpha,s}\right) \circ \delta_\alpha (B)\right\|_{\mathcal{U}} \mathrm{d}\alpha \\ & + \int_s^t \left\|\delta_\alpha^{(L)} (B) - \delta_\alpha (B)\right\|_{\mathcal{U}} \mathrm{d}\alpha . \end{aligned} \quad (112)$$

By using the strong convergence of $\tau_{t,s}^{(L)}$ towards $\tau_{t,s}$ as well as (31) and (32) together with Lebesgue's dominated convergence theorem, one checks that the r.h.s. of (112) vanishes when $B \in \mathcal{U}_0$ and $L \to \infty$. Because of the continuity of the map $t \mapsto \tau_{t,s} \circ \delta_t(B)$, (111) is verified on the dense set $\mathcal{U}_0 \subset \mathrm{Dom}(\delta_t)$.

To prove uniqueness, assume that $\{\hat{\tau}_{t,s}\}_{s,t\in\mathbb{R}}$ is any family of bounded operators on $\mathcal{U}$ satisfying (111) on $\mathcal{U}_0$. By (99) and because $\tau_{t,s}^{(L)}(B) \in \mathcal{U}_0$ for any $B \in \mathcal{U}_0$,

$$\hat{\tau}_{t,s} (B) - \tau_{t,s}^{(L)} (B) = \int_s^t \hat{\tau}_{\alpha,s} \circ \left(\delta_\alpha - \delta_\alpha^{(L)}\right) \circ \tau_{t,\alpha}^{(L)} (B) \mathrm{d}\alpha \quad (113)$$

for any $B \in \mathcal{U}_0$, $L \in \mathbb{R}_0^+$ and $s, t \in \mathbb{R}$. Similar to (36)–(38), we infer from Theorem 4.1 (i) that, for any $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $B \in \mathcal{U}_\Lambda$, $\alpha, t \in \mathbb{R}$ and sufficiently large $L \in \mathbb{R}_0^+$,

$$\left\|\left(\delta_\alpha - \delta_\alpha^{(L)}\right) \circ \tau_{t,\alpha}^{(L)} (B)\right\|_{\mathcal{U}} \leq \|\Psi\|_\infty \mathrm{e}^{2\mathbf{D}|t-\alpha|\|\Psi\|_\infty} \sum_{y\in\Lambda_L^c} \sum_{x\in\Lambda} \mathbf{F}(|x-y|) .$$

In particular, by (39), for any $B \in \mathcal{U}_0$ and $\alpha, t \in \mathbb{R}$,

$$\lim_{L\to\infty} \left\|\left(\delta_\alpha - \delta_\alpha^{(L)}\right) \circ \tau_{t,\alpha}^{(L)} (B)\right\|_{\mathcal{U}} = 0 \quad (114)$$

uniformly for α on compacta. Because of (113) and $\{\hat{\tau}_{t,s}\}_{s,t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{U})$, we then conclude from (114) that, for every $s,t \in \mathbb{R}$, $\hat{\tau}_{t,s}$ coincides on the dense set $\mathcal{U}_0$ with the limit $\tau_{t,s}$ of $\tau_{t,s}^{(L)}$, as $L \to \infty$. By continuity, $\tau_{t,s} = \hat{\tau}_{t,s}$ on $\mathcal{U}$ for any $s,t \in \mathbb{R}$. ∎

The solution of (111) exists under very weak conditions on interactions and potentials, i.e., their continuity, like in the finite–volume case. It yields a fundamental solution for the states of the interacting lattice fermions driven by the time–dependent interaction $\{\Psi^{(t)}\}_{t\in\mathbb{R}}$. More precisely, for any fixed $\rho_s \in \mathcal{U}^*$ at time $s \in \mathbb{R}$, the family $\{\rho_s \circ \tau_{t,s}\}_{t\in\mathbb{R}}$ solves the following ordinary differential equations, for each $B \in \mathcal{U}_0$:

$$\forall t \in \mathbb{R}: \qquad \partial_t \rho_t(B) = \rho_t \circ \delta_t(B) \ . \tag{115}$$

By Corollary 4.2, the initial value problem on $\mathcal{U}^*$ associated to the above infinite system of ordinary differential equations is *well–posed*. Indeed, the solution of (115) is unique: Take any solution $\{\rho_t\}_{t\in\mathbb{R}}$ of (115) and, similar to (113), use the equality

$$\rho_t(B) - \rho_s \circ \tau_{t,s}^{(L)}(B) = \int_s^t \rho_\alpha \left(\left(\delta_\alpha - \delta_\alpha^{(L)} \right) \circ \tau_{t,\alpha}^{(L)}(B) \right) \mathrm{d}\alpha$$

for any $\rho_s \in \mathcal{U}^*$, $B \in \mathcal{U}_0$, $L \in \mathbb{R}_0^+$ and $s,t \in \mathbb{R}$ together with (114) and the weak*–convergence of $\rho_s \circ \tau_{t,s}^{(L)}$ to $\rho_s \circ \tau_{t,s}$, as $L \to \infty$, by Corollary 4.2.

Note that (111) is the evolution equation one formally obtains for automorphisms of the algebra of observables from the Schrödinger equation, in the non–autonomous case. See also [BPH1, Remark 2.1]. A similar remark can be done for the infinite system (115) of ordinary differential equations.

It is a priori unclear whether $\{\tau_{t,s}\}_{s,t\in\mathbb{R}}$ solves the non–autonomous Cauchy initial value problem

$$\forall s,t \in \mathbb{R}: \qquad \partial_s \tau_{t,s} = -\delta_s \circ \tau_{t,s} \ , \qquad \tau_{t,t} = \mathbf{1}_{\mathcal{U}} \ , \tag{116}$$

on some dense domain. The generators $\{\delta_t\}_{t\in\mathbb{R}}$ are generally unbounded operators acting on $\mathcal{U}$ and their domains can additionally depend on time. No unified theory of such linear evolution equations, similar to the Hille–Yosida generation theorems in the autonomous case, is available. See, e.g., [K3, C, S, P, BB] and the corresponding references therein.

By using Lieb–Robinson bounds for multi–commutators, we show below in Theorem 4.5 that the evolution equation (116) *also holds* on the dense set $\mathcal{U}_0$, under conditions like polynomial decays of interactions and boundedness of the external potential. Another example – more restrictive in which concerns the time–dependency of the generator of dynamics, but less restrictive w.r.t. the behavior at large distances of the potential $\mathbf{V}$ – for which (116) holds is given by Theorem 4.7 (i) in Section 4.3.

4.2 Lieb–Robinson Bounds for Multi–Commutators

As explained in Remark 3.11, all results of Section 3.2 depend on Theorem 3.6 (iii). It is the crucial ingredient we need in order to prove Lemma 3.7, from which we derive Lieb–Robinson bounds for multi–commutators. Theorem 4.1 (ii) together with Corollary 4.2 extend Theorem 3.6 (iii) to time–dependent interactions and potentials. This allows us to prove Lemma 3.7 in the non–autonomous case as well. It is then straightforward to extend Lieb–Robinson bounds for multi–commutators to time–dependent interactions and potentials.

Recall that the proof of Lemma 3.7 uses that the space translated finite–volume groups $\{\tau_t^{(n,x)}\}_{t\in\mathbb{R}}$, $x \in \mathfrak{L}$, have all the same limit $\{\tau_t\}_{t\in\mathbb{R}}$, as $n \to \infty$. This also holds in the non–autonomous case. Indeed, for any $n \in \mathbb{N}_0$, $x \in \mathfrak{L}$, every bounded family $\Psi \doteq \{\Psi^{(t)}\}_{t\in\mathbb{R}}$ on $\mathcal{W}$ (i.e., $\|\Psi\|_\infty < \infty$), and each collection $\{\mathbf{V}^{(t)}\}_{t\in\mathbb{R}}$ of potentials, consider the (space) translated family $\{\tau_{t,s}^{(n,x)}\}_{s,t\in\mathbb{R}}$ of finite–volume $*$–automorphisms generated (cf. (99) and (100)) by the symmetric bounded derivation

$$\delta_t^{(n,x)}(B) \doteq i \sum_{\Lambda\subseteq\Lambda_n+x} \left[\Psi_\Lambda^{(t)}, B\right] + i \sum_{y\in\Lambda_n+x} \left[\mathbf{V}_{\{y\}}^{(t)}, B\right] , \qquad B \in \mathcal{U} .$$

In the autonomous case the strong convergence of these evolution families towards $\{\tau_{t,s}\}_{s,t\in\mathbb{R}}$ easily follows from the second Trotter–Kato approximation theorem [EN, Chap. III, Sect. 4.9]. We use the Lieb–Robinson bound of Theorem 4.1 (i) to prove it in the non–autonomous case:

Lemma 4.3 (Limit of translated dynamics)
Let $\Psi \doteq \{\Psi^{(t)}\}_{t\in\mathbb{R}}$ be a bounded family on $\mathcal{W}$ (i.e., $\|\Psi\|_\infty < \infty$) and $\{\mathbf{V}^{(t)}\}_{t\in\mathbb{R}}$ a collection of potentials. For any $y \in \mathfrak{L}$ and $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, assume $\Psi_\Lambda, \mathbf{V}_{\{y\}} \in C(\mathbb{R};\mathcal{U})$. Then

$$\lim_{n\to\infty} \tau_{t,s}^{(n,x)}(B) = \tau_{t,s}(B) , \qquad B \in \mathcal{U} , \ x \in \mathfrak{L}, \ s,t \in \mathbb{R} .$$

Proof: For any $n \in \mathbb{N}_0$ and $x \in \mathfrak{L}$, the translated finite–volume family $\{\tau_{s,t}^{(n,x)}\}_{s,t\in\mathbb{R}}$ solves non–autonomous evolution equations like (99)–(100). Therefore, similar to (113), for any $n \in \mathbb{N}_0$, $x \in \mathfrak{L}$, $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $B \in \mathcal{U}_\Lambda$ and $s, t \in \mathbb{R}$,

$$\tau_{t,s}^{(n,x)}(B) - \tau_{t,s}^{(n,0)}(B) = \int_s^t \tau_{\alpha,s}^{(n,x)} \circ \left(\delta_\alpha^{(n,x)} - \delta_\alpha^{(n,0)}\right) \circ \tau_{t,\alpha}^{(n,0)}(B) \, \mathrm{d}\alpha \, . \tag{117}$$

For sufficiently large $n \in \mathbb{N}_0$ such that $\Lambda \subset (\Lambda_n + x) \cap \Lambda_n$, note that

$$\left\| \left(\delta_\alpha^{(n,x)} - \delta_\alpha^{(n,0)}\right) \circ \tau_{t,\alpha}^{(n,0)}(B) \right\|_{\mathcal{U}} \leq \sum_{\mathcal{Z}\in\mathcal{P}_f(\mathfrak{L}),\ \mathcal{Z}\cap((\Lambda_n+x)^c\cup\Lambda_n^c)\neq\emptyset} \left\| \left[\Psi_\Lambda^{(t)}, \tau_{t,\alpha}^{(n,0)}(B)\right] \right\|_{\mathcal{U}}$$

with $\mathcal{Z}^c \doteq \mathfrak{L}\backslash\mathcal{Z}$ being the complement of any set $\mathcal{Z} \in \mathcal{P}_f(\mathfrak{L})$. Then, similar to Inequality (38), by using Theorem 4.1 (i), one verifies that, for any $x \in \mathfrak{L}$, $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $B \in \mathcal{U}_\Lambda$, $\alpha, t \in \mathbb{R}$, and sufficiently large $n \in \mathbb{N}_0$,

$$\begin{aligned} & \left\| \left(\delta_\alpha^{(n,x)} - \delta_\alpha^{(n,0)}\right) \circ \tau_{t,\alpha}^{(n,0)}(B) \right\|_{\mathcal{U}} \\ \leq \ & 2 \|B\|_{\mathcal{U}} \|\Psi\|_\infty \ \mathrm{e}^{2\mathbf{D}|t-\alpha|\|\Psi\|_\infty} \sum_{y\in(\Lambda_n+x)^c\cup\Lambda_n^c} \sum_{z\in\Lambda} \mathbf{F}(|z-y|) \ , \end{aligned} \tag{118}$$

while

$$\lim_{n\to\infty} \sum_{y\in(\Lambda_n+x)^c\cup\Lambda_n^c} \sum_{z\in\Lambda} \mathbf{F}(|z-y|) = 0 \, , \tag{119}$$

because of (6). We thus deduce from (118)–(119) that

$$\lim_{n\to\infty} \left\| \left(\delta_\alpha^{(n,x)} - \delta_\alpha^{(n,0)}\right) \circ \tau_{t,\alpha}^{(n,0)}(B) \right\|_{\mathcal{U}} = 0$$

uniformly for α on compacta. Combined with (117) and Corollary 4.2, this uniform limit implies the assertion. ∎

With the above result and the introducing remarks of this subsection, it is now straightforward to extend Theorem 3.8 to the non–autonomous case:

Theorem 4.4 (Lieb–Robinson bounds for multi–commutators – Part I)
Let $\Psi \doteq \{\Psi^{(t)}\}_{t\in\mathbb{R}}$ *be a bounded family on* $\mathcal{W}$ *(i.e.,* $\|\Psi\|_\infty < \infty$ *),* $\{\mathbf{V}^{(t)}\}_{t\in\mathbb{R}}$ *a collection of potentials, and* $s \in \mathbb{R}$*. For any* $y \in \mathfrak{L}$ *and* $\Lambda \in \mathcal{P}_f(\mathfrak{L})$*, assume* $\Psi_\Lambda, \mathbf{V}_{\{y\}} \in C(\mathbb{R};\mathcal{U})$*. Then, for any integer* $k \in \mathbb{N}$*,* $\{m_j\}_{j=0}^k \subset \mathbb{N}_0$*, times*

$\{s_j\}_{j=1}^k \subset \mathbb{R}$, lattice sites $\{x_j\}_{j=0}^k \subset \mathfrak{L}$, and elements $B_0 \in \mathcal{U}_0$, $\{B_j\}_{j=1}^k \subset \mathcal{U}_0 \cap \mathcal{U}^+$ such that $B_j \in \mathcal{U}_{\Lambda_{m_j}}$ for $j \in \{0, \ldots, k\}$,

$$\begin{aligned} & \left\| \left[\tau_{s_k,s} \circ \chi_{x_k}(B_k), \ldots, \tau_{s_1,s} \circ \chi_{x_1}(B_1), \chi_{x_0}(B_0) \right]^{(k+1)} \right\|_{\mathcal{U}} \\ \leq & \;\; 2^k \prod_{j=0}^{k} \|B_j\|_{\mathcal{U}} \sum_{T \in \mathcal{T}_{k+1}} \left(\varkappa_T \left(\{(m_j, x_j)\}_{j=0}^k \right) + \Re_{T, \|\Psi\|_\infty} \right) , \end{aligned}$$

where $\varkappa_T$ and $\Re_{T,\alpha}$ are respectively defined by (51) and (53) for $T \in \mathcal{T}_{k+1}$ and $\alpha \in \mathbb{R}_0^+$, the times $\{s_j\}_{j=1}^k$ in (53) being replaced with $\{(s_j - s)\}_{j=1}^k$.

Proof: One easily checks that Theorem 4.1 (ii) holds for $\{\tau_{t,s}^{(n,x)}\}_{s,t\in\mathbb{R}}$ at any fixed $x \in \mathfrak{L}$ and $n \in \mathbb{N}_0$. By Lemma 4.3, Lemma 3.7 also holds in the non–autonomous case and the assertion follows from (57) with the $*$–automorphism τ_{s_j} being replaced by $\tau_{s_j,s}$ for every $j \in \{1, \ldots, k\}$. ■

By Theorems 3.9 and 4.4, we obtain Lieb–Robinson bounds for multi–commutators as well as a version of Corollary 3.10 in the *non–autonomous* case. I.e., interacting and non–autonomous systems also satisfy the so–called tree–decay bounds.

Another application of Theorems 3.9 and 4.4 is a proof of existence of a fundamental solution for the non–autonomous abstract Cauchy initial value problem for observables

$$\forall s \in \mathbb{R} : \qquad \partial_s B_s = -\delta_s(B_s) \ , \qquad B_t = B \in \mathcal{U}_0 \ , \tag{120}$$

in the Banach space $\mathcal{U}$, i.e., a proof of existence of a solution of the evolution equation (116). The latter is a non–trivial statement, as previously discussed, among other things because the domain of δ_s depends, in general, on the time $s \in \mathbb{R}$. [Here, $t \in \mathbb{R}$ is the "initial" time.]

