From the 2nd Law of Thermodynamics to AC–Conductivity Measures of Interacting Fermions in Disordered Media

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Abstract

We study the dynamics of interacting lattice fermions with random hopping amplitudes and random static potentials, in presence of time–dependent electromagnetic fields. The interparticle interaction is short–range and translation invariant. Electromagnetic fields are compactly supported in time and space. In the limit of infinite space supports (macroscopic limit) of electromagnetic fields, we derive Ohm and Joule's laws in the AC–regime. An important outcome is the extension to interacting fermions of the notion of macroscopic AC–conductivity measures, known so far only for free fermions with disorder. Such excitation measures result from the 2nd law of thermodynamics and turn out to be Lévy measures. As compared to the Drude (Lorentz–Sommerfeld) model, widely used in Physics, the quantum many–body problem studied here predicts a much smaller AC–conductivity at large frequencies. This indicates (in accordance with experimental results) that the relaxation time of the Drude model, seen as an effective parameter for the conductivity, should be highly frequency–dependent. We conclude by proposing an alternative effective description – using Lévy Processes in Fourier space – of the phenomenon of electrical conductivity.

Contents

1 Introduction

The present paper belongs to a succession of works on electrical conductivity, starting with [BPK1, BPK2, BPK3, BPK4, BP1].

As claimed in the famous paper [So, p. 505], "*it must be admitted that there is no entirely rigorous quantum theory of conductivity*." Concerning AC–conductivity, however, in the last years significant mathematical progress has been made. See [KLM, KM1, KM2, BC, BPK1, BPK2, BPK3, BPK4] for examples of mathematically rigorous derivations of linear conductivity from first principles of quantum mechanics in the AC–regime. These results indicate a picture of the microscopic origin of Ohm and Joule's laws which differs from usual explanations coming from the Drude (Lorentz–Sommerfeld) model. This is discussed with more details in [BPK2].

As electrical resistance of conductors may result from the presence of interactions between charge carriers, the main drawback of these studies is their restriction to non–interacting systems. Therefore, we aim to extend [BPK1, BPK2, BPK3, BPK4] to fermion systems with interactions. As a first step, [BP1] proves all assertions of [BPK1, BPK2] for fermion systems with short range interactions. We perform here the second part of this program by extending the main results of [BPK3] to interacting fermion systems:

- Like in [BP1], we investigate some non–autonomous C^*–dynamical system on the CAR C^*–algebra of cubic infinite lattices of any dimension. The (non–autonomous) dynamics is generated by short–range and translation invariant interactions between particles, random static potentials, and also random next neighbor hopping amplitudes in presence of local and time–dependent electromagnetic fields. Disorder is here defined via ergodic distributions of random potentials and hopping amplitudes.

- We study the linear response of interacting fermions at thermal equilibrium in disordered media to *macroscopic* electric fields that are time– and space–dependent. In particular, we obtain Ohm's law with a (charge) transport coefficient that is a continuous function (of time) naturally called macroscopic conductivity.

- The Fourier transform of the conductivity is named macroscopic AC–conductivity measure. The fact that this Fourier transform is indeed a measure follows from the 2nd law of thermodynamics. The latter corresponds here to the Kelvin–Planck statement while avoiding the concept of "cooling" [LY1, p. 49]. In particular, the 2nd law yields the positivity of the heat production for cyclic processes on equilibrium states. The concept of conductivity measure, introduced by Klein, Lenoble and Müller, has already been used in [KLM, KM1, KM2, BPK2, BPK3, BPK4] for the *non*–interacting case.

- We give a comparison of our results and the Drude (Lorentz–Sommerfeld) model, widely used in Physics [So, LTW] to describe the phenomenon of electrical conductivity. See also [BP2] for a historical perspective of this subject. In particular, we show that the Drude model and its refinements (like the Drude–Lorentz and the Lorentz–Sommerfeld models) *always* overestimate the in–phase conductivity at high frequencies. This indicates that the relaxation time of the Drude model, seen as an effective parameter for the conductivity, should be frequency–dependent, as already observed for instance in [T, NS1, NS2, SE, YRMK]. In fact, it should either vanish or diverge at large frequencies.

- We show that the AC–conductivity measure of the system under consideration is always a Lévy measure. An alternative effective description of the phenomenon of linear conductivity by using Lévy Processes in Fourier space is discussed. This is reminiscent of Boltzmann equation for collective oscillatory modes of charge excitations.

Note that also new results not presented in [BPK3] are obtained here, even for non–interacting fermions. For instance, in contrast with [BPK3], the hopping amplitudes are allowed to be non–homogeneous in space.

Like in [BP1], these results are made possible by *Lieb–Robinson bounds for multi–commutators*. See [BP3] for more details. As explained in [BP1], this method requires short–range interactions. Our setting includes density–density interactions resulting from the second quantization of two–body interactions defined via a real–valued and summable (in a convenient sense) function $v(r) : [0, \infty) \to \mathbb{R}$. For instance, the celebrated Hubbard model (and any other system with finite range interactions) or models with Yukawa–type potentials are all possible choices, but the Coulomb potential is excluded because it is not summable in space. For more details, see [BP1, Section 2.4].

Our main assertions are Theorems 4.1, 4.2, 5.1, 5.2 and 5.6. The paper is organized as follows:

- Section 2 is a preliminary conceptual review on the notion of thermal equilibrium state in relation to the 2nd law of thermodynamics. In this context, the mathematical results of [PW] are discussed.

- Section 3 formulates the mathematical setting used to study charge transport properties of fermions. We define in particular a Banach space of short–range interactions.

- Section 4 states Ohm's law for macroscopic electromagnetic fields as well as Green–Kubo relations for current Duhamel fluctuation increments.

- In Section 5 we derive the macroscopic AC–conductivity measure from Joule's law and the 2nd law of thermodynamics. Its relations with microscopic AC–conductivity measures and the Drude model are discussed. In

Section 5.4 we propose a notion of time–reversal symmetry for fermion systems on the lattice in presence of disorder and discuss its consequences for the corresponding charge transport coefficients.

- Section 6 proposes an effective description of the phenomenon of linear conductivity by using Lévy Processes.
- Section 7 gathers technical proofs on which Sections 4–6 are based. The arguments strongly use the results of [BP1, BP3].

Notation 1.1
To simplify notation, we denote by D positive and finite constants. These constants do not need to be the same from one statement to another. A norm on a generic vector space $\mathcal{X}$ is denoted by $\|\cdot\|_{\mathcal{X}}$ and the identity map of $\mathcal{X}$ by $\mathbf{1}_{\mathcal{X}}$. To avoid ambiguity, scalar products in $\mathcal{X}$ are sometimes denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$.

2 2nd Law of Thermodynamics and Thermal States

It is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.

[Lord Kelvin, 1851]

See [K]. This is the celebrated *2nd law of thermodynamics*, the history of which starts with Carnot's works in 1824. It is "*one of the most perfect laws in physics*" [LY1, Section 1] and it has never been faulted by reproducible experiments. As explained in [LY1, LY2], different popular formulations of the same principle have been stated by Clausius, Kelvin (and Planck), and Carathéodory. Our study is based on the Kelvin–Planck statement while avoiding the concept of "cooling" [LY1, p. 49]:

No process is possible, the sole result of which is a change in the energy of a simple system (without changing the work coordinates) and the raising of a weight.

The celebrated formulations of Clausius, Kelvin–Planck and Carathéodory are all about impossible processes and let largely open what is possible. This is useful to define the concept of *thermal equilibrium* states in a simple way. Note that Lieb and Yngvason's work [LY1] on the 2nd law is an important structural approach which involves possible processes, instead.

We mathematically implement the Kelvin–Planck principle by using algebraic quantum mechanics. It is a well–known approach – originally due to von Neumann (cf. von Neumann algebras, C^*–algebras) – to quantum mechanics. One important result of the theory of C^*–algebras, performed in the forties, is the celebrated GNS (Gel'fand–Naimark–Segal) representation of states, which permits a natural relation between the new algebraic formulation and the usual Hilbert space based formulation of quantum mechanics to be established. Indeed, I.E. Segal proposed to leave the Hilbert space approach and considered quantum observables as elements of some involutive Banach algebras, now known as C^*–algebras. The GNS representation has also led to very important applications of the Tomita–Takesaki theory, developed in 1970, to quantum field theory and quantum statistical mechanics. These developments gave a solid mathematical basis to the algebraic approach to quantum mechanics and quantum field theory.

Indeed, the algebraic formulation turned out to be extremely fruitful for the mathematical foundations of quantum statistical mechanics and have been an important branch of research during decades with lots of works, in particular on quantum spin. See, e.g., [BR1, BR2, I, S] (quantum spins) and [AM, BP4] (lattice fermions). Basically, it uses some C^*–algebra $\mathcal{X}$, the self–adjoint elements of which are the so–called observables of the physical system. States on the C^*–algebra $\mathcal{X}$ are, by definition, linear functionals $\rho \in \mathcal{X}^*$ which are normalized and positive, i.e., $\rho(\mathbf{1}) = 1$ and $\rho(B^*B) \geq 0$ for all $B \in \mathcal{X}$. They represent the state of the physical system. In the commutative case of classical physics states are usual probability measures.

To define equilibrium states, [PW] is pivotal because it mathematically implements the Kelvin–Planck physical notion of equilibrium:

Systems in the equilibrium are unable to perform mechanical work in cyclic processes.

Note at this point that the above principle (2nd law) defining equilibrium can possibly be violated.

As explained in [PW, p. 276], the above formulation of the 2nd law of thermodynamics is directly related to the notion of *passive* states. Indeed, one defines a (unperturbed) dynamics of the system by a strongly continuous one–parameter group $\tau \equiv \{\tau_t\}_{t\in\mathbb{R}}$ of $*$–automorphisms of $\mathcal{X}$ with (generally unbounded) generator δ. The latter is a dissipative and closed derivation of $\mathcal{X}$. If the state of the system at $t = t_0 \in \mathbb{R}$ is $\rho \in \mathcal{X}^*$, then it evolves as $\rho_t = \rho \circ \tau_{t-t_0}$ for any $t \geq t_0$. On this system, one produces "excitations" by perturbing the generator of dynamics with bounded time–dependant symmetric derivations

$$B \mapsto i\,[A_t, B] := i\,(A_t B - B A_t)\ , \qquad B \in \mathcal{X}\ ,\ t \in \mathbb{R}\ ,$$

for arbitrary differentiable families $\{A_t\}_{t\geq t_0} \subset \mathcal{X}$ of self–adjoint elements of $\mathcal{X}$. In particular, this defines a strongly continuous two–parameter family $\{\tau_{t,t_0}\}_{t\geq t_0}$ of $*$–automorphisms of $\mathcal{X}$ as the solution of a non–autonomous evolution equation defined, for any $B \in \mathrm{Dom}(\delta)$, by

$$\forall t_0, t \in \mathbb{R},\ t \geq t_0 : \quad \partial_t \tau_{t,t_0}(B) = \tau_{t,t_0}(\delta(B) + i\,[A_t, B])\ , \quad \tau_{t_0,t_0}(B) := B\ .$$

The state of the system evolves now as $\rho_t = \rho \circ \tau_{t,t_0}$ for any $t \geq t_0$.

As explained in [PW, p. 276], the energy exchanged between the external device and the perturbed system at time $t \geq t_0$ is equal to

$$L_t^A(\rho) := \int_{t_0}^{t} \rho \circ \tau_{t,t_0}(\partial_t A_t)\,\mathrm{d}t\ . \tag{1}$$

If $L_t^A(\rho) \geq 0$ then work is performed on the system, while $L_t^A(\rho) < 0$ means that one decreases the energy of the system. A *cyclic* process of time length $T \geq 0$ is, by definition, a differentiable family $\{A_t\}_{t\geq t_0} \subset \mathcal{X}$ of self–adjoint elements of $\mathcal{X}$ such that $A_t = 0$ for all $t \leq t_0$ and $t \geq t_1 := T + t_0$. Then, the 2nd law of thermodynamics can be formulated in this mathematical framework as follows (cf. [PW, Definition 1.1]):

Definition 2.1 (2nd law of thermodynamics – Passivity)
Let $(\mathcal{X}, \tau)$ be a C^*–dynamical system. A state $\rho \in \mathcal{X}^*$ is passive iff $L_T^A(\rho) \geq 0$ for all cyclic processes $\{A_t\}_{t\geq t_0} \subset \mathcal{X}$ of any time length $T \geq 0$.

By [PW, Theorem 2.1], passive states ρ of a dynamical system $(\mathcal{X}, \tau)$ can be equivalently defined as states satisfying

$$-i\rho(U^*\delta(U)) \geq 0$$

for all unitaries $U \in \mathcal{X}$ both in the domain of definition of the generator δ of the group τ and in the connected component of the identity of the group of all unitary elements of $\mathcal{X}$ with the norm topology. See, e.g., [BR2, Definition 5.3.21]. This last condition is strongly related with internal energy increments and the 1st law of thermodynamics, see, e.g., [BP1, Theorem 3.2].

By [PW, Theorem 1.1], such states are invariant with respect to (w.r.t.) the unperturbed dynamics: any passive state $\rho \in \mathcal{X}^*$ satisfies

$$\rho = \rho \circ \tau_t , \qquad t \in \mathbb{R} .$$

Physically, it means that the dynamics of the system at equilibrium *cannot* be observed unless one performs external perturbations $\{A_t\}_{t\geq t_0}$ to extract some *excitation spectrum*. This last notion will be discussed in detail in a companion paper and the conductivity measure is one notable example of application.

Moreover, for any $\beta \in \mathbb{R}_0^+$, all (τ, β)–KMS states $\varrho^{(\beta)}$ are passive, see [PW, Theorem 1.2]. The same holds true for $\beta = \infty$, that is, for ground states of $(\mathcal{X}, \tau)$. Any convex combination of passive states is also passive. In particular, for any $n \in \mathbb{N}$, $\beta_1, \ldots, \beta_n, \mu_1, \ldots, \mu_n \in \mathbb{R}^+$ with $\Sigma_{j=1}^n \mu_j = 1$, the state

$$\rho = \sum_{j=1}^{n} \mu_j \varrho^{(\beta_j)} \tag{2}$$

is passive, but it is neither a KMS nor a ground state of $(\mathcal{X}, \tau)$, in general.

We impose another natural condition related to the physical notion [LY1, Definition p. 55] of thermal equilibrium in thermodynamics that excludes such convex combinations. A minimal requirement for the system to be in thermal equilibrium is indeed that it cannot produce work by interacting with any of its copy. To be more precise, prepare $n \in \mathbb{N}$ copies $(\mathcal{X}^{(1)}, \tau^{(1)}, \rho^{(1)}), \ldots, (\mathcal{X}^{(n)}, \tau^{(n)}, \rho^{(n)})$ of the original system defined by $(\mathcal{X}, \tau, \rho)$ and consider the compound system

$$(\otimes_{j=1}^n \mathcal{X}^{(j)}, \otimes_{j=1}^n \tau^{(j)}, \otimes_{j=1}^n \rho^{(j)}) .$$

If $(\mathcal{X}, \tau, \rho)$ is at *thermal equilibrium*, the compound system should also be at equilibrium and it must not be possible to extract any energy from cyclic processes, by the 2nd law of thermodynamics. Therefore, $\otimes_{j=1}^n \rho$ should also be passive for all $n \in \mathbb{N}$. Such states are named in the literature *completely passive* states:

Definition 2.2 (Thermal equilibrium states)
Let $(\mathcal{X}, \tau)$ be a C^*–dynamical system. A state $\varrho \in \mathcal{X}^*$ is completely passive iff $\otimes_{j=1}^n \varrho$ is a passive state of $(\otimes_{j=1}^n \mathcal{X}^{(j)}, \otimes_{j=1}^n \tau^{(j)})$ for all $n \in \mathbb{N}$. We name them thermal equilibrium states of $(\mathcal{X}, \tau)$.

[PW, Theorem 1.4] gives an explicit characterization of thermal equilibrium states:

Theorem 2.3 (Pusz–Woronowicz)
Let $(\mathcal{X},\tau)$ be a C^*–dynamical system. ϱ is a thermal equilibrium state of $(\mathcal{X},\tau)$ iff it is a (τ,β)–KMS state of $(\mathcal{X},\tau)$ for some $\beta\in[0,\infty]$.

The parameter $\beta\in[0,\infty]$ is named *inverse temperature* of the system and is a *consequence* of the 2nd law of thermodynamics. It is a universal parameter of the (possibly infinite) system. In fact, β tunes the value of the internal energy density of the system. Equivalently, *it fixes a time scale* since ϱ is a (τ_t,β)–KMS state iff ϱ is a $(\tau_{\beta t},1)$–KMS state ($\beta<\infty$). The boundary case $\beta=0$ corresponds to the τ–invariant traces, also called chaotic states, whereas (τ,∞)–KMS states are by definition ground states. [$(\tau,-\beta)$–KMS states correspond to (τ,β)–KMS states with a reversal of time.]

The notion of local (relative) entropy seems to be more natural than the concept of local temperature. Indeed, the 2nd law of thermodynamics as expressed in Definitions 2.1–2.2 is a formal expression of the unavoidable lost while one interacts with an object, which is at equilibrium before the interaction. Entropy is only a quantitative counterpart of this lost. It corresponds to heat production in thermodynamics which we study in the context of electricity theory. The positivity of the heat production, which is the content of the 2nd law of thermodynamics, implies the existence of the AC–conductivity measure. See Section 5.

Remark 2.4 (Dynamics versus thermal equilibrium states)
A state $\varrho\in\mathcal{X}^*$ with GNS representation $(\mathcal{H},\pi,\Psi)$ is a KMS state for a strongly continuous one–parameter group $\tau\equiv\{\tau_t\}_{t\in\mathbb{R}}$ of $*$–automorphisms of $\mathcal{X}$ iff its normal extension to $\pi(\mathcal{X})''$ is faithful. See, e.g., [BR2, p. 85]. In this case, the group τ is unique. The faithfulness of states is a physically natural property: By definition, an observable exists iff the corresponding physical property can be observed. Therefore, one could fix a state $\varrho\in\mathcal{X}^*$ of the system that must be, by definition, a thermal equilibrium state, i.e., a KMS state. This assumption implicitly imposes the existence of some (unique) dynamics given by a group $\tau^{(\varrho)}$ and is justified a posteriori via the 2nd law. Constructing KMS states $\varrho^{(\tau)}$ from a given dynamics τ may be technically more involved. It is however the approach we use because the dynamics is fixed by microscopic interactions between particles.

3 C^*–Dynamical Systems for Interacting Fermions

The mathematical framework used here is exactly the one of [BP1]. It is concisely described below. The only additional information is the exact definition of the probability space modelling disorder.

3.1 Disordered Media within Electromagnetic Fields

Disorder in the crystal is modeled by a random variable with distribution $\mathfrak{a}_\Omega$ taking values in the measurable space $(\Omega, \mathfrak{A}_\Omega)$. The probability space $(\Omega, \mathfrak{A}_\Omega, \mathfrak{a}_\Omega)$ is defined as follows:

Ω : Let $\mathfrak{L} := \mathbb{Z}^d$ ($d \in \mathbb{N}$) and

$$\mathfrak{b} := \{\{x, x'\} \subset \mathfrak{L} \ : \ |x - x'| = 1\} \tag{3}$$

be the set of non–oriented bonds of the cubic lattice $\mathfrak{L}$. Then,

$$\Omega := [-1,1]^{\mathfrak{L}} \times \mathbb{D}^{\mathfrak{b}} \quad \text{with} \quad \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\} \ .$$

I.e., any element of Ω is a pair $\omega = (\omega_1, \omega_2) \in \Omega$, where ω_1 is a function on lattice sites with values in $[-1,1]$ and ω_2 is a function on bonds with values in the complex closed unit disc $\mathbb{D}$.

$\mathfrak{A}_\Omega$: Let $\Omega_x^{(1)}$, $x \in \mathfrak{L}$, be an arbitrary element of the Borel σ–algebra $\mathfrak{A}_x^{(1)}$ of the interval $[-1,1]$ w.r.t. the usual metric topology. Define

$$\mathfrak{A}_{[-1,1]^{\mathfrak{L}}} := \bigotimes_{x \in \mathfrak{L}} \mathfrak{A}_x^{(1)} \ ,$$

i.e., $\mathfrak{A}_{[-1,1]^{\mathfrak{L}}}$ is the σ–algebra generated by the cylinder sets $\prod_{x \in \mathfrak{L}} \Omega_x^{(1)}$, where $\Omega_x^{(1)} = [-1,1]$ for all but finitely many $x \in \mathfrak{L}$. In the same way, let

$$\mathfrak{A}_{\mathbb{D}^{\mathfrak{b}}} := \bigotimes_{\mathbf{x} \in \mathfrak{b}} \mathfrak{A}_{\mathbf{x}}^{(2)} \ ,$$

where $\mathfrak{A}_{\mathbf{x}}^{(2)}$, $\mathbf{x} \in \mathfrak{b}$, is the Borel σ–algebra of the complex closed unit disc $\mathbb{D}$ w.r.t. the usual metric topology. Then

$$\mathfrak{A}_\Omega := \mathfrak{A}_{[-1,1]^{\mathfrak{L}}} \otimes \mathfrak{A}_{\mathbb{D}^{\mathfrak{b}}} \ .$$

$\mathfrak{a}_\Omega$: The measure $\mathfrak{a}_\Omega$ is an arbitrary *ergodic* probability measure on the measurable space $(\Omega, \mathfrak{A}_\Omega)$: It is invariant under the action

$$(\omega_1, \omega_2) \longmapsto \chi_x^{(\Omega)}(\omega_1, \omega_2) := \left(\chi_x^{(\mathfrak{L})}(\omega_1), \chi_x^{(\mathfrak{b})}(\omega_2)\right) \ , \qquad x \in \mathbb{Z}^d \ , \tag{4}$$

of the group $(\mathbb{Z}^d, +)$ of translations on Ω and, for any $\mathcal{X} \in \mathfrak{A}_\Omega$ such that $\chi_x^{(\Omega)}(\mathcal{X}) = \mathcal{X}$ for all $x \in \mathbb{Z}^d$, one has $\mathfrak{a}_\Omega(\mathcal{X}) \in \{0, 1\}$. Here, for any $\omega = (\omega_1, \omega_2) \in \Omega$, $x \in \mathbb{Z}^d$ and $y, y' \in \mathfrak{L}$ with $|y - y'| = 1$,

$$\chi_x^{(\mathfrak{L})}(\omega_1)(y) := \omega_1(y + x) \ , \ \chi_x^{(\mathfrak{b})}(\omega_2)(\{y, y'\}) := \omega_2(\{y + x, y' + x\}) \ . \tag{5}$$

We denote by $\mathbb{E}[\,\cdot\,]$ the expectation value associated with $\mathfrak{a}_\Omega$.

