Analysis of a Non-Autonomous Non-Linear Operator-Valued Evolution Equation to Diagonalize Quadratic Operators in Boson Quantum Field Theory

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Abstract. We study a non–autonomous, non-linear evolution equation on the space of operators on a complex Hilbert space. We specify assumptions that ensure the global existence of its solutions and allow us to derive its asymptotics at temporal infinity. We demonstrate that these assumptions are optimal in a suitable sense and more general than those used before. The evolution equation derives from the Brocket–Wegner flow that was proposed to diagonalize matrices and operators by a strongly continuous unitary flow. In fact, the solution of the non–linear flow equation leads to a diagonalization of Hamiltonian operators in boson quantum field theory which are quadratic in the field.

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Contents

I Introduction

In theoretical physics, the second quantization formalism is crucial to treat many–particle problems. In the first quantization formalism, i.e., in the canonical ensemble, the number of particles of the corresponding wave function stays fixed. Whereas in second quantization, i.e., in the grand–canonical ensemble, the particle number is not fixed and their boson or fermion statistics is incorporated in the well–known creation/annihilation operators acting on Fock space. Within this latter framework, we are interested in boson systems, the simplest being the perfect Bose gas, i.e., a system with no interaction defined by a particle number–conserving quadratic Hamiltonian. By quadratic Hamiltonians we refer to self–adjoint operators which are quadratic in the creation and annihilation operators. It is known since Bogoliubov and his celebrated theory of superfluidity [1] that such quadratic operators are reduced to a perfect gas by a suitable unitary transformation, see also [2, Appendix B.2].

Diagonalizations of quadratic boson operators are generally not trivial, and in this paper, we investigate this question under weaker conditions as before. Indeed, after Bogoliubov with his **u**–**v** unitary transformation [1, 2], general results on this problem have been obtained for real quadratic operators with bounded one–particle spectrums by Friedrichs [3, Part V], Berezin [4, Theorem 8.1], Kato and Mugibayashi [5, Theorem 2]. In the present paper we generalize previous analyses of real quadratic boson Hamiltonians with positive one–particle spectrum bounded above and below away from zero to complex,

unbounded one–particle operators without a gap above zero. This generalization is obviously important since, for most physically interesting applications, the one–particle spectrum is neither bounded above nor bounded away from zero. Moreover, our proof is completely different. Its mathematical novelty lies in the use of *non–autonomous evolution equations* as a key ingredient.
More specifically, we employ the Brocket–Wegner flow, originally proposed by Brockett for symmetric matrices in 1991 [6] and, in a different variant, by Wegner in 1994 for self–adjoint operators [7]. This flow leads to unitarily equivalent operators via a non–autonomous hyperbolic evolution equation. The mathematical foundation of such flows [6, 7] has recently been given in [8]. Unfortunately, the results [8] do not apply to the models. In this paper we prove the well–posedness of the Brocket–Wegner flow $\partial_t \mathrm{H}_t = \big[\mathrm{H}_t, [\mathrm{H}_t, \mathrm{N}]\big]$, where for $t \geq 0$, $[\mathrm{H}_t, \mathrm{N}] := \mathrm{H}_t\mathrm{N} - \mathrm{N}\mathrm{H}_t$ is the commutator of a quadratic Hamiltonian H_t and the particle number operator N acting on the boson Fock space. Establishing well–posedness is non–trivial here because the Brocket–Wegner flow is a (quadratically) non–linear first–order differential equation for unbounded operators. It is solved by an auxiliary non–autonomous parabolic evolution equation.
Indeed, non–autonomous evolution equations turn out to be crucial at two different stages of our proof:

(a) To show for $t \geq 0$ the well–posedness of the Brocket–Wegner flow $\partial_t \mathrm{H}_t = \big[\mathrm{H}_t, [\mathrm{H}_t, \mathrm{N}]\big]$ for a quadratic operator H_0 via an auxiliary system of non–linear first–order differential equations for operators;

(b) To rigorously define a family of unitarily equivalent quadratic operators $\mathrm{H}_t = \mathrm{U}_t \mathrm{H}_0 \mathrm{U}_t^*$ as a consequence of the Brocket–Wegner flow.

To be more specific, (a) uses the theory of non–autonomous *parabolic* evolution equations via an auxiliary systems of non–linear first–order differential equation for operators. Whereas the second step (b) uses the theory of non–autonomous *hyperbolic* evolution equations to define the unitary operator U_t by $\mathrm{U}_0 := \mathbf{1}$ and $\partial_t \mathrm{U}_t = -i\mathrm{G}_t\mathrm{U}_t$, for all $t > 0$, with generator $\mathrm{G}_t := i\,[\mathrm{N}, \mathrm{H}_t]$. For bounded generators, the existence, uniqueness and even an explicit form of their solution is given by the Dyson series, as it is well–known. It is much more delicate for unbounded generators, which is what we are dealing with here. It has been studied, after the first result of Kato in 1953 [9], for decades by many authors (Kato again [10, 11] but also Yosida, Tanabe, Kisynski, Hackman, Kobayasi, Ishii, Goldstein, Acquistapace, Terreni, Nickel, Schnaubelt, Caps, Tanaka, Zagrebnov, Neidhardt), see, e.g., [12, 13, 14, 15, 16] and the corresponding references cited therein. Yet, no unified theory of such linear evolution equations that gives a complete characterization analogously to the Hille–Yosida generation theorems is known. By using the Yosida approximation, we simplify Ishii's proof [17, 18] and obtain the well–posedness of this Cauchy problem in the hyperbolic case in order to define the unitary operator U_t.

Next, by taking the limit $t \to \infty$ of unitarily equivalent quadratic operators $\mathrm{H}_t = \mathrm{U}_t \mathrm{H}_0 \mathrm{U}_t^*$, under suitable conditions on H_0, we demonstrate that the limit operator H_∞ is also unitarily equivalent to H_0. This is similar to scattering in quantum field theory since we analyze the strong limit U_∞ of the unitary operator U_t, as $t \to \infty$. The limit operator $\mathrm{H}_\infty = \mathrm{U}_\infty \mathrm{H}_0 \mathrm{U}_\infty^*$ is a quadratic boson Hamiltonian which commutes with the particle number operator N, i.e., $[\mathrm{H}_\infty, \mathrm{N}] = 0$ – a fact which we refer to as H_∞ being N–diagonal. In particular, it can be diagonalized by a unitary on the one–particle Hilbert space, only. Consequently, we provide in this paper a new mathematical application of evolution equations as well as some general results on quadratic operators, which are also interesting for mathematical physicists.

The paper is structured as follows. In Section II, we present our results and discuss them in the context of previously known facts. Section III contains a guideline to our approach in terms of theorems, whereas Section IV illustrates it on an explicit and concrete case, showing, in particular, that a pathological behavior of the Brocket–Wegner flow is not merely a possibility, but does occur. Sections V–VI are the core of our paper, as all important proofs are given here. Finally, Section VII is an appendix with a detailed analysis in Section VII.1 of evolution equations for unbounded operators of hyperbolic type on Banach spaces, and with, in Sections VII.2 and VII.3, some comments on Bogoliubov transformations and Hilbert–Schmidt operators. In particular, we clarify in Section VII.1 Ishii's approach [17, 18] to non–autonomous hyperbolic evolution equations.

II Diagonalization of Quadratic Boson Hamiltonians

In this section we describe our main results on quadratic boson Hamiltonians. First, we define quadratic operators in Section II.1 and present our findings in Section II.2 without proofs. The latter is sketched in Section III and given in full detail in Sections V–VI. Section II.3 is devoted to a historical overview on the diagonalization of quadratic operators.

II.1 Quadratic Boson Operators

To fix notation, let $\mathfrak{h} := L^2(\mathcal{M})$ be a separable complex Hilbert space which we assume to be realized as a space of square–integrable functions on a measure space $(\mathcal{M}, \mathfrak{a})$. The scalar product on $\mathfrak{h}$ is given by

$$\langle f|g\rangle := \int_{\mathcal{M}} \overline{f(x)} g(x) \, \mathrm{d}\mathfrak{a}(x) \ . \tag{II.1}$$

For $f \in \mathfrak{h}$, we define its complex conjugate $\bar{f} \in \mathfrak{h}$ by $\bar{f}(x) := \overline{f(x)}$, for all $x \in \mathcal{M}$. For any bounded (linear) operator X on $\mathfrak{h}$, we define its transpose X^{t} and its complex conjugate $\bar{X}$ by $\langle f|X^{\mathrm{t}} g\rangle := \langle \bar{g}|X\bar{f}\rangle$ and $\langle f|\bar{X} g\rangle := \overline{\langle \bar{f}|X\bar{g}\rangle}$ for $f, g \in \mathfrak{h}$, respectively. Note that the adjoint of the operator X equals

$X^* = \overline{X^{\mathrm{t}}} = \overline{X}^{\mathrm{t}}$, where it exists. The Banach space of bounded operators acting on $\mathfrak{h}$ is denoted by $\mathcal{B}(\mathfrak{h})$, whereas $\mathcal{L}^1(\mathfrak{h})$ and $\mathcal{L}^2(\mathfrak{h})$ are the spaces of trace–class and Hilbert–Schmidt operators, respectively. Norms in $\mathcal{L}^1(\mathfrak{h})$ and $\mathcal{L}^2(\mathfrak{h})$ are respectively denoted by

$$\|X\|_1 := \mathrm{tr}(|X|) \ , \quad \text{for } X \in \mathcal{L}^1(\mathfrak{h}) \ , \tag{II.2}$$

and

$$\|X\|_2 := \sqrt{\mathrm{tr}(X^*X)} \ , \quad \text{for } X \in \mathcal{L}^2(\mathfrak{h}) \ . \tag{II.3}$$

Note that, if there exists a constant $\mathrm{K} < \infty$ such that

$$\left| \sum_{k=1}^{\infty} \langle \eta_k | X \psi_k \rangle \right| \leq \mathrm{K} \ , \tag{II.4}$$

for all orthonormal bases $\{\eta_k\}_{k=1}^{\infty}, \{\psi_k\}_{k=1}^{\infty} \subseteq \mathfrak{h}$, then

$$\mathrm{tr}(X) = \sum_{k=1}^{\infty} \langle \varphi_k | X \varphi_k \rangle \tag{II.5}$$

for any orthonormal basis $\{\varphi_k\}_{k=1}^{\infty} \subseteq \mathfrak{h}$. Finally, we denote by $\mathbf{1}$ the identity operator on various spaces. Assume now the following conditions:

A1: $\Omega_0 = \Omega_0^* \geq 0$ is a positive operator on $\mathfrak{h}$.

A2: $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$ is a (non–zero) Hilbert–Schmidt operator.

A3: The operator Ω_0 is invertible on $\mathrm{Ran} B_0$ and satisfies:

$$\Omega_0 \geq 4 B_0 (\Omega_0^{\mathrm{t}})^{-1} \bar{B}_0 = 4 \overline{B_0^* \Omega_0^{-1} B_0} \ . \tag{II.6}$$

Note that neither the operator Ω_0 in A1 nor its inverse Ω_0^{-1} are necessarily bounded. Observe also that $\Omega_0^2 \geq 4 B_0 \bar{B}_0$ implies A3: Use A1–A2, two times $\Omega_0^2 \geq 4 B_0 \bar{B}_0$ in

$$\left(4 B_0 (\Omega_0^{\mathrm{t}})^{-1} \bar{B}_0\right) \left(4 B_0 (\Omega_0^{\mathrm{t}})^{-1} \bar{B}_0\right) \leq 4 B_0 \bar{B}_0 \leq \Omega_0^2 \ , \tag{II.7}$$

and the fact that the map $X \mapsto X^{1/2}$ is operator monotone. However, the converse does *not* hold, see for instance Remark 26 in Section IV where a trivial example is given.

Next, take some *real* orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ in the dense domain $\mathcal{D}(\Omega_0) \subseteq \mathfrak{h}$ of Ω_0 and, for any $k \in \mathbb{N}$, let $a_k := a(\varphi_k)$ be the corresponding boson annihilation operator acting on the boson Fock space

$$\mathcal{F}_b := \bigoplus_{n=0}^{\infty} \mathcal{S}_n \left(\mathfrak{h}^{\otimes n}\right) \ , \tag{II.8}$$

whose scalar product is again denoted by $\langle\cdot|\cdot\rangle$. Here, $\mathcal{S}_n$ is the orthogonal projection onto the subspace of totally symmetric n–particle wave functions in $\mathfrak{h}^{\otimes n}$, the n–fold tensor product of $\mathfrak{h}$. Then, for any fixed $C_0 \in \mathbb{R}$, the quadratic boson operator is defined through the operators Ω_0 and B_0 by

$$\mathrm{H}_0 := \sum_{k,\ell} \{\Omega_0\}_{k,\ell} \, a_k^* a_\ell + \{B_0\}_{k,\ell} \, a_k^* a_\ell^* + \{\bar{B}_0\}_{k,\ell} \, a_k a_\ell + C_0 \ , \tag{II.9}$$

with $\{X\}_{k,\ell} := \langle \varphi_k | X \varphi_\ell \rangle$ for all operators X acting on $\mathfrak{h}$. Note that a continuous integral $\int \mathrm{d}k \ \mathrm{d}\ell$ could also replace the discrete sum $\sum_{k,\ell}$ in this definition since *all results also hold in the continuous case*.
We first establish the self–adjointness of H_0.

Proposition 1 (Self–adjointness of quadratic operators)
Under Conditions A1–A2, the quadratic operator H_0 *is essentially self–adjoint on the domain*

$$\mathcal{D}(\mathrm{H}_0) := \bigcup_{N=1}^{\infty} \left(\bigoplus_{n=0}^{N} \mathcal{S}_n \left(\mathcal{D}(\Omega_0)^{\otimes n} \right) \right) \ . \tag{II.10}$$

Detailed proofs of self–adjointness of quadratic operators are given in [4, Thm 6.1] and, more recently and also richer in detail, in [19, Thm 5.3]. See also [20, 21, 22].
The aim of this paper is to diagonalize the quadratic Hamiltonian H_0. To this end, it suffices to find a unitary transformation leading to a quadratic operator of the form (II.9) with $B_0 = 0$, after conjugation with this unitary. Such quadratic operators are called *N–diagonal* since they commute with the particle number operator

$$\mathrm{N} := \sum_k a_k^* a_k \ . \tag{II.11}$$

In particular, the N–diagonal part of the quadratic Hamiltonian H_0 equals

$$\Gamma_0 := \sum_{k,\ell} \{\Omega_0\}_{k,\ell} \, a_k^* a_\ell + C_0 \ , \tag{II.12}$$

because $[\Gamma_0, \mathrm{N}] = 0$, for any self–adjoint operator Ω_0 acting on $\mathfrak{h}$.
Note that we diagonalize semi–bounded quadratic boson operators. Indeed, under Conditions A1–A4 (see below), Theorem 7 implies that the Hamiltonian H_0 is bounded from below by $\mathrm{H}_0 \geq C_0$. See [19, Thm 5.4 and Cor. 5.1]. It is a typical situation encountered in quantum mechanics because the N–diagonal part Γ_0 usually corresponds to the kinetic energy operator, whereas the non–N–diagonal part W_0 of H_0 represents interactions.

II.2 Main Results

In this paper, we prove the N–diagonalization of quadratic operators in a more general setting as before, by using the Brocket–Wegner flow [6, 7] for quadratic boson operators. Theories of non–autonomous evolution equations are crucial to define this flow. For more details about our approach, we refer to Section III, particularly Section III.1. Here, we define this unitary flow by the following assertion:

Theorem 2 (Local unitary flow on quadratic operators)
Under Conditions A1–A3, there exist $T_+ \in (0,\infty]$, *two operator families* $(\Omega_t)_{t\in[0,T_+)}$ *and* $(B_t)_{t\in[0,T_+)}$, *satisfying A1–A2, and a strongly continuous family* $(\mathrm{U}_t)_{t\in[0,T_+)}$ *of unitary operators acting on the boson Fock space* $\mathcal{F}_b$ *such that*

$$\mathrm{U}_t\mathrm{H}_0\mathrm{U}_t^* = \sum_{k,\ell} \{\Omega_t\}_{k,\ell}\, a_k^* a_\ell + \mathrm{W}_t + C_t \ , \tag{II.13}$$

*where the non–*N*–diagonal part of the quadratic operator* $\mathrm{U}_t\mathrm{H}_0\mathrm{U}_t^*$ *equals*

$$\mathrm{W}_t := \sum_{k,\ell} \{B_t\}_{k,\ell}\, a_k^* a_\ell^* + \left\{\bar{B}_t\right\}_{k,\ell} a_k a_\ell \quad \text{and} \quad C_t := C_0 + 8\int_0^t \|B_\tau\|_2^2 \ \mathrm{d}\tau \ . \tag{II.14}$$

Furthermore, the map $t \mapsto \|B_t\|_2$ *from* $[0,T_+)$ *to* $\mathbb{R}_0^+$ *is monotonically decreasing,* $\Omega_t \leq \Omega_0$, $\Omega_t^2 - 8B_t\bar{B}_t \geq \Omega_0^2 - 8B_0\bar{B}_0$, *and*

$$\mathrm{tr}\left(\Omega_t^2 - 4B_t\bar{B}_t - \Omega_0^2 + 4B_0\bar{B}_0\right) = 0 \ . \tag{II.15}$$

If $\Omega_0 B_0 = B_0\Omega_0^{\mathrm{t}}$ *then* $\Omega_t B_t = B_t \Omega_t^{\mathrm{t}}$ *and*

$$\Omega_t = \{\Omega_0^2 - 4B_0\bar{B}_0 + 4B_t\bar{B}_t\}^{1/2} \ . \tag{II.16}$$

Remark 3 *If A1–A2 hold, but not A3, then* Ω_t *and* B_t *exist within a small time interval. However, we omit this case to simplify our discussions, see as an example the proof of Proposition 32. In fact, A3 ensures the positivity of* Ω_t *and the decay of the Hilbert–Schmidt norm* $\|B_t\|_2$ *for* $t \in [0,T_+)$, *whereas* Ω_t, B_t, *and* U_t *exist for* $t \in [0,T_{\max})$ *with* $T_{\max} \in (T_+,\infty]$.

Theorem 2 follows from Theorems 11, 14, and 18 (Section III.2). In particular, by Theorem 14, the unitary operator U_t realizes a (time–dependent) Bogoliubov **u–v** transformation:

$$\forall t \in [0,T_+): \qquad \mathrm{U}_t a_k \mathrm{U}_t^* = \sum_{\ell} \{\mathbf{u}_t\}_{k,\ell}\, a_\ell + \{\mathbf{v}_t\}_{k,\ell}\, a_\ell^* \ , \tag{II.17}$$

with $\mathbf{u}_t \in \mathcal{B}(\mathfrak{h})$ and $\mathbf{v}_t \in \mathcal{L}^2(\mathfrak{h})$ satisfying $\mathbf{u}_t\mathbf{u}_t^* - \mathbf{v}_t\mathbf{v}_t^* = \mathbf{1}$, $\mathbf{u}_t^*\mathbf{u}_t - \mathbf{v}_t^{\mathrm{t}}\bar{\mathbf{v}}_t = \mathbf{1}$, $\mathbf{u}_t\mathbf{v}_t^{\mathrm{t}} = \mathbf{v}_t\mathbf{u}_t^{\mathrm{t}}$, and $\mathbf{u}_t^*\mathbf{v}_t = \mathbf{v}_t^{\mathrm{t}}\bar{\mathbf{u}}_t$.
A necessary condition to N–diagonalize the quadratic operator H_0 is thus the convergence of the Hilbert–Schmidt operator $B_t \in \mathcal{L}^2(\mathfrak{h})$ of Theorem 2 to zero in some topology, in fact, the Hilbert–Schmidt topology to be precise. Unfortunately, we cannot exclude, a priori, the existence of a *blow–up* at a finite time, that is, the fact that the Hilbert–Schmidt norm $\|B_t\|_2$ diverges as

$t \nearrow T_{\max}$ with $T_{\max} \in (T_+, \infty)$, see Lemma 43 for more details. Moreover, even if we assume that $T_+ = \infty$, we can only infer from Theorem 2 the convergence of the Hilbert–Schmidt norm $\|B_t\|_2$ as $t \to \infty$ to some positive constant $\mathrm{K} \in [0, \|B_0\|_2]$. In other words, Conditions A1–A3 are not sufficient, a priori, to obtain the convergence of $\|B_t\|_2$ to $\mathrm{K} = 0$, even if we a priori assume that $T_+ = \infty$. As a consequence, we proceed by progressively strengthening A1–A3, using the additional assumptions A4–A6[1] defined as follows:

A4: $\Omega_0^{-1/2} B_0 \in \mathcal{L}^2(\mathfrak{h})$ is a Hilbert–Schmidt operator.

A5: $\mathbf{1} > 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ and $\Omega_0^{-1-\varepsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$ for some constant $\varepsilon > 0$.

A6: $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 + \mu\mathbf{1}$ for some constant $\mu > 0$.

It is easy to show that A5 yields A4, whereas A6 is stronger than Conditions A3–A4 as it obviously implies $\Omega_0^{-\alpha} B_0 \in \mathcal{L}^2(\mathfrak{h})$ for any $\alpha > 0$ because, in this case, $\Omega_0 \geq \mu\mathbf{1}$. In particular, A6 yields A5, up to the inequality $\mathbf{1} > 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$, and this sixth assumption should thus be seen as the most restrictive condition. Opposed to A6, Conditions A4–A5 accommodate the *most difficult cases* where Ω_0^{-1} can be unbounded.

Remark 4 *Condition A5 yields the existence of a constant* $\mathrm{r} > 0$ *such that*

$$\mathbf{1} \geq (4 + \mathrm{r})\, B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0 \ . \tag{II.18}$$

Indeed, $\Omega_0^{-1-\varepsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$ *implies that the operator* $B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ *is compact and thus, by combining* $\mathbf{1} > 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ *with an orthonormal basis of eigenvectors of* $B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$*, one directly gets (II.18). In fact, the assumptions* $\Omega_0^{-1-\varepsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$ *and (II.18) for some constants* $\mathrm{r}, \varepsilon > 0$ *are equivalent to Condition A5 and is what is really used in all proofs or assertions invoking A5.*

Remark 5 *The necessity of Conditions A3–A6 is discussed in Section IV.1 via explicit examples that generalize Bogoliubov's result [1, 2]. In fact, A1–A6 are not technical artefacts but optimal in a suitable sense.*

Henceforth we assume at least A1–A4, from which we derive some partial results (see Theorems 11 (v), 12 (i), and 19 (i)):

Theorem 6 (Global unitary flow on quadratic operators)
Conditions A1–A4 implies $T_+ = \infty$ *as well as the square–integrability of the map* $t \mapsto \|B_t\|_2$ *of Theorem 2 on* $[0, \infty)$*. In this case,* Ω_t *and* B_t *satisfy A1–A4 for all* $t \geq 0$*.*

Since, by Lemma 66,

$$\|\mathrm{W}_t\, (\mathrm{N} + \mathbf{1})^{-1}\,\|_{\mathrm{op}} \leq (4 + \sqrt{2})\|B_t\|_2 \ , \tag{II.19}$$

[1] $\mathbf{1} > 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ means that, for any $\varphi \in \mathfrak{h}$, $\varphi \neq 0$, $\langle\varphi|(\mathbf{1} - 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0)\varphi\rangle > 0$.

Theorem 6 asserts a *quasi* N–*diagonalization* of H_0, in the sense that the non–N–diagonal part W_t of the quadratic operator $\mathrm{U}_t\mathrm{H}_0\mathrm{U}_t^*$, compared to the particle number operator N, tends to zero, as $t \to \infty$. Moreover, this theorem yields the convergence in $\mathcal{L}^1(\mathfrak{h})$ of the (possibly unbounded) operators Ω_t to a positive operator $\Omega_\infty = \Omega_\infty^* \geq 0$ because, for any $t \in \mathbb{R}_0^+ \cup \{\infty\}$,

$$\Omega_t = \Omega_0 - 16 \int_0^t B_\tau \bar{B}_\tau \mathrm{d}\tau \ , \tag{II.20}$$

see Theorem 11 (i). Similarly,

$$C_\infty := \lim_{t\to\infty} C_t = C_0 + 8 \int_0^\infty \|B_\tau\|_2^2 \mathrm{d}\tau < \infty \ . \tag{II.21}$$

Observe also that the properties of the operator family $(\Omega_t)_{t\geq 0}$ described in Theorem 2 can be extended to $t = \infty$: $\Omega_\infty \leq \Omega_0$ whereas $\Omega_\infty^2 \geq \Omega_0^2 - 8B_0\bar{B}_0$ and

$$\mathrm{tr}\left(\Omega_\infty^2 - \Omega_0^2 + 4B_0\bar{B}_0\right) = 0 \ . \tag{II.22}$$

Moreover, in the specific cases where $\Omega_0 B_0 = B_0 \Omega_0^{\mathrm{t}}$, the limit operator

$$\Omega_\infty = \{\Omega_0^2 - 4B_0\bar{B}_0\}^{1/2} \tag{II.23}$$

is *explicit*, and the limit constant equals

$$C_\infty = C_0 + \frac{1}{2}\mathrm{tr}\left(\Omega_0 - \{\Omega_0^2 - 4B_0\bar{B}_0\}^{1/2}\right) < \infty \ . \tag{II.24}$$

The fact that the trace in the last equation is finite under Conditions A1–A4 is not trivial. This property is proven under different assumptions on Ω_0 and B_0 in [4, Lemma 8.1]. In the present paper, however, the finiteness of C_∞ is never used elsewhere, in contrast to Berezin's method where [4, Lemma 8.1] is crucial. In fact, in our case, it is an obvious corollary of Theorem 22 (i).
Using the second assertion (ii) of Theorem 22 we can also make the quasi N–diagonalization of H_0 more precise by controlling the convergence of the operator family $(\mathrm{U}_t\mathrm{H}_0\mathrm{U}_t^*)_{t\geq 0}$ to the well–defined N–diagonal quadratic operator

$$\mathrm{H}_\infty := \sum_{k,\ell} \{\Omega_\infty\}_{k,\ell}\, a_k^* a_\ell + C_\infty \ . \tag{II.25}$$

THEOREM 7 (QUASI N–DIAGONALIZATION OF QUADRATIC OPERATORS)
Conditions A1–A4 imply convergence in the strong resolvent sense and as $t \to \infty$, of the operator family $(\mathrm{U}_t\mathrm{H}_0\mathrm{U}_t^*)_{t\geq 0}$ of Theorem 2 to H_∞.

However, under Conditions A1–A4, the unitary operator U_t itself may not converge, as $t \to \infty$, in general. In particular, it is not clear that H_0 and H_∞ are unitarily equivalent, unfortunately. Therefore, in order to obtain a complete N–diagonalization of H_0, we use either A5 or A6 which separately lead to the strong convergence of the unitary operator U_t to U_∞ and, thus, to the N–diagonalization of quadratic boson operators (Theorem 23).

Theorem 8 (N–diagonalization of quadratic operators)
Under Conditions A1–A3 and either A5 or A6, there exists a Bogoliubov $\mathbf{u}$–$\mathbf{v}$ *unitary transformation* U_∞ *on* $\mathcal{F}_b$ *such that* $\mathrm{H}_\infty = \mathrm{U}_\infty \mathrm{H}_0 \mathrm{U}_\infty^*$.

Observe that the assumptions of Theorem 8 are sufficient, but possibly not necessary, see Remarks 5 and 24. Also, we do not obtain an explicit form of Ω_∞ in the general case. However, under the additional assumption that $\Omega_0 B_0 = B_0 \Omega_0^{\mathrm{t}}$, we derive the limit operator Ω_∞ explicitly, see (II.23). This is a new result and the assumptions of Theorem 8 are also more general as the ones previously used, especially since they include the most difficult cases where both operators Ω_0 and Ω_0^{-1} are unbounded.
In Section III we describe our method in detail, but we also refer to Section IV, where an instructive and fully explicit example is given. In particular, we show in Section IV that (II.23) does not generally hold if $\Omega_0 B_0 \neq B_0 \Omega_0^{\mathrm{t}}$.

II.3 Historical Overview

A diagonalization of quadratic boson operators has been first performed by Bogoliubov [1] in 1947, see also [2, Appendix B.2]. This result is a key ingredient of the Bogoliubov theory of superfluidity, and it defines the borderline to what could possibly be expected. It is restrictive, however, since $\Omega_0 = \Omega_0^{\mathrm{t}}$ and $\Omega_0 B_0 = B_0 \Omega_0$ in his case. In fact, it is one of the simplest examples to diagonalize since this problem is reduced to the N–diagonalization of simple quadratic operators defined, for any $k \in \mathbb{Z}^{\nu \geq 1}/\{-1, 1\}$, by 2×2 real matrices $\Omega_{0,k}$ and $B_{0,k}$ (IV.2) satisfying $\Omega_{0,k} B_{0,k} = B_{0,k} \Omega_{0,k}$ and

$$\Omega_{0,k}^2 \geq 4 B_{0,k} \bar{B}_{0,k} + \mu \mathbf{1} \ , \tag{II.26}$$

see (IV.23).
During the 1950ies and 1960ies, Friedrichs [3, Part V] and Berezin [4, Theorem 8.1] gave a first general result under the condition that $\Omega_0 \in \mathcal{B}(\mathfrak{h})$ and $B_0 \in \mathcal{L}^2(\mathfrak{h})$ are both real symmetric operators satisfying the gap condition

$$\Omega_0 \pm 2 B_0 \geq \mu \mathbf{1} \ , \quad \text{for some } \mu > 0 \ . \tag{II.27}$$

To be precise, under these assumptions, they prove Theorem 8. These hypotheses are stronger than Conditions A1, A2, and A6, which we use to prove Theorem 8:

Proposition 9 (On the Friedrichs–Berezin assumptions)
Let $\Omega_0 \in \mathcal{B}(\mathfrak{h})$ *and* $B_0 \in \mathcal{L}^2(\mathfrak{h})$ *be real symmetric operators satisfying (II.27). Then* Ω_0 *and* B_0 *satisfy Conditions A1, A2, and A6.*

Proof. Using (II.27) we obtain, for any $\varphi \in \mathfrak{h}$, that

$$\langle \varphi | \Omega_0 \varphi \rangle = \frac{1}{2} \sum_{\sigma = \pm 1} \langle \varphi | \left(\Omega_0 + 2\sigma B_0 \right) \varphi \rangle \geq \mu \|\varphi\|^2 \ , \tag{II.28}$$

i.e., $\Omega_0 = \Omega_0^{\mathrm{t}} \geq \mu \mathbf{1} > 0$. For any $\tilde{\mu} \in (0, \mu)$,

$$\tilde{\Omega} := \Omega_0 - \tilde{\mu}\mathbf{1} \geq (\mu - \tilde{\mu})\,\mathbf{1} > 0 \tag{II.29}$$

is globally invertible. By (II.27), observe also that

$$\tilde{\Omega} \geq \Omega_0 - \mu\mathbf{1} \geq \pm 2B_0\ . \tag{II.30}$$

The last inequality together with the existence and the positivity of the inverse operator $\tilde{\Omega}^{-1} \in \mathcal{B}(\mathfrak{h})$ for any $\tilde{\mu} \in (0, \mu)$ yields

$$\pm 2\tilde{\Omega}^{-1/2} B_0 \tilde{\Omega}^{-1/2} \leq \mathbf{1}\ , \tag{II.31}$$

which in turn implies

$$4\tilde{\Omega}^{-1/2} B_0 \tilde{\Omega}^{-1} B_0 \tilde{\Omega}^{-1/2} \leq \mathbf{1}\ . \tag{II.32}$$

From the inequalities $\Omega_0 \geq \tilde{\Omega} \geq (\mu - \tilde{\mu})\,\mathbf{1}$ we infer that $\Omega_0^{-1} \leq \tilde{\Omega}^{-1}$ for any $\tilde{\mu} \in (0, \mu)$ because the map $X \mapsto X^{-1}$ is operator anti–monotone, for positive X. By (II.32), it follows that

$$4B_0\Omega_0^{-1}B_0 \leq 4B_0\tilde{\Omega}^{-1}B_0 \leq \tilde{\Omega} := \Omega_0 - \tilde{\mu}\mathbf{1}\ , \tag{II.33}$$

for any $\tilde{\mu} \in (0, \mu)$. Consequently,

$$\Omega_0 \geq 4B_0\Omega_0^{-1}B_0 + \mu\mathbf{1} \geq \mu\mathbf{1}\ . \tag{II.34}$$

Since $\Omega_0^{\mathrm{t}} = \Omega_0 \geq 0$ and $B_0 = B_0^{\mathrm{t}} = \bar{B}_0 \in \mathcal{L}^2(\mathfrak{h})$, Conditions A1–A2 and A6 hold. □

Later, in 1967, Kato and Mugibayashi [5, Theorem 2] have relaxed the hypothesis (II.27) to accommodate the equality $\Omega_0 = 2B_0$ or $\Omega_0 = -2B_0$ on some n–dimensional subspace of $\mathfrak{h}$. In particular, Theorem 8 was proven in [5], replacing the assumption (II.27) by the two inequalities[2]

$$\Omega_0 + 2B_0 \geq \mu_-\mathbf{1} \qquad \text{and} \qquad \Omega_0 - 2B_0 + P^{(n)} \geq \mu_+\mathbf{1}\ , \tag{II.35}$$

where $\mu_\pm > 0$ are two strictly positive constants and

$$P^{(n)} := \mathbf{1}\left[\Omega_0 = 2B_0\right]$$

is the projection onto the n–dimensional subspace of $\mathfrak{h}$ where $\Omega_0 = 2B_0$.
However, after elementary manipulations, the hypotheses (II.35) yields again a gap condition and the boundedness of the inverse operator $\Omega_0^{-1} \in \mathcal{B}(\mathfrak{h})$. Indeed, the equality $\Omega_0\varphi = 2B_0\varphi = 0$ for some $\varphi \in \varphi$ would contradict the first inequality of (II.35), as it is confirmed by the following proposition:

[2]Their result [5] is also valid by substituting $-B_0$ for B_0 in (II.35).

Proposition 10 (On the Kato–Mugibayashi assumption)
Let $\Omega_0 \in \mathcal{B}(\mathfrak{h})$ *and* $B_0 \in \mathcal{L}^2(\mathfrak{h})$ *be real symmetric operators satisfying the assumption (II.35). Then* Ω_0 *satisfies the gap equation*

$$\Omega_0 \geq \frac{\mu_-}{2}\mathbf{1} > 0 \tag{II.36}$$

and its inverse $\Omega_0^{-1} \in \mathcal{B}(\mathfrak{h})$ *is thus bounded.*

Proof. Let $\mathfrak{h}_0 = \mathrm{Ran}P^{(n)}$ be the finite dimensional subspace of $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_0^{\perp}$. For any $\varphi \in \mathfrak{h}$, there are $\varphi_0 \in \mathfrak{h}_0$ and $\varphi_1 \in \mathfrak{h}_0^{\perp}$ such that $\varphi = \varphi_0 + \varphi_1$. By using $\Omega_0^{\mathrm{t}} = \Omega_0 = \Omega_0^*$ and $B_0 = B_0^{\mathrm{t}} = B_0^*$, the first inequality of (II.35) applied on such $\varphi \in \mathfrak{h}$ implies that

$$\langle \varphi_0 | \Omega_0 \varphi_1 \rangle + \langle \varphi_1 | \Omega_0 \varphi_0 \rangle \geq \frac{\mu_-}{2} \|\varphi\|^2 - \langle \varphi_1 | B_0 \varphi_1 \rangle - \langle \varphi_0 | \Omega_0 \varphi_0 \rangle - \frac{1}{2} \langle \varphi_1 | \Omega_0 \varphi_1 \rangle \ . \tag{II.37}$$

It follows that

$$\begin{aligned} \langle \varphi | \Omega_0 \varphi \rangle &= \langle \varphi_1 | \Omega_0 \varphi_1 \rangle + \langle \varphi_0 | \Omega_0 \varphi_0 \rangle + \langle \varphi_0 | \Omega_0 \varphi_1 \rangle + \langle \varphi_1 | \Omega_0 \varphi_0 \rangle \\ &\geq \frac{1}{2} \left(\langle \varphi_1 | \left(\Omega_0 - 2B_0 \right) \varphi_1 \rangle + \mu_- \|\varphi\|^2 \right) \ , \end{aligned} \tag{II.38}$$

which, combined with the second inequality of (II.35), gives

$$\langle \varphi | \Omega_0 \varphi \rangle \geq \frac{1}{2} \left(\mu_+ \|\varphi_1\|^2 + \mu_- \|\varphi\|^2 \right) \geq \frac{\mu_-}{2} \|\varphi\|^2 \tag{II.39}$$

for any $\varphi \in \mathfrak{h}$. □

Therefore, the assumption (II.35) can be used to define the Brocket–Wegner flow for quadratic boson operators as explained in the next section. In particular, it can be shown that the condition (II.35) is preserved by the flow for all times and, because of Proposition 10 which yields $\|B_t\|_2 = \mathcal{O}\left(\mathrm{e}^{-t\mu_-}\right)$, we can N–diagonalize in this way the quadratic operator H_0, as explained in Theorem 8. This hypothesis can even be relaxed as, for instance, the assumption $\Omega_0 \in \mathcal{B}(\mathfrak{h})$ is not necessary. This result is, however, not proven here as it would lead to adding new assumptions without providing new conceptual ideas.

III Brocket–Wegner Flow for Quadratic Boson Operators

The novelty of our approach lies in the use of a non–autonomous evolution equation, called the Brocket–Wegner flow [6, 7, 8], to diagonalize quadratic boson operators. On a formal level, it is easy to describe, but the mathematically rigorous treatment is rather involved due to the unboundedness of the operators we are dealing with. More specifically, we can distinguish three sources of unboundedness in the quadratic Hamiltonian H_0 (II.9):

- The first one is related to the unboundedness of creation/annihilation operators, and it can be controlled without much efforts by our methods. This was not a problem for Friedrichs [3, Part V], Berezin [4] or Kato and Mugibayashi [5], either.
- The second one is more serious and corresponds to the unboundedness of the self–adjoint operator Ω_0, i.e., the ultraviolet divergence features of the N–diagonal part Γ_0 (II.12) of the Hamiltonian H_0. This situation is already out of the scope of previous statements [3, 4, 5]. Only Bogoliubov's result [1, 2] can still be used but in a very restricted way, rather far from being satisfactory, see Sections II.3 and IV.
- Even worse is the infrared property[3] of the N–diagonal operator Γ_0 making the inverse Ω_0^{-1} of the positive operator $\Omega_0 \geq 0$ also unbounded. This last problem turns out to be rather non–trivial and all previous approaches required an infrared cutoff of the form (II.27) or (II.36). This infrared cutoff condition is shown to be unnecessary, see Section III.3.

We start by a heuristic description of the Brocket–Wegner flow in Section III.1, which is then rigorously formulated in Section III.2. It leads to a (time–dependent) Bogoliubov **u**–**v** transformation, which allows us to diagonalize the quadratic Hamiltonian in the limit of infinite time, cf. Section III.3.

III.1 Setup of the Brocket–Wegner Flow

The Brocket–Wegner flow is a method proposed two decades ago by Brockett [6] to solve linear programming problems and independently by Wegner [7] to diagonalize Hamiltonians. It is defined as the following (quadratically) non–linear first–order differential equation for positive times:

$$\forall t \geq 0 : \qquad \partial_t Y_t = [Y_t, [Y_t, X]] \ , \quad Y_{t=0} := Y_0 \ , \tag{III.1}$$

with (possibly unbounded) operators X and Y_0 acting on a Hilbert or Banach space, and with

$$[Y_t, X] := Y_t X - X Y_t \tag{III.2}$$

being the commutator between operators Y_t and X, as usual. This flow is closely related to non–autonomous evolution equations, see Section VII.1. Indeed, let $U_{t,s}$ be an evolution operator, that is, a jointly strongly continuous in s and t operator family $(U_{t,s})_{t \geq s \geq 0}$ satisfying $U_{s,s} := \mathbf{1}$ and the cocycle property, also called Chapman–Kolmogorov property:

$$\forall t \geq x \geq s \geq 0 : \qquad U_{t,s} = U_{t,x} U_{x,s} \ . \tag{III.3}$$

Suppose that $U_{t,s}$ solves of the non–autonomous evolution equation

$$\forall t \geq s \geq 0 : \qquad \partial_t U_{t,s} = -iG_t U_{t,s} \ , \quad U_{s,s} := \mathbf{1} \ , \tag{III.4}$$

[3]Absence of a spectral gap for $\Omega_0 = \Omega_0^* \geq 0$.

with infinitesimal generator $G_t = i\,[X, Y_t]$. Then the operator

$$Y_t = U_{t,s} Y_s U_{t,s}^{-1} = U_t Y_0 U_t^{-1} \tag{III.5}$$

is a solution of (III.1), where $U_t := U_{t,0}$ and U_t^{-1} is its right inverse. In the context of self–adjoint operators X and Y_0 on a Hilbert space, the Brocket–Wegner flow generates a family $(Y_t)_{t\geq 0}$ of mutually unitarily equivalent operators. Furthermore, Brockett's observation is that a solution Y_t of (III.1) converges, at least for real symmetric matrices Y_0 and X, to a symmetric matrix Y_∞ such that $G_\infty = i\,[X, Y_\infty] = 0$. In other words, Y_∞ is X–diagonal[4]. A thorough treatment of the foundations of the Brocket–Wegner flow can be found in [8]. Now, by using the particle number operator N (II.11) and the quadratic boson operator H_0 (II.9), the infinitesimal generator G_t at $t = 0$ equals

$$\mathrm{G}_0 = i\,[\mathrm{N}, \mathrm{H}_0] = 2i \sum_{k,\ell} \{B_0\}_{k,\ell}\, a_k^* a_\ell^* - \{\bar{B}_0\}_{k,\ell}\, a_k a_\ell \ . \tag{III.6}$$

In particular, G_0 is again a quadratic boson operator. Therefore, in (III.1) we take $Y_0 := \mathrm{H}_0$ and the particle number operator N for X. A solution of the Brocket–Wegner flow for this example has the following form

$$\mathrm{H}_t := \sum_{k,\ell} \{\Omega_t\}_{k,\ell}\, a_k^* a_\ell + \{B_t\}_{k,\ell}\, a_k^* a_\ell^* + \{\bar{B}_t\}_{k,\ell}\, a_k a_\ell + C_t \ , \tag{III.7}$$

with $\Omega_t = \Omega_t^*$ and $B_t = B_t^{\mathrm{t}}$ both acting on $\mathfrak{h}$ and where $C_t \in \mathbb{R}$ is a real number. It follows that the time–dependent infinitesimal generator equals

$$\mathrm{G}_t := i\,[\mathrm{N}, \mathrm{H}_t] = 2i \sum_{k,\ell} \{B_t\}_{k,\ell}\, a_k^* a_\ell^* - \{\bar{B}_t\}_{k,\ell}\, a_k a_\ell \tag{III.8}$$

for any $t \geq 0$. Consequently, the strong convergence of the infinitesimal generator G_t on some domain to 0 heuristically means that $B_t \to 0$ in some topology. The limit operator H_∞ of H_t is then N–diagonal. In particular, it can then be diagonalized by using another unitary operator acting only on the one–particle Hilbert space $\mathfrak{h}$.

III.2 Mathematical Foundations of our Method

As shown in [8, Theorem 2], the Brocket–Wegner flow (III.1) has a unique solution for unbounded operators Y_0 on a Hilbert space provided its iterated commutators with X define relatively bounded operators whose norm tends to zero, as the order increases sufficiently fast. Until now, it is the only known result about the well–posedness of the Brocket–Wegner flow for unbounded operators. In the special case where $Y_0 = \mathrm{H}_0$ (II.9) and $X = \mathrm{N}$ (II.11), however, the infinitesimal generator G_0 (III.6) is unbounded, and we cannot invoke [8, Theorem 2] to ensure the existence of a solution H_t of (III.1).

[4] An operator Y is called *X–diagonal* iff $[Y, X] = 0$.

Such a proof in the general case is difficult since the Brocket–Wegner flow is a (quadratically) non–linear differential equation on operators. However, if H_t takes the form (III.7) and also satisfies (III.1) with $Y_0 = \mathrm{H}_0$ and $X = \mathrm{N}$, then a straightforward computation shows that the operators $\Omega_t = \Omega_t^*$ and $B_t = B_t^{\mathrm{t}}$ in (III.7) satisfy[5]

$$\forall t \geq 0 : \qquad \begin{cases} \partial_t \Omega_t = -16 B_t \bar{B}_t & , \quad \Omega_{t=0} := \Omega_0 \ , \\ \partial_t B_t = -2\left(\Omega_t B_t + B_t \Omega_t^{\mathrm{t}}\right) & , \quad B_{t=0} := B_0 \ , \end{cases} \tag{III.9}$$

whereas the real number C_t equals

$$C_t = C_0 + 8 \int_0^t \|B_\tau\|_2^2 \, \mathrm{d}\tau \ . \tag{III.10}$$

So, the alternative route is to prove the existence of a solution of the system (III.9) and then to define H_t by (III.7) and (III.10). The equations of (III.9) form a system of (quadratically) non–linear first–order differential equations. The existence and uniqueness of a solution $(\Omega_t, B_t)_{t\geq 0}$ for such a problem is also not obvious, either, especially for unbounded operators Ω_t, but still easier to derive since an explicit solution of B_t as a function of Ω_t can be found by using again non–autonomous evolution equations. This is carried out in Sections V.1–V.2 under Conditions A1–A4, i.e., $\Omega_0 \geq 0$, $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$, $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0$, $\Omega_0^{-1/2} B_0 \in \mathcal{L}^2(\mathfrak{h})$. We summarize these results in the following theorem about the well–posedness of the system (III.9) of differential equations:

Theorem 11 (Existence of operators Ω_t and B_t)
Under Conditions A1–A3, there are $T_+ \in (0, \infty]$ and a unique family $(\Omega_t, B_t)_{t\in[0,T_+)}$ satisfying:
(i) *The (possibly unbounded) operator Ω_t is positive and self–adjoint. The family $(\Omega_t - \Omega_0)_{t\in[0,T_+)} \in C^1[[0, T_+); \mathcal{L}^2(\mathfrak{h})]$ is Lipschitz continuous and is a solution of $\partial_t \Omega_t = -16 B_t \bar{B}_t$ in the Hilbert–Schmidt topology.*
(ii) *The family $(B_t)_{t\in[0,T_+)} \in C[[0, T_+); \mathcal{L}^2(\mathfrak{h})]$ is a solution[6] of the non–autonomous parabolic evolution equation $\partial_t B_t = -2(\Omega_t B_t + B_t \Omega_t^{\mathrm{t}})$ in $\mathcal{L}^2(\mathfrak{h})$, provided $t > 0$.*
(iii) *The constant of motion of the flow is given by*

$$\forall t \in [0, T_+) : \qquad \mathrm{tr}\left(\Omega_t^2 - 4B_t\bar{B}_t - \Omega_0^2 + 4B_0\bar{B}_0\right) = 0 \ . \tag{III.11}$$

(iv) *Furthermore, for all $t, s \in [0, T_+)$,*

$$\Omega_t^2 - 8B_t\bar{B}_t = \Omega_s^2 - 8B_s\bar{B}_s + 32 \int_s^t B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau \mathrm{d}\tau \ , \tag{III.12}$$

[5] Recall that the operator Ω_t^{t} denotes the transpose of Ω_t and $\bar{B}_t$ the complex conjugate of B_t, see Section II.1.

[6] The integral equation $B_t = B_0 - 2\int_0^t \left(\Omega_\tau B_\tau + B_\tau \Omega_\tau^{\mathrm{t}}\right) \mathrm{d}\tau$ is valid for any $t \in [0, T_+)$ on $\mathfrak{h}$.

and if $\Omega_0 B_0 = B_0 \Omega_0^{\mathrm{t}}$ *then* $\Omega_t B_t = B_t \Omega_t^{\mathrm{t}}$ *and* $\Omega_t^2 - 4B_t\bar{B}_t = \Omega_0^2 - 4B_0\bar{B}_0$.
(v) *If additionally Condition A4 holds, that is,* $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$, *then* $T_+ = \infty$.

The system (III.9) is therefore well–posed, at least for small times. In Lemmata 49, 51, and 61, we give additional properties on the flow and we summarize them in the theorem below.

THEOREM 12 (INEQUALITIES CONSERVED BY THE FLOW)
Assume Conditions A1–A4 and that $(\Omega_t, B_t)_{t\geq 0}$ *is the solution of (III.9).*
(i) *If at initial time* $t = 0$,

$$\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h}) \quad \textit{and} \quad \Omega_0 \geq (4+\mathrm{r})B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 + \mu\mathbf{1} \tag{III.13}$$

for some $\mu, \mathrm{r} \geq 0$, *then these assumptions are conserved for all times:*

$$\forall t \geq 0: \quad \Omega_t^{-1/2}B_t \in \mathcal{L}^2(\mathfrak{h}) \quad \textit{and} \quad \Omega_t \geq (4+\mathrm{r})B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t + \mu\mathbf{1}\ . \tag{III.14}$$

(ii) *If at initial time* $t = 0$,

$$\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h}) \in \mathcal{L}^2(\mathfrak{h})\ ,\ \Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0\ ,\ \mathbf{1} \geq (4+\mathrm{r})\, B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0 \tag{III.15}$$

for some $\mathrm{r} \geq 0$, *then these assumptions are conserved for all times:*

$$\forall t \geq 0: \quad \Omega_t^{-1}B_t \in \mathcal{L}^2(\mathfrak{h})\ ,\ \Omega_t \geq 4B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t\ ,\ \mathbf{1} \geq (4+\mathrm{r})\, B_t(\Omega_t^{\mathrm{t}})^{-2}\bar{B}_t\ . \tag{III.16}$$

REMARK 13 *Theorem 12 also holds if the operators* Ω_t, $B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t$, *and* $B_t(\Omega_t^{\mathrm{t}})^{-2}\bar{B}_t$ *are replaced by* Ω_t^{t}, $\bar{B}_t\Omega_t^{-1}B_t$, *and* $\bar{B}_t\Omega_t^{-2}B_t$, *respectively. Furthermore, strict inequalities in (III.13) and (III.15), respectively, are also preserved for all times* $t > 0$.

The inequalities of (III.13) and (III.15), which are conserved by the flow (III.9), are crucial to derive the behavior at infinite time of the solution $(\Omega_t, B_t)_{t\geq 0}$ of (III.9), see Section V.3.
The Hamiltonian H_t defined by (III.7), with $(\Omega_t, B_t)_{t\geq 0}$ solution of (III.9) and C_t defined by (III.10), should satisfy the Brocket–Wegner flow (III.1) with $Y_0 = \mathrm{H}_0$ (II.9) and $X = \mathrm{N}$ (II.11). Note, however, that we still do not know whether H_t belongs to the unitary orbit of H_0, i.e., whether (III.5) is satisfied in this case. Consequently, one has to prove the existence of the unitary propagator[7] $\mathrm{U}_{t,s}$ as a solution of (III.4) with infinitesimal generator G_t defined by (III.8). For bounded generators G_t, this is established by standard methods involving the Dyson series (VII.6) as an explicit solution of (III.4). For unbounded G_t, the problem is more delicate, see for instance [12, 13, 14, 15, 16] and the corresponding references cited therein. This problem is solved in Section VII.1 by using the Yosida approximation. More precisely, we give another proof of the

[7]It is a (strongly continuous) evolution operator which is unitary for all times.

well–posedness of the Cauchy problem (III.4) in the hyperbolic case in terms of standard, sufficient conditions on the generator G_t, see Conditions B1–B3 in Section VII.1. Since the generator G_t (III.8) is self–adjoint, B1 is directly satisfied, see (VII.4). But B2–B3 require at least one auxiliary closed operator Θ. In fact, we use the particle number operator N to define the auxiliary self–adjoint operator Θ by setting $\Theta = \mathrm{N} + \mathbf{1}$. This proof is laid out in detail in Section VI.1, see the introduction of Section VI about A1–A4 and Lemmata 66–72. The assertion is the following:

THEOREM 14 (EXISTENCE AND UNIQUENESS OF THE OPERATOR $\mathrm{U}_{t,s}$)
Under Conditions A1–A3, there is a unique unitary propagator $\mathrm{U}_{t,s}$ satisfying, for any $s \in [0, T_+)$ and $t \in [s, T_+)$, the following properties:
(i) $\mathrm{U}_{t,s}$ *conserves the domain $\mathcal{D}(\mathrm{N})$ and is the strong solution on $\mathcal{D}(\mathrm{N})$ of the non–autonomous evolution equations*

$$\forall s \in [0, T_+),\ t \in [s, T_+) : \qquad \begin{cases} \partial_t \mathrm{U}_{t,s} = -i\mathrm{G}_t \mathrm{U}_{t,s} & , \quad \mathrm{U}_{s,s} := \mathbf{1} \quad . \\ \partial_s \mathrm{U}_{t,s} = i\mathrm{U}_{t,s}\mathrm{G}_s & , \quad \mathrm{U}_{t,t} := \mathbf{1} \quad . \end{cases} \tag{III.17}$$

(ii) $\mathrm{U}_{t,s}$ *realizes a Bogoliubov* $\mathbf{u}$–$\mathbf{v}$ *transformation: there are $\mathbf{u}_{t,s} \in \mathcal{B}(\mathfrak{h})$ and $\mathbf{v}_{t,s} \in \mathcal{L}^2(\mathfrak{h})$ such that*

$$\mathbf{u}_{t,s}\mathbf{u}_{t,s}^* - \mathbf{v}_{t,s}\mathbf{v}_{t,s}^* = \mathbf{1} \ , \qquad \mathbf{u}_{t,s}\mathbf{v}_{t,s}^{\mathrm{t}} = \mathbf{v}_{t,s}\mathbf{u}_{t,s}^{\mathrm{t}} \ , \tag{III.18}$$

$$\mathbf{u}_{t,s}^*\mathbf{u}_{t,s} - \mathbf{v}_{t,s}^{\mathrm{t}}\bar{\mathbf{v}}_{t,s} = \mathbf{1} \ , \qquad \mathbf{u}_{t,s}^*\mathbf{v}_{t,s} = \mathbf{v}_{t,s}^{\mathrm{t}}\bar{\mathbf{u}}_{t,s} \ , \tag{III.19}$$

and, on the domain $\mathcal{D}(\mathrm{N}^{1/2})$,

$$\forall k \in \mathbb{N},\ s,t \in [0, T_+),\ t \geq s : \mathrm{U}_{t,s} a_{s,k} \mathrm{U}_{t,s}^* = \sum_{\ell} \{\mathbf{u}_{t,s}\}_{k,\ell}\, a_{s,\ell} + \{\mathbf{v}_{t,s}\}_{k,\ell}\, a_{s,\ell}^* \ , \tag{III.20}$$

where $a_{s,k} := \mathrm{U}_s a_k \mathrm{U}_s^*$ with $\mathrm{U}_s := \mathrm{U}_{s,0}$. In particular, there exists a self–adjoint quadratic boson operator $\mathbb{Q}_{t,s} = \mathbb{Q}_{t,s}^*$ such that $\mathrm{U}_{t,s} = \exp(i\mathbb{Q}_{t,s})$.

REMARK 15 *The existence of $\mathbb{Q}_{t,s}$ in the second assertion (ii) is a direct consequence of the fact that $\mathrm{U}_{t,s}$ realizes a Bogoliubov* $\mathbf{u}$–$\mathbf{v}$ *transformation. For the reader's convenience, we explain this in Section VII.2 (Theorem 96).*

REMARK 16 *The family $(\mathrm{U}_{t,s})_{t\geq s\geq 0}$ can naturally be extended to all $s, t \in \mathbb{R}_0^+$. For more details, see Theorem 71.*

REMARK 17 *Because $(\mathbf{u}_{t,s} - \mathbf{1}) \in \mathcal{L}^2(\mathfrak{h})$,*

$$m := \dim\ker(\mathbf{u}_{t,s}) < \infty \quad \textit{and} \quad n := \dim\ker\left(\mathbf{u}_{t,s}^*\right) < \infty \ .$$

The vacuum of $\mathcal{F}_b$ and its image under $\mathrm{U}_{t,s}$ are orthogonal to each other iff $(m,n) \neq (0,0)$. Furthermore, the new vacuum becomes charged if $m \neq n$. For more details, see [23, p. 56-58].

It remains to establish the link between Ω_t, B_t, C_t and the operator $\mathrm{U}_t\mathrm{H}_0\mathrm{U}_t^*$ in terms of (III.7), where we recall that $\mathrm{U}_t := \mathrm{U}_{t,0}$. This is done in Section VI.2, see the introduction of Section VI about A1–A4 and Lemma 80. It proves the following theorem:

THEOREM 18 (UNITARLY EQUIVALENCE BETWEEN H_s AND H_t)
Under Conditions A1–A3, the self–adjoint operator H_t *defined by (III.7) equals* $\mathrm{H}_t = \mathrm{U}_{t,s}\mathrm{H}_s\mathrm{U}_{t,s}^*$, *for all* $s \in [0, T_+)$ *and* $t \in [s, T_+)$.

III.3 Asymptotic Properties of the Brocket–Wegner Flow

We devote this subsection to the study of the limits of the time–dependent quadratic operator H_t and the unitary propagator $\mathrm{U}_{t,s}$ as $t \to \infty$, which require an analysis of Ω_t, B_t, C_t, $\mathbf{u}_{t,s}$, and $\mathbf{v}_{t,s}$ in the limit of infinite time. All asymptotics of Ω_t, C_t, $\mathbf{u}_{t,s}$, and $\mathbf{v}_{t,s}$ depend on the behavior of the Hilbert–Schmidt norm $\|B_t\|_2$ for large times. Therefore, we first summarize in the next theorem few possible asymptotic behaviors of the map $t \mapsto \|B_t\|_2$, which are taken from Lemmata 58, 59 and Corollary 64.

THEOREM 19 (INTEGRABILITY PROPERTIES OF THE FLOW)
Assume Conditions A1–A4.
(i) *The map* $t \mapsto \|B_t\|_2$ *is square–integrable on* $[0, \infty)$.
(ii) *If A5 holds, that is,* $\mathbf{1} > 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ *and* $\Omega_0^{-1-\varepsilon}B_0 \in \mathcal{L}^2(\mathfrak{h})$ *for some* $\varepsilon > 0$, *then the map* $t \mapsto \|B_t\|_2$ *is integrable on* $[0, \infty)$.
(iii) *If A6 holds, that is,* $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 + \mu\mathbf{1}$ *for some* $\mu > 0$, *then* $\|B_t\|_2 = \mathcal{O}\left(\mathrm{e}^{-2t\mu}\right)$ *decays exponentially to zero in the limit* $t \to \infty$.

REMARK 20 *More precise asymptotics are given in Section V.3: See Corollaries 62 and 64 as well as the discussions after Lemma 65.*

REMARK 21 *Under the Kato–Mugibayashi assumption (II.35), it can be shown from Proposition 10 that* $\|B_t\|_2 = \mathcal{O}\left(\mathrm{e}^{-t\mu_-}\right)$, *similar to Theorem 19 (iii).*

By Theorem 19 (i), the Hilbert–Schmidt norm $\|B_t\|_2$ is square–integrable on $[0, \infty)$ provided A1–A4 hold. From Section VI.3 (Lemmata 81–82 and 84), this assertion implies the existence of the limits of Ω_t, C_t, and H_t, as $t \to \infty$:

THEOREM 22 (LIMITS OF Ω_t, C_t, AND H_t AS $t \to \infty$)
Assume Conditions A1–A4.
(i) *The operator* $(\Omega_0 - \Omega_t) \in \mathcal{L}^1(\mathfrak{h})$ *converges in trace–norm to* $(\Omega_0 - \Omega_\infty)$, *where*

$$\Omega_\infty := \Omega_0 - 16 \int_0^\infty B_\tau \bar{B}_\tau \mathrm{d}\tau = \Omega_\infty^* \geq 0 \tag{III.21}$$

on $\mathcal{D}(\Omega_0)$, *and* $2C_\infty = \mathrm{tr}(\Omega_0 - \Omega_\infty) + 2C_0 < \infty$. *Furthermore,*

$$\Omega_\infty^2 = \Omega_0^2 - 8B_0\bar{B}_0 + 32 \int_0^\infty B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau \mathrm{d}\tau \tag{III.22}$$

with $\mathrm{tr}\left(\Omega_\infty^2 - \Omega_0^2 + 4B_0\bar{B}_0\right) = 0$. *If* $\Omega_0 B_0 = B_0 \Omega_0^{\mathrm{t}}$ *then*

$$\Omega_\infty = \{\Omega_0^2 - 4B_0\bar{B}_0\}^{1/2} \ . \tag{III.23}$$

(ii) *The operator* H_t *converges in the strong resolvent sense to*

$$\mathrm{H}_\infty := \sum_{k,\ell} \{\Omega_\infty\}_{k,\ell} a_k^* a_\ell + C_\infty \ . \tag{III.24}$$

To obtain the limits of $\mathbf{u}_{t,s}$, $\mathbf{v}_{t,s}$, and $\mathrm{U}_{t,s}$, as $t \to \infty$, the square–integrability of the function $t \mapsto \|B_t\|_2$ is not sufficient. In fact, we need its integrability which is ensured by either A5 or A6, see Theorem 19 (ii)–(iii). Indeed, from Theorem 19 and Lemmata 85–88 we infer the existence of the bounded operators $\mathbf{u}_{t,s}$, $\mathbf{v}_{t,s}$, and $\mathrm{U}_{t,s}$ for all $t \in [s,\infty]$:

THEOREM 23 (LIMITS OF $\mathbf{u}_{t,s}$, $\mathbf{v}_{t,s}$, AND $\mathrm{U}_{t,s}$ AS $t \to \infty$)
Assume Conditions A1–A3 and either A5 or A6.
(i) $\mathbf{u}_{t,s}$ *and* $\mathbf{v}_{t,s}$ *converge in* $\mathcal{L}^2(\mathfrak{h})$ *to* $\mathbf{u}_{\infty,s} \in \mathcal{B}(\mathfrak{h})$ *and* $\mathbf{v}_{\infty,s} \in \mathcal{L}^2(\mathfrak{h})$ *respectively*[8], *with* $\mathbf{u}_{\infty,\infty} = \mathbf{1}$, $\mathbf{v}_{\infty,\infty} = 0$, *and*

$$\mathbf{u}_{\infty,s}\mathbf{u}_{\infty,s}^* - \mathbf{v}_{\infty,s}\mathbf{v}_{\infty,s}^* = \mathbf{1} \ , \qquad \mathbf{u}_{\infty,s}\mathbf{v}_{\infty,s}^{\mathrm{t}} = \mathbf{v}_{\infty,s}\mathbf{u}_{\infty,s}^{\mathrm{t}} \ , \tag{III.25}$$

$$\mathbf{u}_{\infty,s}^*\mathbf{u}_{\infty,s} - \mathbf{v}_{\infty,s}^{\mathrm{t}}\bar{\mathbf{v}}_{\infty,s} = \mathbf{1} \ , \qquad \mathbf{u}_{\infty,s}^*\mathbf{v}_{\infty,s} = \mathbf{v}_{\infty,s}^{\mathrm{t}}\bar{\mathbf{u}}_{\infty,s} \ . \tag{III.26}$$

(ii) $\mathrm{U}_{t,s}$ *(resp.* $\mathrm{U}_{t,s}^*$*) converges strongly to a unitary operator* $\mathrm{U}_{\infty,s}$ *(resp.* $\mathrm{U}_{\infty,s}^*$*) which is strongly continuous in* s *and satisfies* $\partial_s \mathrm{U}_{\infty,s} = i\mathrm{U}_{\infty,s}\mathrm{G}_s$ *on* $\mathcal{D}(\mathrm{N})$, $\mathrm{U}_{\infty,s} = \mathrm{U}_{\infty,x}\mathrm{U}_{x,s}$ *for any* $x \geq s \geq 0$, *and* $\mathrm{U}_{\infty,\infty} := \lim_{s\to\infty} \mathrm{U}_{\infty,s} = \mathbf{1}$ *in the strong topology.*
(iii) *For any* $s \in \mathbb{R}_0^+ \cup \{\infty\}$, $\mathrm{U}_{\infty,s}$ *realizes a Bogoliubov* $\mathbf{u}$–$\mathbf{v}$ *transformation:*

$$\forall k \in \mathbb{N},\ \forall s \in \mathbb{R}_0^+ \cup \{\infty\} : \ \mathrm{U}_{\infty,s} a_{s,k} \mathrm{U}_{\infty,s}^* = \sum_\ell \{\mathbf{u}_{\infty,s}\}_{k,\ell}\, a_{s,\ell} + \{\mathbf{v}_{\infty,s}\}_{k,\ell}\, a_{s,\ell}^* \tag{III.27}$$

with $a_{s,k} := \mathrm{U}_s a_k \mathrm{U}_s^*$. *In particular,* $\mathrm{U}_{\infty,s} = \exp(i\mathbb{Q}_{\infty,s})$ *with* $\mathbb{Q}_{\infty,s} = \mathbb{Q}_{\infty,s}^*$ *being some self–adjoint quadratic boson operator.*
(iv) *For any* $s \in \mathbb{R}_0^+ \cup \{\infty\}$, *the unitary operator* $\mathrm{U}_{\infty,s}$ *realizes a* N–*diagonalization of the quadratic boson operator* H_s *as* $\mathrm{H}_\infty = \mathrm{U}_{\infty,s}\mathrm{H}_s\mathrm{U}_{\infty,s}^*$.

In other words, we have N–diagonalized the quadratic operator H_0 (II.9) under the assumptions of Theorem 23.

REMARK 24 *Theorem 23 only depends on the integrability of the map* $t \mapsto \|B_t\|_2$ *on* $[0,\infty)$. *This property is ensured by A5 or A6, or the Kato–Mugibayashi assumption (II.35).*

REMARK 25 *Theorem 23 (ii)–(iii) means that we can continuously extend the definition of the unitary propagator* $\mathrm{U}_{t,s}$ *to* $\mathbb{R}_0^+ \cup \{\infty\}$ *by setting* $\mathrm{G}_\infty := 0$.

[8] See (VI.57) and (VI.60) with $t = \infty$.

IV Illustration of the Method

In this section we apply the Brocket–Wegner flow on Bogoliubov's example, which is the simplest quadratic boson operator one can study. This example is a crucial ingredient of his celebrated microscopic theory of superfluidity [1] for liquid ^{4}He as its diagonalization by the Bogoliubov **u**–**v** transformation shows a Landau–type excitation spectrum, see also [2, 24, 25, 26] for more details. In this simple case, the flow can explicitly be computed and we then generalize Bogoliubov's result by relaxing his gap condition (IV.23). All this study is the subject of Section IV.1. In Section IV.2 we use this example to show a *blow–up* of a solution of the Brocket–Wegner flow. This second subsection is thus a strong warning on the possible pathological behavior of the Brocket–Wegner flow for unbounded operators.

IV.1 The Brocket–Wegner Flow on Bogoliubov's Example

We illustrate first our method on the simplest quadratic boson operator which is of the form

$$\mathrm{H}_0 = \omega_- a_-^* a_- + \omega_+ a_+^* a_+ + \mathrm{b} a_+^* a_-^* + \mathrm{b} a_- a_+ \ . \tag{IV.1}$$

Here,

$$\Omega_0 = \begin{pmatrix} \omega_- & 0 \\ 0 & \omega_+ \end{pmatrix} \qquad \text{and} \qquad B_0 = \begin{pmatrix} 0 & \mathrm{b} \\ \mathrm{b} & 0 \end{pmatrix} \tag{IV.2}$$

with strictly positive $\omega_-, \omega_+ \in \mathbb{R}^+$ and $\mathrm{b} \in \mathbb{R}$. We assume without loss of generality that $\omega_+ \geq \omega_-$.
For any positive real ω_- and ω_+, the Brocket–Wegner flow (III.1) when $Y_0 = \mathrm{H}_0$ and

$$X = \mathrm{N} = a_-^* a_- + a_+^* a_+ \tag{IV.3}$$

yields a time–dependant quadratic operator

$$\mathrm{H}_t = \omega_{-,t} a_-^* a_- + \omega_{+,t} a_+^* a_+ + b_t a_+^* a_-^* + b_t a_- a_+ \ . \tag{IV.4}$$

More precisely,

$$\Omega_t = \begin{pmatrix} \omega_{-,t} & 0 \\ 0 & \omega_{+,t} \end{pmatrix} \quad \text{and} \quad B_t = \begin{pmatrix} 0 & b_t \\ b_t & 0 \end{pmatrix} \tag{IV.5}$$

are real symmetric 2×2–matrices with matrix elements satisfying the differential equations (III.9), that is in this elementary example,

$$\forall t \geq 0 : \quad \begin{cases} \partial_t \omega_{-,t} = -16 b_t^2 & , \quad \omega_{-,0} := \omega_- \quad . \\ \partial_t \omega_{+,t} = -16 b_t^2 & , \quad \omega_{+,0} := \omega_+ \quad . \\ \partial_t b_t = -2 \left(\omega_{-,t} + \omega_{+,t} \right) b_t & , \quad b_0 := \mathrm{b} \quad . \end{cases} \tag{IV.6}$$

This system of differential equations can *explicitly* be solved under the assumption that $\Omega_0 \geq 4 B_0 (\Omega_0^{\mathrm{t}})^{-1} \bar{B}_0$ (cf. A3), i.e., $\omega_+ \omega_- \geq 4\mathrm{b}^2$.

REMARK 26 *In this example, $\Omega_0^2 \geq 4B_0\bar{B}_0$ is equivalent to $\omega_\pm^2 \geq 4\mathrm{b}^2$, which is different from $\omega_+\omega_- \geq 4\mathrm{b}^2$, unless $[\Omega_0, B_0] = 0$, i.e., $\omega_+ = \omega_-$. In fact, the condition $\Omega_0^2 \geq 4B_0\bar{B}_0$, together with A1–A2, yields A3 but the converse does not hold: If $\omega_- = 1$, $\omega_+ = 2$ and $\mathrm{b} \in (1/2, 1/\sqrt{2})$ then A1–A3 hold true but $\Omega_0^2 \geq 4B_0\bar{B}_0$ is false.*

We start this explicit computation with the case $\omega_+\omega_- = 4\mathrm{b}^2$.