To this end, like in (80)–(83), we add the following condition on interactions Φ:

- *Polynomial decay.* Assume (59) and the existence of constants $\upsilon, D \in \mathbb{R}^+$ such that

$$\sup_{x\in\mathfrak{L}} \sum_{\Lambda \in \mathcal{D}(x,m)} \|\Phi_\Lambda\|_{\mathcal{U}} \leq D\,(m+1)^{-\upsilon} \ , \qquad m \in \mathbb{N}_0 \ , \tag{121}$$

while the sequence $\{\mathbf{u}_{n,m}\}_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$ of (59) satisfies

$$\sum_{m,n\in\mathbb{N}} m^{-\upsilon} |\mathbf{u}_{n,m}| < \infty \ . \tag{122}$$

As $\mathbf{F}(|x|) > 0$ for all $x \in \mathfrak{L}$, note that (59) implies

$$\sum_{n\in\mathbb{N}} |\mathbf{u}_{n,m}| \geq Dm^{\varsigma}$$

for some $D \in \mathbb{R}^+$ and all $m \in \mathbb{N}_0$. Hence, the inequality (122) imposes $\upsilon > \varsigma + 1$.

Then, one gets the following assertion:

Theorem 4.5 (Infinite dynamics and non–autonomous evolution equations) *Let $\Psi \doteq \{\Psi^{(t)}\}_{t\in\mathbb{R}} \in C(\mathbb{R};\mathcal{W})$ be a bounded family on $\mathcal{W}$ (i.e., $\|\Psi\|_\infty < \infty$) and $\{\mathbf{V}^{(t)}_{\{x\}}\}_{x\in\mathfrak{L},t\in\mathbb{R}}$ a bounded family on $\mathcal{U}$ of potentials with $\mathbf{V}_{\{x\}} \in C(\mathbb{R};\mathcal{U})$ for any $x \in \mathfrak{L}$. Assume (59) with $\varsigma > 2d$ and that (121)–(122) with $\Phi = \Psi^{(t)}$ and $\nu > \varsigma + 1$ hold uniformly for $t \in \mathbb{R}$. Then, for any $s,t \in \mathbb{R}$, $\tau_{t,s}(\mathcal{U}_0) \subset \mathrm{Dom}(\delta_s)$ and $\{\tau_{t,s}\}_{s,t\in\mathbb{R}}$ solves the non–autonomous evolution equation*

$$\forall s,t \in \mathbb{R}: \qquad \partial_s \tau_{t,s} = -\delta_s \circ \tau_{t,s} \ , \qquad \tau_{t,t} = \mathbf{1}_{\mathcal{U}} \ , \tag{123}$$

in the strong sense on the dense set $\mathcal{U}_0$.

Proof: **1.** Let $s,t \in \mathbb{R}$, $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ and take any element $B \in \mathcal{U}_\Lambda$. As a preliminary step, we prove that $\{\delta_s \circ \tau^{(L)}_{t,s}(B)\}_{L\in\mathbb{R}_0^+}$ converges to $\delta_s \circ \tau_{t,s}(B)$, as $L \to \infty$. In particular, $\tau_{t,s}(\mathcal{U}_0) \subset \mathrm{Dom}(\delta_s)$. By using similar arguments as in the proof of Theorem 4.1 (ii), it suffices to study the limit of $\{\delta_s \circ \tilde{\tau}^{(L)}_{t,s}(B)\}_{L\in\mathbb{R}_0^+}$, see (108).

Similar to (26), from (104)–(109) and straightforward computations, for any $L_1, L_2 \in \mathbb{R}_0^+$ with $\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$,

$$\begin{aligned} &\left\| \delta_s \circ \left(\tilde{\tau}^{(L_2)}_{t,s}(B) - \tilde{\tau}^{(L_1)}_{t,s}(B) \right) \right\|_{\mathcal{U}} \\ &\leq \int_{\min\{s,t\}}^{\max\{s,t\}} \sum_{\mathcal{Z}\in\mathcal{P}_f(\mathfrak{L})} \left\| \left[\hat{\tau}^{(L_1,L_2)}_{s,\alpha}(\Psi^{(s)}_{\mathcal{Z}}), B^{(L_1,L_2)}_\alpha, \tau^{(L_1)}_{t,\alpha}(\tilde{B}_t) \right]^{(3)} \right\|_{\mathcal{U}} \mathrm{d}\alpha \ , \end{aligned} \tag{124}$$

where $\tilde{B}_t \doteq \mathcal{V}_{t,s}(\mathbf{V}_\Lambda) B \mathcal{V}_{s,t}(\mathbf{V}_\Lambda)$,

$$\hat{\tau}^{(L_1,L_2)}_{s,\alpha}(B) \doteq \mathcal{V}_{s,\alpha}(\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}}) \tau^{(L_2)}_{s,\alpha}(B) \mathcal{V}_{\alpha,s}(\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}}) \ , \quad B \in \mathcal{U}, \ s,\alpha \in \mathbb{R} \ , \tag{125}$$

and

$$B_{\alpha}^{(L_1,L_2)} \doteq \sum_{\mathcal{Z}\subseteq\Lambda_{L_2},\ \mathcal{Z}\cap(\Lambda_{L_2}\backslash\Lambda_{L_1})\neq\emptyset} \mathcal{V}_{\alpha,s}(\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}})\Psi_{\mathcal{Z}}\mathcal{V}_{s,\alpha}(\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}}) \in \mathcal{U}^+\cap\mathcal{U}_{\Lambda_{L_2}} .$$

Using (110), observe that, for all $\mathcal{Z} \subseteq \Lambda_{L_2}$ and $\alpha, s \in \mathbb{R}$,

$$\mathcal{V}_{\alpha,s}(\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}})\Psi_{\mathcal{Z}}\mathcal{V}_{s,\alpha}(\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}}) \in \mathcal{U}^+ \cap \mathcal{U}_{\mathcal{Z}} \tag{126}$$

with

$$\left\|\mathcal{V}_{\alpha,s}(\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}})\Psi_{\mathcal{Z}}\mathcal{V}_{s,\alpha}(\mathbf{V}_{\Lambda_{L_2}\backslash\Lambda_{L_1}})\right\|_{\mathcal{U}} = \|\Psi_{\mathcal{Z}}\|_{\mathcal{U}} \ . \tag{127}$$

Similarly, for all $t \in \mathbb{R}$,

$$\tilde{B}_t \in \mathcal{U}_\Lambda \qquad \text{and} \qquad \|\tilde{B}_t\|_{\mathcal{U}} = \|B\|_{\mathcal{U}} \ . \tag{128}$$

In order to bound the sum

$$\sum_{\mathcal{Z}\in\mathcal{P}_f(\mathfrak{L})} \left[\hat{\tau}_{s,\alpha}^{(L_1,L_2)}(\Psi_{\mathcal{Z}}^{(s)}), B_{\alpha}^{(L_1,L_2)}, \tau_{t,\alpha}^{(L_1)}(\tilde{B}_t)\right]^{(3)} \tag{129}$$

of multi–commutators of order three we represent it as a convenient series, whose summability is uniform w.r.t. $L_1, L_2 \in \mathbb{R}_0^+$ ($\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$). To this end, first develop $\tau_{t,\alpha}^{(L_1)}(\tilde{B}_t)$ as a telescoping series: Let $m_0 \in \mathbb{N}_0$ be the smallest integer such that $\Lambda \subset \Lambda_{m_0}$. Then, similar to Lemma 3.7 (autonomous case) and as explained in the proof of Theorem 4.4, for any $\alpha, t \in \mathbb{R}$ and $L_1 \in \mathbb{R}_0^+$,

$$\tau_{t,\alpha}^{(L_1)}(\tilde{B}_t) = \sum_{n=m_0}^{\infty} \tilde{\mathfrak{B}}_{t,\alpha}(n) \ .$$

Here, for all integers $n \geq m_0$, $\tilde{\mathfrak{B}}_{t,\alpha}(n) \in \mathcal{U}_{\Lambda_n}$ where $\|\tilde{\mathfrak{B}}_{t,\alpha}(m_0)\|_{\mathcal{U}} = \|B\|_{\mathcal{U}}$ (see (128)) and, for all $n \in \mathbb{N}$ with $n > m_0$,

$$\|\tilde{\mathfrak{B}}_{t,\alpha}(n)\|_{\mathcal{U}} \leq 2\|B\|_{\mathcal{U}} \|\Psi\|_{\infty} |t-\alpha| \, \mathrm{e}^{4\mathbf{D}|t-\alpha|\|\Psi\|_{\infty}} \frac{\mathbf{u}_{n,m_0}}{(1+n)^{\varsigma}} \ , \tag{130}$$

by Theorem 4.1 (ii) and Assumption (59). Of course, $\tilde{\mathfrak{B}}_{t,\alpha}(n) = 0$ for any integer $n > L_1$ and $\alpha, t \in \mathbb{R}$ because $\{\tau_{t,s}^{(L_1)}\}_{s,t\in\mathbb{R}}$ is a finite–volume dynamics. Meanwhile, because of (110), Theorem 4.4 holds by replacing $\{\tau_{t,s}\}_{s,t\in\mathbb{R}}$ with $\{\hat{\tau}_{t,s}^{(L_1,L_2)}\}_{s,t\in\mathbb{R}}$ at sufficiently large $L_1, L_2 \in \mathbb{R}_0^+$ ($\Lambda_{L_1} \subsetneq \Lambda_{L_2}$). Using this together with (121)–(122) for $\Phi = \Psi^{(t)}$, Equations (126)–(128), Theorem 3.9, as well as the assumptions $\nu > \varsigma + 1$ and $\varsigma > 2d$,

$$\sum_{n_0=m_0}^{\infty} \sum_{x_2 \in \mathfrak{L}} \sum_{m_2 \in \mathbb{N}_0} \sum_{\mathcal{Z}_2 \in \mathcal{D}(x_2,m)} \sum_{x_1 \in \mathfrak{L}} \sum_{m_1 \in \mathbb{N}_0} \sum_{\mathcal{Z}_1 \in \mathcal{D}(x_1,m)} \tag{131}$$

$$\left\| \left[\hat{\tau}_{s,\alpha}^{(L_1,L_2)}(\Psi_{\mathcal{Z}_2}^{(s)}), \mathcal{V}_{\alpha,s}(\mathbf{V}_{\Lambda_{L_2} \backslash \Lambda_{L_1}}) \Psi_{\mathcal{Z}_1} \mathcal{V}_{s,\alpha}(\mathbf{V}_{\Lambda_{L_2} \backslash \Lambda_{L_1}}), \tilde{\mathfrak{B}}_{t,\alpha}(n) \right]^{(3)} \right\|_{\mathcal{U}}$$

$$\leq D \|B\|_{\mathcal{U}} \|\mathbf{u}_{\cdot,m_0}\|_{\ell^1(\mathbb{N})} \left(\sum_{m_1 \in \mathbb{N}_0} (m_1+1)^{\varsigma - \upsilon} \right)$$

$$\times \sum_{m_2 \in \mathbb{N}_0} (m_2+1)^{-\upsilon} \left(\sum_{n_2 \in \mathbb{N}} \mathbf{u}_{n_2,m_2} + (m_2+1)^{\varsigma} \right) < \infty .$$

Similar to (130) and because (121)–(122) with $\Phi = \Psi^{(t)}$ hold uniformly for $t \in \mathbb{R}$, the strictly positive constant $D \in \mathbb{R}^+$ is uniformly bounded for s, t, α on compacta and $L_1, L_2 \in \mathbb{R}_0^+$ ($\Lambda \subset \Lambda_{L_1} \subsetneq \Lambda_{L_2}$). The last sum is an *upper bound* of the integrand of the r.h.s. of (124). Indeed, we deduce from (82) that

$$B_{\alpha}^{(L_1,L_2)} = \sum_{x \in \Lambda_{L_2} \backslash \Lambda_{L_1}} \sum_{m \in \mathbb{N}_0} \sum_{\mathcal{Z} \subseteq \Lambda_{L_2},\, \mathcal{Z} \in \mathcal{D}(x,m)} \frac{1}{|\mathcal{Z} \cap \Lambda_{L_2} \backslash \Lambda_{L_1}|} \mathcal{V}_{\alpha,s}(\mathbf{V}_{\Lambda_{L_2} \backslash \Lambda_{L_1}}) \Psi_{\mathcal{Z}} \mathcal{V}_{s,\alpha}(\mathbf{V}_{\Lambda_{L_2} \backslash \Lambda_{L_1}})$$

and

$$\sum_{\mathcal{Z} \in \mathcal{P}_f(\mathfrak{L}),\, \mathcal{Z} \cap \Lambda_{L_2} \neq \emptyset} \hat{\tau}_{s,\alpha}^{(L_1,L_2)}(\Psi_{\mathcal{Z}}^{(s)}) = \sum_{x \in \Lambda_{L_2}} \sum_{m \in \mathbb{N}_0} \sum_{\mathcal{Z} \in \mathcal{D}(x,m)} \frac{1}{|\mathcal{Z} \cap \Lambda_{L_2}|} \hat{\tau}_{s,\alpha}^{(L_1,L_2)}(\Psi_{\mathcal{Z}}^{(s)}) .$$

[Compare this last sum with (129) by using (126) and (128) to restrict the whole sum over $\mathcal{Z} \in \mathcal{P}_f(\mathfrak{L})$ to finite sets $\mathcal{Z}$ so that $\mathcal{Z} \cap \Lambda_{L_2} \neq \emptyset$.]

As a consequence, for any $s, t \in \mathbb{R}$ and $B \in \mathcal{U}_0$, we infer from (124), (131), and Lebesgue's dominated convergence theorem that $\{\delta_s \circ \tilde{\tau}_{t,s}^{(L)}(B)\}_{L \in \mathbb{R}_0^+}$, and hence $\{\delta_s \circ \tau_{t,s}^{(L)}(B)\}_{L \in \mathbb{R}_0^+}$, are Cauchy nets within the complete space $\mathcal{U}$. By Corollary 4.2, $\{\tau_{t,s}^{(L)}\}_{L \in \mathbb{R}_0^+}$ converges strongly to $\tau_{t,s}$ for every $s, t \in \mathbb{R}$. Recall meanwhile that the operator δ_s is the *closed* operator described in Theorem 3.6 for the interaction $\Psi^{(s)} \in \mathcal{W}$ and the potential $\mathbf{V}^{(s)}$ at fixed $s \in \mathbb{R}$. Therefore,

$\tau_{t,s}(B) \in \mathrm{Dom}(\delta_s)$ and the family $\{\delta_s \circ \tau_{t,s}^{(L)}(B)\}_{L\in\mathbb{R}_0^+}$ converges to $\delta_s \circ \tau_{t,s}(B)$, i.e.,

$$\lim_{L\to\infty} \left\| \delta_s \circ \left(\tau_{t,s}(B) - \tau_{t,s}^{(L)}(B) \right) \right\|_{\mathcal{U}} = 0 \ . \tag{132}$$

In particular, $\tau_{t,s}(\mathcal{U}_0) \subset \mathrm{Dom}(\delta_s)$.

Now, by using (99) one gets that, for $L \in \mathbb{R}_0^+$, $s,t,h \in \mathbb{R}$, $h \neq 0$, and $B \in \mathcal{U}_0$,

$$\begin{aligned} & \left\| |h|^{-1} \left(\tau_{t,s+h}(B) - \tau_{t,s}(B) \right) + \delta_s \circ \tau_{t,s}(B) \right\|_{\mathcal{U}} \\ \leq \ & \left\| \delta_s \circ \left(\tau_{t,s}(B) - \tau_{t,s}^{(L)}(B) \right) \right\|_{\mathcal{U}} \\ & + \sup_{\alpha\in[s-|h|,s+|h|]} \left\| \delta_s^{(L)} \circ \tau_{t,s}^{(L)}(B) - \delta_\alpha^{(L)} \circ \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}} \\ & + \left\| \left(\delta_s^{(L)} - \delta_s \right) \circ \tau_{t,s}^{(L)}(B) \right\|_{\mathcal{U}} \\ & + 2|h|^{-1} \sup_{\alpha\in[s-|h|,s+|h|]} \left\| \tau_{t,\alpha}(B) - \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}} \ . \end{aligned} \tag{133}$$

We proceed by estimating the four terms in the upper bound of (133). The first one is already analyzed, see (132). So, we start with the second. If nothing is explicitly mentioned, the parameters $L \in \mathbb{R}_0^+$, $s,t,h \in \mathbb{R}$, $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ and $B \in \mathcal{U}_\Lambda$ are fixed.