For any $\omega = (\omega_1, \omega_2) \in \Omega$, $V_\omega \in \mathcal{B}(\ell^2(\mathfrak{L}))$ is by definition the self–adjoint multiplication operator with the function $\omega_1 : \mathfrak{L} \to [-1, 1]$. It represents a bounded static potential. To all $\omega \in \Omega$ and strength $\vartheta \in \mathbb{R}_0^+$ of hopping disorder, we also associate another self–adjoint operator $\Delta_{\omega,\vartheta} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ describing the hoppings of a single particle in the lattice:

$$\begin{aligned}[][\Delta_{\omega,\vartheta}(\psi)](x) \ := \ & 2d\psi(x) - \sum_{j=1}^{d} \Big((1 + \vartheta\overline{\omega_2(\{x, x - e_j\})})\, \psi(x - e_j) \\ & + \psi(x + e_j)(1 + \vartheta\omega_2(\{x, x + e_j\})) \Big) \end{aligned} \tag{6}$$

for any $x \in \mathfrak{L}$ and $\psi \in \ell^2(\mathfrak{L})$, with $\{e_k\}_{k=1}^d$ being the canonical orthonormal basis of the Euclidian space $\mathbb{R}^d$. In the case of vanishing hopping disorder $\vartheta = 0$ (up to a minus sign) $\Delta_{\omega,0}$ is the usual d–dimensional discrete Laplacian. Since the hopping amplitudes are complex–valued (ω_2 takes values in $\mathbb{D}$), note additionally that random electromagnetic potentials can be implemented in our model.

Then, for any realization $\omega \in \Omega$ of disorder and parameters $\vartheta, \lambda \in \mathbb{R}_0^+$, the Hamiltonian of a single quantum particle within a bounded static potential is the discrete Schrödinger operator $(\Delta_{\omega,\vartheta} + \lambda V_\omega)$ acting on the Hilbert space $\ell^2(\mathfrak{L})$. The coupling constants $\vartheta, \lambda \in \mathbb{R}_0^+$ represent the strength of disorder of respectively the external static potential and hopping amplitudes.

The time–dependent electromagnetic potential is defined by a compactly supported time–dependent vector potential

$$\mathbf{A} \in \mathbf{C}_0^\infty := \bigcup_{l \in \mathbb{R}^+} C_0^\infty(\mathbb{R} \times [-l, l]^d ; (\mathbb{R}^d)^*) \ ,$$

where $(\mathbb{R}^d)^*$ is the set of one–forms[1] on $\mathbb{R}^d$ that take values in $\mathbb{R}$. I.e., for some $l \in \mathbb{R}^+$, $\mathbf{A} \in C_0^\infty(\mathbb{R} \times [-l, l]^d ; (\mathbb{R}^d)^*)$ and we use the convention $\mathbf{A}(t, x) \equiv 0$

[1]In a strict sense, one should take the dual space of the tangent spaces $T(\mathbb{R}^d)_x$, $x \in \mathbb{R}^d$.

whenever $x \notin [-l,l]^d$. Since $\mathbf{A} \in \mathbf{C}_0^\infty$, $\mathbf{A}(t,x) = 0$ for all $t \leq t_0$, where $t_0 \in \mathbb{R}$ is some initial time. The smoothness of $\mathbf{A}$ is not essential in the proofs and is only assumed for simplicity.

Remark 3.1 *To simplify notation, we identify in the sequel* $(\mathbb{R}^d)^*$ *with* $\mathbb{R}^d$ *via the canonical scalar product of* $\mathbb{R}^d$.

We use the Weyl gauge (also named temporal gauge) for the electromagnetic field and, as a consequence,

$$E_{\mathbf{A}}(t,x) := -\partial_t \mathbf{A}(t,x) \ , \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d \ , \tag{7}$$

is the electric field associated with $\mathbf{A}$. We also define the integrated electric field (or electric tension) along the oriented bond $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$ at time $t \in \mathbb{R}$ by

$$\mathbf{E}_t^{\mathbf{A}}(\mathbf{x}) := \int_0^1 \left[E_{\mathbf{A}}(t, \alpha x^{(2)} + (1-\alpha) x^{(1)}) \right] (x^{(2)} - x^{(1)}) \mathrm{d}\alpha \ . \tag{8}$$

Since $\mathbf{A}$ is by assumption compactly supported, the corresponding electric field satisfies the AC–condition

$$\int_{t_0}^{t} E_{\mathbf{A}}(s,x) \mathrm{d}s = 0 \ , \quad x \in \mathbb{R}^d \ , \tag{9}$$

for sufficiently large times $t \geq t_1 \geq t_0$. From (9),

$$t_1 := \min \left\{ t \geq t_0 : \quad \int_{t_0}^{t'} E_{\mathbf{A}}(s,x) \mathrm{d}s = 0 \quad \text{for all } x \in \mathbb{R}^d \text{ and } t' \geq t \right\} \tag{10}$$

is the time at which the electric field is turned off. In other words, we consider *cyclic* electromagnetic processes.

To simplify notation and without loss of generality (w.l.o.g.), fermions are spinless and have negative charge. The cases of particles with spin and positively charged particles can be treated by exactly the same methods. Thus, using the (minimal) coupling of $\mathbf{A} \in \mathbf{C}_0^\infty$ to the discrete Laplacian, the discrete magnetic Laplacian is (up to a minus sign) the self–adjoint operator

$$\Delta_{\omega,\vartheta}^{(\mathbf{A})} \equiv \Delta_{\omega,\vartheta}^{(\mathbf{A}(t,\cdot))} \in \mathcal{B}(\ell^2(\mathfrak{L})) \ , \qquad t \in \mathbb{R} \ ,$$

defined by

$$\langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}^{(\mathbf{A})} \mathfrak{e}_y \rangle = \exp\left(-i \int_0^1 \left[\mathbf{A}(t, \alpha y + (1-\alpha)x) \right] (y-x) \mathrm{d}\alpha \right) \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta} \mathfrak{e}_y \rangle \tag{11}$$

for all $t \in \mathbb{R}$, $\omega \in \Omega$, $\vartheta \in \mathbb{R}_0^+$ and $x, y \in \mathfrak{L}$. Here, $\langle \cdot, \cdot \rangle$ is the scalar product in $\ell^2(\mathfrak{L})$ and $\{\mathfrak{e}_x\}_{x \in \mathfrak{L}}$ is the canonical orthonormal basis $\mathfrak{e}_x(y) \equiv \delta_{x,y}$ of $\ell^2(\mathfrak{L})$. In (11), similar to (8), $\alpha y + (1-\alpha)x$ and $y - x$ are seen as vectors in $\mathbb{R}^d$. In presence of an electromagnetic field associated to an arbitrary vector potential $\mathbf{A} \in \mathbf{C}_0^\infty$, the one–particle Hamiltonian $(\Delta_{\omega,\vartheta} + \lambda V_\omega)$ at fixed $\omega \in \Omega$ and $\vartheta, \lambda \in \mathbb{R}_0^+$ is replaced with the time–dependent one

$$\Delta_{\omega,\vartheta}^{(\mathbf{A})} + \lambda V_\omega \equiv \Delta_{\omega,\vartheta}^{(\mathbf{A}(t,\cdot))} + \lambda V_\omega , \qquad t \in \mathbb{R} . \tag{12}$$

3.2 Banach Space of Short–Range Interactions

Let $\mathcal{P}_f(\mathfrak{L}) \subset 2^{\mathfrak{L}}$ be the set of all finite subsets of $\mathfrak{L}$. For all $\Lambda \in \mathcal{P}_f(\mathfrak{L})$, $\mathcal{U}_\Lambda$ is the finite dimensional C^*–algebra generated by $\mathbf{1}$ and generators $\{a_{x,\mathrm{s}}\}_{x \in \Lambda, \mathrm{s} \in \mathrm{S}}$ satisfying the canonical anti–commutation relations, S being some finite set of spins. As just explained above, the spin dependence of $a_{x,\mathrm{s}} \equiv a_x$ is irrelevant in our proofs (up to trivial modifications) and, w.l.o.g., we only consider spinless fermions, i.e., the case $\mathrm{S} = \{0\}$.

We denote by $\mathcal{U}$ the CAR C^*–algebra $\mathcal{U}$ of the infinite system, that is, the inductive limit of the finite dimensional C^*–algebras $\{\mathcal{U}_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathfrak{L})}$. The C^*–algebra of all even elements of $\mathcal{U}$ is denoted by $\mathcal{U}^+$ and

$$\mathcal{U}_0 := \bigcup_{\Lambda \in \mathcal{P}_f(\mathfrak{L})} \mathcal{U}_\Lambda \subset \mathcal{U}$$

is the subset of local elements. See [BP1, Section 2.2] for more details. Finally, let $\{\chi_x\}_{x \in \mathfrak{L}}$ be the family of $*$–automorphisms of $\mathcal{U}$ uniquely defined by the conditions

$$\chi_x(a_y) = a_{y+x} , \qquad x, y \in \mathfrak{L} = \mathbb{Z}^d . \tag{13}$$

An *interaction* is a family $\Phi = \{\Phi_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathfrak{L})}$ of even and self–adjoint local elements $\Phi_\Lambda = \Phi_\Lambda^* \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$ with $\Phi_\emptyset = 0$. We define Banach spaces $\mathcal{W}$ of short–range interactions by introducing norms that take into account space decay of interactions. To this end, we use positive–valued and non–increasing decay functions $\mathbf{F} : \mathbb{R}_0^+ \to \mathbb{R}^+$. Like in [BP1], we impose the following conditions on $\mathbf{F}$:

- *Summability on $\mathfrak{L}$.*

$$\|\mathbf{F}\|_{1,\mathfrak{L}} := \sup_{y\in\mathfrak{L}} \sum_{x\in\mathfrak{L}} \mathbf{F}\left(|x-y|\right) = \sum_{x\in\mathfrak{L}} \mathbf{F}\left(|x|\right) < \infty \ . \tag{14}$$

- *Bounded convolution constant.*

$$\mathbf{D} := \sup_{x,y\in\mathfrak{L}} \sum_{z\in\mathfrak{L}} \frac{\mathbf{F}\left(|x-z|\right)\mathbf{F}\left(|z-y|\right)}{\mathbf{F}\left(|x-y|\right)} < \infty \ . \tag{15}$$

Examples of functions $\mathbf{F} : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfying (14)–(15) are given by

$$\mathbf{F}\left(r\right) = (1+r)^{-(d+\epsilon)} \quad \text{and} \quad \mathbf{F}\left(r\right) = \mathrm{e}^{-\varsigma r}(1+r)^{-(d+\epsilon)} \tag{16}$$

for any $\varsigma, \epsilon \in \mathbb{R}^+$. In all the paper, (14)–(15) are assumed to be satisfied.

Then, the norm of any interaction Φ is defined by

$$\|\Phi\|_{\mathcal{W}} := \sup_{x,y\in\mathfrak{L}} \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L}),\ \Lambda\supset\{x,y\}} \frac{\|\Phi_\Lambda\|_{\mathcal{U}}}{\mathbf{F}\left(|x-y|\right)} \ . \tag{17}$$

The real separable Banach space $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is the space of interactions Φ with $\|\Phi\|_{\mathcal{W}} < \infty$. Elements $\Phi \in \mathcal{W}$ are named *short–range* interactions.

3.3 Interacting Fermion Systems in Disordered Media

To any $\omega \in \Omega$ and strength $\vartheta \in \mathbb{R}_0^+$ of hopping disorder, we associate a short–range interaction $\Psi^{(\omega,\vartheta)} \in \mathcal{W}$ defined as follows: Fix an interparticle (IP) interaction $\Psi^{\mathrm{IP}} \in \mathcal{W}$. Then,

$$\Psi_\Lambda^{(\omega,\vartheta)} := \langle \mathfrak{e}_x, \Delta_{\omega,\vartheta}\mathfrak{e}_y\rangle a_x^* a_y + \langle \mathfrak{e}_y, \Delta_{\omega,\vartheta}\mathfrak{e}_x\rangle a_y^* a_x + \Psi_{\{x,y\}}^{\mathrm{IP}} \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$$

whenever $\Lambda = \{x,y\}$ for $x,y \in \mathfrak{L}$, and $\Psi_\Lambda^{(\omega,\vartheta)} := \Psi_\Lambda^{\mathrm{IP}}$ otherwise.

Let

$$\Lambda_l := \{(x_1,\ldots,x_d) \in \mathfrak{L} \ : \ |x_1|,\ldots,|x_d| \leq l\} \ , \qquad l \in \mathbb{R}_0^+ \ , \tag{18}$$

and, for all $x \in \mathfrak{L}$ and $m \in \mathbb{N}$,

$$\mathcal{D}\left(x,m\right) := \{\Lambda \in \mathcal{P}_f(\mathfrak{L}) : x \in \Lambda,\ \Lambda \subseteq \Lambda_m + x,\ \Lambda \nsubseteq \Lambda_{m-1} + x\} \subset 2^{\mathfrak{L}} \ , \tag{19}$$

while $\mathcal{D}\left(x,0\right) := \{\{x\}\}$. We then assume two additional properties of Ψ^{IP}:

- *Translation invariance*. For all $x \in \mathfrak{L}$,

$$\Psi^{\mathrm{IP}}_{\Lambda+x} = \chi_x \left(\Psi^{\mathrm{IP}}_{\Lambda}\right) , \qquad \Lambda \in \mathcal{P}_f(\mathfrak{L}) . \tag{20}$$

- *Polynomial decay.* There is a constant $\varsigma > 2d$ and, for all $m \in \mathbb{N}_0$, an absolutely summable sequence $\{\mathbf{u}_{n,m}\}_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$ such that, for all $n \in \mathbb{N}$ with $n > m$,

$$|\Lambda_n \backslash \Lambda_{n-1}| \sum_{z\in\Lambda_m} \max_{y\in\Lambda_n\backslash\Lambda_{n-1}} \mathbf{F}\left(|z-y|\right) \leq \frac{\mathbf{u}_{n,m}}{(1+n)^{\varsigma}} . \tag{21}$$

  Further, there are constants $\upsilon, D \in \mathbb{R}^+$ such that, for any $m \in \mathbb{N}_0$,

$$\sup_{x\in\mathfrak{L}} \sum_{\Lambda\in\mathcal{D}(x,m)} \left\|\Psi^{\mathrm{IP}}_{\Lambda}\right\|_{\mathcal{U}} \leq D\,(m+1)^{-\upsilon} \quad \text{and} \quad \sum_{m,n\in\mathbb{N}} m^{-\upsilon}\mathbf{u}_{n,m} < \infty . \tag{22}$$

If Ψ^{IP} is translation invariant, note that the supremum in (22) is attained at any fixed $x \in \mathfrak{L}$. Examples of functions $\mathbf{F} : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfying (14)–(15) and (21)–(22) are obviously given by (16), for sufficiently large $\epsilon \in \mathbb{R}^+$ in the polynomial case.

Conditions (14)–(15) and [BP1, Theorem 2.2] ensure the existence of a (non–autonomous) infinite volume dynamics $\{\tau^{(\omega,\vartheta,\lambda,\mathbf{A})}_{t,s}\}_{s,t\in\mathbb{R}}$, in presence of electromagnetic fields and static potentials (cf. (12)). Indeed, any realization $\omega \in \Omega$, disorder strengths $\vartheta, \lambda \in \mathbb{R}_0^+$, interparticle interaction $\Psi^{\mathrm{IP}} \in \mathcal{W}$ and electromagnetic potential $\mathbf{A} \in \mathbf{C}_0^\infty$ naturally define a family $\{\delta^{(\omega,\vartheta,\lambda,\mathbf{A})}_{t}\}_{t\in\mathbb{R}}$ of derivations on the subset $\mathcal{U}_0$ of local elements of $\mathcal{U}$. Then, $\{\tau^{(\omega,\vartheta,\lambda,\mathbf{A})}_{t,s}\}_{s,t\in\mathbb{R}}$ is the unique strongly continuous two–parameter family of $*$–automorphisms of $\mathcal{U}$ satisfying, in the strong sense on the dense domain $\mathcal{U}_0 \subset \mathcal{U}$,

$$\forall s,t \in \mathbb{R} : \qquad \partial_t \tau^{(\omega,\vartheta,\lambda,\mathbf{A})}_{t,s} = \tau^{(\omega,\vartheta,\lambda,\mathbf{A})}_{t,s} \circ \delta^{(\omega,\vartheta,\lambda,\mathbf{A})}_{t} , \qquad \tau^{(\omega,\vartheta,\lambda,\mathbf{A})}_{s,s} = \mathbf{1}_{\mathcal{U}} .$$

See [BP1, Section 2.5] for more details. At $\mathbf{A} = 0$, the (unperturbed) dynamics is autonomous and we denote the corresponding group of $*$–automorphisms by

$$\tau^{(\omega,\vartheta,\lambda)} := \{\tau^{(\omega,\vartheta,\lambda)}_{t}\}_{t\in\mathbb{R}} . \tag{23}$$

Then, as explained in Section 2, thermal equilibrium states are defined to be completely passive states, see Definition 2.2. This definition is based on the 2nd law of thermodynamics. By Theorem 2.3, they are $(\tau^{(\omega,\vartheta,\lambda)},\beta)$–KMS states for some inverse temperature, or time scale, $\beta\in[0,\infty]$. For simplicity, we exclude the boundary cases $\beta=0,+\infty$. As discussed in [BP1, Section 2.6], the set of $(\tau^{(\omega,\vartheta,\lambda)},\beta)$–KMS states is non–empty for any $\beta\in\mathbb{R}^{+}$, $\omega\in\Omega$ and $\vartheta,\lambda\in\mathbb{R}_{0}^{+}$. Here, $\varrho^{(\beta,\omega,\vartheta,\lambda)}$ denotes one element of this set.

For any $\beta\in\mathbb{R}^{+}$ and $\vartheta,\lambda\in\mathbb{R}_{0}^{+}$, we impose two natural conditions on the map

$$\omega\mapsto\varrho^{(\beta,\omega,\vartheta,\lambda)} \tag{24}$$

from the set Ω to the dual space $\mathcal{U}^{*}$:

- *Translation invariance.* Recall that $\{\chi_{x}\}_{x\in\mathfrak{L}}$ is the family of $*$–automorphisms of $\mathcal{U}$ uniquely defined by (13). It implements the action of the group $(\mathbb{Z}^{d},+)$ of lattice translations on the CAR C^{*}–algebra $\mathcal{U}$. On the set Ω this action is represented by the family $\{\chi_{x}^{(\Omega)}\}_{x\in\mathfrak{L}}$, see (4)–(5). Then, we assume that
$$\varrho^{(\beta,\chi_{x}^{(\Omega)}(\omega),\vartheta,\lambda)}=\varrho^{(\beta,\omega,\vartheta,\lambda)}\circ\chi_{x}\ ,\qquad x\in\mathfrak{L}=\mathbb{Z}^{d}\ . \tag{25}$$

- *Measurability.* Thermal equilibrium states are supposed to be random variables. Hence, for any $\beta\in\mathbb{R}^{+}$ and $\vartheta,\lambda\in\mathbb{R}_{0}^{+}$, we assume that the map (24) is measurable w.r.t. to the σ–algebra $\mathfrak{A}_{\Omega}$ on Ω and the Borel σ–algebra $\mathfrak{A}_{\mathcal{U}^{*}}$ of $\mathcal{U}^{*}$ generated by the weak*–topology. Observe that a similar assumption is also necessary for classical disordered systems at equilibrium, see, e.g., [Bo].

These conditions yield the following definition:

Definition 3.2 (Random invariant states)
Let $\omega\mapsto\varrho^{(\omega)}$ be a map from Ω to the set of states on $\mathcal{U}$. We say that this map is a random invariant state when it is measurable w.r.t. to $\mathfrak{A}_{\Omega}$ and $\mathfrak{A}_{\mathcal{U}^{*}}$ and translation invariant in the above sense.

The map (24) is thus a random invariant state. This implies in particular that, for any $\beta\in\mathbb{R}^{+}$ and $\vartheta,\lambda\in\mathbb{R}_{0}^{+}$, the averaged state $\bar{\varrho}^{(\beta,\lambda)}\in\mathcal{U}^{*}$ defined by

$$\bar{\varrho}^{(\beta,\vartheta,\lambda)}(B):=\mathbb{E}\left[\varrho^{(\beta,\omega,\vartheta,\lambda)}(B)\right]\ ,\qquad B\in\mathcal{U}\ , \tag{26}$$

is *translation invariant*, i.e.,

$$\bar{\varrho}^{(\beta,\vartheta,\lambda)}=\bar{\varrho}^{(\beta,\vartheta,\lambda)}\circ\chi_{x}\ ,\qquad x\in\mathfrak{L}=\mathbb{Z}^{d}\ . \tag{27}$$

Recall indeed that $\mathfrak{a}_\Omega$ is also a translation invariant probability measure. [It is even ergodic.]