PROPOSITION 27 (EXAMPLE OF A FLOW WHEN $\omega_+\omega_- = 4\mathrm{b}^2$)
Let Ω_0 and B_0 be the real symmetric 2×2–matrices (IV.2). If

$$\omega_+\omega_- = 4\mathrm{b}^2 \quad \textit{and} \quad \delta := \omega_+ - \omega_- \geq 0 \ , \tag{IV.7}$$

then a solution of (IV.6) is given, for all $t \geq 0$, by either

$$\omega_{-,t} = \frac{\omega_-\delta}{\omega_+ e^{4\delta t} - \omega_-} \ , \ \omega_{+,t} = \frac{\omega_+\delta}{\omega_+ - \omega_- e^{-4\delta t}} \ , \ b_t^2 = \frac{\mathrm{b}^2\delta^2 e^{-4t\delta}}{(\omega_+ - \omega_- e^{-4t\delta})^2} \ , \tag{IV.8}$$

provided $\delta > 0$ is strictly positive, or by

$$\omega_{-,t} = \frac{\omega_-}{4t\omega_- + 1} \ , \ \omega_{+,t} = \frac{\omega_+}{4t\omega_+ + 1} \ , \ b_t^2 = \frac{\omega_+^2}{4\left(4t\omega_+ + 1\right)^2} \ , \tag{IV.9}$$

in case that $\delta = 0$.

PROOF. The constant of motion given in Theorem 11 (iii) implies that

$$\omega_{-,t}^2 + \omega_{+,t}^2 - 8b_t^2 = \omega_-^2 + \omega_+^2 - 8\mathrm{b}^2 \ . \tag{IV.10}$$

Together with (IV.6) it yields the differential equation

$$\partial_t\omega_{+,t} = 2(\omega_-^2 + \omega_+^2 - 8\mathrm{b}^2 - \omega_{-,t}^2 - \omega_{+,t}^2) \ . \tag{IV.11}$$

Since the system (IV.6) of differential equations imposes the equality

$$\omega_{-,t} = \omega_{+,t} + \omega_- - \omega_+ \ , \tag{IV.12}$$

we infer from (IV.11) that

$$\partial_t\omega_{+,t} = 4\left(\omega_-\omega_+ - 4\mathrm{b}^2\right) - 4\left(\omega_{+,t}\left(\omega_- - \omega_+\right) + \omega_{+,t}^2\right) \ . \tag{IV.13}$$

The proposition then follows from (IV.7) together with elementary computations using (IV.10) and (IV.12)–(IV.13). □

Proposition 27 means that the quadratic boson operator H_t (IV.4) converges in the strong resolvent sense to $\mathrm{H}_\infty = \delta a_+^* a_+$, because the matrix element b_t is square–integrable at infinity, see Lemma 84. If $\delta = 0$ then the quadratic operator H_0 defined by (IV.1) is obviously not unitarily equivalent to $\mathrm{H}_\infty = 0$. This is confirmed by the non–integrability of the function b_t at infinity. If $\delta > 0$ then the matrix element b_t is integrable on $[0, \infty)$ and, by Lemma 88, H_0 and H_∞ are unitarily equivalent.
Next we give the explicit solution of the system (IV.6) of differential equations under the assumption that $\omega_+\omega_- > 4\mathrm{b}^2$.

PROPOSITION 28 (EXAMPLE OF A FLOW WHEN $\omega_{+}\omega_{-} > 4\mathrm{b}^2$)
Let Ω_0 and B_0 be the real symmetric 2×2–matrices (IV.2). If $\omega_{+}\omega_{-} > 4\mathrm{b}^2$ then a solution of (IV.6) is given, for all $t \geq 0$, by

$$\omega_{-,t} = \mathrm{h}_t\left(-\delta\right) \ , \ \omega_{+,t} = \mathrm{h}_t\left(\delta\right) \ , \ b_t^2 = \frac{4\varsigma^2 \mathrm{b}^2 \mathrm{e}^{-4\varsigma t}}{\left(\sigma + \varsigma + \left(\varsigma - \sigma\right)\mathrm{e}^{-4\varsigma t}\right)^2} \ , \tag{IV.14}$$

where

$$\mathrm{h}_t\left(\delta\right) := \frac{\left(\varsigma+\delta\right)\left(\varsigma+\sigma\right) + \left(\varsigma-\delta\right)\left(\sigma-\varsigma\right)\mathrm{e}^{-4\varsigma t}}{2\left(\varsigma+\sigma+\left(\varsigma-\sigma\right)\mathrm{e}^{-4\varsigma t}\right)} \tag{IV.15}$$

and

$$\delta := \omega_{+} - \omega_{-},\ \sigma := \omega_{+} + \omega_{-},\ \varsigma := \{\sigma^2 - 16\mathrm{b}^2\}^{1/2} > 0 \ . \tag{IV.16}$$

In particular, in the limit $t \rightarrow \infty$, the functions $\omega_{\pm,t}$ and b_t converge exponentially to $\left(\varsigma \pm \delta\right)/2$ and 0, respectively.

PROOF. The proof is obtained by a combination of Equations (IV.10), (IV.12), and (IV.13) with direct computations using $\omega_{+}\omega_{-} > 4\mathrm{b}^2$, which implies $\varsigma > 0$. We omit the details. □

The explicit solution given in Proposition 28 obviously yields a solution H_t (IV.4) of the Brocket–Wegner flow. In this case, the matrix element b_t is integrable at infinity and, by Lemma 88, the quadratic boson operators H_0 and

$$\mathrm{H}_\infty = \frac{1}{2}\left\{\left(\varsigma+\delta\right)a_{+}^{*}a_{+} + \left(\varsigma-\delta\right)a_{-}^{*}a_{-}\right\} \tag{IV.17}$$

are unitarily equivalent.

To compare with Bogoliubov's result [1, 2], we extend our example to the quadratic boson operator

$$\mathrm{H}_0 = \sum_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \omega_{-k}a_{-k}^{*}a_{-k} + \omega_k a_k^{*}a_k + \mathrm{b}_k a_k^{*}a_{-k}^{*} + \mathrm{b}_k a_{-k}a_k \ , \tag{IV.18}$$

where the positive self–adjoint operator Ω_0 and the Hilbert–Schmidt operator $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2\left(\mathfrak{h}\right)$ can be written as direct sums

$$\Omega_0 = \bigoplus_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \begin{pmatrix} \omega_{-k} & 0 \\ 0 & \omega_k \end{pmatrix} \ , \quad B_0 = \bigoplus_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \begin{pmatrix} 0 & \mathrm{b}_k \\ \mathrm{b}_k & 0 \end{pmatrix} \tag{IV.19}$$

of real symmetric 2×2–matrices, at fixed dimension $\nu \in \mathbb{N}$. Here, $\{-1,1\}$ are the point reflections defined on $\mathbb{Z}^\nu$ by $k \rightarrow \pm k$ and $\mathbb{Z}^\nu/\{-1,1\}$ is by definition the set $\{[k] := \{k,-k\}\}_{k\in\mathbb{Z}^\nu}$ of equivalence classes. Note also that ω_k and b_k are real numbers such that $\Omega_0 = \Omega_0^{*} > 0$, i.e.,

$$\forall [k] \in \mathbb{Z}^\nu/\{-1,1\} : \qquad \omega_{\pm k} \in \mathbb{R}^{+} \ , \tag{IV.20}$$

$B_0 \in \mathcal{L}^2(\mathfrak{h})$, i.e.,

$$\|B_0\|_2^2 = \sum_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \mathrm{b}_k^2 < \infty \ , \tag{IV.21}$$

and $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0$, i.e.,

$$\forall [k] \in \mathbb{Z}^\nu/\{-1,1\} : \qquad \omega_k\omega_{-k} \geq 4\mathrm{b}_k^2 \ . \tag{IV.22}$$

The case where $\omega_k = \omega_{-k}$ for $[k] \in \mathbb{Z}^\nu/\{-1,1\}$, i.e., $[\Omega_0, B_0] = 0$, was solved by Bogoliubov in [1] through the so–called Bogoliubov **u**–**v** transformation under the assumptions (IV.20)–(IV.21) and $\Omega_0^2 \geq 4B_0\bar{B}_0 + \mu_B\mathbf{1}$ for some constant $\mu_B > 0$, i.e.,

$$\forall [k] \in \mathbb{Z}^\nu/\{-1,1\} : \qquad \omega_k^2 = \omega_{-k}^2 \geq 4\mathrm{b}_k^2 + \mu_B > 0 \ , \tag{IV.23}$$

Indeed, in Bogoliubov's theory of superfluidity, the sum is over the set $(2\pi\mathbb{Z}^3/L)/\{-1,1\}$ with $L > 0$, $\mathrm{b}_k = \hat{\varphi}(k) = \hat{\varphi}(-k)$ is the Fourier transform of an absolutely integrable (real) two–body interaction potential $\varphi(x) = \varphi(\|x\|)$, $x \in \mathbb{R}^3$, and $\omega_k = \hbar^2k^2/2m$ is the one–particle energy spectrum in the modes $k \in (2\pi\mathbb{Z}^3/L)$ of free bosons of mass m enclosed in a cubic box $\Lambda = L \times L \times L \subset \mathbb{R}^3$ of side length L. ($\hbar$ is the Planck constant divided by 2π.) See [2, Appendix B.2] for more details.

Note that A6, i.e., $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 + \mu\mathbf{1}$ for some constant $\mu > 0$, corresponds in this example to

$$\forall [k] \in \mathbb{Z}^\nu/\{-1,1\} : \qquad \omega_{-k}\omega_k \geq 4\mathrm{b}_1^2 + \mu\omega_{\pm k} \ . \tag{IV.24}$$

When $\omega_k = \omega_{-k}$, Condition A6 is thus equivalent to (IV.23): On the one hand, (IV.24) implies $\omega_{\pm k} \geq \mu$ for all $[k] \in \mathbb{Z}^\nu/\{-1,1\}$ and (IV.23) with $\mu_B = \mu^2$. On the other hand, since $\mu x + 4\mathrm{b}_1^2 = o(x^2)$, as $x \to \infty$, we can assume without loss of generality that $\omega_k \leq \omega_{\max} < \infty$ for all $[k] \in \mathbb{Z}^\nu/\{-1,1\}$ and (IV.23) yields in this case (IV.24) with $\mu \leq \mu_B\omega_{\max}^{-1}$.

We generalize Bogoliubov's result for $\mu_B = 0$ in (IV.23) by using Proposition 28 because the N–diagonalization of H_0 can be done on each 2×2–block separately. Indeed, the time–dependent quadratic operator H_t computed from the Brocket–Wegner flow (III.1) (Y_0 being equal to (IV.18) and $X = \mathrm{N}$) equals

$$\mathrm{H}_t = \sum_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \omega_{-k,t}a_{-k}^*a_{-k} + \omega_{k,t}a_k^*a_k + b_{k,t}a_k^*a_{-k}^* + b_{k,t}a_{-k}a_k \ . \tag{IV.25}$$

In particular,

$$\Omega_t = \bigoplus_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \begin{pmatrix} \omega_{-k,t} & 0 \\ 0 & \omega_{k,t} \end{pmatrix} \quad \text{and} \quad B_t = \bigoplus_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \begin{pmatrix} 0 & b_{k,t} \\ b_{k,t} & 0 \end{pmatrix} , \tag{IV.26}$$

where the matrix elements $\omega_{\pm k,t}$ and $b_{k,t}$ are solutions of the system (IV.6) of differential equations with initial values $\omega_{\pm k,0} = \omega_{\pm k} \in \mathbb{R}^+$ and $b_{k,0} = \mathrm{b}_k \in \mathbb{R}$ for all $[k] \in \mathbb{Z}^\nu/\{-1,1\}$.

To analyze the N–diagonalization of H_0, the asymptotics of the Hilbert–Schmidt norm $\|B_t\|_2$ is pivotal as explained in Section III.3. In fact, by Theorem 19 (i), $t \mapsto \|B_t\|_2$ is square–integrable on $[0,\infty)$ provided A4 holds, that is,

$$\|\Omega_0^{-1/2} B_0\|_2^2 = \sum_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \mathrm{b}_k^2 \left(\frac{1}{\omega_k} + \frac{1}{\omega_{-k}} \right) < \infty \ . \tag{IV.27}$$

Together with Theorem 22 (ii) and Proposition 28, the quadratic boson operator H_t defined by (IV.25) converges in the strong resolvent sense to

$$\begin{aligned} \mathrm{H}_\infty \ = \ & \frac{1}{2} \sum_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} (\omega_{-k} - \omega_k)\, a_{-k}^* a_{-k} + (\omega_k - \omega_{-k})\, a_k^* a_k \\ & + \{(\omega_k + \omega_{-k})^2 - 16\mathrm{b}_k^2\}^{1/2} \left(a_k^* a_k + a_{-k}^* a_{-k} \right) \ , \end{aligned} \tag{IV.28}$$

under the assumptions (IV.20)–(IV.22) and (IV.27).
Observe that the limit operator Ω_∞ is equal in this elementary example to

$$\Omega_\infty = S_{0,-} + (S_{0,+}^2 - 4 B_0 \bar{B}_0)^{1/2} \tag{IV.29}$$

with $S_{0,\pm} := (\Omega_0 \pm \hat{\Omega}_0)/2$ and

$$\hat{\Omega}_0 := \bigoplus_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \begin{pmatrix} \omega_k & 0 \\ 0 & \omega_{-k} \end{pmatrix} . \tag{IV.30}$$

(The operator $\hat{\Omega}_0$ satisfies the equality $\hat{\Omega}_0 B_0 = B_0 \Omega_0$.) It explicitly shows that Ω_∞ is generally not equal to

$$(\Omega_0^2 - 4 B_0 \bar{B}_0)^{1/2} \ , \tag{IV.31}$$

when $\Omega_0 B_0 \neq B_0 \Omega_0^{\mathrm{t}}$. See also Theorem 52 (ii). Also, the explicit form (IV.29)–(IV.30) cannot be generalized. It is only due to peculiar properties of the operators Ω_0 and B_0 chosen in (IV.19). Indeed, from (III.9) combined with direct computations using (IV.19),

$$\forall t \geq 0 : \quad \left\{ \begin{array}{lcll} \partial_t S_{t,-} = 0 & , & S_{t=0,-} := S_{0,-} & . \\ \partial_t S_{t,+} = -16 B_t \bar{B}_t & , & S_{t=0,+} := S_{0,+} & . \\ \partial_t B_t = -2 \left(S_{t,+} B_t + B_t S_{t,+}^{\mathrm{t}} \right) & , & B_{t=0} := B_0 & . \end{array} \right. \tag{IV.32}$$

In this special example, $S_{0,+} B_0 = B_0 S_{0,+}^{\mathrm{t}}$. Therefore, if (IV.20)–(IV.22) and (IV.27) hold, then we directly obtain the explicit form (IV.29)–(IV.30) of Ω_∞ by applying Theorem 22 (i) and substituting $S_{0,+}$ for Ω_0.
Finally, Theorem 23 (iv) shows that the quadratic boson operators H_0 (IV.18) and H_∞ (IV.28) are unitarily equivalent provided that either A5 or A6 is satisfied. In the case where A5 holds, that is, $\mathbf{1} > 4 B_0 (\Omega_0^{\mathrm{t}})^{-2} \bar{B}_0$, i.e.,

$$\forall [k] \in \mathbb{Z}^\nu/\{-1,1\} : \qquad \omega_k^2 > 4\mathrm{b}_k^2 \ , \ \omega_{-k}^2 > 4\mathrm{b}_k^2 \ , \tag{IV.33}$$

and $\Omega_0^{-1-\varepsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$ for some constant $\varepsilon > 0$, i.e.,

$$\|\Omega_0^{-1-\varepsilon} B_0\|_2^2 = \sum_{[k]\in\mathbb{Z}^\nu/\{-1,1\}} \mathrm{b}_k^2 \left(\frac{1}{\omega_k^{2+2\varepsilon}} + \frac{1}{\omega_{-k}^{2+2\varepsilon}}\right) < \infty \ , \tag{IV.34}$$

this result is already out of the scope of other studies because all previous approaches required an infrared cutoff of the form (II.27) or (II.35), see Section II.3. This example generalizes Bogoliubov's result [1, 2] for $\mu = 0$ in (IV.23). On the other hand, this example can also be used to study the mathematical necessity of Condition A5, that is, (IV.33)–(IV.34). To this end, we study A5 with respect to the integrability of the map $t \mapsto \|B_t\|_2$ on $[0, \infty)$ that is *crucial* to prove Theorem 23, see Remark 24.
We first show below that the property $\Omega_0^{-1} B_0 \in \mathcal{L}^2(\mathfrak{h})$ is *pivotal*:

Proposition 29 ($\Omega_0^{-1} B_0 \in \mathcal{L}^2(\mathfrak{h})$ as a pivotal condition)
There is a choice (Ω_0, B_0) defined by (IV.19), for $\nu = 1$ and satisfying A1–A4, (IV.33), and $\Omega_0^{-1+\epsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$, for any $\epsilon > 0$, such that $\|B_t\|_2 \geq \mathcal{O}\left(t^{-1}\right)$, as $t \to \infty$. In particular, the map $t \mapsto \|B_t\|_2$ is not integrable on $[0, \infty)$. Here, $(\Omega_t, B_t)_{t\geq 0}$ is the solution of (III.9).

Proof. In (IV.19) take $\mathrm{b}_k := 1/2^k$ and $\omega_{-k} := \omega_k := 2\sqrt{2}\mathrm{b}_k$, for $k \in \mathbb{N}$, and $\mathrm{b}_0 := \omega_0 := 0$. Then, Ω_0 obviously satisfies A1. Moreover,

$$\|B_0\|_2^2 = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} = \frac{2}{3} \tag{IV.35}$$

and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$, see Condition A2. By (IV.22) and (IV.33), one gets A3. Finally, by (IV.34), for any $\epsilon > 0$,

$$\left\|\Omega_0^{-1+\epsilon} B_0\right\|_2^2 = \frac{2^{3\epsilon}}{4} \sum_{k=1}^{\infty} \frac{1}{2^{2\epsilon k}} = \frac{2^{3\epsilon}}{4\left(2^{2\epsilon} - 1\right)} < \infty \ , \tag{IV.36}$$

i.e., $\Omega_0^{-1+\epsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$. By (IV.26) and Proposition 28, straightforward computations show from (IV.21) that

$$\|B_t\|_2^2 \geq \left(12 - 8\sqrt{2}\right) \frac{\mathrm{e}^{-\frac{16}{2^k} t}}{2^{2k}} \ , \tag{IV.37}$$

for any $k \in \mathbb{N}$, discarding all terms in the sum but one. In particular, choosing $k \in \mathbb{N}$ such that $2^{k-1} \leq 16t < 2^k$ one gets that, as $t \to \infty$,

$$\|B_t\|_2^2 \geq \left(12 - 8\sqrt{2}\right) \frac{\mathrm{e}^{-\frac{16}{2^k} t}}{2^{2k}} = \mathcal{O}\left(t^{-2}\right) \ .$$

□

Therefore, Proposition 29 shows that the condition $\Omega_0^{-1-\varepsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$ for some constant $\varepsilon > 0$ in A5 is almost optimal to get the integrability of $t \mapsto \|B_t\|_2$

on $[0,\infty)$. The borderline is given by the condition $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$ for which the behavior of $\|B_t\|_2$, as $t \to \infty$, is unclear: $t \mapsto \|B_t\|_2$ may or may not be integrable – both cases can occur.
We now study the inequality $\mathbf{1} > 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ of A5 with respect to the integrability of the Hilbert–Schmidt norm $\|B_t\|_2$ on $[0,\infty)$:

Proposition 30 ($\mathbf{1} > 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ and integrability of $\|B_t\|_2$)
There is a choice (Ω_0, B_0) *defined by (IV.19), for* $\nu = 1$ *and satisfying A1–A4,* $\Omega_0^{-2}B_0 \in \mathcal{L}^2(\mathfrak{h})$*, but not (IV.33), such that the map* $t \mapsto \|B_t\|_2$ *is integrable on* $[0,\infty)$*. Here,* $(\Omega_t, B_t)_{t\geq 0}$ *is the solution of (III.9).*

Proof. In (IV.19) take $\omega_{-1} := 1$, $\omega_1 := 2$ and $\mathrm{b}_1 \in (1/2, 1/\sqrt{2})$, while for $k \in \mathbb{N}\backslash\{1\}$, choose $\mathrm{b}_k := 1/2^k$, $\omega_{-k} := \omega_k := (3/4)^k$. Similar to the proof of Proposition 29, Conditions A1–A3 and $\Omega_0^{-2}B_0 \in \mathcal{L}^2(\mathfrak{h})$ are clearly satisfied for this choice. In particular, A3 for $k = 1$ holds because $\mathrm{b}_1^2 < 1/2$, see (IV.22). Inequality (IV.33) requires for $k = 1$ that $1-\mathrm{b}_1^2 > 0$ and $1-4\mathrm{b}_1^2 > 0$. Therefore, if $\mathrm{b}_1 > 1/2$ then (IV.33) does *not* hold.
We analyze the flow (IV.6) for $k = 1$ and $k \in \mathbb{N}\backslash\{1\}$ separately. For $k = 1$, we study the real symmetric 2×2–matrices

$$\Omega_0^{(1)} = \begin{pmatrix} \omega_{-1} & 0 \\ 0 & \omega_1 \end{pmatrix} \quad \text{and} \quad B_0^{(1)} = \begin{pmatrix} 0 & \mathrm{b}_1 \\ \mathrm{b}_1 & 0 \end{pmatrix} . \tag{IV.38}$$

$(\Omega_0^{(1)}, B_0^{(1)})$ satisfies A1–A3, and A6, see in particular (IV.24) for $k = 1$ and $\mu < 1-2\mathrm{b}_1^2$, where $\mathrm{b}_1^2 < 1/2$. The other cases $k \in \mathbb{N}\backslash\{1\}$ also defines operators $(\Omega_0^{(2)}, B_0^{(2)})$, similar to (IV.26), that satisfy A1–A3 and A5. The assertion then follows from Theorem 19 (ii)–(iii). We omit the details. □
By Proposition 30, Inequality (IV.33) of A5 is not necessary to get the integrability of $\|B_t\|_2$ on $[0,\infty)$, but it is partially replaced by A6 (cf. (IV.24)) in the example given in the proof of Proposition 30. Indeed, the example above is very special and *trivial* in some sense because it is an independent combination of two flows defined from two couples of operators $(\Omega_0^{(1)}, B_0^{(1)})$ and $(\Omega_0^{(2)}, B_0^{(2)})$, respectively: $(\Omega_0^{(1)}, B_0^{(1)})$ does *not* satisfy

$$\mathbf{1} > 4B_0^{(1)}((\Omega_0^{(1)})^{\mathrm{t}})^{-2}\bar{B}_0^{(1)} , \tag{IV.39}$$

but Condition A6. The second one $(\Omega_0^{(2)}, B_0^{(2)})$ satisfies A5 but not A6. Then, the example of Proposition 30 is constructed by using

$$\Omega_0 = \Omega_0^{(1)} \oplus \Omega_0^{(2)} \quad \text{and} \quad B_0 = B_0^{(1)} \oplus B_0^{(2)} \tag{IV.40}$$

because neither A5 nor A6 is satisfied for this choice of operators (Ω_0, B_0). In fact, if one imposes A1–A3 and $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$, then in the example (IV.18) there is *no other way* than the method explained above to break (IV.33) while keeping the integrability of $\|B_t\|_2$ on $[0,\infty)$.

Up to trivial cases presented above, the inequality $\mathbf{1} > 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ really appears in a natural way in our proofs and it is not a technical artefact. Moreover, similar to Proposition 29, it is easy to see that Condition A4 is optimal with respect to the square–integrability of the map $t \mapsto \|B_t\|_2$ on $[0,\infty)$. See also Propositions 27–28 and the proof of Proposition 32 that illustrate A3.

IV.2 Blow–up of the Brocket–Wegner Flow

For bounded operators, the Brocket–Wegner flow (III.1), that is,

$$\forall t \geq 0 : \qquad \partial_t Y_t = [Y_t, [Y_t, X]] \ , \quad Y_{t=0} := Y_0 \ , \tag{IV.41}$$

has a unique *global* solution $(Y_t)_{t\geq 0}$. This assertion is [8, Theorem 1]:

THEOREM 31 (GLOBAL EXISTENCE OF THE BROCKET–WEGNER FLOW)
Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach subalgebra of the Banach algebra $\mathcal{B}(\mathcal{H}) \supset \mathcal{X}$ of bounded operators on a separable Hilbert space $\mathcal{H}$ such that $\|\cdot\|_{\mathcal{X}}$ is a unitarily invariant norm. Suppose that $Y_0 = Y_0^*, X = X^* \in \mathcal{X}$ are two self–adjoint operators such that $X \geq 0$. Then the Brocket–Wegner flow has a unique solution $(Y_t)_{t\geq 0} \in C^\infty[\mathbb{R}_0^+; \mathcal{X}]$ and Y_t is unitarily equivalent to Y_0 for all $t > 0$.

For unbounded operators, the well–posedness of the Brocket–Wegner flow is much more delicate and in [8, Theorem 2] we used a Nash–Moser type of estimate to show that the Brocket–Wegner flow has a unique, smooth *local* unbounded solution $(Y_t)_{t\in[0,T_*]}$ under some restricted conditions on iterated commutators. Here $T_* < \infty$ is finite and the existence of such a solution for larger times $t > T_*$ is, a priori, not excluded in this case. Nevertheless, [8, Theorem 2] suggests that there is generally *no global* solution of (IV.41) for unbounded operators, by opposition to the bounded case. This is confirmed by using the quadratic boson operator of the previous subsection:

PROPOSITION 32 (BLOW–UP OF THE BROCKET–WEGNER FLOW)
There exist two unbounded self-adjoint operators $Y_0 = Y_0^*$ and $X \geq 0$ acting on a separable Hilbert space such that the Brocket–Wegner flow has a (unbounded) local solution $(Y_t)_{t\in[0,T_{\max})}$ with domain $\mathcal{D}(Y_t) = \mathcal{D}(Y_0)$ and whose norm $\|Y_t\varphi\|$, $\varphi \in \mathcal{D}(Y_0)$, diverges at a finite time $T_{\max} < \infty$.

REMARK 33 *Note that in this example the one–particle Hilbert space $\mathfrak{h}$ is isomorphic to $\mathbb{C}^2$.*

PROOF. Take the unbounded self–adjoint operator $Y_0 = H_0$ (IV.1) with $\omega_+ = \omega_- = 0$ and $\mathrm{b} > 0$, that is, a quadratic boson operator satisfying A1–A2 but not A3. As before, the unbounded positive operator X is the particle number operator $X = \mathrm{N} \geq 0$, see (IV.3). The separable Hilbert space is obviously the boson Fock space $\mathcal{F}_b$ defined by (II.8). In this case, it is easy to verify that $Y_t = H_t$ (IV.4) is a solution of the Brocket–Wegner flow, provided the three matrix coefficients $\omega_{\pm,t}$ and b_t solve the system (IV.6) of differential equations. If $\omega_+ = \omega_- = 0$ and $\mathrm{b} > 0$ then b_t is solution of the initial value problem

$$\forall t \geq 0: \qquad \partial_t b_t = 64 b_t \int_0^t b_\tau^2 \mathrm{d}\tau \ , \quad b_0 := \mathrm{b} \ . \tag{IV.42}$$

It is easy to check that the solution of this ODE is

$$\forall t \in [0, T_{\max}): \qquad b_t = \mathrm{b}\sqrt{\tan^2(8\mathrm{b}t) + 1} \ , \tag{IV.43}$$

where $T_{\max} = \pi/(16\mathrm{b})$. By (IV.6), it follows that

$$\forall t \in [0, T_{\max}): \qquad \omega_{-,t} = \omega_{+,t} = -16 \int_0^t b_\tau^2 \mathrm{d}\tau = -2\mathrm{b}\tan(8\mathrm{b}t) \ , \tag{IV.44}$$

whereas

$$\lim_{t \nearrow T_{\max}} |\omega_{-,t}| = \lim_{t \nearrow T_{\max}} |\omega_{+,t}| = \lim_{t \nearrow T_{\max}} b_t = \infty \ . \tag{IV.45}$$

In other words, the Brocket–Wegner flow has a (unbounded) solution $(H_t)_{t\in[0,T_{\max})}$ defined by (IV.4), which *blows up* on its domain in the limit $t \nearrow T_{\max}$. □

Therefore, this rather elementary example using a quadratic boson operator shows a pathological behavior of the Brocket–Wegner flow and we thus has to be really careful while using this flow for unbounded operators. More generally, Proposition 32 provides a strong warning which should discourage us from performing sloppy manipulations with the Brocket–Wegner flow, for instance by producing perturbation series under different ansätze on unbounded Hamiltonians.

V Technical Proofs on the One–Particle Hilbert Space

Together with Section VI, it is the heart of our work. To be precise, here we give the following proofs: the proof of Theorems 11 and 12 (i) in Sections V.1–V.2 as well as the proof of Theorems 12 (ii) and 19 in Section V.3. All these proofs are broken up into several lemmata or theorems, which often yield information beyond the contents of the above theorems.

V.1 Well–Posedness of the Flow

Observe that a solution B_t of (III.9) can be written as a function of Ω_t by using a non–autonomous evolution equation. Indeed, if we assume the existence of a (bounded positive) evolution family $(W_{t,s})_{t\geq s\geq 0}$ acting on $\mathfrak{h}$ and solving the non–autonomous evolution equation

$$\forall t > s \geq 0: \qquad \partial_t W_{t,s} = -2\Omega_t W_{t,s} \ , \quad W_{s,s} := \mathbf{1} \ , \tag{V.1}$$

then, by using the notation $W_t := W_{t,0}$, the operator

$$B_t = B_t^{\mathrm{t}} = W_t B_0 W_t^{\mathrm{t}} \tag{V.2}$$

is a solution of the second differential equation of (III.9). Consequently, we prove below the existence, uniqueness, and strong continuity of the bounded operator family $(\Delta_t)_{t\geq 0}$ solution of the initial value problem

$$\forall t \geq 0: \qquad \partial_t \Delta_t = 16 W_t B_0 W_t^{\mathrm{t}} \left(W_t^{\mathrm{t}}\right)^* \bar{B}_0 W_t^* \ , \quad \Delta_0 := 0 \ , \tag{V.3}$$

with $W_t := W_{t,0}$ satisfying (V.1) for $\Omega_t := \Omega_0 - \Delta_t$.
For this purpose, we first need to define $W_{t,s}$ for any uniformly bounded operator family $(\Delta_t)_{t\geq 0}$. Then, we can use the contraction mapping principle, first for small times $t \in [0, T_0]$ $(T_0 > 0)$. Next, we prove the positivity of the operator Ω_t for small times, the uniqueness of a solution of (V.3), and the so–called *blow–up alternative* of the flow in order to extend – via *an a priori estimate* – the domain of existence $[0, T_0]$ to the positive real line $\mathbb{R}_0^+$ provided A4 holds. The proofs of Theorems 11–12 are broken up into several lemmata or theorems. But first, we need some additional notation.
For any fixed $T > 0$, we define $\mathfrak{C} := C[[0,T];\mathcal{B}(\mathfrak{h})]$ to be the Banach space of strongly continuous maps $t \mapsto \Delta_t$ from $[0,T] \subseteq \mathbb{R}_0^+$ to the set $\mathcal{B}(\mathfrak{h})$ of bounded operators acting on a separable, complex Hilbert space $\mathfrak{h}$. The Banach space $\mathcal{B}(\mathfrak{h})$ is equipped with the usual operator norm $\|\cdot\|_{\mathrm{op}}$, whereas the norm on $\mathfrak{C}$ is defined by

$$\|\Delta\|_\infty := \sup_{t\in[0,T]} \|\Delta_t\|_{\mathrm{op}} \ . \tag{V.4}$$

Also, by $\mathbf{B}_r(X) \subset \mathfrak{C}$ we denote the open ball of radius $r > 0$ centered at $X := (X_t)_{t\in[0,T]} \in \mathfrak{C}$, and below we take $(\Delta_t)_{t\in[0,T]} \in \mathbf{B}_r(0)$ for some finite constant $r > 0$.
To define the operator $W_{t,s}$ the existence of a solution of (V.1) cannot be deduced from Section VII.1 or from existence theorems of parabolic evolution equations. In both cases, we would have to add additional assumptions on Δ_t, for instance some regularity conditions to apply results on parabolic equations. However, our problem is much easier to analyze since the operator family $(\Delta_t)_{t\in[0,T]}$ is uniformly bounded in norm and $\Omega_0 \geq 0$. Indeed, for any $t \geq s \geq 0$, we explicitly define the operator $W_{t,s}$ by the series

$$\begin{aligned} W_{t,s} \equiv W_{t,s}(\Delta) &:= \mathrm{e}^{-2(t-s)\Omega_0} \\ &+ \sum_{n=1}^{\infty} 2^n \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n \mathrm{e}^{-2(t-\tau_1)\Omega_0} \left(\prod_{j=1}^{n} \Delta_{\tau_j} \mathrm{e}^{-2(\tau_j - \tau_{j+1})\Omega_0} \right) , \end{aligned} \tag{V.5}$$

where $\tau_{n+1} := s$ inside the product. In other words, we have chosen $W_{t,s}$ to be the unique solution of the integral equation $W = \mathcal{T}(W)$, with

$$[\mathcal{T}(W)]_{t,s} = \mathrm{e}^{-2(t-s)\Omega_0} + 2\int_s^t \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \ , \tag{V.6}$$

which follows from a standard contraction mapping principle argument. It is important to remark that the operator $W_{t,s}$ is well–defined by (V.5) since

$$\max\left\{ \|W_{t,s}\|_{\mathrm{op}}, \left\|W_{t,s}^{\mathrm{t}}\right\|_{\mathrm{op}} \right\} \leq \mathrm{e}^{2r(t-s)} , \tag{V.7}$$

by $\|\Delta_t\|_{\mathrm{op}} = \|\Delta_t^{\mathrm{t}}\|_{\mathrm{op}} \leq r$ and $\Omega_0 \geq 0$. In fact, it is standard to prove that $W_{t,s}$ is an evolution operator:

Lemma 34 (Properties of the operator $W_{t,s}$)
Assume that $\Omega_0 = \Omega_0^* \geq 0$ and $(\Delta_t)_{t\in[0,T]} \in \mathfrak{C}$ for some $T > 0$. Then, for any $s, x, t \in [0, T]$ so that $t \geq x \geq s$:
(i) *$W_{t,s}$ satisfies the cocycle property $W_{t,x}W_{x,s} = W_{t,s}$.*
(ii) *$W_{t,s}$ is jointly strongly continuous in s and t.*
Similar properties as (i)–(ii) also hold for $W_{t,s}^*$, $W_{t,s}^{\mathrm{t}}$, and $\left(W_{t,s}^{\mathrm{t}}\right)^*$.

Proof. Note that there exists $r > 0$ such that $(\Delta_t)_{t\in[0,T]} \in \mathbf{B}_r(0)$.
(i): Observing that

$$W_{t,s} = [\mathcal{T}(W)]_{t,s} = \mathrm{e}^{-2(t-x)\Omega_0} W_{x,s} + 2\int_x^t \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \tag{V.8}$$

for $t \geq x \geq s \geq 0$, the cocycle property $W_{t,x}W_{x,s} = W_{t,s}$ is proven by multiplying the series (V.5).
(ii): Its strong continuity is standard to verify. Indeed, for any $t' \geq t \geq s$, the properties (V.6)–(V.7), combined with the cocycle property, the uniform bound $\|\Delta_t\|_{\mathrm{op}} \leq r$, and $\Omega_0 \geq 0$, imply after some elementary computations that, for all $\varphi \in \mathfrak{h}$,

$$\|\{W_{t',s} - W_{t,s}\}\varphi\| \leq \left\|\left\{\mathbf{1} - \mathrm{e}^{-2(t'-t)\Omega_0}\right\} W_{t,s}\varphi\right\| + \left(\mathrm{e}^{2r(t'-s)} - \mathrm{e}^{2r(t-s)}\right)\|\varphi\| . \tag{V.9}$$

For any $\alpha > 0$, the semigroup $\mathrm{e}^{2\alpha\Omega_0}$ has a dense domain because $\Omega_0 = \Omega_0^*$. Hence, since $\Omega_0 \geq 0$ and

$$\begin{aligned} \left\|\left\{\mathbf{1} - \mathrm{e}^{-2(t'-t)\Omega_0}\right\} \mathrm{e}^{-2\alpha\Omega_0}\right\|_{\mathrm{op}} &\leq \sup_{\omega\geq 0}\left\{\left(1 - \mathrm{e}^{-2(t'-t)\omega}\right)\mathrm{e}^{-2\alpha\omega}\right\} \\ &\leq |t' - t|\,\alpha^{-1}, \end{aligned} \tag{V.10}$$

then, for any $\delta > 0$, there is $\psi \in \mathcal{D}\left(\mathrm{e}^{2\alpha\Omega_0}\right)$ such that $\|W_{t,s}\varphi - \psi\| \leq \delta$ and

$$\left\|\left\{\mathbf{1} - \mathrm{e}^{-2(t'-t)\Omega_0}\right\} W_{t,s}\varphi\right\| \leq |t' - t|\,\alpha^{-1}\left\|\mathrm{e}^{2\alpha\Omega_0}\psi\right\| + 2\delta . \tag{V.11}$$

In other words, from (V.9) we get the limits

$$\lim_{t'\to t}\|\{W_{t',s} - W_{t,s}\}\varphi\| = \lim_{t\to t'}\|\{W_{t',s} - W_{t,s}\}\varphi\| = 0 \tag{V.12}$$

for all $t' \geq t \geq s$. □

Note that we do not need to know whether the evolution family $(W_{t,s})_{t\geq s\geq 0}$ solves the non–autonomous evolution equation (V.1) to prove the existence of a solution $(\Delta_t)_{t\in[0,T]}$ of (V.3) for sufficiently small times $T > 0$:

Lemma 35 (Existence of Ω_t for small times)
Assume $\Omega_0 = \Omega_0^* \geq 0$ *and* $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$. *Then, for*

$$T_0 := (128 \|B_0\|_2)^{-1} \ , \tag{V.13}$$

there exists a unique strongly continuous operator family $(\Omega_t)_{t\in[0,T_0]}$ *that is a strong solution in* $\mathfrak{h}$ *of the initial value problem*

$$\forall t \in [0, T_0] : \qquad \partial_t \Omega_t = -16 B_t \bar{B}_t \ , \quad \Omega_{t=0} = \Omega_0 \ , \tag{V.14}$$

with $(B_t)_{t\in[0,T_0]} \subset \mathcal{L}^2(\mathfrak{h})$ *defined by (V.2) and (V.5). Furthermore,* $(\Omega_t)_{t\in[0,T_0]}$ *is self–adjoint, with domain* $\mathcal{D}(\Omega_0)$*, and Lipschitz continuous in the norm topology.*

Proof. Observe that the initial value problem (V.3) can be rewritten as the integral equation

$$\forall t \geq 0 : \quad \Delta_t = \mathfrak{T}(\Delta)_t = \{\mathfrak{T}(\Delta)\}_t^* := 16 \int_0^t W_\tau B_0 W_\tau^{\mathrm{t}} \left(W_\tau^{\mathrm{t}}\right)^* \bar{B}_0 W_\tau^* \mathrm{d}\tau \tag{V.15}$$

with $W_t := W_{t,0}$ defined by (V.5). If $r > 0$ and $\Delta = \Delta^* \in \mathbf{B}_r(0)$ then, thanks to (V.7), we obtain

$$\|\mathfrak{T}(\Delta)\|_\infty \leq 16 \int_0^T \left\| W_\tau B_0 W_\tau^{\mathrm{t}} \left(W_\tau^{\mathrm{t}}\right)^* \bar{B}_0 W_\tau^* \right\|_{\mathrm{op}} \mathrm{d}\tau \leq \frac{2}{r} \left(\mathrm{e}^{8rT} - 1\right) \|B_0\|_2^2 \ . \tag{V.16}$$

Now, let $\Delta^{(1)}, \Delta^{(2)} \in \mathbf{B}_r(0)$ which define via (V.5) two evolution operators $W_t^{(1)} := W_{t,0}^{(1)}$ and $W_t^{(2)} := W_{t,0}^{(2)}$. Again from (V.7),

$$\|\mathfrak{T}(\Delta^{(1)}) - \mathfrak{T}(\Delta^{(2)})\|_\infty \leq \frac{16}{3r} \left(\mathrm{e}^{6rT} - 1\right) \|B_0\|_2^2 \Big\{ \|\Lambda\|_\infty + \|\Lambda^{\mathrm{t}}\|_\infty \Big\} \tag{V.17}$$

with $\Lambda_t := W_t^{(1)} - W_t^{(2)}$ for any $t \in [0, T]$. Since the operator $W_{t,s}$ solves the integral equation $W = \mathcal{T}(W)$ (cf. (V.6)),

$$\Lambda_t = 2 \int_0^t \mathrm{e}^{-2(t-\tau)\Omega_0} \Big\{ \Delta_\tau^{(1)} \Lambda_\tau + (\Delta_\tau^{(1)} - \Delta_\tau^{(2)}) W_\tau^{(2)} \Big\} \mathrm{d}\tau \ , \tag{V.18}$$

which, together with (V.7), implies that

$$\|\Lambda\|_\infty \leq 2rT \|\Lambda\|_\infty + \frac{1}{r} \left(\mathrm{e}^{2rT} - 1\right) \|\Delta^{(1)} - \Delta^{(2)}\|_\infty \ . \tag{V.19}$$

That is equivalent when $2rT < 1$ to

$$\|\Lambda\|_\infty \leq \frac{\mathrm{e}^{2rT} - 1}{r(1 - 2rT)} \|\Delta^{(1)} - \Delta^{(2)}\|_\infty \ . \tag{V.20}$$

Inserting this and a similar bound for $\|\Lambda^{\mathrm{t}}\|_\infty$ in (V.17), we obtain

$$\|\mathfrak{T}(\Delta^{(1)}) - \mathfrak{T}(\Delta^{(2)})\|_\infty \leq \frac{32\left(\mathrm{e}^{6rT}-1\right)\left(\mathrm{e}^{2rT}-1\right)}{3r^2\left(1-2rT\right)}\|B_0\|_2^2 \ \|\Delta^{(1)} - \Delta^{(2)}\|_\infty \ . \tag{V.21}$$

Therefore, upon choosing

$$r = r_0 := \sqrt{32}\|B_0\|_2 \qquad \text{and} \qquad T = T_0 := (128\|B_0\|_2)^{-1} \ , \tag{V.22}$$

we observe that

$$\|\mathfrak{T}\left(\Delta\right)\|_\infty \leq \frac{\sqrt{2}}{4}(\mathrm{e}^{\frac{\sqrt{2}}{4}}-1)\|B_0\|_2 < r_0 \ , \tag{V.23}$$

by (V.16), whereas we infer from (V.21) that

$$\|\mathfrak{T}(\Delta^{(1)}) - \mathfrak{T}(\Delta^{(2)})\|_\infty < \frac{1}{2}\|\Delta^{(1)} - \Delta^{(2)}\|_\infty \ . \tag{V.24}$$

In other words, the map

$$\mathfrak{T} : \overline{\mathbf{B}_{r_0}\left(0\right)} \rightarrow \mathbf{B}_{r_0}\left(0\right) \subset \overline{\mathbf{B}_{r_0}\left(0\right)} \tag{V.25}$$

is a contraction. Consequently, the contraction mapping principle on the closed set defined by $\overline{\mathbf{B}_{r_0}\left(0\right)} \subset \mathfrak{C}$, which is equipped with the norm $\|\cdot\|_\infty$, yields a unique fixed point $\Delta = \mathfrak{T}\left(\Delta\right)$ with $\Delta = \Delta^* \in \overline{\mathbf{B}_{r_0}\left(0\right)}$, for $t \leq T_0$. In other words, the lemma follows by defining $\Omega_t := \Omega_0 - \Delta_t = \Omega_t^*$ for $t \leq T_0$. Indeed, since $\|\Delta_t\|_{\mathrm{op}} \leq r_0$, it is clear that $\mathcal{D}\left(\Omega_t\right) = \mathcal{D}\left(\Omega_0\right)$ and, thanks to (V.7),

$$\|\Omega_t - \Omega_s\|_{\mathrm{op}} \leq 16\|B_0\|_2^2\mathrm{e}^{\frac{\sqrt{2}}{4}}|t-s| \tag{V.26}$$

for any $t, s \in [0, T_0]$, i.e., the family $(\Omega_t)_{t\in[0,T_0]}$ is Lipschitz continuous in the norm topology for any $t \in [0, T_0]$. □

Now we observe that the evolution family $(W_{t,s})_{t\geq s\geq 0}$ solves the non–autonomous evolution equation (V.1), provided $(\Delta_t)_{t\in[0,T_0]}$ is a solution of (V.15). From now, we set

$$\forall t \in [0, T_0]: \qquad \Delta_t \equiv 16\int_0^t B_\tau \bar{B}_\tau \mathrm{d}\tau \quad \text{and} \quad \Omega_t := \Omega_0 - \Delta_t \ , \tag{V.27}$$

(cf. Lemma 35) and prove additional properties on $(W_{t,s})_{t\geq s\geq 0}$ in the next lemma:

LEMMA 36 ($W_{t,s}$ AS SOLUTION OF A PARABOLIC EVOLUTION EQUATION) *Assume $\Omega_0 = \Omega_0^* \geq 0$ and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2\left(\mathfrak{h}\right)$, define $T_0 > 0$ by (V.13) and let $(\Omega_t)_{t\in[0,T_0]}$ be the unique solution of (V.14).*
(i) *$\Omega_t W_{t,s} \in \mathcal{B}\left(\mathfrak{h}\right)$ is a bounded operator provided $t > s$:*

$$\forall s \in [0, T_0), \ t \in (s, T_0]: \qquad \|\Omega_t W_{t,s}\|_{\mathrm{op}} \leq \mathrm{C}_0 + \mathrm{D}_0\left(t-s\right)^{-1} \tag{V.28}$$

with $C_0, \mathrm{D}_0 < \infty$. *Moreover,* $(\Omega_t W_{t,s})_{t>s\geq 0}$ *is strongly continuous in* $t > s$.
(ii) *The evolution family* $(W_{t,s})_{t\geq s\geq 0}$ *is the solution of the non–autonomous evolution equations*

$$\left\{ \begin{array}{lllll} \forall s \in [0,T_0),\ t\in (s,T_0] & : & \partial_t W_{t,s} = -2\Omega_t W_{t,s} & , & W_{s,s} := \mathbf{1} \ . \\ \forall t \in (0,T_0],\ s\in [0,t] & : & \partial_s W_{t,s} = 2 W_{t,s}\Omega_s & , & W_{t,t} := \mathbf{1} \ . \end{array} \right. \tag{V.29}$$

The derivatives with respect to t *and* s *are in the strong sense in* $\mathfrak{h}$ *and* $\mathcal{D}(\Omega_0)$, *respectively.*
(iii) *For any* $s \in [0, T_0)$ *and every* $\epsilon \in [0, T_0 - s)$, *the map* $t \mapsto W_{t,s}$ *is Lipschitz continuous on* $[s+\epsilon, T_0]$ *in the norm topology:*

$$\forall t,t' \in [s+\epsilon, T_0] : \qquad \|W_{t',s} - W_{t,s}\|_{\mathrm{op}} \leq 2\left(\mathrm{C}_0 + \mathrm{D}_0\epsilon^{-1}\right)|t'-t| \ . \tag{V.30}$$

Similar properties as (i)–(iii) also hold for $W_{t,s}^*$, $W_{t,s}^{\mathrm{t}}$, *and* $(W_{t,s}^{\mathrm{t}})^*$.

Proof. The operator Ω_t has domain $\mathcal{D}(\Omega_t) = \mathcal{D}(\Omega_0)$ and is continuous in the norm topology for $t \leq T_0$, with T_0 defined by (V.13), see (V.26). Because $\Omega_0 \geq 0$, there is $m > -\infty$ such that $\Omega_t \geq m$ and $-2\Omega_t$ is the generator of an analytic semigroup $(\mathrm{e}^{-2\alpha\Omega_t})_{\alpha\geq 0}$ for any fixed $t \in [0, T_0]$. Consequently, the Cauchy problem (V.29) is a parabolic evolution equation. In particular, existence and uniqueness of its solution is standard in this case, see, e.g., [14, p. 407, Theorem 2.2] or [15, Chap. 5, Theorem 6.1]. Nevertheless, we give here a complete proof as our case is a specific parabolic evolution equation which is easier to study.
Before starting the proof, note that, for any positive operator $X = X^* \geq 0$, $\beta > 0$, and $\alpha \geq 0$, the operator $X^\alpha \mathrm{e}^{-\beta X}$ is bounded in operator norm by

$$\|X^\alpha \mathrm{e}^{-\beta X}\|_{\mathrm{op}} \leq \sup_{x\geq 0}\left\{x^\alpha \mathrm{e}^{-\beta x}\right\} = \left(\frac{\alpha}{\mathrm{e}\beta}\right)^\alpha \ . \tag{V.31}$$

Since $\Omega_0 = \Omega_0^* \geq 0$, we then conclude that

$$\forall t > s \geq 0 : \qquad \|\Omega_0 \mathrm{e}^{-2(t-s)\Omega_0}\|_{\mathrm{op}} \leq \frac{1}{2\mathrm{e}\,(t-s)} \ . \tag{V.32}$$

Moreover, $\Omega_0 \geq 0$ is the generator of an analytic semigroup for $t \in [0, T_0]$. In particular, for any $t' \geq t > s$,

$$\forall t' \geq t > s \geq 0 : \qquad \left\| \mathrm{e}^{-2\left(t'-s\right)\Omega_0} - \mathrm{e}^{-2(t-s)\Omega_0} \right\|_{\mathrm{op}} \leq (t'-t)\,(t-s)^{-1} \ , \tag{V.33}$$

using (V.10) with $\alpha = t - s$. Meanwhile, as $\Omega_0 \geq 0$,

$$\left\| \mathrm{e}^{-2\left(t'-s\right)\Omega_0} - \mathrm{e}^{-2(t-s)\Omega_0} \right\|_{\mathrm{op}} \leq 2 \ . \tag{V.34}$$

As consequence, we can interpolate the estimates (V.33) and (V.34) to get

$$\left\| \mathrm{e}^{-2\left(t'-s\right)\Omega_0} - \mathrm{e}^{-2(t-s)\Omega_0} \right\|_{\mathrm{op}} \leq 2^{1-\nu}\,(t'-t)^\nu\,(t-s)^{-\nu} \tag{V.35}$$

for any $t' \geq t > s$ and $\nu \in [0,1]$. We then conclude these preliminary remarks on the semigroup $\left\{ \mathrm{e}^{-2(t-s)\Omega_0} \right\}_{t\geq s}$ by observing that, for any $t' > t > s$,

$$\left\| \Omega_0 \left(\mathrm{e}^{-2\left(t'-s\right)\Omega_0} - \mathrm{e}^{-2(t-s)\Omega_0} \right) \right\|_{\mathrm{op}} \leq 2 \int_{t-s}^{t'-s} \left\| \Omega_0^2 \mathrm{e}^{-2\tau\Omega_0} \right\|_{\mathrm{op}} \mathrm{d}\tau \leq \frac{(t'-t)}{\mathrm{e}^2 \left(t'-s\right)(t-s)} , \qquad \text{(V.36)}$$

using (V.31) with $X = \Omega_0$, $\alpha = 2$, and $\beta = 2\tau$. Now, we are in position to prove the lemma by following similar arguments as in [15, p. 157-159].
Recall that the operator family $(\Delta_t)_{t\in[0,T_0]} \subset \mathbf{B}_r(0)$ is Lipschitz continuous in the norm topology. See, e.g., (V.26)–(V.27). By using two times (V.35) together with $\Omega_0 \geq 0$, $\Delta = \Delta^* \in \mathbf{B}_r(0)$, (V.6) and (V.7), we then deduce, for any $t' \geq t > s$ and $\nu \in [0,1)$, that

$$\begin{aligned} \|W_{t',s} - W_{t,s}\|_{\mathrm{op}} \leq\ & 2^{1-\nu} \left(t'-t\right)^{\nu} (t-s)^{-\nu} + 2r\left(t'-t\right) \mathrm{e}^{2r\left(t'-s\right)} \\ & + \frac{2^{2-\nu} r}{1-\nu} \mathrm{e}^{2r(t-s)} \left(t'-t\right)^{\nu} (t-s)^{1-\nu} . \end{aligned} \qquad \text{(V.37)}$$

Now, for any $s \in [0, T_0]$ and sufficiently small $\epsilon > 0$, let

$$\forall t \in (s+\epsilon, T_0] : \qquad \mathcal{W}_{t,s}^{(\epsilon)} := 2 \int_s^{t-\epsilon} \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau . \qquad \text{(V.38)}$$

Recall that $-2\Omega_0 \leq 0$ is the generator of an analytic semigroup $(\mathrm{e}^{-2\alpha\Omega_0})_{\alpha\geq 0}$. Consequently, as $\epsilon \to 0$, $\mathcal{W}_{t,s}^{(\epsilon)}$ strongly converges to $\mathcal{W}_{t,s}^{(0)}$ and $\mathcal{W}_{t,s}^{(\epsilon)}$ is strongly differentiable with respect to $t > s+\epsilon$ with derivative

$$\partial_t \mathcal{W}_{t,s}^{(\epsilon)} = 2\mathrm{e}^{-2\epsilon\Omega_0} \Delta_{t-\epsilon} W_{t-\epsilon,s} - 4 \int_s^{t-\epsilon} \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \qquad \text{(V.39)}$$

for any $t \in (s+\epsilon, T_0]$. Therefore, using again $\Omega_0 \geq 0$, $\Delta = \Delta^* \in \mathbf{B}_r(0)$, and (V.7) together with

$$\begin{aligned} -4 \int_s^{t-\epsilon} \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau &= -4 \int_s^{t-\epsilon} \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \left(\Delta_\tau W_{\tau,s} - \Delta_t W_{t,s} \right) \mathrm{d}\tau \\ &\quad -4 \int_s^{t-\epsilon} \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_t W_{t,s} \mathrm{d}\tau \\ &= -4 \int_s^{t-\epsilon} \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \left(\Delta_\tau W_{\tau,s} - \Delta_t W_{t,s} \right) \mathrm{d}\tau \\ &\quad +2 \left(\mathrm{e}^{-2(t-s)\Omega_0} - \mathrm{e}^{-2\epsilon\Omega_0} \right) \Delta_t W_{t,s} , \end{aligned} \qquad \text{(V.40)}$$

we find that

$$\begin{aligned} \|\partial_t \mathcal{W}_{t,s}^{(\epsilon)}\|_{\mathrm{op}} \leq\ & 2r\mathrm{e}^{2r(t-s)} + 4r\mathrm{e}^{2r(t-s)} \\ & +4 \int_s^{t-\epsilon} \|\Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0}\|_{\mathrm{op}} \|\Delta_\tau W_{\tau,s} - \Delta_t W_{t,s}\|_{\mathrm{op}} \mathrm{d}\tau . \end{aligned} \qquad \text{(V.41)}$$

We use now (V.7), (V.32), (V.37) for $\nu \in (0,1)$ and the Lipschitz continuity of the operator family $(\Delta_t)_{t\in[0,T_0]}$ to get

$$\int_s^{t-\epsilon} \|\Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0}\|_{\mathrm{op}} \|\Delta_\tau W_{\tau,s} - \Delta_t W_{t,s}\|_{\mathrm{op}} \mathrm{d}\tau \leq \mathrm{C} < \infty \tag{V.42}$$

for some constant $\mathrm{C} \in (0,\infty)$ which is independent of $s \in [0,T_0]$, $\epsilon > 0$, and $t \in (s+\epsilon, T_0]$. As a consequence, by (V.41), we arrive at the inequality

$$\|\partial_t \mathcal{W}_{t,s}^{(\epsilon)}\|_{\mathrm{op}} \leq \mathrm{C} < \infty \tag{V.43}$$

for some constant $\mathrm{C} \in (0,\infty)$ not depending on $s \in [0,T_0]$, $\epsilon > 0$, and $t \in (s+\epsilon, T_0]$. Moreover, as we prove (V.43), one can verify that $\partial_t \mathcal{W}_{t,s}^{(\epsilon)}$ strongly converges to

$$\mathcal{V}_{t,s} := \partial_t \mathcal{W}_{t,s}^{(\epsilon)}|_{\epsilon=0} = 2\Delta_t W_{t,s} - 4\int_s^t \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \ , \tag{V.44}$$

as $\epsilon \to 0$, for all $s \in [0,T_0]$ and $t \in (s,T_0]$, and

$$\|\mathcal{V}_{t,s}\|_{\mathrm{op}} \leq \mathrm{C} < \infty \ , \tag{V.45}$$

by (V.43). The operator $\mathcal{V}_{t,s}$ is also strongly continuous in $t > s$, see (V.44). Since $\mathcal{W}_{t,s}^{(\epsilon)}$ strongly converges to $\mathcal{W}_{t,s}^{(0)}$, we can take the limit $\epsilon \to 0$ in

$$\mathcal{W}_{t',s}^{(\epsilon)} - \mathcal{W}_{t,s}^{(\epsilon)} = \int_t^{t'} \partial_\tau \mathcal{W}_{\tau,s}^{(\epsilon)} \mathrm{d}\tau \tag{V.46}$$

to arrive at the equality

$$\mathcal{W}_{t',s}^{(0)} - \mathcal{W}_{t,s}^{(0)} = \int_t^{t'} \mathcal{V}_{\tau,s} \mathrm{d}\tau \ , \tag{V.47}$$

with $s \in [0,T_0]$ and $t \in (s,T_0]$. Because $\mathcal{V}_{t,s}$ is strongly continuous in $t > s$, it follows that

$$\forall t \in (s,T_0] : \qquad \mathcal{V}_{t,s} = \partial_t \mathcal{W}_{t,s}^{(0)} \ , \tag{V.48}$$

i.e., $\mathcal{W}_{t,s}^{(0)}$ is strongly differential with respect to $t > s$. Since the operator $W_{t,s}$ solves the integral equation

$$W_{t,s} = [\mathcal{T}(W)]_{t,s} = \mathrm{e}^{-2(t-s)\Omega_0} + \mathcal{W}_{t,s}^{(0)} \tag{V.49}$$

(cf. (V.6)), $W_{t,s}$ is strongly differential with respect to $t > s$ with derivative

$$\forall t \in (s,T_0] : \qquad \partial_t W_{t,s} = -2\Omega_0 \mathrm{e}^{-2(t-s)\Omega_0} + \mathcal{V}_{t,s} \ . \tag{V.50}$$

In particular, by (V.32) and (V.45),

$$\forall t > s \geq 0 : \qquad \|\partial_t W_{t,s}\|_{\mathrm{op}} \leq \mathrm{e}^{-1}(t-s)^{-1} + \mathrm{C} < \infty \ . \tag{V.51}$$

Finally, for any $s \in [0, T_0]$ and sufficiently small $\epsilon > 0$, we define the bounded operator

$$\forall t \in (s + \epsilon, T_0] : \qquad W_{t,s}^{(\epsilon)} := \mathrm{e}^{-2(t-s)\Omega_0} + \mathcal{W}_{t,s}^{(\epsilon)} \ . \tag{V.52}$$

Using (V.39) and the fact that $W_{t,s}^{(\epsilon)} \mathfrak{h} \subset \mathcal{D}(\Omega_0)$ for $t > s + \epsilon$ and $\epsilon > 0$ we then observe that

$$\partial_t W_{t,s}^{(\epsilon)} + 2\Omega_t W_{t,s}^{(\epsilon)} = -2\Delta_t \mathrm{e}^{-2(t-s)\Omega_0} - 2\Delta_t \mathcal{W}_{t,s}^{(\epsilon)} + 2\mathrm{e}^{-2\epsilon\Omega_0} \Delta_{t-\epsilon} W_{t-\epsilon,s} \ , \tag{V.53}$$

where we recall that $\Omega_t := \Omega_0 - \Delta_t$. In the limit $\epsilon \to 0$, the right hand side strongly converges to zero because of (V.49). On the other hand, $\partial_t W_{t,s}^{(\epsilon)}$ strongly converges to $\partial_t W_{t,s} = \mathcal{V}_{t,s}$, as $\epsilon \to 0$. Therefore, the operator $2\Omega_t W_{t,s}^{(\epsilon)}$ converges strongly, as $\epsilon \to 0$, whereas $W_{t,s}^{(\epsilon)}$ converges strongly to $W_{t,s}$ when $\epsilon \to 0$. As Ω_t is a closed operator, it follows that $W_{t,s} \mathfrak{h} \subset \mathcal{D}(\Omega_0)$ and

$$\forall t \in (s, T_0] : \qquad \partial_t W_{t,s} = -2\Omega_0 \mathrm{e}^{-2(t-s)\Omega_0} + \mathcal{V}_{t,s} = -2\Omega_t W_{t,s} \ . \tag{V.54}$$

On the other hand, the assertion

$$\forall t \in (0, T_0], \ s \in [0, t] : \qquad \partial_s W_{t,s} = 2W_{t,s}\Omega_s \ , \tag{V.55}$$

directly follows from the norm continuity of the family $(\Omega_t)_{t \in [0,T_0]}$ together with the equality

$$\begin{aligned} W_{t,s} \quad = \quad & \mathrm{e}^{-2(t-s)\Omega_0} + \sum_{n=1}^{\infty} 2^n \int_s^t \mathrm{d}\tau_1 \cdots \int_{\tau_{n-1}}^t \mathrm{d}\tau_n \\ & \left(\prod_{j=1}^{n} \mathrm{e}^{-2(\tau_{n+2-j} - \tau_{n+1-j})\Omega_0} \Delta_{\tau_{n+1-j}} \right) \mathrm{e}^{-2(\tau_1 - s)\Omega_0} \ , \end{aligned} \tag{V.56}$$

with $\Delta_t := \Omega_0 - \Omega_t \in \mathcal{B}(\mathfrak{h})$ and $\tau_{n+1} := t$. This last equation can easily be derived from (V.5) and Fubini's theorem.

Therefore, by (V.51) and (V.54)–(V.55), we have proven Assertion (ii) as well as the upper bound of (i). Additionally, (V.32), (V.45) and (V.54) yield (V.28) and $(\Omega_t W_{t,s})_{t > s \geq 0}$ is strongly continuous in $t > s$ because of (V.36), (V.54) and the strong continuity of the operator $\mathcal{V}_{t,s}$ with respect to $t > s$. Uniqueness of the solution of (V.54) (or (V.29)) is standard to verify. We omit the details. This concludes the proof of (i)–(ii), which in turn imply (iii).

Note that similar properties as (i)–(iii) also hold for $W_{t,s}^*$, $W_{t,s}^{\mathrm{t}}$, and $(W_{t,s}^{\mathrm{t}})^*$. For instance, let us consider the operator family $(W_{t,s}^*)_{t \geq s \geq 0}$. Since $\Omega_0 = \Omega_0^*$ and $\Delta_t = \Delta_t^*$ for any $t \in [0, T_0]$ (see (V.27)), one verifies from (V.56) that $W_{t,s}^*$ has a representation in term of a series constructed from the integral equation

$$W_{t,s}^* = \mathrm{e}^{-2(t-s)\Omega_0} + 2 \int_s^t \mathrm{e}^{-2(\tau - s)\Omega_0} \Delta_\tau W_{t,\tau}^* \mathrm{d}\tau \ , \tag{V.57}$$

for any $t \in [0, T_0]$ and $s \in [0, t]$. Therefore, similar properties as (i)–(iii) hold for $W_{t,s}^*$ which follow in a similar way as for $W_{t,s}$, but interchanging the role of s and t (compare (V.57) to (V.6)). Analogous observations can be done for $W_{t,s}^{\mathrm{t}}$ and $(W_{t,s}^{\mathrm{t}})^*$ with $(\Omega_0^{\mathrm{t}}, \Delta_t^{\mathrm{t}})$ replacing (Ω_0, Δ_t). We omit the details. □

Remark 37 *One can verify that $(\Omega_t W_{t,s})_{t\geq s+\epsilon\geq 0}$ is also Hölder continuous in the norm topology for every $\epsilon > 0$. See for instance similar arguments done to prove [15, Chap. 5, Theorem 6.9.].*

By Lemmata 34 (ii) and 36, the bounded operator $B_t = B_t^{\mathrm{t}} \in \mathcal{B}(\mathfrak{h})$ of Lemma 35 satisfies (III.9), that is:

Corollary 38 (Properties of B_t for small times)
Assume $\Omega_0 = \Omega_0^* \geq 0$ and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$. Then, for $T_0 := (128\|B_0\|_2)^{-1}$, the bounded operator family $(B_t)_{t\in[0,T_0]}$, defined by (V.2) and (V.5), is strongly continuous and satisfies

$$B_t = B_0 - 2\int_0^t \left(\Omega_\tau B_\tau + B_\tau \Omega_\tau^{\mathrm{t}}\right) \mathrm{d}\tau \ . \tag{V.58}$$

Moreover, $(B_t)_{t\in[0,T_0]} \in \mathfrak{C}$ is the unique strong solution on the domain $\mathcal{D}(\Omega_0^{\mathrm{t}})$ of $\partial_t B_t = -2(\Omega_t B_t + B_t \Omega_t^{\mathrm{t}})$ for $t \in (0, T_0]$ as $B_{t>0}\mathfrak{h} \subseteq \mathcal{D}(\Omega_0)$. $(B_t)_{t\in(0,T_0]}$ is also locally Lipschitz continuous in the norm topology.

Proof. The only non–trivial statement to prove is the uniqueness of a strong solution of the differential equation

$$\forall t \in (0, T_0]: \qquad \partial_t B_t = -2(\Omega_t B_t + B_t \Omega_t^{\mathrm{t}}) \ , \quad B_{t=0} = B_0 \ , \tag{V.59}$$

on $\mathcal{D}(\Omega_0^{\mathrm{t}})$. To this end, let $(\tilde{B}_t)_{t\in[0,T_0]} \subset \mathcal{B}(\mathfrak{h})$ be a strongly continuous family of bounded operators obeying (V.59) on $\mathcal{D}(\Omega_0^{\mathrm{t}})$. It means, in particular, that $\tilde{B}_t \mathcal{D}(\Omega_0^{\mathrm{t}}) \subset \mathcal{D}(\Omega_0)$, whereas $W_{t,s}\mathfrak{h} \subseteq \mathcal{D}(\Omega_0)$ for any $t > s$ (Lemma 36 (i)). Therefore, using Lemma 36 (ii) we observe that

$$\forall t \in (0, T_0],\ s \in (0, t): \qquad \partial_s\{W_{t,s}\tilde{B}_s W_{t,s}^{\mathrm{t}}\} = 0 \ . \tag{V.60}$$

In other words,

$$\forall t \in (0, T_0],\ s \in (0, t): \qquad \tilde{B}_t = W_{t,s}\tilde{B}_s W_{t,s}^{\mathrm{t}} \ . \tag{V.61}$$

By passing to the limit $s \to 0^+$ (cf. Lemma 36 (iii)), it follows that

$$\forall t \in (0, T_0]: \qquad \tilde{B}_t = W_t B_0 W_t^{\mathrm{t}} = B_t \ , \tag{V.62}$$

see (V.2). □

We now prove that Condition A6 is preserved under the dynamics. This preliminary result is crucial to get the positivity of operators Ω_t under A3.