2. For any $\alpha \in \mathbb{R}$, observe that

$$\begin{aligned} \left\| \delta_s^{(L)} \circ \tau_{t,s}^{(L)}(B) - \delta_\alpha^{(L)} \circ \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}} \leq \ & \left\| \left(\delta_s^{(L)} - \delta_\alpha^{(L)} \right) \circ \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}} \\ & + \left\| \delta_s^{(L)} \circ \left(\tau_{t,s}^{(L)} - \tau_{t,\alpha}^{(L)} \right)(B) \right\|_{\mathcal{U}} \ . \end{aligned} \tag{134}$$

By using first (30) for the interaction $\Psi^{(s)}$ and potential $\mathbf{V}^{(s)}$ and then Lieb–Robinson bounds (Theorem 4.1 (i)) in the same way as (38), one verifies that, for any $\alpha \in \mathbb{R}$ and $B \neq 0$,

$$\begin{aligned} & \frac{\left\| \left(\delta_s^{(L)} - \delta_\alpha^{(L)} \right) \circ \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}}}{2\|B\|_{\mathcal{U}}} \\ \leq \ & \left\| \Psi^{(s)} - \Psi^{(\alpha)} \right\|_{\mathcal{W}} \mathrm{e}^{2\mathbf{D}|t-\alpha|\|\Psi\|_\infty} |\Lambda| \, \|\mathbf{F}\|_{1,\mathfrak{L}} + \sum_{x\in\Lambda} \|\mathbf{V}_{\{x\}}^{(\alpha)} - \mathbf{V}_{\{x\}}^{(s)}\|_{\mathcal{U}} \\ & + \mathbf{D}^{-1} \left(\mathrm{e}^{2\mathbf{D}|t-\alpha|\|\Psi\|_\infty} - 1 \right) \sum_{x\in\mathfrak{L}\backslash\Lambda} \|\mathbf{V}_{\{x\}}^{(\alpha)} - \mathbf{V}_{\{x\}}^{(s)}\|_{\mathcal{U}} \sum_{y\in\Lambda} \mathbf{F}(|x-y|) \ . \end{aligned} \tag{135}$$

By assumption, $\Psi \in C(\mathbb{R};\mathcal{W})$, $\{\mathbf{V}_{\{x\}}^{(t)}\}_{x\in\mathfrak{L},t\in\mathbb{R}}$ is a bounded family in $\mathcal{U}$, and $\mathbf{V}_{\{x\}} \in C(\mathbb{R};\mathcal{U})$ for any $x \in \mathfrak{L}$. So, by Lebesgue's dominated convergence theorem, it follows from (135) that

$$\lim_{h\to 0} \sup_{\alpha\in[s-|h|,s+|h|]} \left\| \left(\delta_s^{(L)} - \delta_\alpha^{(L)}\right) \circ \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}} = 0 \,. \tag{136}$$

On the other hand, by (99),

$$\sup_{\alpha\in[s-|h|,s+|h|]} \left\| \delta_s^{(L)} \circ \left(\tau_{t,s}^{(L)} - \tau_{t,\alpha}^{(L)}\right)(B) \right\|_{\mathcal{U}} \leq \int_{s-|h|}^{s+|h|} \left\| \delta_s^{(L)} \circ \delta_\alpha^{(L)} \circ \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}} \mathrm{d}\alpha \,, \tag{137}$$

where

$$\begin{aligned} \left\| \delta_s^{(L)} \circ \delta_\alpha^{(L)} \circ \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}} \leq & \sum_{\mathcal{Z}_1,\mathcal{Z}_2\in\mathcal{P}_f(\mathfrak{L})} \left\| \left[\Psi_{\mathcal{Z}_1}^{(s)}, \Psi_{\mathcal{Z}_2}^{(\alpha)}, \tau_{t,\alpha}^{(L)}(B)\right]^{(3)} \right\|_{\mathcal{U}} \\ & + \sum_{\mathcal{Z}\in\mathcal{P}_f(\mathfrak{L})} \sum_{x\in\mathfrak{L}} \left\| \left[\Psi_{\mathcal{Z}}^{(s)}, \mathbf{V}_{\{x\}}^{(\alpha)}, \tau_{t,\alpha}^{(L)}(B)\right]^{(3)} \right\|_{\mathcal{U}} \\ & + \sum_{\mathcal{Z}\in\mathcal{P}_f(\mathfrak{L})} \sum_{x\in\mathfrak{L}} \left\| \left[\mathbf{V}_{\{x\}}^{(s)}, \Psi_{\mathcal{Z}}^{(\alpha)}, \tau_{t,\alpha}^{(L)}(B)\right]^{(3)} \right\|_{\mathcal{U}} \\ & + \sum_{x,y\in\mathfrak{L}} \left\| \left[\mathbf{V}_{\{x\}}^{(s)}, \mathbf{V}_{\{y\}}^{(\alpha)}, \tau_{t,\alpha}^{(L)}(B)\right]^{(3)} \right\|_{\mathcal{U}} . \end{aligned} \tag{138}$$

Similar to (131), we use Theorems 3.9 (i) and 4.4 for $k = 2$ to derive an upper bound for the r.h.s. of (138), uniformly w.r.t. large $L \in \mathbb{R}_0^+$ and $\alpha \in [s-1, s+1]$. By (137), it follows that

$$\lim_{h\to 0} \sup_{\alpha\in[s-|h|,s+|h|]} \left\| \delta_s^{(L)} \circ \left(\tau_{t,s}^{(L)} - \tau_{t,\alpha}^{(L)}\right)(B) \right\|_{\mathcal{U}} = 0 \,.$$

Combined with (134) and (136) this yields

$$\lim_{h\to 0} \sup_{\alpha\in[s-|h|,s+|h|]} \left\| \delta_s^{(L)} \circ \tau_{t,s}^{(L)}(B) - \delta_\alpha^{(L)} \circ \tau_{t,\alpha}^{(L)}(B) \right\|_{\mathcal{U}} = 0 \,. \tag{139}$$

3. Similar to (38), one gets from Lieb–Robinson bounds (Theorem 4.1 (i)) that

$$\left\| \left(\delta_s^{(L)} - \delta_s\right) \circ \tau_{t,s}^{(L)}(B) \right\|_{\mathcal{U}} \leq \|\Psi\|_\infty \; \mathrm{e}^{2\mathbf{D}|t-s|\|\Psi\|_\infty} \sum_{y\in\Lambda_L^c} \sum_{x\in\Lambda} \mathbf{F}(|x-y|) \;,$$

which combined with (39) gives

$$\lim_{L\to\infty}\left\|\left(\delta_s^{(L)}-\delta_s\right)\circ\tau_{t,s}^{(L)}(B)\right\|_{\mathcal{U}}=0\,. \tag{140}$$

4. In the limit $h\to 0$, we take $L_h\to\infty$ such that

$$\lim_{h\to 0}|h|^{-1}\sup_{\alpha\in[s-|h|,s+|h|]}\left\|\tau_{t,\alpha}(B)-\tau_{t,\alpha}^{(L_h)}(B)\right\|_{\mathcal{U}}=0\,. \tag{141}$$

This is possible because $\tau_{t,s}^{(L)}(B)$ converges to $\tau_{t,s}(B)$, uniformly for t,s on compacta, by Corollary 4.2. We eventually combine (132), (139), (140), and (141) with Inequality (133) to arrive at the assertion. ■

Note that uniqueness of the solution of the non–autonomous evolution equation (123) cannot be proven as done for the proof of uniqueness in Corollary 4.2 (iii). Indeed, take any family $\{\hat{\tau}_{t,s}\}_{s,t\in\mathbb{R}}$ of bounded operators on $\mathcal{U}$ satisfying (123) on $\mathcal{U}_0$. Then, as before in the proof of Corollary 4.2 (iii), for any $B\in\mathcal{U}_0$, $L\in\mathbb{R}_0^+$ and $s,t\in\mathbb{R}$,

$$\tau_{t,s}^{(L)}(B)-\hat{\tau}_{t,s}(B)=\int_s^t\tau_{\alpha,s}^{(L)}\circ\left(\delta_\alpha^{(L)}-\delta_\alpha\right)\circ\hat{\tau}_{t,\alpha}(B)\,\mathrm{d}\alpha\,, \tag{142}$$

by using (100). However, it is not clear this time whether the norm

$$\left\|\tau_{\alpha,s}^{(L)}\circ\left(\delta_\alpha^{(L)}-\delta_\alpha\right)\circ\hat{\tau}_{t,\alpha}(B)\right\|_{\mathcal{U}}=\left\|\left(\delta_\alpha-\delta_\alpha^{(L)}\right)\circ\hat{\tau}_{t,\alpha}(B)\right\|_{\mathcal{U}}$$

vanishes, as $L\to\infty$, even if (32) for δ_α and $\delta_\alpha^{(L)}$ holds true, because $\hat{\tau}_{t,\alpha}(B)\in\mathrm{Dom}(\delta_\alpha)$ can be *outside* $\mathcal{U}_0$. The strong convergence of $\delta_\alpha^{(L)}$ to δ_α on some core of δ_α does not imply, in general, the strong convergence on any core of δ_α. The equality (142) with $\tau_{t,s},\delta_s$ replacing $\tau_{t,s}^{(L)},\delta_s^{(L)}$ is also not clear because (111) only known to hold true on $\mathcal{U}_0$ and *a priori not* on the whole domain $\mathrm{Dom}(\delta_\alpha)$ of δ_α.

The non–autonomous evolution equation (120) of Theorem 4.5 is not parabolic because the symmetric derivation δ_t, $t\in\mathbb{R}$, is generally *not* the generator of an analytic semigroup. Note also that no Hölder continuity condition is imposed on $\{\delta_t\}_{t\in\mathbb{R}}$, like in the class of parabolic evolution equations introduced in [AT, Hypotheses I–II]. See also [S] or [P, Sect. 5.6.] for more simplified studies.

In fact, (120) is rather related to Kato's *hyperbolic* evolution equations [K1, K2, K3]. The so–called *Kato quasi–stability* is satisfied by the family of generators $\{\delta_t\}_{t\in\mathbb{R}}$ because they are always dissipative operators, by Lemma 3.3. $\{\delta_t\}_{t\in\mathbb{R}}$

is also strongly continuous on the dense set $\mathcal{U}_0$, which is a common core of all δ_t, $t \in \mathbb{R}$. However, in general, even for *finite* range interactions $\Psi \in \mathcal{W}$, the strongly continuous two–parameter family $\{\tau_{t,s}\}_{s,t\in\mathbb{R}}$ does *not* conserve the dense set $\mathcal{U}_0$, i.e., $\tau_{t,s}(\mathcal{U}_0) \nsubseteq \mathcal{U}_0$ for any $s \neq t$. In some specific situations one can directly show that the completion of the core $\mathcal{U}_0$ w.r.t. a conveniently chosen norm defines a so–called admissible Banach space $\mathcal{Y} \supset \mathcal{U}_0$ of the generator at any time, which satisfies further technical conditions leading to Kato's hyperbolic conditions [K1, K2, K3]. See also [P, Sect. 5.3.] and [BB, Sect. VII.1], which is used in the proof of Theorem 4.7 (i). Nevertheless, the existence of such a Banach space $\mathcal{Y}$ is a priori unclear in the general case treated in Theorem 4.5. See for instance the uniqueness problem explained just above.

Note that we only assume in Theorem 4.5 some polynomial decay for the interaction with (59) and (121)–(122) (uniformly in time). Recall that these assumptions are fulfilled for any interaction $\Psi \in \mathcal{W}$ with (8), provided the parameter $\epsilon \in \mathbb{R}^+$ is sufficiently large. In the case of exponential decays, stronger results can be deduced from Lieb–Robinson bounds for multi–commutators. For the interested reader, we give below one example, which is based on interactions Φ satisfying the following condition:

- *Exponential decay.* Assume (60) and the existence of constants $\upsilon > \varsigma$ and $D \in \mathbb{R}^+$ such that

$$\sup_{x\in\mathfrak{L}} \sum_{\Lambda\in\mathcal{D}(x,m)} \|\Phi_\Lambda\|_{\mathcal{U}} \leq D\mathrm{e}^{-\upsilon m} , \qquad m \in \mathbb{N}_0 , \tag{143}$$

while

$$\sum_{m\in\mathbb{N}} \mathbf{C}_m \mathrm{e}^{-(\varsigma+\upsilon)m} < \infty . \tag{144}$$

Theorem 4.6 (Graph norm convergence and Gevrey vectors)
Let $\Psi \doteq \{\Psi^{(t)}\}_{t\in\mathbb{R}}$ be a bounded family on $\mathcal{W}$ (i.e., $\|\Psi\|_\infty < \infty$), $\{\mathbf{V}^{(t)}\}_{t\in\mathbb{R}}$ a collection of potentials, and $B \in \mathcal{U}_0$. For any $x \in \mathfrak{L}$ and $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $\Psi_\Lambda, \mathbf{V}_{\{x\}} \in C(\mathbb{R};\mathcal{U})$. Assume that (60) and (143)–(144) hold for $\Phi = \Psi^{(t)}$, uniformly in time.
(i) *Graph norm convergence. As $L \to \infty$, $\tau_{t,s}^{(L)}(B)$ converges, uniformly for s,t on compacta, to $\tau_{t,s}(B)$ within the normed space $(\mathrm{Dom}(\delta_s^m), \|\cdot\|_{\delta_s^m})$, where, for all $m \in \mathbb{N}_0$, $\|\cdot\|_{\delta_s^m}$ stands for the graph norm of the densely defined operator δ_s^m.*
(ii) *Gevrey vectors. If $\{\mathbf{V}_{\{x\}}^{(t)}\}_{x\in\mathfrak{L},t\in\mathbb{R}}$ is a bounded family on $\mathcal{U}$ then, for any*

$\mathrm{T} \in \mathbb{R}_0^+$, *there exist* $r \equiv r_{d,\mathrm{T},\Psi,\mathbf{V},\mathbf{F}} \in \mathbb{R}^+$ *and* $D \equiv D_{\mathrm{T},\Psi,\mathbf{V}} \in \mathbb{R}^+$ *such that, for all* $s,t \in [-\mathrm{T},\mathrm{T}]$, $m_0 \in \mathbb{N}_0$ *and* $B \in \mathcal{U}_{\Lambda_{m_0}}$,

$$\sum_{m\in\mathbb{N}} \frac{r^m}{(m!)^d} \|\delta_s^m \circ \tau_{t,s}(B)\|_{\mathcal{U}} \leq D\mathrm{e}^{m_0\varsigma} \|B\|_{\mathcal{U}} \ .$$

Proof: (i) The case $m=0$ follows from Corollary 4.2. Let $m \in \mathbb{N}$ and $B \in \mathcal{U}_0$. Similar to (124), for any sufficiently large $L_1, L_2 \in \mathbb{R}_0^+$, $\Lambda_{L_1} \subsetneq \Lambda_{L_2}$,

$$\begin{aligned} &\left\| \delta_s^m \circ \left(\tilde{\tau}_{t,s}^{(L_2)}(B) - \tilde{\tau}_{t,s}^{(L_1)}(B) \right) \right\|_{\mathcal{U}} \\ &\leq \int_{\min\{s,t\}}^{\max\{s,t\}} \sum_{\mathcal{Z}_1,\ldots,\mathcal{Z}_m \in \mathcal{P}_f(\mathfrak{L})} \Big\| \Big[\hat{\tau}_{s,\alpha}^{(L_1,L_2)}(\Psi_{\mathcal{Z}_m}^{(s)}), \ldots, \hat{\tau}_{s,\alpha}^{(L_1,L_2)}(\Psi_{\mathcal{Z}_1}^{(s)}), \\ &\qquad , B_\alpha^{(L_1,L_2)}, \tau_{t,\alpha}^{(L_1)}(\tilde{B}_t) \Big]^{(m+2)} \Big\|_{\mathcal{U}} \mathrm{d}\alpha \ , \end{aligned} \tag{145}$$

see (125). From a straightforward generalization of (131) for multi–commutators of degree $m+2$ and the same kind of arguments used in point **1.** of the proof of Theorem 4.5, the r.h.s. of the above inequality tends to zero in the limit of large $L_1, L_2 \in \mathbb{R}_0^+$ ($\Lambda_{L_1} \subsetneq \Lambda_{L_2}$). This holds for every $m \in \mathbb{N}$ because the interaction has, by assumption, exponential decay, see (60) and (143)–(144).