The existence of such random invariant equilibrium states is not clear in general, similar to the classical case. If the $(\tau^{(\omega,\vartheta,\lambda)},\beta)$–KMS state is unique and (20) is satisfied, then it turns out that the (unique) map (24) is a random invariant state. Indeed, in this case, the map (24) is even continuous w.r.t. the pointwise convergence in Ω and the weak*–topology of $\mathcal{U}^*$. This can be proven by using [BR2, Proposition 5.3.23.]. Uniqueness of KMS states appears for instance when either $\Psi^{\mathrm{IP}}, \vartheta = 0$, or at small $\beta \in \mathbb{R}^+$, or in dimension $d = 1$. Moreover, by using methods of constructive quantum field theory, one can also verify the existence of such random invariant thermal equilibrium states at arbitrary $\beta \in \mathbb{R}^+$ and dimension $d \in \mathbb{N}$ if the interparticle interaction $\left\|\Psi^{\mathrm{IP}}\right\|_{\mathcal{W}}$ is small enough and (20) holds.

Now, in presence of electromagnetic fields, the time evolution of the state of the system equals

$$\rho_t^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})} := \begin{cases} \varrho^{(\beta,\omega,\vartheta,\lambda)} & , \qquad t \leq t_0 \ , \\ \varrho^{(\beta,\omega,\vartheta,\lambda)} \circ \tau_{t,t_0}^{(\omega,\vartheta,\lambda,\mathbf{A})} & , \qquad t \geq t_0 \ , \end{cases} \tag{28}$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$. Recall here that $\mathbf{A}(t,x) = 0$ for all $t \leq t_0$.

Remark 3.3 (Time–dependent states as stochastic processes)
Under the above assumptions, by using Lieb–Robinson bounds as in [BP3, Lemma 4.3], it is possible to show that the family $\{\omega \mapsto \rho_t^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}\}_{t\in\mathbb{R}}$ *defines a stochastic process with values in* $\mathcal{U}^*$. *More precisely, for any* $t \in \mathbb{R}$, *the map* $\omega \mapsto \rho_t^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ *is measurable w.r.t. to* $\mathfrak{A}_\Omega$ *and* $\mathfrak{A}_{\mathcal{U}^*}$. *This fact is not essential in the sequel.*

4 Macroscopic Ohm's Law and Green–Kubo Relations

4.1 Macroscopic Charge Transport Coefficients

Fix $\omega \in \Omega$, $\vartheta \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$ and time $t \in \mathbb{R}$. For any oriented bond $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$, we define the paramagnetic and diamagnetic current observables $I_{\mathbf{x}}^{(\omega,\vartheta)}$ and $\mathrm{I}_{\mathbf{x}}^{(\omega,\vartheta,\mathbf{A})}$ respectively by

$$I_{\mathbf{x}}^{(\omega,\vartheta)} := -2\,\mathrm{Im}\left(\langle \mathfrak{e}_{x^{(1)}}, \Delta_{\omega,\vartheta}\mathfrak{e}_{x^{(2)}}\rangle a_{x^{(1)}}^* a_{x^{(2)}}\right) \ , \qquad \mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2 \ . \tag{29}$$

and

$$\mathrm{I}_{\mathbf{x}}^{(\omega,\vartheta,\mathbf{A})} \equiv \mathrm{I}_{\mathbf{x}}^{(\omega,\vartheta,\mathbf{A}(t,\cdot))} \quad := \quad -2\,\mathrm{Im}\left(\left(\mathrm{e}^{i\int_0^1[\mathbf{A}(t,\alpha x^{(2)}+(1-\alpha)x^{(1)})](x^{(2)}-x^{(1)})\mathrm{d}\alpha}-1\right)\right.$$
$$\left.\times\langle\mathfrak{e}_{x^{(1)}},\Delta_{\omega,\vartheta}\mathfrak{e}_{x^{(2)}}\rangle a_{x^{(1)}}^{*}a_{x^{(2)}}\right)\ . \quad (30)$$

If the interparticle interaction Ψ^{IP} is locally gauge invariant, that is, for all $x \in \mathfrak{L}$,

$$\sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})}\left[\Psi_{\Lambda}^{\mathrm{IP}},a_x^*a_x\right]=0\ ,$$

then, in absence of external electromagnetic potentials, $I_{\mathbf{x}}^{(\omega,\vartheta)}$ is the observable related to the flow of particles from the lattice site $x^{(1)}$ to the lattice site $x^{(2)}$ or the current from $x^{(2)}$ to $x^{(1)}$. $\mathrm{I}_{\mathbf{x}}^{(\omega,\vartheta,\mathbf{A})}$ corresponds to a correction, engendered by the presence of an external electromagnetic potential, to the current $I_{\mathbf{x}}^{(\omega,\vartheta)}$. See [BP1, Section 3.2]. Let

$$P_{\mathbf{x}}^{(\omega,\vartheta)} := -2\,\mathrm{Re}\left(\langle\mathfrak{e}_{x^{(1)}},\Delta_{\omega,\vartheta}\mathfrak{e}_{x^{(2)}}\rangle a_{x^{(1)}}^{*}a_{x^{(2)}}\right)\ , \qquad \mathbf{x}:=(x^{(1)},x^{(2)})\in\mathfrak{L}^2\ . \quad (31)$$

Now, for any $\beta\in\mathbb{R}^+$, $\omega\in\Omega$ and $\vartheta,\lambda\in\mathbb{R}_0^+$, we define two important functions associated with these observables:

(p) The paramagnetic transport coefficient $\sigma_{\mathrm{p}}^{(\omega)}\equiv\sigma_{\mathrm{p}}^{(\beta,\omega,\vartheta,\lambda)}$ is defined, for any $\mathbf{x},\mathbf{y}\in\mathfrak{L}^2$ and $t\in\mathbb{R}$, by

$$\sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x},\mathbf{y},t):=\int_0^t\varrho^{(\beta,\omega,\vartheta,\lambda)}\left(i[I_{\mathbf{y}}^{(\omega,\vartheta)},\tau_s^{(\omega,\vartheta,\lambda)}(I_{\mathbf{x}}^{(\omega,\vartheta)})]\right)\mathrm{d}s\ . \quad (32)$$

(d) The diamagnetic transport coefficient $\sigma_{\mathrm{d}}^{(\omega)}\equiv\sigma_{\mathrm{d}}^{(\beta,\omega,\vartheta,\lambda)}$ is defined by

$$\sigma_{\mathrm{d}}^{(\omega)}(\mathbf{x}):=\varrho^{(\beta,\omega,\vartheta,\lambda)}\left(P_{\mathbf{x}}^{(\omega,\vartheta)}\right)\ , \qquad \mathbf{x}\in\mathfrak{L}^2\ . \quad (33)$$

For boxes Λ_l (18), we then define the space–averaged paramagnetic transport coefficient

$$t\mapsto\Xi_{\mathrm{p},l}^{(\omega)}(t)\equiv\Xi_{\mathrm{p},l}^{(\beta,\omega,\vartheta,\lambda)}(t)\in\mathcal{B}(\mathbb{R}^d)$$

w.r.t. the canonical orthonormal basis $\{e_k\}_{k=1}^d$ of the Euclidian space $\mathbb{R}^d$ by

$$\left\{\Xi_{\mathrm{p},l}^{(\omega)}(t)\right\}_{k,q}:=\frac{1}{|\Lambda_l|}\sum_{x,y\in\Lambda_l}\sigma_{\mathrm{p}}^{(\omega)}(x+e_q,x,y+e_k,y,t) \quad (34)$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$. See [BP1, Theorem 3.4, Corollary 3.5] for details on the properties of $\Xi_{\mathrm{p},l}^{(\omega)}$. The space–averaged diamagnetic transport coefficient

$$\Xi_{\mathrm{d},l}^{(\omega)} \equiv \Xi_{\mathrm{d},l}^{(\beta,\omega,\vartheta,\lambda)} \in \mathcal{B}(\mathbb{R}^d)$$

corresponds (w.r.t. $\{e_k\}_{k=1}^d$) to the diagonal matrix defined by

$$\left\{\Xi_{\mathrm{d},l}^{(\omega)}\right\}_{k,q} := \frac{\delta_{k,q}}{|\Lambda_l|} \sum_{x \in \Lambda_l} \sigma_{\mathrm{d}}^{(\omega)}\left(x + e_k, x\right) \in \left[-2\left(\vartheta + 1\right), 2\left(\vartheta + 1\right)\right] . \tag{35}$$

Both random coefficients turn out to be the paramagnetic and diamagnetic (in–phase) conductivities.

We define the deterministic paramagnetic transport coefficient

$$t \mapsto \mathbf{\Xi}_{\mathrm{p}}\left(t\right) \equiv \mathbf{\Xi}_{\mathrm{p}}^{(\beta,\vartheta,\lambda)}\left(t\right) \in \mathcal{B}(\mathbb{R}^d)$$

by

$$\mathbf{\Xi}_{\mathrm{p}}\left(t\right) := \lim_{l \to \infty} \mathbb{E}\left[\Xi_{\mathrm{p},l}^{(\omega)}\left(t\right)\right] \tag{36}$$

for any $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$. It is well–defined, by Theorem 7.1. Furthermore, the convergence is uniform for times t in compact sets. Analogously, we also introduce the deterministic diamagnetic transport coefficient

$$\mathbf{\Xi}_{\mathrm{d}} \equiv \mathbf{\Xi}_{\mathrm{d}}^{(\beta,\vartheta,\lambda)} \in \mathcal{B}(\mathbb{R}^d)$$

defined, for any $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$, by

$$\mathbf{\Xi}_{\mathrm{d}} := \lim_{l \to \infty} \mathbb{E}\left[\Xi_{\mathrm{d},l}^{(\omega)}\right] . \tag{37}$$

Indeed, by translation invariance (27) of $\bar{\varrho}^{(\beta,\vartheta,\lambda)}$, we have, for all $l \in \mathbb{R}^+$,

$$\mathbf{\Xi}_{\mathrm{d}} = \mathbb{E}\left[\Xi_{\mathrm{d},l}^{(\omega)}\right] .$$

Clearly, $\{\mathbf{\Xi}_{\mathrm{d}}\}_{k,k} \in [-2(\vartheta + 1), 2(\vartheta + 1)]$ for any $k \in \{1, \ldots, d\}$.

By using the Akcoglu–Krengel ergodic theorem we show that the limits $l \to \infty$ of $\Xi_{\mathrm{p},l}^{(\omega)}$ and $\Xi_{\mathrm{d},l}^{(\omega)}$ converge almost surely to $\mathbf{\Xi}_{\mathrm{p}}$ and $\mathbf{\Xi}_{\mathrm{d}}$, respectively.

Theorem 4.1 (Macroscopic charge transport coefficients)
Assume (14)–(15), (20) and that the map (24) is a random invariant state (see Definition 3.2). Let $\beta \in \mathbb{R}^+$ *and* $\vartheta, \lambda \in \mathbb{R}_0^+$*. Then, there is a measurable subset* $\tilde{\Omega} \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)} \subset \Omega$ *of full measure (that is,* $\tilde{\Omega} \in \mathfrak{A}_\Omega$ *and* $\mathfrak{a}_\Omega(\tilde{\Omega}) = 1$*) such that, for any* $\omega \in \tilde{\Omega}$*, one has:*
(p) *Paramagnetic transport coefficient: For all* $t \in \mathbb{R}$*,*

$$\Xi_{\mathrm{p}}(t) = \lim_{l \to \infty} \Xi_{\mathrm{p},l}^{(\omega)}(t) \ .$$

The above limit is uniform for times t *on compact sets.*
(d) *Diamagnetic transport coefficient:*

$$\Xi_{\mathrm{d}} = \lim_{l \to \infty} \Xi_{\mathrm{d},l}^{(\omega)} \ .$$

Proof: Assertion (p) is proven in a similar way as Theorem 7.9. See Equation (108) and the arguments thereafter. Note only that the pointwise convergence of any equicontinuous family of functions on $\mathbb{R}$ implies its uniform convergence on compacta. The proof of Assertion (d) is even simpler because there is no time dependency. We omit the details. ∎

4.2 Macroscopic Ohm's Law

For any $l \in \mathbb{R}^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$, we consider now the space–rescaled vector potential

$$\mathbf{A}_l(t,x) := \mathbf{A}(t, l^{-1}x) \ , \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d \ . \tag{38}$$

Since Ohm's law is a linear response to electric fields, we also rescale the strength of the electromagnetic potential $\mathbf{A}_l$ by a real parameter $\eta \in \mathbb{R}$ and study the behavior of current densities in the limit $\eta \to 0$.

Exactly like in [BPK2, Section 3] and [BP1, Section 3.3], w.l.o.g. we consider space–homogeneous (though time–dependent) electric fields in the box Λ_l defined by (18) for $l \in \mathbb{R}^+$. More precisely, let $\vec{w} \in \mathbb{R}^d$ be any (normalized w.r.t. the usual Euclidian norm) vector, $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$ and set $\mathcal{E}_t := -\partial_t \mathcal{A}_t$ for all $t \in \mathbb{R}$. Then, $\bar{\mathbf{A}} \in \mathbf{C}_0^\infty$ is defined to be the electromagnetic potential such that the electric field equals $\mathcal{E}_t \vec{w}$ at time $t \in \mathbb{R}$ for all $x \in [-1,1]^d$ and $(0,0,\ldots,0)$ for $t \in \mathbb{R}$ and $x \notin [-1,1]^d$. This choice yields rescaled electromagnetic potentials $\eta \bar{\mathbf{A}}_l$ as defined by (38) for $l \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$.

For any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\vartheta, \lambda \in \mathbb{R}_0^+$, $\eta \in \mathbb{R}$, $\vec{w} \in \mathbb{R}^d$, $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$ and $t \geq t_0$, the total current density is the sum of three currents defined from (29) and (30):

(th) The (thermal) current density

$$\mathbb{J}_{\mathrm{th}}^{(\omega,l)} \equiv \mathbb{J}_{\mathrm{th}}^{(\beta,\omega,\vartheta,\lambda,l)} \in \mathbb{R}^d$$

at thermal equilibrium inside the box Λ_l is defined, for any $k \in \{1, \ldots, d\}$, by

$$\left\{\mathbb{J}_{\mathrm{th}}^{(\omega,l)}\right\}_k := |\Lambda_l|^{-1} \sum_{x\in\Lambda_l} \varrho^{(\beta,\omega,\vartheta,\lambda)}\left(I_{(x+e_k,x)}^{(\omega,\vartheta)}\right) . \tag{39}$$

(p) The paramagnetic current density is the map

$$t \mapsto \mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) \equiv \mathbb{J}_{\mathrm{p}}^{(\beta,\omega,\vartheta,\lambda,\eta\bar{\mathbf{A}}_l)}(t) \in \mathbb{R}^d$$

defined by the space average of the current increment vector inside the box Λ_l at time $t \geq t_0$, that is, for any $k \in \{1, \ldots, d\}$,

$$\left\{\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t)\right\}_k := |\Lambda_l|^{-1} \sum_{x\in\Lambda_l} \rho_t^{(\beta,\omega,\vartheta,\lambda,\eta\bar{\mathbf{A}}_l)}\left(I_{(x+e_k,x)}^{(\omega,\vartheta)}\right) - \left\{\mathbb{J}_{\mathrm{th}}^{(\omega,l)}\right\}_k . \tag{40}$$

(d) The diamagnetic (or ballistic) current density

$$t \mapsto \mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) \equiv \mathbb{J}_{\mathrm{d}}^{(\beta,\omega,\vartheta,\lambda,\eta\bar{\mathbf{A}}_l)}(t) \in \mathbb{R}^d$$

is defined analogously, for any $t \geq t_0$ and $k \in \{1, \ldots, d\}$, by

$$\left\{\mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t)\right\}_k := |\Lambda_l|^{-1} \sum_{x\in\Lambda_l} \rho_t^{(\beta,\omega,\vartheta,\lambda,\eta\bar{\mathbf{A}}_l)}\left(\mathrm{I}_{(x+e_k,x)}^{(\omega,\vartheta,\eta\bar{\mathbf{A}}_l)}\right) . \tag{41}$$

For more details on the physical interpretation of these currents, see [BPK2, Section 3.4].

By [BP1, Theorem 3.7] and Conditions (14)–(15) and (21)–(22), the current densities behave, at small $|\eta|$ and uniformly w.r.t. the size of the box, linearly w.r.t. η: For any $\vartheta_0 \in \mathbb{R}_0^+$, $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$ and $\eta \in \mathbb{R}$,

$$\begin{aligned} \mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) &= \eta J_{\mathrm{p},l}^{(\omega,\mathcal{A})}(t) + \mathcal{O}\left(\eta^2\right) , \quad J_{\mathrm{p},l}^{(\omega,\mathcal{A})}(t) := \int_{t_0}^t \left(\Xi_{\mathrm{p},l}^{(\omega)}(t-s)\,\vec{w}\right)\mathcal{E}_s \mathrm{d}s , \\ \mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) &= \eta J_{\mathrm{d},l}^{(\omega,\mathcal{A})}(t) + \mathcal{O}\left(\eta^2\right) , \quad J_{\mathrm{d},l}^{(\omega,\mathcal{A})}(t) := \left(\Xi_{\mathrm{d},l}^{(\omega)}\vec{w}\right)\int_{t_0}^t \mathcal{E}_s \mathrm{d}s , \end{aligned}$$

uniformly for $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\vartheta \in [0, \vartheta_0]$, $\lambda \in \mathbb{R}_0^+$, $\vec{w} \in \mathbb{R}^d$ (normalized) and $t \geq t_0$.

The $\mathbb{R}^d$–valued linear coefficients

$$J_{\mathrm{p},l}^{(\omega,\mathcal{A})} \equiv J_{\mathrm{p},l}^{(\beta,\omega,\vartheta,\lambda,\vec{w},\mathcal{A})} \qquad \text{and} \qquad J_{\mathrm{d},l}^{(\omega,\mathcal{A})} \equiv J_{\mathrm{d},l}^{(\beta,\omega,\vartheta,\lambda,\vec{w},\mathcal{A})}$$

of the paramagnetic and diamagnetic current densities, respectively, become deterministic for large boxes. They are directly related to Ξ_{p} and Ξ_{d} via Ohm's law:

Theorem 4.2 (Macroscopic Ohm's law)
Assume (14)–(15), (20)–(22) and that the map (24) is a random invariant state. Let $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$. Then, there is a measurable subset $\tilde{\Omega} \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)} \subset \Omega$ of full measure such that, for any $\omega \in \tilde{\Omega}$, $\vec{w} \in \mathbb{R}^d$, $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$ and $t \geq t_0$, the following assertions hold true:
(th) *Thermal current density:*

$$\lim_{l\to\infty} \left\{ \mathbb{J}_{\mathrm{th}}^{(\omega,l)} \right\}_k = \mathbb{E}\left[\varrho^{(\beta,\omega,\vartheta,\lambda)}(I_{(e_k,0)}^{(\omega,\vartheta)}) \right] , \qquad k \in \{1,\ldots,d\} .$$

(p) *Paramagnetic current density:*

$$\lim_{l\to\infty} J_{\mathrm{p},l}^{(\omega,\mathcal{A})}(t) = \lim_{l\to\infty} \left(\partial_\eta \mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) \Big|_{\eta=0} \right) = \int_{t_0}^{t} \left(\Xi_{\mathrm{p}}(t-s)\, \vec{w} \right) \mathcal{E}_s \mathrm{d}s .$$

(d) *Diamagnetic current density:*

$$\lim_{l\to\infty} J_{\mathrm{d},l}^{(\omega,\mathcal{A})}(t) = \lim_{l\to\infty} \left(\partial_\eta \mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) \Big|_{\eta=0} \right) = (\Xi_{\mathrm{d}}\vec{w}) \int_{t_0}^{t} \mathcal{E}_s \mathrm{d}s .$$

Proof: (th) is similar to [BPK3, Corollary 5.7 (th)]. Assertions (p) and (d) are deduced from Theorem 4.1 and Lebesgue's dominated convergence theorem. Note that the intersection of three measurable sets of full measure has full measure. ■

Like [BP1, Theorem 3.7], Theorem 4.2 can also be extended to space–inhomogeneous macroscopic electromagnetic fields, that is, for space–rescaled vector potentials $\mathbf{A}_l$ (38) with arbitrary $\mathbf{A} \in \mathbf{C}_0^\infty$.

4.3 Green–Kubo Relations

Because of Theorem 4.2 (p)–(d), Ξ_{p} and Ξ_{d} are both *charge* transport coefficents. Thus, they are also named here *paramagnetic* and *diamagnetic (in–phase) conductivities*, respectively. From (36) we can deduce Green–Kubo relations for Ξ_{p} via current Duhamel fluctuations as follows.

Fix in all the subsection $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$. The Duhamel two–point function $(\cdot,\cdot)_\sim^{(\omega)}$ is defined by

$$(B_1, B_2)_\sim^{(\omega)} \equiv (B_1, B_2)_\sim^{(\beta,\omega,\vartheta,\lambda)} := \int_0^\beta \varrho^{(\beta,\omega,\vartheta,\lambda)} \left(B_1^* \tau_{i\alpha}^{(\omega,\vartheta,\lambda)}(B_2) \right) \mathrm{d}\alpha$$

for any $B_1, B_2 \in \mathcal{U}$ and $\omega \in \Omega$. See for instance [BPK2, Section A] and references therein for more details. For any $l \in \mathbb{R}^+$ and $B \in \mathcal{U}$, set

$$\mathbb{F}^{(l)}(B) := \frac{1}{|\Lambda_l|^{1/2}} \sum_{x \in \Lambda_l} \left\{ \chi_x(B) - \varrho^{(\beta,\omega,\vartheta,\lambda)}(\chi_x(B)) \mathbf{1}_{\mathcal{U}} \right\} . \tag{42}$$

We name it the *fluctuation observable* of the element $B \in \mathcal{U}$ in the box Λ_l. Recall that $\{\chi_x\}_{x\in\mathfrak{L}}$ implements the action of the group $(\mathbb{Z}^d, +)$ of lattice translations on the CAR C^*–algebra $\mathcal{U}$, see (13).