LEMMA 39 (CONSERVATION OF CONDITION A6)
Let $\Omega_0 = \Omega_0^* \geq 0$ *and* $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$ *such that* Ω_0 *is invertible on* $\mathrm{Ran}B_0$ *and* $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 + \mu\mathbf{1}$ *for some* $\mu > 0$. *Then, the operator family* $(\Omega_t)_{t\in[0,T_0]}$ *of Lemma 35 satisfies the operator inequality*

$$\forall t \in [0,T_0]: \qquad \Omega_t \geq 4B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t + \mu\mathbf{1} \ . \tag{V.63}$$

PROOF. From Lemma 35 and Corollary 38, there is a strong solution $(\Omega_t, B_t)_{t\in[0,T_0]}$ of (III.9). Let

$$\varkappa := \sup\left\{T \in [0,T_0] \ \middle| \ \forall t \in [0,T]: \quad \Omega_t^{\mathrm{t}} \geq \frac{\mu}{2}\,\mathbf{1}\right\} > 0 \tag{V.64}$$

and observe, by Lemma 36 (ii), that

$$\partial_t \left\{W_{t,s}^* W_{t,s}\right\} = -4W_{t,s}^*\Omega_t W_{t,s} \leq 0 \ , \tag{V.65}$$

for all $s \in (0,\varkappa)$ and $t \in (s,\varkappa)$. Integrating this we obtain $W_{t,s}^*W_{t,s} \leq \mathbf{1}$ for $s \in (0,\varkappa)$ and $t \in [s,\varkappa)$. In the same way, $W_{t,s}^{\mathrm{t}}\left(W_{t,s}^{\mathrm{t}}\right)^* \leq \mathbf{1}$ since $\Omega_t \geq 0$ implies $\Omega_t^{\mathrm{t}} \geq 0$. In fact, the inequality $\Omega_x \geq 0$ for all $x \in [s,t] \subset [0,\varkappa)$ yields

$$\max\left\{ \|W_{t,s}\|_{\mathrm{op}}, \left\|W_{t,s}^{\mathrm{t}}\right\|_{\mathrm{op}} \right\} \leq 1 \ . \tag{V.66}$$

For all $t \in [0,\varkappa)$, let us consider two important operators:

$$\mathfrak{B}_t := B_t\left(\Omega_t^{\mathrm{t}}\right)^{-1}\bar{B}_t \qquad \text{and} \qquad \mathfrak{D}_t := \Omega_t - 4\mathfrak{B}_t \ . \tag{V.67}$$

Note that $\mathfrak{B}_t$ is bounded for all $t \in [0,\varkappa)$ and $\mu > 0$ because (V.2) and (V.66) imply that

$$\forall t \in [0,\varkappa): \qquad \|\mathfrak{B}_t\|_{\mathrm{op}} \leq 2\|B_0\|_2^2\,\mu^{-1} < \infty \ . \tag{V.68}$$

Additionally, by Lemma 35 and Corollary 38, on the domain $\mathcal{D}(\Omega_0)$ we have

$$\forall t \in (0,T_0]: \qquad \partial_t\mathfrak{B}_t = -2\left(\mathfrak{B}_t\mathfrak{D}_t + \mathfrak{D}_t\mathfrak{B}_t\right) - 4B_t\bar{B}_t \ . \tag{V.69}$$

This derivative is justified by combining Lemma 35 and Corollary 38 with the upper bound

$$\begin{aligned} &\Big\|\big(\epsilon^{-1}(\mathfrak{B}_{t+\epsilon} - \mathfrak{B}_t) - \partial_t\{B_t\}(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t \\ &\quad -B_t\partial_t\{(\Omega_t^{\mathrm{t}})^{-1}\}\bar{B}_t - B_t(\Omega_t^{\mathrm{t}})^{-1}\partial_t\{\bar{B}_t\}\big)\,\varphi\Big\| \\ \leq\ & 2\|B_0\|_2\,\mu^{-1}\left\|\left(\epsilon^{-1}(\bar{B}_{t+\epsilon} - \bar{B}_t) - \partial_t\{\bar{B}_t\}\right)\varphi\right\| + 128\|B_0\|_2^3\,\mu^{-2}\left\|\bar{B}_{t+\epsilon} - \bar{B}_t\right\|_{\mathrm{op}} \\ &+2\mu^{-1}\|B_{t+\epsilon} - B_t\|_{\mathrm{op}}\left\|\epsilon^{-1}(\bar{B}_{t+\epsilon} - \bar{B}_t)\varphi\right\| \\ &+\|B_0\|_2\left\|\left(\epsilon^{-1}((\Omega_{t+\epsilon}^{\mathrm{t}})^{-1} - (\Omega_t^{\mathrm{t}})^{-1}) - \partial_t\{(\Omega_t^{\mathrm{t}})^{-1}\}\right)\bar{B}_t\varphi\right\| \\ &+\left\|\left(\epsilon^{-1}(B_{t+\epsilon} - B_t) - \partial_t\{B_t\}\right)(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t\varphi\right\| \ , \end{aligned} \tag{V.70}$$

for any $\varphi \in \mathcal{D}(\Omega_0)$, $t \in (0,T_0]$, and sufficiently small $|\epsilon| > 0$. Note that it is easy to show that $(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t\mathfrak{h} \subset \mathcal{D}(\Omega_0)$. Therefore, by Lemma 35 and (V.69), we observe that $\mathfrak{D}_t$ is the strong solution on the domain $\mathcal{D}(\Omega_0)$ of the initial value problem

$$\forall t \in (0, T_0] : \qquad \partial_t \mathfrak{D}_t = 8\left(\mathfrak{B}_t \mathfrak{D}_t + \mathfrak{D}_t \mathfrak{B}_t\right) \ , \quad \mathfrak{D}_0 := \Omega_0 - 4\mathfrak{B}_0 \ . \tag{V.71}$$

Let the operator $\mathfrak{V}_{t,s}$ be the strong solution in $\mathcal{B}(\mathfrak{h})$ of the non–autonomous evolution equation

$$\forall s, t \in [0, \varkappa) : \qquad \partial_t \mathfrak{V}_{t,s} = 8\mathfrak{B}_t \mathfrak{V}_{t,s} \ , \quad \mathfrak{V}_{s,s} := \mathbf{1} \ . \tag{V.72}$$

By (V.68), the operator $\mathfrak{B}_t$ is bounded for any $t \in [0, \varkappa)$ and the evolution operator $\mathfrak{V}_{t,s}$ is of course well–defined, for any $s, t \in [0, \varkappa)$, by the Dyson series

$$\mathfrak{V}_{t,s} := \mathbf{1} + \sum_{n=1}^{\infty} 8^n \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n \prod_{j=1}^{n} \mathfrak{B}_{\tau_j} \ , \tag{V.73}$$

whose operator norm is bounded from above, for any $\mu > 0$, by

$$\|\mathfrak{V}_{t,s}\|_{\mathrm{op}} \leq \exp\left\{16 \left\|B_0\right\|_2^2 \mu^{-1} \left|t - s\right|\right\} < \infty \ . \tag{V.74}$$

In particular, $\mathfrak{V}_{t,s}$ satisfies the cocycle property and its adjoint is a strong solution in $\mathcal{B}(\mathfrak{h})$ of the non–autonomous evolution equation

$$\forall s, t \in [0, \varkappa) : \qquad \partial_t \mathfrak{V}_{t,s}^* = 8\mathfrak{V}_{t,s}^* \mathfrak{B}_t \ , \quad \mathfrak{V}_{s,s}^* := \mathbf{1} \ , \tag{V.75}$$

as $\mathfrak{B}_t^* = \mathfrak{B}_t$ is self–adjoint. Furthermore, $\mathfrak{V}_{t,s}$ and $\mathfrak{V}_{t,s}^*$ satisfy the non–autonomous evolution equations

$$\forall s, t \in [0, \varkappa) : \qquad \partial_s \mathfrak{V}_{t,s} = -8\mathfrak{V}_{t,s} \mathfrak{B}_s \ , \quad \mathfrak{V}_{t,t} := \mathbf{1} \ , \tag{V.76}$$

and

$$\forall s, t \in [0, \varkappa) : \qquad \partial_s \mathfrak{V}_{t,s}^* = -8\mathfrak{B}_s \mathfrak{V}_{t,s}^* \ , \quad \mathfrak{V}_{t,t} := \mathbf{1} \ , \tag{V.77}$$

respectively. In fact, for any $\epsilon > 0$ the bounded operator family $(\mathfrak{B}_t)_{t \in [\epsilon, \varkappa)}$ is Lipschitz continuous in the norm topology, by Lemma 35 and Corollary 38. A detailed proof of basic properties of $\left(\mathfrak{V}_{t,s}\right)_{t \in [0, \varkappa)}$ is thus given for instance by [15, Chap. 5, Thm 5.2]. In particular, $\left(\mathfrak{V}_{t,s}\right)_{t \in [\epsilon, \varkappa)}$ is also Lipschitz norm continuous. We additionally observe that $\mathfrak{V}_{t,s}^*$ conserves the domain $\mathcal{D}(\Omega_0)$, i.e.,

$$\forall s, t \in (0, \varkappa) : \qquad \mathfrak{V}_{t,s}^* \mathcal{D}(\Omega_0) \subset \mathcal{D}(\Omega_0) \ . \tag{V.78}$$

Indeed, by (V.77), the evolution operator $\mathfrak{V}_{t,s}^*$ satisfies the integral equation

$$\forall s, t \in [0, \varkappa) : \qquad \mathfrak{V}_{t,s}^* := \mathbf{1} + 8 \int_s^t \mathfrak{B}_\tau \mathfrak{V}_{t,\tau}^* \mathrm{d}\tau \tag{V.79}$$

from which we deduce that, for any $s, t \in (0, \varkappa)$,

$$\begin{aligned} \Omega_0 \left(\mathfrak{V}_{t,s}^* - \mathbf{1}\right) &= 8\Omega_0 \int_s^t \left(\Omega_0 + \mathbf{1}\right)^{-1} \left(\Omega_0 + \mathbf{1}\right) \mathfrak{B}_\tau \mathfrak{V}_{t,\tau}^* \mathrm{d}\tau \\ &= 8 \frac{\Omega_0}{\Omega_0 + \mathbf{1}} \int_s^t \left(\Omega_0 + \mathbf{1}\right) \mathfrak{B}_\tau \mathfrak{V}_{t,\tau}^* \mathrm{d}\tau \ , \end{aligned} \tag{V.80}$$

because the closed operator $(\Omega_0 + \mathbf{1})^{-1} \in \mathcal{B}(\mathfrak{h})$ is bounded, the integrands are continuous (see in particular Lemma 36 (i)) and one only has Riemann integrals. It follows that, for all $s,t \in (0,\varkappa)$ and $\mu > 0$,

$$\begin{aligned}\left\|\Omega_0\left(\mathfrak{V}^*_{t,s} - \mathbf{1}\right)\right\|_{\mathrm{op}} &\leq 8\int_{\min\{s,t\}}^{\max\{s,t\}} \left\|\mathfrak{B}_\tau \mathfrak{V}^*_{t,\tau}\right\|_{\mathrm{op}} \mathrm{d}\tau \\ &\quad +8\int_{\min\{s,t\}}^{\max\{s,t\}} \left\|\Delta_\tau \mathfrak{B}_\tau \mathfrak{V}^*_{t,\tau}\right\|_{\mathrm{op}} \mathrm{d}\tau \\ &\quad +8\int_{\min\{s,t\}}^{\max\{s,t\}} \left\|\Omega_\tau \mathfrak{B}_\tau \mathfrak{V}^*_{t,\tau}\right\|_{\mathrm{op}} \mathrm{d}\tau \\ &< \infty ,\end{aligned} \tag{V.81}$$

using (V.2) together with Lemma 36 (i), (V.66), and the same upper bound on the operator norm $\|\mathfrak{V}^*_{s,t}\|_{\mathrm{op}}$ as the one (V.74) on $\|\mathfrak{V}_{t,s}\|_{\mathrm{op}}$. Therefore, (V.78) holds and the (possibly unbounded) operator $\mathfrak{V}_{t,s}\mathfrak{D}_s\mathfrak{V}^*_{t,s}$ is well–defined on the domain $\mathcal{D}(\Omega_0)$ whenever $s,t \in (0,\varkappa)$. Moreover, using (V.71) and (V.76)–(V.77) one verifies that its time derivative on $\mathcal{D}(\Omega_0)$ vanishes, i.e.,

$$\forall s,t \in (0,\varkappa),\ \varphi \in \mathcal{D}(\Omega_0): \qquad \partial_s\{\mathfrak{V}_{t,s}\mathfrak{D}_s\mathfrak{V}^*_{t,s}\}\varphi = 0 . \tag{V.82}$$

To prove (V.82), one needs to know on the domain $\mathcal{D}(\Omega_0)$ that

$$\partial_s\{\mathfrak{D}_s\mathfrak{V}^*_{t,s}\} = \partial_s\{\mathfrak{D}_s\}\mathfrak{V}^*_{t,s} + \mathfrak{D}_s\partial_s\{\mathfrak{V}^*_{t,s}\} , \tag{V.83}$$

as well as

$$\partial_s\{\mathfrak{V}_{t,s}\mathfrak{D}_s\mathfrak{V}^*_{t,s}\} = \partial_s\{\mathfrak{V}_{t,s}\}\mathfrak{D}_s\mathfrak{V}^*_{t,s} + \mathfrak{V}_{t,s}\partial_s\{\mathfrak{D}_s\mathfrak{V}^*_{t,s}\} . \tag{V.84}$$

To prove (V.83), take $s \in (0,\varkappa)$, some sufficiently small parameter $|\epsilon| > 0$, and $\varphi \in \mathcal{D}(\Omega_0)$. Then, observe that

$$\begin{aligned}&\left\|\left(\epsilon^{-1}(\mathfrak{D}_{s+\epsilon}\mathfrak{V}^*_{t,s+\epsilon} - \mathfrak{D}_s\mathfrak{V}^*_{t,s}) - \partial_s\{\mathfrak{D}_s\}\mathfrak{V}^*_{t,s} - \mathfrak{D}_s\partial_s\{\mathfrak{V}^*_{t,s}\}\right)\varphi\right\| \\ \leq\ &\left\|\left(\epsilon^{-1}(\mathfrak{D}_{s+\epsilon} - \mathfrak{D}_s) - \partial_s\mathfrak{D}_s\right)\mathfrak{V}^*_{t,s}\varphi\right\| + \left\|(\mathfrak{D}_{s+\epsilon} - \mathfrak{D}_s)\epsilon^{-1}(\mathfrak{V}^*_{t,s+\epsilon} - \mathfrak{V}^*_{t,s})\varphi\right\| \\ &+\left\|\mathfrak{D}_s\left(\epsilon^{-1}(\mathfrak{V}^*_{t,s+\epsilon} - \mathfrak{V}^*_{t,s}) - \partial_s\mathfrak{V}^*_{t,s}\right)\varphi\right\| .\end{aligned} \tag{V.85}$$

By (V.71) and (V.78), for any $s \in (0,\varkappa)$ and $\varphi \in \mathcal{D}(\Omega_0)$,

$$\lim_{\epsilon\to 0}\left\|\left(\epsilon^{-1}(\mathfrak{D}_{s+\epsilon} - \mathfrak{D}_s) - \partial_s\mathfrak{D}_s\right)\mathfrak{V}^*_{t,s}\varphi\right\| = 0 . \tag{V.86}$$

Recall meanwhile that, for every $\delta \in (0,\varkappa)$, $(\mathfrak{B}_t)_{t\in[\delta,\varkappa)}$ and $(\Omega_t)_{t\in[0,T_0)}$ are both (Lipschitz) norm continuous, by Lemma 35 and Corollary 38. Therefore, for any $s \in (0,\varkappa)$ and $\varphi \in \mathcal{D}(\Omega_0)$,

$$\lim_{\epsilon\to 0}\left\|(\mathfrak{D}_{s+\epsilon} - \mathfrak{D}_s)\epsilon^{-1}(\mathfrak{V}^*_{t,s+\epsilon} - \mathfrak{V}^*_{t,s})\varphi\right\| = 0 , \tag{V.87}$$

using also (V.67)–(V.68) and (V.77). Similar to Equation (V.80), we get from (V.79) that

$$\Omega_0\left(\mathfrak{V}^*_{t,s+\epsilon}-\mathfrak{V}^*_{t,s}\right)=-\frac{8\Omega_0}{\Omega_0+\mathbf{1}}\int_s^{s+\epsilon}\left(\Omega_0+\mathbf{1}\right)\mathfrak{B}_\tau\mathfrak{V}^*_{t,\tau}\mathrm{d}\tau\ , \tag{V.88}$$

which, by Lemma 35, implies that

$$\begin{aligned}\Omega_0\left(\mathfrak{V}^*_{t,s+\epsilon}-\mathfrak{V}^*_{t,s}\right) &= -\frac{8\Omega_0}{\Omega_0+\mathbf{1}}\int_s^{s+\epsilon}\Omega_\tau\mathfrak{B}_\tau\mathfrak{V}^*_{t,\tau}\mathrm{d}\tau\\ &\quad -\frac{8\Omega_0}{\Omega_0+\mathbf{1}}\int_s^{s+\epsilon}\left(\Delta_\tau+\mathbf{1}\right)\mathfrak{B}_\tau\mathfrak{V}^*_{t,\tau}\mathrm{d}\tau\ .\end{aligned} \tag{V.89}$$

Since, by (V.2) and (V.67),

$$\forall t\in[0,\varkappa):\quad \mathfrak{B}_t=W_t\hat{\mathfrak{B}}_t\ ,\quad \hat{\mathfrak{B}}_t:=B_0W_t^{\mathrm{t}}\left(\Omega_t^{\mathrm{t}}\right)^{-1}\left(W_t^{\mathrm{t}}\right)^*\bar{B}_0W_t^*\ , \tag{V.90}$$

note that, for any $s\in(0,\varkappa)$ and $\varphi\in\mathcal{D}(\Omega_0)$,

$$\begin{aligned}&\left\|\epsilon^{-1}\int_s^{s+\epsilon}\left(\Omega_\tau\mathfrak{B}_\tau\mathfrak{V}^*_{t,\tau}-\Omega_s\mathfrak{B}_s\mathfrak{V}^*_{t,s}\right)\varphi\mathrm{d}\tau\right\|\\ \leq\ &\epsilon^{-1}\int_s^{s+\epsilon}\left\|(\Omega_\tau W_\tau-\Omega_sW_s)\hat{\mathfrak{B}}_s\mathfrak{V}^*_{t,s}\varphi\right\|\mathrm{d}\tau\\ &+\epsilon^{-1}\int_s^{s+\epsilon}\left\|\Omega_\tau W_\tau(\hat{\mathfrak{B}}_\tau-\hat{\mathfrak{B}}_s)\mathfrak{V}^*_{t,s}\varphi\right\|\mathrm{d}\tau\ .\end{aligned} \tag{V.91}$$

Therefore, we invoke Lemmata 35–36 and Corollary 38 to deduce from (V.67), (V.77), (V.89), and (V.91) that, for any $s\in(0,\varkappa)$ and $\varphi\in\mathcal{D}(\Omega_0)$,

$$\lim_{\epsilon\to0}\left\|\mathfrak{D}_s\left(\epsilon^{-1}(\mathfrak{V}^*_{t,s+\epsilon}-\mathfrak{V}^*_{t,s})-\partial_s\mathfrak{V}^*_{t,s}\right)\varphi\right\|=0\ . \tag{V.92}$$

Thus, (V.83) results from (V.85)–(V.87) and (V.92). To prove (V.84), take again $s\in(0,\varkappa)$, some sufficiently small $|\epsilon|>0$, and $\varphi\in\mathcal{D}(\Omega_0)$, and consider the upper bound

$$\begin{aligned}&\left\|\left(\epsilon^{-1}(\mathfrak{V}_{t,s+\epsilon}\mathfrak{D}_{s+\epsilon}\mathfrak{V}^*_{t,s+\epsilon}-\mathfrak{V}_{t,s}\mathfrak{D}_s\mathfrak{V}^*_{t,s})\right.\right.\\ &\left.\left.-\partial_s\{\mathfrak{V}_{t,s}\}\mathfrak{D}_s\mathfrak{V}^*_{t,s}-\mathfrak{V}_{t,s}\partial_s\{\mathfrak{D}_s\mathfrak{V}^*_{t,s}\}\right)\varphi\right\|\\ \leq\ &\|\mathfrak{V}_{t,s}\|_{\mathrm{op}}\left\|\left(\epsilon^{-1}(\mathfrak{D}_{s+\epsilon}\mathfrak{V}^*_{t,s+\epsilon}-\mathfrak{D}_s\mathfrak{V}^*_{t,s})-\partial_s\{\mathfrak{D}_s\mathfrak{V}^*_{t,s}\}\right)\varphi\right\|\\ &+\|\mathfrak{V}_{t,s+\epsilon}-\mathfrak{V}_{t,s}\|_{\mathrm{op}}\left\|\epsilon^{-1}(\mathfrak{D}_{s+\epsilon}\mathfrak{V}^*_{t,s+\epsilon}-\mathfrak{D}_s\mathfrak{V}^*_{t,s})\varphi\right\|\\ &+\left\|\left(\epsilon^{-1}(\mathfrak{V}_{t,s+\epsilon}-\mathfrak{V}_{t,s})-\partial_s\mathfrak{V}_{t,s}\right)\mathfrak{D}_s\mathfrak{V}^*_{t,s}\varphi\right\|\ .\end{aligned} \tag{V.93}$$

In the limit $\epsilon\to0$ the three terms of the upper bound vanish because of (V.83), the norm continuity of $(\mathfrak{V}_{t,s})_{t,s\in[\epsilon,\varkappa)}\subset\mathcal{B}(\mathfrak{h})$ for $\epsilon>0$ and (V.76).
We thus obtain (V.83) and (V.84). In other words, (V.82) holds and implies the equality

$$\forall s,t\in(0,\varkappa):\qquad \mathfrak{D}_t=\mathfrak{V}_{t,s}\mathfrak{D}_s\mathfrak{V}^*_{t,s}\ . \tag{V.94}$$

Since from Lemma 35

$$\forall s \in [0, \varkappa) : \qquad \Omega_s = \Omega_0 - 16 \int_0^s B_\tau \bar{B}_\tau \mathrm{d}\tau \ , \tag{V.95}$$

we get, for any $s, t \in (0, \varkappa)$, that

$$\mathfrak{D}_t = \mathfrak{V}_{t,s} \mathfrak{D}_0 \mathfrak{V}_{t,s}^* + 4\mathfrak{V}_{t,s} \mathfrak{B}_0 \mathfrak{V}_{t,s}^* - 4\mathfrak{V}_{t,s} \mathfrak{B}_s \mathfrak{V}_{t,s}^* - 16 \int_0^s \mathfrak{V}_{t,s} B_\tau \bar{B}_\tau \mathfrak{V}_{t,s}^* \mathrm{d}\tau \ , \tag{V.96}$$

where the interchange of the Riemann integral on $[0, s]$ with $\mathfrak{V}_{t,s}$ is justified by the fact that $\mathfrak{V}_{t,s} \in \mathcal{B}(\mathfrak{h})$. As, by assumption,

$$\Omega_0 \geq 4B_0 (\Omega_0^{\mathrm{t}})^{-1} \bar{B}_0 + \mu \mathbf{1} \ , \tag{V.97}$$

i.e., $\mathfrak{D}_0 \geq \mu \mathbf{1}$, it follows that

$$\mathfrak{D}_t \geq \mu \mathfrak{V}_{t,s} \mathfrak{V}_{t,s}^* + 4\mathfrak{V}_{t,s} \mathfrak{B}_0 \mathfrak{V}_{t,s}^* - 4\mathfrak{V}_{t,s} \mathfrak{B}_s \mathfrak{V}_{t,s}^* - 16 \int_0^s \mathfrak{V}_{t,s} B_\tau \bar{B}_\tau \mathfrak{V}_{t,s}^* \mathrm{d}\tau \ , \tag{V.98}$$

for any $s, t \in (0, \varkappa)$.

We proceed by taking the limit $s \rightarrow 0^+$ in (V.98). First, for all $\varphi \in \mathfrak{h}$, $s \in [0, \varkappa)$, and $\mu > 0$,

$$\lim_{s \rightarrow 0^+} \langle \varphi | \left(\mathfrak{B}_s - \mathfrak{B}_0 \right) \varphi \rangle = 0 \ , \tag{V.99}$$

because the operator family $(\bar{B}_t)_{t \in [0, T_0]}$ is strongly continuous (Corollary 38) and for any $s \in [0, \varkappa)$,

$$\langle \varphi | \left(\mathfrak{B}_s - \mathfrak{B}_0 \right) \varphi \rangle \leq \frac{4}{\mu} \left\| B_0 \right\|_2 \left\| \left(\bar{B}_s - \bar{B}_0 \right) \varphi \right\| + \frac{64s}{\mu^2} \left\| B_0 \right\|_2^4 \ , \tag{V.100}$$

see (V.2), (V.66), and (V.95). Since the bounded operator family $(\mathfrak{V}_{t,s})_{s \in [0, \varkappa)}$ is strongly continuous, it is then easy to prove, for $\mu > 0$, that

$$\forall t \in [0, \varkappa) : \qquad \mathfrak{D}_t \geq \mu \mathfrak{V}_{t,0} \mathfrak{V}_{t,0}^* \ , \tag{V.101}$$

by passing to the limit $s \rightarrow 0^+$ in (V.98) with the help of (V.68) and (V.99). From (V.76)–(V.77) combined with $\mathfrak{B}_t = \mathfrak{B}_t^* \geq 0$, for $t \in [0, \varkappa)$, we observe that

$$\partial_s \left\{ \mathfrak{V}_{t,s} \mathfrak{V}_{t,s}^* \right\} = -16 \mathfrak{V}_{t,s} \mathfrak{B}_s \mathfrak{V}_{t,s}^* \leq 0 \ , \tag{V.102}$$

which yields in this case the inequality

$$\forall t \in [0, \varkappa) : \qquad \mathfrak{V}_{t,0} \mathfrak{V}_{t,0}^* \geq \mathbf{1} \ . \tag{V.103}$$

Therefore, we use (V.67) and (V.101), and obtain

$$\forall t \in [0, \varkappa) : \qquad \mathfrak{D}_t := \Omega_t - 4B_t \left(\Omega_t^{\mathrm{t}} \right)^{-1} \bar{B}_t \geq \mu \mathbf{1} \ , \tag{V.104}$$

provided that $\mu > 0$. In particular, for any $t \in [0, \varkappa)$ we have $\Omega_t \geq \mu \mathbf{1}$. This implies $\Omega_\varkappa \geq \mu \mathbf{1}$ by norm continuity of $(\Omega_t)_{t\in[0,T_0]}$ (Lemma 35), and hence

$$\varkappa = T_0 \ . \tag{V.105}$$

It is thus easy to see that (V.104) holds for any $t \in [0, T_0]$, i.e., for $t = \varkappa = T_0$ included. The latter proves the lemma. □

Observe that the (possibly unbounded) operator Ω_t is bounded from below as already mentioned in the proof of Lemma 36. If Condition A3 also holds then, by using Lemmata 35, 39 and Corollary 38, we prove next that Ω_t is a positive operator:

LEMMA 40 (POSITIVITY OF THE OPERATOR Ω_t FOR SMALL TIMES)
Let $\Omega_0 = \Omega_0^* \geq 0$ and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$ such that Ω_0 is invertible on RanB_0 *and $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0$. Let $(\Omega_t)_{t\in[0,T_0]}$ be the operator family of Lemma 35. Then, for all $t \in [0, T_0]$, $\Omega_t \geq 0$ is a positive operator.*

PROOF. Let us consider the general case where it is assumed that

$$\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 \ . \tag{V.106}$$

Pick an arbitrary real parameter $\mu > 0$. From Lemma 35 and Corollary 38, there is a strong solution $(\Omega_{t,\mu}, B_{t,\mu})_{t\in[0,T_0]}$ of (III.9) with initial values $\Omega_{0,\mu} := \Omega_0 + \mu\mathbf{1}$ and $B_{0,\mu} = B_0$. Note that r_0 and T_0 are defined by (V.22) and so, they do not depend on $\mu > 0$. Now, we perform the limit $\mu \to 0^+$ in order to prove that $\Omega_t \geq 0$ in all cases. In fact, by using (III.9) with initial values $\Omega_{0,\mu} = \Omega_0 + \mu\mathbf{1}$ and $B_{0,\mu} = B_0$, for $\mu \geq 0$ and $t \in [0, T_0]$, one gets that

$$\|\Omega_{t,\mu} - \Omega_t\|_{\mathrm{op}} \leq \mu + 16 \int_0^t \left\|B_{\tau,\mu}\bar{B}_{\tau,\mu} - B_\tau \bar{B}_\tau\right\|_{\mathrm{op}} \mathrm{d}\tau \ . \tag{V.107}$$

Here,

$$B_{t,\mu} := W_t(\mu) B_0 W_t^{\mathrm{t}}(\mu) \ , \tag{V.108}$$

where the evolution operator $W_t(\mu) := W_{t,0}(\mu)$ is the strong solution of (V.1) with $\Omega_{t,\mu}$ replacing Ω_t. Therefore, using (V.7) we proceed from the previous inequality to obtain

$$\|\Omega_{t,\mu} - \Omega_t\|_{\mathrm{op}} \leq \mu + 64\mathrm{e}^{6r_0T_0}\|B_0\|_2^2 \int_0^{T_0} \|W_\tau(\mu) - W_\tau\|_{\mathrm{op}} \mathrm{d}\tau \ . \tag{V.109}$$

For any $t \in [0, T_0]$, recall that $W_{t,s}$ is the unique solution of the integral equation $W = \mathcal{T}(W)$, with $\mathcal{T}$ defined by (V.6). Since

$$\left\|\mathrm{e}^{-2\alpha\Omega_0} - \mathrm{e}^{-2\alpha(\Omega_0+\mu)}\right\|_{\mathrm{op}} \leq \left|1 - \mathrm{e}^{-2\alpha\mu}\right| \tag{V.110}$$

for any $\alpha, \mu > 0$, we combine $W = \mathcal{T}(W)$ with (V.22) and the estimate (V.66) applied to $W_t(\mu)$ in order to deduce that

$$\begin{aligned}\sup_{t\in[0,T_0]} \|W_t(\mu) - W_t\|_{\mathrm{op}} \leq\ & \left|1 - \mathrm{e}^{-2T_0\mu}\right| \left(1 + 2T_0^2 \|B_0\|_2^2\right) \\ & +2T_0 \sup_{t\in[0,T_0]} \|\Omega_{t,\mu} - \Omega_t\|_{\mathrm{op}} \\ & +2r_0 T_0 \sup_{t\in[0,T_0]} \|W_t(\mu) - W_t\|_{\mathrm{op}} \ .\end{aligned} \tag{V.111}$$

By (V.22), note that $2r_0T_0 < 1/2$ and so, for any $t \in [0, T_0]$,

$$\begin{aligned}\sup_{t\in[0,T_0]} \|W_t(\mu) - W_t\|_{\mathrm{op}} \leq\ & 2\left|1 - \mathrm{e}^{-2T_0\mu}\right| \left(1 + 2T_0^2 \|B_0\|_2^2\right) \\ & +4T_0 \sup_{t\in[0,T_0]} \|\Omega_{t,\mu} - \Omega_t\|_{\mathrm{op}} \ .\end{aligned} \tag{V.112}$$

Consequently, we infer from (V.22), (V.109), and (V.112) that

$$\sup_{t\in[0,T_0]} \|\Omega_{t,\mu} - \Omega_t\|_{\mathrm{op}} \leq \mu + 2\left|1 - \mathrm{e}^{-2T_0\mu}\right| \ \|B_0\|_2 + \frac{1}{2} \sup_{t\in[0,T_0]} \|\Omega_{t,\mu} - \Omega_t\|_{\mathrm{op}} \ , \tag{V.113}$$

i.e.,

$$\sup_{t\in[0,T_0]} \|\Omega_{t,\mu} - \Omega_t\|_{\mathrm{op}} \leq 2\mu + 4\left|1 - \mathrm{e}^{-2T_0\mu}\right| \ \|B_0\|_2 \ . \tag{V.114}$$

In particular, for any $t \in [0, T_0]$,

$$\lim_{\mu\rightarrow 0^+} \left\{ \sup_{t\in[0,T_0]} \|\Omega_{t,\mu} - \Omega_t\|_{\mathrm{op}} \right\} = 0 \ , \tag{V.115}$$

which implies, for all $\varphi \in \mathfrak{h}$, that

$$\forall t \in [0, T_0] : \qquad (\varphi, \Omega_t\varphi) = \lim_{\mu\rightarrow 0^+} (\varphi, \Omega_{t,\mu}\varphi) \geq 0 \ , \tag{V.116}$$

because of Lemma 39 applied to the operator family $(\Omega_{t,\mu})_{t\in[0,T_0]}$. In other words, the operator Ω_t is positive for any $t \in [0, T_0]$. □

By Lemma 35, there is a unique solution $(\Delta_t)_{t\in[0,T_0]} \in \mathfrak{C}$ of the initial value problem (V.3) for small times and it is thus natural to define the (possibly infinite) maximal time for which such a solution exists, that is,

$$T_{\max} := \sup\left\{ T \geq 0 \ \middle|\ \exists \text{ a solution } (\Delta_t)_{t\in[0,T]} \in \mathfrak{C} \text{ of (V.3)} \right\} \in (0, \infty] \ . \tag{V.117}$$

As already mentioned, Lemma 35 gives an explicit lower bound

$$T_{\max} \geq T_0 := (128\|B_0\|_2)^{-1} > 0 \tag{V.118}$$

on this maximal time. Then, we can clearly extend Lemma 35 and Corollary 38 to all times $t \in [0, T_{\max})$:

THEOREM 41 (LOCAL EXISTENCE OF (Ω_t, B_t))
Assume $\Omega_0 = \Omega_0^* \geq 0$ *and* $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$*. Then there exists an operator family* $(\Omega_t, B_t)_{t\in[0,T_{\max})}$ *satisfying:*
(i) $(\Omega_t)_{t\in[0,T_{\max})}$ *is a family of self–adjoint operators with all the same domain* $\mathcal{D}(\Omega_0)$*. Moreover, it satisfies the initial value problem*

$$\forall t \in [0, T_{\max}) : \qquad \partial_t \Omega_t = -16 B_t \bar{B}_t \ , \quad \Omega_{t=0} = \Omega_0 \ , \tag{V.119}$$

in the strong topology and it is Lipschitz continuous in the norm topology on any compact set $[0, T]$ *with* $T \in (0, T_{\max})$*.*
(ii) $(B_t)_{t\in[0,T_{\max})} = (B_t^{\mathrm{t}})_{t\in[0,T_{\max})}$*, defined by (V.2) and (V.5), is a family of Hilbert–Schmidt operators. It is strongly continuous and*

$$\forall t \in [0, T_{\max}) : \qquad B_t = B_0 - 2\int_0^t \left(\Omega_\tau B_\tau + B_\tau \Omega_\tau^{\mathrm{t}}\right) \mathrm{d}\tau \ , \tag{V.120}$$

with $B_{t>0}\mathfrak{h} \subseteq \mathcal{D}(\Omega_0)$*. Furthermore,* $(B_t)_{t\in(0,T_{\max})}$ *is locally Lipschitz norm continuous.*

PROOF. (i) The first assertion trivially follows from the definition (V.117) of $T_{\max}$ as $\Omega_t := \Omega_0 - \Delta_t$ for all $t \in [0, T_{\max})$. Only the Lipschitz continuity of $(\Omega_t)_{t\in[0,T_0]}$ must be proven. In fact, for any $T \in [0, T_{\max})$,

$$r = \|\Delta\|_\infty := \sup_{t\in[0,T]} \|\Delta_t\|_{\mathrm{op}} < \infty \tag{V.121}$$

is obviously finite as $(\Delta_t)_{t\in[0,T]} \in \mathfrak{C}$. Therefore, $\mathcal{D}(\Omega_t) = \mathcal{D}(\Omega_0)$ and, thanks to (V.7),

$$\|\Omega_t - \Omega_s\|_{\mathrm{op}} \leq 16\|B_0\|_2^2 \mathrm{e}^{8r(t-s)}|t-s| \tag{V.122}$$

for any $t, s \in [0, T]$ and $T \in [0, T_{\max})$, i.e., the operator family $(\Omega_t)_{t\in[0,T_{\max})}$ is Lipschitz continuous on $[0, T]$.
(ii) The second assertion results from Lemma 36 which can clearly be extended to all $t, s \in [0, T]$ and $T \in [0, T_{\max})$ because of (V.122). Note that $B_t \in \mathcal{L}^2(\mathfrak{h})$ is clearly a Hilbert–Schmidt operator as

$$\|B_t\|_2 \leq \mathrm{e}^{8rt}\|B_0\|_2 < \infty \tag{V.123}$$

for any $t \in [0, T]$ and $T \in [0, T_{\max})$, because of (V.2), (V.7), and (V.121). □
This theorem says nothing about the uniqueness of the solutions $(\Omega_t)_{t\in[0,T_{\max})}$ and $(B_t)_{t\in[0,T_{\max})}$. In our next lemma, we show a partial result, that is, the existence of a unique solution $(\Delta_t)_{t\in[0,T_{\max})}$ of (V.3):

LEMMA 42 (UNIQUENESS OF THE OPERATOR FAMILY $(\Omega_t)_{t\in[0,T_{\max})}$)
Assume $\Omega_0 = \Omega_0^* \geq 0$ *and* $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$*. Then there exists a unique operator family* $(\Omega_t)_{t\in[0,T_{\max})}$*, with all the same domain* $\mathcal{D}(\Omega_0)$*, solving in the strong topology the initial value problem*

$$\forall t \in [0, T_{\max}) : \qquad \partial_t \Omega_t = -16 B_t \bar{B}_t \ , \quad \Omega_{t=0} = \Omega_0 \ , \tag{V.124}$$

where $(B_t)_{t\in[0,T_{\max})} \subset \mathcal{L}^2(\mathfrak{h})$ *is defined by (V.2) and (V.5).*

PROOF. For any $T \in (0, T_{\max})$, let $\Delta^{(1)}, \Delta^{(2)} \in \mathfrak{C}$ be two solutions of the initial value problem (V.3), which also define via (V.5) two evolution operators $W_t^{(1)} := W_{t,0}^{(1)}$ and $W_t^{(2)} := W_{t,0}^{(2)}$. Pick

$$r = \sup \left\{ \|\Delta^{(1)}\|_\infty, \|\Delta^{(2)}\|_\infty \right\} < \infty \ . \tag{V.125}$$

If

$$0 < \delta \leq \min \left\{ \frac{1}{6r}, \frac{1}{50\|B_0\|_2}, T_{\max} \right\} \ , \tag{V.126}$$

then we do similar estimates as those used to prove (V.109) and (V.112), and deduce the inequality

$$\begin{aligned} \sup_{t\in[0,\delta]} \|\Delta^{(1)} - \Delta^{(2)}\|_{\mathrm{op}} &\leq \frac{128\mathrm{e}^{6r\delta}}{(1-2r\delta)} \delta^2 \|B_0\|_2^2 \sup_{t\in[0,\delta]} \|\Delta^{(1)} - \Delta^{(2)}\|_{\mathrm{op}} \\ &< \frac{1}{2} \sup_{t\in[0,\delta]} \|\Delta^{(1)} - \Delta^{(2)}\|_{\mathrm{op}} \ , \end{aligned} \tag{V.127}$$

i.e., $\Delta_t^{(1)} = \Delta_t^{(2)}$ for all $t \in [0, \delta]$. Now, we can shift the starting point from $t = 0$ to any fixed $x \in (0, \delta]$ to deduce that $\Delta_t^{(1)} = \Delta_t^{(2)}$ for any $t \in [x, x + \delta] \cap [0, T]$. In other words, by recursively using these arguments, there is a unique solution $(\Delta_t)_{t\in[0,T]}$ in $\mathfrak{C}$ of the initial value problem (V.3) for any $T \in (0, T_{\max})$. This concludes the proof of the lemma as $\Omega_t := \Omega_0 - \Delta_t$ for all $t \in [0, T_{\max})$. □

To diagonalize quadratic boson operators, we need the existence of the operator family $(\Omega_t, B_t)_{t\geq 0}$ for all times, i.e., we want to prove that $T_{\max} = \infty$ is infinite, see (V.117). Unfortunately, *a blow–up is generally not excluded*, that is, the Hilbert–Schmidt norm $\|B_t\|_2$ may diverge in a finite time. Indeed, if $T_{\max} < \infty$ then the continuous map $t \mapsto \|B_t\|_2$ is unbounded on the set $[0, T_{\max})$:

LEMMA 43 (THE BLOW–UP ALTERNATIVE)
Assume $\Omega_0 = \Omega_0^* \geq 0$ and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$. Then, either $T_{\max} = \infty$ and we have a global solution $(\Omega_t)_{t\geq 0}$ of (V.124) or $T_{\max} < \infty$ and

$$\lim_{t\nearrow T_{\max}} \|B_t\|_2 = \infty \ . \tag{V.128}$$

PROOF. By contradiction, assume the finiteness of $T_{\max} < \infty$ and the existence of a sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0, T_{\max})$ converging to $T_{\max}$ such that

$$\kappa := \sup_{n\in\mathbb{N}} \|B_{t_n}\|_2 < \infty \ . \tag{V.129}$$

Consider the integral equation

$$\mathfrak{T}_{t_n}(X)_t = \{\mathfrak{T}_{t_n}(X)\}_t^* := 16 \int_{t_n}^t W_{\tau,t_n} B_{t_n}^{\mathrm{t}} W_{\tau,t_n}^{\mathrm{t}} \left(W_{\tau,t_n}^{\mathrm{t}}\right)^* \bar{B}_{t_n} W_{\tau,t_n}^* \mathrm{d}\tau \tag{V.130}$$

for $t \geq t_n$, where the evolution operator W_{t,t_n} is defined by (V.5) with $(X_t)_{t\in[t_n,t_n+T]} \in \mathfrak{C}$ replacing $(\Delta_t)_{t\in[0,T]}$. Obviously, the proof of Lemma 35 can also be performed for any starting point $t_n \in [0, T_{\max})$ and by (V.129), there is a solution $(X_t)_{t\in[t_n,t_n+T]} \in \mathbf{B}_r(0)$ of $\mathfrak{T}_{t_n}(X) = X$, where the real positive parameters

$$r = \sqrt{32}\kappa > 0 \qquad \text{and} \qquad T = (128\kappa)^{-1} > 0 \tag{V.131}$$

do not depend on $n \in \mathbb{N}$. Using the cocycle property (Lemma 34 (i)) as well as (V.2), one can thus check that

$$\forall t \in [t_n, t_n + T] : \qquad X_t + \Delta_{t_n} = \mathfrak{T}(X + \Delta_{t_n})_t \tag{V.132}$$

for any $n \in \mathbb{N}$, whereas we have the equality

$$\forall t \in [t_n, t_n + T] \cap [0, T_{\max}) : \qquad \Delta_t = X_t + \Delta_{t_n} \ , \tag{V.133}$$

by uniqueness of the solution of $\Delta = \mathfrak{T}(\Delta)$ for any $t \in [0, T_{\max})$, see Lemma 42. Because of (V.117), these two last assertions cannot hold when $t_n + T > T_{\max}$ for any sufficiently large $n \in \mathbb{N}$. We thus conclude that either $T_{\max} = \infty$ or $\kappa = \infty$. □

A sufficient condition to prevent from having a blow–up is given by the gap condition, i.e., Condition A6. Indeed, we can in this case extend the domain of existence of the operator family $(\Omega_t, B_t)_{t\in[0,T_0]}$ to any time $t \geq 0$.

LEMMA 44 (GLOBAL EXISTENCE OF (Ω_t, B_t) UNDER THE GAP CONDITION) *Let $\Omega_0 = \Omega_0^* \geq 0$ and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$ such that Ω_0 is invertible on* $\mathrm{Ran}B_0$ *and $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 + \mu\mathbf{1}$, for some $\mu > 0$. Then $T_{\max} = \infty$ and*

$$\forall t \geq 0 : \qquad \Omega_t \geq 4B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t + \mu\mathbf{1} \ . \tag{V.134}$$

PROOF. The arguments from (V.64) to (V.105) show that the gap equation holds on $[0, T_{\max})$:

$$\forall t \in [0, T_{\max}) : \qquad \Omega_t := \Omega_0 - \Delta_t \geq 4B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t + \mu\mathbf{1} \ , \tag{V.135}$$

In particular, $\Omega_t \geq 0$ is a positive operator for all $t \in [0, T_{\max})$. This property yields (V.66) from which we infer the global estimate

$$\forall t \in [0, T_{\max}) : \qquad \|B_t\|_2 \leq \|B_0\|_2 < \infty \ , \tag{V.136}$$

see (V.2). Using Lemma 43 we deduce that $T_{\max} = \infty$. □

If $\Omega_0 \geq 0$ does not satisfy the gap equation then the absence of a *blow–up* is far from being clear. Another weaker, sufficient condition to prevent from having this behavior is given by Condition A4, that is, $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$ is a Hilbert–Schmidt operator. However, this proof is rather non–trivial and we need an *a priori estimate* to control the Hilbert–Schmidt norm $\|B_t\|_2$. To this end, assume Conditions A1–A3, define the time

$$T_+ := \sup\left\{T \in [0, T_{\max}) \;\middle|\; \forall t \in [0,T]: \quad \Omega_t \geq 0\right\} \in (0, \infty] \tag{V.137}$$

and note that

$$T_+ \geq T_0 := (128\|B_0\|_2)^{-1} > 0 \ , \tag{V.138}$$

by Lemma 40. Furthermore, we observe that (V.66) is clearly satisfied for any $t \in [0, T_+)$, that is,

$$\forall t \in [0, T_+): \qquad \max\left\{ \|W_{t,s}\|_{\mathrm{op}}, \left\|W_{t,s}^{\mathrm{t}}\right\|_{\mathrm{op}} \right\} \leq 1 \ , \tag{V.139}$$

which in turn implies the a priori estimate

$$\forall s \in [0, T_+), \ t \in [s, T_+): \qquad \|B_t\|_2 \leq \|B_s\|_2 \ , \tag{V.140}$$

see (V.2). We now proceed by proving few crucial properties which hold for all $t \in [0, T_+)$.
First, we aim at establishing the differential equation

$$\forall t \in (0, T_+): \qquad \partial_t B_t = -2\left(\Omega_t B_t + B_t \Omega_t^{\mathrm{t}}\right) \ , \quad B_{t=0} := B_0 \ , \tag{V.141}$$

to hold in the Hilbert–Schmidt topology. This fact is important for our study on the boson Fock space in Section VI. To show this, we need to prove that both $\Omega_t B_t$ and $B_t \Omega_t^{\mathrm{t}}$ are Hilbert–Schmidt operators. This is a direct consequence of (V.2) and Lemma 36 (i):

LEMMA 45 ($\Omega_t B_t$ AND $B_t \Omega_t^{\mathrm{t}}$ AS HILBERT–SCHMIDT OPERATORS)
Let $\Omega_0 = \Omega_0^* \geq 0$ and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$ such that Ω_0 is invertible on $\mathrm{Ran}B_0$ *and $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0$. Then, for any $t \in (0, T_+)$, both $\Omega_t B_t$ and $B_t \Omega_t^{\mathrm{t}}$ are Hilbert–Schmidt operators and $\|\Omega_t B_t\|_2 = \|B_t \Omega_t^{\mathrm{t}}\|_2 < \infty$.*

PROOF. By using (V.2), (V.139), Lemma 36 (i) extended to all $t, s \in [0, T]$ for any $T \in [0, T_{\max})$, and the cyclicity of the trace (cf. Lemma 101 (i)), we deduce that

$$\forall t \in (0, T_+): \quad \|\Omega_t B_t\|_2 \leq \|\Omega_t W_t\|_{\mathrm{op}} \|B_0\|_2 \leq \tilde{\mathrm{C}}_0 + \frac{\tilde{\mathrm{D}}_0}{t} < \infty \tag{V.142}$$

with $\tilde{\mathrm{C}}_0, \tilde{\mathrm{D}}_0 < \infty$. Moreover, note that

$$\mathrm{tr}(B_t^* \Omega_t^2 B_t) = \mathrm{tr}(\bar{B}_t \Omega_t^2 B_t) = \mathrm{tr}(B_t (\Omega_t^{\mathrm{t}})^2 \bar{B}_t) = \mathrm{tr}(B_t \left(\Omega_t^{\mathrm{t}}\right)^2 B_t^*) < \infty \ , \tag{V.143}$$

because of Lemma 102 and $B_t^{\mathrm{t}} = B_t$. From the cyclicity of the trace (Lemma 101 (iii)), it follows that

$$\forall t \in (0, T_+): \qquad \|B_t \Omega_t^{\mathrm{t}}\|_2 = \|\Omega_t^{\mathrm{t}} B_t^*\|_2 = \|\Omega_t B_t\|_2 < \infty \ . \tag{V.144}$$

□

As a consequence, since $B_t \in \mathcal{L}^2(\mathfrak{h})$ is Hilbert–Schmidt, one can use the isomorphism $\mathcal{J}$, defined in Section VII.3, between $\mathcal{L}^2(\mathfrak{h})$ and $\mathfrak{h}\otimes\mathfrak{h}^*$. In this space, $(\Omega_t B_t + B_t \Omega_t^{\mathrm{t}})_{t>0}$ is seen as vectors of $\mathcal{L}^2(\mathfrak{h})$, instead of operators acting on the one–particle Hilbert space $\mathfrak{h}$. In particular, in the Hilbert space $\mathfrak{h}\otimes\mathfrak{h}^*$,

$$\mathcal{J}\left(\Omega_t X + X\Omega_t^{\mathrm{t}}\right) = \mathbf{\Omega}_t \mathcal{J}(X) \ , \tag{V.145}$$

provided $\mathcal{J}(X) \in \mathcal{D}(\mathbf{\Omega}_t)$, where, for all $t \in (0, T_+)$,

$$\mathbf{\Omega}_t := \Omega_t \otimes \mathbf{1} + \mathbf{1} \otimes \Omega_t^{\mathrm{t}} \ . \tag{V.146}$$

The differential equation (V.141) is, in $\mathcal{L}^2(\mathfrak{h})$ or $\mathfrak{h}\otimes\mathfrak{h}^*$, a non–autonomous *parabolic* evolution equation as

$$\forall t \in (0,T_+): \qquad \partial_t\{\mathcal{J}(B_t)\} = -2\mathbf{\Omega}_t \mathcal{J}(B_t) \ , \quad \mathcal{J}(B_{t=0}) := \mathcal{J}(B_0) \ . \tag{V.147}$$

Therefore, to establish (V.141) in $\mathcal{L}^2(\mathfrak{h})$, or (V.147) in $\mathfrak{h}\otimes\mathfrak{h}^*$, we proceed by strengthening the regularity properties satisfied by the operators Ω_t and B_t:

Theorem 46 (Continuity properties of Ω_t and B_t)
Let $\Omega_0 = \Omega_0^* \geq 0$ and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$ such that Ω_0 is invertible on $\mathrm{Ran}B_0$ *and $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0$.*
(i) *The operator family $(\Omega_t)_{t\in[0,T_+)}$ is globally Lipschitz continuous in $\mathcal{L}^1(\mathfrak{h})$ and $\mathcal{L}^2(\mathfrak{h})$.*
(ii) *The Hilbert–Schmidt operator family $(B_t)_{t\in[0,T_+)}$ is continuous in $\mathcal{L}^2(\mathfrak{h})$ and even locally Lipschitz continuous in $\mathcal{L}^2(\mathfrak{h})$ on $(0,T_+)$.*
(iii) *The Hilbert–Schmidt operator families $(\Omega_t B_t)_{t\in(0,T_+)}$ and $(B_t\Omega_t^{\mathrm{t}})_{t\in(0,T_+)}$ are continuous in $\mathcal{L}^2(\mathfrak{h})$.*

Proof. (i): For all $s \in [0,T_+)$ and $t \in [s,T_+)$,

$$\|\Omega_t - \Omega_s\|_2 \leq 16\,(t-s)\,\|B_0\|_2^2 \ . \tag{V.148}$$

The latter means that $(\Omega_t)_{t\geq 0}$ is globally Lipschitz continuous in the Hilbert–Schmidt topology. The same trivially holds in $\mathcal{L}^1(\mathfrak{h})$.
(ii): By using (V.2), that is,

$$B_t := W_t B_0 W_t^{\mathrm{t}} = W_{t,s} B_s W_{t,s}^{\mathrm{t}} \ , \tag{V.149}$$

we have, for all $s \in [0,T_+)$ and $t \in [s,T_+)$ that

$$\|B_t - B_s\|_2 \leq \|\left(W_{t,s} - \mathbf{1}\right) B_s\|_2 + \|W_{t,s}B_s\left(W_{t,s}^{\mathrm{t}} - \mathbf{1}\right)\|_2 \ . \tag{V.150}$$

Therefore, we need to analyze both terms of this upper bound. Starting with the first one we observe that

$$\|\left(W_{t,s} - \mathbf{1}\right) B_s\|_2^2 = \mathrm{tr}(\bar{B}_s \left(W_{t,s} - \mathbf{1}\right)^* \left(W_{t,s} - \mathbf{1}\right) B_s) = \sum_{n=1}^{\infty} h_{t,s}(n) \ , \tag{V.151}$$

where

$$h_{t,s}(n) := \| (W_{t,s} - \mathbf{1}) B_s \varphi_n \|^2 \tag{V.152}$$

and $\{\varphi_n\}_{n=1}^{\infty} \subset \mathfrak{h}$ is any orthonormal basis of the Hilbert space $\mathfrak{h}$. Since, thanks to (V.139), $\|W_{t,s}\|_{\mathrm{op}} \leq 1$, the coefficient $h_{t,s}(n)$ is uniformly bounded in t by

$$h_{t,s}(n) \leq 4\|B_s \varphi_n\|^2 \tag{V.153}$$

and, as $B_s \in \mathcal{L}^2(\mathfrak{h})$ is a Hilbert–Schmidt operator,

$$\sum_{n=1}^{\infty} 4\|B_s \varphi_n\|^2 = 4\|B_s\|_2^2 < \infty \ . \tag{V.154}$$

Therefore, by Lebesgue's dominated convergence theorem, we obtain that

$$\lim_{t \to s^+} \|(W_{t,s} - \mathbf{1}) B_s\|_2^2 = \sum_{n=1}^{\infty} \left(\lim_{t \to s^+} h_{t,s}(n) \right) \ , \tag{V.155}$$

provided $h_{t,s}(n)$ has a limit for all $n \in \mathbb{N}$, as $t \to s^+$. By Lemma 34 (ii), the operator $W_{t,s}$ is strongly continuous in t with $W_{s,s} = \mathbf{1}$. Hence, for all $n \in \mathbb{N}$,

$$\lim_{t \to s^+} h_{t,s}(n) = 0 \ , \tag{V.156}$$

which, together with (V.155), implies that

$$\lim_{t \to s^+} \| (W_{t,s} - \mathbf{1}) B_s\|_2^2 = 0 \ . \tag{V.157}$$

The limit

$$\lim_{t \to s^+} \|W_{t,s} B_s \left(W_{t,s}^{\mathrm{t}} - \mathbf{1}\right) \|_2^2 = 0 \tag{V.158}$$

is obtained in the same way as (V.157) because, using $\|W_{t,s}\|_{\mathrm{op}} \leq 1$ and the cyclicity of the trace (Lemma 101 (i)),

$$\|W_{t,s} B_s (W_{t,s}^{\mathrm{t}} - \mathbf{1})\|_2^2 \leq \|((W_{t,s}^{\mathrm{t}})^* - \mathbf{1}) \bar{B}_s\|_2^2 \tag{V.159}$$

and $\left(W_{t,s}^{\mathrm{t}}\right)^*$ is also strongly continuous in $t \geq s \geq 0$ with $\left(W_{s,s}^{\mathrm{t}}\right)^* = \mathbf{1}$ and $\|W_{t,s}^{\mathrm{t}}\|_{\mathrm{op}} \leq 1$.

Consequently, the limits (V.157) and (V.158), together with the upper bound (V.150), yield the right continuity in $\mathcal{L}^2(\mathfrak{h})$ of the Hilbert–Schmidt operator family $(B_t)_{t \in [0,T_+)}$. Furthermore, since the evolution operators $W_{t,s}$ and $W_{t,s}^{\mathrm{t}}$ are also strongly continuous in $s \leq t$, the left continuity in $\mathcal{L}^2(\mathfrak{h})$ of $(B_t)_{t \in [0,T_+)}$ is verified in the same way. In other words, the Hilbert–Schmidt operator family $(B_t)_{t \in [0,T_+)}$ is continuous in $\mathcal{L}^2(\mathfrak{h})$.

We prove now that B_t is locally Lipschitz continuous in $\mathcal{L}^2(\mathfrak{h})$ for $t \in (0, T_+)$. Using the integral equation of Theorem 41 (ii) we directly derive the inequality

$$\|B_t - B_s\|_2 \leq 2 \int_s^t \left(\|\Omega_\tau B_\tau\|_2 + \|B_t \Omega_t^{\mathrm{t}}\|_2 \right) \mathrm{d}\tau \ . \tag{V.160}$$

From Lemma 45 both $\Omega_t B_t$ and $B_t \Omega_t^{\mathrm{t}}$ are Hilbert–Schmidt operators for any $t \in (0, T_+)$ and so, the operator family $(B_t)_{t\in(0,T_+)}$ is locally Lipschitz continuous in the Hilbert–Schmidt topology.

(iii): Finally, we can again use (V.149) to obtain that

$$\begin{aligned} \|\Omega_t B_t - \Omega_s B_s\|_2 \quad &\leq \quad \|\Omega_t W_t\|_{\mathrm{op}} \|B_0 W_s^{\mathrm{t}} \left(W_{t,s}^{\mathrm{t}} - \mathbf{1}\right)\|_2 \\ &\quad + \| \left(\Omega_t W_t - \Omega_s W_s\right) B_0 W_s^{\mathrm{t}} \|_2 \ , \end{aligned} \tag{V.161}$$

for all $s \in [0, T_+)$ and $t \in [s, T_+)$. By Lemma 36 (i) extended to all $t, s \in [0, T]$ for any $T \in [0, T_{\max})$, the bounded operator family $(\Omega_t W_t)_{t>0}$ is strongly continuous. Therefore, similar to (V.157) and (V.158), we can use Lebesgue's dominated convergence theorem to deduce that, for all $s \in (0, T_+)$,

$$\lim_{t\to s^+} \| \left(\Omega_t W_t - \Omega_s W_s\right) B_0 W_s^{\mathrm{t}} \|_2 = 0 \tag{V.162}$$

whereas

$$\lim_{t\to s^+} \left\{ \|\Omega_t W_t\|_{\mathrm{op}} \|B_0 W_s^{\mathrm{t}} \left(W_{t,s}^{\mathrm{t}} - \mathbf{1}\right)\|_2 \right\} = 0 \ . \tag{V.163}$$

Therefore, by (V.161), we arrive at the right continuity in $\mathcal{L}^2(\mathfrak{h})$ of the Hilbert–Schmidt operator family $(\Omega_t B_t)_{t\in(0,T_+)}$. The left continuity in $\mathcal{L}^2(\mathfrak{h})$ of $(\Omega_t B_t)_{t\in(0,T_+)}$ is proven in the same way. Furthermore, using the cyclicity of the trace (Lemma 101), Lemma 102, $B_t^{\mathrm{t}} = B_t$, and Lemma 45, one shows that

$$\begin{aligned} \|B_t\Omega_t^{\mathrm{t}} - B_s\Omega_s^{\mathrm{t}}\|_2^2 \quad &= \quad \|B_t\Omega_t^{\mathrm{t}}\|_2^2 + \|B_s\Omega_s^{\mathrm{t}}\|_2^2 - \mathrm{tr}(\Omega_s^{\mathrm{t}}\bar{B}_s B_t \Omega_t^{\mathrm{t}}) - \mathrm{tr}(\Omega_t^{\mathrm{t}}\bar{B}_t B_s \Omega_s^{\mathrm{t}}) \\ &= \quad \|\Omega_t B_t\|_2^2 + \|\Omega_s B_s\|_2^2 - \mathrm{tr}(\Omega_t B_t \bar{B}_s \Omega_s) - \mathrm{tr}(\Omega_s B_s \bar{B}_t \Omega_t) \\ &= \quad \|\Omega_t B_t\|_2^2 + \|\Omega_s B_s\|_2^2 - \mathrm{tr}(\bar{B}_s \Omega_s \Omega_t B_t) - \mathrm{tr}(\bar{B}_t \Omega_t \Omega_s B_s) \\ &= \quad \|\Omega_t B_t - \Omega_s B_s\|_2^2 \end{aligned} \tag{V.164}$$

for any $s \in (0, T_+)$ and $t \in [s, T_+)$. In other words, the family $(B_t\Omega_t^{\mathrm{t}})_{t\in(0,T_+)}$ of Hilbert–Schmidt operators is also continuous in $\mathcal{L}^2(\mathfrak{h})$. □

As a consequence, the differential equation (V.141) holds true in the Hilbert–Schmidt topology:

COROLLARY 47 (WELL–POSEDNESS OF THE FLOW ON $\mathcal{L}^2(\mathfrak{h})$)
Let $\Omega_0 = \Omega_0^* \geq 0$ and $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$ such that Ω_0 is invertible on $\mathrm{Ran} B_0$ *and $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0$. Then the operator family $(\Omega_t - \Omega_0, B_t)_{t\in[0,T_+)} \in C[[0,T_+); \mathcal{L}^2(\mathfrak{h}) \oplus \mathcal{L}^2(\mathfrak{h})]$ is the unique solution of the system of differential equations*

$$\left\{ \begin{array}{lcll} \forall t \in [0,T_+) & : & \partial_t \Omega_t = -16 B_t \bar{B}_t & , \quad \Omega_{t=0} = \Omega_0 \ , \\ \forall t \in (0,T_+) & : & \partial_t B_t = -2\left(\Omega_t B_t + B_t \Omega_t^{\mathrm{t}}\right) & , \quad B_{t=0} = B_0 \ , \end{array} \right. \tag{V.165}$$

both in $\mathcal{L}^2(\mathfrak{h})$, i.e., in the Hilbert–Schmidt topology.

PROOF. We use Theorem 41 and the a priori estimate (V.140) to deduce, for all $t \in [0, T_+)$ and $\delta \in [-t, T_+ - t)$, the two inequalities

$$\left\|\delta^{-1}\left\{\Omega_{t+\delta} - \Omega_t\right\} + 16B_t\bar{B}_t\right\|_2 \leq 32\delta^{-1}\int_t^{t+\delta} \|B_t - B_\tau\|_2 \|B_t\|_2 \, \mathrm{d}\tau \tag{V.166}$$

and

$$\left\|\delta^{-1}\left\{B_{t+\delta} - B_t\right\} + 2\Omega_t B_t + 2B_t\Omega_t^{\mathrm{t}}\right\|_2 \tag{V.167}$$
$$\leq 2\delta^{-1}\int_t^{t+\delta} \left\{\|\Omega_t B_t - \Omega_\tau B_\tau\|_2 + \left\|B_t\Omega_t^{\mathrm{t}} - B_\tau\Omega_\tau^{\mathrm{t}}\right\|_2\right\} \mathrm{d}\tau \ .$$

Therefore, from (V.140) and Theorem 46 (ii)–(iii) combined with the inequalities (V.166)–(V.167) in the limit $\delta \to 0$, we conclude that $(\Omega_t - \Omega_0, B_t)_{t\in[0,T_+)}$ is a solution of the differential equations stated in the corollary.
Uniqueness of the family $(\Omega_t, B_t)_{t\in[0,T_+)}$ defined in Theorem 41 is then a direct corollary of Lemmata 42 and 100: As soon as B_t is defined by (V.2), the operator Ω_t is unique, see Lemma 42. Uniqueness of $B_t \in \mathcal{L}^2(\mathfrak{h})$ solution of the integral equation of Theorem 41 (ii) directly follows from Lemma 100 for $T = T_+$, $\alpha = 2$, $Z_0 = \Omega_0 = Z_0^*$, and

$$Q_t = -16\int_0^t B_\tau\bar{B}_\tau \mathrm{d}\tau = Q_t^* \ . \tag{V.168}$$

Indeed, the self–adjoint operator $Q_t \in \mathcal{B}(\mathfrak{h})$ is bounded for any $t \in [0, T_+)$, as the a priori estimate (V.140) yields

$$\|Q_t\|_2 \leq 16t\|B_0\|_2 < \infty \tag{V.169}$$

for all $t \in [0, T_+)$. Note that the uniqueness can also be directly deduced from Corollary 38 extended to $[0, T_+)$.
Therefore, assume that some family $(\tilde{\Omega}_t - \Omega_0, \tilde{B}_t)_{t\in[0,T_+)} \in C[[0, T_+); \mathcal{L}^2(\mathfrak{h}) \oplus \mathcal{L}^2(\mathfrak{h})]$ solves (V.165). Then, $\tilde{B}_t$ is equal to (V.2) with $\tilde{\Omega}_t$ replacing Ω_t, by Lemma 100. As a consequence, $(\tilde{\Omega}_t - \Omega_0)_{t\in[0,T_+)} \in C[[0, T_+); \mathcal{L}^2(\mathfrak{h})]$ solves (V.3) and $\tilde{\Omega}_t = \Omega_t$ for all $t \in [0, T_+)$, which in turn implies $\tilde{B}_t = B_t$ for all $t \in [0, T_+)$. □
We now come back to the problem of the global existence of $(\Omega_t, B_t)_{t\geq 0}$. We would like to prevent from having any blow–up, which means that $T_{\max} = \infty$, see (V.117). In the proof of Lemma 44, the gap condition, i.e.,

$$\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 + \mu\mathbf{1} \ , \quad \mu > 0 \ , \tag{V.170}$$

was crucial and is preserved for all times $t \in [0, T_{\max})$ in order to prove next that $T_{\max} = \infty$. However, when $\mu = 0$, the inequality

$$\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 \tag{V.171}$$

is not necessarily preserved for all times $t \in [0, T_{\max})$.

REMARK 48 *From Theorem 52 (i) combined with* $\Omega_t^{\mathrm{t}} \geq 0$*,*

$$\Omega_0^2 \geq 8B_0\bar{B}_0 \Longrightarrow \Omega_t^2 \geq 8B_t\bar{B}_t \ , \tag{V.172}$$

which implies $\Omega_t \geq 8B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t$ *for all* $t \in [0, T_+)$*. However, this proof of the first statement (i) of Theorem 12 only works for this special example. Note that one can also prove in this specific case that* $\Omega_t \geq \frac{\Omega_0}{2t\Omega_0 + \mathbf{1}}$ *for any* $t \in [0, T_+)$*.*

At first sight, one could use similar arguments performed from (V.71) to (V.104) for $\mu = 0$. Indeed, these arguments, which use the system (III.9), suggest that

$$\mathfrak{D}_t := \Omega_t - 4\mathfrak{B}_t = \mathfrak{V}_t\mathfrak{D}_0\mathfrak{V}_t^* \geq 0 \ , \tag{V.173}$$

where

$$\mathfrak{B}_t := B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t = \mathfrak{B}_t^* \geq 0 \tag{V.174}$$

and $\mathfrak{V}_t := \mathfrak{V}_{t,0}$ is the strong solution of the non–autonomous evolution equation (V.72), that is,

$$\forall s, t \in [0, T_+) : \qquad \partial_t \mathfrak{V}_{t,s} = 8\mathfrak{B}_t\mathfrak{V}_{t,s} \ , \quad \mathfrak{V}_{s,s} := \mathbf{1} \ . \tag{V.175}$$

Unfortunately, since $(\Omega_t^{\mathrm{t}})^{-1}$ is possibly unbounded, it is not clear that the evolution operator $\mathfrak{V}_{t,s}$ is well–defined. For instance, the boundedness of $\mathfrak{B}_0 \in \mathcal{B}(\mathfrak{h})$ does not imply, a priori, the boundedness of the operator $\mathfrak{B}_t$ for all times $t \in [0, T_+)$. However, if $\mathfrak{B}_0 \in \mathcal{L}^1(\mathfrak{h})$ is not only bounded but also trace–class, i.e., $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$ (Condition A4), then we show below that the operator $\mathfrak{B}_t \in \mathcal{L}^1(\mathfrak{h})$ stays trace–class for all times $t \in [0, T_+)$ and Inequality (V.173) can thus be justified.