Consequently, $\{\delta_s^m \circ \tilde{\tau}_{t,s}^{(L)}(B)\}_{L\in\mathbb{R}_0^+}$, and hence $\{\delta_s^m \circ \tau_{t,s}^{(L)}(B)\}_{L\in\mathbb{R}_0^+}$, are Cauchy nets in $\mathcal{U}$ for any fixed $s,t \in \mathbb{R}$ and $m \in \mathbb{N}$. At $m=0$, the limit is $\tau_{t,s}(B)$. As the operator δ_s is closed, by induction, for any $m \in \mathbb{N}$ and $s,t \in \mathbb{R}$, $\tau_{t,s}(B) \in \mathrm{Dom}(\delta_s^m)$ and $\delta_s^m \circ \tau_{t,s}^{(L)}(B)$ converges to $\delta_s^m \circ \tau_{t,s}(B)$, as $L \to \infty$.
(ii) For any $m \in \mathbb{N}$, $B \in \mathcal{U}_0$, and sufficiently large $L \in \mathbb{R}_0^+$,

$$\begin{aligned} &\left\| \delta_s^m \circ \tau_{t,s}^{(L)}(B) \right\|_{\mathcal{U}} \\ \leq\ & \sum_{\ell=0}^{m} \sum_{\pi \in \mathcal{S}_{\ell,m}} \sum_{x_{\pi(\ell)}\in\mathfrak{L}} \cdots \sum_{x_{\pi(m)}\in\mathfrak{L}} \sum_{\mathcal{Z}_1\in\mathcal{P}_f(\mathfrak{L})} \cdots \sum_{\mathcal{Z}_{\pi(\ell)-1}\in\mathcal{P}_f(\mathfrak{L})} \sum_{\mathcal{Z}_{\pi(\ell)+1}\in\mathcal{P}_f(\mathfrak{L})} \cdots \\ &\cdots \sum_{\mathcal{Z}_{\pi(m)-1}\in\mathcal{P}_f(\mathfrak{L})} \sum_{\mathcal{Z}_{\pi(m)+1}\in\mathcal{P}_f(\mathfrak{L})} \cdots \sum_{\mathcal{Z}_m\in\mathcal{P}_f(\mathfrak{L})} \\ &\quad \Big\| \Big[\Psi_{\mathcal{Z}_1}^{(s)}, \ldots, \Psi_{\mathcal{Z}_{\pi(\ell)-1}}^{(s)}, \mathbf{V}_{\{x_{\pi(\ell)}\}}^{(s)}, \Psi_{\mathcal{Z}_{\pi(\ell)+1}}^{(s)}, \\ &\qquad \ldots, \Psi_{\mathcal{Z}_{\pi(m)-1}}^{(s)}, \mathbf{V}_{\{x_{\pi(m)}\}}^{(s)}, \Psi_{\mathcal{Z}_{\pi(m)+1}}^{(s)}, \ldots, \Psi_{\mathcal{Z}_m}^{(s)}, \tau_{t,s}^{(L)}(B) \Big]^{(m+1)} \Big\|_{\mathcal{U}} \ , \end{aligned}$$

with $\mathcal{S}_{\ell,m}$ being defined by (52) for $\ell \in \{1, \ldots, m\}$. For $\ell = 0$, we use here the convention $\mathcal{S}_{0,m} \doteq \emptyset$ and all sums involving the maps π in the r.h.s. of the above inequality disappear in this case. Similar to (145), Lieb–Robinson bounds for multi–commutators imply that, if $B \in \mathcal{U}_{\Lambda_{m_0}}$, $m_0 \in \mathbb{N}_0$, then the r.h.s. of the above inequality is bounded by $D(m!)^d r^m \mathrm{e}^{m_0 \varsigma} \|B\|_{\mathcal{U}}$, uniformly for s, t on compacta, where $r \equiv r_{d,\mathrm{T},\Psi,\mathbf{V},\mathbf{F}} \in \mathbb{R}^+$ and $D \equiv D_{\mathrm{T},\Psi,\mathbf{V}} \in \mathbb{R}^+$. We omit the details. By Assertion (i), the same bound thus holds for the norm $\|\delta_s^m \circ \tau_{t,s}(B)\|_{\mathcal{U}}$ of the limiting vector. ■

The assumptions of Theorem 4.6 are satisfied for interactions $\Psi^{(t)} \in \mathcal{W}$ with (61). Note additionally that Theorem 4.6 for $s = t$ shows that

$$\mathcal{U}_0 \subseteq \bigcap_{s \in \mathbb{R}, m \in \mathbb{N}} \mathrm{Dom}\,(\delta_s^m) \subset \mathcal{U} \ .$$

In fact, $\mathcal{U}_0$ is a common core for $\{\delta_s\}_{s \in \mathbb{R}}$ and thus the intersection of domains

$$\bigcap_{s \in \mathbb{R}, m \in \mathbb{N}} \mathrm{Dom}\,(\delta_s^m) \subset \mathcal{U}$$

is also a common core of $\{\delta_s\}_{s \in \mathbb{R}}$. Observe that, at fixed $s \in \mathbb{R}$, the dense space

$$\mathrm{Dom}\,(\delta_s^\infty) \doteq \bigcap_{m \in \mathbb{N}} \mathrm{Dom}\,(\delta_s^m) \subset \mathcal{U}$$

is always a core of δ_s. See, e.g., [EN, Chap. II, 1.8 Proposition].

4.3 Application to Response Theory

In the present subsection we extend to the time–dependent case the assertions of Section 3.3. As previously discussed, these results can be proven, also in the non–autonomous case, for more general (time–dependent) perturbations of the form (73). See also proofs of Inequality (131) and Theorem 4.6. Similar to Section 3.3, the case of perturbations considered below is the relevant one to study linear and non–linear responses of interacting fermions to time–dependent external electromagnetic fields.

Let $\Psi \in \mathcal{W}$ and $\mathbf{V}$ be a potential. [So, these objects do *not* depend on time.] For any $l \in \mathbb{R}_0^+$, we consider a map $(\eta, t) \mapsto \mathbf{W}_t^{(l,\eta)}$ from $\mathbb{R}^2$ to the subspace of self–adjoint elements of $\mathcal{U}_{\Lambda_l}$. Like (71), we consider elements of the form

$$\mathbf{W}_t^{(l,\eta)} \doteq \sum_{x \in \Lambda_l} \sum_{z \in \mathfrak{L}, |z| \leq 1} \mathbf{w}_{x,x+z}(\eta, t) a_x^* a_{x+z} \ , \qquad (\eta, t) \in \mathbb{R}^2, \ l \in \mathbb{R}_0^+ \ , \quad (146)$$

where $\{\mathbf{w}_{x,y}\}_{x,y\in\mathfrak{L}}$ are complex–valued functions of $(\eta,t)\in\mathbb{R}^2$ with

$$\overline{\mathbf{w}_{x,y}(\eta,t)}=\mathbf{w}_{y,x}(\eta,t) \qquad \text{and} \qquad \mathbf{w}_{x,y}(0,t)=0 \tag{147}$$

for all $x,y\in\mathfrak{L}$ and $(\eta,t)\in\mathbb{R}^2$. We assume that $\{\mathbf{w}_{x,y}(\eta,\cdot)\}_{x,y\in\mathfrak{L},\eta\in\mathbb{R}}$ is a family of continuous and uniformly bounded functions (of time): There is $K_1\in\mathbb{R}^+$ such that

$$\sup_{x,y\in\mathfrak{L}}\sup_{\eta,t\in\mathbb{R}}|\mathbf{w}_{x,y}(\eta,t)|\leq K_1\ . \tag{148}$$

The self–adjoint elements $\mathbf{W}_t^{(l,\eta)}$ of $\mathcal{U}$ are related to perturbations of dynamics caused by time–dependent external electromagnetic fields that vanish outside the box Λ_l. By the conditions above on $\mathbf{w}_{x,y}$, for all $l,\eta\in\mathbb{R}$, $t\mapsto\mathbf{W}_t^{(l,\eta)}$ is a continuous map from $\mathbb{R}$ to $\mathcal{B}(\mathcal{U})$.

We now denote the perturbed dynamics by the family $\{\tilde{\tau}_{t,s}^{(l,\eta)}\}_{s,t\in\mathbb{R}}$ of $*$–automorphisms generated by the symmetric derivation

$$\delta_t^{(l,\eta)}\doteq\delta+i\left[\mathbf{W}_t^{(l,\eta)},\ \cdot\ \right]\ , \qquad l\in\mathbb{R}_0^+,\ \eta\in\mathbb{R}\ , \tag{149}$$

in the sense of Corollary 4.2. [This family of $*$–automorphisms has *nothing* to do with (108).] Recall that δ is the symmetric derivation of Theorem 3.6. The last term in the r.h.s. of (149) is clearly a perturbation of δ which depends continuously on time, in the sense of the operator norm on $\mathcal{B}(\mathcal{U})$. It is easy to prove in this case that $\{\tilde{\tau}_{t,s}^{(l,\eta)}\}_{s,t\in\mathbb{R}}$ is the unique *fundamental solution* of (116). It means that $\{\tilde{\tau}_{t,s}^{(l,\eta)}\}_{s,t\in\mathbb{R}}$ is strongly continuous, conserves the domain

$$\mathrm{Dom}(\delta_t^{(l,\eta)})=\mathrm{Dom}(\delta)\ ,$$

satisfies

$$\tilde{\tau}_{t,\cdot}^{(l,\eta)}(B)\in C^1(\mathbb{R};(\mathrm{Dom}(\delta),\|\cdot\|_{\mathcal{U}}))\ ,\quad \tilde{\tau}_{\cdot,s}^{(l,\eta)}(B)\in C^1(\mathbb{R};(\mathrm{Dom}(\delta),\|\cdot\|_{\mathcal{U}}))$$

for all $B\in\mathrm{Dom}(\delta)$, and solves the abstract Cauchy initial value problem (116) on $\mathrm{Dom}(\delta)$.

To explicitly verify this, define the family $\{\mathfrak{V}_{t,s}\}_{s,t\in\mathbb{R}}\subset\mathcal{U}$ of unitary elements by the absolutely summable series

$$\mathfrak{V}_{t,s}\doteq\mathbf{1}_{\mathcal{U}}+\sum_{k\in\mathbb{N}}i^k\int_s^t\mathrm{d}s_1\cdots\int_s^{s_{k-1}}\mathrm{d}s_k\mathbf{W}_{s_k,s_k}^{(l,\eta)}\cdots\mathbf{W}_{s_1,s_1}^{(l,\eta)}\ , \tag{150}$$

where

$$\mathbf{W}_{t,s}^{(l,\eta)} \doteq \tau_t(\mathbf{W}_s^{(l,\eta)}) \in \mathrm{Dom}(\delta) \ , \quad l \in \mathbb{R}_0^+, \ \eta, s, t \in \mathbb{R} \ .$$

By using this unitary family, we obtain the following additional properties of the perturbed dynamics:

Theorem 4.7 (Properties of the perturbed dynamics)
Let $\Psi \in \mathcal{W}$, $l \in \mathbb{R}_0^+$, $\eta, \eta_0 \in \mathbb{R}$, and $\mathbf{V}$ be a potential. Assume Conditions (147)–(148) with $\{\mathbf{w}_{x,y}(\eta,\cdot)\}_{x,y\in\mathfrak{L},\eta\in\mathbb{R}}$ being a family of continuous functions (of time). Then, the family $\{\tilde{\tau}_{t,s}^{(l,\eta)}\}_{s,t\in\mathbb{R}}$ of $*$–automorphisms has the following properties:
(i) *Non–autonomous evolution equation. It is the unique fundamental solution of*

$$\forall s,t \in \mathbb{R}: \qquad \partial_s \tilde{\tau}_{t,s}^{(l,\eta)} = -\delta_s^{(l,\eta)} \circ \tilde{\tau}_{t,s}^{(l,\eta)} \ , \qquad \tilde{\tau}_{t,t}^{(l,\eta)} = \mathbf{1}_{\mathcal{U}} \ .$$

(ii) *Interaction picture. For any $s,t \in \mathbb{R}$,*

$$\tilde{\tau}_{t,s}^{(l,\eta)}(B) = \tau_{-s}\left(\mathfrak{V}_{t,s}\tau_t(B)\mathfrak{V}_{t,s}^*\right) \ , \qquad B \in \mathcal{U} \ .$$

(iii) *Dyson–Phillips series. For any $s,t \in \mathbb{R}$ and $B \in \mathcal{U}$,*

$$\begin{aligned} \tilde{\tau}_{t,s}^{(l,\eta)}(B) &= \tilde{\tau}_{t,s}^{(l,\eta_0)}(B) + \sum_{k\in\mathbb{N}} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \\ &\qquad \left[\mathbf{X}_{s_k,s,s_k}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s,s_1}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t,s}^{(l,\eta_0)}(B)\right]^{(k+1)} . \end{aligned} \tag{151}$$

Here, the series absolutely converges and

$$\mathbf{X}_{t,s,\alpha}^{(l,\eta_0,\eta)} \doteq \tilde{\tau}_{t,s}^{(l,\eta_0)}\left(\mathbf{W}_\alpha^{(l,\eta)} - \mathbf{W}_\alpha^{(l,\eta_0)}\right) \ , \qquad l \in \mathbb{R}_0^+, \ \alpha, s, t, \eta_0, \eta \in \mathbb{R} \ . \tag{152}$$

Proof: Before starting, note that Assertion (i) cannot be deduced from Theorem 4.5 because the cases for which (10) holds for some time $t \in \mathbb{R}$ is excluded by assumptions of that theorem.

1. Assertion (i) could be deduced from [K1, Theorem 6.1]. Here, we use [BB, Theorem 88] because it is proven from three conditions (B1–B3) that are elementary to verify:

B1 *(Kato quasi–stability).* For any $t \in \mathbb{R}$, the generator $\delta_t^{(l,\eta)}$ is conservative, by Lemma 3.3, and Condition B1 of [BB, Section VII.1] is clearly satisfied for $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$, even with *non–ordered* and all real times $t_1, \ldots, t_n \in \mathbb{R}$. Indeed, $\{\delta_t^{(l,\eta)}\}_{t\in\mathbb{R}}$, $l \in \mathbb{R}_0^+$, generate strongly continuous groups, and not only C_0–semigroups.

B2 *(Domains and continuity).* $\{\mathbf{w}_{x,y}(\eta,\cdot)\}_{x,y\in\mathfrak{L},\eta\in\mathbb{R}}$ is by assumption a family of continuous functions (of time) and thus, the map $t \mapsto [\mathbf{W}_t^{(l,\eta)},\cdot]$ from $\mathbb{R}$ to $\mathcal{B}(\mathcal{U})$ is continuous in operator norm. It follows that Condition B2 of [BB, Section VII.1] holds with the Banach space

$$\mathcal{Y} \doteq (\mathrm{Dom}(\delta), \|\cdot\|_\delta) , \tag{153}$$

$\|\cdot\|_\delta$ being the graph norm of the closed operator δ.

B3 *(Intertwining condition).* Since δ is a symmetric derivation with core $\mathcal{U}_0$ (Theorem 3.6 (ii)) and $\mathbf{W}_t^{(l,\eta)} \in \mathcal{U}_{\Lambda_l}$, for any $l \in \mathbb{R}_0^+$, $\eta \in \mathbb{R}$, $t \in \mathbb{R}$ and $B \in \mathrm{Dom}(\delta)$,

$$\delta\left(\left[\mathbf{W}_t^{(l,\eta)},B\right]\right) - \left[\mathbf{W}_t^{(l,\eta)},\delta\left(B\right)\right] = \left[\delta\left(\mathbf{W}_t^{(l,\eta)}\right),B\right] \in \mathcal{U}$$

while, by using (31), one verifies that

$$\begin{aligned}\left\|\left[\delta\left(\mathbf{W}_t^{(l,\eta)}\right),B\right]\right\|_{\mathcal{U}} \;&\leq\; 4\|B\|_{\mathcal{U}}\|\mathbf{W}_t^{(l,\eta)}\|_{\mathcal{U}} \\ &\quad\times\left(|\Lambda_l|\,\mathbf{F}\left(0\right)\|\Psi\|_{\mathcal{W}} + \sum_{x\in\Lambda_l}\left\|\mathbf{V}_{\{x\}}\right\|_{\mathcal{U}}\right) .\end{aligned}$$

In particular, Condition B3 of [BB, Section VII.1] holds true with $\Theta = \delta$.

Therefore, similar to [BB, Theorem 70 (v)], we infer from an extension of [BB, Theorem 88], which takes into account the fact that B1 holds with non–ordered real times (see, e.g., the proof of [BB, Lemma 89]), the existence of a unique solution $\{\mathfrak{W}_{s,t}\}_{s,t\in\mathbb{R}}$ of the non–autonomous evolution equation

$$\forall s,t\in\mathbb{R}: \qquad \partial_s\mathfrak{W}_{s,t} = -\delta_s^{(l,\eta)}\circ\mathfrak{W}_{s,t} , \qquad \mathfrak{W}_{t,t} = \mathbf{1}_{\mathcal{U}} , \tag{154}$$

in the strong sense on $\mathrm{Dom}(\delta)\subset\mathcal{U}$. Here, $\{\mathfrak{W}_{s,t}\}_{s,t\in\mathbb{R}}$ is an evolution family of $\mathcal{B}(\mathcal{U})$, that is, a strongly continuous two–parameter family of bounded operators acting on $\mathcal{U}$ that satisfies the cocycle (Chapman–Kolmogorov) property

$$\forall t,r,s\in\mathbb{R}: \qquad \mathfrak{W}_{s,t} = \mathfrak{W}_{s,r}\mathfrak{W}_{r,t} .$$

2. Note now that the family $\{\mathfrak{V}_{t,s}\}_{s,t\in\mathbb{R}}$ was already studied in the proof of [BPH1, Theorem 5.3] for general closed symmetric derivations δ on $\mathcal{U}$: The series (150) absolutely converges in the Banach space $\mathcal{Y}$ (153). Additionally, for any $s,t\in\mathbb{R}$,

$$\partial_t \mathfrak{V}_{t,s} = i\mathfrak{V}_{t,s}\mathbf{W}_{t,t}^{(l,\eta)} \qquad \text{and} \qquad \partial_s \mathfrak{V}_{t,s} = -i\mathbf{W}_{s,s}^{(l,\eta)}\mathfrak{V}_{t,s}$$

hold in the sense of the Banach space $\mathcal{Y}$, and thus also in the sense of $\mathcal{U}$. Therefore, for any $s,t\in\mathbb{R}$,

$$\mathfrak{W}_{s,t}(B) = \tau_{-s}\left(\mathfrak{V}_{t,s}\tau_t(B)\mathfrak{V}_{t,s}^*\right)\ , \qquad B\in\mathcal{U}\ . \tag{155}$$

To show this equality, use the fact that the r.h.s. of this equation defines an evolution family that is a fundamental solution of (154), see [BPH1, Eqs. (5.24)–(5.26)].