Then, by [BPK2, Eq. (103)] together with (36), one obtains *Green–Kubo relations* for the paramagnetic (in–phase) conductivity: For any $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$,

$$\begin{aligned} \{\Xi_{\mathrm{p}}(t)\}_{k,q} \;=\; & \lim_{l\to\infty} \mathbb{E}\left[\left(\mathbb{F}^{(l)}(I_{(e_k,0)}^{(\omega,\vartheta)}), \mathbb{F}^{(l)}(\tau_t^{(\omega,\vartheta,\lambda)}(I_{(e_q,0)}^{(\omega,\vartheta)})) \right)_\sim^{(\omega)} \right. \\ & \left. - \left(\mathbb{F}^{(l)}(I_{(e_k,0)}^{(\omega,\vartheta)}), \mathbb{F}^{(l)}(I_{(e_q,0)}^{(\omega,\vartheta)}) \right)_\sim^{(\omega)} \right] \end{aligned} \tag{43}$$

with the current observable $I_{(x,y)}^{(\omega,\vartheta)}$ defined by (29). The right hand side (r.h.s.) of the above equation is a current Duhamel fluctuation *increment*. If Conditions (14)–(15) and (20) hold and the map (24) is a random invariant state, then the above limit always exists (and is thus finite), by Theorem 7.1.

Note however that, possibly,

$$\limsup_{l\to\infty} \mathbb{E}\left[\left(\mathbb{F}^{(l)}(I_{(e_k,0)}^{(\omega,\vartheta)}), \mathbb{F}^{(l)}(I_{(e_q,0)}^{(\omega,\vartheta)}) \right)_\sim^{(\omega)} \right] = \infty \tag{44}$$

for some $k, q \in \{1, \ldots, d\}$. In other words, it is not a priori clear whether the interacting quantum system has finite current Duhamel fluctuations or not. When it is finite, similar to [BPK4, Section 3], we can construct a Hilbert space of fluctuations, which implies the existence of a finite conductivity measure as a spectral measure. The finiteness of current Duhamel fluctuations is proven in [BPK4, Section 3] for the non–interacting case with random static potentials and space–homogeneous hopping terms. This can also be shown for sufficiently small $\left\|\Psi^{\mathrm{IP}}\right\|_{\mathcal{W}}$ and disorder strengths ϑ, λ, by using methods of constructive quantum field theory.

5 AC–Conductivity Measure From Joule's Law

Similar to [BPK3, Section 4.3], our derivation of a macroscopic (in–phase) AC–conductivity measure is based on the 2nd law of thermodynamics. It dovetails with the celebrated Joule's law of (classical) electricity theory. To this end we start by introducing energy increment densities, in particular the heat production density.

5.1 Energy Increment Densities

The internal energy observable $H_L^{(\omega,\vartheta,\lambda)} \in \mathcal{U}^+ \cap \mathcal{U}_\Lambda$ of the interacting fermion system for the box Λ_L (18) is defined by

$$\begin{aligned} H_L^{(\omega,\vartheta,\lambda)} &:= \sum_{\Lambda \subset \Lambda_L} \Psi_\Lambda^{(\omega,\vartheta)} + \lambda \sum_{x \in \Lambda_L} \omega_1(x) a_x^* a_x \\ &= \sum_{x,y \in \Lambda_L} \langle \mathfrak{e}_x, (\Delta_{\omega,\vartheta} + \lambda V_\omega) \mathfrak{e}_y \rangle a_x^* a_y + \sum_{\Lambda \subset \Lambda_L} \Psi_\Lambda^{\mathrm{IP}} \end{aligned} \tag{45}$$

for $\omega = (\omega_1, \omega_2) \in \Omega$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and $L \in \mathbb{R}^+$. When the electromagnetic field is switched on, i.e., for $t \geq t_0$, the total energy observable for the box Λ_L that includes the region where the electromagnetic field does not vanish equals

$$H_L^{(\omega,\vartheta,\lambda)} + W_t^{(\omega,\vartheta,\mathbf{A})} ,$$

where, for any $\omega \in \Omega$, $\vartheta \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$ and $t \in \mathbb{R}$,

$$W_t^{(\omega,\vartheta,\mathbf{A})} := \sum_{x,y \in \mathfrak{L}} \langle \mathfrak{e}_x, (\Delta_{\omega,\vartheta}^{(\mathbf{A})} - \Delta_{\omega,\vartheta}) \mathfrak{e}_y \rangle a_x^* a_y \in \mathcal{U}^+ \cap \mathcal{U}_0$$

is the electromagnetic potential energy observable.

Like in [BP1, Sections 3.1, 3.4], we now define four sorts of energy increments associated with the fermion system for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$:

(**Q**) The internal energy increment $\mathbf{S}^{(\omega,\mathbf{A})} \equiv \mathbf{S}^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}_0^+$ defined by

$$\mathbf{S}^{(\omega,\mathbf{A})}(t) := \lim_{L\to\infty} \left\{ \rho_t^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}(H_L^{(\omega,\vartheta,\lambda)}) - \varrho^{(\beta,\omega,\vartheta,\lambda)}(H_L^{(\omega,\vartheta,\lambda)}) \right\} .$$

Under Conditions (14)–(15) and (21)–(22), this map has non–negative finite value and is the heat production because of [BP1, Theorem 3.2].

(**P**) The electromagnetic potential energy increment $\mathbf{P}^{(\omega,\mathbf{A})} \equiv \mathbf{P}^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathbf{P}^{(\omega,\mathbf{A})}(t) := \rho_t^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}(W_t^{(\omega,\vartheta,\mathbf{A})}) .$$

(p) The paramagnetic energy increment $\mathfrak{I}_{\mathrm{p}}^{(\omega,\mathbf{A})} \equiv \mathfrak{I}_{\mathrm{p}}^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathfrak{I}_{\mathrm{p}}^{(\omega,\mathbf{A})}(t) := \lim_{L\to\infty} \Big\{ \rho_t^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}(H_L^{(\omega,\vartheta,\lambda)} + W_t^{(\omega,\vartheta,\mathbf{A})}) - \varrho^{(\beta,\omega,\vartheta,\lambda)}(H_L^{(\omega,\vartheta,\lambda)} + W_t^{(\omega,\vartheta,\mathbf{A})}) \Big\} .$$

(d) The diamagnetic energy increment $\mathfrak{I}_{\mathrm{d}}^{(\omega,\mathbf{A})} \equiv \mathfrak{I}_{\mathrm{d}}^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathfrak{I}_{\mathrm{d}}^{(\omega,\mathbf{A})}(t) := \varrho^{(\beta,\omega,\vartheta,\lambda)}(W_t^{(\omega,\vartheta,\mathbf{A})}) .$$

See [BPK2] for more discussions on the physical interpretation of these energies. Note only that all limits exist at all times because the total energy increment equals

$$\mathbf{S}^{(\omega,\mathbf{A})}(t) + \mathbf{P}^{(\omega,\mathbf{A})}(t) = \mathfrak{I}_{\mathrm{p}}^{(\omega,\mathbf{A})}(t) + \mathfrak{I}_{\mathrm{d}}^{(\omega,\mathbf{A})}(t) .$$

This last quantity is shown in [BP1, Theorem 3.2 (ii)] to be the *work* performed by the electric field and is given by an expression like (1).

Under Conditions (14)–(15) and (21)–(22), all increment energies defined above are of order $\mathcal{O}(\eta^2 l^d)$, as $l \to \infty$, by [BP1, Theorem 3.8]. Indeed, because of possibly non–vanishing thermal currents, the energy increments $\mathbf{P}^{(\omega,\mathbf{A})}$ and $\mathfrak{I}_{\mathrm{d}}^{(\omega,\mathbf{A})}$ are rather $\mathcal{O}(|\eta|\, l^d)$ at small $l \in \mathbb{R}_0^+$. As a consequence, for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$, we define four energy densities:

(**Q**) The *heat production* (or internal energy increment) density $\mathbf{s} \equiv \mathbf{s}^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}_0^+$ defined by

$$\mathbf{s}(t) := \lim_{(\eta,l^{-1})\to(0,0)} \left\{ \left(\eta^2 |\Lambda_l|\right)^{-1} \mathbf{S}^{(\omega,\eta\mathbf{A}_l)}(t) \right\} . \tag{46}$$

(**P**) The (electromagnetic) *potential* energy (increment) density $\mathbf{p} \equiv \mathbf{p}^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathbf{p}(t) := \lim_{\eta\to 0} \lim_{l\to\infty} \left\{ \left(\eta^2 |\Lambda_l|\right)^{-1} \mathbf{P}^{(\omega,\eta\mathbf{A}_l)}(t) \right\} . \tag{47}$$

(p) The *paramagnetic* energy (increment) density $\mathfrak{i}_{\mathrm{p}} \equiv \mathfrak{i}_{\mathrm{p}}^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathfrak{i}_{\mathrm{p}}(t) := \lim_{(\eta,l^{-1})\to(0,0)} \left\{ \left(\eta^2 |\Lambda_l|\right)^{-1} \mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t) \right\} . \tag{48}$$

(d) The *diamagnetic* energy (increment) density $\mathfrak{i}_{\mathrm{d}} \equiv \mathfrak{i}_{\mathrm{d}}^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})}$ the map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathfrak{i}_{\mathrm{d}}(t) := \lim_{\eta\to 0} \lim_{l\to\infty} \left\{ \left(\eta^2 |\Lambda_l|\right)^{-1} \mathfrak{I}_{\mathrm{d}}^{(\omega,\eta\mathbf{A}_l)}(t) \right\} . \tag{49}$$

On a measurable subset of full measure, all energy (increment) densities become deterministic functions that are derived in the next subsection. We explain this in the next subsection.

5.2 Macroscopic Joule's Law

Similar to the heuristics presented in [BPK3, Section 4.2], we expect from Theorem 4.2 that, for $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and any (possibly space inhomogeneous) electromagnetic potential $\mathbf{A} \in \mathbf{C}_0^\infty$, the electric field $E_{\mathbf{A}}$ yields *space–dependent* paramagnetic and diamagnetic current linear response coefficients respectively equal to

$$J_{\mathrm{p}}(t,x) \equiv J_{\mathrm{p}}^{(\beta,\vartheta,\lambda,\mathbf{A})}(t,x) := \int_{t_0}^{t} \Xi_{\mathrm{p}}(t-s)\, E_{\mathbf{A}}(s,x)\mathrm{d}s , \tag{50}$$

$$J_{\mathrm{d}}(t,x) \equiv J_{\mathrm{d}}^{(\beta,\vartheta,\lambda,\mathbf{A})}(t,x) := \Xi_{\mathrm{d}} \int_{t_0}^{t} E_{\mathbf{A}}(s,x)\mathrm{d}s , \tag{51}$$

at any position $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}$. These current linear response coefficients yield two electric work or energy densities produced by the paramagnetic and diamagnetic currents. This fact is proven in the following theorem:

Theorem 5.1 (Macroscopic Joule's law)
Assume (14)–(15), (20)–(22) and that the map (24) is a random invariant state. Let $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$. Then, there is a measurable subset $\tilde{\Omega} \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)} \subset \Omega$ of full measure such that, for any $\omega \in \tilde{\Omega}$, $\mathbf{A} \in \mathbf{C}_0^\infty$ and $t \geq t_0$:
(p) *Paramagnetic energy density:*

$$\mathfrak{i}_{\mathrm{p}}(t) = \int_{\mathbb{R}^d} \mathrm{d}^d x \int_{t_0}^{t} \mathrm{d}s \, \langle E_{\mathbf{A}}(s,x), J_{\mathrm{p}}(s,x) \rangle_{\mathbb{R}^d} \ .$$

(d) *Diamagnetic energy density:*

$$\mathfrak{i}_{\mathrm{d}}(t) = \int_{\mathbb{R}^d} \mathrm{d}^d x \int_{t_0}^{t} \mathrm{d}s \, \langle E_{\mathbf{A}}(s,x), J_{\mathrm{d}}(s,x) \rangle_{\mathbb{R}^d} \ .$$

(**Q**) *Heat production density:*

$$\mathbf{s}(t) = \mathfrak{i}_{\mathrm{p}}(t) - \int_{\mathbb{R}^d} \mathrm{d}^d x \int_{t_0}^{t} \mathrm{d}s \, \langle E_{\mathbf{A}}(s,x), J_{\mathrm{p}}(t,x) \rangle_{\mathbb{R}^d} \ .$$

(**P**) *Electromagnetic potential energy density:*

$$\mathbf{p}(t) = \mathfrak{i}_{\mathrm{d}}(t) + \int_{\mathbb{R}^d} \mathrm{d}^d x \int_{t_0}^{t} \mathrm{d}s \, \langle E_{\mathbf{A}}(s,x), J_{\mathrm{p}}(t,x) \rangle_{\mathbb{R}^d} \ .$$

Proof: The proof is very similar to the proof of [BPK3, Theorem 4.1]. It is a consequence of the Akcoglu–Krengel ergodic theorem, Lieb–Robinson bounds [BP3, Theorem 3.6 (iv)] and [BP1, Theorem 3.8]. For the detailed proof of Assertion (p), see Theorem 7.9. We omit the details for Assertions (d), (Q) and (P). ∎

For more discussions on this subject, see [BPK3, Section 4.2]. In fact, the above result is an extension of [BPK3, Theorem 4.1] to fermion systems *with interactions*.

5.3 AC–Conductivity Measure

At $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$, the paramagnetic transport coefficient $\Xi_{\mathrm{p}} \equiv \Xi_{\mathrm{p}}^{(\beta,\vartheta,\lambda)}$ is a well–defined $\mathcal{B}(\mathbb{R}^d)$–valued function of time. See (36). It is also named here paramagnetic (in–phase) conductivity, because of Theorem 4.2. Using Theorem 7.1 we have $\Xi_{\mathrm{p}} \in C^2\left(\mathbb{R}, \mathcal{B}(\mathbb{R}^d)\right)$, provided Conditions (14)–(15) and (20)–(22) hold and that the map (24) is a random invariant state. These properties together with the positivity of the heat production (Theorem 5.1), i.e., the 2nd law of thermodynamics, lead to a Lévy–Khintchine representation of the symmetric part of the paramagnetic (in–phase) conductivity Ξ_{p}.

Indeed, for any $\Upsilon \in \mathcal{B}(\mathbb{R}^d)$, define its symmetric and antisymmetric parts, w.r.t. to the canonical scalar product of $\mathbb{R}^d$, respectively by

$$[\Upsilon]_+ := \frac{1}{2}\left(\Upsilon + \Upsilon^{\mathrm{t}}\right) \qquad \text{and} \qquad [\Upsilon]_- := \frac{1}{2}\left(\Upsilon - \Upsilon^{\mathrm{t}}\right) . \tag{52}$$

Here, $\Upsilon^{\mathrm{t}} \in \mathcal{B}(\mathbb{R}^d)$ stands for the transpose of the operator $\Upsilon \in \mathcal{B}(\mathbb{R}^d)$ (w.r.t. the canonical scalar product of $\mathbb{R}^d$). Then we have:

Theorem 5.2 (Lévy–Khintchine representation of $[\Xi_{\mathrm{p}}]_+$)
Assume (14)–(15), (20)–(22) and that the map (24) is a random invariant state. For any $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$, there is a unique finite and symmetric $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\boldsymbol{\mu} \equiv \boldsymbol{\mu}^{(\beta,\vartheta,\lambda)}$ on $\mathbb{R}$ such that, for any $t \in \mathbb{R}$,

$$[\Xi_{\mathrm{p}}(t)]_+ = -\frac{t^2}{2}\boldsymbol{\mu}(\{0\}) + \int_{\mathbb{R}\backslash\{0\}} (\cos(t\nu) - 1)\, \nu^{-2} \boldsymbol{\mu}(\mathrm{d}\nu) .$$

$\mathcal{B}_+(\mathbb{R}^d) \subset \mathcal{B}(\mathbb{R}^d)$ stands for the set of positive linear operators on $\mathbb{R}^d$, i.e., symmetric operators w.r.t. to the canonical scalar product of $\mathbb{R}^d$ with positive eigenvalues.

Proof: For all $\varphi \in C_0^\infty(\mathbb{R}; \mathbb{R}^d)$, observe that its derivative $\varphi' \in C_0^\infty(\mathbb{R}; \mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}} \varphi'(s)\, \mathrm{d}s = 0 \in \mathbb{R}^d . \tag{53}$$

As a consequence, we infer from Theorem 5.1 (p) and the equality

$$\Xi_{\mathrm{p}}(-t) = \Xi_{\mathrm{p}}(t)^{\mathrm{t}} , \qquad t \in \mathbb{R} , \tag{54}$$

that, for any $\varphi \in C_0^\infty(\mathbb{R};\mathbb{R}^d)$,

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \ \langle \varphi'(s), [\Xi_{\mathrm{p}}(t-s)]_+ \varphi'(t)\rangle_{\mathbb{R}^d} &= \\ \int_{t_0}^{t_1} \mathrm{d}s \int_{t_0}^{s} \mathrm{d}t \ \langle \varphi'(s), \Xi_{\mathrm{p}}(t-s) \varphi'(t)\rangle_{\mathbb{R}^d} &\geq 0 \, . \end{aligned} \tag{55}$$

Note that (54) is a simple consequence of the stationarity of KMS states. By Theorem 7.1, if (14)–(15) and (20)–(22) hold and the map (24) is a random invariant state, then $[\Xi_{\mathrm{p}}]_+ \in C^2\left(\mathbb{R}, \mathcal{B}(\mathbb{R}^d)\right)$. By integration by parts, it follows from (55) that

$$\int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \left\langle \varphi(s), \partial_t^2 [\Xi_{\mathrm{p}}(t-s)]_+ \varphi(t)\right\rangle_{\mathbb{R}^d} \leq 0 \, .$$

By (52) and (54), $[\Xi_{\mathrm{p}}(-t)]_+ = [\Xi_{\mathrm{p}}(t)]_+$. Hence,

$$\partial_t^2 [\Xi_{\mathrm{p}}(-t)]_+ = \partial_t^2 [\Xi_{\mathrm{p}}(t)]_+$$

for any $t \in \mathbb{R}$. Therefore, $-\partial_t^2[\Xi_{\mathrm{p}}]_+ : \mathbb{R} \to \mathcal{B}(\mathbb{R}^d)$ is a weakly positive definite continuous map that is symmetric w.r.t. time reversal. Moreover, for any $t \in \mathbb{R}$, $\partial_t^2[\Xi_{\mathrm{p}}(t)]_+$ is (by definition) a symmetric operator w.r.t. the canonical scalar product of $\mathbb{R}^d$. Then, we can apply Corollary 7.11 with $\Upsilon = -\partial_t^2[\Xi_{\mathrm{p}}]_+$ to deduce the existence of a unique finite and symmetric $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\boldsymbol{\mu}$ on $\mathbb{R}$ such that

$$\partial_t^2 [\Xi_{\mathrm{p}}(t)]_+ = -\int_{\mathbb{R}} \cos(t\nu)\, \boldsymbol{\mu}(\mathrm{d}\nu) \, . \tag{56}$$

Observe that $[\Xi_{\mathrm{p}}(0)]_+ = \partial_t[\Xi_{\mathrm{p}}(0)]_+ = 0$. Therefore, by integrating this last expression twice, we then obtain that

$$[\Xi_{\mathrm{p}}(t)]_+ = -\frac{t^2}{2}\boldsymbol{\mu}(\{0\}) - \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\alpha \int_{\mathbb{R}\backslash\{0\}} \boldsymbol{\mu}(\mathrm{d}\nu) \cos(\alpha\nu) \, . \tag{57}$$

Since $\boldsymbol{\mu}$ is a a finite measure on $\mathbb{R}$, we can apply twice the Fubini (–Tonelli) theorem to deduce that

$$\begin{aligned} [\Xi_{\mathrm{p}}(t)]_+ &= -\frac{t^2}{2}\boldsymbol{\mu}(\{0\}) - \int_0^t \mathrm{d}s \int_{\mathbb{R}\backslash\{0\}} \boldsymbol{\mu}(\mathrm{d}\nu) \left(\nu^{-1}\sin(s\nu)\right) \\ &= -\frac{t^2}{2}\boldsymbol{\mu}(\{0\}) + \int_{\mathbb{R}\backslash\{0\}} \left(\cos(t\nu) - 1\right) \nu^{-2} \boldsymbol{\mu}(\mathrm{d}\nu) \, . \end{aligned}$$

All integrals are of course well–defined because $\sin(\nu) = \mathcal{O}(\nu)$ and $1 - \cos(\nu) = \mathcal{O}(\nu^2)$, as $\nu \to 0$. ∎

From Theorem 5.1 it is easy to see that the restriction of $\nu^{-2}\boldsymbol{\mu}(\mathrm{d}\nu)$ on $\mathbb{R}\backslash\{0\}$ quantifies the heat production per unit volume due to the component of frequency $\nu \in \mathbb{R}\backslash\{0\}$ of the electric field in accordance with Joule's law in the AC–regime. By (55), note at this point that the antisymmetric component $[\Xi_{\mathrm{p}}]_-$ of the paramagnetic conductivity does not contribute to heat production. Therefore, we define this measure to be the (in–phase) *AC–conductivity measure*:

Definition 5.3 (AC–conductivity measure)
We name $\mu_{\mathrm{AC}} \equiv \mu_{\mathrm{AC}}^{(\beta,\vartheta,\lambda)}$, the restriction of $\nu^{-2}\boldsymbol{\mu}(\mathrm{d}\nu)$ to $\mathbb{R}\backslash\{0\}$, the (in–phase) AC–conductivity measure.

The AC–conductivity measure does not vanish in general, see, e.g., [BPK4, Theorem 4.7]. In this case, we even have the following property:

Proposition 5.4 (Non–triviality of μ_{AC} at high frequencies)
Assume all conditions of Theorem 5.2. If $\mu_{\mathrm{AC}}(\mathbb{R}\backslash\{0\}) \neq 0$ then, for any $\nu \in \mathbb{R}^+$, $\mu_{\mathrm{AC}}(\mathbb{R}\backslash[-\nu,\nu]) \neq 0$.