LEMMA 49 ($\mathfrak{B}_t \in \mathcal{L}^1(\mathfrak{h})$ AND POSITIVITY OF $\mathfrak{D}_t \geq 0$)
Assume $\Omega_0 = \Omega_0^* \geq 0$*,* $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$*,* $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0$*,* $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$*, and let* $(\Omega_t, B_t)_{t\in[0,T_+)}$ *be the solution of (III.9).*
(i) *The map* $t \mapsto \|\mathfrak{B}_t\|_1$ *is monotonically decreasing. In particular,* $\|\mathfrak{B}_t\|_1 \leq \|\mathfrak{B}_0\|_1$ *for all* $t \in [0, T_+)$*.*
(ii) *The operator* $\mathfrak{D}_t \geq 0$ *is positive for all* $t \in [0, T_+)$*.*

PROOF. First, observe that $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$ implies that $(\Omega_0^{\mathrm{t}})^{-1/2}\bar{B}_0 \in \mathcal{L}^2(\mathfrak{h})$, by Lemma 102 and $B_0 = B_0^{\mathrm{t}}$. In particular, $\mathfrak{B}_0 \in \mathcal{L}^1(\mathfrak{h})$ as

$$\mathfrak{b}_0 := \|\mathfrak{B}_0\|_1 = \|\Omega_0^{-1/2}B_0\|_2^2 < \infty \ . \tag{V.176}$$

Let $\mu > 0$, $T \in (0, T_+)$, and set

$$\forall t \in [0, T] : \qquad \mathfrak{B}_{t,\mu} := B_t\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-1}\bar{B}_t \geq 0 \ . \tag{V.177}$$

It is a bounded operator for all $t \in [0, T_+)$ as $\mu > 0$ and $\Omega_t^{\mathrm{t}} \geq 0$ is a positive operator by definition of $T_+ > 0$, see (V.137). In fact, $\mathfrak{B}_{t,\mu} \in \mathcal{L}^1(\mathfrak{h})$ is trace–class. Therefore, we introduce the function q_μ defined on $[0, T_+)$ by

$$q_\mu(t) := \mathrm{tr}\{\mathfrak{B}_{t,\mu}\} = \|\mathfrak{B}_{t,\mu}\|_1 \ . \tag{V.178}$$

Using Corollary 47, Lemma 103, the cyclicity of the trace (Lemma 101 (i)), $\mathfrak{B}_{t,\mu} \in \mathcal{L}^1(\mathfrak{h})$ with $\mathfrak{B}_{t,\mu} \geq 0$, and the positivity of the self–adjoint operators $\Omega_t, \Omega_t^{\mathrm{t}} \geq 0$ for all $t \in [0, T_+)$, we observe that its derivative is well–defined for any strictly positive time $t \in (0, T_+)$ and satisfies:

$$\begin{aligned} \partial_t q_\mu(t) &= 16\mathrm{tr}\left\{B_t \left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-1} \bar{B}_t B_t \left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-1} \bar{B}_t\right\} \qquad \text{(V.179)} \\ &\quad -4\mathrm{tr}\left\{\Omega_t B_t \left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-1} \bar{B}_t + B_t \Omega_t^{\mathrm{t}} \left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-1} \bar{B}_t\right\} \\ &\leq 16\mathrm{tr}\{\mathfrak{B}_{t,\mu}^2\} \leq 16\, q_\mu^2(t) \ . \qquad \text{(V.180)} \end{aligned}$$

By majorisation, we thus obtain the inequality

$$q_\mu(t) \leq \frac{1}{q_\mu^{-1}(s) - 16\,(t-s)} \leq 2 q_\mu(s) \ , \tag{V.181}$$

provided that the times $s \in (0, T_+)$ and $t \in [s, T_+)$ satisfy the inequality

$$t - s \leq \frac{1}{32 q_\mu(s)} \ . \tag{V.182}$$

Now, using the resolvent identity

$$(X + \mathbf{1})^{-1} - (Y + \mathbf{1})^{-1} = (X + \mathbf{1})^{-1} (Y - X) (Y + \mathbf{1})^{-1} \tag{V.183}$$

for any positive operator $X, Y \geq 0$, as well as the Cauchy–Schwarz inequality, the cyclicity of the trace (Lemma 101 (i)), the positivity of the operator $\Omega_s^{\mathrm{t}} \geq 0$ for all $s \in [0, T_+)$ and the a priori estimate (V.140), note that

$$\begin{aligned} |q_\mu(t) - q_\mu(s)| &\leq \|\mathfrak{B}_{t,\mu} - \mathfrak{B}_{s,\mu}\|_1 \\ &\leq \mu^{-2} \|B_s\|_2^2 \ \|\Omega_t^{\mathrm{t}} - \Omega_s^{\mathrm{t}}\|_1 \\ &\quad + 2\mu^{-1} \|B_s\|_2 \ \|B_t - B_s\|_2 \qquad \text{(V.184)} \end{aligned}$$

for all $\mu > 0$ and $s, t \in [0, T_+)$. In particular, thanks to Theorem 46 (i)–(ii), the function q_μ is continuous on the whole interval $[0, T_+)$ and in particular at zero:

$$\lim_{s \to 0^+} q_\mu(s) = q_\mu(0) \leq q_0(0) = \mathfrak{b}_0 := \|\mathfrak{B}_0\|_1 \ . \tag{V.185}$$

We thus combine (V.185) with (V.181) and (V.182) to arrive at the inequality

$$\forall \mu > 0, \ \forall t \in [0, T] : \qquad \|\mathfrak{B}_{t,\mu}\|_{\mathrm{op}} \ \leq \ q_\mu(t) \ \leq \ 2\mathfrak{b}_0 \ , \tag{V.186}$$

provided that $T < 1/(32\ \mathfrak{b}_0)$. In particular, since, by (V.184) and Theorem 46 (i)–(ii), the trace–class operator family $(\mathfrak{B}_{t,\mu})_{t \in [0,T]}$ is (at least) norm continuous ($T < 1/(32\ \mathfrak{b}_0)$), the evolution operator defined by the Dyson series

$$\forall s, t \in [0, T] \ : \qquad \mathfrak{W}_{t,s} \ := \mathbf{1} + \sum_{n=1}^{\infty} 8^n \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n \ \mathfrak{B}_{\tau_1,\mu} \cdots \mathfrak{B}_{\tau_n,\mu} \tag{V.187}$$

is the unique, bounded and norm continuous solution of the non–autonomous evolution equation

$$\forall s,t\in[0,T]: \qquad \partial_t \mathfrak{W}_{t,s} = 8\,\mathfrak{B}_{t,\mu}\,\mathfrak{W}_{t,s}\ , \quad \mathfrak{W}_{s,s} = \mathbf{1}\ . \tag{V.188}$$

Next, we set

$$\mathfrak{D}_{t,\mu} := \Omega_t - 4\mathfrak{B}_{t,\mu} \tag{V.189}$$

on the interval $[0,T]$ and observe that, for all strictly positive times $t\in(0,T]$,

$$\partial_t \mathfrak{D}_{t,\mu} = 8\mathfrak{B}_{t,\mu}\big(\mathfrak{D}_{t,\mu}-\mu\mathbf{1}\big) + 8\big(\mathfrak{D}_{t,\mu}-\mu\mathbf{1}\big)\mathfrak{B}_{t,\mu} \tag{V.190}$$

on the domain $\mathcal{D}(\Omega_0)$, with positive initial value

$$\mathfrak{D}_{0,\mu} \ := \Omega_0 - 4B_0\left(\Omega_0^{\mathrm{t}}+\mu\right)^{-1}\bar{B}_0 \geq 0\ . \tag{V.191}$$

The proof of (V.190) uses exactly the same arguments as those proving (V.71). Using (V.188) and also the arguments justifying (V.94) we remark that

$$\forall s,t\in(0,T]: \qquad \mathfrak{D}_{t,\mu}-\mu\mathbf{1} = \mathfrak{W}_{t,s}\big(\mathfrak{D}_{s,\mu}-\mu\mathbf{1}\big)\mathfrak{W}_{t,s}^{*}\ , \tag{V.192}$$

from which we obtain, for $s,t\in(0,T]$, the inequality

$$\begin{aligned}\mathfrak{D}_{t,\mu} \quad \geq \quad & -\mu\left(\mathfrak{W}_{t,s}\mathfrak{W}_{t,s}^{*}-\mathbf{1}\right)+4\mathfrak{W}_{t,s}\mathfrak{B}_{0,\mu}\mathfrak{W}_{t,s}^{*}-4\mathfrak{W}_{t,s}\mathfrak{B}_{s,\mu}\mathfrak{W}_{t,s}^{*}\\ & -16\int_0^s \mathfrak{W}_{t,s}B_\tau\bar{B}_\tau\mathfrak{W}_{t,s}^{*}\mathrm{d}\tau,\end{aligned} \tag{V.193}$$

using (V.95). As it is similarly done to prove (V.101), we take the limit $s\to 0$ in this last inequality to arrive at the following assertion:

$$\forall t\in[0,T]: \qquad \mathfrak{D}_{t,\mu} \geq -\mu\left(\mathfrak{W}_{t,0}\mathfrak{W}_{t,0}^{*}-\mathbf{1}\right)\ , \tag{V.194}$$

with strictly positive arbitrary parameter $\mu>0$. Now, since, by (V.188),

$$\mathfrak{W}_{t,s}\mathfrak{W}_{t,s}^{*}-\mathbf{1} = 16\int_s^t \mathfrak{W}_{t,\tau}\mathfrak{B}_{\tau,\mu}\mathfrak{W}_{t,\tau}^{*} \tag{V.195}$$

for any $s,t\in[0,T]$, one gets the upper bound

$$\|\mathfrak{W}_{t,s}\mathfrak{W}_{t,s}^{*}-\mathbf{1}\|_{\mathrm{op}} \leq 16t \sup_{s\leq\tau\leq t}\{\|\mathfrak{B}_{\tau,\mu}\|_{\mathrm{op}}\}\left(\sup_{s\leq\tau\leq t}\left\{\|\mathfrak{W}_{t,\tau}\mathfrak{W}_{t,\tau}^{*}-\mathbf{1}\|_{\mathrm{op}}\right\}+1\right)\ . \tag{V.196}$$

Introducing the function

$$w_\mu := \sup_{0\leq\tau\leq T}\|\mathfrak{W}_{T,\tau}\mathfrak{W}_{T,\tau}^{*}-\mathbf{1}\|_{\mathrm{op}} \tag{V.197}$$

and using the upper bound

$$\sup_{0\leq\tau\leq T}\{\|\mathfrak{B}_{\tau,\mu}\|_{\mathrm{op}}\} \leq 2\mathfrak{b}_0 \tag{V.198}$$

(cf. (V.186)), we infer from (V.196) that

$$w_{\mu} \ \leq \frac{32T\mathfrak{b}_0}{1-32T\mathfrak{b}_0} =: \mathrm{K}_T < \infty \tag{V.199}$$

because $T < 1/(32\ \mathfrak{b}_0)$. Inserting (V.199) into (V.194), we arrive at the following inequality:

$$\forall t \in [0,T]: \qquad \mathfrak{D}_{t,\mu} \geq -\mu \mathrm{K}_T\ \mathbf{1}\ . \tag{V.200}$$

Since $\Omega_t \geq \mathfrak{D}_{t,\mu}$, this last statement reads

$$\forall \mu > 0,\ \forall t \in [0,T]: \qquad \Omega_t \geq \mathfrak{D}_{t,\mu} \geq -\mu \mathrm{K}_T\ \mathbf{1}\ . \tag{V.201}$$

Therefore, it remains to perform the limit $\mu \to 0^+$ in (V.201). Note first that $\ker(\Omega_t^{\mathrm{t}}) \cap \mathrm{Ran}(B_t) = \emptyset$, by (V.177)–(V.178), (V.181), and (V.185). Since the net $(y_\mu)_{\mu>0}$ of bounded real functions

$$x \mapsto y_\mu\,(x) = \frac{1}{x+\mu} \tag{V.202}$$

from $\mathbb{R}_0^+$ to $\mathbb{R}_0^+$ is monotonically decreasing in μ, the spectral theorem applied to the positive self–adjoint operator $\Omega_t^{\mathrm{t}} \geq 0$ for $t \in [0,T_+)$ and the monotone convergence theorem both yield the limit

$$\lim_{\mu \searrow 0} (\varphi, \mathfrak{B}_{t,\mu}\varphi) = (\varphi, \mathfrak{B}_t \varphi) \in [0, 2\mathfrak{b}_0] \tag{V.203}$$

for all $\varphi \in \mathfrak{h}$. It follows that $\mathfrak{B}_t^{1/2} \in \mathcal{B}\,(\mathfrak{h})$, i.e., $\mathfrak{B}_t \in \mathcal{B}\,(\mathfrak{h})$, and, by (V.201),

$$\forall t \in [0,T]: \qquad \mathfrak{D}_t := \Omega_t - 4\mathfrak{B}_t \geq 0\ , \tag{V.204}$$

provided that $T \in [0,T_+)$ satisfies $T < 1/(32\ \mathfrak{b}_0)$.
In fact, since the functional

$$\varphi \mapsto z_\mu\,(\varphi) = (\varphi, \mathfrak{B}_{t,\mu}\varphi) \tag{V.205}$$

from $\mathfrak{h}$ to $\mathbb{R}_0^+$ is monotonically decreasing in μ, we obtain that

$$\lim_{\mu \to 0^+} \|\mathfrak{B}_{t,\mu}\|_1 = \|\mathfrak{B}_t\|_1 \in [0, 2\mathfrak{b}_0]\ , \tag{V.206}$$

by applying again the monotone convergence theorem together with (V.186) and (V.203).
Now, from (V.179) and knowing that $\Omega_t \geq 4\mathfrak{B}_t$ for all $t \in [0,T]$, we remark that

$$\partial_t q_\mu(t) \leq -4\mathrm{tr}\left\{ B_t \Omega_t^{\mathrm{t}} \left(\Omega_t^{\mathrm{t}} + \mu \mathbf{1}\right)^{-1} \bar{B}_t \right\} \leq 0 \tag{V.207}$$

for any strictly positive time $t \in (0,T]$. It follows that $\|\mathfrak{B}_{t,\mu}\|_1 \leq \|\mathfrak{B}_{s,\mu}\|_1 < \infty$ for all $s \in (0,T]$ and $t \in [s,T]$, which, by (V.185), can be extended by continuity to $s = 0$. In particular, through (V.206), $\|\mathfrak{B}_t\|_1 \leq \|\mathfrak{B}_s\|_1 < \infty$ for any $s \in [0, T]$ and $t \in [s, T]$. As a consequence,

$$T < 1/(32\ \mathfrak{b}_0) \leq 1/(32\ \mathfrak{b}_t) \tag{V.208}$$

for any $t \in [0, T]$. We can thus shift the starting point from $t = 0$ to any fixed $x \in (0, T)$ and use the same arguments. In particular, by using an induction argument, we deduce that $\Omega_t \geq 4\mathfrak{B}_t$ and $\|\mathfrak{B}_t\|_1 \leq \|\mathfrak{B}_s\|_1 < \infty$ for all $s \in [0, T_+)$ and $t \in [s, T_+)$. □

We are now in position to prove that Condition A4, that is, $\Omega_0^{-1/2} B_0 \in \mathcal{L}^2(\mathfrak{h})$, prevents from having a blow–up. In particular, in this case $T_+ = T_{\max} = \infty$, see (V.117) and (V.137).

Theorem 50 (Global existence of (Ω_t, B_t))
Assume Conditions A1–A4, that is, $\Omega_0 = \Omega_0^* \geq 0$, $B_0 = B_0^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$, $\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0$, *and* $\Omega_0^{-1/2} B_0 \in \mathcal{L}^2(\mathfrak{h})$. *Then* $T_+ = T_{\max} = \infty$ *and*

$$\forall t \geq 0 : \qquad \Omega_t \geq 4B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t\ , \qquad \Omega_t^{-1/2} B_t \in \mathcal{L}^2(\mathfrak{h})\ . \tag{V.209}$$

Proof. For any $x \geq 0$ and $\mu > 0$, we consider the integral equation $X_\mu = \mathfrak{T}_x(X_\mu)$, where $\Omega_{x,\mu} := \Omega_x + \mu\mathbf{1}$ and $B_{x,\mu} = B_x$. By Lemma 44, there is a (unique) global solution $(X_{t,\mu})_{t\in[x,\infty)}$ of such integral equation for any $x \in [0, T_+)$ and $\mu > 0$ satisfying

$$\forall x \in \mathbb{R}_0^+, t \in [x, \infty) : \qquad \Omega_{t,\mu} = \Omega_x + \mu\mathbf{1} - X_{t,\mu} \geq \mu\mathbf{1}\ , \tag{V.210}$$

because

$$\forall t \in [0, T_+) : \qquad \Omega_t \geq 4B_t(\Omega_t^{\mathrm{t}})^{-1}\bar{B}_t\ , \tag{V.211}$$

see Lemma 49 (ii). On the other hand, the solution $(\Delta_t)_{t\in[0,T]} \in \mathfrak{C}$ of the initial value problem (V.3) satisfies

$$\forall t \in [x, T) : \qquad \Delta_t - \Delta_x = \mathfrak{T}_x(\Delta - \Delta_x)_t\ , \tag{V.212}$$

for all $x \in [0, T_{\max})$ and $T \in [x, T_{\max})$. Therefore, using (V.140) together with similar arguments as those used to prove Lemma 40 we show that the bounded operator $X_{t,\mu}$ converges in norm to $(\Delta_t - \Delta_x)$ as $\mu \to 0^+$, uniformly for $t \in [x, x + T_0] \cap [0, T_{\max})$ whenever $x \in [0, T_+)$. By (V.210), it follows that $\Omega_t \geq 0$ for all $t \in [x, x + T_0] \cap [0, T_{\max})$ and any $x \in [0, T_+)$. The positive time T_0 does not depend on $x \in [0, T_+)$ and, by (V.137), we conclude that $T_+ = T_{\max}$. The latter yields

$$\forall t \in [0, T_{\max}) : \quad \Omega_t := \Omega_0 - \Delta_t \geq 0\ ,\ \|B_t\|_2 \leq \|B_0\|_2 < \infty\ , \tag{V.213}$$

where $(\Delta_t)_{t\in[0,T]}$ is the solution in $\mathfrak{C}$ of the initial value problem (V.3) for any $T \in (0, T_{\max})$. In particular, by Lemma 43, we arrive at the assertion $T_{\max} = \infty$. Moreover, $\Omega_t^{-1/2} B_t \in \mathcal{L}^2(\mathfrak{h})$ because of Lemma 49 (i) and

$$\forall t \geq 0 : \qquad \|\mathfrak{B}_t\|_1 = \|\Omega_t^{-1/2} B_t\|_2^2\ . \tag{V.214}$$

□

We finally conclude this subsection by some observations on the flow. Assume Conditions A1–A4. By (III.9), we note that, formally,

$$\forall t>0: \qquad \left\{ \begin{array}{ll} \partial_t \mathfrak{D}_t = 8\left(\mathfrak{B}_t \mathfrak{D}_t + \mathfrak{D}_t \mathfrak{B}_t\right) & , \\ \partial_t \mathfrak{B}_t = -2\left(\mathfrak{B}_t \mathfrak{D}_t + \mathfrak{D}_t \mathfrak{B}_t\right) - 4 B_t \bar{B}_t & , \end{array} \right. \tag{V.215}$$

with $\mathfrak{B}_t \in \mathcal{L}^1(\mathfrak{h})$, whereas $\mathfrak{D}_t := \Omega_t - 4\mathfrak{B}_t \geq 0$ is the sum of a positive operator Ω_0 and an operator which is bounded for all $t \geq 0$, by Lemma 49. By Lemma 39 and Theorem 46, the trace–class operator family $(\mathfrak{B}_t)_{t\geq 0}$ is clearly norm continuous when A6 holds and this property should persist under A1–A4. Since $(\Omega_t)_{t\geq 0}$ is at least norm continuous (Theorem 46), it should exist two evolution operators $\mathfrak{U}_{t,s}$ and $\mathfrak{V}_{t,s}$ which are the strong solution in $\mathcal{B}(\mathfrak{h})$ of

$$\forall t>s\geq 0: \qquad \partial_t \mathfrak{U}_{t,s} = -2\mathfrak{D}_t \mathfrak{U}_{t,s}\ , \quad \mathfrak{U}_{s,s} := \mathbf{1}\ , \tag{V.216}$$

and (V.175) (here $T_+ = \infty$), respectively. In particular,

$$\forall t\geq s\geq 0: \qquad \mathfrak{D}_t = \mathfrak{V}_{t,s} \mathfrak{D}_s \mathfrak{V}_{t,s}^* \geq 0\ , \tag{V.217}$$

as heuristically explained before Lemma 49 (cf. (V.173)), whereas we should have

$$\forall t\geq s\geq 0: \qquad \mathfrak{B}_t = \mathfrak{U}_{t,s} \mathfrak{B}_s \mathfrak{U}_{t,s}^* - 4\int_s^t \mathfrak{U}_{t,\tau} B_\tau \bar{B}_\tau \mathfrak{U}_{t,\tau}^* \mathrm{d}\tau\ . \tag{V.218}$$

This equation may be proven by using the operator families $(\mathfrak{B}_{t,\mu})_{t\geq 0}$ and $(\mathfrak{D}_{t,\mu})_{t\geq 0}$ respectively defined, for any arbitrarily constant $\mu > 0$, by (V.177) and (V.189), together with the limit $\mu \rightarrow 0^+$.
Another interesting observation on the flow is the conservation for all times of all inequalities of the type

$$\Omega_0 \geq (4+\mathrm{r}) B_0 (\Omega_0^{\mathrm{t}})^{-1} \bar{B}_0 + \mu \mathbf{1} = (4+\mathrm{r})\mathfrak{B}_0 + \mu\mathbf{1}\ , \tag{V.219}$$

where $\mu, \mathrm{r} \geq 0$ are two nonnegative constants. This is already proven when $\mathrm{r} = 0$ and $\mu \geq 0$, see Lemmata 44 and Theorem 50, but this property can now be generalized to all $\mu, \mathrm{r} \geq 0$:

LEMMA 51 (CONSERVATION BY THE FLOW OF $\Omega_0 \geq (4+\mathrm{r})\mathfrak{B}_0 + \mu\mathbf{1}$)
Assume Conditions A1–A2 and A4. If (V.219) holds for some $\mu, \mathrm{r} \geq 0$*, then*

$$\forall t\geq 0: \qquad \Omega_t \geq (4+\mathrm{r}) B_t (\Omega_t^{\mathrm{t}})^{-1} \bar{B}_t + \mu\mathbf{1}\ . \tag{V.220}$$

PROOF. First note that

$$\Omega_0 \geq (4+\mathrm{r}) B_0 (\Omega_0^{\mathrm{t}})^{-1} \bar{B}_0 + \mu\mathbf{1} \geq B_0 (\Omega_0^{\mathrm{t}})^{-1} \bar{B}_0 \tag{V.221}$$

and, hence, by Theorem 50, there exists a unique solution $(\Omega_t, B_t)_{t\geq 0}$ of (III.9) obeying

$$\forall t\geq 0: \qquad \Omega_t \geq \mathfrak{B}_t := 4 B_t (\Omega_t^{\mathrm{t}})^{-1} \bar{B}_t\ , \tag{V.222}$$

see also Theorem 41 for more details. Moreover, $\mathfrak{B}_t$ is a positive operator with trace uniformly bounded in $t \geq 0$, see Lemma 49.
Now assume that $\mu > 0$. Then, by Theorem 46, the bounded operator family $(\mathfrak{B}_t)_{t\geq 0}$ is norm continuous. (This property should also hold for $\mu = 0$, but it is not necessary for this proof.) Then, we use $\mathfrak{B}_t$ to define a norm continuous evolution operator by the non–autonomous evolution equation

$$\forall t, s \in \mathbb{R}_0^+ : \qquad \partial_t \widetilde{\mathfrak{V}}_{t,s} = 2(4+\mathrm{r})\mathfrak{B}_t \widetilde{\mathfrak{V}}_{t,s} \ , \quad \widetilde{\mathfrak{V}}_{s,s} := \mathbf{1} \ . \tag{V.223}$$

Since $\mathfrak{B}_t$ is bounded, uniformly in $t \geq 0$, $\widetilde{\mathfrak{V}}_{t,s}$ is bounded. Indeed,

$$\|\widetilde{\mathfrak{V}}_{t,s}\|_{\mathrm{op}}^2 \leq \exp\left\{2\,(4+\mathrm{r})\,|t-s|\,\|\mathfrak{B}_0\|_1\right\} \ . \tag{V.224}$$

Furthermore,

$$\forall t, s \in \mathbb{R}_0^+ : \qquad \partial_s \widetilde{\mathfrak{V}}_{t,s} = -2(4+\mathrm{r})\widetilde{\mathfrak{V}}_{t,s}\mathfrak{B}_s \ , \quad \widetilde{\mathfrak{V}}_{t,t} := \mathbf{1} \ . \tag{V.225}$$

We set

$$\widetilde{\mathfrak{D}}_t := \Omega_t - (4+\mathrm{r})\mathfrak{B}_t = \mathfrak{D}_t - \mathrm{r}\mathfrak{B}_t \tag{V.226}$$

and observe that, for all $t > 0$,

$$\partial_t \widetilde{\mathfrak{D}}_t = 2(4+\mathrm{r})(\mathfrak{B}_t\widetilde{\mathfrak{D}}_t + \widetilde{\mathfrak{D}}_t\mathfrak{B}_t) + 4\mathrm{r}B_t\bar{B}_t + 4\mathrm{r}(4+\mathrm{r})\mathfrak{B}_t^2 \ . \tag{V.227}$$

We can use the arguments justifying (V.94) to prove that

$$\forall t, s \in \mathbb{R}^+ : \quad \widetilde{\mathfrak{D}}_t = \widetilde{\mathfrak{V}}_{t,s}\widetilde{\mathfrak{D}}_s\widetilde{\mathfrak{V}}_{t,s}^* + \int_s^t \widetilde{\mathfrak{V}}_{t,\tau}\left(4\mathrm{r}B_\tau\bar{B}_\tau + 4\mathrm{r}(4+\mathrm{r})\mathfrak{B}_\tau^2\right)\widetilde{\mathfrak{V}}_{t,\tau}^*\mathrm{d}\tau \ . \tag{V.228}$$

The proof of this equality clearly uses (V.225) and (V.227), but we omit the details and directly conclude that

$$\forall t \geq s > 0 : \qquad \widetilde{\mathfrak{D}}_t \geq \widetilde{\mathfrak{V}}_{t,s}\widetilde{\mathfrak{D}}_s\widetilde{\mathfrak{V}}_{t,s}^* \ . \tag{V.229}$$

By (V.219), it follows that

$$\widetilde{\mathfrak{D}}_t \geq \mu\widetilde{\mathfrak{V}}_{t,s}\widetilde{\mathfrak{V}}_{t,s}^* + (4+\mathrm{r})\widetilde{\mathfrak{V}}_{t,s}\mathfrak{B}_0\widetilde{\mathfrak{V}}_{t,s}^* - (4+\mathrm{r})\widetilde{\mathfrak{V}}_{t,s}\mathfrak{B}_s\widetilde{\mathfrak{V}}_{t,s}^* - 16\int_0^s \widetilde{\mathfrak{V}}_{t,s}B_\tau\bar{B}_\tau\widetilde{\mathfrak{V}}_{t,s}^*\mathrm{d}\tau \tag{V.230}$$

for all $t \geq s > 0$. Using the norm continuity of the bounded operator families $(\mathfrak{B}_t)_{t\geq 0}$ and $(\widetilde{\mathfrak{V}}_{t,s})_{t,s\in\mathbb{R}^+}$, we take the limit $s \to 0^+$ in (V.230) to obtain that

$$\forall t \geq 0 : \qquad \widetilde{\mathfrak{D}}_t \geq \mu\widetilde{\mathfrak{V}}_{t,0}\widetilde{\mathfrak{V}}_{t,0}^* \ . \tag{V.231}$$

By (V.225), we note that

$$\partial_s\left\{\widetilde{\mathfrak{V}}_{t,s}\widetilde{\mathfrak{V}}_{t,s}^*\right\} = -2(4+\mathrm{r})\widetilde{\mathfrak{V}}_{t,s}^*\mathfrak{B}_t\widetilde{\mathfrak{V}}_{t,s} \leq 0 \ , \tag{V.232}$$

i.e., $\widetilde{\mathfrak{V}}_{t,s}\widetilde{\mathfrak{V}}_{t,s}^{*}\geq \mathbf{1}$ for all $t\geq s\geq 0$. Using this together with (V.231) we then deduce that $\widetilde{\mathfrak{D}}_{t}\geq \mu\mathbf{1}$, which is the asserted estimate (V.220) for $\mu>0$.
Now, if $\mu=0$ then one uses the proof of Lemma 49 by using the operator $\mathfrak{B}_{t,\mu}$ defined, for any arbitrarily constant $\mu>0$, by (V.177) together with the limit $\mu\rightarrow 0^{+}$. We omit the details, especially since similar arguments are also performed many times for other operators, see the proofs of Lemmata 60, 61, 63, and 65 where this strategy is always used. □
Therefore, Theorem 50 and Lemma 51 yield Assertion (i) of Theorem 12.

V.2 Constants of Motion

Theorem 46 and Corollary 47 together with Theorem 50 already imply Assertions (i)–(ii) and (v) of Theorem 11. It remains to prove the third (iii) and fourth (iv) ones, which are carried out in this subsection through four steps.
First, we relate the operator family $(\Omega_t)_{t\geq 0}$ to a "commutator" defined by

$$K_{t}:=\Omega_{t}B_{t}-B_{t}\Omega_{t}^{\mathrm{t}}=-K_{t}^{\mathrm{t}}\ . \tag{V.233}$$

This bounded (linear) operator is well–defined for any strictly positive time $t>0$ on the whole Hilbert space $\mathfrak{h}$ because both $\Omega_t B_t$ and $B_t\Omega_t^{\mathrm{t}}$ are Hilbert–Schmidt operators, by Lemma 45. The relation between the operator families $(K_t)_{t>0}$ and $(\Omega_t)_{t\geq 0}$ is explained in Theorem 52. It yields a *constant of motion*, see Theorem 52 (iii).
Secondly, if one assumes (for simplicity) the condition

$$\Omega_0 B_0(\Omega_0^{\mathrm{t}}+\mathbf{1})^{-1}\in\mathcal{B}(\mathfrak{h})\ , \tag{V.234}$$

then K_0 is well–defined on the domain $\mathcal{D}(\Omega_0^{\mathrm{t}})$ and the operator family $(K_t)_{t\geq 0}$ can be computed like the operator family $(B_t)_{t\geq 0}$, see Equation (V.2). This study requires a preliminary step (Lemma 53) and it is concluded by Lemma 54.
Finally, combining all these results one directly deduces a simple expression for the operator family $(\Omega_t)_{t\geq 0}$ under the condition $K_0=0$. The latter is Corollary 55, which expresses *two* constants of motion in this special case.
So, we start by deriving a first constant of motion of the flow under Conditions A1–A3:

Theorem 52 (Ω_t^2 and constant of motion)
Assume Conditions A1–A3. Let $t,s\in[0,T_{+})$ with T_+ defined by (V.137) and $(\Omega_t,B_t)_{t\in[0,T_+)}$ be defined via Theorem 41. Then the following statements hold true:
(i) *Domains: $\mathcal{D}(\Omega_t^2)=\mathcal{D}(\Omega_s^2)$ and*

$$\Omega_t^2-8B_t\bar{B}_t=\Omega_s^2-8B_s\bar{B}_s+32\int_s^t B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\mathrm{d}\tau\ . \tag{V.235}$$

(ii) $(\Omega_t^2)_{t\geq 0}$ *and the "commutator" family* $(K_t)_{t>0}$ *(see (V.233)) satisfy:*

$$\Omega_t^2 - 4B_t\bar{B}_t = \Omega_s^2 - 4B_s\bar{B}_s + 8\int_s^t \left(B_\tau\bar{K}_\tau + K_\tau^{\mathrm{t}}\bar{B}_\tau\right)\mathrm{d}\tau \ . \tag{V.236}$$

(iii) *Constant of motion of the flow:*

$$\mathrm{tr}\left(\Omega_t^2 - 4B_t\bar{B}_t - \Omega_s^2 + 4B_s\bar{B}_s\right) = 0 \ . \tag{V.237}$$

PROOF. Assume for convenience that $T_+ = \infty$.
(i): For any $t > 0$, Corollary 47 yields

$$\partial_t\left\{B_t\bar{B}_t\right\} = -2\left(\Omega_t B_t\bar{B}_t + B_t\bar{B}_t\Omega_t\right) - 4B_t\Omega_t^{\mathrm{t}}\bar{B}_t \tag{V.238}$$

and

$$\partial_t\left\{\Omega_t^2 - 4B_t\bar{B}_t\right\} = 8\left(B_t\Omega_t^{\mathrm{t}} - \Omega_t B_t\right)\bar{B}_t + 8B_t\left(\Omega_t^{\mathrm{t}}\bar{B}_t - \bar{B}_t\Omega_t\right) \ , \tag{V.239}$$

in the strong sense in $\mathcal{D}(\Omega_0)$. To compute this last derivative, note that one invokes similar arguments as those used to prove (V.92). We omit the details. In particular, thanks to (V.238),

$$2\int_s^t\left(\Omega_\tau B_\tau\bar{B}_\tau + B_\tau\bar{B}_\tau\Omega_\tau\right)\mathrm{d}\tau = B_s\bar{B}_s - B_t\bar{B}_t - 4\int_s^t B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\mathrm{d}\tau \tag{V.240}$$

for any $t \geq s > 0$, which is combined with (V.239) to yield the first statement (i) for strictly positive times $t \geq s > 0$:

$$\forall t \geq s > 0: \quad \Omega_t^2 - 8B_t\bar{B}_t = \Omega_s^2 - 8B_s\bar{B}_s + 32\int_s^t B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\mathrm{d}\tau \ , \tag{V.241}$$

on the domain $\mathcal{D}(\Omega_t^2) = \mathcal{D}(\Omega_s^2) \subseteq \mathcal{D}(\Omega_0^2)$. So, it remains to take the limit $s \to 0^+$ in this equality and to show also that $\mathcal{D}(\Omega_t^2) = \mathcal{D}(\Omega_0^2)$, for any $t \geq 0$. We start with the last integral by observing that the limit

$$\int_0^t B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\mathrm{d}\tau := \lim_{s\to 0^+}\int_s^t B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\mathrm{d}\tau \tag{V.242}$$

defines a bounded operator. Indeed, for any $x, t > 0$, let

$$A_{x,t} := W_x B_0 W_t^{\mathrm{t}}\Omega_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^*\bar{B}_0 W_x^* = A_{x,t}^* \ . \tag{V.243}$$

For any $t > 0$, the operator $W_t^{\mathrm{t}}\Omega_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^*$ is bounded (cf. Lemma 36 (i)) and positive, i.e., $W_t^{\mathrm{t}}\Omega_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^* \geq 0$. Since $B_0 \in \mathcal{L}^2(\mathfrak{h})$, $A_{x,t} \in \mathcal{L}^1(\mathfrak{h})$ is trace–class and, by cyclicity of the trace (Lemma 101 (i)),

$$\mathrm{tr}\left(A_{x,t}\right) = \mathrm{tr}\left(\left(W_t^{\mathrm{t}}\Omega_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^*\right)^{1/2}\bar{B}_0 W_x^* W_x B_0\left(W_t^{\mathrm{t}}\Omega_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^*\right)^{1/2}\right) \ , \tag{V.244}$$

for any $x,t>0$. We thus combine the derivative (V.65) extended to all $t>s\geq 0$ and (V.244) with Lemma 103 to arrive at $\partial_x\left\{\mathrm{tr}\left(A_{x,t}\right)\right\}\leq 0$ for all $x,t>0$, i.e.,

$$\forall t\geq x>0: \qquad \mathrm{tr}\left(B_t\Omega_t^{\mathrm{t}}\bar{B}_t\right)\leq \mathrm{tr}\left(W_xB_0W_t^{\mathrm{t}}\Omega_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^*\bar{B}_0W_x^*\right)\ . \qquad \text{(V.245)}$$

Since $W_t^{\mathrm{t}}\Omega_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^*\in\mathcal{B}(\mathfrak{h})$ for all $t>0$, the upper bound of this last estimate is continuous at $x=0$, similar to (V.157). Therefore, (V.245) also holds for $x=0$ and we conclude that, for any $t>0$ and $s\in[0,t]$,

$$\int_s^t \mathrm{tr}\left(B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\right)\mathrm{d}\tau\leq\int_s^t\mathrm{tr}\left(B_0W_\tau^{\mathrm{t}}\Omega_\tau^{\mathrm{t}}\left(W_\tau^{\mathrm{t}}\right)^*\bar{B}_0\right)\mathrm{d}\tau\ . \qquad \text{(V.246)}$$

We note that, for any $t>0$ and $s\in[0,t]$,

$$\forall\varphi\in\mathfrak{h}: \qquad \int_s^t\left\langle\varphi\middle|W_\tau^{\mathrm{t}}\Omega_\tau^{\mathrm{t}}\left(W_\tau^{\mathrm{t}}\right)^*\varphi\right\rangle\mathrm{d}\tau=\frac{1}{4}\left\langle\varphi\middle|(W_s^{\mathrm{t}}(W_s^{\mathrm{t}})^*-W_t^{\mathrm{t}}(W_t^{\mathrm{t}})^*)\varphi\right\rangle\ , \qquad \text{(V.247)}$$

because W_t^{t} and $\left(W_t^{\mathrm{t}}\right)^*$ are strongly continuous on $[0,\infty)$ and

$$\forall t>0: \qquad \partial_t\left\{W_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^*\right\}=-4W_t^{\mathrm{t}}\Omega_t^{\mathrm{t}}\left(W_t^{\mathrm{t}}\right)^*\leq 0\ , \qquad \text{(V.248)}$$

see Lemma 36 (ii) applied to W_t^{t} and $(W_t^{\mathrm{t}})^*$, and extended to all $t>s\geq 0$. Using (V.247) and taking any orthonormal basis $\{g_k\}_{k=1}^\infty\subseteq\mathfrak{h}$ we now remark that

$$\begin{aligned}\sum_{k=1}^\infty\int_s^t\left\langle\bar{B}_0g_k\middle|W_\tau^{\mathrm{t}}\Omega_\tau^{\mathrm{t}}\left(W_\tau^{\mathrm{t}}\right)^*\bar{B}_0g_k\right\rangle\mathrm{d}\tau &= \frac{1}{4}\mathrm{tr}\left(B_0(W_s^{\mathrm{t}}(W_s^{\mathrm{t}})^*-W_t^{\mathrm{t}}(W_t^{\mathrm{t}})^*)\bar{B}_0\right)\\ &\leq \frac{1}{4}\left\|B_0\right\|_2^2<\infty\ ,\end{aligned} \qquad \text{(V.249)}$$

because of (V.139). Therefore, for any $t>0$ and $s\in[0,t]$, we invoke Fubini's theorem to exchange the trace with the integral in the upper bound of (V.246) and get

$$\int_0^t\mathrm{tr}\left(B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\right)\mathrm{d}\tau\leq\frac{1}{4}\mathrm{tr}\left(B_0(W_s^{\mathrm{t}}(W_s^{\mathrm{t}})^*-W_t^{\mathrm{t}}(W_t^{\mathrm{t}})^*)\bar{B}_0\right)\leq\frac{1}{4}\left\|B_0\right\|_2^2<\infty\ . \qquad \text{(V.250)}$$

As $\Omega_t^{\mathrm{t}}\geq 0$, we can then use a second time Fubini's theorem to infer from (V.250) that

$$\begin{aligned}\left\|\int_s^t B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\mathrm{d}\tau\right\|_1 &= \mathrm{tr}\left(\int_s^t B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\mathrm{d}\tau\right)=\int_s^t\mathrm{tr}\left(B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\right)\mathrm{d}\tau\\ &\leq \frac{1}{4}\mathrm{tr}\left(B_0(W_s^{\mathrm{t}}(W_s^{\mathrm{t}})^*-W_t^{\mathrm{t}}(W_t^{\mathrm{t}})^*)\bar{B}_0\right)\\ &\leq \frac{1}{4}\left\|B_0\right\|_2^2<\infty\ ,\end{aligned} \qquad \text{(V.251)}$$

for any $t > 0$ and $s \in [0, t]$. In other words, the positive operator (V.242) belongs to $\mathcal{L}^1(\mathfrak{h})$ and it is straightforward to verify that

$$\lim_{t \to 0^+} \left\| \int_0^t B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau \mathrm{d}\tau \right\|_1 = 0 , \tag{V.252}$$

because of (V.251) and the strong continuity of W_t^{t} and $(W_t^{\mathrm{t}})^*$ on $[0, \infty)$, see Lemma 34 (ii) applied to W_t^{t} and $(W_t^{\mathrm{t}})^*$.
We study now the (time–independent) domain $\mathcal{D}(\Omega_t^2)$. First, by Corollary 47, recall again that, for any $t \geq 0$,

$$\Omega_t := \Omega_0 - \Delta_t = \Omega_0 - 16 \int_0^t B_\tau \bar{B}_\tau \mathrm{d}\tau \geq 0 . \tag{V.253}$$

Then, a formal calculation implies that, for any $t \geq 0$,

$$\Omega_t^2 = \Omega_0^2 - 16\Omega_0 \int_0^t B_\tau \bar{B}_\tau \mathrm{d}\tau - 16 \int_0^t B_\tau \bar{B}_\tau \Omega_0 \mathrm{d}\tau + 16^2 \left(\int_0^t B_\tau \bar{B}_\tau \mathrm{d}\tau \right)^2 . \tag{V.254}$$

We observe that

$$\int_0^t B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau \mathrm{d}\tau \qquad \text{and} \qquad \int_0^t B_\tau \bar{B}_\tau \Omega_\tau \left(\Omega_0 + \mathbf{1}\right)^{-1} \mathrm{d}\tau \tag{V.255}$$

define trace–class operators because of (V.251) and

$$\left\| \int_s^t B_\tau \bar{B}_\tau \Omega_\tau \left(\Omega_0 + \mathbf{1}\right)^{-1} \mathrm{d}\tau \right\|_1 \leq (t - s) \left\| B_0 \right\|_2^2 + 16 \left(t - s\right)^2 \left\| B_0 \right\|_2^4 < \infty \tag{V.256}$$

for any $t \geq s \geq 0$, by (V.140) and (V.253). Via (V.240) and Theorem 46 (ii), it follows that

$$\int_0^t \Omega_\tau B_\tau \bar{B}_\tau \left(\Omega_0 + \mathbf{1}\right)^{-1} \mathrm{d}\tau := \lim_{s \to 0^+} \int_s^t \Omega_\tau B_\tau \bar{B}_\tau \left(\Omega_0 + \mathbf{1}\right)^{-1} \mathrm{d}\tau \tag{V.257}$$

defines also a trace–class operator.
On the other hand, we can interchange the operator Ω_0 and the first integral of (V.254) – just as we did to prove (V.80) – and use again (V.253) to obtain that, for any $t \geq s > 0$,

$$\begin{aligned} \Omega_0 \int_s^t B_\tau \bar{B}_\tau \mathrm{d}\tau \quad &= \quad \frac{\Omega_0}{\Omega_0 + \mathbf{1}} \int_s^t \Omega_\tau B_\tau \bar{B}_\tau \mathrm{d}\tau + \frac{\Omega_0}{\Omega_0 + \mathbf{1}} \int_s^t B_\tau \bar{B}_\tau \mathrm{d}\tau \\ &\quad + \frac{16\Omega_0}{\Omega_0 + \mathbf{1}} \int_s^t \mathrm{d}\tau_1 \int_s^{\tau_1} \mathrm{d}\tau_2 B_{\tau_2} \bar{B}_{\tau_2} B_{\tau_1} \bar{B}_{\tau_1} . \end{aligned} \tag{V.258}$$

It follows from (V.257) that, for any $t > 0$,

$$\left(\lim_{s \to 0^+} \Omega_0 \int_s^t B_\tau \bar{B}_\tau \left(\Omega_0 + \mathbf{1}\right)^{-1} \mathrm{d}\tau \right) \in \mathcal{L}^1(\mathfrak{h}) . \tag{V.259}$$

Here, the limit is in the strong sense in $\mathfrak{h}$. Since, for any $t>0$,

$$\lim_{s\to 0^+}\int_s^t B_\tau \bar{B}_\tau (\Omega_0+\mathbf{1})^{-1}\,\mathrm{d}\tau=\int_0^t B_\tau \bar{B}_\tau (\Omega_0+\mathbf{1})^{-1}\,\mathrm{d}\tau \tag{V.260}$$

(in the strong topology) and Ω_0 is a closed operator, we infer from (V.259) that

$$\Omega_0\int_0^t B_\tau \bar{B}_\tau (\Omega_0+\mathbf{1})^{-1}\,\mathrm{d}\tau=\lim_{s\to 0^+}\Omega_0\int_s^t B_\tau \bar{B}_\tau (\Omega_0+\mathbf{1})^{-1}\,\mathrm{d}\tau \tag{V.261}$$

defines also a trace–class operator for any $t\geq 0$. It is in particular bounded and we deduce from (V.254) that $\mathcal{D}(\Omega_t^2)=\mathcal{D}(\Omega_0^2)$ for all $t\geq 0$.
We proceed by analyzing the convergence of the operator Ω_s^2 to Ω_0^2, as $s\to 0^+$. So, by using

$$2\left(\Omega_s^2-\Omega_0^2\right)=(\Omega_s-\Omega_0)(\Omega_s+\Omega_0)+(\Omega_s+\Omega_0)(\Omega_s-\Omega_0) \tag{V.262}$$

and the resolvent identity (V.183) (as $\mathcal{D}(\Omega_t^2)=\mathcal{D}(\Omega_0^2)$) together with

$$\left\|(X+\mathbf{1})\left(X^2+\mathbf{1}\right)^{-1}\right\|_{\mathrm{op}}\leq\sup_{x\geq 0}\left(\frac{x+1}{x^2+1}\right)<2 \tag{V.263}$$

for any positive operator $X\geq 0$, one gets

$$\begin{aligned}\left\|\left(\Omega_s^2+\mathbf{1}\right)^{-1}-\left(\Omega_0^2+\mathbf{1}\right)^{-1}\right\|_{\mathrm{op}} &\leq 4\left\|(\Omega_s+\mathbf{1})^{-1}\left(\Omega_s^2-\Omega_0^2\right)(\Omega_0+\mathbf{1})^{-1}\right\|_{\mathrm{op}}\\ &\leq 4\left(2+\|\Delta_s\|_2\right)\|\Omega_s-\Omega_0\|_{\mathrm{op}}\ . \end{aligned}\tag{V.264}$$

The operator family $(\Omega_t)_{t\geq 0}$ is continuous, at least in the norm topology (Theorem 41 (i)). So, we conclude from (V.264) that Ω_s^2 converges in the norm resolvent sense to Ω_0^2, as $s\to 0^+$.
As a consequence, we arrive at Assertion (i) of the lemma by passing to the limit $s\to 0^+$ in (V.241) with the help of (V.242) and the continuity of the Hilbert–Schmidt operator family $(B_t)_{t\geq 0}$, see, e.g., Theorem 46 (ii).
<u>(ii):</u> By using (V.233) and integrating (V.239), we find that, for any $t\geq s>0$,

$$\Omega_t^2-4B_t\bar{B}_t=\Omega_s^2-4B_s\bar{B}_s+8\int_s^t\left(B_\tau\bar{K}_\tau+K_\tau^{\mathrm{t}}\bar{B}_\tau\right)\mathrm{d}\tau \tag{V.265}$$

on the domain $\mathcal{D}(\Omega_0^2)\subset\mathcal{D}(\Omega_0)$. Therefore, since Ω_s^2 converges in the norm resolvent sense to Ω_0^2, as $s\to 0^+$, and $(B_t)_{t\geq 0}$ is continuous in $\mathcal{L}^2(\mathfrak{h})$ (Theorem 46 (ii)), one verifies Equation (V.265) for $s=0$ on the domain $\mathcal{D}(\Omega_0^2)\subset\mathcal{D}(\Omega_0)$ by elementary computations using (V.233) and the fact that the operators in (V.255) and (V.257) are all bounded.
<u>(iii):</u> From the cyclicity of the trace (Lemma 101) combined with $\mathrm{tr}\,(X^{\mathrm{t}})=\mathrm{tr}\,(X)$, $\mathrm{tr}(\overline{X})=\overline{\mathrm{tr}(X)}$ (Lemma 102) and $B_t=B_t^{\mathrm{t}}\in\mathcal{L}^2(\mathfrak{h})$, note that

$$\mathrm{tr}\left(\Omega_t B_t\bar{B}_t\right)=\mathrm{tr}\left(\bar{B}_t\Omega_t B_t\right)=\mathrm{tr}\left(B_t\Omega_t^{\mathrm{t}}\bar{B}_t\right)=\mathrm{tr}\left(B_t\bar{B}_t\Omega_t\right)\ . \tag{V.266}$$

Moreover, because the positive operator (V.242) belongs to $\mathcal{L}^1(\mathfrak{h})$ and satisfies (V.251), we deduce from the first assertion (i) that

$$\mathrm{tr}\left(\Omega_t^2-\Omega_s^2\right)=8\mathrm{tr}\left(B_t\bar{B}_t\right)-8\mathrm{tr}\left(B_s\bar{B}_s\right)+32\int_s^t \mathrm{tr}\left(B_\tau\Omega_\tau^{\mathrm{t}}\bar{B}_\tau\right)\mathrm{d}\tau \qquad \text{(V.267)}$$

for all $t\geq s\geq 0$. In particular, by continuity in $\mathcal{L}^2(\mathfrak{h})$ of $(B_t)_{t\geq 0}$,

$$\lim_{s\to 0^+}\mathrm{tr}\left(\Omega_t^2-\Omega_s^2\right)=\mathrm{tr}\left(\Omega_t^2-\Omega_0^2\right)\ . \qquad \text{(V.268)}$$

Now, we use (V.238) together with Lemma 103 and (V.266)–(V.267) to deduce that

$$\forall t\geq s>0:\qquad \mathrm{tr}\left(\Omega_t^2-\Omega_s^2\right)=4\left(\|B_t\|_2^2-\|B_s\|_2^2\right)\ . \qquad \text{(V.269)}$$

The latter can be extended by continuity to $s=0$ because of the continuity in $\mathcal{L}^2(\mathfrak{h})$ of $(B_t)_{t\geq 0}$ and (V.268). □

Under Condition (V.234), we aim to obtain a simple expression for the operator family $(K_t)_{t\geq 0}$, similar to (V.2) for the operator family $(B_t)_{t\geq 0}$. To this end, we need first to study the operators

$$\mathfrak{d}_t:=(\Omega_0+\mathbf{1})\,\Delta_t\,(\Omega_0+\mathbf{1})^{-1}\quad\text{and}\quad \boldsymbol{V}_{t,s}:=(\Omega_0+\mathbf{1})\,W_{t,s}\,(\Omega_0+\mathbf{1})^{-1}\ , \qquad \text{(V.270)}$$

for any $t\in[0,T_+)$ and $s\in[0,t]$, where $(\Delta_t)_{t\in[0,T_+)}$ is defined by (V.27) extended to $[0,T_+)$.

Lemma 53 ($\boldsymbol{V}_{t,s}$ as an evolution operator)
Assume Conditions A1–A3. Then, for any $s,x,t\in[0,T_+)$ so that $t\geq x\geq s$:
(i) $(\mathfrak{d}_t)_{t\geq 0}\in C[[0,T_+);\mathcal{L}^1(\mathfrak{h})]$ *and* $(\boldsymbol{V}_{t,s})_{t\geq s\geq 0}\subset\mathcal{B}(\mathfrak{h})$.
(ii) $\boldsymbol{V}_{t,s}$ *satisfies the cocycle property* $\boldsymbol{V}_{t,x}\boldsymbol{V}_{x,s}=\boldsymbol{V}_{t,s}$.
(iii) $\boldsymbol{V}_{t,s}$ *is jointly strongly continuous in s and t.*
(iv) *The evolution family $(\boldsymbol{V}_{t,s})_{t\geq s\geq 0}$ is the solution of the non–autonomous evolution equations*

$$\left\{\begin{array}{lll}\forall s\in[0,T_+),\ t\in(s,T_+) & : & \partial_t\boldsymbol{V}_{t,s}=-2\,(\Omega_0-\mathfrak{d}_t)\,\boldsymbol{V}_{t,s}\quad,\quad \boldsymbol{V}_{s,s}:=\mathbf{1}\ .\\ \forall t\in(0,T_+),\ s\in[0,t] & : & \partial_s\boldsymbol{V}_{t,s}=2\boldsymbol{V}_{t,s}\,(\Omega_0-\mathfrak{d}_s)\quad,\quad \boldsymbol{V}_{t,t}:=\mathbf{1}\ .\end{array}\right. \qquad \text{(V.271)}$$

The derivatives with respect to t and s are in the strong sense in $\mathfrak{h}$ and $\mathcal{D}(\Omega_0)$, respectively.
Similar properties as (i)–(iv) also hold for the bounded operators $\boldsymbol{V}_{t,s}^{\mathrm{t}}$ and $\mathfrak{d}_t^{\mathrm{t}}$.

Proof. Without loss of generality assume for convenience that $T_+=\infty$. Since $B_{t>0}\mathfrak{h}\subseteq\mathcal{D}(\Omega_0)$, we use (V.253) and interchange the operator $(\Omega_0+\mathbf{1})$ and the integral (cf. (V.257)–(V.261)) to arrive at

$$\begin{aligned}\mathfrak{d}_t &= 16\int_0^t \Omega_\tau B_\tau\bar{B}_\tau\,(\Omega_0+\mathbf{1})^{-1}\,\mathrm{d}\tau+16\int_0^t B_\tau\bar{B}_\tau\,(\Omega_0+\mathbf{1})^{-1}\,\mathrm{d}\tau\\ &\quad +16^2\int_0^t\mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2\ B_{\tau_2}\bar{B}_{\tau_2}B_{\tau_1}\bar{B}_{\tau_1}\,(\Omega_0+\mathbf{1})^{-1}\ .\end{aligned} \qquad \text{(V.272)}$$

On the domain $\mathcal{D}(\Omega_0)$, Equation (V.240) holds for $s = 0$. Indeed, $(B_t)_{t\geq 0} \in C[\mathbb{R}_0^+; \mathcal{L}^2(\mathfrak{h})]$ (Theorem 46 (ii)) and, in the strong limit $s \to 0^+$, both terms

$$\int_0^t B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau \mathrm{d}\tau \qquad \text{and} \qquad \int_0^t B_\tau \bar{B}_\tau \Omega_\tau \left(\Omega_0 + \mathbf{1}\right)^{-1} \mathrm{d}\tau \tag{V.273}$$

define bounded operators that are continuous in $\mathcal{L}^1(\mathfrak{h})$ for all $t \geq 0$: To obtain the boundedness and continuity of the first operator in (V.273), one uses (V.251) together with Lebesgue's dominated convergence theorem and Lemma 34 (ii) applied to $W_{t,s}^{\mathrm{t}}$ and $\left(W_{t,s}^{\mathrm{t}}\right)^*$. To get the boundedness and continuity of the second operator in (V.273), use (V.256). Therefore,

$$\int_0^t \Omega_\tau B_\tau \bar{B}_\tau \mathrm{d}\tau = B_0 \bar{B}_0 - B_t \bar{B}_t - 4 \int_0^t B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau \mathrm{d}\tau - 2 \int_0^t B_\tau \bar{B}_\tau \Omega_\tau \mathrm{d}\tau \tag{V.274}$$

on the domain $\mathcal{D}(\Omega_0)$. By (V.272) and Theorem 46 (ii), it follows that $(\mathfrak{d}_t)_{t\geq 0} \in C[\mathbb{R}_0^+; \mathcal{L}^1(\mathfrak{h})]$.

Now, for any $t \geq s \geq 0$,

$$\begin{aligned} \boldsymbol{V}_{t,s} &= \mathrm{e}^{-2(t-s)\Omega_0} + 2 \int_s^t \left(\Omega_0 + \mathbf{1}\right) \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \left(\Omega_0 + \mathbf{1}\right)^{-1} \mathrm{d}\tau \\ &= \mathrm{e}^{-2(t-s)\Omega_0} + 2 \int_s^t \mathrm{e}^{-2(t-\tau)\Omega_0} \mathfrak{d}_\tau \boldsymbol{V}_{\tau,s} \mathrm{d}\tau \ , \end{aligned} \tag{V.275}$$

because $W = \mathcal{T}(W)$ with $\mathcal{T}(W)$ defined by (V.6). Note that we again interchange the operator $(\Omega_0 + \mathbf{1})$ and the integral in (V.275). This is justified by using once more the closedness of the operator Ω_0 as follows: For any $t > s \geq 0$ and sufficiently small $\epsilon > 0$,

$$\begin{aligned} \Omega_0 \int_s^{t-\epsilon} \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau &= \Omega_0 \mathrm{e}^{-\epsilon\Omega_0} \int_s^{t-\epsilon} \mathrm{e}^{(-2(t-\tau)+\epsilon)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \\ &= \int_s^{t-\epsilon} \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \end{aligned} \tag{V.276}$$

is a bounded operator. It remains to take the limit $\epsilon \to 0^+$ in the strong topology. On the one hand,

$$\lim_{\epsilon \to 0^+} \int_s^{t-\epsilon} \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau = \int_s^t \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \tag{V.277}$$

for any $t > s \geq 0$, in the strong sense in $\mathfrak{h}$. On the other hand, the strong limit

$$\lim_{\epsilon \to 0^+} \int_s^{t-\epsilon} \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau = \int_s^t \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \tag{V.278}$$

defines again a bounded operator for any $t > s \geq 0$, see (V.39) and (V.44)–(V.45). Therefore, by combining (V.276)–(V.278) with the fact that Ω_0 is a closed operator we obtain

$$\Omega_0 \int_s^t \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau = \int_s^t \Omega_0 \mathrm{e}^{-2(t-\tau)\Omega_0} \Delta_\tau W_{\tau,s} \mathrm{d}\tau \qquad \text{(V.279)}$$

for any $t \geq s \geq 0$.
This last assertion justifies Equation (V.275), which leads to $(\boldsymbol{V}_{t,s})_{t \geq s \geq 0} \subset \mathcal{B}(\mathfrak{h})$, because $(\mathfrak{d}_t)_{t\geq 0} \in C[\mathbb{R}_0^+; \mathcal{L}^1(\mathfrak{h})]$. In other words, we obtain Assertion (i), which combined with (V.275), implies that the statements of Lemmata 34 and 36 hold true for the bounded evolution operator $(\boldsymbol{V}_{t,s})_{t\geq s\geq 0}$. One gets in particular Assertions (ii)–(iv). Like for Lemmata 34 and 36, similar properties as (i)–(iv) also hold for $(\mathfrak{d}_t^{\mathrm{t}})_{t\geq 0}$ and $(\boldsymbol{V}_{t,s}^{\mathrm{t}})_{t\geq s\geq 0}$. □

We are now in position to get a simple expression for the operator family $(K_t)_{t\geq 0}$, similar to (V.2) for $(B_t)_{t\geq 0}$:

Lemma 54 (Explicit expression for the "commutator")
Assume Conditions A1–A3 and (V.234). Then,

$$\forall t, s \in [0, T_+) : \qquad K_t = W_{t,s} K_s W_{t,s}^{\mathrm{t}} \ . \qquad \text{(V.280)}$$

Proof. Without loss of generality assume for convenience that $T_+ = \infty$. By Corollary 47, for any $t > 0$,

$$\partial_t \left\{ B_t \Omega_t^{\mathrm{t}} \right\} = -2(\Omega_t B_t)\Omega_t^{\mathrm{t}} - 2(B_t \Omega_t^{\mathrm{t}})\Omega_t^{\mathrm{t}} - 16 B_t \bar{B}_t B_t \ , \qquad \text{(V.281)}$$

in the strong sense in $\mathcal{D}(\Omega_0^{\mathrm{t}})$. Note indeed that both $\Omega_t B_t$ and $B_t \Omega_t^{\mathrm{t}}$ are Hilbert–Schmidt operators, by Lemma 45. On the other hand, by using (V.2) and the evolution operator $(\boldsymbol{V}_{t,s})_{t\geq s\geq 0}$ with the notation $\boldsymbol{V}_t := \boldsymbol{V}_{t,0}$ (see (V.270) and Lemma 53),

$$\Omega_t B_t (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} = \Omega_t (\Omega_0 + \mathbf{1})^{-1} \boldsymbol{V}_t (\Omega_0 + \mathbf{1}) B_0 (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} \boldsymbol{V}_t^{\mathrm{t}} \ , \qquad \text{(V.282)}$$

for all $t \geq 0$. Since, by (V.140) and (V.253),

$$\left\| \Omega_t (\Omega_0 + \mathbf{1})^{-1} \right\|_{\mathrm{op}} \leq 1 + 16t \left\| B_0 \right\|_2^2 < \infty \ , \qquad \text{(V.283)}$$

and (V.234) holds, we can compute the derivative of the operator (V.282) and infer from Corollary 47, Lemma 53, and (V.270) that, for any $t > 0$,

$$\partial_t \left\{ \Omega_t B_t \right\} = -2\Omega_t^2 B_t - 2\Omega_t B_t \Omega_t^{\mathrm{t}} - 16 B_t \bar{B}_t B_t \ , \qquad \text{(V.284)}$$

in the strong sense in $\mathcal{D}(\Omega_0^{\mathrm{t}})$. Indeed, $B_{t>0}\mathfrak{h} \subseteq \mathcal{D}(\Omega_0)$ (Theorem 41 (ii)) and $(\Omega_0^2 B_t)\mathfrak{h} \subseteq \mathcal{D}(\Omega_0)$ for $t > 0$ because of (V.234), $W_{t>0}^{\mathrm{t}}\mathfrak{h} \subseteq \mathcal{D}(\Omega_0^{\mathrm{t}})$ (Lemma 36 (ii) extended to all $t > s \geq 0$), $\boldsymbol{V}_{t>0}\mathfrak{h} \subseteq \mathcal{D}(\Omega_0)$ (Lemma 53 (iv)) and the equality

$$\Omega_0^2 B_t = \Omega_0 \boldsymbol{V}_t \left\{ (\Omega_0 + \mathbf{1}) B_0 (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} \right\} \left\{ (\Omega_0^{\mathrm{t}} + \mathbf{1}) W_t^{\mathrm{t}} \right\} \ . \qquad \text{(V.285)}$$

(In fact, $(\Omega_t^2 B_t)_{t>0} \subset \mathcal{L}^2(\mathfrak{h})$, by Lemma 56.) We combine now (V.281) and (V.284) with (V.233) to deduce that, for any $t > 0$,

$$\partial_t K_t = -2\left(\Omega_t K_t + K_t \Omega_t^{\mathrm{t}}\right) \ , \tag{V.286}$$

in the strong sense in $\mathcal{D}(\Omega_0^{\mathrm{t}})$. In particular, $K_{t>0}\mathfrak{h} \subseteq \mathcal{D}(\Omega_0)$. Thus, by Lemma 36 applied to $W_{t,s}$ and $W_{t,s}^{\mathrm{t}}$ and extended to all $t > s > 0$, it follows that, for any $t > s > 0$,

$$\partial_s \{W_{t,s} K_s W_{t,s}^{\mathrm{t}}\} = 0 \ , \tag{V.287}$$

in the strong sense in $\mathcal{D}(\Omega_0^{\mathrm{t}})$. In other words,

$$\forall t \geq s > 0 : \qquad K_t = W_{t,s} K_s W_{t,s}^{\mathrm{t}} \ , \tag{V.288}$$

on the dense domain $\mathcal{D}(\Omega_0^{\mathrm{t}})$. Both operators K_t and $W_{t,s} K_s W_{t,s}^{\mathrm{t}}$ are bounded, so we can extend by continuity (V.288) to the whole Hilbert space $\mathfrak{h}$. Hence, it remains to take the limit $s \to 0^+$ in (V.288).
By (V.2), (V.233), and (V.270), for any $t \geq 0$,

$$\begin{aligned} K_t(\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} &= \Omega_t(\Omega_0 + \mathbf{1})^{-1} \boldsymbol{V}_{t,s}(\Omega_0 + \mathbf{1}) B_0 (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} \boldsymbol{V}_{t,s}^{\mathrm{t}} \\ &\quad - B_t \Omega_t^{\mathrm{t}} (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} \ . \end{aligned} \tag{V.289}$$

Therefore, using (V.234), (V.283), Theorem 46 (i)–(ii), and Lemma 53 (cf. (i), (iii)) applied to $\boldsymbol{V}_{t,s}$ and $\boldsymbol{V}_{t,s}^{\mathrm{t}}$, we arrive at

$$\lim_{t \to 0^+} \left\| K_t (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} - K_0 (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} \right\|_{\mathrm{op}} = 0 \ . \tag{V.290}$$

Since, for any $t \geq s \geq 0$,

$$W_{t,s} K_s W_{t,s}^{\mathrm{t}} (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} = W_{t,s} K_s (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} \boldsymbol{V}_{t,s}^{\mathrm{t}} \ , \tag{V.291}$$

we infer from (V.7), (V.290), Lemma 34 (ii), and Lemma 53 (cf. (i), (iii)) that

$$\lim_{s \to 0^+} \left\| W_{t,s} K_s W_{t,s}^{\mathrm{t}} (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} - W_t K_0 W_t (\Omega_0^{\mathrm{t}} + \mathbf{1})^{-1} \right\|_{\mathrm{op}} = 0 \tag{V.292}$$

and (V.288) holds for $s = 0$ on the dense domain $\mathcal{D}(\Omega_0^{\mathrm{t}})$. Again, by (V.234), both operators K_t and $W_t K_0 W_t^{\mathrm{t}}$ are bounded, so (V.288) is also satisfied at $s = 0$ on the whole Hilbert space $\mathfrak{h}$. □
Therefore, by combining Theorem 52 (ii) with Lemma 54 we directly obtain a simple expression for the one–particle Hamiltonian Ω_t under the condition that $K_0 = 0$:

Corollary 55 (Constants of motion when $K_0 = 0$)
Assume Conditions A1–A3 and $\Omega_0 B_0 = B_0 \Omega_0^{\mathrm{t}}$. *Then, for any* $t, s \in [0, T_+)$,

$$\Omega_t B_t = B_t \Omega_t^{\mathrm{t}} \qquad \text{and} \qquad \Omega_t^2 - 4 B_t \bar{B}_t = \Omega_s^2 - 4 B_s \bar{B}_s \ . \tag{V.293}$$

This assertion concludes the proof of the third (iii) and fourth (iv) assertions of Theorem 11: (iii) is Theorem 52 (iii), whereas (iv) corresponds to Theorem 52 (i) and Corollary 55.

V.3 Asymptotics Properties of the Flow

First, since the system (III.9) of differential equations holds in the Hilbert–Schmidt topology (Corollary 47), we start by giving an explicit upper bound on the Hilbert–Schmidt norm of the operator $\Omega_t^\alpha B_t \in \mathcal{L}^2(\mathfrak{h})$:

Lemma 56 (Asymptotics properties of $\|\Omega_t^\alpha B_t\|_2$)
Assume Conditions A1–A4. Then, for all strictly positive numbers $\alpha, t > 0$ and any integer $n \in \mathbb{N}$,

$$\|\Omega_t^\alpha B_t\|_2 \leq \left(\frac{2^{n-1}\alpha}{\mathrm{e}t}\right)^\alpha \|B_0\|_2^{2^{-n}} \|B_t\|_2^{1-2^{-n}} \leq \left(\frac{2^{n-1}\alpha}{\mathrm{e}t}\right)^\alpha \|B_0\|_2 \ . \qquad \text{(V.294)}$$

Proof. Observe that

$$\|\Omega_t^\alpha B_t\|_2 \leq \left\|\Omega_t^{2\alpha}\mathrm{e}^{-4t\Omega_t}\right\|_{\mathrm{op}}^{1/2} \left\{\mathrm{tr}\left(\bar{B}_t \mathrm{e}^{4t\Omega_t} B_t\right)\right\}^{1/2} \ , \qquad \text{(V.295)}$$

using

$$\mathrm{tr}\,(X^*YX) \leq \|Y\|_{\mathrm{op}}\, \mathrm{tr}\,(X^*X) \ . \qquad \text{(V.296)}$$

Since $\Omega_t = \Omega_t^* \geq 0$, we already know that, for all $\alpha, t > 0$,

$$\left\|\Omega_t^{2\alpha}\mathrm{e}^{-4t\Omega_t}\right\|_{\mathrm{op}}^{1/2} \leq \left(\frac{\alpha}{2\mathrm{e}t}\right)^\alpha \ , \qquad \text{(V.297)}$$

see (V.31). To use (V.295) we thus need to prove that the trace

$$\mathrm{tr}\left(\bar{B}_t \mathrm{e}^{4t\Omega_t} B_t\right) \qquad \text{(V.298)}$$

exists and is uniformly bounded for all times $t \geq 0$. The self–adjoint operator Ω_t is possibly unbounded and we consequently study the trace

$$\mathrm{tr}\left(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t\right) \qquad \text{(V.299)}$$

for $t \geq 0$ and $\lambda > 0$, where the positive, bounded operator $\Omega_{t,\lambda} = \Omega_{t,\lambda}^* \geq 0$ is the Yosida approximation

$$\Omega_{t,\lambda} := \frac{\lambda\Omega_t}{\lambda\mathbf{1} + \Omega_t} = \lambda - \lambda^2 \left(\lambda\mathbf{1} + \Omega_t\right)^{-1} = \Omega_{t,\lambda}^* \in \mathcal{B}(\mathfrak{h}) \qquad \text{(V.300)}$$

of the positive self–adjoint operator Ω_t. Since $\Omega_t \geq 0$ and $\|B_t\|_2 \leq \|B_0\|_2 < \infty$, note that $\|\Omega_{t,\lambda}\|_{\mathrm{op}} \leq \lambda$ and

$$\mathrm{tr}\left(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t\right) \leq \mathrm{e}^{4t\lambda}\|B_t\|_2^2 \leq \mathrm{e}^{4t\lambda}\|B_0\|_2^2 < \infty \ . \qquad \text{(V.301)}$$

We analyze now the derivative of $\mathrm{tr}\left(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t\right)$ for any $t > 0$. If $t \mapsto X_t \in \mathcal{B}(\mathfrak{h})$ is strongly differentiable then $t \mapsto \exp(X_t)$ is strongly differentiable and its derivative equals

$$\partial_t \left\{\mathrm{e}^{X_t}\right\} = \int_0^1 \mathrm{e}^{(1-\tau)X_t} \left(\partial_t X_t\right) \mathrm{e}^{\tau X_t} \mathrm{d}\tau \ , \qquad \text{(V.302)}$$

see [27, 28]. Since the operator family $(\Omega_t)_{t\geq 0}$ is (at least) strongly differentiable and

$$\partial_t \Omega_{t,\lambda} = -16\lambda^2 \left(\lambda \mathbf{1} + \Omega_t\right)^{-1} B_t \bar{B}_t \left(\lambda \mathbf{1} + \Omega_t\right)^{-1} \qquad \text{(V.303)}$$

(see Theorem 41 (i)), a straightforward computation using (V.302) shows that

$$\partial_t \left\{ \mathrm{e}^{4t\Omega_{t,\lambda}} \right\} = 4\Omega_{t,\lambda} \mathrm{e}^{4t\Omega_{t,\lambda}} - 64t\lambda^2 \int_0^1 \frac{\mathrm{e}^{4t(1-\tau)\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} B_t \bar{B}_t \frac{\mathrm{e}^{4\tau t\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} \,\mathrm{d}\tau \ . \qquad \text{(V.304)}$$

Note that

$$\left\| \partial_t \left\{ \mathrm{e}^{4t\Omega_{t,\lambda}} \right\} \right\|_{\mathrm{op}} \leq 4\lambda \mathrm{e}^{4t\lambda} + 64t \mathrm{e}^{4t\lambda} \|B_0\|_2^4 \ , \qquad \text{(V.305)}$$

because $\|\Omega_{t,\lambda}\|_{\mathrm{op}} \leq \lambda$ and $\|B_t\|_2 \leq \|B_0\|_2 < \infty$. Therefore, we infer from Corollary 47, Lemma 103, and (V.304) that , for all $t > 0$,

$$\begin{aligned} \partial_t \left\{ \mathrm{tr}\left(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t \right) \right\} \;=\; & -2\mathrm{tr}\left(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t \Omega_t^{\mathrm{t}} \right) - 2\mathrm{tr}\left(\Omega_t^{\mathrm{t}} \bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t \right) \\ & -64t\lambda^2 \int_0^1 \mathrm{tr}\left(\bar{B}_t \frac{\mathrm{e}^{4t(1-\tau)\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} B_t \bar{B}_t \frac{\mathrm{e}^{4\tau t\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} B_t \right) \mathrm{d}\tau \\ & +4\mathrm{tr}\left(\bar{B}_t \mathrm{e}^{2t\Omega_{t,\lambda}} \left(\Omega_{t,\lambda} - \Omega_t \right) \mathrm{e}^{2t\Omega_{t,\lambda}} B_t \right) \ . \end{aligned} \qquad \text{(V.306)}$$

Note that the exchange of the trace with the integral on the finite $[0,1]$, which is performed in (V.306), is justified by Lemma 104 as the operators

$$\frac{\mathrm{e}^{4t(1-\tau)\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} \qquad \text{and} \qquad \frac{\mathrm{e}^{4\tau t\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} \qquad \text{(V.307)}$$

are bounded for any $t, \tau \geq 0$, whereas $B_t \in \mathcal{L}^2(\mathfrak{h})$ is a Hilbert–Schmidt operator. By cyclicity of the trace (cf. Lemma 101),

$$\mathrm{tr}\left(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t \Omega_t^{\mathrm{t}} \right) = \mathrm{tr}\left(\Omega_t^{\mathrm{t}} \bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t \right) = \mathrm{tr}\left(\mathrm{e}^{2t\Omega_{t,\lambda}} B_t \Omega_t^{\mathrm{t}} \bar{B}_t \mathrm{e}^{2t\Omega_{t,\lambda}} \right) \geq 0 \ , \qquad \text{(V.308)}$$

because $\Omega_t^{\mathrm{t}} = (\Omega_t^{\mathrm{t}})^* \geq 0$ and $\mathrm{e}^{2t\Omega_{t,\lambda}} B_t \in \mathcal{L}^2(\mathfrak{h})$. One also has

$$\Omega_{t,\lambda} - \Omega_t = -\frac{\Omega_t^2}{\lambda \mathbf{1} + \Omega_t} \leq 0 \qquad \text{(V.309)}$$

and, by cyclicity of the trace (Lemma 101 (i)),

$$\begin{aligned} & \mathrm{tr}\left(\bar{B}_t \frac{\mathrm{e}^{4t(1-\tau)\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} B_t \bar{B}_t \frac{\mathrm{e}^{4\tau t\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} B_t \right) \qquad \text{(V.310)} \\ = \; & \mathrm{tr}\left(\frac{\mathrm{e}^{2\tau t\Omega_{t,\lambda}}}{\left(\lambda \mathbf{1} + \Omega_t\right)^{1/2}} B_t \bar{B}_t \frac{\mathrm{e}^{4t(1-\tau)\Omega_{t,\lambda}}}{\lambda \mathbf{1} + \Omega_t} B_t \bar{B}_t \frac{\mathrm{e}^{2\tau t\Omega_{t,\lambda}}}{\left(\lambda \mathbf{1} + \Omega_t\right)^{1/2}} \right) \geq 0 \ . \end{aligned}$$

Consequently, we infer from (V.306)–(V.310) that $\partial_t \{ \mathrm{tr}(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t) \} \leq 0$ for any $t > 0$. In particular, by (V.301),

$$\forall t \geq s > 0 : \quad \mathrm{tr}(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t) \leq \mathrm{tr}(\bar{B}_s \mathrm{e}^{4s\Omega_{s,\lambda}} B_s) \leq \mathrm{e}^{4s\lambda} \|B_0\|_2^2 \ , \qquad \text{(V.311)}$$

which in turn implies that

$$\forall t \geq 0, \lambda > 0 : \qquad \mathrm{tr}(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t) \leq \|B_0\|_2^2 \ . \tag{V.312}$$

We proceed by taking the limit $\lambda \to \infty$ in (V.312). Since the family $(y_\lambda)_{\lambda>0}$ of real functions $\mathbb{R}_0^+ \to \mathbb{R}_0^+$,

$$x \mapsto y_\lambda(x) = \frac{\lambda x}{\lambda + x} \tag{V.313}$$

is monotonically increasing in λ, the spectral theorem applied to the positive self–adjoint operator $\Omega_{t,\lambda} = \Omega_{t,\lambda}^* \geq 0$ together with the monotone convergence theorem yields

$$\lim_{\lambda\to\infty} \langle \varphi | \mathrm{e}^{4t\Omega_{t,\lambda}} \varphi \rangle = \begin{cases} \langle \varphi | \mathrm{e}^{4t\Omega_t} \varphi \rangle & , \qquad \text{if } \varphi \in \mathcal{D}\left(\mathrm{e}^{2t\Omega_t}\right) , \\ \infty & , \qquad \text{if } \varphi \notin \mathcal{D}\left(\mathrm{e}^{2t\Omega_t}\right) , \end{cases} \tag{V.314}$$

for all $\varphi \in \mathfrak{h}$. It follows that $B_t \mathfrak{h} \in \mathcal{D}\left(\mathrm{e}^{2t\Omega_t}\right)$ for all $t \geq 0$, because of the inequalities (V.312) and $\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t \geq 0$. In particular, thanks to the limit (V.314), we obtain

$$\lim_{\lambda\to\infty} \langle B_t \varphi | \mathrm{e}^{4t\Omega_{t,\lambda}} B_t \varphi \rangle = \langle B_t \varphi | \mathrm{e}^{4t\Omega_t} B_t \varphi \rangle < \infty \tag{V.315}$$

for all $\varphi \in \mathfrak{h}$. We finally invoke again the monotone convergence theorem to infer from (V.312) and (V.315) that

$$\lim_{\lambda\to\infty} \left\{ \mathrm{tr}(\bar{B}_t \mathrm{e}^{4t\Omega_{t,\lambda}} B_t) \right\} = \mathrm{tr}(\bar{B}_t \mathrm{e}^{4t\Omega_t} B_t) \leq \|B_0\|_2^2 \tag{V.316}$$

for all $t \geq 0$. Combining the inequality (V.295) with the upper bounds (V.297) and (V.316) we arrive at the assertion

$$\forall t > 0 : \qquad \|\Omega_t^\alpha B_t\|_2 \leq \left(\frac{\alpha}{2\mathrm{e}t}\right)^\alpha \|B_0\|_2 < \infty \ . \tag{V.317}$$

This bound can again be improved by recursively using the Cauchy–Schwarz inequality. Indeed,

$$\|\Omega_t^\alpha B_t\|_2 = \{\mathrm{tr}\left(\bar{B}_t \Omega_t^{2\alpha} B_t\right)\}^{1/2} \leq \|\Omega_t^{2\alpha} B_t\|_2^{1/2} \|B_t\|_2^{1/2} \ . \tag{V.318}$$

We proceed by again applying the Cauchy–Schwarz inequality on $\|\Omega_t^{2\alpha} B_t\|_2$ in (V.318) to obtain

$$\|\Omega_t^\alpha B_t\|_2 \leq \|\Omega_t^{4\alpha} B_t\|_2^{\frac{1}{4}} \|B_t\|_2^{\frac{1}{2}+\frac{1}{4}} \ . \tag{V.319}$$

Doing this n times, we obtain

$$\|\Omega_t^\alpha B_t\|_2 \leq \|\Omega_t^{2^n\alpha} B_t\|_2^{2^{-n}} \|B_t\|_2^{1-2^{-n}} \ , \tag{V.320}$$

as

$$\sum_{j=1}^n 2^{-j} = 1 - 2^{-n} \ . \tag{V.321}$$

We now combine (V.317) with (V.320) to show that

$$\|\Omega_t^\alpha B_t\|_2 \leq \left(\frac{2^{n-1}\alpha}{\mathrm{e}t}\right)^\alpha \|B_0\|_2^{2^{-n}} \|B_t\|_2^{1-2^{-n}} \tag{V.322}$$

for any $n \in \mathbb{N}$. □

COROLLARY 57 (ASYMPTOTICS PROPERTIES OF $\|B_t\|_2$)
Assume Conditions A1–A4. Then, for any $t \geq 0$, $n \in \mathbb{N}$, and $\alpha > 0$,

$$\|B_t\|_2 \leq \left(\frac{2^{n-1}\alpha}{\mathrm{e}t}\right)^{\frac{\alpha}{1+2^{-n}}} \|\Omega_t^{-\alpha} B_t\|_2^{\frac{1}{1+2^{-n}}} \|B_0\|_2^{\frac{1}{2^n+1}} , \tag{V.323}$$

provided $\|\Omega_t^{-\alpha} B_t\|_2 < \infty$.