3. Since $\{\tau_t\}_{t\in\mathbb{R}}$ is a group of $*$–automorphisms and $\{\mathfrak{V}_{t,s}\}_{s,t\in\mathbb{R}}$ is a family of unitary elements of $\mathcal{U}$, we deduce from (155) that $\{\mathfrak{W}_{s,t}\}_{s,t\in\mathbb{R}}$ is a collection of $*$–automorphisms of the C^*–algebra $\mathcal{U}$. We also infer from (155) that the two–parameter evolution family $\{\mathfrak{W}_{s,t}\}_{s,t\in\mathbb{R}}$ solves on $\mathrm{Dom}(\delta)$ the abstract Cauchy initial value problem

$$\forall s,t\in\mathbb{R}: \qquad \partial_t\mathfrak{W}_{s,t} = \mathfrak{W}_{s,t}\circ\delta_t^{(l,\eta)}\ , \qquad \mathfrak{W}_{s,s} = \mathbf{1}_{\mathcal{U}}\ . \tag{156}$$

The solution of (156) is unique in $\mathcal{B}(\mathcal{U})$, by Corollary 4.2 (iii). We thus arrive at Assertions (i)–(ii) with the equality

$$\tilde{\tau}_{t,s}^{(l,\eta)} = \mathfrak{W}_{s,t}\ , \qquad l\in\mathbb{R}_0^+,\ \eta,s,t\in\mathbb{R}\ . \tag{157}$$

4. For any $l\in\mathbb{R}_0^+$, $s,t\in\mathbb{R}$, $\eta,\eta_0\in\mathbb{R}$, and $B\in\mathcal{U}$, define

$$\hat{\tau}_{t,s}^{(l,\eta,\eta_0)}(B) \doteq \tilde{\tau}_{t,s}^{(l,\eta_0)}(B) + \sum_{k\in\mathbb{N}} i^k\int_s^t \mathrm{d}s_1\cdots\int_s^{s_{k-1}}\mathrm{d}s_k \tag{158}$$
$$\left[\mathbf{X}_{s_k,s,s_k}^{(l,\eta_0,\eta)},\ldots,\mathbf{X}_{s_1,s,s_1}^{(l,\eta_0,\eta)},\tilde{\tau}_{t,s}^{(l,\eta_0)}(B)\right]^{(k+1)}\ .$$

This series is well–defined and absolutely convergent. Indeed, because of (148), there is a constant $D\in\mathbb{R}^+$ such that, for all $l\in\mathbb{R}_0^+$ and $\eta,\eta_0\in\mathbb{R}$,

$$\sup_{t\in\mathbb{R}}\left\|\delta_t^{(l,\eta)}-\delta_t^{(l,\eta_0)}\right\|_{\mathcal{B}(\mathcal{U})} < D\ .$$

It follows that

$$\left\|\hat{\tau}_{t,s}^{(l,\eta,\eta_0)}\right\|_{\mathcal{B}(\mathcal{U})} \leq \mathrm{e}^{D(t-s)} , \qquad l \in \mathbb{R}_0^+,\ s,t \in \mathbb{R},\ \eta,\eta_0 \in \mathbb{R} . \tag{159}$$

See, e.g., [P, Chap. 5, Theorems 2.3 and 3.1]. Now, for any $l \in \mathbb{R}_0^+$, $s,t \in \mathbb{R}$, $\eta,\eta_0 \in \mathbb{R}$, and $B \in \mathcal{U}$, note that (158) yields

$$\hat{\tau}_{t,s}^{(l,\eta,\eta_0)}(B) = \tilde{\tau}_{t,s}^{(l,\eta_0)}(B) + i\int_s^t \mathrm{d}s_1 \hat{\tau}_{s_1,s}^{(l,\eta,\eta_0)}\left(\left[\mathbf{W}_{s_1}^{(l,\eta)} - \mathbf{W}_{s_1}^{(l,\eta_0)}, \tilde{\tau}_{t,s_1}^{(l,\eta_0)}(B)\right]\right)$$

from which we deduce that $\{\hat{\tau}_{t,s}^{(l,\eta)}\}_{s,t\in\mathbb{R}}$ solves (111), by (156)–(157), (159) and continuity of the maps $t \mapsto \mathbf{W}_t^{(l,\eta)}$ and $t \mapsto \tilde{\tau}_{t,s}^{(l,\eta_0)}(B)$ from $\mathbb{R}$ to $\mathcal{U}$. Hence, by Corollary 4.2 (iii), $\hat{\tau}_{t,s}^{(l,\eta,\eta_0)} = \tilde{\tau}_{t,s}^{(l,\eta)}$ for any $l \in \mathbb{R}_0^+$, $s,t \in \mathbb{R}$ and $\eta,\eta_0 \in \mathbb{R}$. ■

Now, by assuming the uniform Lipschitz continuity of the family

$$\{\mathbf{w}_{x,y}(\cdot,t)\}_{x,y\in\mathfrak{L},t\in\mathbb{R}}$$

of functions (of η), i.e., for all parameters $\eta,\eta_0 \in \mathbb{R}$,

$$\sup_{x,y\in\mathfrak{L}}\sup_{t\in\mathbb{R}} |\mathbf{w}_{x,y}(\eta,t) - \mathbf{w}_{x,y}(\eta_0,t)| \leq K_1 |\eta - \eta_0| , \tag{160}$$

we can extend Theorem 3.13 to the non–autonomous case.

To this end, for some interaction Φ with energy observables $U_{\Lambda_L}^{\Phi}$ defined by (76) we study the increment (77), which now equals

$$\mathbf{T}_{t,s}^{(l,\eta,L)} \doteq \tilde{\tau}_{t,s}^{(l,\eta)}(U_{\Lambda_L}^{\Phi}) - \tau_{t,s}(U_{\Lambda_L}^{\Phi}) , \qquad l,L \in \mathbb{R}_0^+,\ s,t,\eta \in \mathbb{R} . \tag{161}$$

By (147), note again that $\mathbf{T}_{t,s}^{(l,0,L)} = 0$. Exactly like in the proof of Theorem 3.13, we prove a version of Taylor's theorem for increments in the non–autonomous case:

Theorem 4.8 (Taylor's theorem for increments)
Let $l,\mathrm{T} \in \mathbb{R}_0^+$, $s,t \in [-\mathrm{T},\mathrm{T}]$, $\eta,\eta_0 \in \mathbb{R}$, $\Psi \in \mathcal{W}$, and $\mathbf{V}$ be any potential. Assume (59) with $\varsigma > d$, (147)–(148) and (160), with $\{\mathbf{w}_{x,y}(\eta,\cdot)\}_{x,y\in\mathfrak{L},\eta\in\mathbb{R}}$ being a family of continuous functions (of time). Take an interaction Φ satisfying (83) with $\mathbf{v}_m = (1+m)^{\varsigma}$. Then:

(i) *The map* $\eta \mapsto \mathbf{T}_{t,s}^{(l,\eta,L)}$ *converges uniformly on* $\mathbb{R}$*, as* $L \to \infty$*, to a continuous function* $\mathbf{T}_{t,s}^{(l,\eta)}$ *of* η *and*

$$\mathbf{T}_{t,s}^{(l,\eta)} - \mathbf{T}_{t,s}^{(l,\eta_0)} = \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} i \int_s^t \mathrm{d}s_1 \tilde{\tau}_{s_1,s}^{(l,\eta)} \left(\left[\mathbf{W}_{s_1}^{(l,\eta)} - \mathbf{W}_{s_1}^{(l,\eta_0)}, \tilde{\tau}_{t,s_1}^{(l,\eta_0)}(\Phi_\Lambda) \right] \right) .$$

(ii) *For any* $m \in \mathbb{N}$ *satisfying* $d(m+1) < \varsigma$*,*

$$\begin{aligned} & \mathbf{T}_{t,s}^{(l,\eta)} - \mathbf{T}_{t,s}^{(l,\eta_0)} \\ = & \sum_{k=1}^{m} \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k \left[\mathbf{X}_{s_k,s,s_k}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s,s_1}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t,s}^{(l,\eta_0)}(\Phi_\Lambda) \right]^{(k+1)} \\ & + \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} i^{m+1} \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_m} \mathrm{d}s_{m+1} \\ & \tilde{\tau}_{s_{m+1},s}^{(l,\eta)} \left(\left[\mathbf{W}_{s_{m+1}}^{(l,\eta)} - \mathbf{W}_{s_{m+1}}^{(l,\eta_0)}, \mathbf{X}_{s_m,s_{m+1},s_m}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s_{m+1},s_1}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t,s_{m+1}}^{(l,\eta_0)}(\Phi_\Lambda) \right]^{(m+2)} \right) . \end{aligned}$$

(iii) *All the above series in* Λ *absolutely converge: For any* $m \in \mathbb{N}$ *satisfying* $d(m+1) < \varsigma$*,* $k \in \{1, \ldots, m\}$*, and* $\{s_j\}_{j=1}^{m+1} \subset [-\mathrm{T}, \mathrm{T}]$*,*

$$\sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} \left\| \left[\mathbf{X}_{s_k,s,s_k}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s,s_1}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t,s}^{(l,\eta_0)}(\Phi_\Lambda) \right]^{(k+1)} \right\|_{\mathcal{U}} \leq D \left| \Lambda_l \right| \left| \eta - \eta_0 \right|^k$$

and

$$\begin{aligned} \sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} & \left\| \tilde{\tau}_{s_{m+1},s}^{(l,\eta)} \left(\left[\mathbf{W}_{s_{m+1}}^{(l,\eta)} - \mathbf{W}_{s_{m+1}}^{(l,\eta_0)}, \mathbf{X}_{s_m,s_{m+1},s_m}^{(l,\eta_0,\eta)}, \ldots, \mathbf{X}_{s_1,s_{m+1},s_1}^{(l,\eta_0,\eta)}, \tilde{\tau}_{t,s_{m+1}}^{(l,\eta_0)}(\Phi_\Lambda) \right]^{(m+2)} \right) \right\|_{\mathcal{U}} \\ & \leq D \left| \Lambda_l \right| \left| \eta - \eta_0 \right|^{m+1} \end{aligned}$$

for some constant $D \in \mathbb{R}^+$ *depending only on* $m, d, \mathrm{T}, \Psi, K_1, \Phi, \mathbf{F}$*. The last assertion also holds for* $m = 0$*.*

Proof: By Theorems 3.9 and 4.4, Corollary 3.10 holds in the non–autonomous case. Moreover, by Lemma 4.3, Lemma 3.7 is also satisfied in the non–autonomous case. Therefore, the proof is an easy extension of the proof of Theorem 3.13. ■

If the interaction has exponential decay, we show that the map $\eta \mapsto |\Lambda_l|^{-1} \mathbf{T}_{t,s}^{(l,\eta)}$ from $\mathbb{R}$ to $\mathcal{U}$ is bounded in the sense of Gevrey classes, uniformly w.r.t. $l \in \mathbb{R}_0^+$. This corresponds to Theorem 3.14 in the non–autonomous case:

Theorem 4.9 (Increments as Gevrey maps)
Let $l, \mathrm{T} \in \mathbb{R}_0^+$, $s, t \in [-\mathrm{T}, \mathrm{T}]$, $\Psi \in \mathcal{W}$, and $\mathbf{V}$ be any potential. Assume (60) and take an interaction Φ satisfying (83) with $\mathbf{v}_m = \mathrm{e}^{m\varsigma}$. For all $x, y \in \mathfrak{L}$, assume further the real analyticity of the map $\eta \mapsto \mathbf{w}_{x,y}(\eta, \cdot)$ from $\mathbb{R}$ to the Banach space $C(\mathbb{R};\mathbb{C})$, which is equipped with the supremum norm, as well as the existence of $r \in \mathbb{R}^+$ such that

$$K_2 \doteq \sup_{x,y\in\mathfrak{L}} \sup_{m\in\mathbb{N}} \sup_{\eta,t\in\mathbb{R}} \frac{r^m \partial_\eta^m \mathbf{w}_{x,y}(\eta,t)}{m!} < \infty \ .$$

(i) *Smoothness. As a function of $\eta \in \mathbb{R}$, $\mathbf{T}_{t,s}^{(l,\eta)} \in C^\infty(\mathbb{R};\mathcal{U})$ and for any $m \in \mathbb{N}$,*

$$\partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta)} = \sum_{k=1}^{m} \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k$$
$$\partial_\varepsilon^m \left[\mathbf{X}_{s_k,s,s_k}^{(l,\eta,\eta+\varepsilon)}, \ldots, \mathbf{X}_{s_1,s,s_1}^{(l,\eta,\eta+\varepsilon)}, \tilde{\tau}_{t,s}^{(l,\eta)}(\Phi_\Lambda)\right]^{(k+1)}\Bigg|_{\varepsilon=0} .$$

The above series in Λ are absolutely convergent.
(ii) *Uniform boundedness of the Gevrey norm of density of increments. There exist $\tilde{r} \equiv \tilde{r}_{d,\mathrm{T},\Psi,K_2,\mathbf{F}} \in \mathbb{R}^+$ and $D \equiv D_{\mathrm{T},\Psi,K_2,\Phi} \in \mathbb{R}^+$ such that, for all $l \in \mathbb{R}_0^+$, $\eta \in \mathbb{R}$ and $s, t \in [-\mathrm{T}, \mathrm{T}]$,*

$$\sum_{m\in\mathbb{N}} \frac{\tilde{r}^m}{(m!)^d} \sup_{l\in\mathbb{R}_0^+} \left\| |\Lambda_l|^{-1} \partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta)} \right\|_{\mathcal{U}} \leq D \ .$$

Proof: Like for Theorem 4.8, the assertions are easily proven by extending the proof of Theorem 3.14 to the non–autonomous case. ■

This theorem has important consequences in terms of increment density limit

$$\lim_{l\to\infty} |\Lambda_l|^{-1} \rho(\mathbf{T}_{t,s}^{(l,\eta)})$$

at any fixed $s, t \in \mathbb{R}$ and state $\rho \in \mathcal{U}^*$. This limit is to be understood as an accumulation point of the bounded net $\{|\Lambda_l|^{-1}\rho(\mathbf{T}_{t,s}^{(l,\eta)})\}_{l>0}$:

Corollary 4.10 (Increment density limit)
Let $\rho \in \mathcal{U}^*$. Under the conditions of Theorem 4.9, there is a subsequence $\{l_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ such that, for all $s, t \in [-\mathrm{T}, \mathrm{T}]$, the following limit exists

$$\eta \mapsto \mathbf{g}_{t,s}(\eta) \doteq \lim_{n\to\infty} |\Lambda_{l_n}|^{-1} \rho(\mathbf{T}_{t,s}^{(l_n,\eta)})$$

and defines a smooth function $\mathbf{g}_{t,s} \in C^{\infty}(\mathbb{R})$*. Furthermore, there exist* $\tilde{r} \equiv \tilde{r}_{d,\mathrm{T},\Psi,K_2,\mathbf{F}} \in \mathbb{R}^+$ *and* $D \equiv D_{\mathrm{T},\Psi,K_2,\Phi} \in \mathbb{R}^+$ *such that, for all* $\eta \in \mathbb{R}$ *and* $s,t \in [-\mathrm{T},\mathrm{T}]$,

$$\sum_{m\in\mathbb{N}} \frac{\tilde{r}^m}{(m!)^d} \left|\partial_\eta^m \mathbf{g}_{t,s}(\eta)\right| \leq D\ .$$

Proof: Let $\mathrm{T} \in \mathbb{R}_0^+$. By Theorem 4.8 (i) for $\eta_0 = 0$ together with (147) and Corollary 4.2 (ii),

$$\sup_{l\in\mathbb{R}_0^+} \sup_{\eta\in\mathbb{R}} \sup_{s,t\in[-\mathrm{T},\mathrm{T}]} \left\{|\Lambda_l|^{-1} \rho(\mathbf{T}_{t,s}^{(l,\eta)})\right\} < \infty\ . \tag{162}$$

Furthermore, we infer from Theorem 4.9 that, for any $m \in \mathbb{N}$,

$$\sup_{l\in\mathbb{R}_0^+} \sup_{\eta\in\mathbb{R}} \sup_{s,t\in[-\mathrm{T},\mathrm{T}]} \left\{|\Lambda_l|^{-1} \rho(\partial_\eta^m \mathbf{T}_{t,s}^{(l,\eta)})\right\} < \infty\ . \tag{163}$$

By (162) and (163), the assertions are consequences of Theorem 4.9 combined with the mean value theorem and the (Arzelà–) Ascoli theorem [Ru, Theorem A5]. Indeed, $\{l_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ is taken as a so–called diagonal sequence $l_n = l_n^{(n)}$ of a family $\{l_n^{(m)}\}_{n\in\mathbb{N}}$, $m \in \mathbb{N}_0$, of sequences in $\mathbb{R}_0^+$ such that, for all $m \in \mathbb{N}_0$, the m–th derivative $|\Lambda_{l_n}|^{-1} \partial_\eta^m \mathbf{T}_{t,s}^{(l_n^{(m)},\eta)}$ uniformly converges as $n \to \infty$. With this choice,

$$\partial_\eta^m \mathbf{g}_{t,s}(\eta) = \lim_{n\to\infty} |\Lambda_{l_n}|^{-1} \rho(\partial_\eta^m \mathbf{T}_{t,s}^{(l_n,\eta)})\ .$$

∎

From the above corollary, at dimension $d = 1$ and s,t on compacta, the increment density limit $\mathbf{g}_{t,s} \in C^{\infty}(\mathbb{R})$ defines a real analytic function. As a consequence, the increment density limit is never zero for η outside a discrete subset of $\mathbb{R}$, unless $\mathbf{g}_{t,s}$ is *identically* vanishing for *all* $\eta \in \mathbb{R}$.