Proof: By (56), note that $\partial_t^2[\Xi_{\mathrm{p}}(0)]_+ = -\boldsymbol{\mu}(\mathbb{R})$. Hence, in this case, $[\Xi_{\mathrm{p}}]_+$ behaves as

$$[\Xi_{\mathrm{p}}(t)]_+ = -\frac{t^2}{2}\boldsymbol{\mu}(\mathbb{R}) + o\left(t^2\right) , \qquad \text{as } t \to 0 . \tag{58}$$

Of course, if $\mu_{\mathrm{AC}}(\mathbb{R}\backslash\{0\}) \neq 0$ then $\boldsymbol{\mu}(\mathbb{R}) \neq 0$. Moreover, (57) yields

$$\|[\Xi_{\mathrm{p}}(t)]_+\|_{\mathcal{B}(\mathbb{R}^d)} \leq \frac{t^2}{2}\|\boldsymbol{\mu}\|_{\mathcal{B}(\mathbb{R}^d)}(\mathbb{R}) , \qquad t \in \mathbb{R} , \tag{59}$$

where, for any $\mathcal{B}(\mathbb{R}^d)$–valued measure μ on $\mathbb{R}$, $\|\mu\|_{\mathcal{B}(\mathbb{R}^d)}$ denotes the positive measure on $\mathbb{R}$ defined, for any Borel set $\mathcal{X}$, by

$$\|\mu\|_{\mathcal{B}(\mathbb{R}^d)}(\mathcal{X}) := \sup\left\{\sum_{i\in I}\|\mu(\mathcal{X}_i)\|_{\mathcal{B}(\mathbb{R}^d)} : \{\mathcal{X}_i\}_{i\in I} \text{ is a finite Borel partition of } \mathcal{X}\right\} .$$

Take any non–vanishing Schwartz function $\varphi : \mathbb{R} \to \mathbb{R}$ with Fourier transform $\hat{\varphi}$ supported on $[1,2]$. For any $\alpha \in \mathbb{R}^+$, let φ_α be defined by

$$\varphi_\alpha(t) = \alpha^{-1}\varphi\left(\alpha^{-1}t\right) , \qquad t \in \mathbb{R} .$$

Let $\vec{w} \in \mathbb{R}^d$ be any vector such that

$$\langle \vec{w}, \boldsymbol{\mu}(\mathbb{R}) \vec{w} \rangle_{\mathbb{R}^d} > 0 . \tag{60}$$

Then, by (58)–(59) and Lebesgue's dominated convergence theorem, as $\alpha \to 0^+$, the full heat density produced by the electric field $\vec{w}\varphi$ equals

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \ \langle \varphi_\alpha(s) \vec{w}, [\boldsymbol{\Xi}_{\mathrm{p}}(t-s)]_+ \varphi_\alpha(t) \vec{w} \rangle_{\mathbb{R}^d} \\
= \ & -\frac{1}{4} \langle \vec{w}, \boldsymbol{\mu}(\mathbb{R}) \vec{w} \rangle_{\mathbb{R}^d} \int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \, \varphi_\alpha(s) (t-s)^2 \varphi_\alpha(t) + o\left(\alpha^2\right) \\
= \ & \frac{1}{2} \langle \vec{w}, \boldsymbol{\mu}(\mathbb{R}) \vec{w} \rangle_{\mathbb{R}^d} \alpha^2 \left(\int_{\mathbb{R}} s\varphi(s) \, \mathrm{d}s \right)^2 + o\left(\alpha^2\right) .
\end{aligned}$$

For sufficiently small $\alpha \in \mathbb{R}^+$, it follows that

$$\int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \ \langle \varphi_\alpha(s) \vec{w}, [\boldsymbol{\Xi}_{\mathrm{p}}(t-s)]_+ \varphi_\alpha(t) \vec{w} \rangle_{\mathbb{R}^d} > 0 ,$$

using (60). On the other hand, for any $\alpha \in \mathbb{R}^+$,

$$\begin{aligned}
& \int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \ \langle \varphi_\alpha(s) \vec{w}, [\boldsymbol{\Xi}_{\mathrm{p}}(t-s)]_+ \varphi_\alpha(t) \vec{w} \rangle_{\mathbb{R}^d} \\
\leq \ & \|\vec{w}\|_{\mathbb{R}^d}^2 \|\hat{\varphi}\|_\infty^2 \|\mu_{\mathrm{AC}}\left(\left[\alpha^{-1}, \infty\right)\right)\|_{\mathcal{B}(\mathbb{R}^d)} .
\end{aligned}$$

The assertion is then a consequence of the last two inequalities. ■

In the non–interacting case, we show in [BPK4, Theorem 4.1] that $\boldsymbol{\mu}(\{0\}) = 0$ and μ_{AC} is a *finite* measure on $\mathbb{R} \backslash \{0\}$:

$$\|\mu_{\mathrm{AC}}\|_{\mathcal{B}(\mathbb{R}^d)}(\mathbb{R} \backslash \{0\}) = \int_{\mathbb{R} \backslash \{0\}} \nu^{-2} \|\boldsymbol{\mu}\|_{\mathcal{B}(\mathbb{R}^d)}(\mathrm{d}\nu) < \infty .$$

In particular, the measure $\boldsymbol{\mu}([-\nu, \nu])$ is $\mathcal{O}(\nu^2)$ in the limit $\nu \to 0^+$. These properties are directly related with the finiteness of current Duhamel fluctuations in the limit of large space scales, which is not clear in presence of interactions, see (44) and discussion thereafter.

At high frequencies, by finiteness of the positive measure $\boldsymbol{\mu}$, the AC–condutivity measure satisfies

$$\mu_{\mathrm{AC}}([\nu, \infty)) \leq \nu^{-2} \boldsymbol{\mu}([\nu, \infty)) \leq \nu^{-2} \boldsymbol{\mu}(\mathbb{R}) , \qquad \nu \in \mathbb{R}^+ . \tag{61}$$

The same property of course holds for negative frequencies, by symmetry of $\boldsymbol{\mu}$ (w.r.t. ν). We can compare this property with the corresponding one of the celebrated Drude model.

Indeed, the (in–phase) AC–conductivity measure obtained from the Drude model is absolutely continuous w.r.t. the Lebesgue measure with the function

$$\nu \mapsto \vartheta_{\mathrm{T}}(\nu) \sim \frac{\mathrm{T}}{1+\mathrm{T}^2\nu^2} \tag{62}$$

being the corresponding Radon–Nikodym derivative. Here, the *relaxation time* $\mathrm{T} > 0$ is related to the mean time interval between two collisions of a charged carrier with defects in the crystal. This function is the Fourier transform of the in–phase conductivity

$$t \mapsto D \exp\left(-\mathrm{T}^{-1}|t|\right) ,$$

where $D \in \mathbb{R}^+$ is some strictly positive constant. See for instance [BPK4, Section 1] for more discussions.

At high frequencies, Drude's approach *heavily overestimates* the AC–conductivity measure μ_{AC} obtained from the more realistic model studied here. Indeed, we can infer from (61) that, in the limit $\nu \to \infty$ of high frequencies,

$$\lim_{\nu\to\infty}\left\{\nu^2\mu_{\mathrm{AC}}\left([\nu,\infty)\right)\right\} = 0 , \tag{63}$$

whereas, by (62), the corresponding quantity for the Drude model diverges in the same limit:

$$\nu^2\int_\nu^\infty \vartheta_{\mathrm{T}}(z)\,\mathrm{d}z \sim \nu^2 \arctan\left(\frac{1}{\mathrm{T}\nu}\right) = \mathcal{O}\left(\mathrm{T}^{-1}\nu\right) , \qquad \text{as } \nu\to\infty .$$

The same behavior as for the Drude model holds for the AC–conductivity measure obtained from the Lorentz–Drude model.

Hence, the asymptotics (63) motivates the use of the relaxation time as an effective ν–dependent parameter of the Drude model, i.e., one replaces T with $\mathrm{T}(\nu)$ in (62), as observed for instance in [T]. Indeed, with this Ansatz and the asymptotics (63), either $\mathrm{T}(\nu)$ vanishes faster than ν^{-3} or it diverges faster than ν, as $\nu \to \infty$. Note that experimental measurements seem to indicate that

$$\mathrm{T}(\nu) = \frac{\mathrm{T}(0)}{1+D\mathrm{T}(0)\nu^2}$$

in some metals. See for instance [T] for one experimental evidence of this fact and [NS1, NS2, SE, YRMK] for theoretical studies.

The concept of relaxation time or mean free path [So] (of electrons) in the Drude model and its extensions is very intuitive. However, the microscopic interpretation of this classical notion is difficult, in particular if one has to take T as a ν–dependent parameter. Quoting meanwhile [LTW, p. 24]:

Physicists had to wait for the discovery of quantum mechanics to understand why electrons apparently do not scatter from ions that occupy regular lattice sites: The wave character of an electron causes the electron to diffract from an ideal crystal. Resistance appears only when electrons scatter from imperfections in the crystal. With that quantum mechanical revision, the Drude model can still be used, but in the new picture an electron is envisaged as zigzagging between impurities.

Indeed, the average length an electron travels before it seems to collide with an ion or defects in the crystal is experimentally measured in metals to be about *two order of magnitude* larger than the lattice constant. [Note however that defects in our model are allowed to appear on all lattice sites via the probability measure $\mathfrak{a}_\Omega$, see Section 3.1.]

The high frequency asymptotics of the (in–phase) AC–conductivity discussed above makes explicit further problems with this classical picture. Observe moreover that if the interparticle interaction has stronger polynomial decay than in the assumptions of Theorem 5.2, then the asymptotics (63) can be improved by replacing ν^2 with ν^k for an integer $k > 2$. To show this, one uses Lieb–Robinson bounds for multi–commutators [BP3, Theorems 3.8–3.9] of order $k+1 > 3$ to get $\Xi_{\mathrm{p}} \in C^k\left(\mathbb{R}, \mathcal{B}(\mathbb{R}^d)\right)$. See also Remark 7.2. However, we expect the model to physically break down for frequencies ν corresponding to wavelengths (of light) of the order of the lattice spacing. For usual materials, it would dovetail with the frequency range of hard X–rays.

Similar to [BP1, Corollary 3.5], we deduce now general properties of the paramagnetic conductivity from Theorem 5.2:

Corollary 5.5 (Properties of $[\Xi_{\mathrm{p}}]_+$)
Assume all conditions of Theorem 5.2 and let $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$. Then, $[\Xi_{\mathrm{p}}]_+ \in C^2\left(\mathbb{R}, \mathcal{B}(\mathbb{R}^d)\right)$ and the following holds:
(i) *Time–reversal symmetry of* $[\Xi_{\mathrm{p}}]_+$*:* $[\Xi_{\mathrm{p}}(0)]_+ = 0$ *and*

$$[\Xi_{\mathrm{p}}(-t)]_+ = [\Xi_{\mathrm{p}}(t)]_+ \ , \qquad t \in \mathbb{R} \ .$$

(ii) *Negativity of* $[\Xi_{\mathrm{p}}]_+$*:*

$$-[\Xi_{\mathrm{p}}(t)]_+ \in \mathcal{B}_+(\mathbb{R}^d) \ , \qquad t \in \mathbb{R} \ .$$

(iii) *Cesàro mean of* $[\Xi_{\mathrm{p}}]_+$*: If* $\boldsymbol{\mu}(\{0\}) = 0$ *and* $\|\mu_{\mathrm{AC}}\|_{\mathcal{B}(\mathbb{R}^d)}(\mathbb{R}\backslash\{0\}) < \infty$ *then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t [\Xi_{\mathrm{p}}(s)]_+ \mathrm{d}s = -\mu_{\mathrm{AC}}(\mathbb{R}\backslash\{0\}) \ .$$

Proof: (i)–(iii) are direct consequences of Theorem 5.2, the Fubini (–Tonelli) theorem and Lebesgue's dominated convergence theorem. ■

Assuming (14)–(15), note that, for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\vartheta, \lambda \in \mathbb{R}_0^+$, there exists a (generally non–zero) symmetric and finite $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\mu_{\mathrm{p},l}^{(\omega)} \equiv \mu_{\mathrm{p},l}^{(\beta,\omega,\vartheta,\lambda)}$ on $\mathbb{R}$ such that

$$[\Xi_{\mathrm{p},l}^{(\omega)}(t)]_+ = \int_{\mathbb{R}} (\cos(t\nu) - 1)\, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu) \ , \qquad t \in \mathbb{R} \ . \tag{64}$$

Away from $\nu = 0$ and as $l \to \infty$ the finite microscopic conductivity measure $\mu_{\mathrm{p},l}^{(\omega)}$ converges in the weak*–topology to the macroscopic AC–conductivity measure μ_{AC}:

Theorem 5.6 (From microscopic to macroscopic AC–conductivity measures) *Assume Conditions (14)–(15), (20), (21)–(22) with* $\varsigma > 3d$*, and that the map (24) is a random invariant state. Let* $\beta \in \mathbb{R}^+$ *and* $\vartheta, \lambda \in \mathbb{R}_0^+$*. Then,* $\Xi_{\mathrm{p}} \in C^3(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$ *and there is a measurable subset* $\tilde{\Omega} \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)} \subset \Omega$ *of full measure such that, for all* $\omega \in \tilde{\Omega}$*:*
(i) *Tightness: The sequence* $\{\mu_{\mathrm{p},l}^{(\omega)}\}_{l\in\mathbb{R}^2}$ *of finite measures is tight.*
(ii) *Weak*–convergence away from* $\nu = 0$*: For any* $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^d$ *and any bounded continuous function* f *on* $\mathbb{R}\backslash\{0\}$ *with* $0 \notin \overline{\mathrm{supp} f}$*,*

$$\lim_{l\to\infty}\int_{\mathbb{R}} f(\nu)\, \nu^2 \langle \vec{w}_1, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\, \vec{w}_2\rangle_{\mathbb{R}^d} = \int_{\mathbb{R}\backslash\{0\}} f(\nu)\, \nu^2 \langle \vec{w}_1, \mu_{\mathrm{AC}}(\mathrm{d}\nu)\, \vec{w}_2\rangle_{\mathbb{R}^d} \ .$$

Proof: Fix $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$. Under assumptions of the theorem, $\Xi_{\mathrm{p}} \in C^3(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$ and there is a measurable subset $\tilde{\Omega} \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)} \subset \Omega$ of full measure such that

$$\partial_t^2 [\Xi_{\mathrm{p},l}^{(\omega)}(t)]_+ = \lim_{l\to\infty} \partial_t^2 [\Xi_{\mathrm{p},l}^{(\omega)}(t)]_+ \ , \qquad t \in \mathbb{R} \ . \tag{65}$$

The proof is omitted as the arguments are very similar to those proving Theorem 4.1 (p). Note only that Conditions (21)–(22) with $\varsigma > 3d$ are imposed to obtain

Lieb–Robinson bounds for multi–commutators [BP3, Theorems 3.8–3.9] of order *four*. This is needed to obtain the equicontinuity of the family

$$\left\{ t \mapsto \partial_t^2 [\Xi_{\mathrm{p},l}^{(\omega)}(t)]_+ \right\}_{l \in \mathbb{R}^+, \omega \in \Omega}$$

of functions of time. See for instance Remark 7.2, the proofs of Theorems 7.1 and 7.9.

Meanwhile, for $l \in \mathbb{R}^+$, we apply twice the Fubini (–Tonelli) theorem to deduce that

$$\partial_t^2 [\Xi_{\mathrm{p},l}^{(\omega)}(t)]_+ = -\int_{\mathbb{R}} \cos(t\nu)\, \boldsymbol{\mu}_l^{(\omega)}(\mathrm{d}\nu) , \qquad t \in \mathbb{R} , \tag{66}$$

with $\boldsymbol{\mu}_l^{(\omega)} := \nu^2 \mu_{\mathrm{p},l}^{(\omega)}$. Observe from (56) and (65)–(66) that $\boldsymbol{\mu}_l^{(\omega)}$ is a finite measure and

$$\lim_{l \to \infty} \boldsymbol{\mu}_l^{(\omega)}(\mathbb{R}) = \lim_{l \to \infty} \partial_t^2 [\Xi_{\mathrm{p},l}^{(\omega)}(0)]_+ = \partial_t^2 [\Xi_{\mathrm{p}}(0)]_+ = \boldsymbol{\mu}(\mathbb{R}) \in \mathcal{B}_+(\mathbb{R}^d) . \tag{67}$$

Now, take any vector $\vec{w} \in \mathbb{R}^d$. Let $\boldsymbol{\mu}_{l,\vec{w}}^{(\omega)}$ and $\boldsymbol{\mu}_{\vec{w}}$ be the measures on $\mathbb{R}$ respectively defined, for any Borel set $\mathcal{X} \subset \mathbb{R}$, by

$$\boldsymbol{\mu}_{l,\vec{w}}^{(\omega)}(\mathcal{X}) := \langle \vec{w}, \boldsymbol{\mu}_l^{(\omega)}(\mathcal{X})\, \vec{w} \rangle_{\mathbb{R}^d} \quad \text{and} \quad \boldsymbol{\mu}_{\vec{w}}(\mathcal{X}) := \langle \vec{w}, \boldsymbol{\mu}(\mathcal{X})\, \vec{w} \rangle_{\mathbb{R}^d} .$$

Assume w.l.o.g. that $\boldsymbol{\mu}_{\vec{w}}(\mathbb{R}) > 0$. Then, by combining (56) and (64)–(67) with $\partial_t^2 [\Xi_{\mathrm{p}}]_+ \in C(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$ (Theorem 7.1) and [D, Theorems 3.2.3 and 3.3.6], we deduce that, on the subset $\tilde{\Omega}$ of full measure, the sequence $\{\boldsymbol{\mu}_{l,\vec{w}}^{(\omega)}\}_{l \in \mathbb{R}^2}$ is tight and converges in the weak*–topology to $\boldsymbol{\mu}_{\vec{w}}$, as $l \to \infty$. By Definition 5.3, this implies Assertion (ii) for $\vec{w}_1 = \vec{w}_2$. Its extension to arbitrary vectors $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^d$ is a consequence of the polarization identity, see, e.g., (113). Assertion (i) easily follows from the tightness of $\{\boldsymbol{\mu}_{l,\vec{w}}^{(\omega)}\}_{l \in \mathbb{R}^2}$ for $\vec{w} \in \mathbb{R}^d$ and the polarization identity. ■

5.4 Time–Reversal Invariance of Random Equilibrium States

In this subsection we define time–reversal invariance of *random* fermion systems and derive its consequences on conductivity. We do not define this symmetry of random systems as the almost surely time–reversal invariance. But instead, we give a weaker, and hence more general, notion of "time–reversal invariance in average". This is done in the same spirit of what we do above to introduce translation invariance for disordered systems at thermal equilibrium. See, for instance, Definition 3.2. In fact, by doing this, we allow for a large class of random magnetic potentials.

Let $\mathcal{X}$ be a C^*–algebra with unity $\mathbf{1}$ and assume the existence of a map $\Theta : \mathcal{X} \to \mathcal{X}$ with the following properties:

- Θ is antilinear and continuous.
- $\Theta(\mathbf{1}) = \mathbf{1}$ and $\Theta \circ \Theta = \mathrm{Id}_{\mathcal{X}}$.
- $\Theta(B_1 B_2) = \Theta(B_1)\,\Theta(B_2)$ for all $B_1, B_2 \in \mathcal{X}$.
- $\Theta(B^*) = \Theta(B)^*$ for all $B \in \mathcal{X}$.

Such a map is called a *time–reversal* operation of the C^*–algebra $\mathcal{X}$. For $\mathcal{X} = \mathcal{U}$ (CAR C^*–algebra of the lattice $\mathfrak{L}$), there is a natural time–reversal operation $\mathfrak{T}$, which is uniquely defined by the condition

$$\mathfrak{T}(a_x) = a_x \qquad x \in \mathfrak{L} . \tag{68}$$

See also [BPK2, Section 2.1.4].

For any strongly continuous one–parameter group $\tau := \{\tau_t\}_{t\in\mathbb{R}}$ of $*$–automorphisms of $\mathcal{X}$, the family $\tau^\Theta := \{\tau_t^\Theta\}_{t\in\mathbb{R}}$ defined by

$$\tau_t^\Theta := \Theta \circ \tau_t \circ \Theta , \qquad t \in \mathbb{R} ,$$

is again a strongly continuous one–parameter group of automorphisms. Similarly, for any state $\rho \in \mathcal{X}^*$, the linear functional ρ^Θ defined by

$$\rho^\Theta(B) = \overline{\rho \circ \Theta(B)} , \qquad B \in \mathcal{X} ,$$

is again a state. We say that τ and ρ are *time–reversal invariant* w.r.t. Θ if they satisfy $\tau_t^\Theta = \tau_{-t}$ for all $t \in \mathbb{R}$ and $\rho^\Theta = \rho$. If τ is time–reversal invariant then, for all $\beta \in \mathbb{R}^+$, there is at least one time–reversal invariant (τ,β)–KMS state $\varrho \in \mathcal{X}^*$, provided the set of (τ,β)–KMS states is not empty. This follows from the convexity of the set of KMS states, see [BPK2, Lemma A.12].

Now, we introduce a notion of time–reversal invariance for the random system considered here. If Ψ is an interaction, we call it time–reversal invariant whenever

$$\mathfrak{T}(\Psi_\Lambda) = \Psi_\Lambda, \qquad \Lambda \in \mathcal{P}_f(\mathfrak{L}) .$$

For any $\omega = (\omega_1, \omega_2) \in \Omega$, we define $\overline{\omega} := (\omega_1, \overline{\omega_2}) \in \Omega$, where

$$\overline{\omega_2}(b) := \overline{\omega_2(b)}, \qquad b \in \mathfrak{b} \ .$$

We say that the random state (24) is time–reversal symmetric if, for all $\omega \in \Omega$,

$$\rho^{(\beta,\overline{\omega},\vartheta,\lambda)} = [\rho^{(\beta,\omega,\vartheta,\lambda)}]^{\mathfrak{T}} \ .$$

Similarly, we call the random dynamic (23) on $\mathcal{U}$ time–reversal symmetric if, for all $\omega \in \Omega$,

$$\tau_{-t}^{(\overline{\omega},\vartheta,\lambda)} = \mathfrak{T} \circ \tau_t^{(\omega,\vartheta,\lambda)} \circ \mathfrak{T} \ , \qquad t \in \mathbb{R} \ .$$

It is not difficult to see that, if the interparticle interaction Ψ^{IP} is time–reversal invariant then the (unperturbed) random dynamics $\tau^{(\omega,\vartheta,\lambda)}$ is time–reversal symmetric in the above sense for any $\vartheta, \lambda \in \mathbb{R}_0^+$. Further, we say that the Ω–valued random variable ω, the distribution of which is given by the probability space $(\Omega, \mathfrak{A}_\Omega, \mathfrak{a}_\Omega)$, is time–reversal invariant if the map $\omega \mapsto \overline{\omega}$ is measurable w.r.t. $\mathfrak{A}_\Omega$ and preserves the measure $\mathfrak{a}_\Omega$.