PROOF. Using the Cauchy–Schwarz inequality for the trace note that

$$\|B_t\|_2^2 = \mathrm{tr}\left(\bar{B}_t \Omega_t^\alpha \Omega_t^{-\alpha} B_t\right) \leq \|\Omega_t^{-\alpha} B_t\|_2 \|\Omega_t^\alpha B_t\|_2 \tag{V.324}$$

for any $\alpha > 0$. Therefore, by Lemma 56, we find that, for any $n \in \mathbb{N}$,

$$\|B_t\|_2^2 \leq \left(\frac{2^{n-1}\alpha}{\mathrm{e}t}\right)^\alpha \|\Omega_t^{-\alpha} B_t\|_2 \|B_0\|_2^{2^{-n}} \|B_t\|_2^{1-2^{-n}} \tag{V.325}$$

from which we deduce the assertion. □
Therefore, since the map

$$t \mapsto \|\mathfrak{B}_t\|_2 = \|\Omega_t^{-1/2} B_t\|_2^2 \tag{V.326}$$

is monotonously decreasing (Lemma 49 (i)), by Corollary 57 for $\alpha = 1/2$, one deduces a first *explicit* upper bound on Hilbert–Schmidt norm $\|B_t\|_2$:

$$\|B_t\|_2^2 \leq \|\mathfrak{B}_0\|_1^{\frac{1}{1+2^{-n}}} \|B_0\|_2^{\frac{2}{2^n+1}} \left(\frac{2^{n-2}}{\mathrm{e}t}\right)^{\frac{1}{1+2^{-n}}} < \infty \tag{V.327}$$

for any $t > 0$ and $n \in \mathbb{N}$. In particular, for any $\delta \in (0,1)$ and as $t \to \infty$, $\|B_t\|_2^2 = o\left(t^{-\delta}\right)$. Unfortunately, this last estimate is not good enough to obtain the square–integrability of the map $t \mapsto \|B_t\|_2$ on $[0, \infty)$. This is nevertheless proven in the next lemma:

LEMMA 58 (SQUARE–INTEGRABILITY OF $\|B_t\|_2$)
Assume Conditions A1–A4. Then the map $t \mapsto \|B_t\|_2$ is square–integrable on $[0, \infty)$:

$$\forall t \geq s \geq 0 : \qquad 4\int_s^t \|B_\tau\|_2^2 \,\mathrm{d}\tau \leq \|\mathfrak{B}_s\|_1 - \|\mathfrak{B}_t\|_1 \ . \tag{V.328}$$

PROOF. Using $\Omega_t \geq 4\mathfrak{B}_t \geq 4\mathfrak{B}_{t,\mu}$ and $\|\mathfrak{B}_t\|_1 \leq \|\mathfrak{B}_0\|_1 < \infty$ (Lemma 49), for all $t > 0$, Equality (V.179) implies that

$$\partial_t q_\mu(t) \leq -4\mathrm{tr}\left\{B_t \bar{B}_t\right\} + 4\mu\|\mathfrak{B}_t\|_1 \ . \tag{V.329}$$

In the limit $\mu \to 0$ (cf. (V.206)), this differential inequality yields the square–integrability on the whole positive real line $[0, \infty)$ of the map $t \mapsto \|B_t\|_2$ as well as (V.328), which includes $s = 0$ because of (V.185). □

Much stronger decays on the Hilbert–Schmidt norm $\|B_t\|_2$ can be obtained, for instance under Condition A6, that is, under the assumption that

$$\Omega_0 \geq 4B_0(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 + \mu\mathbf{1} = 4\mathfrak{B}_0 + \mu\mathbf{1} \tag{V.330}$$

for some constant $\mu > 0$. Indeed, in this case, the map $t \mapsto \|B_t\|_2$ decays exponentially to zero, as $t \to \infty$, because A6 is conserved for all times, by Lemma 44. More precisely, one gets the following asymptotics on the Hilbert–Schmidt norm $\|B_t\|_2$:

LEMMA 59 (ASYMPTOTICS OF $\|B_t\|_2$ UNDER A GAP CONDITION)
Assume Conditions A1–A2 and A6, that is, $\Omega_0 \geq 4\mathfrak{B}_0 + \mu\mathbf{1}$ *for some* $\mu > 0$. *Then*

$$\forall \alpha \in \mathbb{R}, t \geq \max(0, \tfrac{\alpha}{2\mu}) : \qquad \|\Omega_t^\alpha B_t\|_2 \leq \mu^\alpha \mathrm{e}^{-2t\mu}\|B_0\|_2 \ . \tag{V.331}$$

In particular, $\|B_t\|_2$ *decays exponentially to zero, as* $t \to \infty$.

PROOF. If $\Omega_0 \geq 4\mathfrak{B}_0 + \mu\mathbf{1}$ for some $\mu > 0$, then $\Omega_t \geq \mu\mathbf{1}$ for all $t \geq 0$ because of Lemma 44. Therefore, we get the assertion by using the upper bound

$$\|\Omega_t^{2\alpha}\mathrm{e}^{-4t\Omega_t}\|_{\mathrm{op}} \leq \sup_{\omega \geq \mu}\left\{\omega^{2\alpha}\mathrm{e}^{-4t\omega}\right\} = \mu^{2\alpha}\mathrm{e}^{-4t\mu} \tag{V.332}$$

for any $t \geq \max(0, \frac{\alpha}{2\mu})$ together with Inequalities (V.295) and (V.316). □

Therefore, under the gap condition A6, the map $t \mapsto \|B_t\|_2$ is clearly integrable on $[0, \infty)$. We would like to prove this property under weaker conditions than A6. A minimal requirement is to assume A1–A4. In fact, we strengthen A4 by assuming next that $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$ is a Hilbert–Schmidt operator.

As explained in Section IV.1 via Proposition 29, this property is *pivotal* with respect to the integrability of the map $t \mapsto \|B_t\|_2$ on $[0, \infty)$. It is conserved by the flow, which equivalently means that

$$\forall t \geq 0 : \qquad \mathfrak{C}_t := B_t(\Omega_t^{\mathrm{t}})^{-2}\bar{B}_t \in \mathcal{L}^1(\mathfrak{h}) \tag{V.333}$$

is a trace–class operator for all times. Indeed, the following assertion holds:

LEMMA 60 (CONSERVATION BY THE FLOW OF $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$)
Assume Conditions A1–A3 and $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$. *Then*

$$\forall t \geq s \geq 0 : \qquad \|\mathfrak{C}_t\|_1 \leq \|\mathfrak{C}_s\|_1 \exp\left\{32\int_s^t \|\mathfrak{B}_\tau\|_1 \mathrm{d}\tau\right\} < \infty \ . \tag{V.334}$$

In particular, $(\Omega_t^{-1}B_t)_{t\geq 0} \subset \mathcal{L}^2(\mathfrak{h})$ *is a family of Hilbert–Schmidt operators.*

PROOF. First, observe that $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$ implies $(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 \in \mathcal{L}^2(\mathfrak{h})$, by Lemma 102 and $B_0 = B_0^{\mathrm{t}}$. In particular, $\mathfrak{E}_0 \in \mathcal{L}^1(\mathfrak{h})$ since

$$\|\mathfrak{E}_0\|_1 := \|(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0\|_2^2 = \|\Omega_0^{-1}B_0\|_2^2 < \infty \ . \tag{V.335}$$

Moreover, $B_0 \in \mathcal{L}^2(\mathfrak{h})$ and $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$ also yield $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$ because

$$\begin{aligned}\|\Omega_0^{-1/2}B_0\|_2 &\leq \|\Omega_0^{-1/2}\mathbb{P}_{\Omega_0\leq \mathbf{1}}B_0\|_2 + \|\Omega_0^{-1/2}\left(\mathbf{1}-\mathbb{P}_{\Omega_0\leq\mathbf{1}}\right)B_0\|_2 \\ &\leq \|\Omega_0^{-1}B_0\|_2 + \|B_0\|_2 \ ,\end{aligned} \tag{V.336}$$

where the operator $\mathbb{P}_{\Omega_0\leq\mathbf{1}}$ is the spectral projection of the positive self–adjoint operator $\Omega_0 = \Omega_0^* \geq 0$ on the interval $[0,1]$.
Let $\mu > 0$ and set

$$\forall t \geq 0 : \qquad \mathfrak{E}_{t,\mu} := B_t \left(\Omega_t^{\mathrm{t}} + \mu\right)^{-2} \bar{B}_t \geq 0 \ . \tag{V.337}$$

Similar to (V.177), $\mathfrak{E}_{t,\mu}$ is a bounded operator as $\mu > 0$ and $\Omega_t^{\mathrm{t}} \geq 0$, by Lemma 51. From Corollary 47 and Lemma 103 we also observe that the function

$$p_\mu(t) := \mathrm{tr}\left\{\mathfrak{E}_{t,\mu}\right\} = \|\mathfrak{E}_{t,\mu}\|_1 \tag{V.338}$$

satisfies, for any strictly positive $t > 0$, the differential inequality

$$\begin{aligned}\partial_t p_\mu(t) &= 16\mathrm{tr}\left\{B_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-2}\bar{B}_tB_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-1}\bar{B}_t\right\} \qquad \text{(V.339)}\\ &\quad +16\mathrm{tr}\left\{B_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-1}\bar{B}_tB_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-2}\bar{B}_t\right\} \\ &\quad -4\mathrm{tr}\left\{\Omega_tB_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-2}\bar{B}_t + B_t\Omega_t^{\mathrm{t}}\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-2}\bar{B}_t\right\} \\ &\leq 32\|\mathfrak{B}_t\|_{\mathrm{op}}\ \mathrm{tr}\{\mathfrak{E}_{t,\mu}\} \leq 32\|\mathfrak{B}_t\|_1\ p_\mu(t) \ , \qquad \text{(V.340)}\end{aligned}$$

by using the cyclicity of the trace (Lemma 101), the positivity of the self–adjoint operators $\Omega_t, \Omega_t^{\mathrm{t}} \geq 0$, and Lemma 49 (i). Therefore, thanks to Grønwall's Lemma, we obtain that

$$\forall t \geq s > 0 : \qquad \|\mathfrak{E}_{t,\mu}\|_1 \leq \|\mathfrak{E}_{s,\mu}\|_1 \exp\left\{32\int_s^t \|\mathfrak{B}_\tau\|_1 \mathrm{d}\tau\right\} \ , \tag{V.341}$$

for all $\mu > 0$. Similar to (V.184), note that

$$|p_\mu(t) - p_\mu(s)| \leq \|\mathfrak{E}_{t,\mu} - \mathfrak{E}_{s,\mu}\|_1 \leq 2\mu^{-3}\|\Omega_t^{\mathrm{t}} - \Omega_s^{\mathrm{t}}\|_1\|B_0\|_2^2 + 2\mu^{-2}\|B_0\|_2\|B_t - B_s\|_2 \tag{V.342}$$

for all $\mu > 0$ and $s,t \geq 0$. Combined with Theorem 46 (i)–(ii), it shows the continuity of the function p_μ:

$$\lim_{t\to s} p_\mu(t) = p_\mu(s) \leq p_0(s) = \|\mathfrak{E}_s\|_1 \tag{V.343}$$

for any $t, s \geq 0$. The latter implies (V.341) for $s = 0$:

$$\forall t \geq s \geq 0: \qquad \|\mathfrak{E}_{t,\mu}\|_1 \leq \|\mathfrak{E}_{s,\mu}\|_1 \exp\left\{32 \int_s^t \|\mathfrak{B}_\tau\|_1 \mathrm{d}\tau\right\} < \infty \ , \qquad \text{(V.344)}$$

see (V.335). In particular, $\ker(\Omega_t^{\mathrm{t}}) \cap \mathrm{Ran}(B_t) = \emptyset$ as already mentioned in the proof of Lemma 49. Moreover, similar to (V.203) and (V.206), we infer from (V.344) together with the monotone convergence theorem that, for all $t \geq 0$,

$$\lim_{\mu \to 0^+} \|\mathfrak{E}_{t,\mu} - \mathfrak{E}_{t,0}\|_1 = 0 \ , \qquad \text{(V.345)}$$

with $\mathfrak{E}_t \equiv \mathfrak{E}_{t,0}$. Using this and (V.344) we thus deduce the upper bound of the lemma, which yields $\Omega_t^{-1} B_t \in \mathcal{L}^2(\mathfrak{h})$ as

$$\forall t \geq 0: \qquad \|\mathfrak{E}_t\|_1 := \|(\Omega_t^{\mathrm{t}})^{-1} \bar{B}_t\|_2^2 = \|\Omega_t^{-1} B_t\|_2^2 < \infty \ , \qquad \text{(V.346)}$$

similar to establishing (V.335). □

Lemma 60 indicates that the integrability of the map $t \mapsto \|\mathfrak{B}_t\|_1$ on the positive real line $[0, \infty)$ should also be an important property for a "good" asymptotics of the flow. To obtain this, we need, at least, the assumption

$$\mathbf{1} \geq 4 B_0 (\Omega_0^{\mathrm{t}})^{-2} \bar{B}_0 = 4 \mathfrak{E}_0 \ . \qquad \text{(V.347)}$$

In this case, the behavior of the map $t \mapsto \|\mathfrak{E}_t\|_1$ becomes better as it is monotonically decreasing:

LEMMA 61 (CONSERVATION BY THE FLOW OF $B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0 < \mathbf{1}/(4+\mathrm{r})$)
Assume Conditions A1–A3, $\Omega_0^{-1} B_0 \in \mathcal{L}^2(\mathfrak{h})$, and $\mathfrak{E}_0 \leq \mathbf{1}/(4+\mathrm{r})$ for $\mathrm{r} \geq 0$.
(i) *The family $(\mathfrak{E}_t)_{t\geq 0} \subset \mathcal{L}^1(\mathfrak{h})$ defined by (V.333) satisfies: $\mathfrak{E}_t \leq \mathbf{1}/(4+\mathrm{r})$.*
(ii) *The map $t \mapsto \|\mathfrak{E}_t\|_1$ from $\mathbb{R}_0^+$ to $\mathbb{R}_0^+$ is monotonically decreasing. In particular, $\|\mathfrak{E}_t\|_1 \leq \|\mathfrak{E}_0\|_1$ for all $t \geq 0$.*

PROOF. (i): Recall that $\Omega_0^{-1} B_0 \in \mathcal{L}^2(\mathfrak{h})$ yields Condition A4, that is, $\Omega_0^{-1/2} B_0 \in \mathcal{L}^2(\mathfrak{h})$. By Theorems 46, 50 and Corollary 47 , the trace–class operator $\mathfrak{E}_{t,\mu} \in \mathcal{L}^1(\mathfrak{h})$ defined by (V.337) for any $t \geq 0$ and $\mu > 0$ has derivative (at least in the strong sense in $\mathcal{D}(\Omega_0)$) equal to

$$\forall t \geq s > 0: \qquad \partial_t \mathfrak{E}_{t,\mu} = -4\mathfrak{B}_{t,\mu} - \mathcal{G}_{t,\mu}\mathfrak{E}_{t,\mu} - \mathfrak{E}_{t,\mu}\mathcal{G}_{t,\mu} \ , \qquad \text{(V.348)}$$

where

$$\mathcal{G}_{t,\mu} := 2\Omega_t - 16\mathfrak{B}_{t,\mu} - 2\mu \quad \text{and} \quad \mathfrak{B}_{t,\mu} := B_t \left(\Omega_t^{\mathrm{t}} + \mu \mathbf{1}\right)^{-1} \bar{B}_t \geq 0 \ , \qquad \text{(V.349)}$$

see (V.177). Therefore, we introduce, as before, the evolution operator $\mathfrak{M}_{t,s}$ defined by the non–autonomous evolution equation

$$\forall t > s \geq 0: \qquad \partial_t \mathfrak{M}_{t,s} = -\mathcal{G}_{t,\mu}\mathfrak{M}_{t,s} \ , \quad \mathfrak{M}_{s,s} := \mathbf{1} \ . \qquad \text{(V.350)}$$

Indeed, the (possibly unbounded) generator

$$\mathcal{G}_{t,\mu}=2\Omega_0-32\int_0^t B_\tau \bar{B}_\tau \mathrm{d}\tau-16\mathfrak{B}_{t,\mu}-2\mu \tag{V.351}$$

is the sum of a positive operator $2\Omega_0\geq 0$ and a bounded one

$$\mathcal{C}_{t,\mu}:=-32\int_0^t B_\tau \bar{B}_\tau \mathrm{d}\tau-16\mathfrak{B}_{t,\mu}-2\mu \tag{V.352}$$

with operator norm

$$\|\mathcal{C}_{t,\mu}\|_{\mathrm{op}}\leq 32t\|B_0\|_2+16\|\mathfrak{B}_0\|_1+2\mu\ , \tag{V.353}$$

see (V.140), Lemma 49 (i), and Theorem 50. As $\{\mathcal{C}_{t,\mu}\}_{t\geq 0}$ is also continuous with respect to the norm topology for any $\mu>0$ (Theorem 46 (i)–(ii)), the evolution operator $\mathfrak{M}_{t,s}$ solving (V.350) exists and is unique, bounded uniformly in s,t on compact sets, and norm continuous for any $t>s$, see arguments given in Lemmata 34 and 36. Furthermore, on the domain $\mathcal{D}(\Omega_0)$,

$$\forall t>0,\ t\geq s\geq 0:\qquad \partial_s\mathfrak{M}_{t,s}=\mathfrak{M}_{t,s}\mathcal{G}_{s,\mu}\ ,\quad \mathfrak{M}_{t,t}:=\mathbf{1}\ , \tag{V.354}$$

whereas one verifies that

$$\forall t>s\geq 0:\qquad \partial_s\mathfrak{M}_{t,s}^*=\mathcal{G}_{s,\mu}\mathfrak{M}_{t,s}^*\ ,\quad \mathfrak{M}_{t,t}^*:=\mathbf{1}\ , \tag{V.355}$$

in the strong sense in $\mathfrak{h}$ (as $\mathfrak{M}_{t,s}^*\mathfrak{h}\subseteq\mathcal{D}(\Omega_0)$ for $t>s$). Note that one proves (V.355) by observing that $\mathfrak{M}_{t,s}$ has a representation in term of a series constructed from the integral equation

$$\forall t\geq s\geq 0:\qquad \mathfrak{M}_{t,s}=\mathrm{e}^{-2(t-s)\Omega_0}-\int_s^t\mathfrak{M}_{t,\tau}\mathcal{C}_{\tau,\mu}\mathrm{e}^{-2(\tau-s)\Omega_0}\mathrm{d}\tau\ , \tag{V.356}$$

similar to the series (V.56) with $(-\mathcal{C}_{t,\mu})_{t\geq 0}$ replacing $(2\Delta_t)_{t\geq 0}$. See also Lemma 36. We omit the details.

Using $\mathfrak{E}_{t,\mu}\mathfrak{h}\subseteq\mathcal{D}(\Omega_0)$ for $t>0$ (as $B_{t>0}\mathfrak{h}\subseteq\mathcal{D}(\Omega_0)$), we deduce from (V.348) and (V.354)–(V.355) that, for any $t>s>0$,

$$\partial_s\{\mathfrak{M}_{t,s}\mathfrak{E}_{s,\mu}\mathfrak{M}_{t,s}^*\}=-4\mathfrak{M}_{t,s}\mathfrak{B}_{s,\mu}\mathfrak{M}_{t,s}^*\ . \tag{V.357}$$

This last derivative is verified by using the upper bound

$$\begin{aligned}
&\Big\|\big(\epsilon^{-1}\left(\mathfrak{M}_{t,s+\epsilon}\mathfrak{E}_{s+\epsilon,\mu}\mathfrak{M}_{t,s+\epsilon}^*-\mathfrak{M}_{t,s}\mathfrak{E}_{s,\mu}\mathfrak{M}_{t,s}^*\right)-\mathfrak{M}_{t,s}\mathfrak{E}_{s,\mu}\partial_s\left\{\mathfrak{M}_{t,s}^*\right\}\\
&\quad-\mathfrak{M}_{t,s}\partial_s\left\{\mathfrak{E}_{s,\mu}\right\}\mathfrak{M}_{t,s}^*-\partial_s\left\{\mathfrak{M}_{t,s}\right\}\mathfrak{E}_{s,\mu}\mathfrak{M}_{t,s}^*\big)\varphi\Big\|\\
\leq\ &\|\mathfrak{M}_{t,s+\epsilon}\|_{\mathrm{op}}\|\mathfrak{E}_{s+\epsilon,\mu}-\mathfrak{E}_{s,\mu}\|_{\mathrm{op}}\left\|\epsilon^{-1}\left(\mathfrak{M}_{t,s+\epsilon}^*-\mathfrak{M}_{t,s}^*\right)\varphi\right\|\\
&+\|\mathfrak{M}_{t,s+\epsilon}\|_{\mathrm{op}}\|\mathfrak{E}_{s,\mu}\|_{\mathrm{op}}\left\|\epsilon^{-1}\left(\mathfrak{M}_{t,s+\epsilon}^*-\mathfrak{M}_{t,s}^*\right)-\partial_s\left\{\mathfrak{M}_{t,s}^*\right\}\varphi\right\|\\
&+\left\|\left(\mathfrak{M}_{t,s+\epsilon}-\mathfrak{M}_{t,s}\right)\mathfrak{E}_{s,\mu}\partial_s\left\{\mathfrak{M}_{t,s}^*\right\}\varphi\right\|\\
&+\|\mathfrak{M}_{t,s+\epsilon}\|_{\mathrm{op}}\left\|\left(\epsilon^{-1}\left(\mathfrak{E}_{s+\epsilon,\mu}-\mathfrak{E}_{s,\mu}\right)-\partial_s\left\{\mathfrak{E}_{s,\mu}\right\}\right)\mathfrak{M}_{t,s}^*\varphi\right\|\\
&+\left\|\left(\mathfrak{M}_{t,s+\epsilon}-\mathfrak{M}_{t,s}\right)\partial_s\left\{\mathfrak{E}_{s,\mu}\right\}\mathfrak{M}_{t,s}^*\varphi\right\|\\
&+\left\|\left(\epsilon^{-1}\left(\mathfrak{M}_{t,s+\epsilon}-\mathfrak{M}_{t,s}\right)-\partial_s\left\{\mathfrak{M}_{t,s}\right\}\right)\mathfrak{E}_{s,\mu}\mathfrak{M}_{t,s}^*\varphi\right\|\ ,
\end{aligned} \tag{V.358}$$

for any $\varphi \in \mathfrak{h}$, $t > s > 0$, and sufficiently small $|\epsilon| > 0$. Indeed, by (V.342), $(\mathfrak{E}_{s,\mu})_{s\geq 0}$ is norm continuous, $\mathfrak{M}_{t,s}^*$ and its adjoint are both strongly continuous and uniformally bounded in t, s on compact sets. Therefore, it follows from (V.348), (V.354)–(V.355), and (V.358) in the limit $\epsilon \to 0$ that (V.357) holds and we thus arrive at the equality

$$\forall t \geq s > 0: \qquad \mathfrak{E}_{t,\mu} = \mathfrak{M}_{t,s}\mathfrak{E}_{s,\mu}\mathfrak{M}_{t,s}^* - 4\int_s^t \mathfrak{M}_{t,\tau}\mathfrak{B}_{\tau,\mu}\mathfrak{M}_{t,\tau}^* \mathrm{d}\tau \ . \qquad \text{(V.359)}$$

Combining (V.354)–(V.355) with the operator inequalities $\Omega_t \geq 4\mathfrak{B}_t \geq 4\mathfrak{B}_{t,\mu}$, we obtain that, for any $t > s \geq 0$,

$$\begin{aligned} \partial_s \left\{\mathfrak{M}_{t,s}\mathfrak{M}_{t,s}^*\right\} &= 2\mathfrak{M}_{t,s}\mathcal{G}_{s,\mu}\mathfrak{M}_{t,s}^* \geq -16\mathfrak{M}_{t,s}\mathfrak{B}_{s,\mu}\mathfrak{M}_{t,s}^* - 4\mu\mathfrak{M}_{t,s}\mathfrak{M}_{t,s}^* \\ &\geq (4+\mathrm{r})\left(-4\mathfrak{M}_{t,s}\mathfrak{B}_{s,\mu}\mathfrak{M}_{t,s}^* - \mu\mathfrak{M}_{t,s}\mathfrak{M}_{t,s}^*\right) \ , \end{aligned} \qquad \text{(V.360)}$$

with $\mathrm{r} \geq 0$. Hence, for any $t \geq s \geq 0$ and $\mathrm{r} \geq 0$,

$$-4\int_s^t \mathfrak{M}_{t,\tau}\mathfrak{B}_{\tau,\mu}\mathfrak{M}_{t,\tau}^* \mathrm{d}\tau \leq \frac{1}{(4+\mathrm{r})}\left(\mathbf{1} - \mathfrak{M}_{t,s}\mathfrak{M}_{t,s}^*\right) + \mu\int_s^t \mathfrak{M}_{t,\tau}\mathfrak{M}_{t,\tau}^* \mathrm{d}\tau \ . \qquad \text{(V.361)}$$

Inserting this inequality into (V.359) we get, for any $t \geq s > 0$, that

$$\begin{aligned} (4+\mathrm{r})\mathfrak{E}_{t,\mu} \ &\leq \ \mathfrak{M}_{t,s}\left((4+\mathrm{r})\mathfrak{E}_{s,\mu}\right)\mathfrak{M}_{t,s}^* + \mathbf{1} - \mathfrak{M}_{t,s}\mathfrak{M}_{t,s}^* \\ &\quad +(4+\mathrm{r})\mu\int_s^t \mathfrak{M}_{t,\tau}\mathfrak{M}_{t,\tau}^* \mathrm{d}\tau \ . \end{aligned} \qquad \text{(V.362)}$$

Next, we take the limit $s \to 0$. The evolution operator $\mathfrak{M}_{t,s} \in \mathcal{B}(\mathfrak{h})$, i.e., the unique solution of (V.350), is jointly strongly continuous in s and t (see Lemma 34 (ii)) and is uniformly bounded. Moreover, by (V.342), the trace–class operator $\mathfrak{E}_{t,\mu} \in C[\mathbb{R}_0^+; \mathcal{L}^1(\mathfrak{h})]$ is continuous in $\mathcal{L}^1(\mathfrak{h})$. It follows that Inequality (V.362) also holds for $s = 0$ and using the operator inequalities

$$(4+\mathrm{r})\mathfrak{E}_{0,\mu} \leq (4+\mathrm{r})B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0 \leq \mathbf{1} \qquad \text{(V.363)}$$

we deduce that

$$\forall t \geq 0: \quad (4+\mathrm{r})\mathfrak{E}_{t,\mu} \leq \mathbf{1} + (4+\mathrm{r})\mu\int_0^t \mathfrak{M}_{t,\tau}\mathfrak{M}_{t,\tau}^* \mathrm{d}\tau \ . \qquad \text{(V.364)}$$

Note that (V.353) implies

$$\|\mathfrak{M}_{t,s}\|_{\mathrm{op}} \leq \exp\left(32t^2\|B_0\|_2 + 16t\|\mathfrak{B}_0\|_1 + \mu t\right) \ , \qquad \text{(V.365)}$$

for any $t \geq s \geq 0$ and $\mu > 0$. So, by (V.345) and (V.365), we infer from the limit $\mu \to 0^+$ in Inequality (V.364) that $\mathfrak{E}_t \leq \mathbf{1}/(4+\mathrm{r})$.

<u>(ii):</u> We proceed by rewriting (V.339) as

$$\begin{aligned}\partial_t p_\mu(t) &= 4\mu p_\mu(t) - 4\mathrm{tr}\left\{\left(\mathbf{1} - 4B_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-2}\bar{B}_t\right)B_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-1}\bar{B}_t\right\} \\ &\quad -4\mathrm{tr}\left\{\left(\Omega_t - 4B_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-1}\bar{B}_t\right)B_t\left(\Omega_t^{\mathrm{t}}+\mu\mathbf{1}\right)^{-2}\bar{B}_t\right\} \ . \end{aligned} \tag{V.366}$$

We now use Assertion (i), Lemma 49 (ii), and the cyclicity of the trace (Lemma 101) to infer from (V.366) the differential inequality

$$\forall t>0: \qquad \partial_t p_\mu(t) \le 4\mu p_\mu(t) \ . \tag{V.367}$$

Thanks to Grønwall's Lemma and the continuity of the function p_μ (see (V.343)), we then obtain from (V.367) that

$$\forall t \ge s \ge 0: \qquad p_\mu(t) \le p_\mu(s)\mathrm{e}^{4\mu(t-s)} \ . \tag{V.368}$$

The latter is combined with (V.345) in the limit $\mu \to 0^+$ to deduce the inequality $\|\mathfrak{C}_t\|_1 \le \|\mathfrak{C}_s\|_1$ for all $t \ge s \ge 0$. □

Therefore, using the upper bound of Corollary 57 for $\alpha = 1$ together with Lemma 61 (ii) one gets, for any $t \ge 1$ and $n \in \mathbb{N}$, the upper bound

$$\|B_t\|_2 \le \|\mathfrak{C}_0\|_1^{\frac{1}{2(1+2^{-n})}} \|B_0\|_2^{\frac{1}{2^n+1}} \left(\frac{2^{n-1}}{\mathrm{e}t}\right)^{\frac{1}{1+2^{-n}}} \ . \tag{V.369}$$

In particular, for any $\delta \in (0,1)$ and as $t \to \infty$, $\|B_t\|_2 = o\left(t^{-\delta}\right)$. In fact, one can obtain $\|B_t\|_2 = o\left(t^{-1}\right)$ as well as the integrability of the map $t \mapsto \|\mathfrak{B}_t\|_1$ under a slightly stronger assumption than (V.347):

COROLLARY 62 (INTEGRABILITY OF $\|\mathfrak{B}_t\|_1$)
Assume Conditions A1–A3, $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$, and $\mathbf{1} \ge (4+\mathrm{r})B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$ for some $\mathrm{r}>0$. Then the map $t \mapsto \|\mathfrak{B}_t\|_1$ is integrable on $[0,\infty)$:

$$\forall t \ge s \ge 0: \qquad \frac{4\mathrm{r}}{4+\mathrm{r}}\int_s^t \|\mathfrak{B}_\tau\|_1 \mathrm{d}\tau \le \|\mathfrak{C}_s\|_1 - \|\mathfrak{C}_t\|_1 \ . \tag{V.370}$$

Additionally, $\|B_t\|_2 = o\left(t^{-1}\right)$, as $t \to \infty$.

PROOF. By using (V.366), the positivity of the self–adjoint operators $\Omega_t, \Omega_t^{\mathrm{t}} \ge 0$, the cyclicity of the trace (Lemma 101), and the operator inequalities $\Omega_t \ge 4\mathfrak{B}_t \ge 4\mathfrak{B}_{t,\mu}$ (Lemma 49 (ii), Theorem 50) and $\mathfrak{C}_{t,\mu} \le \mathfrak{C}_t \le \mathbf{1}/(4+\mathrm{r})$ (Lemma 61 (i)), we obtain the differential inequality

$$\forall t \ge s > 0: \qquad \partial_t p_\mu(t) \le 4\mu\|\mathfrak{C}_{t,\mu}\|_1 - \frac{4\mathrm{r}}{4+\mathrm{r}}\|\mathfrak{B}_{t,\mu}\|_1 \ . \tag{V.371}$$

By Lemma 49 (i), Theorem 50 and Lemma 61 (ii), recall that

$$\forall t \ge 0: \quad \|\mathfrak{B}_t\|_1 \le \|\mathfrak{B}_0\|_1 \quad \text{and} \quad \|\mathfrak{C}_{t,\mu}\|_1 \le \|\mathfrak{C}_t\|_1 \le \|\mathfrak{C}_0\|_1 \ , \tag{V.372}$$

whereas p_μ is continuous at zero. Therefore, (V.371) in the limit $\mu \to 0^+$ (cf. (V.206) and (V.345)) implies that the map $t \mapsto \|\mathfrak{B}_t\|_1$ is integrable on $[0,\infty)$ and satisfies (V.370). Moreover, by combining Lemma 58 with (V.370) and (V.372) we obtain, for all $T \geq 0$, that

$$\begin{aligned}\int_T^\infty (s-T)\,\|B_s\|_2^2\,\mathrm{d}s &= \int_T^\infty \int_T^\infty \mathbf{1}[s \geq t]\,\|B_s\|_2^2\,\mathrm{d}t\ \mathrm{d}s \\ &= \int_T^\infty \int_t^\infty \|B_s\|_2^2\,\mathrm{d}s\ \mathrm{d}t \leq \frac{1}{4}\int_T^\infty \|\mathfrak{B}_t\|_1 \mathrm{d}t \\ &\leq \frac{4+\mathrm{r}}{16\mathrm{r}}\|\mathfrak{E}_T\|_1 \leq \frac{4+\mathrm{r}}{16\mathrm{r}}\|\mathfrak{E}_0\|_1\ . \qquad \text{(V.373)}\end{aligned}$$

Since the map $s \mapsto \|B_s\|_2^2$ is monotonically decreasing on $[0,\infty)$, we further observe that

$$\int_n^{n+1} (s-1)\,\|B_s\|_2^2\,\mathrm{d}s \geq (n-1)\,\|B_{n+1}\|_2^2 \qquad \text{(V.374)}$$

for all $n \in \mathbb{N}$. Therefore,

$$\sum_{n=2}^{\infty} (n-2)\,\|B_n\|_2^2 = \sum_{n=1}^{\infty} (n-1)\,\|B_{n+1}\|_2^2 \leq \int_1^\infty (s-1)\,\|B_s\|_2^2\,\mathrm{d}s < \infty\ . \qquad \text{(V.375)}$$

Now, suppose that $\|B_n\|_2 \geq \epsilon n^{-1}$ for some $\epsilon > 0$. Then

$$\sum_{n=2}^{\infty} (n-2)\,\|B_n\|_2^2 \geq \epsilon \sum_{n=2}^{\infty} \frac{n-2}{n^2} = \infty\ , \qquad \text{(V.376)}$$

in contradiction to (V.375), and thus $\|B_t\|_2 = o(t^{-1})$. □

In the last proof, note that we use (V.373) which is equivalent to

$$\int_0^t \mathrm{d}\tau_1 \int_{\tau_1}^\infty \mathrm{d}\tau_2\,\|B_{\tau_2}\|_2^2 \leq \frac{4+\mathrm{r}}{16\mathrm{r}}\|\mathfrak{E}_0\|_1 < \infty\ , \qquad \text{(V.377)}$$

see (V.328) and (V.370). However, this inequality does not necessarily imply the integrability of the Hilbert–Schmidt norm $\|B_t\|_2$. A sufficient condition is $\Omega_0^{-1-\varepsilon}B_0 \in \mathcal{L}^2(\mathfrak{h})$ for some $\varepsilon > 0$. Indeed, by Corollary 57 for $\alpha = 1+\varepsilon$ and $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$,

$$\|B_t\|_2 \leq \|\Omega_t^{-1-\varepsilon}B_t\|_2^{\frac{1}{1+2^{-n}}}\,\|B_0\|_2^{\frac{1}{2^n+1}} \left(\frac{2^{n-1}(1+\varepsilon)}{\mathrm{e}t}\right)^{\frac{1+\varepsilon}{1+2^{-n}}}\ . \qquad \text{(V.378)}$$

The latter implies the integrability of the map $t \mapsto \|B_t\|_2$ on $[0,\infty)$ if, for instance, $\|\Omega_t^{-1-\varepsilon}B_t\|_2$ is uniformly bounded for all times. That is what we prove below under the assumption that $\Omega_0^{-1-\varepsilon}B_0$ is a Hilbert–Schmidt operator:

LEMMA 63 (CONSERVATION BY THE FLOW OF $\Omega_0^{-1-\varepsilon}B_0 \in \mathcal{L}^2(\mathfrak{h})$)
Assume Conditions A1–A3, $\mathbf{1} \geq 4B_0(\Omega_0^{\mathrm{t}})^{-2}\bar{B}_0$*, and* $\Omega_0^{-1-\varepsilon}B_0 \in \mathcal{L}^2(\mathfrak{h})$ *for some* $\varepsilon \in (0,1/2)$*. Then*

$$\forall t \geq s \geq 0: \qquad \|\Omega_t^{-1-\varepsilon}B_t\|_2 \leq \|\Omega_s^{-1-\varepsilon}B_s\|_2 \exp\left\{8\int_s^t \|\mathfrak{B}_\tau\|_1 \mathrm{d}\tau\right\} . \tag{V.379}$$

PROOF. Using an inequality like (V.336) we observe that $B_0 \in \mathcal{L}^2(\mathfrak{h})$ and $\Omega_0^{-1-\varepsilon}B_0 \in \mathcal{L}^2(\mathfrak{h})$ with $\varepsilon > 0$ imply that $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$ and $\Omega_0^{-1}B_0 \in \mathcal{L}^2(\mathfrak{h})$, which in turn implies that $(\Omega_0^{\mathrm{t}})^{-1}\bar{B}_0 \in \mathcal{L}^2(\mathfrak{h})$, by Lemma 102 and $B_0 = B_0^{\mathrm{t}}$. Since $\Omega_t = \Omega_t^* \geq 0$, the function h_μ defined for all $\mu > 0$ and $t \geq 0$ by

$$h_\mu(t) := \|\left(\Omega_t + \mu\mathbf{1}\right)^{-1-\varepsilon} B_t\|_2^2 = \mathrm{tr}(B_t(\Omega_t^{\mathrm{t}} + \mu\mathbf{1})^{-2-2\varepsilon}\bar{B}_t) \tag{V.380}$$

is uniformly bounded in time by

$$h_\mu(t) \leq \mu^{-2-2\varepsilon}\|B_t\|_2^2 \leq \mu^{-2-2\varepsilon}\|B_0\|_2^2 . \tag{V.381}$$

We proceed by using functional calculus to observe that

$$\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2\varepsilon} = \frac{\sin(2\pi\varepsilon)}{\pi}\int_0^\infty \lambda^{-2\varepsilon}\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\mathrm{d}\lambda \tag{V.382}$$

for any $\mu > 0$ and $\varepsilon \in (0,1/2)$, see [13, (1.4.2) and 1.4.7 (d)]. Note that $B_t(\Omega_t^{\mathrm{t}} + \mu\mathbf{1})^{-1} \in \mathcal{L}^2(\mathfrak{h})$ and

$$\int_0^\infty \lambda^{-2\varepsilon}\|\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\|_{\mathrm{op}}\,\mathrm{d}\lambda \leq \int_0^\infty \frac{1}{\lambda^{2\varepsilon}(\lambda+\mu)}\mathrm{d}\lambda < \infty , \tag{V.383}$$

because $B_t \in \mathcal{L}^2(\mathfrak{h})$, $\Omega_t^{\mathrm{t}} \geq 0$, $\varepsilon \in (0,1/2)$, and $\mu > 0$. So, using Lemma 104 we thus obtain that

$$h_\mu(t) = \frac{\sin(2\pi\varepsilon)}{\pi}\int_0^\infty \lambda^{-2\varepsilon}\xi_{\mu,\lambda}(t)\;\mathrm{d}\lambda , \tag{V.384}$$

where $\xi_{\mu,\lambda}$ is the function defined, for any $\mu > 0$ and $\lambda, t \geq 0$, by

$$\xi_{\mu,\lambda}(t) := \mathrm{tr}\left(B_t\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2}\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\bar{B}_t\right) . \tag{V.385}$$

This positive function exists as

$$\forall\mu > 0,\ \lambda, t \geq 0: \qquad 0 \leq \xi_{\mu,\lambda}(t) \leq \mu^{-2}(\lambda+\mu)^{-1}\|B_0\|_2^2 . \tag{V.386}$$

For all $\mu, t > 0$ and $\lambda \geq 0$, its derivative equals

$$\begin{aligned}
\partial_t\xi_{\mu,\lambda}(t) \;=\; & 16\mathrm{tr}\left\{B_t\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2}\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\bar{B}_tB_t\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\bar{B}_t\right\} \\
& +16\mathrm{tr}\left\{B_t\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2}\bar{B}_tB_t\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-1}\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\bar{B}_t\right\} \\
& +16\mathrm{tr}\left\{B_t\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-1}\bar{B}_tB_t\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2}\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\bar{B}_t\right\} \\
& -4\mathrm{tr}\left\{\Omega_tB_t\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2}\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\bar{B}_t\right\} \\
& -4\mathrm{tr}\left\{B_t\Omega_t^{\mathrm{t}}\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2}\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\,\mathbf{1}\right)^{-1}\bar{B}_t\right\} ,
\end{aligned} \tag{V.387}$$

because of Corollary 47, and Lemmata 101 and 103. By using the positivity of the self–adjoint operators $\Omega_t, \Omega_t^{\mathrm{t}} \geq 0$ and the cyclicity of the trace (Lemma 101) together with the inequalities (V.296), $\|Y\|_{\mathrm{op}} \leq \|Y\|_1$, $\Omega_t \geq 4\mathfrak{B}_t \geq 4\mathfrak{B}_{t,\mu}$ (Lemma 49 (ii)), and $\mathfrak{C}_{t,\mu} \leq \mathfrak{C}_t \leq \mathbf{1}/4$ (Lemma 61 (i)), we can then bound from above the derivative of $\xi_{\mu,\lambda}$ by

$$\partial_t \xi_{\mu,\lambda}(t) \leq 4\left(4\left\|\mathfrak{B}_t\right\|_1 + \mu\right) \xi_{\mu,\lambda}(t) \leq 4\left(4\left\|\mathfrak{B}_t\right\|_1 + \mu\right) \xi_{\mu,\lambda}(t) \qquad \text{(V.388)}$$

for all strictly positive $\mu, t > 0$ and $\lambda \geq 0$. By (V.386) and Lemma 49 (i), note that, for any $\mu > 0$ and $\varepsilon \in (0, 1/2)$,

$$\lambda^{-2\varepsilon}\left|\partial_t \xi_{\mu,\lambda}(t)\right| \leq 4\left(4\left\|\mathfrak{B}_0\right\|_1 + \mu\right) \mu^{-2}\left\|B_0\right\|_2^2 \lambda^{-2\varepsilon}(\lambda+\mu)^{-1} \ , \qquad \text{(V.389)}$$

which is an integrable upper bound with respect to $\lambda \in (0, \infty)$, see (V.383). Therefore, we invoke Lebesgue's dominated convergence theorem to deduce that

$$\partial_t h_\mu(t) = \frac{\sin(2\pi\varepsilon)}{\pi} \int_0^\infty \lambda^{-2\varepsilon} \partial_t \xi_{\mu,\lambda}(t) \, \mathrm{d}\lambda \ , \qquad \text{(V.390)}$$

which, combined with (V.384) and (V.388), implies the differential inequality

$$\forall t > 0: \qquad \partial_t h_\mu(t) \leq 4\left(4\left\|\mathfrak{B}_t\right\|_1 + \mu\right) h_\mu(t) \ . \qquad \text{(V.391)}$$

Using again Grønwall's Lemma we thus find the upper bound

$$\forall t \geq s > 0: \qquad h_\mu(t) \leq h_\mu(s) \exp\left\{4 \int_s^t \left(4\left\|\mathfrak{B}_\tau\right\|_1 + \mu\right) \mathrm{d}\tau\right\} \ . \qquad \text{(V.392)}$$

Now, since, by (V.382),

$$\begin{aligned} &\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2-2\varepsilon} - \left(\Omega_0^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2-2\varepsilon} \qquad \text{(V.393)} \\ = \ & \frac{\sin(2\pi\varepsilon)}{\pi} \int_0^\infty \lambda^{-2\varepsilon} \frac{1}{\left(\Omega_t^{\mathrm{t}} + (\lambda+\mu)\mathbf{1}\right)} \left(\Omega_0^{\mathrm{t}} - \Omega_t^{\mathrm{t}}\right) \frac{1}{\left(\Omega_0^{\mathrm{t}} + (\lambda+\mu)\mathbf{1}\right)\left(\Omega_0^{\mathrm{t}} + \mu\mathbf{1}\right)^2} \mathrm{d}\lambda \\ &+ \left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2\varepsilon}\left(\left(\Omega_t^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2} - \left(\Omega_0^{\mathrm{t}} + \mu\mathbf{1}\right)^{-2}\right) \ , \end{aligned}$$

similar to (V.184) or (V.342), one gets

$$|h_\mu(s) - h_\mu(0)| \leq 3\mu^{-3-2\varepsilon}\|\Omega_s^{\mathrm{t}} - \Omega_0^{\mathrm{t}}\|_1 \|B_0\|_2^2 + 2\mu^{-2-2\varepsilon}\|B_0\|_2 \|B_s - B_0\|_2 \qquad \text{(V.394)}$$

for all $\mu, s > 0$. By Theorem 46 (i)–(ii), it demonstrates the continuity of the function h_μ at zero and the inequality (V.392) also holds for $s = 0$:

$$\forall t \geq s \geq 0: \qquad h_\mu(t) \leq h_\mu(s) \exp\left\{4 \int_s^t \left(4\left\|\mathfrak{B}_\tau\right\|_1 + \mu\right) \mathrm{d}\tau\right\} \ , \qquad \text{(V.395)}$$

with $h_\mu(0) \leq h_0(0) < \infty$, by assumption. Similar to (V.203) and (V.206), we infer from (V.395) and the monotone convergence theorem that, for all $t \geq 0$,

$$\lim_{\mu \to 0^+} h_\mu(t) = \|\Omega_t^{-1-\varepsilon} B_t\|_2^2 \leq h_0(0) \exp\left\{16 \int_0^t \left\|\mathfrak{B}_\tau\right\|_1 \mathrm{d}\tau\right\} < \infty \ . \qquad \text{(V.396)}$$

Then, the assertion of this lemma follows by passing to the limit $\mu \to 0^+$ in (V.395). □

Since

$$\int_0^1 \|B_t\|_2 \, \mathrm{d}t \leq \|B_0\|_2 \tag{V.397}$$

(see (V.140) and Theorem 50), by (V.378) for $\varepsilon \in (0, 1/2)$ and Lemma 63 combined with Corollary 62, we directly deduce the integrability of the map $t \mapsto \|B_t\|_2$ on the positive real line $[0, \infty)$:

Corollary 64 (Integrability of $\|B_t\|_2$)
Assume Conditions A1–A3, $\Omega_0^{-1-\varepsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$, and $\mathbf{1} \geq (4 + \mathrm{r})\, B_0(\Omega_0^{\mathrm{t}})^{-2} \bar{B}_0$ for some $\mathrm{r}, \varepsilon > 0$. Then, for any $\delta \in (0, 1+\varepsilon) \cap (0, 3/2)$ and as $t \to \infty$, $\|B_t\|_2 = o\left(t^{-\delta}\right)$. In particular, the map $t \mapsto \|B_t\|_2$ is integrable on $[0, \infty)$.

Of course, the larger the positive parameter $\varepsilon > 0$, the better the asymptotics of the map $t \mapsto \|B_t\|_2$. As an example, we give the study for $\varepsilon = 1/2$ but one should be able to use, recursively, all the arguments of this section to find $\|B_t\|_2 = o\left(t^{-\delta}\right)$ for any $\delta \in (0, 1+\varepsilon)$, provided $\Omega_0^{-1-\varepsilon} B_0 \in \mathcal{L}^2(\mathfrak{h})$ with $\varepsilon > 0$.

Lemma 65 (Conservation by the flow of $\Omega_0^{-3/2} B_0 \in \mathcal{L}^2(\mathfrak{h})$)
Assume Conditions A1–A3, $\Omega_0^{-3/2} B_0 \in \mathcal{L}^2(\mathfrak{h})$, and $\mathbf{1} \geq (4 + \mathrm{r})\, B_0(\Omega_0^{\mathrm{t}})^{-2} \bar{B}_0$ for some $\mathrm{r} > 0$. Then the operator $\mathfrak{S}_t := B_t(\Omega_t^{\mathrm{t}})^{-3} \bar{B}_t \in \mathcal{L}^1(\mathfrak{h})$ is trace–class:

$$\forall t \geq s \geq 0: \qquad \|\mathfrak{S}_t\|_1 \leq \|\mathfrak{S}_s\|_1 \exp\left\{16 \int_s^t \|\mathfrak{B}_\tau\|_1 \mathrm{d}\tau\right\} . \tag{V.398}$$

Proof. Let $\mu > 0$ and set

$$\forall t \geq 0: \qquad \mathfrak{S}_{t,\mu} := B_t \left(\Omega_t^{\mathrm{t}} + \mu\right)^{-3} \bar{B}_t \geq 0 . \tag{V.399}$$

Similar to (V.177) or (V.337), $\mathfrak{S}_{t,\mu}$ is a bounded operator as $\mu > 0$ and $\Omega_t^{\mathrm{t}} \geq 0$, see Lemma 51. Via Corollary 47 and Lemma 103 the function g_μ defined by

$$\forall t \geq 0: \qquad g_\mu(t) := \mathrm{tr}\left\{\mathfrak{S}_{t,\mu}\right\} = \|\mathfrak{S}_{t,\mu}\|_1 = \xi_{\mu,0}(t) \tag{V.400}$$

satisfies the differential inequality

$$\forall t > 0: \qquad \partial_t g_\mu(t) \leq 4\left(4\left\|\mathfrak{B}_t\right\|_1 + \mu\right)\ g_\mu(t) , \tag{V.401}$$

see (V.387)–(V.388) for $\lambda = 0$. So, using as above Grønwall's Lemma, it means that

$$\forall t \geq s > 0: \qquad g_\mu(t) \leq g_\mu(s) \exp\left\{4 \int_s^t \left(4\|\mathfrak{B}_\tau\|_1 + \mu\right) \mathrm{d}\tau\right\} . \tag{V.402}$$

An inequality similar to (V.184) or (V.342) or (V.394) shows the continuity of the function g_μ at zero and (V.402) can be extended by continuity to $s =$

0. Similar to (V.203) and (V.206), we infer from (V.402) together with the monotone convergence theorem that, for all $t \geq s \geq 0$,

$$\lim_{\mu \to 0^+} \|\mathfrak{S}_{t,\mu}\|_1 = \|\mathfrak{S}_t\|_1 < \infty \ , \tag{V.403}$$

with $\mathfrak{S}_t \equiv \mathfrak{S}_{t,0}$. The lemma thus follows from (V.402) and (V.403). □

Therefore, using the upper bound of Corollary 57 for $\alpha = 3/2$ together with Lemma 65 one gets, for any $t \geq 1$ and $\delta \in (0, 3/2)$, that

$$\|B_t\|_2 = o(t^{-\delta}), \qquad \text{as } t \to \infty \ . \tag{V.404}$$

In fact, in the same way the asymptotics $\|B_t\|_2 = o(t^{-1})$ is shown in Corollary 62, one can verify under the assumptions of Lemma 65 that

$$\|B_t\|_2 = o(t^{-3/2}), \qquad \text{as } t \to \infty \ . \tag{V.405}$$

VI Technical Proofs on the Boson Fock Space

Here we give the following proofs: the proof of Theorem 14 in Section VI.1, the proof of Theorem 18 in Section VI.2, the proof of Theorem 22 in Section VI.3, the proof of Theorem 23 in Section VI.4. All these proofs are broken up into several lemmata, which sometime yield information beyond the contents of the above theorems.

Additionally to A1–A2, we always assume without loss of generality Conditions A3–A4. These assumptions are however not necessary in Sections VI.1–VI.2 and only used for convenience. Indeed, A3–A4 ensure the existence of the family $(\Omega_t, B_t)_{t\geq 0}$ solving (III.9) for all times as well as the positivity of $\Omega_t \geq 0$ (i.e., $T_+ = \infty$), which implies

$$\forall t \geq 0 : \qquad \|B_t\|_2 \leq \|B_0\|_2 \ . \tag{VI.1}$$

See (V.140) and Theorem 50. If only A4 is not satisfied then $t \in [0, \infty)$ is replaced by $t \in [0, T_+)$ in Sections VI.1–VI.2, see (V.137)–(V.138). If A3–A4 do not hold then in arguments of Sections VI.1–VI.2 one replaces $t \in [0, \infty)$ and (VI.1) by respectively $[0, T]$ and

$$\forall t \in [0, T] : \qquad \|B_t\|_2 \leq \mathrm{e}^{8rT} \|B_0\|_2 \ , \tag{VI.2}$$

for any $T \in (0, T_{\max})$ and some $r \equiv r_T > 0$. See (V.2), (V.7), and (V.117)–(V.118).

VI.1 Existence and Uniqueness of the Unitary Propagator

To prove the existence of the unitary propagator $\mathrm{U}_{t,s}$ solution of (III.4) with infinitesimal generator G_t (III.8), we need to verify three conditions, namely B1, B2, and B3, see Section VII.1. Condition B1 is directly satisfied because, by Proposition 1, our generator G_t is self–adjoint, see (VII.4). To verify Conditions

B2–B3, a closed auxiliary operator Θ has to be fixed. The simplest choice is to take the particle number operator N (II.11) plus the identity $\mathbf{1}$, i.e., $\Theta = \mathrm{N}+\mathbf{1}$. Then, establishing B2 and B3 amounts to prove that the relative norms $\|\mathrm{G}_t(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}}$ and $\|[\mathrm{N}, \mathrm{G}_t](\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}}$ are uniformly bounded for all times $t \geq 0$, with the function $\|\mathrm{G}_t(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}}$ being continuous on $[0,\infty)$. We thus infer from Theorem 89 the existence and uniqueness of a unitary propagator $\mathrm{U}_{t,s}$ solution, in the strong topology on the domain $\mathcal{D}(\mathrm{N})$, of the non–autonomous evolution equation (III.4) with infinitesimal generator $G_t = \mathrm{G}_t$ (III.8).
We start by giving some general estimates used many times in our paper.

LEMMA 66 (RELATIVE NORMS W.R.T N–DIAGONAL OPERATORS)
Let $\theta = \theta^* \geq 0$ be a positive, invertible operator on $\mathfrak{h}$ and $X, Y = Y^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$ with respective second quantizations

$$\Theta \ : \ = \mathbf{1}+\sum_{k,\ell} \{\theta\}_{k,\ell}\, a_k^* a_\ell \ , \tag{VI.3}$$

$$\mathbb{X} \ : \ = \sum_{k,\ell} \{X\}_{k,\ell}\, a_k^* a_\ell \ , \tag{VI.4}$$

$$\mathbb{Y} \ : \ = \sum_{k,\ell} \{Y\}_{k,\ell}\, a_k a_\ell \ . \tag{VI.5}$$

Then the norms of X, Y, and Y^* relative to Θ are bounded by:

$$\|\mathbb{X}\Theta^{-1}\|_{\mathrm{op}} \leq \|\theta^{-1/2}X^*X\theta^{-1/2}\|_2^{1/2} + \|\theta^{-1/2}X\theta^{-1/2}\|_2 \ , \tag{VI.6}$$

$$\|\mathbb{Y}\Theta^{-1}\|_{\mathrm{op}} \leq \|\theta^{-1/2}Y(\theta^{\mathrm{t}})^{-1/2}\|_2 \ , \tag{VI.7}$$

$$\|\mathbb{Y}^*\Theta^{-1}\|_{\mathrm{op}} \leq \|\theta^{-1/2}Y(\theta^{\mathrm{t}})^{-1/2}\|_2 + 2\|\theta^{-1/2}Y^*Y\theta^{-1/2}\|_2^{1/2} + \sqrt{2}\|Y\|_2 \ . \tag{VI.8}$$

PROOF. By the spectral theorem, there is a representation of $\mathfrak{h}$ as $L^2(\mathcal{A}, \mathrm{d}\mathfrak{a})$, a suitable probability measure $\mathfrak{a}$ on a set $\mathcal{A}$ such that θ is a multiplication operator, i.e., there is a Borel–measurable function $\vartheta : \mathcal{A} \to \mathbb{R}_0^+$ such that

$$\forall \varphi \in L^2(\mathcal{A}, \mathrm{d}\mathfrak{a}),\ x \in \mathcal{A}: \qquad (\theta\varphi)(x) = \vartheta_x \varphi(x) \ .$$

Since all norms in (VI.6)–(VI.8) are unitarily invariant (up to the appearance of the transpose θ^{t} of θ in (VI.7) and (VI.8)), we may assume without loss of generality that (we write $\mathrm{d}\mathfrak{a}_x \mathrm{d}\mathfrak{a}_y =: \mathrm{d}^2\mathfrak{a}_{x,y}$ and $\mathrm{d}\mathfrak{a}_x \mathrm{d}\mathfrak{a}_y \mathrm{d}\mathfrak{a}_w \mathrm{d}\mathfrak{a}_z =: \mathrm{d}^4\mathfrak{a}_{x,y,w,z}$)

$$\Theta \ : \ = \mathbf{1}+\int \vartheta_x a_x^* a_x \mathrm{d}\mathfrak{a}_x \ , \tag{VI.9}$$

$$\mathbb{X} \ : \ = \int X_{x,y} a_x^* a_y \mathrm{d}^2\mathfrak{a}_{x,y} \ , \tag{VI.10}$$

$$\mathbb{Y} \ : \ = \int Y_{x,y} a_x a_y \mathrm{d}^2\mathfrak{a}_{x,y} \ , \tag{VI.11}$$

with $Y^{\mathrm{t}} = Y$. Now, for any $\varphi \in \mathcal{F}_b$, we estimate that

$$
\begin{aligned}
\|\mathbb{Y}\varphi\| &\le \int |Y_{x,y}|\, \|a_x a_y \varphi\|\, \mathrm{d}^2\mathfrak{a}_{x,y} \\
&\le \left(\int \frac{|Y_{x,y}|^2}{\vartheta_x \vartheta_y} \mathrm{d}^2\mathfrak{a}_{x,y} \right)^{1/2} \left(\int \vartheta_x \vartheta_y \, \|a_x a_y \varphi\|^2 \, \mathrm{d}^2\mathfrak{a}_{x,y} \right)^{1/2} \\
&= \|\theta^{-1/2} Y \theta^{-1/2}\|_2 \left(\int \vartheta_y \| (\Theta - \mathbf{1})^{1/2} \, a_y \varphi\|^2 \mathrm{d}\mathfrak{a}_y \right)^{1/2} \\
&\le \|\theta^{-1/2} Y \theta^{-1/2}\|_2 \, \| (\Theta - \mathbf{1}) \, \varphi\| \, , \qquad \text{(VI.12)}
\end{aligned}
$$

which proves (VI.7) for $\theta^{\mathrm{t}} = \theta$. Next, we observe that, by the Canonical Commutations Relations (CCR),

$$
\begin{aligned}
\|\mathbb{X}\varphi\|^2 &= \int \overline{X_{x,y}} X_{w,z} \langle a_y \varphi | a_x a_w^* a_z \varphi \rangle \mathrm{d}^4\mathfrak{a}_{x,y,w,z} \\
&= \int \overline{X_{x,y}} X_{w,z} \langle a_w a_y \varphi | a_x a_z \varphi \rangle \mathrm{d}^4\mathfrak{a}_{x,y,w,z} \\
&\quad + \int (X^* X)_{y,z} \langle a_y \varphi | a_z \varphi \rangle \mathrm{d}^2\mathfrak{a}_{y,z} \\
&\le \left(\int \frac{|X_{x,y}|^2}{\vartheta_x \vartheta_y} \mathrm{d}^2\mathfrak{a}_{x,y} \right) \left(\int \vartheta_x \vartheta_y \, \|a_x a_y \varphi\|^2 \, \mathrm{d}^2\mathfrak{a}_{x,y} \right) \\
&\quad + \left(\int \frac{| (X^* X)_{y,z} |^2}{\vartheta_y \vartheta_z} \mathrm{d}^2\mathfrak{a}_{y,z} \right)^{1/2} \left(\int \vartheta_y \, \|a_y \varphi\|^2 \, \mathrm{d}\mathfrak{a}_y \right) \\
&\le \|\theta^{-1/2} X \theta^{-1/2}\|_2^2 \, \| (\Theta - \mathbf{1}) \, \varphi\|^2 \\
&\quad + \|\theta^{-1/2} X^* X \theta^{-1/2}\|_2 \, \| (\Theta - \mathbf{1})^{1/2} \, \varphi\|^2 \, , \qquad \text{(VI.13)}
\end{aligned}
$$

from which (VI.6) is immediate. Finally, the CCR imply that

$$
\begin{aligned}
\|\mathbb{Y}^*\varphi\|^2 &= \int Y_{x,y} \overline{Y_{w,z}} \langle \varphi | a_x a_y a_w^* a_z^* \varphi \rangle \mathrm{d}^4\mathfrak{a}_{x,y,w,z} \\
&= \int Y_{x,y} \overline{Y_{w,z}} \langle a_z a_w \varphi | a_x a_y \varphi \rangle \mathrm{d}^4\mathfrak{a}_{x,y,w,z} \\
&\quad + \int \left(Y^* Y^{\mathrm{t}} + Y^* Y + \overline{Y} Y^{\mathrm{t}} + \overline{Y} Y \right)_{x,y} \langle a_x \varphi | a_y \varphi \rangle \mathrm{d}^2\mathfrak{a}_{x,y} \\
&\quad + \left(\mathrm{tr}\left(\overline{Y} Y\right) + \mathrm{tr}\left(Y^* Y\right) \right) \|\varphi\|^2 \\
&\le \|\theta^{-1/2} Y \theta^{-1/2}\|_2^2 \, \| (\Theta - \mathbf{1}) \, \varphi\|^2 \\
&\quad + 4\|\theta^{-1/2} Y^* Y \theta^{-1/2}\|_2 \, \| (\Theta - \mathbf{1})^{1/2} \, \varphi\|^2 \\
&\quad + 2\|Y\|_2^2 \, \|\varphi\|^2, \qquad \text{(VI.14)}
\end{aligned}
$$

which yields (VI.8) for $\theta^{\mathrm{t}} = \theta$. □

We analyze now the relative norm estimates related to the generator G_t introduced in (III.8) with respect to the self–adjoint, invertible operator $\Theta = \mathrm{N} + \mathbf{1}$. In the following lemmata, we assume Conditions A1–A4 defined in Sections II.1–II.2, which ensure the existence of operators Ω_t and B_t solution of the system (III.9) of differential equations for all times, see Theorem 11. See also the introduction of Section VI about A1–A4.

LEMMA 67 (VERIFICATION OF CONDITIONS B2–B3 WITH $\Theta = \mathrm{N} + \mathbf{1}$)
Assume Conditions A1–A4 and let

$$\mathcal{Y} := \mathcal{D}(\mathrm{N}) = \left\{ \varphi \in \mathcal{F}_b \;\middle|\; \|\varphi\|_{\mathcal{Y}} := \|(\mathrm{N} + \mathbf{1})\varphi\| < \infty \right\} . \tag{VI.15}$$

Then, for any $t \geq 0$, $\mathcal{Y} \subset \mathcal{D}(\mathrm{G}_t)$ with $\mathrm{G}_t \in C[\mathbb{R}_0^+; \mathcal{B}(\mathcal{Y}, \mathcal{F}_b)]$, i.e., Condition B2 holds true. Furthermore, the norm $\|[\mathrm{N}, \mathrm{G}_t]\|_{\mathcal{B}(\mathcal{Y})}$ is bounded on $[0, \infty)$ by

$$\| [\mathrm{N}, \mathrm{G}_t] \|_{\mathcal{B}(\mathcal{Y})} := \| [\mathrm{N}, \mathrm{G}_t] (\mathrm{N} + \mathbf{1})^{-1} \|_{\mathrm{op}} \leq 22 \, \|B_t\|_2 \leq 22 \, \|B_0\|_2 \ , \tag{VI.16}$$

i.e., Condition B3 holds true.

PROOF. First, we analyze the behavior of the function $\|\mathrm{G}_t(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}}$. From (III.8) observe that $\mathrm{G}_t = 2i(\mathbb{B}_t^* - \mathbb{B}_t)$ with

$$\mathbb{B}_t := \sum_{k,\ell} \left\{ \bar{B}_t \right\}_{k,\ell} a_k a_\ell \ . \tag{VI.17}$$

Hence, the relative norm of G_t with respect to $\Theta = \mathrm{N} + \mathbf{1}$ is bounded by

$$\|\mathrm{G}_t (\mathrm{N} + \mathbf{1})^{-1} \|_{\mathrm{op}} \leq 2\|\mathbb{B}_t (\mathrm{N} + \mathbf{1})^{-1} \|_{\mathrm{op}} + 2\|\mathbb{B}_t^* (\mathrm{N} + \mathbf{1})^{-1} \|_{\mathrm{op}} \ . \tag{VI.18}$$

By using Lemma 66 with $\theta = \mathbf{1}$ and $Y = \bar{B}_t$, the operators $\mathbb{B}_t (\mathrm{N} + \mathbf{1})^{-1}$ and $\mathbb{B}_t^* (\mathrm{N} + \mathbf{1})^{-1}$ are bounded respectively by

$$\|\mathbb{B}_t (\mathrm{N} + \mathbf{1})^{-1} \|_{\mathrm{op}} \leq \|B_t\|_2 \quad \text{and} \quad \|\mathbb{B}_t^* (\mathrm{N} + \mathbf{1})^{-1} \|_{\mathrm{op}} \leq (3 + \sqrt{2})\|B_t\|_2 \ . \tag{VI.19}$$

Combined with (VI.1) and the inequalities of (VI.19), the upper bound (VI.18) implies, for any $t \geq 0$, that

$$\|\mathrm{G}_t (\mathrm{N} + \mathbf{1})^{-1} \|_{\mathrm{op}} \leq 11\|B_t\|_2 \leq 11\|B_0\|_2 \ . \tag{VI.20}$$

(We use $(8 + 2\sqrt{2}) < 11$.) Additionally, by substituting $(B_t - B_s)$ for the operator B_t in the last inequality (VI.20) we also obtain, for all $t, s \geq 0$, that

$$\| \{\mathrm{G}_t - \mathrm{G}_s\} (\mathrm{N} + \mathbf{1})^{-1} \|_{\mathrm{op}} \leq 11\|B_t - B_s\|_2 \ . \tag{VI.21}$$

By Theorem 11 (ii), the operator B_t is continuous in the Hilbert–Schmidt topology for any $t \geq 0$. Consequently $\mathrm{G}_t \in C[\mathbb{R}_0^+; \mathcal{B}(\mathcal{Y}, \mathcal{F}_b)]$ with $\mathcal{Y} = \mathcal{D}(\mathrm{N})$, and Condition B2 is therefore verified.