This mathematical property refers to a physical one. It reflects a generic alternative between either *strictly* positive or *identically* vanishing heat production density, at macroscopic scale, in presence of non–vanishing external electric fields. Indeed, by taking $\Phi = \Psi$ in Theorem 4.9, $\mathbf{T}_{t,s}^{(l,\eta)}$ is related to the heat produced by the presence of an electromagnetic field, encoded in $\mathbf{W}_t^{(l,\eta)}$. If we use cyclic processes, which means here that $\mathbf{W}_t^{(l,\eta)} = 0$ outside some compact set $[t_0, t_1] \subset \mathbb{R}$, then the KMS state $\varrho \in \mathcal{U}^*$ applied on the energy increment $\mathbf{T}_{t_1,t_0}^{(l,\eta)}$ is the total heat production (1st law of Thermodynamics) with increment density limit equal to $\mathbf{g}_{t_1,t_0}(\eta)$. It is non–negative, by the 2nd law of Thermodynamics. See [BP2] for more details on the 1st and 2nd laws for the quantum systems considered here. Now, if $\mathbf{g}_{t_1,t_0}(\eta)$ is *identically* vanishing for *all* $\eta \in \mathbb{R}$ then it means that the external perturbation never produces heat in the system, which is a very strong property. The latter is expected to be the case, for instance, for superconductors driven by electric perturbations. This kind of behavior should highlight major features of the system (like possibly broken symmetry). Hence, if the heat production density is not *identically* vanishing, generically, it is *strictly* positive, at least at dimension $d = 1$, because of properties of real analytic functions mentioned above.

For higher dimensions $d > 1$ and s, t on compacta, Corollary 4.10 implies that the increment density limit $\mathbf{g}_{t,s} \in C^{\infty}(\mathbb{R})$ belongs to the Gevrey class

$$C_d^{\omega}(\mathbb{R}) \doteq \left\{ f \in C^{\infty}(\mathbb{R}) \ : \ \sup_{\eta \in \mathbb{R}} \left| \partial_{\eta}^{m} f\left(\eta\right) \right| \leq D^{m} \left(m!\right)^{d} \text{ for any } m \in \mathbb{N} \right\} \ .$$

If $d > 1$, the elements of $C_d^{\omega}(\mathbb{R})$ are usually neither analytic nor quasi–analytic. In particular, functions of $C_d^{\omega}(\mathbb{R})$ can have arbitrarily small support, while $C_d^{\omega}(\mathbb{R}) \subsetneq C_{d'}^{\omega}(\mathbb{R})$ whenever $d < d'$. Thus, the alternative above, which is related to the heat production density in presence of external electric fields, does not follow from Corollary 4.10 for higher dimensions $d \geq 2$. However, note that, at least for the quasi–free dynamics (also in the presence of a random potential), the heat production density is a real analytic function of η at any dimension $d \in \mathbb{N}$, at least for η near zero. This follows from [BPH1, Theorem 3.4]. Therefore, the above alternative for the heat production density may be true at any dimension, provided the interaction decays fast enough in space (or is finite–range, in the extreme case).

Observe finally that if a Gevrey function $f : \mathbb{R} \rightarrow \mathbb{R}$ is invertible on some open interval $I \subset \mathbb{R}$ then the inverse $f^{-1} : f(I) \rightarrow \mathbb{R}$ is again a Gevrey function. So, the above theorem implies that, if the relation between applied field strength η and the density of increment at $l \rightarrow \infty$ is injective for some range of field strengths η, then the applied field strength in that range is a Gevrey function of the density of increment. For more details on Gevrey classes, see, e.g., [H].

5 Applications to Conductivity Measures

5.1 Charged Transport Properties in Mathematics

Altogether, the classical theory of linear conductivity (including the theory of (Landau) Fermi liquids, see, e.g., [BP1] for a historical perspective) is more like a makeshift theoretical construction than a smooth and complete theory. It is unsatisfactory to use the Drude (or the Drude–Lorentz) model – which does not take into account quantum mechanics – together with certain ad hoc hypotheses as a proper microscopic explanation of conductivity. For instance, in [NS1, NS2, SE, YRMK], the (normally fixed) relaxation time of the Drude model has to be taken as an effective frequency–dependent parameter to fit with experimental data [T] on usual metals like gold. In fact, as claimed in the famous paper [So, p. 505], "*it must be admitted that there is no entirely rigorous quantum theory of conductivity*."

Concerning AC–conductivity, however, in the last years significant mathematical progress has been made. See, e.g., [KLM, KM1, KM2, BC, BPH1, BPH2, BPH3, BPH4, BP2, BP3, W, DG] for examples of mathematically rigorous derivations of linear conductivity from first principles of quantum mechanics in the AC–regime. In particular, the notion of conductivity measure has been introduced for the first time in [KLM], albeit only for non–interacting systems. These results indicate a physical picture of the microscopic origin of Ohm and Joule's laws which differs from usual explanations coming from the Drude (Lorentz–Sommerfeld) model.

As electrical resistance of conductors may result from the presence of interactions between charge carriers, an important issue is to tackle the interacting case. This is first[1] done in [BP2, BP3] for very general systems of interacting quantum particles on lattices, including many important models of condensed matter physics like the celebrated Hubbard model. This was out of scope of [KLM, KM1, KM2, DG, BPH1, BPH2, BPH3, BPH4, W] which strongly rely on properties of quasi–free dynamics and states.

The central issue in [BP2, BP3] is to get estimates on transport coefficients related to electric conduction, which are *uniform* w.r.t. the random parameters and the volume $|\Lambda_l|$ of the box Λ_l where the electromagnetic field lives. This is crucial

[1] With regard to interacting systems, explicit constructions of KMS states are obtained in the Ph.D. thesis [W] for a one–dimensional model of interacting fermions with a finite range pair interaction. But, the author studies in [W, Chap. 9] the linear response theory only for non–interacting fermions, keeping in mind possible generalizations to interacting systems.

to get valuable information on conductivity in the macroscopic limit $l \to \infty$ and otherwise the results presented in [BP2, BP3] would loose almost all their interest. To get such estimates in the non–interacting case [BPH1, BPH2, BPH3, BPH4], we applied tree–decay bounds on multi–commutators in the sense of [BPH1, Section 4]. The latter are based on combinatorial results [BPH1, Theorem 4.1] already used before, for instance in [FMU], and require the dynamics to be implemented by Bogoliubov automorphisms. A solution to the issue for the *interacting* case is made possible by the results of Sections 3.3 and 4.3, which are direct consequences of the Lieb–Robinson bounds for multi–commutators. Detailed discussions on the estimates for the interacting case are found in [BP2]. See also Corollary 3.10, which is an extension of the tree–decay bounds [BPH1, Section 4] to the interacting case.

In [BP3] the existence of macroscopic AC–conductivity measures for interacting systems is derived from the 2nd law of thermodynamics, explained in Section 5.4. The Lieb–Robinson bound for multi–commutators of order 3 implies that it is always a Lévy measure, see [BP3, Theorems 7.1 and 5.2]. We also derive below other properties of the AC–conductivity measures from Lieb–Robinson bounds for multi–commutators of higher orders. See Sections 5.5–5.6. In particular, we study their behavior at high frequencies (Theorems 5.1 and 5.5): in contrast to the prediction of the Drude (Lorentz–Sommerfeld) model, widely used in physics [So, LTW] to describe the phenomenon of electrical conductivity, the conductivity measure stemming from short–range interparticle interactions has to decay rapidly at high frequencies.

The proposed mathematical approach to the problem of deriving macroscopic conductivity properties from the microscopic quantum dynamics of an infinite system of particles also yield new physical insight, beyond classical theories of conduction: a notion of current viscosity related to the interplay of paramagnetic and diamagnetic currents, heat/entropy production via different types of energy and current increments, existence of (AC–) conductivity measures from the 2nd law and (possibly) as a spectral (excitation) measure from current fluctuations are all examples of new physical concepts derived in the course of the studies performed in [BPH2, BPH4, BP2, BP3] and previously not discussed in the literature.

Note, however, that, by now, our results do not give explicit information on the conductivity measure for concrete models (like the Hubbard model, for instance). The latter belongs to "hard analysis", by contrast with our results which are rather on the side of the "soft analysis" (similar to the difference between knowing the spectrum of a concrete self–adjoint operator and knowing the spectral theorem). Moreover, our approach do not directly provide a mathematical understanding from first principles of Ohm's laws as a bulk property in the DC–regime, which is one of the most important and difficult problems in mathematical physics for more than one century. We believe, however, that our results can support further rigorous developments towards a solution of such a difficult problem: one could, for instance, try to show, for some class of models, that the conductivity measure is absolutely continuous w.r.t. to the Lebesgue measure and that its Radon–Nikodym derivative is continuous at low frequencies, having a well-defined zero–frequency limit.

We thus present in the following some central results of [BP2, BP3], with a few complementary studies, as an example of an important application in mathematical physics of Lieb–Robinson bounds for multi–commutators.

5.2 Interacting Fermions in Disordered Media

(i) Kinetic part: Let $\Delta_{\mathrm{d}} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ be (up to a minus sign) the usual d–dimensional discrete Laplacian defined by

$$[\Delta_{\mathrm{d}}(\psi)](x) \doteq 2d\psi(x) - \sum_{z\in\mathfrak{L},\ |z|=1} \psi(x+z) , \qquad x \in \mathfrak{L},\ \psi \in \ell^2(\mathfrak{L}) .$$

This defines a short–range interaction $\Psi^{(\mathrm{d})} \in \mathcal{W}$ by

$$\Psi_\Lambda^{(\mathrm{d})} \doteq \langle \mathfrak{e}_x, \Delta_{\mathrm{d}} \mathfrak{e}_y \rangle a_x^* a_y + (1-\delta_{x,y}) \langle \mathfrak{e}_y, \Delta_{\mathrm{d}} \mathfrak{e}_x \rangle a_y^* a_x \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$$

whenever $\Lambda = \{x, y\}$ for $x, y \in \mathfrak{L}$, and $\Psi_\Lambda^{(\mathrm{d})} \doteq 0$ otherwise.

(ii) Disordered media: Disorder in the crystal is modeled by a random potential associated to a probability space $(\Omega, \mathfrak{A}_\Omega, \mathfrak{a}_\Omega)$ defined as follows: Let $\Omega \doteq [-1,1]^{\mathfrak{L}}$. I.e., any element of Ω is a function on lattice sites with values in $[-1,1]$. For $x \in \mathfrak{L}$, let Ω_x be an arbitrary element of the Borel σ–algebra of the interval $[-1,1]$ w.r.t. the usual metric topology. $\mathfrak{A}_\Omega$ is the σ–algebra generated by the cylinder sets $\prod_{x\in\mathfrak{L}} \Omega_x$, where $\Omega_x = [-1,1]$ for all but finitely many $x \in \mathfrak{L}$. Then, $\mathfrak{a}_\Omega$ is an arbitrary *ergodic* probability measure on the measurable space $(\Omega, \mathfrak{A}_\Omega)$. This means that the probability measure $\mathfrak{a}_\Omega$ is invariant under the action

$$\omega(y) \mapsto \chi_x^{(\Omega)}(\omega)(y) \doteq \omega(y+x) , \qquad x, y \in \mathbb{Z}^d , \tag{164}$$

of the group $(\mathbb{Z}^d, +)$ of lattice translations on Ω and, for any $\mathcal{X} \in \mathfrak{A}_\Omega$ such that $\chi_x^{(\Omega)}(\mathcal{X}) = \mathcal{X}$ for all $x \in \mathbb{Z}^d$, one has $\mathfrak{a}_\Omega(\mathcal{X}) \in \{0,1\}$. We denote by $\mathbb{E}[\,\cdot\,]$ the expectation value associated with $\mathfrak{a}_\Omega$.

Then, any realization $\omega \in \Omega$ and strength $\lambda \in \mathbb{R}_0^+$ of disorder is implemented by the potential $\mathbf{V}^{(\omega)}$ defined by

$$\mathbf{V}_{\{x\}}^{(\omega)} \doteq \lambda\omega(x)\, a_x^* a_x \ , \qquad x \in \mathfrak{L} \ . \tag{165}$$

(iii) Interparticle interactions: They are taken into account by choosing some short–range interaction $\Psi^{\mathrm{IP}} \in \mathcal{W}$ such that $\Psi_\Lambda^{\mathrm{IP}} = 0$ whenever $\Lambda = \{x, y\}$ for $x, y \in \mathfrak{L}$, and

$$\sum_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} \left[\Psi_\Lambda^{\mathrm{IP}}, a_x^* a_x\right] = 0 \ , \qquad \Psi_{\Lambda+x}^{\mathrm{IP}} = \chi_x\left(\Psi_\Lambda^{\mathrm{IP}}\right) \ , \qquad \Lambda \in \mathcal{P}_f(\mathfrak{L}),\ x \in \mathfrak{L} \ . \tag{166}$$

Here, the family $\{\chi_x\}_{x\in\mathfrak{L}}$ of $*$–automorphisms of $\mathcal{U}$ implements the action of the group $(\mathbb{Z}^d, +)$ of lattice translations on the CAR C^*–algebra $\mathcal{U}$, see (42). Observe that this class of interparticle interactions includes all density–density interactions resulting from the second quantization of two–body interactions defined via a real–valued and summable function $v(r) : [0, \infty) \to \mathbb{R}$ such that

$$\sup_{r \in \mathbb{R}_0^+} \left\{ \frac{v(r)}{\mathbf{F}(r)} \right\} < \infty \ .$$

Then, by (i)–(iii), the full interaction

$$\Psi = \Psi^{(\mathrm{d})} + \Psi^{\mathrm{IP}} \in \mathcal{W} \tag{167}$$

and the potential $\mathbf{V}^{(\omega)}$ uniquely defines an infinite volume dynamics corresponding to the C_0–group $\tau^{(\omega)} \doteq \{\tau_t^{(\omega)}\}_{t\in\mathbb{R}}$ of $*$–automorphisms with generator $\delta^{(\omega)}$. See Theorem 3.6.

(iv) Space–homogeneous electromagnetic fields: Let $l \in \mathbb{R}^+$, $\eta \in \mathbb{R}$, and the compactly supported function $\mathcal{A} \in C_0^\infty(\mathbb{R}; \mathbb{R}^d)$ with $\mathcal{A}(t) \doteq 0$ for all $t \leq 0$. Set $E(t) \doteq -\partial_t \mathcal{A}(t)$ for all $t \in \mathbb{R}$. Then, the electric field at time $t \in \mathbb{R}$ equals $\eta E(t)$ inside the cubic box Λ_l and $(0, 0, \ldots, 0)$ outside. Up to negligible terms of order $\mathcal{O}(l^{d-1})$, this leads to a perturbation (of the generator of dynamics) of the form (146), (149) with complex–valued $\{\mathbf{w}_{x,y}\}_{x,y\in\mathfrak{L}}$ functions of $(\eta, t) \in \mathbb{R}^2$ defined by $\mathbf{w}_{x,x+z}(\eta, t) = 0$ for any $x, z \in \mathfrak{L}$ with $|z| > 1$ while

$$\mathbf{w}_{x,x\pm e_q}(\eta, t) \doteq \left(\exp\left(\mp i\eta \int_0^t E_q(s)\ \mathrm{d}s\right) - 1\right) \langle \mathfrak{e}_x, \Delta_{\mathrm{d}} \mathfrak{e}_{x\pm e_q} \rangle = \overline{\mathbf{w}_{x\pm e_q, x}(\eta, t)}$$

for any $q \in \{1, \ldots, d\}$. Here, $E(t) = (E_1(t), \ldots, E_d(t))$ and $\{e_q\}_{q=1}^d$ is the canonical orthonormal basis of the Euclidian space $\mathbb{R}^d$. These functions clearly satisfy Conditions (147)–(148) and (160).