Like in the case of translation invariance, the existence of random invariant thermal equilibrium states which are time–reversal symmetric in the above sense is not clear in general. If the $(\tau^{(\omega,\vartheta,\lambda)}, \beta)$–KMS state is unique and Ψ^{IP} is time–reversal invariant, then the (unique) map (24) is a random state which is time–reversal symmetric. The arguments to prove this are similar to the ones used in the proof of [BPK2, Lemma A.12]. As already discussed, if (20) holds then (24) is, moreover, a random invariant state. See Section 3.3.

Time–reversal invariance implies the following important properties of charge transport coefficients related to the models considered here:

Theorem 5.7 (Consequences of time–reversal symmetry)
Assume (14)–(15), (20)–(22), time–reversal invariance of the interparticle interaction Ψ^{IP} and the (Ω–valued) random variable ω, as well as that the map (24) is a random invariant state which is time–reversal symmetric. Let $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$. Then, the following assertions hold true:
(th) *Vanishing thermal current density:*

$$\lim_{l \to \infty} \left\{ \mathbb{J}_{\mathrm{th}}^{(\omega,l)} \right\}_k = \mathbb{E}\left[\varrho^{(\beta,\omega,\vartheta,\lambda)}(I_{(e_k,0)}^{(\omega,\vartheta)}) \right] = 0 \ , \qquad k \in \{1, \ldots, d\} \ .$$

(p) *Vanishing antisymmetric part of the paramagnetic conductivity:*

$$[\Xi_{\mathrm{p}}(t)]_- = 0 \ , \qquad t \in \mathbb{R} \ .$$

Proof: (th) directly follows form Theorem 4.2 (th), the equality $\mathfrak{T}(I_{(e_k,0)}^{(\omega,\vartheta)}) = -I_{(e_k,0)}^{(\overline{\omega},\vartheta)}$, which is a consequence of (68), $\varrho^{(\beta,\omega,\vartheta,\lambda)}(I_{(e_k,0)}^{(\omega,\vartheta)}) \in \mathbb{R}$, the time–reversal invariance of the random variable ω and the time–reversal symmetry of the random state $\varrho^{(\beta,\omega,\vartheta,\lambda)}$. These facts combined with the time–reversal symmetry of $\tau^{(\omega,\vartheta,\lambda)}$, which follows from the assumptions on Ψ^{IP}, and the stationarity of KMS states imply (p). ∎

6 Epilogue: AC–Conductivity and Lévy Processes

For simplicity, we assume that the paramagnetic conductivity Ξ_{p} is of the form $\boldsymbol{\sigma}_{\mathrm{p}}\mathbf{1}_{\mathbb{R}^d}$ with $\boldsymbol{\sigma}_{\mathrm{p}}$ being a real–valued function of time. In particular, $[\Xi_{\mathrm{p}}]_- = 0$ and, by (54), $\boldsymbol{\sigma}_{\mathrm{p}}(t) = \boldsymbol{\sigma}_{\mathrm{p}}(-t)$ for any $t \in \mathbb{R}$ with $\boldsymbol{\sigma}_{\mathrm{p}}(0) = 0$. This property of Ξ_{p} holds true, for instance, if the random variables $\{(\omega_1(x), \omega_2(b))\}_{x\in\mathfrak{L},b\in\mathfrak{b}}$ are independently and identically distributed and the interparticle interaction $\Psi^{\mathrm{IP}} \in \mathcal{W}$ has the form

$$\Psi_{\Lambda}^{\mathrm{IP}} = v(|x-y|)\, a_x^* a_x a_y^* a_y$$

whenever $\Lambda = \{x, y\}$ for $x, y \in \mathfrak{L}$, and $\Psi_{\Lambda}^{\mathrm{IP}} = 0$ when $|\Lambda| > 2$. Here, $v(r) : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a real–valued function such that

$$\sup_{r\in\mathbb{R}_0^+}\left\{\frac{v(r)}{\mathbf{F}(r)}\right\} < \infty\ .$$

See [BPK3, Lemma 5.23] for more details.

In this case, by Theorem 5.2, for any $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$, there is a unique finite and symmetric $\mathbb{R}$–valued measure $\mathfrak{m}_{\mathrm{AC}}$ on $\mathbb{R}\backslash\{0\}$ such that, for any $\alpha \in \mathbb{R}$,

$$\boldsymbol{\sigma}_{\mathrm{p}}(\alpha) = -\frac{\alpha^2}{2}D_{\{0\}} + \int_{\mathbb{R}\backslash\{0\}} \left(\mathrm{e}^{i\alpha\nu} - 1 - i\alpha\nu\mathbf{1}[|\nu|<1]\right)\mathfrak{m}_{\mathrm{AC}}(\mathrm{d}\nu)\ , \qquad (69)$$

where $\boldsymbol{\mu}(\{0\}) = D_{\{0\}}\mathbf{1}_{\mathbb{R}^d}$ with $D_{\{0\}} \in \mathbb{R}$ and

$$\int_{\mathbb{R}} \left(1 \wedge \nu^2\right)\mathfrak{m}_{\mathrm{AC}}(\mathrm{d}\nu) < \infty\ . \qquad (70)$$

Equations (69)–(70) correspond to the Lévy–Khintchine representation of the function $\boldsymbol{\sigma}_{\mathrm{p}}$. Observe that $\mu_{\mathrm{AC}} = \mathfrak{m}_{\mathrm{AC}}\mathbf{1}_{\mathbb{R}^d}$ and by a slight abuse of terminology, we also name $\mathfrak{m}_{\mathrm{AC}}$ the AC–conductivity measure.

Therefore, by [Ky, Theorem 2.1.], there is a probability space $(\Omega_L, \mathfrak{F}, \mathbb{P})$ on which a $\mathbb{R}$–valued Lévy process $\digamma = \{\digamma_t : t \in \mathbb{R}_0^+\}$ with characteristic exponent $\boldsymbol{\sigma}_{\mathrm{p}}$ (up to a minus sign) exists. More explicitly,

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(i\alpha\digamma_t\right)\right] = \exp\left(t\boldsymbol{\sigma}_{\mathrm{p}}\left(\alpha\right)\right) , \qquad \alpha \in \mathbb{R},\ t \in \mathbb{R}_0^+ ,$$

with $\mathbb{E}_{\mathbb{P}}[\ \cdot\ ]$ being the expectation value associated with the probability measure $\mathbb{P}$. In this context, $\mathfrak{m}_{\mathrm{AC}}$ is called the Lévy measure of $\digamma$. It describes the jumps of $\digamma$. For more details on Lévy processes, see for instance [B, Ky] and references therein.

By (69), $\digamma$ has no drift but a diffusion component when $D_{\{0\}} > 0$. There is also a Poisson random measure N (see, e.g., [Ky, Definition 2.3.]) distributed on

$$\left(\mathbb{R}_0^+ \times \mathbb{R}\backslash\left\{0\right\}, \mathfrak{A}_{\mathbb{R}_0^+ \times \mathbb{R}\backslash\{0\}}\right) ,$$

$\mathfrak{A}_{\mathbb{R}_0^+ \times \mathbb{R}\backslash\{0\}}$ being the Borel σ–algebra of $\mathbb{R}_0^+ \times \mathbb{R}\backslash\left\{0\right\}$, with characteristic measure (or intensity) $\mathfrak{m}_{\mathrm{AC}}$ such that

$$\digamma_t = \sqrt{D_{\{0\}}}B_t + \int_0^t \int_{|\nu|\geq 1} \nu N\left(\mathrm{d}s\mathrm{d}\nu\right) + \int_0^t \int_{|\nu|<1} \nu M\left(\mathrm{d}s\mathrm{d}\nu\right) , \qquad t \in \mathbb{R}_0^+ . \tag{71}$$

Here, M is the associated martingale measure

$$M\left(\mathrm{d}s\mathrm{d}\nu\right) := N\left(\mathrm{d}s\mathrm{d}\nu\right) - \mathrm{d}s\mathfrak{m}_{\mathrm{AC}}\left(\mathrm{d}\nu\right)$$

and B is a Brownian motion. The second term in the r.h.s. of (71) is a compound Poisson process with rate $\mathfrak{m}_{\mathrm{AC}}(\mathbb{R}\backslash(-1,1))$ and jump distribution

$$\left(\mathfrak{m}_{\mathrm{AC}}(\mathbb{R}\backslash(-1,1))\right)^{-1}\mathfrak{m}_{\mathrm{AC}} ,$$

provided $\mathfrak{m}_{\mathrm{AC}}(\mathbb{R}\backslash(-1,1)) > 0$. The third term in the r.h.s. of (71) is another Lévy process, which is a square integrable martingale on the same probability space. It is the uniform limit $\varepsilon \to 0^+$ (along an appropriate deterministic subsequence) on compacta of the compound Poisson process with drift

$$\int_0^t \int_{\varepsilon\leq|\nu|<1} \nu N\left(\mathrm{d}s\mathrm{d}\nu\right) - t\int_{\varepsilon\leq|\nu|<1} \nu\mathfrak{m}_{\mathrm{AC}}\left(\mathrm{d}\nu\right) , \qquad t \in \mathbb{R}_0^+,\ \varepsilon \in (0,1) .$$

The limit Lévy process can also be seen as a superposition of an infinite number of compound Poisson processes with drift, see for instance [Ky, Section 2.5].

When

$$0 < \mathfrak{m}_{\mathrm{AC}}\left(\mathbb{R}\backslash\{0\}\right) < \infty \qquad \text{and} \qquad D_{\{0\}} = 0 \ , \tag{72}$$

$\digamma_t$ is a compound Poisson process with rate $\mathfrak{m}_{\mathrm{AC}}(\mathbb{R}\backslash\{0\})$ and jump distribution

$$(\mathfrak{m}_{\mathrm{AC}}(\mathbb{R}\backslash\{0\}))^{-1}\,\mathfrak{m}_{\mathrm{AC}} \ . \tag{73}$$

See [Ky, Lemma 2.13]. In particular, the AC–conductivity measure $\mathfrak{m}_{\mathrm{AC}}$ describes the jump structure of the symmetric Lévy process $\digamma$ in the frequency domain $\mathbb{R}$.

As an example, we can take the AC–conductivity measure obtained from the Drude model. This measure is absolutely continuous w.r.t. the Lebesgue measure with Radon–Nikodym density ϑ_{T} defined by (62). Recall that the relaxation time $\mathrm{T} > 0$ is the mean time interval between two collisions of a charged carrier with defects in the crystal. For all $\mathrm{T} > 0$, the measure of the full set $\mathbb{R}\backslash\{0\}$ equals $\|\vartheta_{\mathrm{T}}\|_1 = D$. In particular, the mean time between frequency jumps does not depend on $\mathrm{T} > 0$ in this new classical process. In the limit $\mathrm{T} \rightarrow 0^+$ of perfect isolator $\vartheta_{\mathrm{T}} \rightarrow 0$ uniformly on $\mathbb{R}$ while in the limit $\mathrm{T} \rightarrow \infty$ of perfect conductor $\vartheta_{\mathrm{T}} \rightarrow 0$ uniformly on $\mathbb{R}\backslash[-\varepsilon,\varepsilon]$ for any $\varepsilon > 0$. Hence, by a similar expression to (73) for the Drude model and because of (62), the probability of large (frequency) jumps increases in the limit $\mathrm{T} \rightarrow 0^+$ (isolator limit), but decreases when $\mathrm{T} \rightarrow \infty$ (conductor limit). The stochastic process $\digamma$ gives an alternative classical picture to electrical conduction.

7 Technical Proofs

7.1 Study of the Paramagnetic Conductivity

Lieb–Robinson bounds and their extensions [BP3] to multi–commutators are here pivotal mathematical tools.

For any $\vartheta_0, \lambda \in \mathbb{R}_0^+$, $\vartheta \in [0,\vartheta_0]$, $\omega \in \Omega$, $t \in \mathbb{R}$, $B_1 \in \mathcal{U}^+ \cap \mathcal{U}_{\Lambda^{(1)}}$ and $B_2 \in \mathcal{U}_{\Lambda^{(2)}}$ with disjoint sets $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f(\mathfrak{L})$,

$$\begin{aligned}\left\|\left[\tau_t^{(\omega,\vartheta,\lambda)}\left(B_1\right), B_2\right]\right\|_{\mathcal{U}} \ \leq \ & 2\mathbf{D}^{-1}\left\|B_1\right\|_{\mathcal{U}}\left\|B_2\right\|_{\mathcal{U}}\left(\mathrm{e}^{2\mathbf{D}|t|D_{\vartheta_0}}-1\right) \\ & \times \sum_{x\in\Lambda^{(1)}}\sum_{y\in\Lambda^{(2)}}\mathbf{F}\left(|x-y|\right) \ .\end{aligned} \tag{74}$$

This is the usual Lieb–Robinson bound. See, e.g., [BP1, Theorem 2.1 (iii)]. Here, the real constant D_{ϑ_0} is defined, for any $\vartheta_0 \in \mathbb{R}_0^+$, by

$$D_{\vartheta_0} := \sup\left\{\left\|\Psi^{(\omega,\vartheta)}\right\|_{\mathcal{W}} : \omega \in \Omega,\ \vartheta \in [0,\vartheta_0]\right\} < \infty \ . \tag{75}$$

See Sections 3.1 and 3.3. As a consequence, the paramagnetic transport coefficient $\sigma_{\mathrm{p}}^{(\omega)}$ defined by (32) satisfies

$$\begin{aligned}\left|\sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x},\mathbf{y},t)\right| &\leq 8\mathbf{D}^{-1}(1+\vartheta_0)^2|t|\left(\mathrm{e}^{2\mathbf{D}|t|D_{\vartheta_0}}-1\right) \\ &\quad \times \sum_{x\in\{x^{(1)},x^{(2)}\}}\sum_{y\in\{y^{(1)},y^{(2)}\}}\mathbf{F}(|x-y|)\end{aligned} \tag{76}$$

for $t \in \mathbb{R}$ and

$$\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2 , \qquad \mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$$

with $\{x^{(1)}, x^{(2)}\} \cap \{y^{(1)}, y^{(2)}\} = \emptyset$. This inequality implies the existence of the macroscopic paramagnetic conductivity defined by (36) with its first derivative. The existence and continuity of its second derivative follow from Lieb–Robinson bounds for multi–commutators [BP3, Theorems 3.8–3.9] of order three:

Theorem 7.1 (Paramagnetic conductivity)
Assume (14)–(15), (20) and that the map (24) is a random invariant state. Let $\beta \in \mathbb{R}^+$ and $\vartheta, \lambda \in \mathbb{R}_0^+$. Then, there is $\mathbf{\Xi}_{\mathrm{p}} \in C^1(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$ such that, uniformly for times t on compacta,

$$\mathbf{\Xi}_{\mathrm{p}}(t) = \lim_{l\to\infty} \mathbb{E}\left[\Xi_{\mathrm{p},l}^{(\omega)}(t)\right] \quad \textit{and} \quad \partial_t \mathbf{\Xi}_{\mathrm{p}}(t) = \lim_{l\to\infty} \partial_t \mathbb{E}\left[\Xi_{\mathrm{p},l}^{(\omega)}(t)\right] .$$

Moreover, if (21)–(22) also hold, then $\mathbf{\Xi}_{\mathrm{p}} \in C^2(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$ and, uniformly for times t on compacta,

$$\partial_t^2 \mathbf{\Xi}_{\mathrm{p}}(t) = \lim_{l\to\infty} \partial_t^2 \mathbb{E}\left[\Xi_{\mathrm{p},l}^{(\omega)}(t)\right] .$$

Proof: The three limits are proven in the same way. The first two only need (76), which follows from usual Lieb–Robinson bounds. By contrast, the last limit requires Lieb–Robinson bounds for multi–commutators [BP3, Theorems 3.8–3.9] of order three and is thus technically more difficult than the other ones. As a consequence, we focus on the limit of $\partial_t^2 \mathbb{E}[\Xi_{\mathrm{p},l}^{(\omega)}(t)]$, as $l \to \infty$, and we omit the details for the first two.

Fix $\beta \in \mathbb{R}^+$, $\vartheta_0, \lambda \in \mathbb{R}_0^+$, $\vartheta \in [0, \vartheta_0]$, $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$. By [BP1, Theorem 2.1 (i)], $\{\tau_t^{(\omega,\vartheta,\lambda)}\}_{t\in\mathbb{R}}$ is a C_0–group of $*$–automorphisms with generator $\delta^{(\omega,\vartheta,\lambda)}$. We thus compute from Equations (32) and (34) that

$$\partial_t^2 \left\{\mathbb{E}\left[\Xi_{\mathrm{p},l}^{(\omega)}(t)\right]\right\}_{k,q} = \frac{1}{|\Lambda_l|} \sum_{x,y\in\Lambda_l} \varrho^{(\beta,\vartheta,\lambda)}\left(i[I_{(y+e_k,y)}^{(\omega,\vartheta)}, \tau_t^{(\omega,\vartheta,\lambda)} \circ \delta^{(\omega,\vartheta,\lambda)}(I_{(x+e_q,x)}^{(\omega,\vartheta)})]\right) ,$$

where $\bar{\varrho}^{(\beta,\vartheta,\lambda)}$ is the translation invariant state defined by (26). Then, by (27),

$$\partial_t^2 \left\{ \mathbb{E}\left[\Xi_{\mathrm{p},l}^{(\omega)}(t) \right] \right\}_{k,q} = \sum_{x\in\mathfrak{L}} \xi_l(x)\, \bar{\varrho}^{(\beta,\vartheta,\lambda)} \left(i[I_{(e_k,0)}^{(\omega,\vartheta)}, \tau_t^{(\omega,\vartheta,\lambda)} \circ \delta^{(\omega,\vartheta,\lambda)}(I_{(x+e_q,x)}^{(\omega,\vartheta)})] \right) \tag{77}$$

with

$$\xi_l(x) := \frac{1}{|\Lambda_l|} \sum_{y\in\Lambda_l} \mathbf{1}_{\{x\in\Lambda_l - y\}} \in [0,1] \ , \qquad x\in\mathfrak{L} \ , \ l\in\mathbb{R}^+ \ . \tag{78}$$

For any $l\in\mathbb{R}^+$, the map $x\mapsto\xi_l(x)$ on $\mathfrak{L}$ has finite support and, for any $x\in\mathfrak{L}$,

$$\lim_{l\to\infty} \xi_l(x) = 1 \ . \tag{79}$$

Paramagnetic current observables (29) are obviously local elements, i.e., $I_{\mathbf{x}}^{(\omega,\vartheta)} \in \mathcal{U}_0$ for any $\mathbf{x}\in\mathfrak{L}^2$, while from [BP1, Theorem 2.1 (ii)]

$$\delta^{(\omega,\vartheta,\lambda)}(B) = i \sum_{z,u\in\mathfrak{L}} \langle \mathfrak{e}_z, (\Delta_{\omega,\vartheta} + \lambda V_\omega)\mathfrak{e}_u \rangle \, [a_z^* a_u, B] + i \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})} \left[\Psi_\Lambda^{\mathrm{IP}}, B \right]$$

for any $B\in\mathcal{U}_0$. Therefore, we get that

$$\begin{aligned} & \sum_{x\in\mathfrak{L}} \xi_l(x) \left(i[I_{(e_k,0)}^{(\omega,\vartheta)}, \tau_t^{(\omega,\vartheta,\lambda)} \circ \delta^{(\omega,\vartheta,\lambda)}(I_{(x+e_q,x)}^{(\omega,\vartheta)})] \right) \\ = \ & i\sum_{x\in\mathfrak{L}} \xi_l(x) \sum_{z,u\in\mathfrak{L}} \langle \mathfrak{e}_z, (\vartheta\Delta_\omega + \lambda V_\omega)\mathfrak{e}_u \rangle [I_{(e_k,0)}^{(\omega,\vartheta)}, \tau_t^{(\omega,\vartheta,\lambda)}([a_z^* a_u, I_{(x+e_q,x)}^{(\omega,\vartheta)}])] \\ & + i\sum_{x\in\mathfrak{L}} \xi_l(x) \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})} [I_{(e_k,0)}^{(\omega,\vartheta)}, \tau_t^{(\omega,\vartheta,\lambda)}([\Phi_\Lambda, I_{(x+e_q,x)}^{(\omega,\vartheta)}])] \ . \end{aligned} \tag{80}$$

The most delicate term in this equation is the last one. In fact, by using (22), $\varsigma > 2d$, $\upsilon > \varsigma + 1$ and

$$\mathcal{P}_f(\mathfrak{L}) = \bigcup_{x\in\mathfrak{L},\ m\in\mathbb{N}_0} \mathcal{D}(x,m)$$

(cf. (19)), together with Lieb–Robinson bounds for multi–commutators of order three [BP3, Corollary 3.10] (tree–decay bounds), one gets that, for $\omega\in\Omega$, $\vartheta_0, \lambda \in$

$\mathbb{R}_0^+$, $\vartheta \in [0, \vartheta_0]$, $k, q \in \{1, \ldots, d\}$ and $T \in \mathbb{R}^+$,

$$\begin{aligned}
&\sum_{x\in\mathfrak{L}} \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})} \sup_{t\in[-T,T]} \left\| [I_{(e_k,0)}^{(\omega,\vartheta)}, \tau_t^{(\omega,\vartheta,\lambda)}([\Phi_\Lambda, I_{(x+e_q,x)}^{(\omega,\vartheta)}])] \right\|_{\mathcal{U}} \\
&= \sum_{x\in\mathfrak{L}} \sum_{\Lambda\in\mathcal{P}_f(\mathfrak{L})} \sup_{t\in[-T,T]} \left\| [\tau_{-t}^{(\omega,\vartheta,\lambda)}(I_{(e_k,0)}^{(\omega,\vartheta)}), [I_{(x+e_q,x)}^{(\omega,\vartheta)}, \Phi_\Lambda]] \right\|_{\mathcal{U}} \\
&\leq D(1+\vartheta_0)^2 d^\varsigma \left(2D_{\vartheta_0} \|\mathbf{u}_{\cdot,1}\|_{\ell^1(\mathbb{N})} |T| \,\mathrm{e}^{4\mathbf{D}|T|D_{\vartheta_0}} + 2^\varsigma\right)^2 \sum_{m\in\mathbb{N}_0} (m+1)^{\varsigma-\upsilon} < \infty \,.
\end{aligned}$$

Here, the positive constant $D \in \mathbb{R}^+$ does not depend on $\omega \in \Omega$, $\vartheta_0, \lambda \in \mathbb{R}_0^+$, $\vartheta \in [0, \vartheta_0]$ and $k, q \in \{1, \ldots, d\}$. Note that $\upsilon > \varsigma + 1$ is a consequence of (22). The same kind of inequality holds for the 1st term in the r.h.s. of (80). Then, using Lebesgue's dominated convergence theorem, one gets from (77)–(79) that the map

$$t \mapsto \partial_t^2 \mathbb{E}\left[\Xi_{\mathrm{p},l}^{(\omega)}(t)\right] = \mathbb{E}\left[\partial_t^2 \Xi_{\mathrm{p},l}^{(\omega)}(t)\right]$$

converges uniformly on compacta, as $l \to \infty$, to a continuous function $\partial_t^2 \Xi_{\mathrm{p}} \in C(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$. ∎

Remark 7.2 (Conductivity and space decays of interactions)
Under stronger assumptions like in the case of exponential decays of interactions, much stronger results can be deduced from Lieb–Robinson bounds for multi–commutators. In particular, under assumptions of [BP3, Theorem 4.6] in the autonomous case, one verifies that $\Xi_{\mathrm{p}} \in C^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$ is a Gevrey map of order d. In particular, for $d = 1$, Ξ_{p} is in this case a real analytic map. Recall that $d \in \mathbb{N}$ is the space dimension of the lattice $\mathfrak{L} = \mathbb{Z}^d$.