Finally, straightforward computations using the CCR show that the commutator

$$\frac{1}{i}\left[\mathrm{N},\mathrm{G}_t\right] = 4\sum_{k,\ell}\left\{B_t\right\}_{k,\ell} a_k^* a_\ell^* + \left\{\bar{B}_t\right\}_{k,\ell} a_k a_\ell = 4\left(\mathbb{B}_t^* + \mathbb{B}_t\right) \qquad \text{(VI.22)}$$

is also a well–defined self–adjoint quadratic operator (Proposition 1). Therefore, we can substitute $2B_t$ for B_t in (VI.20) to get, for all $t \geq 0$, that

$$\|\left[\mathrm{N},\mathrm{G}_t\right]\left(\mathrm{N}+\mathbf{1}\right)^{-1}\|_{\mathrm{op}} \leq 22\|B_t\|_2 \leq 22\|B_0\|_2 \ , \qquad \text{(VI.23)}$$

i.e., Condition B3. □

The above lemma concludes the preliminary steps necessary to apply Theorem 89.

LEMMA 68 (EXISTENCE AND UNIQUENESS OF $\mathrm{U}_{t,s}$)
Under Conditions A1–A4, there is a unique, bounded evolution operator $(\mathrm{U}_{t,s})_{t\geq s\geq 0}$ satisfying on the domain $\mathcal{D}(\mathrm{N})$ the non–autonomous evolutions

$$\forall t \geq s \geq 0 : \qquad \begin{cases} \partial_t \mathrm{U}_{t,s} = -i\mathrm{G}_t\mathrm{U}_{t,s} & , \quad \mathrm{U}_{s,s} := \mathbf{1} \ . \\ \partial_s \mathrm{U}_{t,s} = i\mathrm{U}_{t,s}\mathrm{G}_s & , \quad \mathrm{U}_{t,t} := \mathbf{1} \ . \end{cases} \qquad \text{(VI.24)}$$

Moreover, $\mathrm{U}_{t,s}$ conserves the domains $\mathcal{D}(\mathrm{N})$ and $\mathcal{D}\left(\mathrm{N}^2\right)$ for all $t \geq s \geq 0$ as

$$\|(\mathrm{N}+\mathbf{1})\mathrm{U}_{t,s}(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} \leq \exp\left\{22\int_s^t \|B_\tau\|_2 \mathrm{d}\tau\right\} < \infty \ , \qquad \text{(VI.25)}$$

$$\|(\mathrm{N}+\mathbf{1})^2\mathrm{U}_{t,s}(\mathrm{N}+\mathbf{1})^{-2}\|_{\mathrm{op}} \leq \exp\left\{132\int_s^t \|B_\tau\|_2 \mathrm{d}\tau\right\} < \infty \ . \qquad \text{(VI.26)}$$

The bounded operator family

$$\left\{(\mathrm{N}+\mathbf{1})\mathrm{U}_{t,s}(\mathrm{N}+\mathbf{1})^{-1}\right\}_{t\geq s\geq 0} \qquad \text{(VI.27)}$$

is also jointly strongly continuous in s and t.

PROOF. From Lemma 67, Conditions B2–B3 are satisfied with $\mathcal{Y} = \mathcal{D}(\mathrm{N})$, whereas Condition B1 with $m = 1$ and $\beta_0(t) = 0$ is a direct consequence of the self–adjointness of G_t, see Proposition 1 and (VII.4). Therefore, we can apply Theorem 89 with the closed operator Θ being the unbounded self–adjoint operator $(\mathrm{N}+\mathbf{1})$ and the Banach space $\mathcal{X}$ being the Hilbert space $\mathcal{F}_b$. In this specific case, the bounded operator $\mathrm{U}_{\lambda,t,s}$ has

$$G_{t,\lambda} = \mathrm{G}_{t,\lambda} := \frac{\lambda\mathrm{G}_t}{\lambda\mathbf{1} + i\mathrm{G}_t} \in \mathcal{B}(\mathcal{F}_b) \qquad \text{(VI.28)}$$

as infinitesimal generator, which is the Yosida approximation of the unbounded self–adjoint operator G_t, see (VII.5)–(VII.6). This evolution operator $\mathrm{U}_{\lambda,t,s}$

converges, as $\lambda \to \infty$, to an evolution operator $\mathrm{U}_{t,s}$ in the strong sense on the boson Fock space $\mathcal{F}_b$ with $\|\mathrm{U}_{t,s}\| \leq 1$. This strong convergence on $\mathcal{F}_b$ becomes a norm convergence on the dense domain $\mathcal{D}(\mathrm{N}) \subset \mathcal{F}_b$ since, by Lemma 91,

$$\lim_{\lambda \to \infty} \|\{\mathrm{U}_{\lambda,t,s} - \mathrm{U}_{t,s}\}(\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} = 0 \tag{VI.29}$$

for all $t \geq s \geq 0$. By using Lemma 93 with $m = 1$, $\beta_0(t) = 0$, and

$$\gamma_1(t) := \|[\mathrm{N}, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \leq 22\|B_t\|_2 \leq 22\|B_0\|_2 \tag{VI.30}$$

(see (VI.16)), the bounded operator $\mathrm{U}_{t,s}$ also satisfies Inequality (VI.25), i.e., it conserves the domain $\mathcal{D}(\mathrm{N})$. Furthermore, for all $t \geq s \geq 0$, the evolution operator $\mathrm{U}_{t,s}$ is a solution in the strong topology on $\mathcal{D}(\mathrm{N})$ of the non–autonomous evolution equation (VI.24), by Theorem 89. The strong continuity of the bounded operator family (VI.27) results from Lemma 93.
Now, we prove (VI.26), which means that $\mathrm{U}_{t,s}$ conserves the domain $\mathcal{D}(\mathrm{N}^2)$. The latter can be seen by using again Lemma 93 provided we show that

$$[(\mathrm{N} + \mathbf{1})^2, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2} \in \mathcal{B}(\mathcal{F}_b) \ . \tag{VI.31}$$

Note that

$$[(\mathrm{N} + \mathbf{1})^2, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2} = [\mathrm{N}^2, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2} + 2[\mathrm{N}, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2} \ . \tag{VI.32}$$

In view of (VI.30), it suffices to bound the norm

$$\|[\mathrm{N}^2, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2}\|_{\mathrm{op}} \tag{VI.33}$$

and we shall use the equality

$$[\mathrm{N}^2, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2} = \mathrm{N}[\mathrm{N}, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2} + [\mathrm{N}, \mathrm{G}_t]\mathrm{N}(\mathrm{N} + \mathbf{1})^{-2} \ . \tag{VI.34}$$

The commutator equality

$$[XY, Z] = X[Y, Z] + [X, Z]Y \tag{VI.35}$$

for any unbounded operators X, Y, Z only holds, a priori, on a common core for the three operators in the equality. It turns out that both operators in the r.h.s of (VI.34) are, for all $t \geq 0$, bounded respectively by

$$\|[\mathrm{N}, \mathrm{G}_t]\mathrm{N}(\mathrm{N} + \mathbf{1})^{-2}\|_{\mathrm{op}} \leq \|[\mathrm{N}, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \leq 22\|B_t\|_2 \tag{VI.36}$$

and

$$\|\mathrm{N}[\mathrm{N}, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2}\|_{\mathrm{op}} \leq 66\|B_t\|_2 \ . \tag{VI.37}$$

Indeed, (VI.36) corresponds to (VI.30). The proof of the second bound uses the equality

$$\mathrm{N}[\mathrm{N}, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-2} = [\mathrm{N}, \mathrm{G}_t]\mathrm{N}(\mathrm{N} + \mathbf{1})^{-2} + [\mathrm{N}, [\mathrm{N}, \mathrm{G}_t]]\mathrm{N}(\mathrm{N} + \mathbf{1})^{-2} \ . \tag{VI.38}$$

From straightforward computations using (VI.22) and the CCR, we observe that

$$\frac{1}{i}\,[\mathrm{N},[\mathrm{N},\mathrm{G}_t]] = 8\sum_{k,\ell} \{B_t\}_{k,\ell}\, a_k^* a_\ell^* - \{\bar{B}_t\}_{k,\ell}\, a_k a_\ell = 8\,(\mathbb{B}_t^* - \mathbb{B}_t) \qquad \text{(VI.39)}$$

is a well–defined self–adjoint quadratic operator (Proposition 1) satisfying

$$\|\,[\mathrm{N},[\mathrm{N},\mathrm{G}_t]]\,(\mathrm{N}+\mathbf{1})^{-1}\,\|_{\mathrm{op}} \leq 44\|B_t\|_2 \leq 44\|B_0\|_2\ , \qquad \text{(VI.40)}$$

see (VI.20) when we substitute $4B_t$ for B_t. Using (VI.36), (VI.38), and (VI.40) we arrive at (VI.37), which is combined with (VI.34) and (VI.36) to obtain the upper bound

$$\|[\mathrm{N}^2,\mathrm{G}_t](\mathrm{N}+\mathbf{1})^{-2}\|_{\mathrm{op}} \leq 88\|B_t\|_2\ . \qquad \text{(VI.41)}$$

By (VI.30) and (VI.32), it follows that

$$\gamma_2(t) := \|[(\mathrm{N}+\mathbf{1})^2,\mathrm{G}_t](\mathrm{N}+\mathbf{1})^{-2}\|_{\mathrm{op}} \leq 132\|B_t\|_2 \leq 132\|B_0\|_2\ , \qquad \text{(VI.42)}$$

because of (VI.1). By Lemma 93, the latter in turn implies (VI.26). □

Observe that Lemma 68 neither implies that the adjoint $\mathrm{U}_{t,s}^*$ of the evolution operator $\mathrm{U}_{t,s}$ conserves the dense domain $\mathcal{D}(\mathrm{N})$ of the particle number operator N, nor that it is jointly strongly continuous in s and t for all $t \geq s \geq 0$. Since these properties are necessary below, in particular, to prove the unitarity of $\mathrm{U}_{t,s}$, we establish them in the next lemma.

LEMMA 69 (PROPERTIES OF THE BOUNDED OPERATOR $\mathrm{U}_{t,s}^*$)
Under Conditions A1–A4, the adjoint $\mathrm{U}_{t,s}^*$ of the evolution operator $\mathrm{U}_{t,s}$ is jointly strongly continuous in s and t for all $t \geq s \geq 0$ and a strong solution on the domain $\mathcal{D}(\mathrm{N})$ of the non–autonomous evolution equations

$$\forall t \geq s \geq 0: \qquad \begin{cases} \partial_s \mathrm{U}_{t,s}^* = -i\mathrm{G}_s \mathrm{U}_{t,s}^* & , \quad \mathrm{U}_{t,t} := \mathbf{1} \quad . \\ \partial_t \mathrm{U}_{t,s}^* = i\mathrm{U}_{t,s}^* \mathrm{G}_t & , \quad \mathrm{U}_{s,s}^* := \mathbf{1} \quad . \end{cases} \qquad \text{(VI.43)}$$

Moreover, $\mathrm{U}_{t,s}^*$ conserves the domains $\mathcal{D}(\mathrm{N})$ and $\mathcal{D}(\mathrm{N}^2)$ for all $t \geq s \geq 0$ as

$$\|(\mathrm{N}+\mathbf{1})\mathrm{U}_{t,s}^*(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} \leq \exp\left\{22\int_s^t \|B_\tau\|_2 \mathrm{d}\tau\right\} < \infty\ , \qquad \text{(VI.44)}$$

$$\|(\mathrm{N}+\mathbf{1})^2\mathrm{U}_{t,s}^*(\mathrm{N}+\mathbf{1})^{-2}\|_{\mathrm{op}} \leq \exp\left\{132\int_s^t \|B_\tau\|_2 \mathrm{d}\tau\right\} < \infty\ . \qquad \text{(VI.45)}$$

PROOF. We first observe that $\mathrm{U}_{\lambda,t,s}^*$ has a norm convergent representation as a Dyson series similar to (VII.6):

$$\begin{aligned} \mathrm{U}_{\lambda,t,s}^* &= \mathbf{1} + \sum_{n=1}^{\infty} i^n \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n \mathrm{G}_{\tau_n,\lambda}^* \cdots \mathrm{G}_{\tau_1,\lambda}^* \\ &= \mathbf{1} + \sum_{n=1}^{\infty} i^n \int_s^t \mathrm{d}\tau_1 \cdots \int_{\tau_{n-1}}^t \mathrm{d}\tau_n \mathrm{G}_{\tau_1,\lambda}^* \cdots \mathrm{G}_{\tau_n,\lambda}^*\ . \end{aligned} \qquad \text{(VI.46)}$$

Since the bounded generator

$$\mathrm{G}_{t,\lambda}^{*}=\frac{\lambda \mathrm{G}_{t}}{\lambda \mathbf{1}-i\mathrm{G}_{t}}\in \mathcal{B}\left(\mathcal{F}_{b}\right) \tag{VI.47}$$

fulfills the same estimates as $\mathrm{G}_{t,\lambda}$, there exists a bounded operator $\tilde{\mathrm{U}}_{t,s}\in \mathcal{B}\left(\mathcal{F}_{b}\right)$ such that

$$\lim_{\lambda\rightarrow\infty}\Vert\{\mathrm{U}_{\lambda,t,s}^{*}-\tilde{\mathrm{U}}_{t,s}\}(\mathrm{N}+\mathbf{1})^{-1}\Vert_{\mathrm{op}}=0\ , \tag{VI.48}$$

just as we did for $\mathrm{U}_{\lambda,t,s}$ and $\mathrm{U}_{t,s}$ in Lemma 68 and

$$\begin{aligned}
\Vert(\mathrm{N}+\mathbf{1})\tilde{\mathrm{U}}_{t,s}(\mathrm{N}+\mathbf{1})^{-1}\Vert_{\mathrm{op}} &\leq \exp\left\{ 22\int_{s}^{t}\gamma_{1}\left(\tau\right) \mathrm{d}\tau\right\}\ , &\text{(VI.49)}\\
\Vert(\mathrm{N}+\mathbf{1})^{2}\tilde{\mathrm{U}}_{t,s}(\mathrm{N}+\mathbf{1})^{-2}\Vert_{\mathrm{op}} &\leq \exp\left\{ 132\int_{s}^{t}\gamma_{2}\left(\tau\right) \mathrm{d}\tau\right\}\ , &\text{(VI.50)}
\end{aligned}$$

see (VI.30) and (VI.42). Moreover, $(\tilde{\mathrm{U}}_{t,s})_{t\geq s\geq0}$ is jointly strongly continuous in s and t (cf. Lemma 92) and it is a strong solution on the domain $\mathcal{D}\left(\mathrm{N}\right)$ of (VI.43), similar to Lemma 94 by inverting the role of s and t, see (VI.46).
It remains to prove that $\tilde{\mathrm{U}}_{t,s}=\mathrm{U}_{t,s}^{*}$ for all $t\geq s\geq0$. It is in this case trivial. Indeed, for any $t\geq s\geq0$ and $\varphi,\psi\in\mathcal{F}_{b}$,

$$\langle(\tilde{\mathrm{U}}_{t,s}-\mathrm{U}_{t,s}^{*})\varphi|\psi\rangle=\langle(\tilde{\mathrm{U}}_{t,s}-\mathrm{U}_{\lambda,t,s}^{*})\varphi|\psi\rangle+\langle\varphi|(\mathrm{U}_{\lambda,t,s}-\mathrm{U}_{t,s})\psi\rangle\ . \tag{VI.51}$$

As explained in the proof of Lemma 68, the evolution operator $\mathrm{U}_{\lambda,t,s}$ strongly converges, as $\lambda\rightarrow\infty$, to the evolution operator $\mathrm{U}_{t,s}$, whereas $\mathrm{U}_{\lambda,t,s}^{*}$ converges strongly to $\tilde{\mathrm{U}}_{t,s}$. Therefore, we infer from (VI.51) that $\tilde{\mathrm{U}}_{t,s}=\mathrm{U}_{t,s}^{*}$ for any $t\geq s\geq0$. □
Now, we are in position to prove the unitarity of the operators $\mathrm{U}_{t,s}$ for $t\geq s\geq0$.

LEMMA 70 (UNITARITY OF THE EVOLUTION OPERATOR $\mathrm{U}_{t,s}$)
Under Conditions A1–A4, the evolution operator $\mathrm{U}_{t,s}$ *is unitary for all* $t\geq s\geq 0$, *i.e., the family* $(\mathrm{U}_{t,s})_{t\geq s\geq0}$ *forms a unitary propagator.*

PROOF. We infer from Lemmata 68–69 that, for any $t\geq s\geq0$,

$$(\mathbf{1}-\mathrm{U}_{t,s}\mathrm{U}_{t,s}^{*})(\mathrm{N}+\mathbf{1})^{-1}=\int_{s}^{t}\partial_{\tau}\left\{ \mathrm{U}_{t,\tau}\mathrm{U}_{t,\tau}^{*}\right\} (\mathrm{N}+\mathbf{1})^{-1}\mathrm{d}\tau=0\ , \tag{VI.52}$$

and

$$(\mathbf{1}-\mathrm{U}_{t,s}^{*}\mathrm{U}_{t,s})(\mathrm{N}+\mathbf{1})^{-1}=-\int_{s}^{t}\partial_{\tau}\left\{ \mathrm{U}_{\tau,s}^{*}\mathrm{U}_{\tau,s}\right\} (\mathrm{N}+\mathbf{1})^{-1}\mathrm{d}\tau=0\ . \tag{VI.53}$$

Thus, for all $t\geq s\geq0$, $\mathrm{U}_{t,s}^{*}\mathrm{U}_{t,s}=\mathrm{U}_{t,s}\mathrm{U}_{t,s}^{*}=\mathbf{1}$ on the dense domain $\mathcal{D}\left(\mathrm{N}\right) \subset\mathcal{F}_{b}$, which immediately implies the unitarity of $\mathrm{U}_{t,s}\in\mathcal{B}\left(\mathcal{F}_{b}\right)$. □
The proof of Theorem 14 (i) is now complete by combining Lemma 68 with Lemma 70. See also the introduction of Section VI about A1–A4. In fact, by Lemmata 68–70 one deduces the following theorem:

THEOREM 71 (NATURAL EXTENSION OF THE EVOLUTION OPERATOR $\mathrm{U}_{t,s}$)
Under Conditions A1–A4, let $\mathbf{U}_{t,s} := \mathrm{U}_{t,s}$ *for* $t \geq s \geq 0$, *whereas for* $s \geq t \geq 0$, $\mathbf{U}_{t,s} := \mathrm{U}_{s,t}^*$. *Then,* $(\mathbf{U}_{t,s})_{s,t\in\mathbb{R}_0^+}$ *forms a unitary propagator:*
(i) *For any* $s,t \in \mathbb{R}_0^+$, $\mathbf{U}_{t,s} = \mathbf{U}_{s,t}^*$ *is a unitary operator.*
(ii) *It satisfies the cocycle property* $\mathbf{U}_{t,x}\mathbf{U}_{x,s} = \mathbf{U}_{t,s}$ *for any* $s,x,t \in \mathbb{R}_0^+$.
(iii) *It is jointly strongly continuous in* s *and* t *for all* $s,t \in \mathbb{R}_0^+$.
(iv) *It conserves the domains* $\mathcal{D}(\mathrm{N})$ *and* $\mathcal{D}(\mathrm{N}^2)$ *for all* $t \geq s \geq 0$, *see (VI.25)–(VI.26) and (VI.44)–(VI.45).*
(v) *It solves, in the strong sense in* $\mathcal{D}(\mathrm{N})$, *the non–autonomous evolution equations*

$$\forall s,t \in \mathbb{R}_0^+ : \qquad \begin{cases} \partial_t \mathbf{U}_{t,s} = -i\mathrm{G}_t\mathbf{U}_{t,s} & , \quad \mathbf{U}_{s,s} := \mathbf{1} \quad . \\ \partial_s \mathbf{U}_{t,s} = i\mathbf{U}_{t,s}\mathrm{G}_s & , \quad \mathbf{U}_{t,t} := \mathbf{1} \quad . \end{cases} \tag{VI.54}$$

PROOF. Combine Lemmata 68–70. In fact, the family $(\mathbf{U}_{t,s})_{s,t\in\mathbb{R}_0^+}$ can directly be derived from arguments proving Lemmata 90–94. Indeed, the generator $\mathrm{G}_t = \mathrm{G}_t^*$ is self–adjoint and so, it satisfies the Kato quasi–stability condition B1 (Section VII.1) even with *non–ordered* times, see (VII.4). □
Note that this theorem is only given here for the interested reader, but it is not essential for our proofs. Actually, we only use below the families $(\mathrm{U}_{t,s})_{t\geq s\geq 0}$ and $(\mathrm{U}_{t,s}^*)_{t\geq s\geq 0}$ to keep explicit the unitarity of operators $\mathrm{U}_{t,s}, \mathrm{U}_{t,s}^*$ – instead of the cocycle property of $(\mathbf{U}_{t,s})_{s,t\in\mathbb{R}_0^+}$ – because we focus on the diagonalization of *self–adjoint* quadratic boson operators.
We conclude the proof of Theorem 14 by analyzing the effect of the unitary operator $\mathrm{U}_{t,s}$, on annihilation/creation operators a_k, a_k^*. The aim is to relate our method to the so–called Bogoliubov **u**–**v** transformation [1, 2], but in a more general setting (see [4] and Section VII.2). The main assertion, which yields Theorem 14 (ii), is the following lemma:

LEMMA 72 (THE BOGOLIUBOV **u**–**v** TRANSFORMATION)
Under Conditions A1–A4, for all $t \geq s \geq 0$ *and* $k \in \mathbb{N}$, *the operators*

$$a_{t,k} \quad := \quad \mathrm{U}_{t,s} a_{s,k} \mathrm{U}_{t,s}^* = \sum_{\ell} \{\mathbf{u}_{t,s}\}_{k,\ell}\, a_{s,\ell} + \{\mathbf{v}_{t,s}\}_{k,\ell}\, a_{s,\ell}^* \ , \tag{VI.55}$$

$$a_{s,k} \quad := \quad \mathrm{U}_{t,s}^* a_{t,k} \mathrm{U}_{t,s} = \sum_{\ell} \left\{\mathbf{u}_{t,s}^*\right\}_{k,\ell} a_{t,\ell} - \left\{\mathbf{v}_{t,s}^{\mathrm{t}}\right\}_{k,\ell} a_{t,\ell}^* \tag{VI.56}$$

are well–defined on $\mathcal{D}(\mathrm{N}^{1/2})$ *and satisfy the CCR. Here,* $a_{0,k} := a_k$ *and the operators*

$$\begin{aligned} \mathbf{u}_{t,s} &:= \mathbf{1} + \sum_{n=1}^{\infty} 4^{2n} \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{2n-1}} \mathrm{d}\tau_{2n} B_{\tau_{2n}} \bar{B}_{\tau_{2n-1}} \cdots B_{\tau_2}\bar{B}_{\tau_1} \ , \\ \mathbf{v}_{t,s} &:= -\sum_{n=0}^{\infty} 4^{2n+1} \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{2n}} \mathrm{d}\tau_{2n+1} B_{\tau_{2n+1}} \bar{B}_{\tau_{2n}} \cdots \bar{B}_{\tau_2} B_{\tau_1} \ , \end{aligned} \tag{VI.57}$$

with $\tau_0 := t$, satisfy for all $t \geq s \geq 0$: $\mathbf{v}_{t,s} \in \mathcal{L}^2(\mathfrak{h})$ and

$$\mathbf{u}_{t,s}\mathbf{u}_{t,s}^* - \mathbf{v}_{t,s}\mathbf{v}_{t,s}^* = \mathbf{1} \ , \qquad \mathbf{u}_{t,s}\mathbf{v}_{t,s}^{\mathrm{t}} = \mathbf{v}_{t,s}\mathbf{u}_{t,s}^{\mathrm{t}} \ , \tag{VI.58}$$

$$\mathbf{u}_{t,s}^*\mathbf{u}_{t,s} - \mathbf{v}_{t,s}^{\mathrm{t}}\bar{\mathbf{v}}_{t,s} = \mathbf{1} \ , \qquad \mathbf{u}_{t,s}^*\mathbf{v}_{t,s} = \mathbf{v}_{t,s}^{\mathrm{t}}\bar{\mathbf{u}}_{t,s} \ . \tag{VI.59}$$

Proof. First, we observe that $(\mathbf{u}_{t,s} - \mathbf{1}) \in \mathcal{L}^2(\mathfrak{h})$ and $\mathbf{v}_{t,s} \in \mathcal{L}^2(\mathfrak{h})$, for all $t \geq s \geq 0$, because one directly checks from (VI.57) that

$$1 + \|\mathbf{u}_{t,s} - \mathbf{1}\|_2 \leq \cosh\left\{4\int_s^t \|B_\tau\|_2 \mathrm{d}\tau\right\} \ , \quad \|\mathbf{v}_{t,s}\|_2 \leq \sinh\left\{4\int_s^t \|B_\tau\|_2 \mathrm{d}\tau\right\} \ . \tag{VI.60}$$

Furthermore, because $(B_t)_{t\geq 0} \in C[\mathbb{R}_0^+; \mathcal{L}^2(\mathfrak{h})]$ (see Theorem 11 (ii), (v)), the operator family $(\mathbf{u}_{t,s} - \mathbf{1}, \mathbf{v}_{t,s})_{t\geq s}$ is a solution in $\mathcal{L}^2(\mathfrak{h}) \times \mathcal{L}^2(\mathfrak{h})$ of the system of differential equations

$$\forall t \geq s \geq 0: \qquad \left\{ \begin{array}{lcl} \partial_t \mathbf{u}_{t,s} = -4\mathbf{v}_{t,s}\bar{B}_t & , & \mathbf{u}_{s,s} := \mathbf{1} \ . \\ \partial_t \mathbf{v}_{t,s} = -4\mathbf{u}_{t,s}B_t & , & \mathbf{v}_{s,s} := 0 \ . \end{array} \right. \tag{VI.61}$$

It follows that

$$\partial_t \left\{\mathbf{u}_{t,s}\mathbf{u}_{t,s}^* - \mathbf{v}_{t,s}\mathbf{v}_{t,s}^*\right\} = 0 \quad \text{and} \quad \partial_t \left\{\mathbf{u}_{t,s}\mathbf{v}_{t,s}^{\mathrm{t}} - \mathbf{v}_{t,s}\mathbf{u}_{t,s}^{\mathrm{t}}\right\} = 0 \ , \tag{VI.62}$$

for all $t \geq s \geq 0$, which imply (VI.58). On the other hand, observe that

$$\begin{aligned} \mathbf{u}_{t,s} &= \mathbf{1} + \sum_{n=1}^{\infty} 4^{2n} \int_s^t \mathrm{d}\tau_1 \cdots \int_{\tau_{2n-1}}^t \mathrm{d}\tau_{2n} B_{\tau_1}\bar{B}_{\tau_2} \cdots B_{\tau_{2n-1}}\bar{B}_{\tau_{2n}} \ , \\ \mathbf{v}_{t,s} &= -\sum_{n=0}^{\infty} 4^{2n+1} \int_s^t \mathrm{d}\tau_1 \cdots \int_{\tau_{2n}}^t \mathrm{d}\tau_{2n+1} B_{\tau_1}\bar{B}_{\tau_2} \cdots \bar{B}_{\tau_{2n}} B_{\tau_{2n+1}} \ . \end{aligned} \tag{VI.63}$$

It yields on $\mathcal{L}^2(\mathfrak{h})$ and for all $t \geq s \geq 0$ the system of differential equations

$$\forall t \geq s \geq 0: \qquad \left\{ \begin{array}{lcl} \partial_s \mathbf{u}_{t,s} = 4B_s\bar{\mathbf{v}}_{t,s} & , & \mathbf{u}_{t,t} := \mathbf{1} \ . \\ \partial_s \mathbf{v}_{t,s} = 4B_s\bar{\mathbf{u}}_{t,s} & , & \mathbf{v}_{t,t} := 0 \ . \end{array} \right. \tag{VI.64}$$

Therefore, for all $t \geq s \geq 0$,

$$\partial_t \left\{\mathbf{u}_{t,s}^*\mathbf{u}_{t,s} - \mathbf{v}_{t,s}^{\mathrm{t}}\bar{\mathbf{v}}_{t,s}\right\} = 0 \quad \text{and} \quad \partial_t \left\{\mathbf{u}_{t,s}^*\mathbf{v}_{t,s} - \mathbf{v}_{t,s}^{\mathrm{t}}\bar{\mathbf{u}}_{t,s}\right\} = 0 \tag{VI.65}$$

from which (VI.59) follows.

Now, for each $k \in \mathbb{N}$, we define the operator $\tilde{a}_{t,k}$ acting on the boson Fock space $\mathcal{F}_b$ by

$$\tilde{a}_{t,k} := \sum_{\ell} \{\mathbf{u}_t\}_{k,\ell}\, a_\ell + \{\mathbf{v}_t\}_{k,\ell}\, a_\ell^* = a(\mathbf{u}_t^*\varphi_k) + a^*(\mathbf{v}_t^{\mathrm{t}}\bar{\varphi}_k) \tag{VI.66}$$

with $\mathbf{u}_t := \mathbf{u}_{t,0}$ and $\mathbf{v}_t := \mathbf{v}_{t,0}$. Recall that $\{\varphi_k\}_{k=1}^{\infty}$ is some real orthonormal basis in $\mathcal{D}(\Omega_0) \subseteq \mathfrak{h}$ and $a_k := a(\varphi_k)$ is the standard boson annihilation operator acting on the boson Fock space $\mathcal{F}_b$. Because of (VI.58)–(VI.60), the operator family $\{\tilde{a}_{t,k}, \tilde{a}_{t,k}^*\}_{k=1}^{\infty}$ satisfies the CCR and all operators $\tilde{a}_{t,k}$ and $\tilde{a}_{t,k}^*$ are well–defined on the domain $\mathcal{D}\left(\mathrm{N}^{1/2}\right)$. Indeed, a straightforward computation shows that

$$\begin{aligned}\mathrm{N}_t \quad := \quad & \sum_k \tilde{a}_{t,k}^* \tilde{a}_{t,k} = \mathrm{N} + \|\mathbf{v}_t\|_2^2 + \sum_{\ell,m} \{\mathbf{v}_t \mathbf{v}_t^* + \mathbf{u}_t^* \mathbf{u}_t - \mathbf{1}\}_{m,\ell}\, a_m^* a_\ell \\ & + \sum_{\ell,m} \{\mathbf{u}_t^* \mathbf{v}_t\}_{m,\ell}\, a_m^* a_\ell^* + \left\{\mathbf{u}_t^{\mathrm{t}} \bar{\mathbf{v}}_t\right\}_{m,\ell} a_\ell a_m \ , \qquad \text{(VI.67)}\end{aligned}$$

which, by similar arguments as in Lemma 66, implies that

$$\begin{aligned} & \| \left(\mathrm{N}_t + \mathbf{1}\right)^{1/2} \left(\mathrm{N} + \mathbf{1}\right)^{-1/2} \|_{\mathrm{op}} \qquad \text{(VI.68)} \\ \leq \quad & 2 \|\mathbf{v}_t\|_2 + (1 + \|\mathbf{u}_t - 1\|_2^{1/2})(1 + \|(\mathbf{u}_t - \mathbf{1})\|_2^{1/2} + 2 \|\mathbf{v}_t\|_2^{1/2}) \ ,\end{aligned}$$

where $\|\mathbf{u}_t - \mathbf{1}\|_2$ and $\|\mathbf{v}_t\|_2$ are bounded by (VI.60), see also Lemma 102. In particular, (VI.68) yields

$$\|\tilde{a}_{t,k}^{\#} \left(\mathrm{N} + \mathbf{1}\right)^{-1/2} \|_{\mathrm{op}} \leq \| \left(\mathrm{N}_t + \mathbf{1}\right)^{1/2} \left(\mathrm{N} + \mathbf{1}\right)^{-1/2} \|_{\mathrm{op}} < \infty \ , \qquad \text{(VI.69)}$$

where $\tilde{a}_{t,k}^{\#}$ denotes either $\tilde{a}_{t,k}$ or $\tilde{a}_{t,k}^*$ for any $k \in \mathbb{N}$ and $t \geq 0$.
Next, we define the strong derivative $\partial_t \tilde{a}_{t,k}$ for any positive time $t \geq 0$ and all $k \in \mathbb{N}$. To this end, we set

$$R_{k,\ell}(\delta) \quad := \quad \delta^{-1} \int_0^\delta \left\{\mathbf{v}_{t+\tau} \bar{B}_{t+\tau} - \mathbf{v}_t \bar{B}_t\right\}_{k,\ell} \mathrm{d}\tau \ , \qquad \text{(VI.70)}$$

$$\tilde{R}_{k,\ell}(\delta) \quad := \quad \delta^{-1} \int_s^\delta \left\{\mathbf{u}_{t+\tau} B_{t+\tau} - \mathbf{u}_t B_t\right\}_{k,\ell} \mathrm{d}\tau \ , \qquad \text{(VI.71)}$$

where $\delta > 0$, and observe that

$$\begin{aligned} & \delta^{-1}\left(\tilde{a}_{t+\delta,k} - \tilde{a}_{t,k}\right) + 4 \sum_\ell \left\{\mathbf{v}_t \bar{B}_t\right\}_{k,\ell} a_\ell + \left\{\mathbf{u}_t B_t\right\}_{k,\ell} a_\ell^* \\ & = -4 \sum_\ell \left(R_{k,\ell}(\delta)\, a_\ell + \tilde{R}_{k,\ell}(\delta)\, a_\ell^* \right) \ , \qquad \text{(VI.72)}\end{aligned}$$

by the fundamental theorem of calculus. Introducing the definitions

$$F_\delta := \sum_\ell R_{k,\ell}(\delta)\, a_\ell \ , \qquad \tilde{F}_\delta^* := \sum_\ell \tilde{R}_{k,\ell}(\delta)\, a_\ell^* \ , \qquad \text{(VI.73)}$$

and using the equality

$$\tilde{F}_\delta \tilde{F}_\delta^* = \sum_\ell \left|\tilde{R}_{k,\ell}(\delta)\right|^2 \cdot \mathbf{1} + \tilde{F}_\delta^* \tilde{F}_\delta \ , \qquad \text{(VI.74)}$$

we obtain, as in Lemma 66, that

$$\|(F_\delta + \tilde{F}_\delta^*)(\mathrm{N}+\mathbf{1})^{-1/2}\|_{\mathrm{op}} \leq 2\left(\|R(\delta)\|_2 + \|\tilde{R}(\delta)\|_2\right) , \tag{VI.75}$$

where $R(\delta)$ and $\tilde{R}(\delta)$ are the Hilbert–Schmidt operators defined by their coefficients (VI.70) and (VI.71), respectively. Since $\mathbf{u}_{t,s}$, $\mathbf{v}_{t,s}$, and B_t are all continuous in t with respect to the Hilbert–Schmidt topology, (VI.75) inserted into (VI.72) implies that the limit

$$\partial_t \tilde{a}_{t,k} := \lim_{\delta\to 0}\left\{\delta^{-1}\left(\tilde{a}_{t+\delta,k} - \tilde{a}_{t,k}\right)\right\} = -4\sum_{\ell}\left\{\mathbf{v}_t \bar{B}_t\right\}_{k,\ell} a_\ell + \left\{\mathbf{u}_t B_t\right\}_{k,\ell} a_\ell^* \tag{VI.76}$$

holds true on $\mathcal{D}(\mathrm{N}^{1/2})$. Moreover, a simple computation using (VI.69) and

$$\begin{aligned}&\|(\mathrm{N}+\mathbf{1})^{1/2}\,\mathrm{G}_t\,(\mathrm{N}+\mathbf{1})^{-2}\|_{\mathrm{op}} \\ &\leq \|\mathrm{G}_t\,(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} + \|\,[\mathrm{N},\mathrm{G}_t]\,(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} < \infty\end{aligned} \tag{VI.77}$$

(cf. (VI.20) and (VI.23)) shows that the evolution equations

$$\forall k\in\mathbb{N},\ t\geq 0: \qquad \partial_t \tilde{a}_{t,k} = i\,[\tilde{a}_{t,k},\mathrm{G}_t] \ , \quad \tilde{a}_{0,k} = a_k \ , \tag{VI.78}$$

hold true on $\mathcal{D}(\mathrm{N}^2)\subset\mathcal{D}(\mathrm{N}^{1/2})$.
Observe now that, for all $k\in\mathbb{N}$ and $t\geq 0$,

$$[\mathrm{N},\tilde{a}_{t,k}] = \sum_{\ell}\left\{-\mathbf{u}_t\right\}_{k,\ell} a_\ell + \left\{\mathbf{v}_t\right\}_{k,\ell} a_\ell^* \ , \tag{VI.79}$$

and it is thus straightforward to check that

$$\begin{aligned}\left\|(\mathrm{N}+\mathbf{1})\tilde{a}_{t,k}(\mathrm{N}+\mathbf{1})^{-2}\right\|_{\mathrm{op}} &\leq \|\tilde{a}_{t,k}(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} + \left\|[\mathrm{N},\tilde{a}_{t,k}]\,(\mathrm{N}+\mathbf{1})^{-2}\right\|_{\mathrm{op}} \\ &< \infty ,\end{aligned} \tag{VI.80}$$

using (VI.69). By (VI.78), for all vectors $\psi\in\mathcal{D}(\mathrm{N}^2)$, it implies that

$$\partial_t\left\{\mathrm{U}_{t,s}^*\tilde{a}_{t,k}\mathrm{U}_{t,s}\psi\right\} = \mathrm{U}_{t,s}^*\left(\partial_t\tilde{a}_{t,k} - i\,[\tilde{a}_{t,k},\mathrm{G}_t]\right)\mathrm{U}_{t,s}\psi = 0 \tag{VI.81}$$

for all $t\geq s\geq 0$, using Lemmata 68–69. Hence, we deduce from (VI.81) that, for all $\psi\in\mathcal{D}(\mathrm{N}^2)$,

$$\forall k\in\mathbb{N},\ t\geq s\geq 0: \qquad \tilde{a}_{t,k}\psi = \mathrm{U}_{t,s}\tilde{a}_{s,k}\mathrm{U}_{t,s}^*\psi \ . \tag{VI.82}$$

The domain $\mathcal{D}(\mathrm{N}^2)$ is a core for each closed operator of the family $(\tilde{a}_{t,k})_{k\in\mathbb{N},t\geq 0}$, see (VI.66). Therefore, since, by (VI.69),

$$\tilde{a}_{t,k}(\mathrm{N}+\mathbf{1})^{-1/2}\in\mathcal{B}(\mathcal{F}_b) \ , \tag{VI.83}$$

for any vector $\varphi\in\mathcal{D}(\mathrm{N}^{1/2})$ there is a sequence $\{\psi_n\}_{n=0}^\infty\subset\mathcal{D}(\mathrm{N}^2)$ converging to φ such that $\{\tilde{a}_{t,k}\psi_n\}_{n=0}^\infty\subset\mathcal{F}_b$ converges to $\tilde{a}_{t,k}\varphi\in\mathcal{F}_b$. On the other hand, since $U_{t,s}$ is unitary and a_k is defined as a closed operator, $(U_{t,s}\tilde{a}_{s,k}U^*_{t,s})_{k\in\mathbb{N},t\geq s\geq 0}$ is a family of closed operators and (VI.82) implies the equalities

$$\forall n,k\in\mathbb{N},\ t\geq s\geq 0: \qquad \tilde{a}_{t,k}\psi_n = U_{t,s}\tilde{a}_{s,k}U^*_{t,s}\psi_n \ . \tag{VI.84}$$

It follows that $\varphi\in\mathcal{D}(U_{t,s}\tilde{a}_{s,k}U^*_{t,s})$ for all $k\in\mathbb{N}$, and

$$\forall k\in\mathbb{N},\ t\geq s\geq 0: \qquad \tilde{a}_{t,k}\varphi = U_{t,s}\tilde{a}_{s,k}U^*_{t,s}\varphi \ , \tag{VI.85}$$

for any $\varphi\in\mathcal{D}(N^{1/2})$. □

In other words, the isospectral flow defined via $U_{t,s}$ is a (time–dependent) Bogoliubov **u**–**v** unitary transformation, see also Theorem 96. This last lemma concludes the proof of Theorem 14.

In the last proof, note that we have obtained the inclusion

$$\mathcal{D}(N^{1/2})\subset\mathcal{D}(U_t a_k U^*_t) \ , \tag{VI.86}$$

by closedness of the operators $\tilde{a}_{t,k}$ and $U_t a_k U^*_t$ together with (VI.69) and (VI.82). In fact, $U_{t,s}$ and $U^*_{t,s}$ conserve the domain $\mathcal{D}\left(N^{1/2}\right)$ for all $t\geq s\geq 0$, as

$$\|(N+\mathbf{1})^{1/2}U^{\#}_{t,s}(N+\mathbf{1})^{-1/2}\|_{\mathrm{op}} \leq \exp\left\{88\int_s^t \|B_\tau\|_2 \mathrm{d}\tau\right\} < \infty \ , \tag{VI.87}$$

where $U^{\#}_{t,s}$ denotes either $U_{t,s}$ or $U^*_{t,s}$. It directly implies, again, that $\mathcal{D}(N^{1/2})\subset\mathcal{D}(U_t a_k U^*_t)$. Inequality (VI.87) can be seen by using Lemma 90 provided we show that

$$[(N+\mathbf{1})^{1/2},G_t](N+\mathbf{1})^{-1/2}\in\mathcal{B}\left(\mathcal{F}_b\right) \ . \tag{VI.88}$$

However, the latter demands several estimations using the equality

$$(N+\mathbf{1})^{1/2} = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}\lambda}{\sqrt{\lambda}}\frac{N+\mathbf{1}}{N+(\lambda+1)\,\mathbf{1}} \tag{VI.89}$$

on the domain $\mathcal{D}\left(N^{1/2}\right)$, see [13, (1.4.2) and 1.4.7 (e)].

VI.2 Brocket–Wegner Flow on Quadratic Boson Operators

The proof of Theorem 18 is carried out by using the fact that

$$\partial_t\left(U^*_{t,s}\left(H_t+i\lambda\mathbf{1}\right)^{-1}U_{t,s}\right) = 0 \tag{VI.90}$$

for positive times $t\geq s$ and $\lambda\in\mathbb{R}$, because formally $\partial_t H_t = i[H_t,G_t]$ and $\partial_t U_{t,s} = -iG_tU_{t,s}$. Equation (VI.90) yields

$$\left(H_t+i\lambda\mathbf{1}\right)^{-1} = U_{t,s}\left(H_s+i\lambda\mathbf{1}\right)^{-1}U^*_{t,s} \ , \tag{VI.91}$$

which in turn implies the sought equality $\mathrm{H}_t = \mathrm{U}_{t,s}\mathrm{H}_s\mathrm{U}_{t,s}^*$.
Unfortunately, the proof is not as straightforward as it looks like. Indeed, the infinitesimal generator G_t (III.8) of the unitary propagator $\mathrm{U}_{t,s}$ is unbounded. The derivative $\partial_t\mathrm{H}_t$ of the time–dependent quadratic operator H_t (III.7) is also unbounded because it is a quadratic operator formally given by a commutator $[\mathrm{H}_t, \mathrm{G}_t]$ of two unbounded operators. Therefore, the computation of the derivative in (VI.90) has to be performed carefully. In fact, Equality (VI.90) is only proven for strictly positive times $t \geq s > 0$ on the dense domain $\mathcal{D}(\mathrm{N})$ of the particle number operator N (II.11) and the statement (VI.91) for $s = 0$ is shown by strong continuity.
We break the proof of Theorem 18 into several lemmata. For the reader's convenience we start by giving two trivial statements for unbounded operators, as they are used several times.

Lemma 73 (Resolvent and unbounded self–adjoint operators)
Let $\lambda \in \mathbb{R}$, Θ be any invertible operator on a Hilbert space $\mathcal{X}$, and X, Y be two self–adjoint operators on $\mathcal{X}$ with domains $\mathcal{D}(X), \mathcal{D}(Y) \subseteq \mathcal{X}$, respectively. Then one has:
(i) $\Theta(i\lambda\mathbf{1} + X)^{-1}\Theta^{-1} = (i\lambda\mathbf{1} + \Theta X\Theta^{-1})^{-1}$.
(ii) *If Θ is unitary and $(i\lambda\mathbf{1} + X)^{-1} = \Theta(i\lambda\mathbf{1} + Y)^{-1}\Theta^*$, then $X = \Theta Y\Theta^*$.*

We give now three simple but pivotal lemmata. First, we show that the bounded resolvent of H_t defined in (III.7) conserves the dense domain $\mathcal{D}(\mathrm{N})$ of the particle number operator N (II.11). Secondly, we prove the boundedness of the difference $(\mathrm{H}_t - \mathrm{H}_s)$ of quadratic operators with respect to the particle number operator N. Thirdly, we prove the resolvent identity

$$(\mathrm{H}_t + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_s + i\lambda\mathbf{1})^{-1} = (\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{H}_s - \mathrm{H}_t)(\mathrm{H}_s + i\lambda\mathbf{1})^{-1} . \tag{VI.92}$$

Observe that this last statement is not completely obvious. Indeed, recall that, for all $\varphi \in \mathcal{F}_b$,

$$(\mathrm{H}_t + i\lambda\mathbf{1})(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\varphi = \varphi , \tag{VI.93}$$

but obviously, the converse equality

$$(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{H}_t + i\lambda\mathbf{1})\varphi = \varphi \tag{VI.94}$$

only holds for all $\varphi \in \mathcal{D}(\mathrm{H}_t)$.

Lemma 74 (Conservation of $\mathcal{D}(\mathrm{N})$ by $(\mathrm{H}_t + i\mathbf{1})^{-1}$, $\mathrm{H}_t(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}$)
Under Conditions A1–A4 and for any $\lambda \in \mathbb{R}$ satisfying $|\lambda| > 11\|B_t\|_2$ at $t \geq 0$,

$$\|(\mathrm{N} + \mathbf{1})(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \leq \frac{1}{|\lambda| - 11\|B_t\|_2} , \tag{VI.95}$$

whereas

$$\|(\mathrm{N} + \mathbf{1})\mathrm{H}_t(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \leq \frac{|\lambda|(1 + 11\|B_t\|_2)}{|\lambda| - 11\|B_t\|_2} . \tag{VI.96}$$

Proof. From (III.8) and Lemma 73 (i), applied to $\Theta = \mathrm{N}+\mathbf{1}$ and to the self-adjoint (Proposition 1) operator $X = \mathrm{H}_t$ defined by (III.7), we obtain

$$(\mathrm{N}+\mathbf{1})(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{N}+\mathbf{1})^{-1} = \Big(\mathrm{H}_t - i\mathrm{G}_t(\mathrm{N}+\mathbf{1})^{-1} + i\lambda\mathbf{1}\Big)^{-1} . \quad \text{(VI.97)}$$

Because of the upper bound (VI.20), the operator $\mathrm{G}_t\,(\mathrm{N}+\mathbf{1})^{-1}$ is bounded and the inequality $|\lambda| > 11\|B_t\|_2$ implies that

$$|\lambda| > 11\|B_t\|_2 \geq \|\mathrm{G}_t\,(\mathrm{N}+\mathbf{1})^{-1}\,\|_{\mathrm{op}} . \quad \text{(VI.98)}$$

Since

$$\sum_{n=0}^{\infty} \frac{\|\mathrm{G}_t\,(\mathrm{N}+\mathbf{1})^{-1}\,\|_{\mathrm{op}}^n}{|\lambda|^{n+1}} = \frac{1}{|\lambda| - \|\mathrm{G}_t\,(\mathrm{N}+\mathbf{1})^{-1}\,\|_{\mathrm{op}}} , \quad \text{(VI.99)}$$

under condition (VI.98), we obtain the norm convergence of the Neumann series

$$(\mathrm{H}_t - i\mathrm{G}_t(\mathrm{N}+\mathbf{1})^{-1} + i\lambda\mathbf{1})^{-1} = \sum_{n=0}^{\infty} (\mathrm{H}_t + i\lambda\mathbf{1})^{-1} \Big\{ i\mathrm{G}_t\,(\mathrm{N}+\mathbf{1})^{-1}\,(\mathrm{H}_t + i\lambda\mathbf{1})^{-1} \Big\}^n \quad \text{(VI.100)}$$

for any $t \geq 0$. In particular, by (VI.97), we obtain the first statement of the lemma.

Additionally, we infer from (VI.97) that

$$(\mathrm{N}+\mathbf{1})\mathrm{H}_t(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{N}+\mathbf{1})^{-1} = \frac{\mathrm{H}_t - i\mathrm{G}_t(\mathrm{N}+\mathbf{1})^{-1}}{\mathrm{H}_t - i\mathrm{G}_t(\mathrm{N}+\mathbf{1})^{-1} + i\lambda\mathbf{1}} . \quad \text{(VI.101)}$$

Hence, by using (VI.100) together with the obvious inequality

$$\|\mathrm{H}_t(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\|_{\mathrm{op}} \leq 1 , \quad \text{(VI.102)}$$

we obtain the second upper bound of the lemma under condition (VI.98). □

Lemma 75 (Boundedness of the difference $(\mathrm{H}_t - \mathrm{H}_s)$ in $\mathcal{D}(\mathrm{N})$)
Assume Conditions A1–A4. Then, for all $t, s \geq 0$,

$$\|(\mathrm{H}_t - \mathrm{H}_s)(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} \leq 40 \int_s^t \|B_\tau\|_2^2 \mathrm{d}\tau + (4+\sqrt{2})\|B_t - B_s\|_2 . \quad \text{(VI.103)}$$

Proof. By combining the definition (III.7) with (III.9)–(III.10) (cf. Theorem 11), we get the equality

$$\begin{aligned} \mathrm{H}_t - \mathrm{H}_s \;&=\; 8\int_s^t \|B_\tau\|_2^2\,\mathrm{d}\tau - 16\sum_{k,\ell} \int_s^t \left\{B_\tau \bar{B}_\tau\right\}_{k,\ell} \mathrm{d}\tau\ a_k^* a_\ell \\ &\quad + \sum_{k,\ell} \left\{B_t - B_s\right\}_{k,\ell} a_k^* a_\ell^* + \left\{\bar{B}_t - \bar{B}_s\right\}_{k,\ell} a_k a_\ell \end{aligned} \quad \text{(VI.104)}$$

for all $t \geq s \geq 0$. Therefore, applying Lemma 66 with $\theta = \mathbf{1}$,

$$X = X^* = \Omega_t - \Omega_s = -16 \int_s^t B_\tau \bar{B}_\tau \mathrm{d}\tau \ , \tag{VI.105}$$

and the Hilbert–Schmidt operator $Y = \bar{B}_t - \bar{B}_s \in \mathcal{L}^2(\mathfrak{h})$, we obtain (VI.103). □

LEMMA 76 (ANALYSIS OF THE DIFFERENCE OF RESOLVENTS IN $\mathcal{D}(\mathrm{N})$)
Under Conditions A1–A4 and for any $\lambda \in \mathbb{R}$ satisfying $|\lambda| > 11\|B_0\|_2$, all times $t, s \geq 0$, and $\varphi \in \mathcal{D}(\mathrm{N})$,

$$\{(\mathrm{H}_t + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_s + i\lambda\mathbf{1})^{-1}\}\varphi = (\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{H}_s - \mathrm{H}_t)(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}\varphi \ . \tag{VI.106}$$

PROOF. Clearly, by (VI.93),

$$(\mathrm{H}_t + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_s + i\lambda\mathbf{1})^{-1} = \{(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{H}_s + i\lambda\mathbf{1}) - \mathbf{1}\}(\mathrm{H}_s + i\lambda\mathbf{1})^{-1} . \tag{VI.107}$$

By Lemma 74, the resolvent $(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}$ conserves the domain $\mathcal{D}(\mathrm{N})$ under the condition that $|\lambda| > 11\|B_0\|_2 \geq 11\|B_t\|_2$ (cf. (VI.1)). In particular, for any $s \geq 0$ and $\varphi \in \mathcal{F}_b$,

$$(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}(\mathrm{N} + \mathbf{1})^{-1}\varphi \in \mathcal{D}(\mathrm{N}) \cap \mathcal{D}(\mathrm{H}_s) \ . \tag{VI.108}$$

On the other hand, observe from Lemma 75 that, for all $t, s \geq 0$,

$$\mathrm{H}_t(\mathrm{N} + \mathbf{1})^{-1} = \mathrm{H}_s(\mathrm{N} + \mathbf{1})^{-1} + \mathbb{D}_t \ , \tag{VI.109}$$

with the *bounded* operator $\mathbb{D}_t \in \mathcal{B}(\mathcal{F}_b)$ defined by

$$\mathbb{D}_t := (\mathrm{H}_t - \mathrm{H}_s)(\mathrm{N} + \mathbf{1})^{-1} \ . \tag{VI.110}$$

In other words, for all $t, s \geq 0$,

$$\mathcal{D}(\mathrm{N}) \cap \mathcal{D}(\mathrm{H}_s) = \mathcal{D}(\mathrm{N}) \cap \mathcal{D}(\mathrm{H}_t) \ . \tag{VI.111}$$

Therefore, for all $t, s \geq 0$ and any $\varphi \in \mathcal{F}_b$, (VI.108) and (VI.111) obviously yield

$$(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}(\mathrm{N} + \mathbf{1})^{-1}\varphi \in \mathcal{D}(\mathrm{N}) \cap \mathcal{D}(\mathrm{H}_t) \ , \tag{VI.112}$$

which, by (VI.94), in turn implies that

$$\begin{aligned} & (\mathrm{H}_s + i\lambda\mathbf{1})^{-1}(\mathrm{N} + \mathbf{1})^{-1}\varphi \\ = \ & (\mathrm{H}_t + i\lambda\mathbf{1})^{-1}(\mathrm{H}_t + i\lambda\mathbf{1})(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}(\mathrm{N} + \mathbf{1})^{-1}\varphi \ . \end{aligned} \tag{VI.113}$$

Combining this last equality with (VI.107) we obtain

$$\begin{aligned} & \{(\mathrm{H}_t + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_s + i\lambda\mathbf{1})^{-1}\}(\mathrm{N} + \mathbf{1})^{-1}\varphi \\ & = -(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\mathbb{D}_t(\mathrm{N} + \mathbf{1})(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}(\mathrm{N} + \mathbf{1})^{-1}\varphi \end{aligned} \tag{VI.114}$$

for all $\varphi \in \mathcal{F}_b$, any $t, s \geq 0$, and under the condition that $|\lambda| > 11\|B_0\|_2$. □

REMARK 77 *The operator on the right side of (VI.106) is bounded because the operator on the left side is. In particular, Lemma 76 can be extended to all* $\varphi \in \mathcal{F}_b$. *This is however not necessary for our proofs.*

We are now in position to show the norm continuity of the time–dependent resolvent $(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}$ on the dense domain $\mathcal{D}(\mathrm{N})$ of the particle number operator N.

LEMMA 78 (CONTINUITY OF $(\mathrm{H}_t + i\mathbf{1})^{-1}$)
Under Conditions A1–A4 and for any $\lambda \in \mathbb{R}$ *satisfying* $|\lambda| > 11\|B_0\|_2$, *the bounded operator*

$$(\mathrm{H}_t + i\lambda\mathbf{1})^{-1} \in C[\mathbb{R}_0^+; \mathcal{B}(\mathcal{Y}, \mathcal{F}_b)] \tag{VI.115}$$

with $\mathcal{Y} = \mathcal{D}(\mathrm{N})$ *being the dense domain of* N. *In particular, the resolvent* $(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}$ *is strongly continuous for all* $t \geq 0$.

PROOF. By Theorem 11 (ii), we already know that the operator $B_t \in \mathcal{L}^2(\mathfrak{h})$ is continuous in the Hilbert–Schmidt topology for any $t \geq 0$. Therefore, Lemma 75 yields in the limit $t \to s$ $(t, s \geq 0)$ that

$$\lim_{t\to s} \|(\mathrm{H}_t - \mathrm{H}_s)(\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} = 0 \ . \tag{VI.116}$$

On the other hand, Lemma 76 leads, for any $t \geq s \geq 0$, to the inequality

$$\begin{aligned} &\|\{(\mathrm{H}_t + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_s + i\lambda\mathbf{1})^{-1}\}(\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \\ &\leq \|(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\| \, \|(\mathrm{H}_t - \mathrm{H}_s)(\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \\ &\|(\mathrm{N} + \mathbf{1})(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}(\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \ . \end{aligned} \tag{VI.117}$$

Combined with the limit (VI.116) and Lemma 74 for $|\lambda| > 11\|B_0\|_2 \geq 11\|B_t\|_2$ (cf. (VI.1)), Inequality (VI.117) in turn implies in the limit $t \to s$ $(t, s \geq 0)$ that

$$\lim_{t\to s} \|\{(\mathrm{H}_t + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_s + i\lambda\mathbf{1})^{-1}\}(\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} = 0 \ . \tag{VI.118}$$

Since the domain $\mathcal{D}(\mathrm{N})$ is dense and the resolvent $(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}$ is bounded under the condition that $\lambda \in \mathbb{R}$ is not zero, the previous limit yields the strong continuity of the resolvent $(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}$ for all $t \geq 0$. □

We now prove that the quadratic form associated with the resolvent $(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}$, for any real number λ that fulfills $|\lambda| > 11\|B_0\|_2$, satisfies a differential equation on the set $\mathcal{D}(\mathrm{N}) \times \mathcal{D}(\mathrm{N})$.

LEMMA 79 (EVOLUTION EQUATION ON RESOLVENTS)
Under Conditions A1–A4 and for $\lambda \in \mathbb{R}$ *such that* $|\lambda| > 11\|B_0\|_2$, *the resolvent* $(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}$ *satisfies, for any* $\varphi, \psi \in \mathcal{D}(\mathrm{N})$, *the differential equation*

$$\forall t \geq s > 0: \qquad \langle\psi|(\partial_t Y_t)\varphi\rangle = \langle\psi|(i\,[Y_t, \mathrm{G}_t])\varphi\rangle \ , \quad Y_s := (\mathrm{H}_s + i\lambda\mathbf{1})^{-1} \ . \tag{VI.119}$$

PROOF. For any $\varphi \in \mathcal{F}_b$ and all strictly positive times $t > 0$, we define the derivative $\partial_t \mathrm{H}_t$ on the domain $\mathcal{D}(\mathrm{N})$ to be the limit operator

$$\partial_t(\mathrm{H}_t (\mathrm{N}+\mathbf{1})^{-1} \varphi) := \lim_{\epsilon \to 0} \left\{ \epsilon^{-1} (\mathrm{H}_{t+\epsilon} - \mathrm{H}_t) (\mathrm{N}+\mathbf{1})^{-1} \varphi \right\} . \tag{VI.120}$$

This derivative is well–defined. Indeed, from Theorem 46 (ii) the Hilbert–Schmidt operator B_t is locally Lipschitz continuous in $\mathcal{L}^2(\mathfrak{h})$ on $(0,\infty)$. Consequently, we infer from Lemma 75 that, for all $t > 0$,

$$\lim_{\epsilon \to 0} \| \epsilon^{-1} (\mathrm{H}_{t+\epsilon} - \mathrm{H}_t)(\mathrm{N}+\mathbf{1})^{-1} \|_{\mathrm{op}} < \infty . \tag{VI.121}$$

In fact, we can apply Corollary 47 and Lemma 66 with $\theta = \mathbf{1}$,

$$X = X^* = \epsilon^{-1} \{\Omega_{t+\epsilon} - \Omega_t\} + 16 B_t \bar{B}_t \tag{VI.122}$$

and the Hilbert–Schmidt operator

$$Y = \epsilon^{-1} \left\{ \bar{B}_{t+\epsilon} - \bar{B}_t \right\} + 2 \left\{ \Omega_t^{\mathrm{t}} \bar{B}_t + \bar{B}_t \Omega_t \right\} \in \mathcal{L}^2(\mathfrak{h}) , \tag{VI.123}$$

for $t > 0$ and $\epsilon \geq -t$, in order to show that the derivative $\partial_t \mathrm{H}_t$ defined by (VI.120) is equal on the dense domain $\mathcal{D}(\mathrm{N})$ to

$$\begin{aligned} \partial_t \mathrm{H}_t \quad = \quad & -2 \sum_{k,\ell} \left\{ \Omega_t B_t + B_t \Omega_t^{\mathrm{t}} \right\}_{k,\ell} a_k^* a_\ell^* + \left\{ \Omega_t^{\mathrm{t}} \bar{B}_t + \bar{B}_t \Omega_t \right\}_{k,\ell} a_k a_\ell \\ & -16 \sum_{k,\ell} \left\{ B_t \bar{B}_t \right\}_{k,\ell} a_k^* a_\ell + 8 \| B_t \|_2^2 \ . \end{aligned} \tag{VI.124}$$

This operator is self–adjoint for any $t > 0$, by Proposition 1, because $\Omega_t B_t, B_t \Omega_t^{\mathrm{t}} \in \mathcal{L}^2(\mathfrak{h})$, see Lemma 45.
Now, from (III.7)–(III.8) combined with Equation (VI.124), the derivative $\partial_t \mathrm{H}_t$ is formally equal to

$$\partial_t \mathrm{H}_t = i \left[\mathrm{H}_t, \mathrm{G}_t \right] \tag{VI.125}$$

and furthermore, $(\partial_t \mathrm{H}_t)(\mathrm{N}+\mathbf{1})^{-1} \in \mathcal{B}(\mathcal{F}_b)$ is a bounded operator for all $t > 0$. Therefore,

$$\begin{aligned} & \mathrm{H}_t \mathrm{G}_t (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} (\mathrm{N}+\mathbf{1})^{-1} \\ & = \mathrm{G}_t \mathrm{H}_t (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} (\mathrm{N}+\mathbf{1})^{-1} + (\partial_t \mathrm{H}_t) (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} (\mathrm{N}+\mathbf{1})^{-1} , \end{aligned} \tag{VI.126}$$

where both operators in the r.h.s. of the equation are bounded for any $t > 0$ and $|\lambda| > 11 \|B_0\|_2 \geq 11 \|B_t\|_2$, because of (VI.1), (VI.20), and Lemma 74. In particular, the operator

$$\mathrm{H}_t \mathrm{G}_t (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} (\mathrm{N}+\mathbf{1})^{-1} \in \mathcal{B}(\mathcal{F}_b)$$

is also bounded, and Equality (VI.126) yields

$$(\mathrm{H}_t + i\lambda \mathbf{1})^{-1} (\partial_t \mathrm{H}_t) (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} (\mathrm{N}+\mathbf{1})^{-1} = \left[(\mathrm{H}_t + i\lambda \mathbf{1})^{-1}, \mathrm{G}_t \right] (\mathrm{N}+\mathbf{1})^{-1} \tag{VI.127}$$

for all strictly positive times $t > 0$.
On the other hand, for any $\varphi, \psi \in \mathcal{F}_b$ and all $t > 0$, we define the derivative

$$
\begin{aligned}
&(\langle(\mathrm{N}+\mathbf{1})^{-1}\psi|\partial_t\{(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}\}(\mathrm{N}+\mathbf{1})^{-1}\varphi\rangle) \qquad \text{(VI.128)}\\
&:= \lim_{\epsilon\to 0}\langle(\mathrm{N}+\mathbf{1})^{-1}\psi|\epsilon^{-1}\{(\mathrm{H}_{t+\epsilon}+i\lambda\mathbf{1})^{-1}-(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}\}(\mathrm{N}+\mathbf{1})^{-1}\varphi\rangle\ .
\end{aligned}
$$

For any $\varphi, \psi \in \mathcal{F}_b$, Lemma 76 yields the inequality

$$
\begin{aligned}
&|\langle(\mathrm{N}+\mathbf{1})^{-1}\psi|\{(\mathrm{H}_{t+\epsilon}+i\lambda\mathbf{1})^{-1}-(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}\}(\mathrm{N}+\mathbf{1})^{-1}\varphi\rangle\\
&+\langle(\mathrm{N}+\mathbf{1})^{-1}\psi|(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}(\mathrm{H}_{t+\epsilon}-\mathrm{H}_t)(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}(\mathrm{N}+\mathbf{1})^{-1}\varphi\rangle|\\
&\le \|(\mathrm{H}_{t+\epsilon}-\mathrm{H}_t)(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}}\|(\mathrm{N}+\mathbf{1})(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}}\\
&\quad\|\{(\mathrm{H}_{t+\epsilon}-i\lambda\mathbf{1})^{-1}-(\mathrm{H}_t-i\lambda\mathbf{1})^{-1}\}(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}}\|\psi\|\,\|\varphi\|\ . \qquad \text{(VI.129)}
\end{aligned}
$$

Therefore, since $|\lambda| > 11\|B_0\|_2 \geq 11\|B_t\|_2$ for any $t \geq 0$, we use the upper bound (VI.129) in the limit $\epsilon \to 0$ together with Lemmata 74, 78, and (VI.120)–(VI.121), in order to prove that the derivative (VI.128) at any strictly positive time $t > 0$ is equal to

$$
\begin{aligned}
&\langle(\mathrm{N}+\mathbf{1})^{-1}\psi|\partial_t\{(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}\}(\mathrm{N}+\mathbf{1})^{-1}\varphi\rangle \qquad \text{(VI.130)}\\
&=-\langle(\mathrm{N}+\mathbf{1})^{-1}\psi|(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}(\partial_t\mathrm{H}_t)(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}(\mathrm{N}+\mathbf{1})^{-1}\varphi\rangle\ ,
\end{aligned}
$$

which in turn implies the lemma because of Equation (VI.127). □
Now, we prove Theorem 18 by using the resolvents of H_t and H_s and Lemma 73 (ii).

Lemma 80 (Unitary transformation of H_s via $\mathrm{U}_{t,s}$)
Under Conditions A1–A4, $\mathrm{H}_t = \mathrm{U}_{t,s}\mathrm{H}_s\mathrm{U}_{t,s}^*$*, where* $\mathrm{U}_{t,s}$ *is the unitary propagator defined in Theorem 14 for any* $t \geq s \geq 0$.

Proof. Let $|\lambda| > 11\|B_0\|_2 \geq 11\|B_t\|_2$ (cf. (VI.1)) and $t \geq s > 0$. For any vectors $\varphi, \psi \in \mathcal{F}_b$, observe that

$$
\begin{aligned}
&\left\langle(\mathrm{N}+\mathbf{1})^{-1}\psi\right|\partial_t\left(\mathrm{U}_{t,s}^*(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}\mathrm{U}_{t,s}(\mathrm{N}+\mathbf{1})^{-1}\right)\varphi\Big\rangle\\
=\ &\left\langle(\mathrm{N}+\mathbf{1})^{-1}\psi\right|\partial_t(\mathrm{U}_{t,s}^*(\mathrm{N}+\mathbf{1})^{-1})\\
&\left\{(\mathrm{N}+\mathbf{1})(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}(\mathrm{N}+\mathbf{1})^{-1}\right\}\left\{(\mathrm{N}+\mathbf{1})\mathrm{U}_{t,s}(\mathrm{N}+\mathbf{1})^{-1}\right\}\varphi\Big\rangle\\
&+\left\langle(\mathrm{N}+\mathbf{1})^{-1}\psi\right|\left\{\mathrm{U}_{t,s}^*(\mathrm{H}_t+i\lambda\mathbf{1})^{-1}\right\}\partial_t(\mathrm{U}_{t,s}(\mathrm{N}+\mathbf{1})^{-1})\varphi\Big\rangle\\
&+\left\langle(\mathrm{N}+\mathbf{1})^{-1}\left\{(\mathrm{N}+\mathbf{1})\mathrm{U}_{t,s}(\mathrm{N}+\mathbf{1})^{-1}\right\}\psi\right|\partial_t((\mathrm{H}_t+i\lambda\mathbf{1})^{-1})\\
&(\mathrm{N}+\mathbf{1})^{-1}\left\{(\mathrm{N}+\mathbf{1})\mathrm{U}_{t,s}(\mathrm{N}+\mathbf{1})^{-1}\right\}\varphi\Big\rangle\ . \qquad \text{(VI.131)}
\end{aligned}
$$

Each of the five operators in braces $\{\cdot\}$ within this equation is *bounded* because of (VI.25), (VI.44), and Lemma 74. Moreover, Lemmata 68–69 tells us that

$$\forall t \geq s \geq 0 : \qquad \left\{ \begin{array}{l} \partial_t \{ \mathrm{U}_{t,s} \left(\mathrm{N} + \mathbf{1} \right)^{-1} \} = -i \mathrm{G}_t \mathrm{U}_{t,s} \left(\mathrm{N} + \mathbf{1} \right)^{-1} \quad , \\ \partial_t \{ \mathrm{U}_{t,s}^* \left(\mathrm{N} + \mathbf{1} \right)^{-1} \} = i \mathrm{U}_{t,s}^* \mathrm{G}_t \left(\mathrm{N} + \mathbf{1} \right)^{-1} \quad , \end{array} \right. \tag{VI.132}$$

in the strong sense in $\mathcal{F}_b$, whereas, by Lemma 79,

$$\begin{aligned} & \langle (\mathrm{N} + \mathbf{1})^{-1} \psi | (\partial_t \{ (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} \}) \, (\mathrm{N} + \mathbf{1})^{-1} \varphi \rangle \\ & = \langle (\mathrm{N} + \mathbf{1})^{-1} \psi | (i[(\mathrm{H}_t + i\lambda \mathbf{1})^{-1}, \mathrm{G}_t]) \, (\mathrm{N} + \mathbf{1})^{-1} \varphi \rangle, \end{aligned} \tag{VI.133}$$

for all vectors $\varphi, \psi \in \mathcal{F}_b$ and all strictly positive times $t > 0$. Therefore, it follows from Equation (VI.131) combined with the derivatives (VI.132)–(VI.133) that

$$\langle (\mathrm{N} + \mathbf{1})^{-1} \psi | \partial_t \{ \mathrm{U}_{t,s}^* \, (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} \, \mathrm{U}_{t,s} \} \, (\mathrm{N} + \mathbf{1})^{-1} \varphi \rangle = 0 \tag{VI.134}$$

for any $\varphi, \psi \in \mathcal{F}_b$ and all $t \geq s > 0$, i.e.,

$$\langle (\mathrm{N} + \mathbf{1})^{-1} \psi | (\mathrm{U}_{t,s}^* \, (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} \, \mathrm{U}_{t,s} - (\mathrm{H}_s + i\lambda \mathbf{1})^{-1}) \, (\mathrm{N} + \mathbf{1})^{-1} \varphi \rangle = 0 \, . \tag{VI.135}$$

The vector

$$(\mathrm{N} + \mathbf{1}) \left\{ \mathrm{U}_{t,s}^* \, (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} \, \mathrm{U}_{t,s} - (\mathrm{H}_s + i\lambda \mathbf{1})^{-1} \right\} (\mathrm{N} + \mathbf{1})^{-1} \varphi \tag{VI.136}$$

is well–defined because of Inequalities (VI.25), (VI.44), and Lemma 74. Consequently, by taking $\psi \in \mathcal{F}_b$ equal to (VI.136) in (VI.135) we obtain

$$\mathrm{U}_{t,s}^* \, (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} \, \mathrm{U}_{t,s} \, (\mathrm{N} + \mathbf{1})^{-1} = (\mathrm{H}_s + i\lambda \mathbf{1})^{-1} \, (\mathrm{N} + \mathbf{1})^{-1} \, . \tag{VI.137}$$

Since the domain $\mathcal{D}(\mathrm{N})$ is dense and the resolvent $(\mathrm{H}_t + i\lambda \mathbf{1})^{-1}$ as well as the unitary operator $\mathrm{U}_{t,s}$ are bounded, we infer from (VI.137) that

$$\mathrm{U}_{t,s}^* \, (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} \, \mathrm{U}_{t,s} = (\mathrm{H}_s + i\lambda \mathbf{1})^{-1} \tag{VI.138}$$

for all strictly positive times $t \geq s > 0$. It thus remains to prove this equality for all $t \geq s = 0$.