Thus, the system of fermions in disordered medium, the interaction of which is encoded by (167), is perturbed from $t = 0$ onwards by space–homogeneous electromagnetic fields, leading to a well–defined family $\{\tilde{\tau}_{t,s}^{(\omega,l,\eta)}\}_{s,t\in\mathbb{R}}$ of $*$–automorphisms, as explained in Theorem 4.7.

5.3 Paramagnetic Conductivity

(i) Paramagnetic currents: For any pair $(x, y) \in \mathfrak{L}^2$, we define the current observable by

$$I_{(x,y)} \doteq i(a_y^* a_x - a_x^* a_y) = I_{(x,y)}^* \in \mathcal{U}_0 \ . \tag{168}$$

It is seen as a current because it satisfies a discrete continuity equation. See, e.g., [BP2, Section 3.2]. For any $\mathcal{A} \in C_0^\infty(\mathbb{R}; \mathbb{R}^d)$, $l \in \mathbb{R}^+$, $\omega \in \Omega$, $\eta \in \mathbb{R}$ and $t \in \mathbb{R}_0^+$, these observables are used to define a paramagnetic current increment density observable $\mathbb{J}_{\mathrm{p},l}^{(\omega)}(t, \eta) \in \mathcal{U}^d$:

$$\left\{\mathbb{J}_{\mathrm{p},l}^{(\omega)}(t,\eta)\right\}_k \doteq |\Lambda_l|^{-1} \sum_{x\in\Lambda_l} \left\{\tilde{\tau}_{t,0}^{(\omega,l,\eta)}\left(I_{(x+e_k,x)}\right) - \tau_t^{(\omega)}\left(I_{(x+e_k,x)}\right)\right\} \ .$$

Compare with Equation (161).

Note that electric fields accelerate charged particles and induce so–called diamagnetic currents, which correspond to the ballistic movement of particles. In fact, as explained in [BPH2, Sections III and IV], this component of the total current creates a kind of "wave front" that destabilizes the whole system by changing its state. The presence of diamagnetic currents leads then to the progressive appearance of paramagnetic currents which are responsible for heat production and the in–phase AC–conductivity of the system. Diamagnetic currents are not relevant for the present purpose and are thus not defined here. For more details, see [BPH2, BP2, BP3].

(ii) Paramagnetic conductivity: We define the space–averaged paramagnetic transport coefficient observable $\mathcal{C}_{\mathrm{p},l}^{(\omega)} \in C^1(\mathbb{R}; \mathcal{B}(\mathbb{R}^d; \mathcal{U}^d))$, w.r.t. the canonical orthonormal basis $\{e_q\}_{q=1}^d$ of the Euclidian space $\mathbb{R}^d$, by the corresponding matrix entries

$$\left\{\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t)\right\}_{k,q} \doteq \frac{1}{|\Lambda_l|} \sum_{x,y\in\Lambda_l} \int_0^t i[\tau_{-s}^{(\omega)}(I_{(y+e_q,y)}), I_{(x+e_k,x)}]\mathrm{d}s \tag{169}$$

for any $l \in \mathbb{R}^+$, $\omega \in \Omega$, $t \in \mathbb{R}$ and $k, q \in \{1, \ldots, d\}$.

By (i)–(ii), if Ψ^{IP} satisfies (59) with $\varsigma > 2d$ (polynomial decay) then we infer from Theorem 4.8 that, for any $\mathcal{A} \in C_0^\infty(\mathbb{R}; \mathbb{R}^d)$,

$$\mathbb{J}_{\mathrm{p},l}^{(\omega)}(t, \eta) = \eta \mathbf{J}_{\mathrm{p},l}^{(\omega)}(t) + \mathcal{O}\left(\eta^2\right) . \tag{170}$$

The correction terms of order $\mathcal{O}(\eta^2)$ are uniformly bounded in $l \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda, t \in \mathbb{R}_0^+$. By explicit computations, one checks that

$$\mathbf{J}_{\mathrm{p},l}^{(\omega)}(t) = \int_0^t \tau_t^{(\omega)}\left(\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t-s)\right) E(s)\, \mathrm{d}s \tag{171}$$

for any $\mathcal{A} \in C_0^\infty(\mathbb{R}; \mathbb{R}^d)$, $l \in \mathbb{R}^+$, $\omega \in \Omega$ and $t \in \mathbb{R}_0^+$. The latter is the paramagnetic *linear response current*. For more details, see also [BP2, Theorem 3.7]. Here, for any $\mathbf{D} \in \mathcal{B}(\mathbb{R}^d; \mathcal{U}^d)$, $\tau_t^{(\omega)}(\mathbf{D}) \in \mathcal{B}(\mathbb{R}^d; \mathcal{U}^d)$ is, by definition, the linear operator on $\mathbb{R}^d$ defined by the matrix entries w.r.t. the canonical orthonormal basis $\{e_q\}_{q=1}^d$ of the Euclidian space $\mathbb{R}^d$

$$\left\{\tau_t^{(\omega)}(\mathbf{D})\right\}_{k,q} \doteq \tau_t^{(\omega)}\left(\{\mathbf{D}\}_{k,q}\right) , \qquad k, q \in \{1, \ldots, d\} .$$

5.4 2nd law of Thermodynamics and Equilibrium States

<u>(i) States:</u> $\rho \in \mathcal{U}^*$ is a state if $\rho \geq 0$, that is, $\rho(B^*B) \geq 0$ for all $B \in \mathcal{U}$, and $\rho(\mathbf{1}) = 1$. States encode the statistical distribution of all physical quantities associated to observables $B = B^* \in \mathcal{U}$.

For any $\mathbf{D} \in \mathcal{B}(\mathbb{R}^d; \mathcal{U}^d)$, $\rho(\mathbf{D}) \in \mathcal{B}(\mathbb{R}^d)$ is, by definition, the linear operator defined, w.r.t. the canonical orthonormal basis $\{e_q\}_{q=1}^d$ of $\mathbb{R}^d$, by

$$\{\rho(\mathbf{D})\}_{k,q} \doteq \rho\left(\{\mathbf{D}\}_{k,q}\right) , \qquad k, q \in \{1, \ldots, d\} .$$

<u>(ii) 2nd law of thermodynamics:</u> As explained in [LY1, LY2], different formulations of the same principle have been stated by Clausius, Kelvin (and Planck), and Carathéodory. Our study is based on the Kelvin–Planck statement while avoiding the concept of "cooling" [LY1, p. 49]. It can be expressed as follows [PW, p. 276]:

Systems in the equilibrium are unable to perform mechanical work in cyclic processes.

(iii) Passive states: To define equilibrium states, the 2nd law, as expressed in [PW], is pivotal because it leads to a clear mathematical formulation of the Kelvin–Planck notion of equilibrium: For any strongly continuous one–parameter group $\tau \equiv \{\tau_t\}_{t\in\mathbb{R}}$ of $*$–automorphisms of $\mathcal{U}$, one obtains a well–defined strongly continuous two–parameter family $\{\tau_{t,t_0}^{(\mathbf{W})}\}_{t\geq t_0}$ of $*$–automorphisms of $\mathcal{U}$ by perturbing the generator of dynamics with bounded time–dependant symmetric derivations

$$B \mapsto i\left[\mathbf{W}_t, B\right] \ , \qquad B \in \mathcal{U} \ , \ t \in \mathbb{R} \ ,$$

for any arbitrary *cyclic process* $\{\mathbf{W}_t\}_{t\geq t_0}$ of time length $T \geq 0$, that is, a differentiable family $\{\mathbf{W}_t\}_{t\geq t_0} \subset \mathcal{U}$ of self–adjoint elements of $\mathcal{X}$ such that $\mathbf{W}_t = 0$ for all real times $t \notin [t_0, T + t_0]$. Then, a state $\varrho \in \mathcal{U}^*$ is *passive* (cf. [PW, Definition 1.1]) iff the work

$$\int_{t_0}^{t} \varrho \circ \tau_{t,t_0}^{(\mathbf{W})} \left(\partial_t \mathbf{W}_t\right) \mathrm{d}t$$

performed on the system is non–negative for all cyclic processes $\{\mathbf{W}_t\}_{t\geq t_0}$ of any time length $T \geq 0$. By [PW, Theorem 1.1], such states are invariant w.r.t. the unperturbed dynamics: $\varrho = \varrho \circ \tau_t$ for any $t \in \mathbb{R}$.

If $\tau = \tau^{(\omega)}$ with $\omega \in \Omega$ then, as explained in [BP2, Section 2.6], at least one passive state $\varrho^{(\omega)}$ exists. It represents an equilibrium state of the system (in a broad sense), the mathematical definition of which encodes the 2nd law.

(iv) Random invariant passive states: We impose two natural conditions on the map $\omega \mapsto \varrho^{(\omega)}$ from the set Ω to the dual space $\mathcal{U}^*$:

- Translation invariance. Using definitions (42) and (164), we assume that

$$\varrho^{(\chi_x^{(\Omega)}(\omega))} = \varrho^{(\omega)} \circ \chi_x \ , \qquad x \in \mathfrak{L} = \mathbb{Z}^d \ . \tag{172}$$

- Measurability. The map $\omega \mapsto \varrho^{(\omega)}$ is measurable w.r.t. to the σ–algebra $\mathfrak{A}_\Omega$ on Ω and the Borel σ–algebra $\mathfrak{A}_{\mathcal{U}^*}$ of $\mathcal{U}^*$ generated by the weak*–topology. Note that a similar assumption is also used to define equilibrium for classical systems in disordered media, see, e.g., [Bo].

A map satisfying such properties is named here *a random invariant state* [BP3, Definition 3.1]. Such maps alway exist in the one–dimension case if the norm $\left\|\Psi^{\mathrm{IP}}\right\|_{\mathcal{W}}$ of the interparticle interaction is finite. The same is true in any dimension if the inverse temperature $\beta \in \mathbb{R}^+$ is small enough. This is a consequece of the uniqueness of KMS, which is implied by the mentioned conditions. By using methods of constructive quantum field theory, one can also verify the existence of such random invariant passive states $\varrho^{(\omega)}$, $\omega \in \Omega$, at arbitrary dimension and any fixed $\beta \in \mathbb{R}^+$, if the interparticle interaction $\left\|\Psi^{\mathrm{IP}}\right\|_{\mathcal{W}}$ is small enough and (166) holds. See, for instance, [FU, Theorem 2.1] (together with [PW, Theorem 1.4]) for the small β case in quantum spin systems. See also [BP3, Section 3.3] for further discussions on this topic.

5.5 Macroscopic Paramagnetic Conductivity

For any short–range interaction $\Psi^{\mathrm{IP}} \in \mathcal{W}$, the limit

$$\Xi_{\mathrm{p}}(t) \doteq \lim_{l\to\infty} \mathbb{E}\left[\varrho^{(\omega)}(\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t))\right] \in \mathcal{B}(\mathbb{R}^d) \tag{173}$$

exists and is uniform for t on compacta. To see this, use the usual Lieb–Robinson bounds (Theorem 3.6 (iv)) to estimate (169) in the limit $l \to \infty$. Here, for any measurable $\mathbf{D}^{(\omega)} \in \mathcal{B}(\mathbb{R}^d)$, the expectation value $\mathbb{E}[\mathbf{D}^{(\omega)}] \in \mathcal{B}(\mathbb{R}^d)$ (associated with $\mathfrak{a}_\Omega$) is defined, w.r.t. the canonical orthonormal basis $\{e_q\}_{q=1}^d$ of $\mathbb{R}^d$, by the matrix entries

$$\left\{\mathbb{E}\left[\mathbf{D}^{(\omega)}\right]\right\}_{k,q} \doteq \mathbb{E}\left[\{\mathbf{D}\}_{k,q}\right], \qquad k,q \in \{1,\ldots,d\}.$$

The function $\Xi_{\mathrm{p}} \in C^1(\mathbb{R};\mathcal{B}(\mathbb{R}^d))$ can be directly related to a linear response current, as suggested by (170)–(171). See [BP3, Theorem 4.2 (p)] for more details. [If one does not take expectation values of currents, one can also show that the limit $l \to \infty$ of $\varrho^{(\omega)}(\mathbf{J}_{\mathrm{p},l}^{(\omega)})$ almost everywhere exists and equals the expectation value, in the same limit, by using the Akcoglu–Krengel ergodic theorem, see [BPH3, BP3].]

[BP3, Theorem 7.1] asserts that

$$\Xi_{\mathrm{p}} \in C^2(\mathbb{R};\mathcal{B}(\mathbb{R}^d))$$

if $\Psi^{\mathrm{IP}} \in \mathcal{W}$ and (59) holds with $\varsigma > 2d$. Now, we give a stronger version of this result which is an application of Lieb–Robinson bounds for multi–commutators (Theorems 3.8–3.9) of high orders. This new result on the regularity of the function Ξ_{p} of time has important consequences on the asymptotics of AC–Conductivity measures at high frequencies, see Theorem 5.5.

Theorem 5.1 (Regularity of the paramagnetic conductivity)
Let $\lambda \in \mathbb{R}_0^+$ and assume that the map $\omega \mapsto \varrho^{(\omega)}$ is a random invariant passive state and $\Psi^{\mathrm{IP}} \in \mathcal{W}$ satisfies (166).
(i) *Polynomial decay: Assume Ψ^{IP} satisfies (59). Then, for any $m \in \mathbb{N}$ satisfying $d(m+1) < \varsigma$, $\Xi_{\mathrm{p}} \in C^{m+1}(\mathbb{R}; \mathcal{B}(\mathbb{R}^d))$ and, uniformly for t on compacta,*

$$\partial_t^{m+1} \Xi_{\mathrm{p}}(t) = \lim_{l \to \infty} \partial_t^{m+1} \mathbb{E}\left[\varrho^{(\omega)}(\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t))\right] . \tag{174}$$

(ii) *Exponential decay: Assume Ψ^{IP} satisfies (60). Then, for all $m \in \mathbb{N}$, $\Xi_{\mathrm{p}} \in C^{\infty}(\mathbb{R}; \mathcal{B}(\mathbb{R}^d))$ and (174) holds true with the limit being uniform for t on compacta.*

Remark 5.2 (Fermion systems with random Laplacians)
The same assertion holds for the random models treated in [BP3], i.e., for fermions on the lattice with short–range and translation invariant (cf. (166)) interaction $\Psi^{\mathrm{IP}} \in \mathcal{W}$, random potentials (cf. (165)) and, additionally, random next neighbor hopping amplitudes. [So, Δ_{d} is replaced in [BP3] with a random Laplacian $\Delta_{\omega,\vartheta}$.] Similar to what is done here, disorder is defined in [BP3] via ergodic distributions of random potentials and hopping amplitudes.

The proof of this statement is a consequence of the following general lemma:

Lemma 5.3
Let $\Psi \in \mathcal{W}$ and $\mathbf{V}$ be any potential such that

$$\sup_{x \in \mathfrak{L}} \left\| \mathbf{V}_{\{x\}} \right\|_{\mathcal{U}} < \infty . \tag{175}$$

Take $T \in \mathbb{R}_0^+$ and $B_0, B_1 \in \mathcal{U}_0$.
(i) *Polynomial decay: Assume (59). Then, for any $m \in \mathbb{N}$ satisfying $dm < \varsigma$, $\mathcal{U}_0 \subseteq \mathrm{Dom}(\delta^m)$. Moreover, if $d(m+1) < \varsigma$,*

$$\sum_{y \in \mathfrak{L}} \sup_{t \in [-T,T]} \sup_{x \in \mathfrak{L}} \left\| \left[\tau_t \circ \chi_x(B_1), \delta^m \circ \chi_y(B_0))\right] \right\|_{\mathcal{U}} < \infty . \tag{176}$$

(ii) *Exponential decay: Assume (60). Then,*

$$\mathcal{U}_0 \subseteq \bigcap_{m \in \mathbb{N}} \mathrm{Dom}(\delta^m) \subset \mathcal{U}$$

and (176) holds true for all $m \in \mathbb{N}$.