7.2 Study of the Paramagnetic Energy Increment

The aim of this subsection is to derive the paramagnetic energy density i_{p} defined by (48). This is achieved in various lemmata which then yield two theorems and one corollary. The derivation ends with Theorem 7.9, which serves as springboard to obtain Theorem 5.1.

First, by assuming (14)–(15) and (21)–(22), [BP1, Theorem 3.8 (p)] says that, for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\vartheta_0, \lambda \in \mathbb{R}_0^+$, $\vartheta \in [0, \vartheta_0]$, $\eta \in \mathbb{R}$, $\mathbf{A} \in \mathbf{C}_0^\infty$ and $t \geq t_0$,

$$\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t) = \eta^2 l^d \int_{t_0}^{t} \int_{t_0}^{s_1} \mathbf{X}_l^{(\omega)}(s_1, s_2)\, \mathrm{d}s_2 \mathrm{d}s_1 + \mathcal{O}(\eta^3 l^d) \,, \tag{81}$$

where, for any $s_1, s_2 \in \mathbb{R}$,

$$\mathbf{X}_l^{(\omega)}(s_1, s_2) \equiv \mathbf{X}_l^{(\beta,\omega,\vartheta,\lambda,\mathbf{A})} := \frac{1}{|\Lambda_l|} \sum_{\mathbf{x},\mathbf{y} \in \mathfrak{K}} \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, s_1 - s_2)\, \mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{x}) \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{y}) \ . \tag{82}$$

The subleading term in the r.h.s. of (81) is order $\mathcal{O}(\eta^3 l^d)$, uniformly for $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\vartheta \in [0, \vartheta_0]$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$. Here,

$$\mathfrak{K} := \left\{ \mathbf{x} = (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2 \ : \ |x^{(1)} - x^{(2)}| = 1 \right\} \tag{83}$$

is the set of oriented bonds of nearest neighbors. Note also that the integral in (81) can be exchanged with the (finite) sum (82) because $\mathbf{A} \in \mathbf{C}_0^\infty$.

The first important result of the present subsection is a proof that the random variable $\mathbf{X}_l^{(\omega)}$ almost surely converges to a constant function, as $l \to \infty$. See Corollary 7.8. To prove this, Conditions (21)–(22) are not anymore necessary. Then, Lebesgue's dominated convergence theorem yields the paramagnetic energy increment $\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t)$ in the limit $(\eta, l^{-1}) \to (0, 0)$, see Theorem 7.9.

We use the same strategy of proof as the one of [BPK3, Section 5.4] for the non–interacting case with homogeneous hopping terms. However, in spite of interactions, we strongly simplify the corresponding technical arguments by using Lieb–Robinson bounds. In particular, we do not anymore need complex times. But like in [BPK3, Section 5.4], the (compact) support $\mathrm{supp}(\mathbf{A}(t, \cdot)) \subset \mathbb{R}^d$ of the vector potential $\mathbf{A}(t, \cdot)$ at $t \in \mathbb{R}$ is divided in small regions to use the piecewise–constant approximation of the smooth electric field $E_{\mathbf{A}}$. To do this, we assume w.l.o.g. that, for all $t \in \mathbb{R}$,

$$\mathrm{supp}(\mathbf{A}(t, \cdot)) \subset [-1/2, 1/2]^d \ . \tag{84}$$

From now on we fix the parameters $\beta \in \mathbb{R}^+$, $\vartheta_0, \lambda \in \mathbb{R}_0^+$, $\vartheta \in [0, \vartheta_0]$ and $\mathbf{A} \in \mathbf{C}_0^\infty$ with (84).

Then, for every $n \in \mathbb{N}$, we divide the elementary box $[-1/2, 1/2]^d$ in n^d boxes $\{b_j\}_{j \in \mathcal{D}_n}$ of side–length $1/n$, where

$$\mathcal{D}_n := \{-(n-1)/2, -(n-3)/2, \cdots, (n-3)/2, (n-1)/2\}^d \ . \tag{85}$$

Explicitly, for any $j \in \mathcal{D}_n$,

$$b_j := jn^{-1} + n^{-1}[-1/2, 1/2]^d \quad \text{and} \quad [-1/2, 1/2]^d = \bigcup_{j \in \mathcal{D}_n} b_j \ . \tag{86}$$

For any $l \in \mathbb{R}^+$, $\omega \in \Omega$, $n \in \mathbb{N}$ and $s_1, s_2 \in \mathbb{R}$, let

$$\mathbf{Y}_{l,n}^{(\omega)}(s_1, s_2) := \frac{1}{|\Lambda_l|} \sum_{j \in \mathcal{D}_n} \sum_{\mathbf{x},\mathbf{y} \in \mathfrak{K} \cap (lb_j)^2} \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, s_1 - s_2) \, \mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{x}) \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{y}) \, . \tag{87}$$

We show now that the accumulation points of $\mathbf{Y}_{l,n}^{(\omega)}$, as $l \to \infty$, do not depend on $n \in \mathbb{N}$ and coincide with those of $\mathbf{X}_l^{(\omega)}$:

Lemma 7.3 (Approximation I)
Assume (14)–(15). Let $n \in \mathbb{N}$. Then,

$$\lim_{l \to \infty} \left| \mathbf{X}_l^{(\omega)}(s_1, s_2) - \mathbf{Y}_{l,n}^{(\omega)}(s_1, s_2) \right| = 0 \, ,$$

uniformly for $\omega \in \Omega$ and $s_1, s_2 \in \mathbb{R}$.

Proof: We observe from (82), (86) and (87) that

$$\begin{aligned} & \left| \mathbf{X}_l^{(\omega)}(s_1, s_2) - \mathbf{Y}_{l,n}^{(\omega)}(s_1, s_2) \right| \qquad (88) \\ \leq \; & \frac{1}{|\Lambda_l|} \sum_{j,k \in \mathcal{D}_n, j \neq k} \sum_{\mathbf{x} \in \mathfrak{K} \cap (lb_j)^2} \sum_{\mathbf{y} \in \mathfrak{K} \cap (lb_k)^2} \left| \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, s_1 - s_2) \, \mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{x}) \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{y}) \right| \\ & + \frac{1}{|\Lambda_l|} \sum_{j \in \mathcal{D}_n} \sum_{\mathbf{x} \in \partial(lb_j)} \sum_{\mathbf{y} \in \mathfrak{K}} \left| \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, s_1 - s_2) \, \mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{x}) \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{y}) \right. \\ & \qquad \left. + \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{y}, \mathbf{x}, s_1 - s_2) \, \mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{y}) \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{x}) \right| \, , \end{aligned}$$

where, for any $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ with complement $\Lambda^c \subset \mathfrak{L}$,

$$\partial \Lambda := \left\{ \mathbf{x} = (x^{(1)}, x^{(2)}) \in \mathfrak{K} : \{x^{(1)}, x^{(2)}\} \cap \Lambda \neq 0, \; \{x^{(1)}, x^{(2)}\} \cap \Lambda^c \neq 0 \right\} \, .$$

Because $\mathbf{A} \in \mathbf{C}_0^\infty$, note that

$$\left\| \mathbf{E}^{\mathbf{A}} \right\|_\infty := \sup \left\{ |E_{\mathbf{A}}(t, x)| \; : \; (t, x) \in \mathrm{supp}(A) \right\} < \infty \, . \tag{89}$$

Therefore, using (14), (76), (89) and the fact that $\mathbf{A}(t, \cdot) = 0$ for any $t \notin [t_0, t_1]$ (cf. (10)), we deduce from Inequality (88) that

$$\begin{aligned} \left| \mathbf{X}_l^{(\omega)}(s_1, s_2) - \mathbf{Y}_{l,n}^{(\omega)}(s_1, s_2) \right| \; \leq \; & 8 (1 + \vartheta_0)^2 (t_1 - t_0) \, \| \mathbf{E}^{\mathbf{A}} \|_\infty^2 \qquad (90) \\ & \times \left(\mathrm{e}^{2\mathbf{D}(t_1 - t_0) D_{\vartheta_0}} \mathbf{D}^{-1} \mathbf{K}_l + \tilde{\mathbf{K}}_l \right) \end{aligned}$$

for all $l \in \mathbb{R}^+$, $\omega \in \Omega$, $n \in \mathbb{N}$ and $s_1, s_2 \in \mathbb{R}$, where

$$\mathbf{K}_l := \frac{1}{|\Lambda_l|} \sum_{j,k \in \mathcal{D}_n, j \neq k} \sum_{x \in \mathfrak{L} \cap (lb_j)} \sum_{z_{1,2} \in \mathfrak{L}, |z_{1,2}|=1} \sum_{y \in \mathfrak{L} \cap (lb_k)} \mathbf{F}\left(|x + z_1 + z_2 - y|\right) \quad (91)$$

and

$$\tilde{\mathbf{K}}_l := 2dn^d \left(8 + \|\mathbf{F}\|_{1,\mathfrak{L}}\right) \sum_{z \in \mathfrak{L}, |z|=1} \frac{1}{|\Lambda_l|} \sum_{x \in \mathfrak{L}} \mathbf{1}\left[(x, x+z) \in \partial(lb_0)\right] . \quad (92)$$

Clearly, one has

$$\lim_{l \to \infty} \tilde{\mathbf{K}}_l = 0 . \quad (93)$$

Therefore, it remains to prove that $\mathbf{K}_l$ vanishes when $l \to \infty$ in order to prove the lemma.

To this end, for any $l \in \mathbb{R}^+$ and $\delta \in [0, 1]$, define two constants:

$$\mathbf{K}_{l,\delta}^{\lesseqgtr} := \frac{1}{|\Lambda_l|} \sum_{j,k \in \mathcal{D}_n, j \neq k} \sum_{x \in \mathfrak{L} \cap (lb_j)} \sum_{z_{1,2} \in \mathfrak{L}, |z_{1,2}|=1} \sum_{y \in \mathfrak{L} \cap (lb_k)} \mathbf{1}\left[|x + z_1 + z_2 - y| \lesseqgtr \delta l\right] \mathbf{F}\left(|x + z_1 + z_2 - y|\right) .$$

Obviously, by (91), for any $\delta, l \in \mathbb{R}^+$,

$$\mathbf{K}_l = \mathbf{K}_{l,\delta}^{\leq} + \mathbf{K}_{l,\delta}^{>} . \quad (94)$$

Recall that $\mathbf{F} : \mathbb{R}_0^+ \to \mathbb{R}^+$, which encodes the short range property of interactions, is a non–increasing function, by assumption. As a consequence, explicit estimates using $\mathbf{F}(r) \leq \mathbf{F}(0)$ show that, for any $\delta, l \in \mathbb{R}^+$,

$$\left|\mathbf{K}_{l,\delta}^{\leq}\right| = \mathcal{O}\left(\delta^{d+1} l^d\right) , \quad (95)$$

while

$$\left|\mathbf{K}_{l,\delta}^{>}\right| \leq 4d^2 n^d \sum_{x \in \mathfrak{L}, |x| > \delta l} \mathbf{F}\left(|x|\right) . \quad (96)$$

Take $\delta = l^{-\frac{(d+1/2)}{d+1}}$. Then, by (95), $|\mathbf{K}_{l,\delta}^{\leq}| = \mathcal{O}(l^{-1/2})$ and $\delta l = l^{\frac{1}{2(d+1)}}$, which combined with (14), (94) and (96) yield

$$\lim_{l \to \infty} \mathbf{K}_l = 0 .$$

By (90) and (93), we thus arrive at the assertion. ■

We now consider piecewise–constant approximations of the (smooth) electric field $E_{\mathbf{A}}$ (7), that is,

$$E_{\mathbf{A}}(t,x) := -\partial_t \mathbf{A}(t,x) \ , \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d \ . \tag{97}$$

For any $j \in \mathcal{D}_n$, let $z^{(j)} \in b_j$ be any fixed point of the box b_j. Then, we define the function

$$\begin{aligned}\bar{\mathbf{Y}}_{l,n}^{(\omega)}(s_1,s_2) \ := \ & \frac{1}{|\Lambda_l|} \sum_{j\in\mathcal{D}_n} \sum_{\mathbf{x},\mathbf{y}\in\mathfrak{K}\cap(lb_j)^2} \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x},\mathbf{y},s_1-s_2) \\ & \times \left[E_{\mathbf{A}}(s_1,z^{(j)})\right](x^{(1)}-x^{(2)}) \left[E_{\mathbf{A}}(s_2,z^{(j)})\right](y^{(1)}-y^{(2)})\end{aligned} \tag{98}$$

for any $l \in \mathbb{R}^+$, $\omega \in \Omega$, $n \in \mathbb{N}$ and $s_1,s_2 \in \mathbb{R}$, where $\mathbf{x} := (x^{(1)},x^{(2)})$ and $\mathbf{y} := (y^{(1)},y^{(2)})$, see (83). This new function approximates (87) arbitrarily well, as $l \to \infty$ and $n \to \infty$:

Lemma 7.4 (Approximation II)
Assume (14)–(15). Then

$$\lim_{n\to\infty}\left\{\limsup_{l\to\infty}\left|\mathbf{Y}_{l,n}^{(\omega)}(s_1,s_2)-\bar{\mathbf{Y}}_{l,n}^{(\omega)}(s_1,s_2)\right|\right\}=0\ ,$$

uniformly for $\omega \in \Omega$ and $s_1,s_2 \in \mathbb{R}$.

Proof: By taking the canonical orthonormal basis $\{e_k\}_{k=1}^d$ of $\mathbb{R}^d$, we directly infer from (8), (38) and (97) that, for any $l \in \mathbb{R}^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$, $j \in \mathcal{D}_n$, $t \in \mathbb{R}$, $k \in \{1,\cdots,d\}$ and $x \in lb_j$,

$$\begin{aligned}&\left|\mathbf{E}_t^{\mathbf{A}_l}(x,x\pm e_k)-\left[E_{\mathbf{A}}(t,z^{(j)})\right](\pm e_k)\right| \\ \leq\ & \int_0^1 \left|\left[\partial_t\mathbf{A}(t,z^{(j)})\right](e_k)-\left[\partial_t\mathbf{A}_l(t,x\pm(1-\alpha)e_k)\right](e_k)\right|\mathrm{d}\alpha \\ \leq\ & \sup_{y\in\tilde{b}_{j,l}}\left|\left[\partial_t\mathbf{A}(t,z^{(j)})\right](e_k)-\left[\partial_t\mathbf{A}(t,y)\right](e_k)\right|<\infty\ ,\end{aligned}$$

where

$$\tilde{b}_{j,l} := \left\{ y \in \mathbb{R}^d \ : \ \min_{x\in b_j}|y-x| \leq l^{-1} \right\} \ .$$

In particular, since $\mathbf{A} \in \mathbf{C}_0^\infty$, there is a finite constant $D_{\mathbf{A}} \in \mathbb{R}^+$ not depending on $j \in \mathcal{D}_n$, $t \in \mathbb{R}$, $k \in \{1, \cdots, d\}$ and $x \in b_j$ such that

$$\left| \mathbf{E}_t^{\mathbf{A}_l}(x, x \pm e_k) - \left[E_{\mathbf{A}}(t, z^{(j)}) \right] (\pm e_k) \right| \leq D_{\mathbf{A}}(n^{-1} + l^{-1}) . \tag{99}$$

Therefore, using (14), (76), (89), (99) and the fact that $\mathbf{A}(t, \cdot) = 0$ for any $t \notin [t_0, t_1]$ (cf. (10)), like in (90), we deduce from (87) and (98) that

$$\begin{aligned}\left| \mathbf{Y}_{l,n}^{(\omega)}(s_1, s_2) - \bar{\mathbf{Y}}_{l,n}^{(\omega)}(s_1, s_2) \right| \leq\ & 64 d^2 D_{\mathbf{A}} \left\| \mathbf{E}^{\mathbf{A}} \right\|_\infty (1 + \vartheta_0)^2 (t_1 - t_0) (n^{-1} + l^{-1}) \\ & \times \left(\| \mathbf{F} \|_{1, \mathfrak{L}} \mathbf{D}^{-1} \mathrm{e}^{2\mathbf{D}(t_1 - t_0) D_{\vartheta_0}} + 2 \right) .\end{aligned}$$

This upper bound implies the lemma. ■

By taking the canonical orthonormal basis $\{e_k\}_{k=1}^d$ of $\mathbb{R}^d$ and setting $e_{-k} := -e_k$ for each $k \in \{1, \cdots, d\}$, we rewrite the function (98) as

$$\begin{aligned}\bar{\mathbf{Y}}_{l,n}^{(\omega)}(s_1, s_2) := \ & \frac{1}{n^d} \sum_{j \in \mathcal{D}_n} \sum_{k,q \in \{1, -1, \cdots, d, -d\}} \mathbf{Z}_{l,j,k,q}^{(\omega)}(s_1 - s_2) \\ & \times \left[E_{\mathbf{A}}(s_1, z^{(j)}) \right] (e_k) \left[E_{\mathbf{A}}(s_2, z^{(j)}) \right] (e_q)\end{aligned} \tag{100}$$

for any $l \in \mathbb{R}^+$, $\omega \in \Omega$, $n \in \mathbb{N}$ and $s_1, s_2 \in \mathbb{R}$, where, for $l \in \mathbb{R}^+$, $\omega \in \Omega$, $j \in \mathcal{D}_n$, $k, q \in \{1, -1, \cdots, d, -d\}$ and $t \in \mathbb{R}$,

$$\mathbf{Z}_{l,j,k,q}^{(\omega)}(t) := \frac{n^d}{|\Lambda_l|} \sum_{x,y \in \mathfrak{L} \cap (l b_j)} \sigma_{\mathrm{p}}^{(\omega)}(y, y - e_q, x, x - e_k, t) . \tag{101}$$

Notice that, as compared to (98), we have added in (100) terms related to x, y on the boundary of $\mathfrak{L} \cap (l b_j)$, but we use the same notation $\bar{\mathbf{Y}}_{l,n}^{(\omega)}$ for simplicity. These terms have indeed vanishing contribution in the limit $l \to \infty$. Here, for any $\omega \in \Omega$, $t \in \mathbb{R}$, $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$ and $\mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$,

$$\sigma_{\mathrm{p}}^{(\omega)}(x^{(1)}, x^{(2)}, y^{(1)}, y^{(2)}, t) \equiv \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, t) , \tag{102}$$

see (32). Hence, it remains to analyze the limit of (101), as $l \to \infty$. But before doing this study, observe that, for all $x, y \in \mathfrak{L}$, $k, q \in \{1, -1, \cdots, d, -d\}$ and $t \in \mathbb{R}$, the map

$$\omega \mapsto \sigma_{\mathrm{p}}^{(\omega)}(y, y - e_q, x, x - e_k, t)$$

is bounded and measurable w.r.t. the σ–algebra $\mathfrak{A}_\Omega$, by assumption. Indeed, the map (24) is a random invariant state (Definition 3.2). Recall also that $\mathbb{E}[\,\cdot\,]$ is the expectation value associated with the probability measure $\mathfrak{a}_\Omega$, see Section 3.1.