The unitary operator $\mathrm{U}_{t,s}$ satisfies the cocycle property $\mathrm{U}_{t,x}\mathrm{U}_{x,s} = \mathrm{U}_{t,s}$ for any $t \geq x \geq s \geq 0$ and so, Equation (VI.138) is equivalent to the equality

$$\mathrm{U}_t^* \, (\mathrm{H}_t + i\lambda \mathbf{1})^{-1} \, \mathrm{U}_t = \mathrm{U}_s^* \, (\mathrm{H}_s + i\lambda \mathbf{1})^{-1} \, \mathrm{U}_s \tag{VI.139}$$

for all $t \geq s > 0$, where we recall that $\mathrm{U}_t := \mathrm{U}_{t,0}$. To perform the limit $s \to 0$ in the previous equation, note that, for any $\varphi \in \mathcal{F}_b$,

$$\begin{aligned} & \| \{ \mathrm{U}_s^* \, (\mathrm{H}_s + i\lambda \mathbf{1})^{-1} \, \mathrm{U}_s - (\mathrm{H}_0 + i\lambda \mathbf{1})^{-1} \} \varphi \| \\ \leq \; & \| \mathrm{U}_s^* \|_{\mathrm{op}} \, \| \, (\mathrm{H}_s + i\lambda \mathbf{1})^{-1} \, \|_{\mathrm{op}} \| \, (\mathrm{U}_s - \mathbf{1}) \, \varphi \| \\ & + \| \mathrm{U}_s^* \|_{\mathrm{op}} \, \| \{ (\mathrm{H}_s + i\lambda \mathbf{1})^{-1} - (\mathrm{H}_0 + i\lambda \mathbf{1})^{-1} \} \varphi \| \\ & + \| \, (\mathrm{U}_s^* - \mathbf{1}) \, (\mathrm{H}_0 + i\lambda \mathbf{1})^{-1} \, \varphi \| \, . \end{aligned} \tag{VI.140}$$

The unitary operators $\mathrm{U}_{t,s}$ and $\mathrm{U}_{t,s}^{*}$ are jointly strongly continuous in s and t with $\mathrm{U}_{s,s}^{*} = \mathrm{U}_{s,s} := \mathbf{1}$ for all $t \geq s \geq 0$, see Lemmata 68–70. Furthermore, the resolvent $(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}$ is strongly continuous for any $t \geq 0$, by Lemma 78. Consequently, we combine Equality (VI.139) in the limit $s \to 0$ with Inequality (VI.140) to verify (VI.138) for all $t \geq s = 0$. In other words, we obtain in this way the desired statement:

$$\forall t \geq s \geq 0: \qquad \mathrm{U}_{t,s}^{*} (\mathrm{H}_t + i\lambda\mathbf{1})^{-1} \mathrm{U}_{t,s} = (\mathrm{H}_s + i\lambda\mathbf{1})^{-1} . \tag{VI.141}$$

The lemma then follows from Lemma 73 (ii) because, as already explained, $\mathrm{U}_{t,s}$ is unitary, by Lemma 70. □

VI.3 Quasi N–Diagonalization of Quadratic Boson Operators

The limits of Ω_t, C_t, and H_t, as $t \to \infty$, can all be obtained by using the square integrability of $\|B_t\|_2$ on $[0, \infty)$. Sufficient conditions to get this property are given by Theorem 19 (i). Therefore, the proof of Theorem 22 is broken in three lemmata by always using the assumption that $\|B_t\|_2$ is square–integrable on $[0, \infty)$.

Lemma 81 (Limits of Ω_t and C_t for infinite times)
Assume Conditions A1–A4 and the square–integrability of $t \mapsto \|B_t\|_2$. Then $(\Omega_0 - \Omega_t)_{t\geq 0} \in \mathcal{L}^1(\mathfrak{h})$ converges in $\mathcal{L}^1(\mathfrak{h})$ to $(\Omega_0 - \Omega_\infty)$, where

$$\Omega_\infty := \Omega_0 - 16 \int_0^\infty B_\tau \bar{B}_\tau \mathrm{d}\tau = \Omega_\infty^{*} \geq 0 \tag{VI.142}$$

on the domain $\mathcal{D}(\Omega_0)$, and the limit $t \to \infty$ of real numbers $(C_t)_{t\geq 0}$ (cf. (III.10)) equals

$$C_\infty := 8 \int_0^\infty \|B_\tau\|_2^2 \, \mathrm{d}\tau + C_0 < \infty . \tag{VI.143}$$

Proof. The operators $\Delta_t := \Omega_0 - \Omega_t \in \mathcal{L}^1(\mathfrak{h})$ for all $t \geq 0$ form a Cauchy sequence on the Banach space $\mathcal{L}^1(\mathfrak{h})$, as $t \to \infty$, because the map $t \mapsto \|B_t\|_2$ is square–integrable on $[0, \infty)$. Indeed, integrating (III.9) observe that

$$\forall t, s \in \mathbb{R}_0^+ : \qquad \Omega_t = \Omega_s - 16 \int_s^t B_\tau \bar{B}_\tau \mathrm{d}\tau \geq 0 , \tag{VI.144}$$

which implies that

$$\forall t \geq s \geq 0: \qquad \|\Delta_t - \Delta_s\|_1 \leq 16 \int_s^t \|B_\tau\|_2^2 \, \mathrm{d}\tau . \tag{VI.145}$$

In other words, by completeness of $\mathcal{L}^1(\mathfrak{h})$, there is a trace–class operator

$$\Delta_\infty := 16 \int_0^\infty B_\tau \bar{B}_\tau \mathrm{d}\tau \in \mathcal{L}^1(\mathfrak{h}) \tag{VI.146}$$

to which Δ_t converges in trace–norm. In particular, the domain of $\Omega_\infty := \Omega_0 - \Delta_\infty$ equals $\mathcal{D}(\Omega_0)$. The positivity of Ω_∞ is also a direct consequence of $\Omega_t \geq 0$. The existence of C_∞ as well as its explicit form are obvious because of (III.10). □

The convergence of the operator Ω_t to Ω_∞ given in Lemma 81 means via (III.9) that one can extend (VI.144) to infinite times:

$$\forall t,s \in \mathbb{R}_0^+ \cup \{\infty\} : \qquad \Omega_t = \Omega_s - 16 \int_s^t B_\tau \bar{B}_\tau \mathrm{d}\tau \ . \tag{VI.147}$$

Next, we analyze Theorem 52 (i), (iii), and Corollary 55 in the limit $t \to \infty$, when A4 holds (i.e., $T_+ = \infty$). In particular, the limit operator Ω_∞ given in Lemma 81 can explicitly be computed in the special case $\Omega_0 B_0 = B_0 \Omega_0^{\mathrm{t}}$.

Lemma 82 (Limit operator Ω_∞ and constant of motion)
Assume Conditions A1–A4 and the square–integrability of $t \mapsto \|B_t\|_2$. Let $s \in \mathbb{R}_0^+ \cup \{\infty\}$ and $B_\infty := 0$.
(i)

$$\Omega_\infty^2 = \Omega_s^2 - 8 B_s \bar{B}_s + 32 \int_s^\infty B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau \mathrm{d}\tau \ . \tag{VI.148}$$

(ii) *If $\Omega_0 B_0 = B_0 \Omega_0^{\mathrm{t}}$ then*

$$\Omega_s B_s = B_s \Omega_s^{\mathrm{t}}, \quad \Omega_\infty = \{\Omega_s^2 - 4 B_s \bar{B}_s\}^{1/2} \ . \tag{VI.149}$$

(iii) *The constant of motion of the flow in the limit $t \to \infty$ equals*

$$\mathrm{tr}\left(\Omega_\infty^2 - \Omega_s^2 + 4 B_s \bar{B}_s\right) = 0 \ . \tag{VI.150}$$

Proof. (i): Using Theorem 52 (i) we observe that

$$\left\|\Omega_t^2 - \Omega_s^2\right\|_1 \leq 8 \left\|B_s\right\|_2^2 + 8 \left\|B_t\right\|_2^2 + 32 \int_s^t \left\|B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau\right\|_1 \mathrm{d}\tau \tag{VI.151}$$

for any $t \geq s \geq 0$. Since, for all $t > 0$,

$$\left\|B_t \Omega_t^{\mathrm{t}} \bar{B}_t\right\|_1 = \left\|\bar{B}_t \Omega_t B_t\right\|_1 = \|\Omega_t^{1/2} B_t\|_2^2 \tag{VI.152}$$

(Lemma 102), we apply the upper bound of Lemma 56 for $\alpha = 1/2$ and $n = 1$ to get the inequality

$$\forall t > 0 : \qquad \left\|B_t \Omega_t^{\mathrm{t}} \bar{B}_t\right\|_1 \leq \frac{1}{2\mathrm{e}t} \|B_0\|_2 \|B_t\|_2 \ . \tag{VI.153}$$

Therefore, since $t \mapsto \|B_t\|_2$ is square–integrable on $[0,\infty)$, the map $t \mapsto \|B_t \Omega_t^{\mathrm{t}} \bar{B}_t\|_2$ is also integrable as $t \to \infty$ and, by (VI.151), the operator family $(\Omega_t^2 - \Omega_s^2)_{t \geq s} \in \mathcal{L}^1(\mathfrak{h})$ for fixed $s > 0$ is a Cauchy sequence on the Banach space $\mathcal{L}^1(\mathfrak{h})$, as $t \to \infty$. By completeness of $\mathcal{L}^1(\mathfrak{h})$ together with Theorem 52

(i) (here $T_+ = \infty$) and the resolvent identity (V.183), the positive operator family $(\Omega_t^2)_{t\geq 0}$ converges in the norm resolvent sense to the positive operator, defined for all $s \geq 0$, by

$$\Xi_\infty := \Omega_s^2 - 8B_s\bar{B}_s + 32\int_s^\infty B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau \mathrm{d}\tau \geq 0 \ , \tag{VI.154}$$

with domain $\mathcal{D}(\Xi_\infty) = \mathcal{D}\left(\Omega_0^2\right)$.
On the other hand, we observe that, for any $t \geq 0$,

$$\left(16\int_t^\infty B_\tau \bar{B}_\tau \mathrm{d}\tau\right)\mathfrak{h} \subset \mathcal{D}(\Omega_0) \ . \tag{VI.155}$$

Indeed, using (VI.147) and the closedness of the bounded resolvent $(\Omega_\infty + \mathbf{1})^{-1}$ together with Lemmata 101 (i) and 102 we arrive at the upper bound

$$\begin{aligned}\left\|\Omega_\infty \int_t^\infty B_\tau \bar{B}_\tau \mathrm{d}\tau\right\|_{\mathrm{op}} \leq\ & \left\|\Omega_\infty (\Omega_\infty + \mathbf{1})^{-1}\right\|_{\mathrm{op}} \int_t^\infty \left\|B_\tau \Omega_\tau^{\mathrm{t}} \bar{B}_\tau\right\|_1 \mathrm{d}\tau \\ & +16\left\|\Omega_\infty (\Omega_\infty + \mathbf{1})^{-1}\right\|_{\mathrm{op}} \left(\int_t^\infty \|B_\tau\|_2^2 \,\mathrm{d}\tau\right)^2 \\ & + \left\|\Omega_\infty (\Omega_\infty + \mathbf{1})^{-1}\right\|_{\mathrm{op}} \int_t^\infty \|B_\tau\|_2^2 \,\mathrm{d}\tau \ .\end{aligned} \tag{VI.156}$$

Since the maps $t \mapsto \|B_t\|_2^2$ and $t \mapsto \left\|B_t \Omega_t^{\mathrm{t}} \bar{B}_t\right\|_1$ are both integrable on $[0, \infty)$, the last upper bound yields Assertion (VI.155). Therefore, we can use (V.183), (V.262)–(V.263), and (VI.147) together with (VI.155) and obtain a similar upper bound as (V.264), that is,

$$\left\|\left(\Omega_\infty^2 + \mathbf{1}\right)^{-1} - \left(\Omega_t^2 + \mathbf{1}\right)^{-1}\right\|_{\mathrm{op}} \leq 128\int_t^\infty \|B_\tau\|_2^2 \,\mathrm{d}\tau \left(1 + 8\int_t^\infty \|B_\tau\|_2^2 \,\mathrm{d}\tau\right) \ . \tag{VI.157}$$

By square–integrability of the map $t \mapsto \|B_t\|_2$ on $[0, \infty)$, it follows that the positive operator family $(\Omega_t^2)_{t\geq 0}$ converges in the norm resolvent sense to Ω_∞^2, which is also defined on the domain $\mathcal{D}\left(\Omega_0^2\right)$, by (VI.147) and (VI.155). As a consequence, Ω_∞^2 equals the operator Ξ_∞ defined by (VI.154).

(ii): If $\Omega_0 B_0 = B_0 \Omega_0^{\mathrm{t}}$ then, for all $t, s \geq 0$, $\Omega_t B_t = B_t \Omega_t^{\mathrm{t}}$ and $\Omega_t^2 - 4B_t\bar{B}_t = \Omega_s^2 - 4B_s\bar{B}_s$, see Corollary 55 with $T_+ = \infty$. Since $\|B_t\|_2$ converges to zero in the limit $t \to \infty$ and the sequence $(\Omega_\infty^2 - \Omega_t^2)_{t\geq 0} \in \mathcal{L}^1(\mathfrak{h})$ converges in $\mathcal{L}^1(\mathfrak{h})$ to the zero operator (cf. (i)), we deduce the second assertion using the trivial equality

$$\Omega_\infty^2 - \Omega_s^2 + 4B_s\bar{B}_s = \{\Omega_\infty^2 - \Omega_t^2\} + \{\Omega_t^2 - \Omega_s^2 + 4B_s\bar{B}_s\} \tag{VI.158}$$

for all $t, s \geq 0$.

(iii): Similarly, (VI.158) together with Theorem 52 (iii) and the trace–class convergence of $(\Omega_\infty^2 - \Omega_t^2)_{t\geq 0}$ (cf. (i)) also implies (VI.150). □

Therefore, Theorem 22 (i) is a direct consequence of Lemmata 81–82 combined with Theorem 19 (i). It remains to prove its second statement (ii). This is easily performed in the next two lemmata, which conclude the proof of Theorem 22.

LEMMA 83 (LIMIT OF H_t AS $t \to \infty$ ON THE DOMAIN $\mathcal{D}(\mathrm{N})$)
Assume Conditions A1–A4 and the square–integrability of $t \mapsto \|B_t\|_2$. *Then*

$$\lim_{t\to\infty} \left\|(\mathrm{H}_\infty - \mathrm{H}_t)(\mathrm{N}+\mathbf{1})^{-1}\right\|_{\mathrm{op}} = 0\ , \tag{VI.159}$$

where the self–adjoint quadratic boson operator H_∞ *is defined by*

$$\mathrm{H}_\infty := \sum_{k,\ell} \{\Omega_\infty\}_{k,\ell} a_k^* a_\ell + C_\infty\ . \tag{VI.160}$$

PROOF. Because of (VI.147) and the square–integrability of the map $t \mapsto \|B_t\|_2$ on $[0,\infty)$, we can extend Lemma 75 to all $t, s \in \mathbb{R}_0^+ \cup \{\infty\}$ with $B_\infty := 0$ and obviously deduce the assertion. □

LEMMA 84 ($\mathrm{H}_t \to \mathrm{H}_\infty$ IN THE STRONG RESOLVENT SENSE)
Assume Conditions A1–A4 and the square–integrability of $t \mapsto \|B_t\|_2$. *Then, for any non–zero real* $\lambda \neq 0$ *and any vector* $\varphi \in \mathcal{F}_b$,

$$\lim_{t\to\infty} \left\|\{(\mathrm{H}_\infty + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\}\varphi\right\| = 0\ . \tag{VI.161}$$

PROOF. For any non–zero real $\lambda \neq 0$, there is $T_\lambda \geq 0$ such that $|\lambda| > 11\|B_t\|_2$ for all $t > T_\lambda$ because, by assumption, the Hilbert–Schmidt norm $\|B_t\|_2$ vanishes when $t \to \infty$. If the map $t \mapsto \|B_t\|_2$ is square–integrable on $[0,\infty)$ then Lemmata 74–76 are satisfied for $t = \infty$ and all $s \geq 0$ because of Lemma 81, see also (VI.147). Therefore, similar to (VI.117), we obtain for any $t > T_\lambda$ the inequality

$$\begin{aligned} &\left\|\{(\mathrm{H}_\infty + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\}(\mathrm{N}+\mathbf{1})^{-1}\right\|_{\mathrm{op}} \\ \leq\ & \|\,(\mathrm{H}_\infty + i\lambda\mathbf{1})^{-1}\,\|_{\mathrm{op}} \|\,(\mathrm{H}_\infty - \mathrm{H}_t)\,(\mathrm{N}+\mathbf{1})^{-1}\,\|_{\mathrm{op}} \\ & \|\,(\mathrm{N}+\mathbf{1})\,(\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\,(\mathrm{N}+\mathbf{1})^{-1}\,\|_{\mathrm{op}}\ , \end{aligned} \tag{VI.162}$$

which in turn yields the limit

$$\lim_{t\to\infty} \left\|\{(\mathrm{H}_\infty + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\}(\mathrm{N}+\mathbf{1})^{-1}\right\|_{\mathrm{op}} = 0 \tag{VI.163}$$

for any non–zero real $\lambda \neq 0$, see Lemma 83. The domain $\mathcal{D}(\mathrm{N})$ is dense in $\mathcal{F}_b$ and both resolvent are bounded. It is then straightforward to verify the strong convergence of resolvents from (VI.163). □

VI.4 N–Diagonalization of Quadratic Boson Operators

To obtain the limits $t \to \infty$ of the bounded operators $\mathbf{u}_{t,s}$, $\mathbf{v}_{t,s}$ (see (VI.57)) and $\mathrm{U}_{t,s}$ (Theorem 14), as well as the equality $\mathrm{H}_\infty = \mathrm{U}_{\infty,s}\mathrm{H}_s\mathrm{U}^*_{\infty,s}$, one needs the integrability of the map $t \mapsto \|B_t\|_2$ on $[0,\infty)$. Sufficient conditions to ensure this are given by Theorem 19. So, we break up the proof of Theorem 23 into several lemmata, always using the integrability of the map $t \mapsto \|B_t\|_2$ on $[0,\infty)$.

Lemma 85 (Limits of $\mathbf{u}_{t,s}$ and $\mathbf{v}_{t,s}$ as $t \to \infty$)
Assume Conditions A1–A4 and the integrability of $t \mapsto \|B_t\|_2$. Then $\mathbf{u}_{t,s}$ and $\mathbf{v}_{t,s}$ converge in $\mathcal{L}^2(\mathfrak{h})$ respectively to

$$\mathbf{u}_{\infty,s} := \mathbf{1} + \sum_{n=1}^{\infty} 4^{2n} \int_s^\infty \mathrm{d}\tau_1 \cdots \int_s^{\tau_{2n-1}} \mathrm{d}\tau_{2n} B_{\tau_{2n}} \bar{B}_{\tau_{2n-1}} \cdots B_{\tau_2}\bar{B}_{\tau_1} \quad \text{(VI.164)}$$

and

$$\mathbf{v}_{\infty,s} := -\sum_{n=0}^{\infty} 4^{2n+1} \int_s^\infty \mathrm{d}\tau_1 \cdots \int_s^{\tau_{2n}} \mathrm{d}\tau_{2n+1} B_{\tau_{2n+1}} \bar{B}_{\tau_{2n}} \cdots \bar{B}_{\tau_2} B_{\tau_1}\ , \quad \text{(VI.165)}$$

with $\mathbf{u}_{\infty,\infty} = \mathbf{1}$, $\mathbf{v}_{\infty,\infty} = 0$,

$$\mathbf{u}_{\infty,s}\mathbf{u}^*_{\infty,s} - \mathbf{v}_{\infty,s}\mathbf{v}^*_{\infty,s} = \mathbf{1}\ , \qquad \mathbf{u}_{\infty,s}\mathbf{v}^{\mathrm{t}}_{\infty,s} = \mathbf{v}_{\infty,s}\mathbf{u}^{\mathrm{t}}_{\infty,s}\ , \quad \text{(VI.166)}$$

$$\mathbf{u}^*_{\infty,s}\mathbf{u}_{\infty,s} - \mathbf{v}^{\mathrm{t}}_{\infty,s}\bar{\mathbf{v}}_{\infty,s} = \mathbf{1}\ , \qquad \mathbf{u}^*_{\infty,s}\mathbf{v}_{\infty,s} = \mathbf{v}^{\mathrm{t}}_{\infty,s}\bar{\mathbf{u}}_{\infty,s}\ , \quad \text{(VI.167)}$$

and

$$\|\mathbf{u}_{\infty,s} - \mathbf{1}\|_2 \leq \cosh\left\{4\int_s^\infty \|B_\tau\|_2\ \mathrm{d}\tau\right\} - 1\ , \quad \text{(VI.168)}$$

$$\|\mathbf{v}_{\infty,s}\|_2 \leq \sinh\left\{4\int_s^\infty \|B_\tau\|_2\ \mathrm{d}\tau\right\}\ . \quad \text{(VI.169)}$$

Proof. By using (VI.61), written as two integral equations, and (VI.60) we get the upper bound

$$\begin{aligned} &\|\mathbf{u}_{t_1,s} - \mathbf{u}_{t_2,s}\|_2 + \|\mathbf{v}_{t_1,s} - \mathbf{v}_{t_2,s}\|_2 \\ \leq\ & \left(\exp\left\{4\int_{t_1}^{t_2} \|B_\tau\|_2\ \mathrm{d}\tau\right\} - 1\right)\exp\left\{4\int_s^{t_1}\|B_\tau\|_2\ \mathrm{d}\tau\right\} \quad \text{(VI.170)}\end{aligned}$$

for any $t_2 \geq t_1 \geq s \geq 0$. In fact, by (VI.57) and (VI.164)–(VI.165), this inequality also holds for $t_2 = \infty$, provided the map $t \mapsto \|B_t\|_2$ is integrable on $[0,\infty)$. Consequently, the Hilbert–Schmidt operators $(\mathbf{u}_{t,s} - \mathbf{1})$ and $\mathbf{v}_{t,s}$ defined by (VI.57) converge in $\mathcal{L}^2(\mathfrak{h})$, respectively to the operators $(\mathbf{u}_{\infty,s} - \mathbf{1})$ and $\mathbf{v}_{\infty,s}$ defined above. The properties of $\mathbf{u}_{\infty,s}$ and $\mathbf{v}_{\infty,s}$ given in the lemma are straightforward to verify and we omit the details. Also, the operators

$\mathbf{u}_{\infty,\infty}$ and $\mathbf{v}_{\infty,\infty}$ are defined by taking the limit $s \rightarrow \infty$. It is easy to check that $\mathbf{u}_{\infty,\infty} = \mathbf{1}$ and $\mathbf{v}_{\infty,\infty} = 0$. □

This lemma corresponds to the first statement (i) of Theorem 23, using Theorem 19. Now, we prove its second statement (ii), again assuming the integrability of the map $t \mapsto \|B_t\|_2$ on $[0, \infty)$.

Lemma 86 (Limits of $\mathrm{U}_{t,s}$ and $\mathrm{U}_{t,s}^*$ as $t \rightarrow \infty$)
Assume Conditions A1–A4 and the integrability of $t \mapsto \|B_t\|_2$. Then, as $t \rightarrow \infty$, the unitary operator $\mathrm{U}_{t,s}$ strongly converges to a strongly continuous in $s \in \mathbb{R}_0^+ \cup \{\infty\}$ unitary operator $\mathrm{U}_{\infty,s}$ satisfying the non–autonomous evolution equation

$$\forall s \geq 0 : \qquad \partial_s \mathrm{U}_{\infty,s} = i \mathrm{U}_{\infty,s} \mathrm{G}_s \ , \quad \mathrm{U}_{\infty,\infty} = \mathbf{1} \ , \tag{VI.171}$$

on the domain $\mathcal{D}(\mathrm{N})$ and the cocycle property $\mathrm{U}_{\infty,s} = \mathrm{U}_{\infty,x} \mathrm{U}_{x,s}$ for $x \geq s \geq 0$. Moreover, its adjoint $\mathrm{U}_{t,s}^*$ also converges in the strong topology to $\mathrm{U}_{\infty,s}^*$ when $t \rightarrow \infty$.

Proof. From straightforward estimations using Lemmata 68, 70, and (VI.20) we obtain that, for any $t_2 \geq s_2 \geq s_1$ and $t_2 \geq t_1 \geq s_1$,

$$\begin{aligned} \|(\mathrm{U}_{t_2,s_2} - \mathrm{U}_{t_1,s_1})(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} \quad \leq \quad & 11 \int_{s_1}^{s_2} \|B_\tau\|_2 \ \mathrm{d}\tau \\ & + \frac{1}{2} \exp\left\{ 22 \int_{s_1}^{t_1} \|B_\tau\|_2 \ \mathrm{d}\tau \right\} \\ & \times \left(\exp\left\{ 22 \int_{t_1}^{t_2} \|B_\tau\|_2 \ \mathrm{d}\tau \right\} - 1 \right) \ . \end{aligned} \tag{VI.172}$$

In particular, if the map $t \mapsto \|B_t\|_2$ is integrable on $[0, \infty)$ then by taking the limit $\min\{t_1, t_2\} \rightarrow \infty$ and choosing $s_1 := s_2 := s$ in (VI.172) we show that the operator

$$\{\mathrm{U}_{t_2,s} - \mathrm{U}_{t_1,s}\}(\mathrm{N}+\mathbf{1})^{-1} \tag{VI.173}$$

converges to zero in norm. Hence, by standard arguments (cf. (VII.34)–(VII.38)), there exists a bounded operator $\mathrm{U}_{\infty,s}$ defined, for any $\varphi \in \mathcal{F}_b$ and $s \geq 0$, by the strong limit

$$\mathrm{U}_{\infty,s}\varphi := \lim_{t \rightarrow \infty} \{\mathrm{U}_{t,s}\varphi\} \ . \tag{VI.174}$$

In particular, by the unitarity of the operator $\mathrm{U}_{t,s}$ for all $t \geq s \geq 0$,

$$\forall s \geq 0 : \qquad \|\mathrm{U}_{\infty,s}\|_{\mathrm{op}} \leq 1 \ . \tag{VI.175}$$

From (VI.172) we also obtain

$$\mathrm{U}_{\infty,\infty}\varphi := \lim_{s \rightarrow \infty} \mathrm{U}_{\infty,s}\varphi = \lim_{s \rightarrow \infty} \left\{ \lim_{t \rightarrow \infty} \mathrm{U}_{t,s}\varphi \right\} = \lim_{t \rightarrow \infty} \mathrm{U}_{t,t}\varphi = \mathbf{1} \ . \tag{VI.176}$$

By using similar arguments as in the proof of Lemma 92, one additionally verifies that $\mathrm{U}_{\infty,s}$ is strongly continuous in s and $\mathrm{U}_{\infty,s} = \mathrm{U}_{\infty,x}\mathrm{U}_{x,s}$ for any $x \geq s \geq 0$. Moreover, for any sufficiently small parameter $|\epsilon| > 0$, $t > s + \epsilon$, and $\varphi \in \mathcal{D}\left(\mathrm{N}^2\right) \subseteq \mathcal{D}\left(\mathrm{G}_s\right)$ (cf. (VI.20)),

$$\begin{aligned}
& \|\{\epsilon^{-1}(\mathrm{U}_{\infty,s+\epsilon} - \mathrm{U}_{\infty,s})\varphi - i\mathrm{U}_{\infty,s}\mathrm{G}_s\}\varphi\| \\
\leq\ & \|\{\mathrm{U}_{\infty,s} - \mathrm{U}_{t,s}\}\mathrm{G}_s\varphi\| + \epsilon^{-1}\int_s^{s+\epsilon} \|\{\mathrm{G}_\tau - \mathrm{G}_s\}\varphi\|\mathrm{d}\tau \\
& +\epsilon^{-1}\int_s^{s+\epsilon} \|\{\mathrm{U}_{t,\tau} - \mathrm{U}_{t,s}\}\left(\mathrm{N}+\mathbf{1}\right)^{-1}\|_{\mathrm{op}}\|\mathrm{G}_s\left(\mathrm{N}+\mathbf{1}\right)^{-1}\|_{\mathrm{op}}\mathrm{d}\tau \\
& +\epsilon^{-1}\int_s^{s+\epsilon} \|\{\mathrm{U}_{t,\tau} - \mathrm{U}_{t,s}\}\left(\mathrm{N}+\mathbf{1}\right)^{-1}\|_{\mathrm{op}}\|\left[\mathrm{N},\mathrm{G}_s\right]\left(\mathrm{N}+\mathbf{1}\right)^{-1}\|_{\mathrm{op}}\mathrm{d}\tau \\
& +\epsilon^{-1}\|\{\mathrm{U}_{\infty,s} - \mathrm{U}_{t,s}\}\varphi\| + \epsilon^{-1}\|\{\mathrm{U}_{\infty,s+\epsilon} - \mathrm{U}_{t,s+\epsilon}\}\varphi\| \ .
\end{aligned} \tag{VI.177}$$

Since $\mathrm{G}_t \in C[\mathbb{R}_0^+;\mathcal{B}(\mathcal{Y},\mathcal{F}_b)]$ and

$$\left[\mathrm{N},\mathrm{G}_s\right]\left(\mathrm{N}+\mathbf{1}\right)^{-1} \in \mathcal{B}\left(\mathcal{F}_b\right) \tag{VI.178}$$

(cf. Lemma 67), we take $t = t(\epsilon) \to \infty$ as $\epsilon \to 0$ in (VI.177) and use (VI.172) together with standard arguments to get that $\partial_s \mathrm{U}_{\infty,s} = i\mathrm{U}_{\infty,s}\mathrm{G}_s$ for any $\varphi \in \mathcal{D}\left(\mathrm{N}^2\right) \subseteq \mathcal{D}\left(\mathrm{G}_s\right)$. Remark that $\mathcal{D}\left(\mathrm{N}^2\right)$ is clearly a core for the particle number operator N. As a consequence, for any vector $\varphi \in \mathcal{D}(\mathrm{N})$ there is a sequence $\{\varphi_n\}_{n=0}^\infty \subset \mathcal{D}(\mathrm{N}^2)$ converging to φ such that $\{\mathrm{N}\varphi_n\}_{n=0}^\infty \subset \mathcal{F}_b$ converges to $\mathrm{N}\varphi \in \mathcal{F}_b$. Since, for any $n \in \mathbb{N}$, $s \geq 0$, and sufficiently small $|\epsilon| > 0$,

$$\begin{aligned}
\|\left(\mathrm{U}_{\infty,s+\epsilon} - \mathrm{U}_{\infty,s}\right)\left(\varphi - \varphi_n\right)\| \ \leq\ & \|\left(\mathrm{U}_{\infty,s+\epsilon} - \mathrm{U}_{\infty,s}\right)\left(\mathrm{N}+\mathbf{1}\right)^{-1}\|_{\mathrm{op}} \\
& \times\left(\|\mathrm{N}\left(\varphi - \varphi_n\right)\| + \|\varphi - \varphi_n\|\right)
\end{aligned} \tag{VI.179}$$

and

$$\|\mathrm{U}_{\infty,s}\mathrm{G}_s\left(\varphi - \varphi_n\right)\| \leq \|\mathrm{G}_s\left(\mathrm{N}+\mathbf{1}\right)^{-1}\|_{\mathrm{op}} + \left(\|\mathrm{N}\left(\varphi - \varphi_n\right)\| + \|\varphi - \varphi_n\|\right) \ , \tag{VI.180}$$

we infer from $\partial_s \mathrm{U}_{\infty,s}\varphi_n = i\mathrm{U}_{\infty,s}\mathrm{G}_s\varphi_n$, Lemma 67, and (VI.172) that the derivative $\partial_s \mathrm{U}_{\infty,s} = i\mathrm{U}_{\infty,s}\mathrm{G}_s$ holds on the domain $\mathcal{D}\left(\mathrm{N}\right)$, for any $s \geq 0$.
Now, the fact that $\mathrm{U}_{t,s}^*$ strongly converges to a bounded operator $\mathrm{U}_{\infty,s}^*$ results from standard arguments using the inequality

$$\|(\mathrm{U}_{t_2,s}^* - \mathrm{U}_{t_1,s}^*)(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} \leq 11\int_{t_1}^{t_2} \|B_\tau\|_2 \ \mathrm{d}\tau \ , \tag{VI.181}$$

which is deduced, for any $t_2 \geq t_1 \geq s$, from (VI.20) and Lemma 69. Moreover, similar to (VI.176),

$$\mathrm{U}_{\infty,\infty}^*\varphi := \lim_{s\to\infty} \mathrm{U}_{\infty,s}^*\varphi = \mathbf{1} \ . \tag{VI.182}$$

See also similar arguments to establishing Lemma 69, in particular (VI.51). Finally, the unitarity of $\mathrm{U}_{\infty,s}$ is straightforward to verify. Indeed, using the unitary of the operator $\mathrm{U}_{t,s}$ for all $t \geq s \geq 0$,

$$\begin{aligned} \left\| \left(\mathrm{U}_{\infty,s}^{*} \mathrm{U}_{\infty,s} - \mathbf{1} \right) \varphi \right\| &= \left\| \left(\mathrm{U}_{\infty,s}^{*} \mathrm{U}_{\infty,s} - \mathrm{U}_{t,s}^{*} \mathrm{U}_{t,s} \right) \varphi \right\| \\ &\leq \left\| \left(\mathrm{U}_{\infty,s} - \mathrm{U}_{t,s} \right) \varphi \right\| + \left\| \left(\mathrm{U}_{\infty,s}^{*} - \mathrm{U}_{t,s}^{*} \right) \mathrm{U}_{\infty,s} \varphi \right\| \ , \end{aligned} \tag{VI.183}$$

for any $\varphi \in \mathcal{F}_b$, and

$$\begin{aligned} \left\| \left(\mathrm{U}_{\infty,s} \mathrm{U}_{\infty,s}^{*} - \mathbf{1} \right) \varphi \right\| &= \left\| \left(\mathrm{U}_{\infty,s} \mathrm{U}_{\infty,s}^{*} - \mathrm{U}_{t,s} \mathrm{U}_{t,s}^{*} \right) \varphi \right\| \\ &\leq \left\| \mathrm{U}_{\infty,s}^{*} - \mathrm{U}_{t,s}^{*} \varphi \right\| + \left\| \left(\mathrm{U}_{\infty,s} - \mathrm{U}_{t,s} \right) \mathrm{U}_{\infty,s}^{*} \varphi \right\| \ . \end{aligned} \tag{VI.184}$$

Therefore, since, as $t \to \infty$, $\mathrm{U}_{t,s}$ (resp. $\mathrm{U}_{t,s}^{*}$) strongly converges to $\mathrm{U}_{\infty,s}$ (resp. $\mathrm{U}_{t,s}^{*}$), Equations (VI.176), (VI.182), (VI.183) and (VI.184) yield

$$\forall s \in \mathbb{R}_0^+ \cup \{\infty\} : \qquad \mathrm{U}_{\infty,s}^{*} \mathrm{U}_{\infty,s} = \mathrm{U}_{\infty,s} \mathrm{U}_{\infty,s}^{*} = \mathbf{1} \ . \tag{VI.185}$$

□

We continue the proof of Theorem 23 by showing that $\mathrm{U}_{\infty,s}$ realizes a Bogoliubov **u**–**v** transformation for any $s \in \mathbb{R}_0^+ \cup \{\infty\}$ (Theorem 23 (iii)).

Lemma 87 (The Bogoliubov **u**–**v** transformation)
Assume Conditions A1–A4 and the integrability of $t \mapsto \|B_t\|_2$. Then, for all $s \in \mathbb{R}_0^+ \cup \{\infty\}$, the unitary operator $\mathrm{U}_{\infty,s}$ satisfies on $\mathcal{D}(\mathrm{N}^{1/2})$:

$$\forall k \in \mathbb{N} : \quad \mathrm{U}_{\infty,s} a_{s,k} \mathrm{U}_{\infty,s}^{*} = \sum_{\ell} \left\{ \mathbf{u}_{\infty,s} \right\}_{k,\ell} a_\ell + \left\{ \mathbf{v}_{\infty,s} \right\}_{k,\ell} a_\ell^{*} \ , \tag{VI.186}$$

where the annihilation operator $a_{s,k}$ is defined in Lemma 72.

Proof. Inequality (VI.16) tells us that

$$\|[\mathrm{N}, \mathrm{G}_t](\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \leq 22 \|B_t\|_2 \tag{VI.187}$$

and the upper bound (VI.44) yields

$$\|(\mathrm{N} + \mathbf{1}) \mathrm{U}_{t,s}^{*} (\mathrm{N} + \mathbf{1})^{-1}\|_{\mathrm{op}} \leq \exp\left\{ 22 \int_s^\infty \|B_\tau\|_2 \, \mathrm{d}\tau \right\} \ , \tag{VI.188}$$

provided the map $t \mapsto \|B_t\|_2$ is integrable on $[0, \infty)$. So, we can follow similar arguments as the ones given in the proof of Lemma 93. In particular, we can express the uniformly bounded operator

$$(\mathrm{N} + \mathbf{1}) \mathrm{U}_{t,s}^{*} (\mathrm{N} + \mathbf{1})^{-1} \tag{VI.189}$$

in terms of $\mathrm{U}_{t,s}^{*}$ and the bounded operator

$$[\mathrm{N},\mathrm{G}_t](\mathrm{N}+\mathbf{1})^{-1}\in\mathcal{B}(\mathcal{F}_b)\ . \tag{VI.190}$$

Indeed, using (VI.187)–(VI.188) and the non–autonomous evolution equations (VI.43), we observe that the bounded operators

$$\mathrm{U}_{t,s}^{*}\qquad\text{and}\qquad(\mathrm{N}+\mathbf{1})\,\mathrm{U}_{t,s}^{*}\,(\mathrm{N}+\mathbf{1})^{-1} \tag{VI.191}$$

satisfy the equality

$$\begin{aligned}&\Big((\mathrm{N}+\mathbf{1})\,\mathrm{U}_{t,s}^{*}\,(\mathrm{N}+\mathbf{1})^{-1}-\mathrm{U}_{t,s}^{*}\Big)\,(\mathrm{N}+\mathbf{1})^{-1}\\ =\;&-\int_s^t\partial_\tau\left\{\mathrm{U}_{\tau,s}^{*}\,(\mathrm{N}+\mathbf{1})^{-1}\,(\mathrm{N}+\mathbf{1})^{2}\,\mathrm{U}_{t,\tau}^{*}\,(\mathrm{N}+\mathbf{1})^{-2}\right\}\mathrm{d}\tau\\ =\;&i\int_s^t\mathrm{U}_{\tau,s}^{*}\,[\mathrm{N},\mathrm{G}_\tau]\,(\mathrm{N}+\mathbf{1})^{-1}\,(\mathrm{N}+\mathbf{1})\,\mathrm{U}_{t,\tau}^{*}\,(\mathrm{N}+\mathbf{1})^{-2}\,\mathrm{d}\tau\ .\end{aligned} \tag{VI.192}$$

In other words, as $\mathcal{D}(\mathrm{N})$ is a dense subset of $\mathcal{F}_b$, one obtains

$$\begin{aligned}(\mathrm{N}+\mathbf{1})\mathrm{U}_{t,s}^{*}\,(\mathrm{N}+\mathbf{1})^{-1}\;=\;&\mathrm{U}_{t,s}^{*}+\sum_{n=1}^{\infty}i^{n}\int_s^t\mathrm{d}\tau_1\cdots\int_s^{\tau_{n-1}}\mathrm{d}\tau_n\\ &\prod_{j=1}^{n}\mathrm{U}_{\tau_j,s}^{*}\left\{\left[\mathrm{N},\mathrm{G}_{\tau_j}\right](\mathrm{N}+\mathbf{1})^{-1}\right\}\mathrm{U}_{\tau_{j-1},\tau_j}^{*}\end{aligned} \tag{VI.193}$$

on the whole Hilbert space $\mathcal{F}_b$, where $\tau_0:=t$. By (VI.187) and the unitarity of $\mathrm{U}_{t,s}^{*}$ for all $t\geq s\geq 0$, note that

$$\begin{aligned}&\left\|\sum_{n=N}^{\infty}i^{n}\int_s^t\mathrm{d}\tau_1\cdots\int_s^{\tau_{n-1}}\mathrm{d}\tau_n\prod_{j=1}^{n}\mathrm{U}_{\tau_j,s}^{*}\left\{\left[\mathrm{N},\mathrm{G}_{\tau_j}\right](\mathrm{N}+\mathbf{1})^{-1}\right\}\mathrm{U}_{\tau_{j-1},\tau_j}^{*}\right\|_{\mathrm{op}}\\ \leq\;&\frac{\left(22\int_s^\infty\|B_\tau\|_2\mathrm{d}\tau\right)^N}{N!}\exp\left\{22\int_s^\infty\|B_\tau\|_2\mathrm{d}\tau\right\}\end{aligned} \tag{VI.194}$$

for any $N\geq 1$. In other words, the series (VI.193) is norm convergent, uniformly in $t>s$. Observe that the particle number operator N is a closed operator. Consequently, using the limit (VI.174) and the upper bound (VI.187) together with the unitarity of $\mathrm{U}_{t,s}^{*}$ and (VI.193) we get

$$(\mathrm{N}+\mathbf{1})\mathrm{U}_{\infty,s}^{*}\,(\mathrm{N}+\mathbf{1})^{-1}\,\varphi=\lim_{t\to\infty}\left\{(\mathrm{N}+\mathbf{1})\,\mathrm{U}_{t,s}^{*}\,(\mathrm{N}+\mathbf{1})^{-1}\,\varphi\right\} \tag{VI.195}$$

for all $s\geq 0$ and any vector $\varphi\in\mathcal{F}_b$, where

$$\|\,(\mathrm{N}+\mathbf{1})\,\mathrm{U}_{\infty,s}^{*}\,(\mathrm{N}+\mathbf{1})^{-1}\,\|_{\mathrm{op}}\leq\exp\left\{22\int_s^\infty\|B_\tau\|_2\ \mathrm{d}\tau\right\}<\infty\ . \tag{VI.196}$$

Now, for any $k \in \mathbb{N}$, we define on $\mathcal{D}\left(\mathrm{N}^{1/2}\right)$ the asymptotic annihilation operator $a_{\infty,k}$ to be

$$a_{\infty,k} := \sum_{\ell} \{\mathbf{u}_{\infty,s}\}_{k,\ell}\, a_\ell + \{\mathbf{v}_{\infty,s}\}_{k,\ell}\, a_\ell^* = a(\mathbf{u}_\infty^* \varphi_k) + a^*(\mathbf{v}_\infty^{\mathrm{t}} \bar{\varphi}_k) \ , \quad \text{(VI.197)}$$

where the bounded operators $\mathbf{u}_{\infty,s}$ and $\mathbf{v}_{\infty,s}$ are defined in Lemma 85 and $\mathbf{u}_\infty := \mathbf{u}_{\infty,0}$, $\mathbf{v}_\infty := \mathbf{v}_{\infty,0}$. Recall that $\{\varphi_k\}_{k=1}^\infty$ is some real orthonormal basis in $\mathcal{D}(\Omega_0) \subseteq \mathfrak{h}$ and $a_k := a(\varphi_k)$ is the standard boson annihilation operator acting on the boson Fock space $\mathcal{F}_b$. We observe next that, for all $\varphi \in \mathcal{F}_b$ and $k \in \mathbb{N}$,

$$\begin{aligned} & \|(a_{\infty,k} - \mathrm{U}_{\infty,s} a_{s,k} \mathrm{U}_{\infty,s}^*)(\mathrm{N}+\mathbf{1})^{-1}\varphi\| \\ \leq \ & \|(\mathrm{U}_{t,s} - \mathrm{U}_{\infty,s}) a_{s,k} (\mathrm{N}+\mathbf{1})^{-1}(\mathrm{N}+\mathbf{1})\mathrm{U}_{\infty,s}^*(\mathrm{N}+\mathbf{1})^{-1}\varphi\| \\ & + \|\mathrm{U}_{t,s}\|_{\mathrm{op}} \|a_{s,k}(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} \|(\mathrm{N}+\mathbf{1})(\mathrm{U}_{t,s}^* - \mathrm{U}_{\infty,s}^*)(\mathrm{N}+\mathbf{1})^{-1}\varphi\| \\ & + \|(a_{\infty,k} - \mathrm{U}_{t,s} a_{s,k} \mathrm{U}_{t,s}^*)(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} \ . \end{aligned} \quad \text{(VI.198)}$$

Because of (VI.69), for all $s \geq 0$ and $k \in \mathbb{N}$,

$$\|a_{s,k}(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} < \infty \ . \quad \text{(VI.199)}$$

Furthermore, by (VI.55), that is,

$$a_{t,k} := a(\mathbf{u}_t^* \varphi_k) + a^*(\mathbf{v}_t^{\mathrm{t}} \bar{\varphi}_k) = \mathrm{U}_{t,s} a_{s,k} \mathrm{U}_{t,s}^* \quad \text{(VI.200)}$$

on $\mathcal{D}(\mathrm{N}^{1/2})$, straightforward estimations as in Lemma 66 imply that, for all $t \geq s \geq 0$ and $k \in \mathbb{N}$,

$$\|(a_{\infty,k} - \mathrm{U}_{t,s} a_{s,k} \mathrm{U}_{t,s}^*)(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} \leq \|\mathbf{u}_{\infty,s} - \mathbf{u}_{t,s}\|_2 + 2\|\mathbf{v}_{\infty,s} - \mathbf{v}_{t,s}\|_2 \ . \quad \text{(VI.201)}$$

Hence, for all $s \geq 0$ and $k \in \mathbb{N}$, we combine Lemma 85 with Inequality (VI.201) and get the limit

$$\lim_{t\to\infty} \|(a_{\infty,k} - \mathrm{U}_{t,s} a_{s,k} \mathrm{U}_{t,s}^*)(\mathrm{N}+\mathbf{1})^{-1}\|_{\mathrm{op}} = 0 \ , \quad \text{(VI.202)}$$

which, by the upper bounds (VI.196), (VI.198), and (VI.199), together with Lemma 86 and (VI.195), implies that

$$\forall \varphi \in \mathcal{D}(\mathrm{N}) \, , \ s \geq 0, \ k \in \mathbb{N} : \qquad a_{\infty,k}\varphi = \mathrm{U}_{\infty,s} a_{s,k} \mathrm{U}_{\infty,s}^* \varphi \ , \quad \text{(VI.203)}$$

provided the map $t \mapsto \|B_t\|_2$ is integrable on $[0,\infty)$. The domain $\mathcal{D}(\mathrm{N})$ is a core for each element of the family $(a_{\infty,k})_{k\in\mathbb{N},t\geq 0}$ of closed operators, which satisfy

$$a_{\infty,k}(\mathrm{N}+\mathbf{1})^{-1/2} \in \mathcal{B}(\mathcal{F}_b) \ , \quad \text{(VI.204)}$$

see (VI.197), Lemma 85, and the arguments used to prove (VI.69). Furthermore, $\mathrm{U}_{\infty,s}$ is unitary and $a_{s,k}$ is a closed operator, see Lemmata 72 and 86.

Therefore, (VI.203) can be extended by continuity to $\mathcal{D}(\mathrm{N}^{1/2})$, just as we did for (VI.82). □

Note that the previous lemma shows that

$$\mathcal{D}(\mathrm{N}^{1/2}) \subset \mathcal{D}(\mathrm{U}_{\infty,s} a_{s,k} \mathrm{U}_{\infty,s}^{*}) \ . \tag{VI.205}$$

This fact can directly be seen from the inequality

$$\|(\mathrm{N}+\mathbf{1})^{1/2}\mathrm{U}_{\infty,s}^{*}(\mathrm{N}+\mathbf{1})^{-1/2}\|_{\mathrm{op}} \leq \exp\left\{88\int_{s}^{\infty}\|B_{\tau}\|_{2}\mathrm{d}\tau\right\} \tag{VI.206}$$

for all $s \geq 0$. Assuming (VI.87) one gets this upper bound in the same way we have proven (VI.196).

We are in position to conclude that H_s and H_∞ are unitarily equivalent, i.e., to prove the fourth statement (iv) of Theorem 23.

Lemma 88 (Unitary equivalence of H_s and H_∞)
Assume Conditions A1–A4 and the integrability of $t \mapsto \|B_t\|_2$. Then

$$\forall s \geq 0 : \qquad \mathrm{H}_\infty = \mathrm{U}_{\infty,s}\mathrm{H}_s\mathrm{U}_{\infty,s}^{*} \ ,$$

where $\mathrm{U}_{\infty,s}$ *is the unitary operator defined in Lemma 86.*

Proof. Let $\lambda > 11\|B_0\|_2$. Because $\mathrm{U}_{t,s}$ is unitary and

$$\forall t \geq s \geq 0 : \qquad (\mathrm{H}_t + i\lambda\mathbf{1})^{-1} = \mathrm{U}_{t,s}\left(\mathrm{H}_s + i\lambda\mathbf{1}\right)^{-1}\mathrm{U}_{t,s}^{*} \ , \tag{VI.207}$$

(see (VI.141)), one gets, for any $\varphi \in \mathcal{F}_b$ and $t \geq s \geq 0$,

$$\begin{aligned} & \|\{(\mathrm{H}_\infty + i\lambda\mathbf{1})^{-1} - \mathrm{U}_{\infty,s}(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}\mathrm{U}_{\infty,s}^{*}\}\varphi\| \\ \leq\ & \|\{(\mathrm{H}_\infty + i\lambda\mathbf{1})^{-1} - (\mathrm{H}_t + i\lambda\mathbf{1})^{-1}\}\varphi\| \\ & +\|\left(\mathrm{H}_s + i\lambda\mathbf{1}\right)^{-1}\|_{\mathrm{op}}\ \|(\mathrm{U}_{t,s}^{*} - \mathrm{U}_{\infty,s}^{*})\varphi\| \\ & +\|(\mathrm{U}_{t,s} - \mathrm{U}_{\infty,s})(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}\mathrm{U}_{\infty,s}^{*}\varphi\| \ . \end{aligned} \tag{VI.208}$$

Consequently, by Lemmata 84 and 86, we obtain that, for any $s \geq 0$,

$$(\mathrm{H}_\infty + i\lambda\mathbf{1})^{-1} = \mathrm{U}_{\infty,s}(\mathrm{H}_s + i\lambda\mathbf{1})^{-1}\mathrm{U}_{\infty,s}^{*} \ , \tag{VI.209}$$

provided $\lambda > 11\|B_0\|_2$. We finally use Lemma 73 (ii) to conclude the proof. □

Theorem 23 (iv) then follows from Lemma 88 combined with Theorem 19.

VII Appendix

For the reader's convenience and because similar arguments are used above, we first give in Section VII.1 a detailed analysis of non–autonomous evolution equations for unbounded operators of hyperbolic type on reflexive Banach spaces. In particular, we clarify Ishii's approach [17, 18] to non–autonomous hyperbolic evolution equations. Then, Section VII.2 is devoted to generators of Bogoliubov **u**–**v** (unitary) transformations, whereas Section VII.3 should be seen as a toolbox where useful, simple results related to Hilbert–Schmidt operators and the trace are proven.

VII.1 Non–Autonomous Evolution Equations on Reflexive Banach Spaces

This section is patterned after [29].
In quantum mechanics, a well–known example of a non–autonomous evolution equation is the time-dependent Schrödinger equation. The self–adjoint generator $G_t = G_t^*$ acting on a Hilbert space is interpreted in this context as a time-dependent Hamiltonian. More generally, the notion of non–autonomous evolution equations on Banach spaces is well–known in the general context of abstract quasi–linear evolution equations, see, e.g., [12, 13, 14, 15] for reviews of this topic. By using two reflexive Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with $\mathcal{Y} \subset \mathcal{X}$ being a dense set, standard sufficient conditions for the well–posedness of non–autonomous evolution equations (III.4), that is,

$$\forall t \geq s \geq 0 : \qquad \partial_t U_{t,s} = -iG_t U_{t,s} \ , \quad U_{s,s} := \mathbf{1} \ , \tag{VII.1}$$

are the following:

B1 *(Kato quasi–stability).* There exist a constant $m \geq 1$ and a real-valued, upper integrable map $t \mapsto \beta_0(t)$ on $\mathbb{R}_0^+$ such that

$$\left\| \prod_{j=1}^{n} \left(\lambda_j \mathbf{1} + iG_{t_j} \right)^{-1} \right\|_{\mathrm{op}} \leq m \prod_{j=1}^{n} \frac{1}{\lambda_j - \beta_0(t_j)} \tag{VII.2}$$

for any family of real numbers $\{t_j, \lambda_j\}$ such that $0 \leq t_1 \leq \ldots \leq t_n$ and $\lambda_1 > \beta_0(t_1), \ldots, \lambda_n > \beta_0(t_n)$.

B2 *(Domains and continuity).* One has $\mathcal{Y} \subset \mathcal{D}(G_t)$ for any $t \geq 0$ with $G_t \in C[\mathbb{R}_0^+; \mathcal{B}(\mathcal{Y}, \mathcal{X})]$.

B3 *(Intertwining condition).* There exists a closed (linear) operator Θ from its dense domain $\mathcal{D}(\Theta) = \mathcal{Y}$ to $\mathcal{X}$ such that the norm

$$\beta_1(t) := \|[\Theta, G_t]\|_{\mathcal{B}(\mathcal{Y},\mathcal{X})} \tag{VII.3}$$

is bounded for any $t \in \mathbb{R}_0^+$.

If G_t is a self–adjoint operator on a Hilbert space then the assumption B1 is directly satisfied with $\beta_0(t) = 0$ and $m = 1$ since, for all $\lambda \in \mathbb{R}$,

$$\left\| (\lambda \mathbf{1} + iG_t)^{-1} \right\|_{\mathrm{op}} = \frac{1}{\inf\limits_{r \in \sigma(G_t)} |\lambda + ir|} \leq |\lambda|^{-1} \ . \tag{VII.4}$$

Within this appendix, we present in this context a detailed analysis of the well–posedness of non–autonomous evolution equations. A first proof was performed by Kato with analogue assumptions [10, 11]. His idea was to discretize the differential equation (VII.1) in order to use the Hille–Yosida generation theorems. Then, by taking the continuous limit he obtained a well–defined solution $U_{t,s}$ of (VII.1). The strategy presented here is different since we use the Yosida approximation, i.e., the sequence $\{G_{t,\lambda}\}_{\lambda\geq 0}$ of strongly continuous maps defined by

$$t \rightarrow G_{t,\lambda} := \frac{\lambda G_t}{\lambda \mathbf{1} + iG_t} \in \mathcal{B}(\mathcal{X}) \ . \tag{VII.5}$$

Indeed, for all $t > s \geq 0$, we already know the solution in the strong topology of the initial value problem (VII.1) with $G_{t,\lambda}$ replacing G_t. It is equal to the evolution operator[9] $U_{\lambda,t,s}$ well–defined by the Dyson series

$$U_{\lambda,t,s} := \mathbf{1} + \sum_{n=1}^{\infty} (-i)^n \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n \prod_{j=1}^{n} G_{\tau_j,\lambda} \tag{VII.6}$$

for all $t \geq s \geq 0$ and $\lambda \in \mathbb{R}^+$. Then, since $G_{t,\lambda}$ approximates in the strong topology the generator G_t for large λ, we analyze the strong limit of operators $U_{\lambda,t,s}$, as $\lambda \rightarrow \infty$, to obtain the existence of a limit operator $U_{t,s}$ with appropriate properties.

This idea was already used in an analogue context for the hyperbolic case by Ishii [17], see also [18]. Nevertheless, we give again this proof in detail since it is technically different from Ishii's ones [17, 18] (compare with Lemma 91) and also, because we use without details several times similar arguments in our proofs above. In comparison with other methods, it uses a simple controlled approximation of $U_{t,s}$ in terms of the Dyson series $U_{\lambda,t,s}$ (VII.6) and Conditions B1–B3 appear in a natural way. Now, we express the main theorem of this subsection.

Theorem 89 ($U_{t,s}$ as operator limit of Dyson series)
Under Conditions B1–B3, there is a unique strong solution $(U_{t,s})_{t\geq s\geq 0} \subset \mathcal{B}(\mathcal{X})$ on $\mathcal{Y} \subset \mathcal{D}(G_t)$ of the non–autonomous evolution equations[10]

$$\forall t \geq s \geq 0 : \qquad \left\{ \begin{array}{lcl} \partial_t U_{t,s} = -iG_t U_{t,s} & , & U_{s,s} := \mathbf{1} \ . \\ \partial_s U_{t,s} = iU_{t,s} G_s & , & U_{t,t} := \mathbf{1} \ . \end{array} \right. \tag{VII.7}$$

In particular, for all $t \geq s \geq 0$, $U_{t,s}\mathcal{Y} \subset \mathcal{Y}$ and $U_{t,s}$ is an evolution operator. Furthermore, $U_{t,s}$ is approximated by a Dyson series since, as $\lambda \rightarrow \infty$, $U_{\lambda,t,s}$ converges in the strong topology on $\mathcal{X}$ and in the norm topology on $\mathcal{Y}$ to $U_{t,s}$.

The detailed proof of this theorem corresponds to Lemmata 90–94. Remark that *additional* properties on $U_{t,s}$ are also proven within these lemmas. To simplify our notations (for example a definition of a norm in $\mathcal{Y}$), we assume that Θ is invertible but our arguments are in fact *independent* of this assumption.

[9] $U_{\lambda,t,s}$ is jointly strongly continuous in s and t with $U_{\lambda,t,s} = U_{\lambda,t,x} U_{\lambda,x,s}$ for $t \geq x \geq s \geq 0$ and $U_{\lambda,s,s} = \mathbf{1}$.

[10] The derivatives ∂_t and ∂_s on the borderline $t = s$ or $s = 0$ have to be understood as either right or left derivatives. See Lemma 94.

Also, to avoid triviality, we assume G_t is unbounded for any $t \geq 0$. Because $G_t \Theta^{-1} \in \mathcal{B}(\mathcal{X})$ (cf. B2), the operator Θ shares with G_t this property, i.e., $\Theta \notin \mathcal{B}(\mathcal{X})$. Therefore, an important ingredient of our proof is to get a uniform upper bound of the operator norms $\|U_{\lambda,t,s}\|_{\mathrm{op}}$ and $\|\Theta U_{\lambda,t,s}\Theta^{-1}\|_{\mathrm{op}}$, see Lemma 90. For this standard question, B1 and the function γ defined, for all $t \geq 0$, by

$$\gamma(t) := \beta_0(t) + m\beta_1(t) = \beta_0(t) + m \left\| [\Theta, G_t]\, \Theta^{-1} \right\|_{\mathrm{op}} \tag{VII.8}$$

(cf. B3) enter into the game. This preliminary work is necessary to prove the convergence of $U_{\lambda,t,s}$, as $\lambda \to \infty$, to a bounded operator $U_{t,s}$ on the Banach space $\mathcal{X}$, see Lemma 91. We proceed by proving that $U_{t,s}$ is an evolution operator, see Lemma 92. Then, a crucial result before obtaining the differentiability of $U_{t,s}$ is the study of the convergence of the bounded operator family $(\Theta U_{\lambda,t,s}\Theta^{-1})_{\lambda \geq 0}$, cf. Lemma 93. Finally, we study the differentiability of $U_{t,s}$, see Lemma 94. This concludes the proof of Theorem 89. Now, we give the promised series of lemmata with their proofs.

LEMMA 90 (PRESERVATION BY $U_{\lambda,t,s}$ OF THE BANACH SPACE $\mathcal{Y}$)
Under Conditions B1–B3, for any $t \geq s \geq 0$ and $\lambda > \gamma(t) \geq \beta_0(t)$, one has:

$$\|U_{\lambda,t,s}\|_{\mathrm{op}} \leq m \exp\left\{ \int_s^t \frac{\lambda \beta_0(\tau)}{\lambda - \beta_0(\tau)} \mathrm{d}\tau \right\} , \tag{VII.9}$$

$$\left\|\Theta U_{\lambda,t,s}\Theta^{-1}\right\|_{\mathrm{op}} \leq m \exp\left\{ \int_s^t \frac{\lambda \gamma(\tau)}{\lambda - \gamma(\tau)} \mathrm{d}\tau \right\} . \tag{VII.10}$$

PROOF. First, note that

$$iG_{t,\lambda} := \frac{i\lambda G_t}{\lambda \mathbf{1} + iG_t} = \lambda \mathbf{1} - \frac{\lambda^2}{\lambda \mathbf{1} + iG_t} =: \lambda \mathbf{1} + \tilde{G}_{t,\lambda} . \tag{VII.11}$$

Also, if $\tilde{U}_{\lambda,t,s} := \mathrm{e}^{\lambda(t-s)} U_{\lambda,t,s}$ then $\partial_t\{\tilde{U}_{\lambda,t,s}\} = -\tilde{G}_{t,\lambda}\tilde{U}_{\lambda,t,s}$. Therefore, B1 implies the following estimate for the Dyson expansion (VII.6):

$$\begin{aligned} \|U_{\lambda,t,s}\|_{\mathrm{op}} &= \mathrm{e}^{-\lambda(t-s)} \|\tilde{U}_{\lambda,t,s}\|_{\mathrm{op}} \\ &\leq \mathrm{e}^{-\lambda(t-s)} \left(1 + \sum_{n=1}^{\infty} \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n \left\| \prod_{j=1}^n \tilde{G}_{\tau_j,\lambda} \right\|_{\mathrm{op}} \right) \\ &\leq \mathrm{e}^{-\lambda(t-s)} \left(1 + \sum_{n=1}^{\infty} \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n \prod_{j=1}^n \frac{\lambda^2}{\lambda - \beta_0(\tau_j)} \right) , \end{aligned} \tag{VII.12}$$

i.e., we get (VII.9). Moreover, $[\Theta, G_t]\, \Theta^{-1} \in \mathcal{B}(\mathcal{X})$ and, similar to Lemma 73

(i) and Equation (VI.100),

$$\Theta\left(\lambda\mathbf{1}+iG_t\right)^{-1}\Theta^{-1} = \frac{1}{\lambda\mathbf{1}+iG_t+i\left[\Theta,G_t\right]\Theta^{-1}} \qquad \text{(VII.13)}$$

$$= \sum_{n=0}^{\infty}\left(\lambda\mathbf{1}+iG_t\right)^{-1}\left\{i\left[G_t,\Theta\right]\Theta^{-1}\left(\lambda\mathbf{1}+iG_t\right)^{-1}\right\}^{n} \qquad \text{(VII.14)}$$

for any family of real numbers $\{t_j,\lambda_j\}$ such that $0\leq t_1\leq\ldots\leq t_n$ and $\lambda_1>\gamma(t_1),\ldots,\lambda_n>\gamma(t_n)$. Therefore, it is standard to verify that

$$\left\|\prod_{j=1}^{n}\Theta\left(\lambda_j\mathbf{1}+iG_{t_j}\right)^{-1}\Theta^{-1}\right\|_{\mathrm{op}} \leq m\prod_{j=1}^{n}\frac{1}{\lambda_j-\beta_0(t_j)-m\beta_1(t_j)} = m\prod_{j=1}^{n}\frac{1}{\lambda_j-\gamma(t_j)}\ , \qquad \text{(VII.15)}$$

from which we prove (VII.10) just as we did for (VII.9). □

Lemma 91 (Convergence of $U_{\lambda,t,s}$ when $\lambda\rightarrow\infty$)
Under Conditions B1–B3 and for any $t\geq s\geq 0$, the strong limit $U_{t,s}$ of $U_{\lambda,t,s}$ when $\lambda\rightarrow\infty$ exists and satisfies

$$\lim_{\lambda\rightarrow\infty}\left\|\left\{U_{\lambda,t,s}-U_{t,s}\right\}\Theta^{-1}\right\|_{\mathrm{op}}=0 \quad \text{and} \quad \left\|U_{t,s}\right\|_{\mathrm{op}}\leq m\exp\left\{\int_s^t\beta_0(\tau)\,\mathrm{d}\tau\right\}. \qquad \text{(VII.16)}$$

Proof. Fix an arbitrary large parameter T taken such that $T\geq t$ and let

$$\left\{\begin{array}{ll} M:=\max\limits_{0\leq t\leq T}\left\{\left\|G_t\Theta^{-1}\right\|_{\mathrm{op}}\right\} & , \\ \gamma_{\sup}:=\max\limits_{0\leq t\leq T}\{\gamma(t)\}=\max\limits_{0\leq t\leq T}\{\beta_0(t)+m\beta_1(t)\} & , \\ Z_\epsilon:=\max\left\{\left\|\left(G_t-G_{t'}\right)\Theta^{-1}\right\|_{\mathrm{op}}\ \middle|\ t,t'\in[0,T+1]\,,|t-t'|\leq 2\epsilon\right\} & . \end{array}\right. \qquad \text{(VII.17)}$$

Also, by taking $G_{t<0}:=G_0$ we define the derivable operator $J_{\epsilon,\alpha}(t)$, for any $t\geq 0$, by

$$J_{\epsilon,\alpha}(t):=\frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon}\frac{\alpha}{\alpha\mathbf{1}+iG_\tau}\mathrm{d}\tau\ . \qquad \text{(VII.18)}$$

By Condition B1 combined with (VII.15) and (VII.17) for any $\alpha\geq 2\gamma_{\sup}$, observe that

$$\left\|\left\{\mathbf{1}-J_{\epsilon,\alpha}(t)\right\}\Theta^{-1}\right\|_{\mathrm{op}}\leq 2mM\alpha^{-1}\ . \qquad \text{(VII.19)}$$

Then, via Lemma 90 for $\alpha,\eta,\lambda \geq 2\gamma_{\mathrm{sup}}$, we directly get

$$\left\| \left\{ U_{\lambda,t,s} - U_{\eta,t,s} \right\} \Theta^{-1} \right\|_{\mathrm{op}} \leq \left\| \left\{ J_{\epsilon,\alpha}(t) U_{\lambda,t,s} - U_{\eta,t,s} J_{\epsilon,\alpha}(s) \right\} \Theta^{-1} \right\|_{\mathrm{op}} + \frac{4m^2 M}{\alpha} \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} . \tag{VII.20}$$

Now, by using (VII.1) (with $G_{t,\lambda}$ replacing G_t) one obtains

$$J_{\epsilon,\alpha}(t) U_{\lambda,t,s} - U_{\eta,t,s} J_{\epsilon,\alpha}(s) = \int_s^t \partial_\tau \left\{ U_{\eta,t,\tau} J_{\epsilon,\alpha}(\tau) U_{\lambda,\tau,s} \right\} \mathrm{d}\tau = \mathrm{A} + \mathrm{B} + \mathrm{C} \tag{VII.21}$$

with

$$\mathrm{A} := i \int_s^t U_{\eta,t,\tau} \left\{ G_{\tau,\eta} - G_{\tau,\lambda} \right\} J_{\epsilon,\alpha}(\tau) U_{\lambda,\tau,s} \mathrm{d}\tau , \tag{VII.22}$$

$$\mathrm{B} := \frac{i}{2\epsilon} \int_s^t \mathrm{d}\tau \int_{\tau-\epsilon}^{\tau+\epsilon} \mathrm{d}r\ U_{\eta,t,\tau} \left[\frac{\lambda^2}{\lambda \mathbf{1} + iG_\tau}, \frac{\alpha}{\alpha \mathbf{1} + iG_r} \right] U_{\lambda,\tau,s} , \tag{VII.23}$$

$$\mathrm{C} := \frac{i\alpha}{2\epsilon} \int_s^t U_{\eta,t,\tau} \frac{1}{\alpha \mathbf{1} + iG_{\tau+\epsilon}} \left\{ G_{\tau-\epsilon} - G_{\tau+\epsilon} \right\} \frac{1}{\alpha \mathbf{1} + iG_{\tau-\epsilon}} U_{\lambda,\tau,s} \mathrm{d}\tau . \tag{VII.24}$$

So, we bound below the terms $\|\mathrm{A}\Theta^{-1}\|_{\mathrm{op}}$, $\|\mathrm{B}\Theta^{-1}\|_{\mathrm{op}}$, and $\|\mathrm{C}\Theta^{-1}\|_{\mathrm{op}}$. First, by Lemma 90 for any $\eta,\lambda \geq 2\gamma_{\mathrm{sup}}$, we get that

$$\left\| \mathrm{A}\Theta^{-1} \right\|_{\mathrm{op}} \leq m^2 \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} \int_s^t \left\| \left(G_{\tau,\eta} - G_{\tau,\lambda} \right) J_{\epsilon,\alpha}(\tau) \Theta^{-1} \right\|_{\mathrm{op}} \mathrm{d}\tau . \tag{VII.25}$$

Since we have the equality

$$\left\{ G_{\tau,\eta} - G_{\tau,\lambda} \right\} J_{\epsilon,\alpha}(\tau) = \frac{\alpha(\eta-\lambda) iG_\tau^2}{(\eta \mathbf{1} + iG_\tau)(\lambda \mathbf{1} + iG_\tau)(\alpha \mathbf{1} + iG_\tau)} \left\{ 1 + \frac{i}{2\epsilon} \int_{\tau-\epsilon}^{\tau+\epsilon} (G_r - G_\tau) \left(\frac{1}{\alpha \mathbf{1} + iG_r} \right) \mathrm{d}r \right\} \tag{VII.26}$$

and the estimation

$$\left\| \frac{iG_\tau}{\alpha \mathbf{1} + iG_\tau} \right\|_{\mathrm{op}} \leq 1 + \alpha \left\| (\alpha \mathbf{1} + iG_\tau)^{-1} \right\|_{\mathrm{op}} , \tag{VII.27}$$

by Condition B1 combined with (VII.15) and (VII.17) we obtain

$$\left\| \left\{ G_{\tau,\eta} - G_{\tau,\lambda} \right\} J_{\epsilon,\alpha}(\tau) \Theta^{-1} \right\|_{\mathrm{op}} \leq 12m^3 \left\{ \frac{\alpha M}{\min\{\lambda,\eta\}} + 3mZ_\epsilon \right\} \tag{VII.28}$$

for $\alpha, \eta, \lambda \geq 2\gamma_{\mathrm{sup}}$ and $|\eta - \lambda| \leq \max\{\lambda, \eta\}$. Therefore, we infer from (VII.25) that

$$\left\| \mathrm{A}\Theta^{-1} \right\|_{\mathrm{op}} \leq 12m^5 \left\{ \frac{\alpha M}{\min\{\lambda, \eta\}} + 3mZ_\epsilon \right\} (t-s) \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} . \tag{VII.29}$$

On the other hand, since

$$\left[\frac{1}{\lambda + iG_\tau}, \frac{1}{\alpha + iG_r} \right] = \frac{1}{\lambda + iG_\tau} \frac{1}{\alpha + iG_r} \left[G_r, G_\tau - G_r \right] \frac{1}{\alpha + iG_r} \frac{1}{\lambda + iG_\tau} , \tag{VII.30}$$

then, by using again B1, Lemma 90, (VII.15), (VII.17), and (VII.27), one gets, for any $\alpha, \eta, \lambda \geq 2\gamma_{\mathrm{sup}}$, the upper bounds

$$\left\| \mathrm{B}\Theta^{-1} \right\|_{\mathrm{op}} \leq 48m^4 Z_\epsilon (t-s) \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} , \tag{VII.31}$$

$$\left\| \mathrm{C}\Theta^{-1} \right\|_{\mathrm{op}} \leq \frac{2m^4}{\epsilon\alpha} Z_\epsilon (t-s) \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} . \tag{VII.32}$$

Therefore, we combine (VII.25), (VII.31), and (VII.32) with $m \geq 1$, (VII.20), and (VII.21) to deduce the following upper bound:

$$\begin{aligned} \left\| \{U_{\lambda,t,s} - U_{\eta,t,s}\} \Theta^{-1} \right\|_{\mathrm{op}} \leq\ & 84(M+1)\, m^6 \left\{ \frac{1}{\alpha} + \frac{\alpha}{\min\{\lambda, \eta\}} + \left(1 + \frac{1}{\epsilon\alpha}\right) Z_\epsilon \right\} \\ & \times (t-s+1) \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} . \end{aligned} \tag{VII.33}$$

By taking first $\min\{\lambda, \eta\} \to \infty$, then $\alpha \to \infty$ and $\epsilon \downarrow 0^+$, we conclude that

$$\lim_{\min\{\lambda,\eta\}\to\infty} \left\| \{U_{\lambda,t,s} - U_{\eta,t,s}\} \Theta^{-1} \right\|_{\mathrm{op}} = 0 , \tag{VII.34}$$

where we have used Condition B2 when $\epsilon \downarrow 0^+$. The fact that the bounded operator family $(U_{\lambda,t,s})_{\lambda \geq 0}$ is a Cauchy sequence in the strong topology on $\mathcal{X}$ is now standard to verify: Let $\delta > 0$ and $\varphi \in \mathcal{X}$. By density of $\mathcal{Y} = \mathcal{D}(\Theta)$, there exists $\psi \in \mathcal{Y}$ such that $\|\varphi - \psi\| \leq \delta$. Then, by Lemma 90, one deduces, for $\eta, \lambda \geq 2\gamma_{\mathrm{sup}}$, that

$$\begin{aligned} \left\| \{U_{\lambda,t,s} - U_{\eta,t,s}\} \varphi \right\| \leq\ & \left\| \{U_{\lambda,t,s} - U_{\eta,t,s}\} \Theta^{-1} \right\|_{\mathrm{op}} \|\Theta\psi\| \\ & + 2\delta m \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} . \end{aligned} \tag{VII.35}$$

Hence, we combine (VII.35) with (VII.34) to observe that the family $(U_{\lambda,t,s}\varphi)_{\lambda \geq 2\gamma_{\mathrm{sup}}}$ forms a Cauchy sequence in the Banach space $\mathcal{X}$. Therefore, there exists an operator $U_{t,s}$ defined by

$$\forall \varphi \in \mathcal{X} : \qquad U_{t,s}\varphi := \lim_{\lambda\to\infty} \{U_{\lambda,t,s}\varphi\} \in \mathcal{X} \tag{VII.36}$$

and satisfying

$$\lim_{\lambda\rightarrow\infty}\left\|\{U_{\lambda,t,s}-U_{t,s}\}\,\Theta^{-1}\right\|_{\mathrm{op}}=0\ . \tag{VII.37}$$

Moreover, since

$$\|U_{t,s}\varphi\|=\lim_{\lambda\rightarrow\infty}\|U_{\lambda,t,s}\varphi\|\leq\|\varphi\|\lim_{\lambda\rightarrow\infty}\left\{\|U_{\lambda,t,s}\|_{\mathrm{op}}\right\}\ , \tag{VII.38}$$

we directly obtain the corresponding upper bound on the operator norm $\|U_{t,s}\|_{\mathrm{op}}$ from Lemma 90. □

LEMMA 92 ($U_{t,s}$ AS AN EVOLUTION OPERATOR)
Under Conditions B1–B3, the bounded operator family $(U_{t,s})_{t\geq s\geq 0}$ is jointly strongly continuous in s and t, and satisfies the cocycle property $U_{t,s}=U_{t,x}U_{x,s}$ for $t\geq x\geq s\geq 0$ with $U_{s,s}=\mathbf{1}$.