Proof: (i) Because of (175), assume w.l.o.g. that $\mathbf{V}=0$. Take $t\in\mathbb{R}$, $n_0,n_1\in\mathbb{N}$ and local elements $B_0\in\mathcal{U}_{\Lambda_{n_0}}$ and $B_1\in\mathcal{U}_{\Lambda_{n_1}}$. Then, we infer from Theorem 3.6 (ii) and (80)–(82) that, for any $x,y\in\mathfrak{L}$ and $n\in\mathbb{N}$,

$$\begin{aligned}&\left\|\left[\tau_t\circ\chi_x(B_1),\delta^n\circ\chi_y\left(B_0\right)\right)\right]\right\|_{\mathcal{U}}\\ \leq\ &\sum_{x_n\in\mathfrak{L}}\sum_{m_n\in\mathbb{N}_0}\sum_{\mathcal{Z}_n\in\mathcal{D}(x_n,m_n)}\cdots\sum_{x_1\in\mathfrak{L}}\sum_{m_1\in\mathbb{N}_0}\sum_{\mathcal{Z}_1\in\mathcal{D}(x_1,m_1)}\\ &\qquad\left\|\left[\tau_t\circ\chi_x(B_1),\Psi_{\mathcal{Z}_n},\ldots,\Psi_{\mathcal{Z}_1},\chi_y\left(B_0\right)\right]^{(n+2)}\right\|_{\mathcal{U}}.\end{aligned}\tag{177}$$

Therefore, we can directly use Lieb–Robinson bounds for multi–commutators of order $n+2$ to bound (177): We combine Theorems 3.8 and 3.9 (i) with Equation (68) to deduce from (177) that, for any $x,y\in\mathfrak{L}$ and $n\in\mathbb{N}$,

$$\begin{aligned}&\left\|\left[\tau_t\circ\chi_x(B_1),\delta^n\circ\chi_y\left(B_0\right)\right)\right]\right\|_{\mathcal{U}}\\ \leq\ &2^{n+1}d^{\frac{\varsigma(n+1)}{2}}(1+n_0)^{\varsigma}\left\|B_1\right\|_{\mathcal{U}}\left\|B_0\right\|_{\mathcal{U}}\\ &\times\left(2\|\Psi\|_{\mathcal{W}}\left|t\right|\mathrm{e}^{4\mathbf{D}|t|\|\Psi\|_{\mathcal{W}}}\left\|\mathbf{u}_{\cdot,n_1}\right\|_{\ell^1(\mathbb{N})}+(1+n_1)^{\varsigma}\right)\\ &\times\left(\sup_{x\in\mathfrak{L}}\left(\sum_{m\in\mathbb{N}_0}(1+m)^{\varsigma}\sum_{\mathcal{Z}\in\mathcal{D}(x,m)}\left\|\Psi_{\mathcal{Z}}\right\|_{\mathcal{U}}\right)\right)^n\\ &\times\sum_{x_n\in\mathfrak{L}}\cdots\sum_{x_1\in\mathfrak{L}}\left(\sum_{T\in\mathcal{T}_{n+2}}\prod_{\{j,l\}\in T}\frac{1}{(1+|x_j-x_l|)^{\varsigma(\max\{\mathfrak{d}_T(j),\mathfrak{d}_T(l)\})^{-1}}}\right)\end{aligned}\tag{178}$$

with $x_0\doteq y\in\mathfrak{L}$ and $x_{n+1}\doteq x\in\mathfrak{L}$. If $\Psi\in\mathcal{W}$ and Condition (59) holds true, then one easily verifies (83) with $\mathbf{v}_m=(1+m)^{\varsigma}$. Recall also that the condition $\varsigma>(n+1)\,d$ yields (87) with $k=n+1$. Using these observations, one directly arrives at (176), starting from (178).

Remark that $\mathcal{U}_0\subseteq\mathrm{Dom}(\delta^n)$ is proven exactly in the same way. In fact, it is easier to prove and only requires the condition $\varsigma>nd$ because we have in this case multi–commutators of only order $n+1$.
(ii) The proof is very similar to the polynomial case. We omit the details. See Theorem 3.9 (ii) and (69), and in the case (60) holds and $\Psi\in\mathcal{W}$, note again that Condition (83) is satisfied with $\mathbf{v}_m=\mathrm{e}^{m\varsigma}$. ∎

We are now in position to prove Theorem 5.1.

Proof: Fix $k,q\in\{1,\ldots,d\}$, $t\in\mathbb{R}$ and $m\in\mathbb{N}$. By Theorem 3.6 (i), $\tau^{(\omega)}\doteq\{\tau_t^{(\omega)}\}_{t\in\mathbb{R}}$ is a C_0–group of $*$–automorphisms with generator $\delta^{(\omega)}$. It is, indeed, associated with the interaction (167) and the potential defined by (165). If Ψ^{IP} satisfies (59), then Condition (59) also holds true for the full interaction (167). A similar observation can be made when Ψ^{IP} satisfies (60).

Paramagnetic current observables (168) are local elements, i.e., $I_{(x,y)} \in \mathcal{U}_0$ for any $(x,y) \in \mathfrak{L}^2$. Then, by Lemma 5.3, we thus compute from (169) that, for any $m \in \mathbb{N}$ such that $\mathcal{U}_0 \subseteq \mathrm{Dom}(\delta^m)$,

$$\begin{aligned} & \partial_t^{m+1} \left\{ \mathbb{E}\left[\varrho^{(\omega)}(\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t)) \right] \right\}_{k,q} \\ = & -\frac{1}{|\Lambda_l|} \sum_{x,y\in\Lambda_l} \mathbb{E}\left[\varrho^{(\omega)} \left(i[\tau_{-t}^{(\omega)} \circ (\delta^{(\omega)})^m (I_{(y+e_q,y)}), I_{(x+e_k,x)}] \right) \right] . \end{aligned} \tag{179}$$

The last function of $\omega \in \Omega$ in the expectation value $\mathbb{E}[\,\cdot\,]$ (associated with $\mathfrak{a}_\Omega$) is measurable, because $\omega \mapsto \varrho^{(\omega)}$ is, by definition, a random invariant state while one can check that the map

$$\omega \mapsto i[\tau_{-t}^{(\omega)} \circ (\delta^{(\omega)})^m (I_{(y+e_q,y)}), I_{(x+e_k,x)}]$$

from Ω to $\mathcal{U}$ is continuous, using Theorem 3.6 and the second Trotter–Kato approximation theorem [EN, Chap. III, Sect. 4.9]. Additionally, if $\varrho^{(\omega)}$ is a passive state w.r.t. to $\tau^{(\omega)}$ for any $\omega \in \Omega$ then, $\varrho^{(\omega)} = \varrho^{(\omega)} \circ \tau_t^{(\omega)}$, see [PW, Theorem 1.1]. Therefore, it follows from (179) that

$$\begin{aligned} & \partial_t^{m+1} \left\{ \bar{\varrho}\left(\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t) \right) \right\}_{k,q} \\ = & \ \frac{1}{|\Lambda_l|} \sum_{x,y\in\Lambda_l} \mathbb{E}\left[\varrho^{(\omega)} \left(i[\tau_t^{(\omega)} \left(I_{(x+e_k,x)} \right), (\delta^{(\omega)})^m (I_{(y+e_q,y)})] \right) \right] . \end{aligned} \tag{180}$$

Now, if (166) and (172) hold true, then, by using the fact that $\mathfrak{a}_\Omega$ is also a translation invariant probability measure (it is even ergodic), we obtain from (180) that, for any $m \in \mathbb{N}$ such that $\mathcal{U}_0 \subseteq \mathrm{Dom}(\delta^m)$,

$$\begin{aligned} & \partial_t^{m+1} \left\{ \bar{\varrho}\left(\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t) \right) \right\}_{k,q} \\ = & \sum_{y\in\mathfrak{L}} \xi_l(y)\, \mathbb{E}\left[\varrho^{(\omega)} \left(i[\tau_t^{(\omega)} \left(I_{(e_k,0)} \right), (\delta^{(\omega)})^m \circ \chi_y (I_{(e_q,0)})] \right) \right] \end{aligned} \tag{181}$$

with

$$\xi_l(y) \doteq \frac{1}{|\Lambda_l|} \sum_{x\in\Lambda_l} \mathbf{1}_{\{y\in\Lambda_l - x\}} \in [0,1] , \qquad y \in \mathfrak{L} ,\ l \in \mathbb{R}^+ .$$

For any $l \in \mathbb{R}^+$, the map $y \mapsto \xi_l(y)$ on $\mathfrak{L}$ has finite support and, for any $y \in \mathfrak{L}$,

$$\lim_{l \to \infty} \xi_l(y) = 1 . \tag{182}$$

As a consequence, if (i) Ψ^{IP} satisfies (59) and $d(m+1) < \varsigma$ or (ii) Ψ^{IP} satisfies (60), then, by combining Lemma 5.3 with Lebesgue's dominated convergence theorem, one gets from (173) and (181)–(182) that the map

$$t \mapsto \partial_t^{m+1} \left\{ \mathbb{E}\left[\varrho^{(\omega)}(\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t)) \right] \right\} = \mathbb{E}\left[\partial_t^{m+1} \varrho^{(\omega)}(\mathcal{C}_{\mathrm{p},l}^{(\omega)}(t)) \right]$$

converges uniformly on compacta, as $l \to \infty$, to the continuous function $\partial_t^{m+1} \Xi_{\mathrm{p}} \in C(\mathbb{R}; \mathcal{B}(\mathbb{R}^d))$. ■

5.6 AC–Conductivity Measure

By applying [BP3, Theorems 5.2 and 5.6 (p), Remark 5.3] to the interacting fermion system under consideration we get a *Lévy–Khintchine representation* of the paramagnetic (in–phase) conductivity Ξ_{p}: Assume Ψ^{IP} satisfies (59) with $\varsigma > 2d$ (polynomial decay). Then, there is a unique finite and symmetric $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\boldsymbol{\mu}$ on $\mathbb{R}$ such that, for any $t \in \mathbb{R}$,

$$\Xi_{\mathrm{p}}(t) = -\frac{t^2}{2} \boldsymbol{\mu}(\{0\}) + \int_{\mathbb{R}\backslash\{0\}} (\cos(t\nu) - 1)\, \nu^{-2} \boldsymbol{\mu}(\mathrm{d}\nu) . \tag{183}$$

Here, $\mathcal{B}_+(\mathbb{R}^d) \subset \mathcal{B}(\mathbb{R}^d)$ stands for the set of positive linear operators on $\mathbb{R}^d$, i.e., symmetric operators w.r.t. to the canonical scalar product of $\mathbb{R}^d$ with positive eigenvalues. The (in–phase) *AC–conductivity measure* is defined from the measure $\boldsymbol{\mu}$ as follows:

Definition 5.4 (AC–conductivity measure)
We name the Lévy measure μ_{AC}, the restriction of $\nu^{-2}\boldsymbol{\mu}(\mathrm{d}\nu)$ to $\mathbb{R}\backslash\{0\}$, the (in–phase) AC–conductivity measure.

Indeed, by [BP3, Theorems 5.1 and 5.6 (p)], one checks that μ_{AC} quantifies the energy (or heat) production Q per unit volume due to the component of frequency $\nu \in \mathbb{R}\backslash\{0\}$ of the electric field, in accordance with Joule's law in the AC–regime: Indeed, for any smooth electric field $E(t) = \mathcal{E}(t)\, \vec{w}$ with $\vec{w} \in \mathbb{R}^d$, $\mathcal{E} \doteq -\partial_t \mathcal{A}(t)$

and $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$, the total heat per unit volume produced by the electric field (after being switch off) is equal to

$$Q = \frac{1}{2} \int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2 \mathcal{E}_{s_2} \mathcal{E}_{s_1} \langle \vec{w}, \Xi_{\mathrm{p}} (s_1 - s_2) \vec{w} \rangle_{\mathbb{R}^d} .$$

If the Fourier transform $\hat{\mathcal{E}}$ of $\mathcal{E} \in C_0^\infty(\mathbb{R};\mathbb{R})$ has support away from $\nu = 0$, then

$$Q = \frac{1}{2} \int_{\mathbb{R}\backslash\{0\}} |\hat{\mathcal{E}}(\nu)|^2 \langle \vec{w}, \mu_{\mathrm{AC}}(\mathrm{d}\nu) \vec{w} \rangle_{\mathbb{R}^d} .$$

Moreover, by using [BP3, Theorems 4.2 and 5.6 (p)] together with simple computations, one checks that the in–phase linear response currents J_{in}, which is the component of the total current producing heat, also called active current, is equal in this case to

$$J_{\mathrm{in}}(t) = \int_{\mathbb{R}\backslash\{0\}} \hat{\mathcal{E}}(\nu)\ \mathrm{e}^{i\nu t}\ \mu_{\mathrm{AC}}(\mathrm{d}\nu) \vec{w} .$$

By (183) and Definition 5.4, observe that the AC–conductivity measure μ_{AC} of the system under consideration is a Lévy measure. This is reminiscent of experimental observations of other quantum phenomena like (subrecoil) laser cooling [BBAC]. In fact, an alternative effective description of the phenomenon of linear conductivity by using Lévy processes in Fourier space is discussed in [BP3, Section 6].

The explicit form of the conductivity measure for concrete models (like the Hubbard model, for instance) is still an open problem. However, in [BP3, Section 5.3], we were able to qualitatively compare the AC–conductivity measure associated with the celebrated Drude model with the Lévy measure μ_{AC} given by Definition 5.4. Indeed, the (in–phase) AC–conductivity measure obtained from the Drude model is absolutely continuous w.r.t. the Lebesgue measure with the function

$$\nu \mapsto \vartheta_{\mathrm{T}}(\nu) \sim \frac{\mathrm{T}}{1 + \mathrm{T}^2 \nu^2} \tag{184}$$

being the corresponding Radon–Nikodym derivative. Here, the *relaxation time* $\mathrm{T} > 0$ is related to the mean time interval between two collisions of a charged carrier with defects in the crystal. See for instance [BPH4, Section 1] for more discussions. This measure *heavily overestimates* μ_{AC} at high frequencies. Indeed, as explained in [BP3, Section 5.3], by finiteness of the positive measure $\boldsymbol{\mu}$, the AC–conductivity measure satisfies

$$\mu_{\mathrm{AC}}([\nu,\infty)) \leq \nu^{-2} \boldsymbol{\mu}([\nu,\infty)) \leq \nu^{-2} \boldsymbol{\mu}(\mathbb{R}) , \qquad \nu \in \mathbb{R}^+ , \tag{185}$$

provided Ψ^{IP} satisfies (59) with $\varsigma > 2d$. The same property of course holds for negative frequencies, by symmetry of $\boldsymbol{\mu}$ (w.r.t. ν). Compare (185) with (184). From Theorem 5.1, much stronger results on the frequency decay of μ_{AC} can be obtained if the interaction Ψ^{IP} is fast decaying in space:

Theorem 5.5 (Moments of AC–conductivity measures)
Let $\lambda \in \mathbb{R}_0^+$, $\Psi^{\mathrm{IP}} \in \mathcal{W}$ satisfying (166), and assume that the map $\omega \mapsto \varrho^{(\omega)}$ is a random invariant passive state.
(i) *Polynomial decay: Assume Ψ^{IP} satisfies (59) with $\varsigma > 2d$. Then, for any $m \in \mathbb{N}$ satisfying $d(m+1) < \varsigma$,*

$$\int_{\mathbb{R}\backslash\{0\}} \nu^{m+1} \mu_{\mathrm{AC}}(\mathrm{d}\nu) \in \mathcal{B}_+(\mathbb{R}^d) \ , \tag{186}$$

i.e., the $(m+1)$-th moment of the measure μ_{AC} exists.
(ii) *Exponential decay: Assume Ψ^{IP} satisfies (60). Then, (186) holds true for all $m \in \mathbb{N}$.*

Proof: By (183) and Lebesgue's dominated convergence theorem, for any $t \in \mathbb{R}$,

$$\partial_t^2 \boldsymbol{\Xi}_{\mathrm{p}}(t) = -\int_{\mathbb{R}} \cos(t\nu)\, \boldsymbol{\mu}(\mathrm{d}\nu) = -\int_{\mathbb{R}} \mathrm{e}^{it\nu} \boldsymbol{\mu}(\mathrm{d}\nu) \ ,$$

provided $\varsigma > 2d$ in (59) (with $\Psi = \Psi^{\mathrm{IP}}$). In other words, the finite and symmetric $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\boldsymbol{\mu}$ on $\mathbb{R}$ can be seen as the Fourier transform of $-\partial_t^2 \boldsymbol{\Xi}_{\mathrm{p}}(t)$ or, that is, as the characteristic function of $\boldsymbol{\mu}$. Therefore, by well–known properties of characteristic functions (see, e.g., [D, Theorem 3.3.9.] for the special case $n = 2$ and [Kl, Theorem 15.34] for the general case $n \in 2\mathbb{N}_0$), for any even $n \in \mathbb{N}_0$, $\partial_t^2 \boldsymbol{\Xi}_{\mathrm{p}} \in C^n(\mathbb{R}; \mathcal{B}(\mathbb{R}^d))$ implies that

$$\int_{\mathbb{R}} \nu^n \boldsymbol{\mu}(\mathrm{d}\nu) \in \mathcal{B}_+(\mathbb{R}^d) \ .$$

If $m \in \mathbb{N}_0$ is odd, then, by the above assertion for $n < m$ and the symmetry of the measure $\boldsymbol{\mu}$ (which follows from the symmetry of μ_{AC}), we conclude that

$$\int_{\mathbb{R}} \nu^m \boldsymbol{\mu}(\mathrm{d}\nu) = 0 \in \mathcal{B}_+(\mathbb{R}^d) \ .$$

This observation combined with Theorem 5.1 and Definition 5.4 yields Assertions (i)–(ii). ■

Remark 5.6 (Fermion systems with random Laplacians)
The same assertion holds for the random models treated in [BP3]. See also Remark 5.2.

This last theorem is an significant improvement of the asymptotics (185) of [BP3] and is a straightforward application of Lieb–Robinson bounds for multi–commutators of high orders (Theorems 3.8–3.9), see Lemma 5.3.

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