Lemma 7.5 (Infinite volume limit and ergodicity)
Assume (14)–(15), (20) and that the map (24) is a random invariant state. For any $t \in \mathbb{R}$, there is a measurable subset $\tilde{\Omega}(t) \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)}(t) \subset \Omega$ of full measure such that, for $n \in \mathbb{N}$, $j \in \mathcal{D}_n$, $k, q \in \{1, -1, \cdots, d, -d\}$ and $\omega \in \tilde{\Omega}(t)$,

$$\lim_{l \to \infty} \mathbf{Z}_{l,j,k,q}^{(\omega)}(t) = \{\Xi_{\mathrm{p}}(t)\}_{k,q} = \sum_{x \in \mathfrak{L}} \mathbb{E}\left[\sigma_{\mathrm{p}}^{(\omega)}(x, x - e_q, 0, -e_k, t)\right] \in \mathbb{R} .$$

Proof: For any $\omega \in \Omega$, $t \in \mathbb{R}$, $k, q \in \{1, -1, \cdots, d, -d\}$ and $x \in \mathfrak{L}$, let

$$\mathfrak{F}_{t,k,q}^{(\omega)}(\{x\}) = \sum_{y \in \mathfrak{L}} \sigma_{\mathrm{p}}^{(\omega)}(y, y - e_q, x, x - e_k, t) \ . \tag{103}$$

By the assumptions of the lemma, this sum is uniformly bounded for all $x \in \mathfrak{L}$ and defines a random variable. Indeed, we infer from (14) and (76) that

$$\left|\mathfrak{F}_{t,k,q}^{(\omega)}(\{x\})\right| \leq 8(1+\vartheta_0)^2 |t| \left(\|\mathbf{F}\|_{1,\mathfrak{L}} \mathbf{D}^{-1}\left(\mathrm{e}^{2\mathbf{D}|t|D_{\vartheta_0}} - 1\right) + 2\right) . \tag{104}$$

We now define an additive process $\{\mathfrak{F}_{t,k,q}^{(\omega)}(\Lambda)\}_{\Lambda \in \mathcal{P}_f(\mathfrak{L})}$ by

$$\mathfrak{F}_{t,k,q}^{(\omega)}(\Lambda) = \sum_{x \in \Lambda} \mathfrak{F}_{t,k,q}^{(\omega)}(\{x\}) \tag{105}$$

for any finite subset $\Lambda \in \mathcal{P}_f(\mathfrak{L})$ with cardinality $|\Lambda| < \infty$, see [BPK3, Definition 5.2][2]. Indeed, the map $\omega \mapsto \mathfrak{F}_{t,k,q}^{(\omega)}(\Lambda)$ is bounded and measurable w.r.t. the σ–algebra $\mathfrak{A}_\Omega$ for all $\Lambda \in \mathcal{P}_f(\mathfrak{L})$. Moreover, by Conditions (20) and (25),

$$\mathfrak{F}_{t,k,q}^{(\chi_x^{(\Omega)}(\omega))}(\Lambda) = \mathfrak{F}_{t,k,q}^{(\omega)}(\Lambda + x) \ , \qquad \Lambda \in \mathcal{P}_f(\mathfrak{L}),\ x \in \mathbb{Z}^d . \tag{106}$$

See Section 3.3, in particular Definition 3.2. For any $\Lambda \in \mathcal{P}_f(\mathfrak{L})$,

$$\frac{1}{|\Lambda|}\mathbb{E}\left[\mathfrak{F}_{t,k,q}^{(\omega)}(\Lambda)\right] \leq 8(1+\vartheta_0)^2 |t| \left(\|\mathbf{F}\|_{1,\mathfrak{L}} \mathbf{D}^{-1}\left(\mathrm{e}^{2\mathbf{D}|t|D_{\vartheta_0}} - 1\right) + 2\right) ,$$

because of (104)–(105). Then, by (106) and ergodicity of the measure $\mathfrak{a}_\Omega$, for any $t \in \mathbb{R}$ and $k, q \in \{1, -1, \cdots, d, -d\}$, [BPK3, Theorem 5.5][2] applied on the previous additive process holds and one gets the existence of a measurable subset

$$\hat{\Omega}_{k,q}(t) \equiv \hat{\Omega}_{k,q}^{(\beta,\vartheta,\lambda)}(t) \subset \Omega$$

[2]Replace the product measure of [BPK3] with ergodic measures $\mathfrak{a}_\Omega$, as defined in Section 3.1.

of full measure such that, for all $\omega \in \hat{\Omega}_{k,q}(t)$, $n \in \mathbb{N}$ and $j \in \mathcal{D}_n$,

$$\lim_{l\to\infty} \left\{ \frac{n^d}{|\Lambda_l|} \mathfrak{F}_{t,k,q}^{(\omega)}(lb_j) \right\} = \mathbb{E}\left[\mathfrak{F}_{t,k,q}^{(\omega)}(\{0\}) \right] . \tag{107}$$

In the same way one proves Lemma 7.3,

$$\lim_{l\to\infty} \left\{ \frac{n^d}{|\Lambda_l|} \sum_{x\in\mathfrak{L}\cap(lb_j)} \sum_{y\in\mathfrak{L}\backslash\{\mathfrak{L}\cap(lb_j)\}} \sigma_{\mathrm{p}}^{(\omega)}(y, y-e_q, x, x-e_k, s_1-s_2) \right\} = 0 .$$

Using this with (103), (105) and (107), and observing meanwhile from the proof of Theorem 7.1 that

$$\mathbb{E}\left[\mathfrak{F}_{t,k,q}^{(\omega)}(\{0\}) \right] = \sum_{x\in\mathfrak{L}} \mathbb{E}\left[\sigma_{\mathrm{p}}^{(\omega)}(x, x-e_q, 0, -e_k, t) \right] = \{\Xi_{\mathrm{p}}(t)\}_{k,q}$$

for all $k, q \in \{1, -1, \cdots, d, -d\}$ and any $t \in \mathbb{R}$, we arrive at the assertion for any realization $\omega \in \tilde{\Omega}(t)$ with

$$\tilde{\Omega}(t) := \bigcap_{k,q\in\{1,-1,\cdots,d,-d\}} \hat{\Omega}_{k,q}(t) .$$

[Any countable intersection of measurable sets of full measure has full measure.] ∎

Exactly like in the proof of Lemma 7.5, one shows that, for any $\beta \in \mathbb{R}^+$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$, there is a measurable subset $\tilde{\Omega}(t) \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)}(t) \subset \Omega$ of full measure such that, for any $\omega \in \tilde{\Omega}(t)$,

$$\Xi_{\mathrm{p}}(t) = \lim_{l\to\infty} \Xi_{\mathrm{p},l}^{(\omega)}(t) \in \mathcal{B}(\mathbb{R}^d) . \tag{108}$$

This holds under Conditions (14)–(15) and (20), provided that the map (24) is a random invariant state.

Define the deterministic function

$$\begin{aligned} \mathbf{X}_\infty(s_1, s_2) \quad := \quad & \sum_{k,q\in\{1,-1,\cdots,d,-d\}} \{\Xi_{\mathrm{p}}(s_1-s_2)\}_{k,q} \\ & \times \int_{\mathbb{R}^d} [E_{\mathbf{A}}(s_1, x)](e_k) [E_{\mathbf{A}}(s_2, x)](e_q) \mathrm{d}^d x \end{aligned} \tag{109}$$

for any $s_1, s_2 \in \mathbb{R}$. We show next that the function $\mathbf{X}_l^{(\omega)}$ defined by (82) almost surely converges to $\mathbf{X}_\infty \equiv \mathbf{X}_\infty^{(\beta,\vartheta,\lambda)}$, as $l \to \infty$:

Theorem 7.6 (Infinite volume limit of X–integrands – I)
Assume (14)–(15), (20) and that the map (24) is a random invariant state. Let $\beta \in \mathbb{R}^{+}$, $\vartheta, \lambda \in \mathbb{R}_0^+$ and $s_1, s_2 \in \mathbb{R}$. Then, there is a measurable subset $\tilde{\Omega}(s_1, s_2) \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)}(s_1, s_2) \subset \Omega$ of full measure such that, for any $\mathbf{A} \in \mathbf{C}_0^{\infty}$ and $\omega \in \tilde{\Omega}(s_1, s_2)$,

$$\lim_{l \to \infty} \mathbf{X}_l^{(\omega)}(s_1, s_2) = \mathbf{X}_{\infty}(s_1, s_2) \ .$$

Proof: Let $\beta \in \mathbb{R}^{+}$, $\vartheta_0, \lambda \in \mathbb{R}_0^+$, $\vartheta \in [0, \vartheta_0]$ and $s_1, s_2 \in \mathbb{R}$. Assume w.l.o.g. that (84) holds. Using Lemmata 7.3–7.5 and (100)–(101), we obtain the existence of a measurable subset $\tilde{\Omega}(s_1, s_2) \equiv \tilde{\Omega}^{(\beta,\vartheta,\lambda)}(s_1, s_2) \subset \Omega$ of full measure such that, for any $\omega \in \tilde{\Omega}(s_1, s_2)$,

$$\lim_{l \to \infty} \mathbf{X}_l^{(\omega)}(s_1, s_2) = \sum_{k,q \in \{1,-1,\cdots,d,-d\}} \{\Xi_{\mathrm{p}}(s_1 - s_2)\}_{k,q} \times \lim_{n \to \infty} \left\{ \frac{1}{n^d} \sum_{j \in \mathcal{D}_n} \left[E_{\mathbf{A}}(s_1, z^{(j)})\right](e_k) \left[E_{\mathbf{A}}(s_2, z^{(j)})\right](e_q) \right\} \ .$$

The latter implies the theorem because the term within the limit $n \to \infty$ is a Riemann sum and $E_{\mathbf{A}} \in \mathbf{C}_0^{\infty}$ for any $\mathbf{A} \in \mathbf{C}_0^{\infty}$, see (97). ∎

To find the energy increment $\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t)$ given by (81) in the limit $(\eta, l^{-1}) \to (0,0)$, we use below Lebesgue's dominated convergence theorem and we thus need to remove the dependency of the measurable subset $\tilde{\Omega}(s_1, s_2)$ on $s_1, s_2 \in \mathbb{R}$, see Theorem 7.6. To achieve this, we first show uniform boundedness and equicontinuity of the function $\mathbf{X}_l^{(\omega)}$ defined by (82):

Lemma 7.7 (Uniform Boundedness and Equicontinuity of X–integrands)
Assume (14)–(15). The family

$$\left\{(s_1, s_2) \mapsto \mathbf{X}_l^{(\omega)}(s_1, s_2)\right\}_{l \in \mathbb{R}^+, \omega \in \Omega}$$

of maps from $\mathbb{R}^2$ to $\mathbb{C}$ is uniformly bounded and equicontinuous.

Proof: The uniform boundedness of this collection of maps is an immediate consequence of (76) and (89). The arguments are indeed similar to those proving

Inequality (90): Assume w.l.o.g. that (84) holds. Then, by combining (82) with (76) and (89) one gets

$$\left|\mathbf{X}_{l}^{(\omega)}(s_{1},s_{2})\right| \leq 32d^{2}\left\|\mathbf{E}^{\mathbf{A}}\right\|_{\infty}^{2}(1+\vartheta_{0})^{2}(t_{1}-t_{0})\left(\|\mathbf{F}\|_{1,\mathfrak{L}}\mathbf{D}^{-1}\mathrm{e}^{2\mathbf{D}(t_{1}-t_{0})D_{\vartheta_{0}}}+2\right)$$

for any $\beta\in\mathbb{R}^{+}$, $\vartheta_{0},\lambda\in\mathbb{R}_{0}^{+}$, $\vartheta\in[0,\vartheta_{0}]$ and $s_{1},s_{2}\in\mathbb{R}$.

To prove the uniform equicontinuity, we use [BP1, Theorem 3.6], which is also an immediate consequence of (76). We omit the details. ■

Theorem 7.6 and Lemma 7.7 allows us to eliminate the (s_{1},s_{2})–dependency of the measurable set $\tilde{\Omega}(s_{1},s_{2})$ of Theorem 7.6.

Corollary 7.8 (Infinite volume limit of X–integrands – II)
Assume (14)–(15), (20) and that the map (24) is a random invariant state. Let $\beta\in\mathbb{R}^{+}$ and $\vartheta,\lambda\in\mathbb{R}_{0}^{+}$. Then, there is a measurable subset $\tilde{\Omega}\equiv\tilde{\Omega}^{(\beta,\vartheta,\lambda)}\subset\Omega$ of full measure such that, for any $s_{1},s_{2}\in\mathbb{R}$, $\mathbf{A}\in\mathbf{C}_{0}^{\infty}$ and $\omega\in\tilde{\Omega}$,

$$\lim_{l\rightarrow\infty}\mathbf{X}_{l}^{(\omega)}(s_{1},s_{2})=\mathbf{X}_{\infty}(s_{1},s_{2})\ . \tag{110}$$

Proof: Fix $\beta\in\mathbb{R}^{+}$ and $\vartheta,\lambda\in\mathbb{R}_{0}^{+}$. By Theorem 7.6, for any $s_{1},s_{2}\in\mathbb{Q}$, there is a measurable subset $\hat{\Omega}(s_{1},s_{2})\subset\Omega$ of full measure such that (110) holds. Let $\tilde{\Omega}$ be the intersection of all such subsets $\hat{\Omega}(s_{1},s_{2})$. Since this intersection is countable, $\tilde{\Omega}$ is measurable and has full measure. By Lemma 7.7 and the density of $\mathbb{Q}$ in $\mathbb{R}$, it follows that (110) holds true for any $s_{1},s_{2}\in\mathbb{R}$, $\mathbf{A}\in\mathbf{C}_{0}^{\infty}$ and $\omega\in\tilde{\Omega}$.
■

Therefore, because of (81), Lemma 7.7 and Corollary 7.8, we can now use Lebesgue's dominated convergence theorem to get the paramagnetic energy density i_{p} defined by (48):

Theorem 7.9 (Paramagnetic energy density)
Assume (14)–(15), (20), (21)–(22) and that the map (24) is a random invariant state. Let $\beta\in\mathbb{R}^{+}$ and $\vartheta,\lambda\in\mathbb{R}_{0}^{+}$. Then, there is a measurable subset $\tilde{\Omega}\equiv\tilde{\Omega}^{(\beta,\vartheta,\lambda)}\subset\Omega$ of full measure such that, for any $\omega\in\tilde{\Omega}$, $\mathbf{A}\in\mathbf{C}_{0}^{\infty}$ and $t\geq t_{0}$,

$$\mathrm{i}_{\mathrm{p}}(t):=\lim_{(\eta,l^{-1})\rightarrow(0,0)}\left\{\left(\eta^{2}l^{d}\right)^{-1}\mathfrak{J}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_{l})}(t)\right\}=\int_{t_{0}}^{t}\int_{t_{0}}^{s_{1}}\mathbf{X}_{\infty}(s_{1},s_{2})\,\mathrm{d}s_{2}\mathrm{d}s_{1}\ .$$

This theorem yields Theorem 5.1 (p).

7.3 Appendix: the Bochner Theorem

For completeness, we give in this appendix a proof of the Bochner theorem for weakly positive definite maps Υ from $\mathbb{R}$ to $\mathcal{B}(\mathbb{R}^d)$. By weakly positive definite $\mathcal{B}(\mathbb{R}^d)$–valued map, we mean that, for any $\varphi \in C_0^\infty(\mathbb{R};\mathbb{R}^d)$,

$$\int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \, \langle \varphi(s), \Upsilon(t-s)\varphi(t)\rangle_{\mathbb{R}^d} \geq 0 \, . \tag{111}$$

It is a simple consequence of the usual Bochner theorem for weakly positive definite complex–valued functions:

Theorem 7.10 (The Bochner theorem)
The following are equivalent:
(i) $f : \mathbb{R} \to \mathbb{C}$ *is a weakly positive definite and continuous function, i.e.,*

$$\int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \, \overline{\varphi(s)} f(t-s)\varphi(t) \geq 0 \, , \qquad \varphi \in C_0^\infty(\mathbb{R};\mathbb{C}) \, .$$

(ii) *There is a unique finite positive measure* μ *on* $\mathbb{R}$ *such that*

$$f(t) = \int_{\mathbb{R}} \mathrm{e}^{it\nu} \mu(\mathrm{d}\nu) \, , \qquad t \in \mathbb{R} \, .$$

Proof: See for instance [RS2, Theorem IX.9 and discussion thereafter]. ■

Corollary 7.11 (A Bochner theorem for real matrix–valued maps)
Let $\Upsilon : \mathbb{R} \to \mathcal{B}(\mathbb{R}^d)$ *be a weakly positive definite continuous map. If* $\Upsilon(t) = \Upsilon(-t) \in \mathcal{B}(\mathbb{R}^d)$ *is symmetric w.r.t. the canonical scalar product of* $\mathbb{R}^d$ *for any* $t \in \mathbb{R}$*, then there is a unique finite and symmetric* $\mathcal{B}_+(\mathbb{R}^d)$*–valued measure* μ_Υ *on* $\mathbb{R}$ *such that*

$$\Upsilon(t) = \int_{\mathbb{R}} \cos(t\nu)\, \mu_\Upsilon(\mathrm{d}\nu) \, .$$

Proof: First, for any $t \in \mathbb{R}$, we define $\Upsilon(t)$ as an operator on $\mathbb{C}^d$ by

$$\Upsilon(t)(\vec{w}_R + i\vec{w}_I) = \Upsilon(t)\vec{w}_R + i\Upsilon(t)\vec{w}_I \, , \qquad \vec{w}_R, \vec{w}_I \in \mathbb{R}^d \, .$$

For $\vec{w} \in \mathbb{C}^d$, let $f_{\vec{w}}$ be the complex–valued function on $\mathbb{R}$ defined by

$$f_{\vec{w}}(t) := \langle \vec{w}, \Upsilon(t)\vec{w}\rangle_{\mathbb{C}^d} \, , \qquad t \in \mathbb{R} \, .$$

If $\Upsilon(t) = \Upsilon(-t) \in \mathcal{B}(\mathbb{R}^d)$ is symmetric w.r.t. the canonical scalar product of $\mathbb{R}^d$ for any $t \in \mathbb{R}$, then $f_{\vec{w}}$ is a weakly positive definite and continuous (complex–valued) function. By Theorem 7.10, for any $\vec{w} \in \mathbb{C}^d$, there is a unique finite positive measure $\mu_{\vec{w}}$ on $\mathbb{R}$ such that

$$f_{\vec{w}}(t) = \int_{\mathbb{R}} \mathrm{e}^{it\nu} \mu_{\vec{w}}(\mathrm{d}\nu) \ , \qquad t \in \mathbb{R} \ . \tag{112}$$

Now, we define a $\mathcal{B}(\mathbb{R}^d)$–valued measure μ_Υ on $\mathbb{R}$ by using the polarization identity: For any Borel set $\mathcal{X} \subset \mathbb{R}$, in the canonical orthonormal basis $\{e_k\}_{k=1}^d$ of $\mathbb{R}^d$,

$$\langle e_k, \mu_\Upsilon(\mathcal{X}) e_q \rangle_{\mathbb{R}^d} := \frac{1}{4} \Big(\mu_{e_k+e_q}(\mathcal{X}) - \mu_{e_k-e_q}(\mathcal{X}) \Big) \ . \tag{113}$$

By this definition, $\mu_\Upsilon(\mathcal{X})$ is a symmetric operator on $\mathbb{R}^d$ (w.r.t. the canonical scalar product). Moreover, one can check that, for all $\vec{w} \in \mathbb{R}^d$ and any Borel set $\mathcal{X} \subset \mathbb{R}$,

$$\langle \vec{w}, \mu_\Upsilon(\mathcal{X}) \vec{w} \rangle_{\mathbb{R}^d} = \mu_{\vec{w}}(\mathcal{X}) \ . \tag{114}$$

Indeed, if $\vec{w} := (w_1, \ldots, w_d) \in \mathbb{R}^d$ then, by symmetry of the operator $\Upsilon(t) \in \mathcal{B}(\mathbb{R}^d)$,

$$f_{\vec{w}}(t) = \frac{1}{4} \sum_{k,q=1}^{d} w_k w_q \left(f_{e_k+e_q}(t) - f_{e_k-e_q}(t) \right) \ , \qquad t \in \mathbb{R} \ .$$

Hence, from the injectivity of the Fourier transform of finite measures,

$$\mu_{\vec{w}} = \frac{1}{4} \sum_{k,q=1}^{d} w_k w_q \Big(\mu_{e_k+e_q} - \mu_{e_k-e_q} \Big)$$

and (114) follows. By positivity of $\mu_{\vec{w}}$, μ_Υ is a $\mathcal{B}_+(\mathbb{R}^d)$–valued measure on $\mathbb{R}$. Moreover, we deduce from (112) that

$$\langle \vec{w}, \Upsilon(t) \vec{w} \rangle_{\mathbb{R}^d} = \int_{\mathbb{R}} \mathrm{e}^{it\nu} \langle \vec{w}, \mu_\Upsilon(\mathrm{d}\nu) \vec{w} \rangle_{\mathbb{R}^d} \ , \qquad t \in \mathbb{R} \ , \ \vec{w} \in \mathbb{R}^d \ .$$

If $\Upsilon(t) = \Upsilon(-t)$ for any $t \in \mathbb{R}$, then $\mu_\Upsilon(\mathcal{X}) = \mu_\Upsilon(-\mathcal{X})$ for any Borel set $\mathcal{X} \subset \mathbb{R}$ and hence,

$$\langle \vec{w}, \Upsilon(t) \vec{w} \rangle_{\mathbb{R}^d} = \int_{\mathbb{R}} \cos(t\nu) \langle \vec{w}, \mu_\Upsilon(\mathrm{d}\nu) \vec{w} \rangle_{\mathbb{R}^d} \ , \qquad t \in \mathbb{R} \ , \ \vec{w} \in \mathbb{R}^d \ . \tag{115}$$

By using the symmetry of the operators $\Upsilon(t) \in \mathcal{B}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}} \cos(t\nu)\, \mu_{\Upsilon}(\mathrm{d}\nu) \in \mathcal{B}(\mathbb{R}^d)$$

at any fixed $t \in \mathbb{R}$, we arrive at the assertion from (115). ■

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