PROOF. First, for any $\varphi\in\mathcal{X}$ and $t\geq x\geq s\geq 0$, straightforward estimations using Lemmata 90–91 show that

$$U_{s,s}\varphi=\lim_{\lambda\rightarrow\infty}U_{\lambda,s,s}\varphi=\varphi \tag{VII.39}$$

and

$$U_{t,s}\varphi-U_{t,x}U_{x,s}\varphi=\lim_{\lambda\rightarrow\infty}\{U_{\lambda,t,s}\varphi-U_{\lambda,t,x}U_{\lambda,x,s}\varphi\}=0\ . \tag{VII.40}$$

Now, it remains to prove the strong continuity of $U_{t,s}\Theta^{-1}$. If $s\leq t$, $s'\leq t'$, and $\max\{s',s\}<\min\{t,t'\}$, i.e., $(s,t)\cap(s',t')\neq\varnothing$, then, by rewriting (VII.1) (with $G_{t,\lambda}$ replacing G_t) as an integral equation, one obtains the inequality

$$\begin{aligned}&\left\|\{U_{\lambda,t',s'}-U_{\lambda,t,s}\}\,\Theta^{-1}\right\|_{\mathrm{op}}\\ \leq\quad & mM\{|s-s'|+|t-t'|\}\exp\left\{\int_{\min\{s,s'\}}^{\max\{t,t'\}}2\gamma(\tau)\,\mathrm{d}\tau\right\}\end{aligned} \tag{VII.41}$$

for $\lambda\geq 2\gamma_{\sup}$ (VII.17), see also Lemma 90. In other words, $U_{\lambda,t,s}\Theta^{-1}$ is uniformly norm continuous. Therefore, by density of $\mathcal{Y}=\mathcal{D}(\Theta)$ combined with Lemma 91, $U_{t,s}$ is jointly strongly continuous in s and t. □

LEMMA 93 (CONVERGENCE OF $\Theta U_{\lambda,t,s}\Theta^{-1}$ WHEN $\lambda\rightarrow\infty$)
Under Conditions B1–B3 and for any $t\geq s\geq 0$, the strong limit $V_{t,s}$ of $\Theta U_{\lambda,t,s}\Theta^{-1}$ when $\lambda\rightarrow\infty$ exists, is jointly strongly continuous in s and t, and satisfies $V_{t,s}=\Theta U_{t,s}\Theta^{-1}$ with

$$\left\|\Theta U_{t,s}\Theta^{-1}\right\|_{\mathrm{op}}\leq m\exp\left\{\int_s^t\gamma(\tau)\,\mathrm{d}\tau\right\}\ . \tag{VII.42}$$

Proof. For any $t \geq s \geq 0$ and $\lambda > \gamma(t) \geq \beta_0(t)$, define the bounded operators

$$V_{\lambda,t,s} := \Theta U_{\lambda,t,s} \Theta^{-1} \tag{VII.43}$$

and

$$V_{\lambda,t,s;\tau_1,\ldots,\tau_n}^{(n)} := \prod_{j=1}^{n} U_{\lambda,t,\tau_j} \left\{ \left[\Theta, G_{\tau_j,\lambda}\right] \Theta^{-1} \right\} U_{\lambda,\tau_j,\tau_{j+1}} \ , \tag{VII.44}$$

with $\tau_{n+1} := s$. See Lemma 90. By (VII.1), observe that

$$\begin{aligned} V_{\lambda,t,s} - U_{\lambda,t,s} &= \int_s^t \partial_\tau \left\{ U_{\lambda,t,\tau} V_{\lambda,\tau,s} \right\} \mathrm{d}\tau \\ &= -i \int_s^t U_{\lambda,t,\tau} \left\{ \left[\Theta, G_{\tau,\lambda}\right] \Theta^{-1} \right\} V_{\lambda,\tau,s} \mathrm{d}\tau \ . \end{aligned} \tag{VII.45}$$

Therefore, $V_{\lambda,t,s}$ (VII.43) is directly expressed in terms of $U_{\lambda,t,s}$ and the bounded operator $[\Theta, G_{t,\lambda}]\,\Theta^{-1}$:

$$V_{\lambda,t,s} = U_{\lambda,t,s} + \sum_{n=1}^{\infty} (-i)^n \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n V_{\lambda,t,s;\tau_1,\ldots,\tau_n}^{(n)} \ . \tag{VII.46}$$

Since Condition B1 together with (VII.15) implies that

$$\left\| [\Theta, G_{t,\lambda}]\,\Theta^{-1} \right\|_{\mathrm{op}} \leq 4m^2 \left\| [\Theta, G_t]\,\Theta^{-1} \right\|_{\mathrm{op}} = 4m^2 \beta_1(t) \tag{VII.47}$$

for any $\lambda \geq 2\gamma_{\mathrm{sup}}$, we infer from Lemma 90 that

$$\left\| V_{\lambda,t,s;\tau_1,\ldots,\tau_n}^{(n)} \right\|_{\mathrm{op}} \leq m \exp\left\{ \int_s^t 2\gamma(\tau)\,\mathrm{d}\tau \right\} \left(4m^3\right)^n \prod_{j=1}^{n} \beta_1(\tau_j) \,, \tag{VII.48}$$

cf. (VII.17) and (VII.44). Thus, for any $\lambda \geq 2\gamma_{\mathrm{sup}}$ and every $N > 1$,

$$\begin{aligned} &\left\| \sum_{n=N}^{\infty} (-i)^n \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n V_{\lambda,t,s;\tau_1,\ldots,\tau_n}^{(n)} \right\|_{\mathrm{op}} \\ \leq\ & \frac{m\varsigma^N}{N!} \exp\left\{ \int_s^t \left\{ 2\gamma(\tau) + 4m^3 \beta_1(\tau) \right\} \mathrm{d}\tau \right\} \end{aligned} \tag{VII.49}$$

with $\varsigma := 4m^3 \max\left\{ \beta_1(\tau) : \tau \in [s,t] \right\}$. In other words, the series (VII.46) is norm convergent, uniformly in $\lambda \geq 2\gamma_{\mathrm{sup}}$. Also, for all $\lambda \geq 2\gamma_{\mathrm{sup}}$ and $\varphi \in \mathcal{X}$,

$$\begin{aligned} \left\| [\Theta, G_{\lambda,t} - G_t]\,\Theta^{-1}\varphi \right\| \leq\ & 2m\beta_1(t) \left\| \Theta \frac{G_t}{\lambda \mathbf{1} + iG_t} \Theta^{-1} \varphi \right\| \\ & + \left\| \frac{G_t}{\lambda \mathbf{1} + iG_t} \left\{ [\Theta, G_t]\,\Theta^{-1} \right\} \varphi \right\| \end{aligned} \tag{VII.50}$$

and

$$\left\| \frac{G_t}{\lambda \mathbf{1} + iG_t} \Theta^{-1} \right\|_{\mathrm{op}} \leq 2mM\lambda^{-1} , \tag{VII.51}$$

$$\left\| \Theta \frac{G_t}{\lambda \mathbf{1} + iG_t} \Theta^{-2} \right\|_{\mathrm{op}} \leq 2mM\lambda^{-1} \{1 + \beta_1 (t)\} , \tag{VII.52}$$

see (VII.17). So, by density of the subset $\mathcal{D}(\Theta) = \mathcal{Y}$ of $\mathcal{X}$, one obtains, for all $\varphi \in \mathcal{X}$, that

$$\lim_{\lambda \to \infty} \left\| \frac{G_t}{\lambda \mathbf{1} + iG_t} \varphi \right\| = 0 \tag{VII.53}$$

and

$$\lim_{\lambda \to \infty} \left\| \Theta \frac{G_t}{\lambda \mathbf{1} + iG_t} \Theta^{-1} \varphi \right\| = 0 . \tag{VII.54}$$

Consequently, Inequality (VII.50) implies that, for all $\varphi \in \mathcal{X}$,

$$\lim_{\lambda \to \infty} \left\{ [\Theta, G_{t,\lambda}] \, \Theta^{-1} \varphi \right\} = [\Theta, G_t] \, \Theta^{-1} \varphi . \tag{VII.55}$$

And then, by Lemma 91, we can define the operator $V_{t,s;\tau_1,\ldots,\tau_n}^{(n)}$ by

$$V_{t,s;\tau_1,\ldots,\tau_n}^{(n)} \varphi := \prod_{j=1}^{n} U_{t,\tau_j} \left\{ \left[\Theta, G_{\tau_j} \right] \Theta^{-1} \right\} U_{\tau_j,\tau_{j+1}} = \lim_{\lambda \to \infty} V_{\lambda,t,s;\tau_1,\ldots,\tau_n}^{(n)} \varphi \tag{VII.56}$$

for any $\varphi \in \mathcal{X}$, $n \in \mathbb{N}$, and $t \geq \tau_1 \geq \ldots \geq \tau_n \geq \tau_{n+1} := s \geq 0$. Moreover, by Lebesgue's dominated convergence theorem, there also exists an operator $V_{t,s}$ defined, for any $\varphi \in \mathcal{X}$ and $t \geq s \geq 0$, by

$$V_{t,s} \varphi := \lim_{\lambda \to \infty} \left\{ \Theta U_{\lambda,t,s} \Theta^{-1} \varphi \right\} = U_{t,s} \varphi + \sum_{n=1}^{\infty} \int_s^t \mathrm{d}\tau_1 \cdots \int_s^{\tau_{n-1}} \mathrm{d}\tau_n V_{t,s;\tau_1,\ldots,\tau_n}^{(n)} \varphi . \tag{VII.57}$$

Observe that the operator $V_{t,s}$ is bounded since, for any $\varphi \in \mathcal{X}$ and $t \geq s \geq 0$,

$$\begin{aligned} \|V_{t,s} \varphi\| &= \lim_{\lambda \to \infty} \left\| \Theta U_{\lambda,t,s} \Theta^{-1} \varphi \right\| \\ &\leq \|\varphi\| \lim_{\lambda \to \infty} \left\{ \left\| \Theta U_{\lambda,t,s} \Theta^{-1} \right\|_{\mathrm{op}} \right\} \\ &\leq \|\varphi\| \; m \exp \left\{ \int_s^t \gamma(\tau) \, \mathrm{d}\tau \right\} , \end{aligned} \tag{VII.58}$$

see Lemma 90. Now, let $\varphi \in \mathcal{X}$ and $\psi_\lambda := U_{\lambda,t,s} \Theta^{-1} \varphi \in \mathcal{Y}$. By Lemma 91,

$$\lim_{\lambda \to \infty} \{\psi_\lambda\} = U_{t,s} \Theta^{-1} \varphi , \tag{VII.59}$$

whereas we infer from (VII.57) that

$$\lim_{\lambda \to \infty} \{\Theta \psi_\lambda\} = V_{t,s} \varphi . \tag{VII.60}$$

Since Θ is a closed operator, the last equalities then imply that

$$V_{t,s}\varphi = \Theta U_{t,s}\Theta^{-1}\varphi \qquad \text{(VII.61)}$$

for any $\varphi \in \mathcal{X}$. Finally, from (VII.57) combined with Lemma 91, (VII.17), and (VII.58), observe that

$$\begin{aligned} \|V_{t',s'} - V_{t,s}\|_{\mathrm{op}} &\leq \|U_{t',s'} - U_{t,s}\|_{\mathrm{op}} + m^2\gamma_{\sup}\{|s-s'| + |t-t'|\} \\ &\quad \exp\left\{\int_{\min\{s,s'\}}^{\max\{t,t'\}} (\beta_0(\tau) + 2\gamma(\tau))\,\mathrm{d}\tau\right\} . \end{aligned} \qquad \text{(VII.62)}$$

In other words, the bounded operator family $(V_{t,s})_{t\geq s\geq 0}$ is jointly strongly continuous in s and t. □

LEMMA 94 (DIFFERENTIABILITY OF THE EVOLUTION OPERATOR $U_{t,s}$)
Under Conditions B1–B3, the evolution operator $(U_{t,s})_{t\geq s\geq 0}$ is the unique strong solution on $\mathcal{Y}$ of the initial value problem

$$\forall t > s \geq 0: \qquad \partial_t U_{t,s} = -iG_t U_{t,s}\ , \quad U_{s,s} := \mathbf{1}\ . \qquad \text{(VII.63)}$$

Furthermore, $(U_{t,s})_{t\geq s\geq 0}$ also satisfies the non–autonomous evolution equation

$$\forall t > s > 0: \qquad \partial_s U_{t,s} = iU_{t,s}G_s\ , \quad U_{t,t} := \mathbf{1}\ . \qquad \text{(VII.64)}$$

At $t = s \geq 0$, its right derivative also equals $\partial_t^+ U_{t,s}|_{t=s} = -iG_s$, whereas $\partial_s^- U_{t,s}|_{s=t} = iG_t$ and $\partial_s^+ U_{t,s}|_{s=0} = iU_{t,0}G_0$ for $t > 0$, all in the strong sense in $\mathcal{Y}$.

PROOF. By using Lemma 90 and (VII.17), direct manipulations show that, for any $\psi \in \mathcal{Y}$, $\epsilon > 0$ and $\lambda \geq 2\gamma_{\sup}$,

$$\begin{aligned} &\left\|\left\{\epsilon^{-1}(U_{t+\epsilon,s} - U_{t,s}) + iG_t U_{t,s}\right\}\psi\right\| \\ \leq\ & 2mM\epsilon^{-1}\int_t^{t+\epsilon} (\|\Theta\{U_{\tau,s} - U_{t,s}\}\psi\| + \|\Theta\{U_{\lambda,\tau,s} - U_{\tau,s}\}\psi\|)\,\mathrm{d}\tau \\ &+ Z_\epsilon\|\Theta\psi\|\, m\exp\left\{\int_s^t 2\gamma(\tau)\,\mathrm{d}\tau\right\} + \epsilon^{-1}\int_t^{t+\epsilon} \|\{G_{\tau,\lambda} - G_\tau\}U_{t,s}\psi\|\,\mathrm{d}\tau \\ &+ \epsilon^{-1}\|\{U_{t+\epsilon,s} - U_{\lambda,t+\epsilon,s}\}\psi\| + \epsilon^{-1}\|\{U_{\lambda,t,s} - U_{t,s}\}\psi\|\ . \end{aligned} \qquad \text{(VII.65)}$$

Since, by Condition B1 and (VII.17), one has

$$\left\|\frac{G_t}{\lambda + iG_t}\Theta^{-1}\right\|_{\mathrm{op}} \leq \frac{mM}{\lambda - \beta_0(t)}\ , \qquad \text{(VII.66)}$$

we use again the density of $\mathcal{Y}$ for any $\varphi \in \mathcal{X}$ and $\delta > 0$ by taking $\tilde{\psi} \in \mathcal{Y}$ such that $\|\varphi - \tilde{\psi}\| \leq \delta$ in order to get the inequality

$$\left\| \{G_{t,\lambda} - G_t\} \Theta^{-1}\varphi \right\| \leq M \left\| \frac{G_t}{\lambda + iG_t}\varphi \right\| \leq 3mM\delta + \frac{mM^2}{\lambda - \beta_0(t)} \left\| \Theta\tilde{\psi} \right\| . \tag{VII.67}$$

Through Lemma 93 we observe that

$$\|\Theta U_{t,s}\psi\| \leq \left\|\Theta U_{t,s}\Theta^{-1}\right\|_{\mathrm{op}} \|\Theta\psi\| \leq \|\Theta\psi\| \, m \exp\left\{ \int_s^t \gamma(\tau)\, \mathrm{d}\tau \right\} < \infty . \tag{VII.68}$$

Consequently, by taking the limit $\lambda \to \infty$ in (VII.65) with the use of Lebesgue's dominated convergence theorem combined with Lemmata 90–93 for any $\psi \in \mathcal{Y}$, we get the upper bound

$$\begin{aligned} & \left\| \left\{ \epsilon^{-1}\left(U_{t+\epsilon,s} - U_{t,s}\right) + iG_t U_{t,s} \right\} \psi \right\| \\ \leq \; & 2mM\epsilon^{-1} \int_t^{t+\epsilon} \|\Theta\{U_{\tau,s} - U_{t,s}\}\psi\| \, \mathrm{d}\tau + 3mM\delta \\ & + Z_\epsilon \|\Theta\psi\| \, m \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} . \end{aligned} \tag{VII.69}$$

When the parameters δ and ϵ go to zero, this last inequality associated with Lemma 93 shows that

$$\forall t \geq s \geq 0 : \qquad \partial_t^+ U_{t,s} = -iG_t U_{t,s} \ , \quad U_{s,s} := \mathbf{1} \ , \tag{VII.70}$$

where the right derivative is in the strong sense in $\mathcal{Y}$. The left derivative is obtained in the same way, provided $t > s$. In other words, $U_{t,s}$ is a strong solution on $\mathcal{Y}$ of the initial value problem $\partial_t U_{t,s} = -iG_t U_{t,s}$ for any $t \geq s \geq 0$. Equation (VII.64) is also proven in a similar way as follows: For any $t > s > 0$, $\psi \in \mathcal{Y}$, sufficiently small $|\epsilon| > 0$ and $\lambda \geq 2\gamma_{\mathrm{sup}}$, we use straightforward estimates together with Lemmata 90–91 and (VII.17) to obtain that

$$\begin{aligned} & \left\| \left\{ \epsilon^{-1}\left(U_{t,s+\epsilon} - U_{t,s}\right) - iU_{t,s}G_s \right\} \psi \right\| \\ \leq \; & \epsilon^{-1} \int_s^{s+\epsilon} \left(\|(U_{\lambda,t,\tau} - U_{t,\tau}) G_\tau\psi\| + \|(U_{t,\tau} - U_{t,s}) G_\tau\psi\| \right) \mathrm{d}\tau \\ & + m \exp\left\{ \int_s^t \beta_0(\tau)\, \mathrm{d}\tau \right\} Z_\epsilon \|\Theta\psi\| \\ & + m \exp\left\{ \int_s^t 2\gamma(\tau)\, \mathrm{d}\tau \right\} \epsilon^{-1} \int_s^{s+\epsilon} \|(G_{\tau,\lambda} - G_\tau)\psi\| \, \mathrm{d}\tau \\ & + \left\| \epsilon^{-1}\{U_{t,s+\epsilon} - U_{\lambda,t,s+\epsilon}\}\psi \right\| + \left\| \epsilon^{-1}\{U_{\lambda,t,s} - U_{t,s}\}\psi \right\| . \end{aligned} \tag{VII.71}$$

Now, by using Lemma 92, (VII.67), and Lebesgue's dominated convergence theorem, we take the limit $\lambda \to 0^+$ in (VII.71) to deduce that

$$\begin{aligned} & \left\| \left\{ \epsilon^{-1} \left(U_{t,s+\epsilon} - U_{t,s} \right) - iU_{t,s} G_s \right\} \psi \right\| \\ \leq \; & \epsilon^{-1} \int_s^{s+\epsilon} \left\| \left(U_{t,\tau} - U_{t,s} \right) G_\tau \psi \right\| \mathrm{d}\tau \qquad \text{(VII.72)} \\ & + m \exp \left\{ \int_s^t \beta_0 (\tau) \, \mathrm{d}\tau \right\} Z_\epsilon \left\| \Theta \psi \right\| + 3m^2 M \delta \exp \left\{ \int_s^t 2\gamma (\tau) \, \mathrm{d}\tau \right\} . \end{aligned}$$

Passing to the limits $\delta, \epsilon \to 0$ and using Lemma 92 we thus arrive at

$$\forall t > s > 0 : \qquad \partial_s U_{t,s} = iU_{t,s} G_s \ , \quad U_{t,t} := \mathbf{1} \ , \qquad \text{(VII.73)}$$

where the derivative is in the strong sense in $\mathcal{Y}$. Using exactly the same arguments,

$$\partial_s^+ U_{t,s}|_{s=0} = iU_{t,0} G_0 \qquad \text{and} \qquad \partial_s^- U_{t,s}|_{s=t} = iG_t \ , \qquad \text{(VII.74)}$$

provided $t > 0$.
Note that the uniqueness of the solution of the Cauchy problem (VII.1) results from the fact that any other (bounded) solution $\tilde{U}_{t,s}$ satisfy the equality

$$(\tilde{U}_{t,s} - U_{t,s})\varphi = \int_s^t \partial_\tau \{U_{t,\tau} \tilde{U}_{\tau,s}\} \varphi \ \mathrm{d}\tau = 0 \qquad \text{(VII.75)}$$

for any $\varphi \in \mathcal{Y}$. The set $\mathcal{Y}$ is dense and both operators are bounded, so the previous equality means that $U_{t,s} = \tilde{U}_{t,s}$. □

VII.2 Autonomous Generators of Bogoliubov Transformations

The goal of this subsection is to give a simple proof of the fact that all Bogoliubov **u**–**v** transformations, i.e., all unitary transformations U on the boson Fock space $\mathcal{F}_b$ of the form

$$\forall \varphi \in \mathfrak{h} : \qquad Ua(\varphi)U^* = a(\mathbf{u}^* \varphi) + a^*(\mathbf{v}^{\mathrm{t}} \bar{\varphi}) + \langle \bar{\psi} | \bar{\varphi} \rangle \qquad \text{(VII.76)}$$

with $\mathbf{u}, \mathbf{v} \in \mathcal{B}(\mathfrak{h})$ and $\psi \in \mathfrak{h}$, can be represented as $U = \exp(i\mathbb{Q})$, where

$$\begin{aligned} \mathbb{Q} \; = \; & \sum_{k,\ell} \{X\}_{k,\ell} \, a_k^* a_\ell + \{Y\}_{k,\ell} \, a_k^* a_\ell^* + \left\{\bar{Y}\right\}_{k,\ell} a_k a_\ell \\ & + \sum_k \{c_k a_k^* + \bar{c}_k a_k\} + \text{const} \qquad \text{(VII.77)} \end{aligned}$$

is a self–adjoint quadratic boson operator. In fact, below we show that

$$U = U_1 \left(\mathbf{u}, \mathbf{v} \right) \exp \left[a(\psi) - a^*(\psi) \right] \ , \qquad \text{(VII.78)}$$

where $U_1 \left(\mathbf{u}, \mathbf{v} \right) = \exp(i\mathbb{Q}_1)$, with $\mathbb{Q}_1$ being a self–adjoint quadratic boson operator of the form (VII.77) for $c_k = 0$. See Lemma 98 and (VII.112).
The unitarity of the Bogoliubov transformation $U \in \mathcal{B}(\mathcal{F}_b)$ is directly related to properties of operators $\mathbf{u}, \mathbf{v} \in \mathcal{B}(\mathfrak{h})$, as expressed in the following proposition:

PROPOSITION 95 (PROPERTIES OF OPERATORS $\mathbf{u}$ AND $\mathbf{v}$)
Assume that the Bogoliubov transformation $U \in \mathcal{B}(\mathcal{F}_b)$ defined by (VII.76) with $\mathbf{u}, \mathbf{v} \in \mathcal{B}(\mathfrak{h})$ is unitary. Then $\mathbf{v} \in \mathcal{L}^2(\mathfrak{h})$ is Hilbert–Schmidt and

$$\mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^* = \mathbf{1} , \qquad \mathbf{u}\mathbf{v}^{\mathrm{t}} = \mathbf{v}\mathbf{u}^{\mathrm{t}} , \tag{VII.79}$$
$$\mathbf{u}^*\mathbf{u} - \mathbf{v}^{\mathrm{t}}\bar{\mathbf{v}} = \mathbf{1} , \qquad \mathbf{u}^*\mathbf{v} = \mathbf{v}^{\mathrm{t}}\bar{\mathbf{u}} . \tag{VII.80}$$

Conversely, if $\mathbf{v} \in \mathcal{L}^2(\mathfrak{h})$ and (VII.79)–(VII.80) hold then the Bogoliubov transformation $U \in \mathcal{B}(\mathcal{F}_b)$ defined by (VII.76) is unitary.

PROOF. A Bogoliubov transformation $U \in \mathcal{B}(\mathcal{F}_b)$ is unitary iff the operator family $(Ua_kU^*, U^*a_k^*U)_{k=1}^\infty$ satisfy the Canonical Commutations Relations (CCR), like the operator family $(a_k, a_k^*)_{k=1}^\infty$ defined below by (VII.81), and the vacuum $|0\rangle \in \mathcal{D}(U)$. By (VII.76), this property directly yields the proposition through explicit computations. □
Note that the homogeneous Bogoliubov transformation corresponds to $\psi = 0$ in (VII.76). In this case, by taking any *real* orthonormal basis $\{\varphi_k\}_{k=1}^\infty \subseteq \mathfrak{h}$ and using the definition

$$\forall k \in \mathbb{N} : \qquad a_k := a(\varphi_k) , \tag{VII.81}$$

we observe that

$$\tilde{a}_k := Ua(\varphi_k)U^* = a(\mathbf{u}^*\varphi_k) + a^*(\mathbf{v}^{\mathrm{t}}\varphi_k) = \sum_\ell \{\mathbf{u}\}_{k,\ell}\, a_\ell + \{\mathbf{v}\}_{k,\ell}\, a_\ell^* , \tag{VII.82}$$

as for the Bogoliubov $\mathbf{u}$–$\mathbf{v}$ (unitary) transformation $\mathrm{U}_{t,s}$, see Theorem 14 (ii). It is then natural to ask for the generator of Bogoliubov $\mathbf{u}$–$\mathbf{v}$ transformations when it defines a unitary operator $U \in \mathcal{B}(\mathcal{F}_b)$, i.e., when $\mathbf{v} \in \mathcal{L}^2(\mathfrak{h})$ and (VII.79)–(VII.80) hold. This is the main result of this subsection, expressed in the following theorem:

THEOREM 96 (GENERATOR OF BOGOLIUBOV $\mathbf{u}$–$\mathbf{v}$ TRANSFORMATIONS)
The generator $\mathbb{Q}$ of a Bogoliubov transformation $U = \exp(i\mathbb{Q})$ defined by (VII.76) with $\psi \in \mathfrak{h}$ and operators $\mathbf{u}, \mathbf{v}$ satisfying $\mathbf{v} \in \mathcal{L}^2(\mathfrak{h})$ and (VII.79)–(VII.80) is a self–adjoint quadratic boson operator.

To prove this theorem, we make use of the fact that if $\mathbb{Q}_1 = \mathbb{Q}_1^*$ and $\mathbb{Q}_2 = \mathbb{Q}_2^*$ are two self–adjoint quadratic boson operators then

$$\mathrm{e}^{i\mathbb{Q}_1}\mathrm{e}^{i\mathbb{Q}_2} = \mathrm{e}^{i\mathbb{Q}_3} \tag{VII.83}$$

for some self–adjoint quadratic boson operator $\mathbb{Q}_3 = \mathbb{Q}_3^*$. We also need three elementary lemmata. The first one concerns an explicit computation about an elementary homogeneous Bogoliubov transformation.

LEMMA 97 (ELEMENTARY HOMOGENEOUS BOGOLIUBOV TRANSF.)
Let $\alpha > 0$ and $f \in \mathfrak{h}$ such that $\|f\| = 1$. Then, for all $\varphi \in \mathfrak{h}$,

$$\begin{aligned} & \exp\left[\frac{\alpha}{2}\left(a(f)^2 - a^*(f)^2\right)\right] a(\varphi) \exp\left[-\frac{\alpha}{2}\left(a(f)^2 - a^*(f)^2\right)\right] \\ = \; & a\left(\cosh(\alpha)\langle f|\varphi\rangle f\right) + a^*\left(\sinh(\alpha)\langle \bar{f}|\bar{\varphi}\rangle f\right) + a\left(\varphi - \langle f|\varphi\rangle f\right) \ . \end{aligned} \tag{VII.84}$$

PROOF. We refrain from proving the self–adjointness of $q := ia(f)^2 - ia^*(f)^2$. For any $\varphi \in \mathfrak{h}$ and $t \in [0, \alpha/2]$ with $\alpha > 0$, we introduce the operator

$$a_t(\varphi) := \exp[-itq]\, a(\varphi) \exp[itq] \ . \tag{VII.85}$$

Then its derivative for all $t \in [0, \alpha/2]$ equals

$$\partial_t a_t(\varphi) = i\left[a_t(\varphi), q\right] = \mathrm{e}^{-itq}\left[a(\varphi), a^*(f)^2\right]\mathrm{e}^{itq} = 2\langle\varphi|f\rangle a_t^*(f) \ . \tag{VII.86}$$

So, clearly,

$$\forall \varphi \perp f \ : \qquad \mathrm{e}^{-i\alpha q/2} a(\varphi) \mathrm{e}^{i\alpha q/2} = a(\varphi) \ . \tag{VII.87}$$

Conversely, if $\varphi = f$ and $\|f\| = 1$ then

$$\partial_t \begin{pmatrix} a_t(f) \\ a_t^*(f) \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_t(f) \\ a_t^*(f) \end{pmatrix} , \tag{VII.88}$$

which in turn implies that

$$\begin{pmatrix} a_{\alpha/2}(f) \\ a_{\alpha/2}^*(f) \end{pmatrix} = \exp\left[\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} a(f) \\ a^*(f) \end{pmatrix} . \tag{VII.89}$$

Since

$$\begin{aligned} \exp\left[\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] &= \sum_{n=0}^{\infty} \left\{ \frac{\alpha^{2n}}{(2n)!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\alpha^{2n+1}}{(2n+1)!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \\ &= \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} , \end{aligned} \tag{VII.90}$$

we infer from (VII.89) that

$$a_{\alpha/2}(f) = \cosh(\alpha)\, a(f) + \sinh(\alpha)\, a^*(f) \ , \tag{VII.91}$$

for any $\alpha > 0$ and $f \in \mathfrak{h}$ such that $\|f\| = 1$. The lemma then results from (VII.87) and (VII.91) combined with the antilinearity of $a(\varphi)$ and the linearity of $a^*(\varphi)$ with respect to $\varphi \in \mathfrak{h}$. □

The second step is to analyze an elementary inhomogeneous Bogoliubov transformation where $\mathbf{u} = \mathbf{v} = 0$.

Lemma 98 (Elementary inhomogeneous Bogoliubov transf.)
Let $\psi \in \mathfrak{h}$ *and* $\alpha > 0$. *Then, for all* $\varphi \in \mathfrak{h}$,

$$\exp\left[\alpha\left(a(\psi) - a^*(\psi)\right)\right] a(\varphi) \exp\left[-\alpha\left(a(\psi) - a^*(\psi)\right)\right] = a(\varphi) + \alpha\langle\bar{\psi}|\bar{\varphi}\rangle \ . \tag{VII.92}$$

Proof. For any $t \in [0, \alpha]$ with $\alpha > 0$, define the operators

$$Z := i\left(a(\psi) - a^*(\psi)\right) \qquad \text{and} \qquad a_t\left(\phi\right) := \mathrm{e}^{-itZ} a\left(\varphi\right) \mathrm{e}^{itZ} \ . \tag{VII.93}$$

Then its derivative for all $t \in [0, \alpha]$ equals

$$\partial_t a_t\left(\phi\right) = \mathrm{e}^{-itZ}\left[a\left(\varphi\right), a^*(\psi)\right] \mathrm{e}^{itZ} = \langle\varphi|\psi\rangle = \langle\bar{\psi}|\bar{\varphi}\rangle \ , \tag{VII.94}$$

which clearly implies the assertion. □

The proof of Theorem 96 uses the fact that any unitary operator is the exponential of some self–adjoint operator. This is completely standard and we shortly prove it here for completeness.

Lemma 99 (Generator of unitary operators)
Let $u \in \mathcal{B}(\mathfrak{h})$ *be unitary. Then there exists a self–adjoint operator* $(\mathrm{h}, \mathcal{D}(\mathrm{h}))$ *acting on* $\mathfrak{h}$ *such that* $u = \exp(i\mathrm{h})$.

Proof. Since u is unitary, it is normal and the spectral theorem implies that there is a realization of u on some $L^2(\mathcal{A}, \mathrm{d}\mathfrak{a})$ as a multiplication operator, i.e., there exists a measurable function $\upsilon : \mathcal{A} \to \mathbb{R}$ such that

$$\forall f \in L^2(\mathcal{A}, \mathrm{d}\mathfrak{a}),\ x \in \mathcal{A} : \qquad (uf)(x) = \mathrm{e}^{i\upsilon(x)} f(x) \ . \tag{VII.95}$$

Defining h on $\mathfrak{h}$ by

$$\forall f \in L^2(\mathcal{A}, \mathrm{d}\mathfrak{a}),\ x \in \mathcal{A} : \qquad (\mathrm{h}f)(x) := \upsilon(x) f(x) \ , \tag{VII.96}$$

one gets the self–adjoint operator sought for. □

Now, we are in position to prove Theorem 96:

Proof of Theorem 96. Since $\mathbf{v} \in \mathcal{L}^2(\mathfrak{h})$, this operator is compact and its singular value decomposition is

$$\mathbf{v} = \sum_k \lambda_k |g_k\rangle\langle\bar{f}_k| \ , \tag{VII.97}$$

where $\{f_k\}_{k=1}^\infty, \{g_k\}_{k=1}^\infty \subseteq \mathfrak{h}$ are orthonormal bases and $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{R}$ is a set of real numbers satisfying

$$\|\mathbf{v}\|_2^2 = \mathrm{tr}\left(\mathbf{v}\mathbf{v}^*\right) = \sum_k \lambda_k^2 < \infty \ . \tag{VII.98}$$

Indeed, straightforward computations show that

$$\mathbf{v}\mathbf{v}^* = \sum_k \lambda_k^2 |g_k\rangle\langle g_k| \tag{VII.99}$$

and since $\mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^* = \mathbf{1}$, we also deduce that

$$\mathbf{u}\mathbf{u}^* = \sum_k \left(\lambda_k^2 + 1\right) |g_k\rangle\langle g_k| \ . \tag{VII.100}$$

Writing

$$s_k := \sinh(\alpha_k) := \lambda_k \ , \tag{VII.101}$$

$$c_k := \cosh(\alpha_k) := \sqrt{1+\lambda_k^2} \ , \tag{VII.102}$$

and using some suitable orthonormal basis $\{h_k\}_{k=1}^{\infty} \subseteq \mathfrak{h}$ we infer from Equation (VII.100) that

$$\mathbf{v} = \sum_k s_k |g_k\rangle\langle \bar{f}_k| \qquad \text{and} \qquad \mathbf{u} = \sum_k c_k |g_k\rangle\langle h_k| \ . \tag{VII.103}$$

Next, observe that

$$\mathbf{u}^*\mathbf{v} = \sum_{k,\ell} c_k s_\ell |h_k\rangle\langle g_k|g_\ell\rangle\langle \bar{f}_\ell| = \sum_k c_k s_k |h_k\rangle\langle \bar{f}_k| \ , \tag{VII.104}$$

$$\mathbf{v}^{\mathrm{t}}\bar{\mathbf{u}} = \sum_{k,\ell} c_k s_\ell |f_\ell\rangle\langle \bar{g}_\ell|\bar{g}_k\rangle\langle \bar{h}_k| = \sum_k c_k s_k |f_k\rangle\langle \bar{h}_k| \ . \tag{VII.105}$$

Therefore, $\mathbf{u}^*\mathbf{v} = \mathbf{v}^{\mathrm{t}}\bar{\mathbf{u}}$ implies the equality $f_k = h_k$ for all $k \in \mathbb{N}$ such that $s_k \neq 0$ and we conclude that

$$\mathbf{v} = \sum_k s_k |g_k\rangle\langle \bar{h}_k| \qquad \text{and} \qquad \mathbf{u} = \sum_k c_k |g_k\rangle\langle h_k| \ . \tag{VII.106}$$

Now, we define the unitary operator U_1 to be

$$U_1 := \exp(-i\mathbb{Q}_1) \ , \tag{VII.107}$$

where

$$\mathbb{Q}_1 := \sum_k \frac{i}{2}\alpha_k \left(a(g_k)^2 - a^*(g_k)^2\right) = \mathbb{Q}_1^* \ . \tag{VII.108}$$

By Lemma 97, we have, for all $\varphi \in \mathfrak{h}$, the equality

$$U_1 a(\varphi) U_1^* = \sum_k \left\{ a(c_k\langle g_k|\varphi\rangle g_k) + a^*(s_k\langle \bar{g}_k|\bar{\varphi}\rangle g_k) \right\} \ . \tag{VII.109}$$

We also define a unitary operator $\hat{u}$ by $\hat{u}(g_k) = h_k$ for all $k \in \mathbb{N}$. By Lemma 99, there is a self–adjoint operator h such that

$$\hat{u} := \sum_k |h_k\rangle\langle g_k| = \exp(i\mathrm{h}) \ . \tag{VII.110}$$

Using the second quantization $\mathrm{d}\Gamma(\mathrm{h})$ of h, which is in this case a self–adjoint quadratic boson operator, we thus note that

$$\forall \varphi \in \mathfrak{h}: \qquad \exp\left(i\mathrm{d}\Gamma\left(\mathrm{h}\right)\right) a^*(\varphi) \exp\left(-i\mathrm{d}\Gamma\left(\mathrm{h}\right)\right) = a^*(\hat{u}\varphi) \ . \tag{VII.111}$$

Finally, we set the unitary operator

$$U := \exp\left(i\mathrm{d}\Gamma\left(\mathrm{h}\right)\right) U_1 \exp\left(a(\psi) - a^*(\psi)\right) \ . \tag{VII.112}$$

Then $U = \exp(i\mathbb{Q})$ for some self–adjoint quadratic boson operator $\mathbb{Q} = \mathbb{Q}^*$ and using Lemma 98, (VII.106), (VII.109), and (VII.111), we arrive, for all $\varphi \in \mathfrak{h}$, at the equalities

$$\begin{aligned} Ua(\varphi)U^* &= \mathrm{e}^{i\mathrm{d}\Gamma(\mathrm{h})} U_1 \left(a(\varphi) + \langle \bar{\psi}|\bar{\varphi}\rangle\right) U_1^* \mathrm{e}^{-i\mathrm{d}\Gamma(\mathrm{h})} \\ &= \mathrm{e}^{i\mathrm{d}\Gamma(\mathrm{h})} \left(\sum_k \{a(c_k\langle g_k|\varphi\rangle g_k) + a^*(s_k\langle \bar{g}_k|\bar{\varphi}\rangle g_k)\} + \langle \bar{\psi}|\bar{\varphi}\rangle \right) \mathrm{e}^{-i\mathrm{d}\Gamma(\mathrm{h})} \\ &= \sum_k \{a(c_k\langle g_k|\varphi\rangle h_k) + a^*(s_k\langle \bar{g}_k|\bar{\varphi}\rangle h_k)\} + \langle \bar{\psi}|\bar{\varphi}\rangle \\ &= a(\mathbf{u}^*\varphi) + a^*(\mathbf{v}^{\mathrm{t}}\bar{\varphi}) + \langle \bar{\psi}|\bar{\varphi}\rangle \ , \end{aligned} \tag{VII.113}$$

which prove Theorem 96 because the last equality uniquely defines the unitary operator U up to an overall phase factor. □

VII.3 Trace and Representation of Hilbert–Schmidt Operators

It is well–known that the Hilbert space $\mathcal{L}^2(\mathfrak{h})$ of Hilbert–Schmidt operators can be identified with the tensor product $\mathfrak{h}\otimes\mathfrak{h}^*$ by the natural unitary isomorphism

$$\mathcal{J}\Big[\,|\varphi\rangle\,\langle\psi|\,\Big] := \varphi\otimes\psi \ . \tag{VII.114}$$

The Hilbert space $\mathfrak{h}\otimes\mathfrak{h}^*$ has norm $\|\cdot\|_2$ defined from the usual scalar product

$$\left(\varphi\otimes\psi, \varphi'\otimes\psi'\right)_2 := \langle\varphi|\varphi'\rangle\langle\psi'|\psi\rangle \ , \quad \text{for } \varphi\otimes\psi, \varphi'\otimes\psi' \in \mathfrak{h}\otimes\mathfrak{h}^* \ . \tag{VII.115}$$

The norm on $\mathcal{L}^2(\mathfrak{h})$ is also denoted by $\|\cdot\|_2$ and results from the scalar product

$$(X,Y)_2 := \mathrm{tr}(X^*Y) \ , \qquad \text{for } X,Y \in \mathcal{L}^2(\mathfrak{h}) \ . \tag{VII.116}$$

The unitary isomorphism $\mathcal{J}$ is useful in this paper, albeit not essential, because of the unboundedness of the self–adjoint operator $\Omega_0 = \Omega_0^* \geq 0$. Indeed, the unitary isomorphism $\mathcal{J}$ allows us to interpret the differential equation

$$\forall t \geq 0: \qquad \partial_t Y_t = -\alpha\left(Z_tY_t + Y_tZ_t^{\mathrm{t}}\right) \ , \quad Y_0 \in \mathcal{L}^2(\mathfrak{h}) \ , \tag{VII.117}$$

on $\mathcal{L}^2(\mathfrak{h})$ as a *non–autonomous evolution equation*, even if Z_t is an unbounded operator acting on $\mathfrak{h}$. See as an example Equations (III.9) and (V.147). In particular, the uniqueness of a solution in $\mathcal{L}^2(\mathfrak{h})$ of (VII.117) is an immediate consequence of this fact. It is illustrated for a special parabolic case in the next lemma:

Lemma 100 (Uniqueness of a solution of $\partial_t Y_t = -\alpha \left(Z_t Y_t + Y_t Z_t^{\mathrm{t}} \right)$)
Let

$$\forall t \in [0, T) : \qquad Z_t := Z_0 + Q_t \ , \quad Q_0 := 0 \ , \tag{VII.118}$$

where $T \in (0, \infty]$, $Z_0 = Z_0^* \geq 0$ is a positive (possibly unbounded) operator on $\mathfrak{h}$ and $(Q_t)_{t\in[0,T)}$ is a strongly continuous family of bounded, self–adjoint operators. Then the zero operator family $(0)_{t\in[0,T)} \in C^1[[0, T); \mathcal{L}^2(\mathfrak{h})]$ is the unique solution in the Hilbert–Schmidt topology of the integral equation $X = \mathfrak{F}(X)$, where

$$\forall t \in [0, T) : \qquad [\mathfrak{F}(X)]_t := -\alpha \int_0^t \left(Z_\tau X_\tau + X_\tau Z_\tau^{\mathrm{t}} \right) \mathrm{d}\tau \tag{VII.119}$$

for any $\alpha > 0$. The same assertion holds if Z_τ is replaced by Z_τ^{t} in (VII.119).

Proof. Assume the family $(X_t)_{t\in[0,T)}$ of Hilbert–Schmidt operators solves (VII.119). Then, using the unitary isomorphism $\mathcal{J}$ we note that $\hat{X}_t := \mathcal{J}(X_t)$ obeys

$$\forall t \in [0, T) : \qquad \hat{X}_t = -\alpha \int_0^t \hat{Z}_\tau \hat{X}_\tau \mathrm{d}\tau \ , \tag{VII.120}$$

where

$$\hat{Z}_t := \mathcal{J} \left[Z_t (\cdot) + (\cdot) Z_t^{\mathrm{t}} \right] \mathcal{J}^* = Z_t \otimes \mathbf{1} + \mathbf{1} \otimes Z_t^{\mathrm{t}} \ . \tag{VII.121}$$

The existence and uniqueness of the solution of (VII.120) now follows from standard arguments, since $\hat{Z}_t = \hat{Z}_0 + \hat{Q}_t$ with

$$\hat{Z}_0 := Z_0 \otimes \mathbf{1} + \mathbf{1} \otimes Z_0^{\mathrm{t}} \geq 0 \quad \text{and} \quad \hat{Q}_t := Q_t \otimes \mathbf{1} + \mathbf{1} \otimes Q_t^{\mathrm{t}} \in \mathcal{B}(\mathfrak{h}) \ . \tag{VII.122}$$

It implies that $X_t = \mathcal{J}^*(\hat{X}_t) = 0$ is the unique solution of (VII.119) in $\mathcal{L}^2(\mathfrak{h})$. If Z_τ replaces Z_τ^{t} in (VII.119), then we only need to change Z_t^{t} by $Z_t^* = Z_t$ in (VII.121) and the same arguments as above yield the assertion. □

It clearly follows under the assumptions of Lemma 100 that the solution in $\mathcal{L}^2(\mathfrak{h})$ of the (parabolic) non–autonomous evolution equation (VII.117) is unique. (Existence of a solution of (VII.117) is in fact standard under these assumptions.)

Now, we observe that the cyclicity of the trace

$$\mathrm{tr}(XY) = \mathrm{tr}(YX) < \infty \tag{VII.123}$$

holds when $X, Y \in \mathcal{B}(\mathfrak{h})$ and both $XY, YX \in \mathcal{L}^1(\mathfrak{h})$, see [30, Cor. 3.8], but one must be careful in invoking cyclicity of the trace with unbounded operators like $\Omega_0 = \Omega_0^* \geq 0$. In the next lemma, we get around this problem by extending [30, Cor. 3.8]:

Lemma 101 (Cyclicity of the trace)
(i) *For any $X, Y \in \mathcal{B}(\mathfrak{h})$ such that $XY, YX \in \mathcal{L}^1(\mathfrak{h})$,*

$$\mathrm{tr}(XY) = \mathrm{tr}(YX) \ . \tag{VII.124}$$

In particular, if $X \in \mathcal{L}^1(\mathfrak{h})$ *and* $Y \in \mathcal{B}(\mathfrak{h})$ *then* $XY, YX \in \mathcal{L}^1(\mathfrak{h})$ *and (VII.124) holds.*
(ii) *For any* $X \in \mathcal{B}(\mathfrak{h})$ *and (possibly unbounded) self–adjoint operators* $Y = Y^*$ *on* $\mathfrak{h}$ *such that* $XY, YX \in \mathcal{L}^1(\mathfrak{h})$,

$$\mathrm{tr}(XY) = \mathrm{tr}(YX) \ . \tag{VII.125}$$

(iii) *For any (possibly unbounded) operators* X *and* Y *on* $\mathfrak{h}$,

$$\|XY\|_2^2 := \mathrm{tr}(Y^*X^*XY) = \mathrm{tr}(XYY^*X^*) =: \|Y^*X^*\|_2^2 \ . \tag{VII.126}$$

PROOF. (i): The first assertion is [30, Cor. 3.8] and results from the fact that the spectrums $\sigma(XY)$ and $\sigma(YX)$, respectively of the trace–class operators XY and YX, satisfy

$$\sigma(XY) \cup \{0\} = \sigma(YX) \cup \{0\} \tag{VII.127}$$

with the same multiplicity for each non–zero element of the spectrums. The fact that $X \in \mathcal{L}^1(\mathfrak{h})$ and $Y \in \mathcal{B}(\mathfrak{h})$ yield $XY, YX \in \mathcal{L}^1(\mathfrak{h})$ is deduced from [31, Thm VI.19 (b)].
(ii): If Y is bounded then the assertion follows directly from (i), so we may assume that Y is unbounded. For $m > 0$, observe that

$$\mathrm{tr}\left(XY - XY\mathbf{1}\left[|Y| \leq m\right]\right) = \sum_{k=1}^{\infty} \lambda_k \langle \varphi_k | \mathbf{1}\left[|Y| > m\right] \psi_k \rangle \ , \tag{VII.128}$$

using the singular value decomposition

$$XY = \sum_{k=1}^{\infty} \lambda_k |\psi_k\rangle\langle\varphi_k| \tag{VII.129}$$

of the trace–class operator XY, where the singular values $\{\lambda_k\}_{k=1}^{\infty}$ are absolutely summable and $\{\varphi_k\}_{k=1}^{\infty}, \{\psi_k\}_{k=1}^{\infty} \subset \mathfrak{h}$ are orthonormal bases. Since

$$\lim_{m\to\infty} \langle \varphi_k | \mathbf{1}\left[|Y| > m\right] \psi_k \rangle = 0 \ , \tag{VII.130}$$

for all $k \in \mathbb{N}$, Lebesgue's dominated convergence theorem implies that

$$\mathrm{tr}\left(XY\right) = \lim_{m\to\infty} \mathrm{tr}\left(XY\mathbf{1}\left[|Y| \leq m\right]\right) \ . \tag{VII.131}$$

Similarly,

$$\mathrm{tr}\left(YX\right) = \lim_{m\to\infty} \mathrm{tr}\left(Y\mathbf{1}\left[|Y| \leq m\right] X\right) \ , \tag{VII.132}$$

and thus (i) implies that

$$\begin{aligned} \mathrm{tr}\left(XY\right) &= \lim_{m\to\infty} \mathrm{tr}\left(XY\mathbf{1}\left[|Y| \leq m\right]\right) \\ &= \lim_{m\to\infty} \mathrm{tr}\left(Y\mathbf{1}\left[|Y| \leq m\right] X\right) = \mathrm{tr}\left(YX\right) \ . \end{aligned} \tag{VII.133}$$

(iii): Suppose that $\|XY\|_2^2 < \infty$. Then, $XY \in \mathcal{L}^2(\mathfrak{h})$ and also $Y^*X^* = (XY)^* \in \mathcal{L}^2(\mathfrak{h})$. Hence, both $(Y^*X^*)(XY), (XY)(Y^*X^*) \in \mathcal{L}^1(\mathfrak{h})$, and (i) implies that

$$\|XY\|_2^2 := \mathrm{tr}((Y^*X^*)(XY)) = \mathrm{tr}((XY)(Y^*X^*)) =: \|Y^*X^*\|_2^2 \ . \quad \text{(VII.134)}$$

□

We conclude the paper by three other elementary lemmata which are extensively used in our proofs and are related to properties of the trace.

Lemma 102 (Trace, transpose and complex conjugate)
For any operator X on $\mathfrak{h}$ and $n \in \mathbb{N}$, $(X^{\mathrm{t}})^n = (X^n)^{\mathrm{t}}$, $\bar{X}^n = \overline{X^n}$. This property can be extended to any $n \in \mathbb{Z}$ whenever X is invertible. Moreover, $X \in \mathcal{L}^1(\mathfrak{h})$ iff $X^{\mathrm{t}} \in \mathcal{L}^1(\mathfrak{h})$; $X \in \mathcal{L}^1(\mathfrak{h})$ iff $\bar{X} \in \mathcal{L}^1(\mathfrak{h})$. In particular, if $X \in \mathcal{L}^1(\mathfrak{h})$ then

$$\mathrm{tr}(X^{\mathrm{t}}) = \mathrm{tr}(X), \quad \mathrm{tr}(\bar{X}) = \overline{\mathrm{tr}(X)} \ . \quad \text{(VII.135)}$$

Proof. Recall that the scalar product on $\mathfrak{h}$ is given by

$$\langle f|g\rangle := \int_{\mathcal{M}} \overline{f(x)} g(x) \, \mathrm{d}\mathfrak{a}(x) \ , \quad \text{(VII.136)}$$

where, for every $f \in \mathfrak{h}$, we define its complex conjugate $\bar{f} \in \mathfrak{h}$ by $\bar{f}(x) := \overline{f(x)}$, for all $x \in \mathcal{M}$. See (II.1). For any operator X on $\mathfrak{h}$, we define its transpose X^{t} and its complex conjugate $\bar{X}$ by $\langle f|X^{\mathrm{t}}g\rangle := \langle \bar{g}|X\bar{f}\rangle$ and $\langle f|\bar{X}g\rangle := \overline{\langle \bar{f}|X\bar{g}\rangle}$ for all $f, g \in \mathfrak{h}$, respectively. So, for any $f, g \in \mathfrak{h}$ and every $n \in \mathbb{N}$,

$$\langle f| \left(X^{\mathrm{t}}\right)^2 g\rangle = \langle \overline{X^{\mathrm{t}}g}|X\bar{f}\rangle = \langle \overline{X\bar{f}}|X^{\mathrm{t}}g\rangle = \langle \bar{g}|X^2\bar{f}\rangle = \langle f| \left(X^2\right)^{\mathrm{t}} g\rangle \ . \quad \text{(VII.137)}$$

Then, the assertion $(X^{\mathrm{t}})^n = (X^n)^{\mathrm{t}}$ for all $n \in \mathbb{N}$ follows by induction. If X is invertible then, for any $f, g \in \mathfrak{h}$,

$$\langle f|X^{\mathrm{t}} \left(X^{-1}\right)^{\mathrm{t}} g\rangle = \langle \overline{(X^{-1})^{\mathrm{t}} g}|X\bar{f}\rangle = \langle \overline{X\bar{f}}| \left(X^{-1}\right)^{\mathrm{t}} g\rangle = \langle \bar{g}|X^{-1}X\bar{f}\rangle = \langle f|g\rangle \quad \text{(VII.138)}$$

and, similar to this,

$$\langle f| \left(X^{-1}\right)^{\mathrm{t}} X^{\mathrm{t}} g\rangle = \langle f|g\rangle \ . \quad \text{(VII.139)}$$

Therefore, $\left(X^{-1}\right)^{\mathrm{t}} = (X^{\mathrm{t}})^{-1}$. As a consequence, $(X^{\mathrm{t}})^n = (X^n)^{\mathrm{t}}$ for all $n \in \mathbb{Z}$. The proof of $\bar{X}^n = \overline{X^n}$ is performed in the same way. We omit the details. Furthermore, by taking any orthonormal bases $\{\eta_k\}_{k=1}^\infty, \{\psi_k\}_{k=1}^\infty \subseteq \mathfrak{h}$ and $m \in \mathbb{N}$ one finds

$$\sum_{k=1}^m \langle \eta_k|X^{\mathrm{t}}\psi_k\rangle = \sum_{k=1}^m \langle \bar{\psi}_k|X\bar{\eta}_k\rangle \ . \quad \text{(VII.140)}$$

It follows that $X \in \mathcal{L}^1(\mathfrak{h})$ iff $X^{\mathrm{t}} \in \mathcal{L}^1(\mathfrak{h})$. Furthermore, if $X \in \mathcal{L}^1(\mathfrak{h})$ then we choose a *real* orthonormal basis $\{g_k\}_{k=1}^{\infty} \subseteq \mathfrak{h}$ to deduce that

$$\mathrm{tr}(X^{\mathrm{t}}) = \sum_{k=1}^{\infty} \langle g_k | X^{\mathrm{t}} g_k \rangle = \sum_{k=1}^{\infty} \langle g_k | X g_k \rangle = \mathrm{tr}(X) \ . \tag{VII.141}$$

Similar to this, $X \in \mathcal{L}^1(\mathfrak{h})$ iff $\bar{X} \in \mathcal{L}^1(\mathfrak{h})$, and if $X \in \mathcal{L}^1(\mathfrak{h})$ then $\mathrm{tr}(\bar{X}) = \overline{\mathrm{tr}(X)}$.
□

In the next lemma, we use a strongly differentiable family $(Y_t)_{t>0} \subset \mathcal{B}(\mathfrak{h})$ such that $(\partial_t Y_t)_{t>0} \subset \mathcal{B}(\mathfrak{h})$ is locally uniformly bounded. This means that $\|\partial_t Y_t\|_{\mathrm{op}}$ is uniformly bounded on any compact set of $(0,\infty)$.

Lemma 103 (Trace and time derivatives)
Let $(X_t)_{t>0} \in C[\mathbb{R}^+; \mathcal{L}^2(\mathfrak{h})]$ be a family of Hilbert–Schmidt operators which is differentiable in $\mathcal{L}^2(\mathfrak{h})$ and $(Y_t)_{t>0} \subset \mathcal{B}(\mathfrak{h})$ be a strongly differentiable family such that $(\partial_t Y_t)_{t>0} \subset \mathcal{B}(\mathfrak{h})$ is locally uniformly bounded. Then, for any $t > 0$,

$$\partial_t \{\mathrm{tr}(X_t^* Y_t X_t)\} = \mathrm{tr}(\partial_t \{X_t^*\} Y_t X_t) + \mathrm{tr}(X_t^* Y_t \partial_t \{X_t\}) + \mathrm{tr}(X_t^* \partial_t \{Y_t\} X_t) \ . \tag{VII.142}$$

Proof. For any $\epsilon > -t$, the equality

$$\begin{aligned} & \epsilon^{-1}\left(X_{t+\epsilon}^* Y_{t+\epsilon} X_{t+\epsilon} - X_t^* Y_t X_t\right) \\ & -\partial_t \{X_t^*\} Y_t X_t - X_t^* \partial_t \{Y_t\} X_t - X_t^* Y_t \partial_t \{X_t\} \\ = \ & X_t^* \left(\epsilon^{-1}\left(Y_{t+\epsilon} - Y_t\right) - \partial_t Y_t\right) X_t \\ & + \left(X_{t+\epsilon}^* - X_t^*\right)\left\{\epsilon^{-1}\left(Y_{t+\epsilon} - Y_t\right)\right\} X_{t+\epsilon} \\ & + X_t^* \left\{\epsilon^{-1}\left(Y_{t+\epsilon} - Y_t\right)\right\}\left(X_{t+\epsilon} - X_t\right) \\ & + \left(\epsilon^{-1}\left(X_{t+\epsilon}^* - X_t^*\right) - \partial_t X_t^*\right) Y_t X_{t+\epsilon} \\ & + \partial_t \{X_t^*\} Y_t \left(X_{t+\epsilon} - X_t\right) \\ & + X_t^* Y_t \left(\epsilon^{-1}\left(X_{t+\epsilon} - X_t\right) - \partial_t X_t\right) \ , \end{aligned} \tag{VII.143}$$

combined with the cyclicity of the trace (Lemma 101 (i)), the Cauchy–Schwarz inequality, and the continuity of the map $Z \mapsto Z^*$ in the Hilbert–Schmidt topology, implies the upper bound

$$\begin{aligned} & \left|\epsilon^{-1}\left(\mathrm{tr}(X_{t+\epsilon}^* Y_{t+\epsilon} X_{t+\epsilon}) - \mathrm{tr}(X_t^* Y_t X_t)\right)\right. \\ & \left. -\mathrm{tr}(\partial_t \{X_t^*\} Y_t X_t) - \mathrm{tr}(X_t^* Y_t \partial_t \{X_t\}) - \mathrm{tr}(X_t^* \partial_t \{Y_t\} X_t)\right| \\ \leq \ & \left(\left\|\epsilon^{-1}\left(Y_{t+\epsilon} - Y_t\right)\right\|_{\mathrm{op}} \left(\|X_{t+\epsilon}\|_2 + \|X_t\|_2\right) + \|\partial_t X_t\|_2 \|Y_t\|_{\mathrm{op}}\right) \|X_{t+\epsilon} - X_t\|_2 \\ & + \|Y_t\|_{\mathrm{op}} \left(\|X_{t+\epsilon}\|_2 + \|X_t\|_2\right) \left\|\epsilon^{-1}\left(X_{t+\epsilon} - X_t\right) - \partial_t X_t\right\|_2 \\ & + \left|\mathrm{tr}\left(X_t^* \left(\epsilon^{-1}\left(Y_{t+\epsilon} - Y_t\right) - \partial_t Y_t\right) X_t\right)\right| \ . \end{aligned} \tag{VII.144}$$

We observe that there is a constant $\mathrm{K} > 0$ such that

$$\left\|\epsilon^{-1}\left(Y_{t+\epsilon} - Y_t\right)\right\|_{\mathrm{op}} \leq \epsilon^{-1} \int_t^{t+\epsilon} \|\partial_\tau Y_\tau\|_{\mathrm{op}} \, \mathrm{d}\tau < \mathrm{K} \tag{VII.145}$$

for sufficiently small $|\epsilon|$, because $(\partial_t Y_t)_{t>0} \subset \mathcal{B}(\mathfrak{h})$ is locally uniformly bounded. Then, by computing the trace with some orthonormal basis and using Lebesgue's dominated convergence theorem, we obtain that

$$\lim_{\epsilon \to 0} \mathrm{tr}\left(X_t^* \left\{\epsilon^{-1}\left(Y_{t+\epsilon} - Y_t\right)\right\} X_t\right) = \mathrm{tr}(X_t^* \partial_t \{Y_t\} X_t) \ . \tag{VII.146}$$

Moreover, by assumption,

$$\lim_{\epsilon \to 0} \|X_{t+\epsilon} - X_t\|_2 = \lim_{\epsilon \to 0} \left\|\epsilon^{-1}\left(X_{t+\epsilon} - X_t\right) - \partial_t X_t\right\|_2 = 0 \ . \tag{VII.147}$$

Therefore, we arrive at the assertion by using (VII.144)–(VII.147) and the boundedness of the families $(X_t)_{t>0} \subset \mathcal{L}^2(\mathfrak{h})$ and $(Y_t)_{t>0} \subset \mathcal{B}(\mathfrak{h})$, respectively in the Hilbert–Schmidt and norm topologies. □

LEMMA 104 (TRACE AND FUBINI'S THEOREM)
Let $\mathcal{I} \subseteq [0,\infty)$ and $(X_\lambda)_{\lambda \in \mathcal{I}} \subset \mathcal{B}(\mathfrak{h})$ be any family of bounded operators satisfying

$$\int_{\mathcal{I}} \|X_\lambda\|_{\mathrm{op}} \ \mathrm{d}\lambda < \infty \ . \tag{VII.148}$$

Then the operator

$$\left(Y := \int_{\mathcal{I}} X_\lambda \ \mathrm{d}\lambda\right) \in \mathcal{B}(\mathfrak{h}) \tag{VII.149}$$

is bounded and furthermore,

$$\forall Z \in \mathcal{L}^2(\mathfrak{h}) : \qquad \mathrm{tr}\left(Z^* Y Z\right) = \int_{\mathcal{I}} \mathrm{tr}\left(Z^* X_\lambda Z\right) \mathrm{d}\lambda \ . \tag{VII.150}$$

PROOF. Note first that the boundedness of Y is an immediate consequence of (VII.148) and the triangle inequality satisfied by the operator norm. Now, by using the Cauchy–Schwarz inequality and (VII.148) we observe that, for any $f, g \in \mathfrak{h}$,

$$\begin{aligned} \int_{\mathcal{I}} \left(\int_{\mathcal{M}} \left|\overline{f(x)}\left(X_\lambda g\right)(x)\right| \mathrm{d}\mathfrak{a}(x)\right) \mathrm{d}\lambda &= \int_{\mathcal{I}} \langle\, |f| \,|\, |X_\lambda g| \,\rangle \mathrm{d}\lambda \leq \|f\| \int_{\mathcal{I}} \|X_\lambda g\| \,\mathrm{d}\lambda \\ &\leq \|f\| \, \|g\| \int_{\mathcal{I}} \|X_\lambda\|_{\mathrm{op}} \mathrm{d}\lambda < \infty \ . \end{aligned} \tag{VII.151}$$

Therefore, Fubini's theorem implies that

$$\forall f, g \in \mathfrak{h} : \qquad \langle f | Y g \rangle = \int_{\mathcal{I}} \langle f | X_\lambda g \rangle \mathrm{d}\lambda \ . \tag{VII.152}$$

Taking any orthonormal basis $\{g_k\}_{k=1}^{\infty} \subseteq \mathfrak{h}$ we also infer from (VII.148) that

$$\forall Z \in \mathcal{L}^2(\mathfrak{h}) : \quad \int_{\mathcal{I}} \sum_{k=1}^{\infty} |\langle Z g_k | X_\lambda Z g_k \rangle| \ \mathrm{d}\lambda \leq \|Z\|_2^2 \int_{\mathcal{I}} \|X_\lambda\|_{\mathrm{op}} \mathrm{d}\lambda < \infty \ . \tag{VII.153}$$

Consequently, we combine (VII.152)–(VII.153) with Fubini's theorem to arrive at the equalities

$$\mathrm{tr}\,(Z^{*}YZ) = \int_{\mathcal{I}} \sum_{k=1}^{\infty} \langle Zg_k | X_\lambda Z g_k \rangle \mathrm{d}\lambda = \int_{\mathcal{I}} \mathrm{tr}\,(Z^{*}X_\lambda Z)\,\mathrm{d}\lambda\ , \qquad \text{(VII.154)}$$

for any $Z \in \mathcal{L}^2(\mathfrak{h})$. □

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