ATTRACTORS MET X-ELLIPTIC OPERATORS

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Abstract. We consider degenerate parabolic and damped hyperbolic equations involving an operator $\mathcal{L}$, that is X-elliptic with respect to a family of locally Lipschitz continuous vector fields $X = \{X_1, \dots, X_m\}$. The local well-posedness is established under subcritical growth restrictions on the nonlinearity f, which are determined by the geometry and functional setting naturally associated to the family of vector fields X. Assuming additionally that f is dissipative, the global existence of solutions follows, and we can characterize their longtime behavior using methods from the theory of infinite dimensional dynamical systems.

1. Introduction

Our aim is to show that the theory of semigroups and global attractors extends to a large class of semilinear evolution equations involving degenerate elliptic operators. To this end we consider as sample problems the semilinear heat and the semilinear damped wave equation, where the classical Laplace operator is replaced by the second order partial differential operator in divergence form

$$\mathcal{L}u := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}\partial_{x_j}u),$$

that is X-elliptic with respect to the family of vector fields $X = \{X_1, \dots, X_m\}$. The notion of X-elliptic operators, which will be recalled in Subsection 1.1, was first introduced in 2000 in the paper [26]. However, several families of operators that fall into this class were already present in literature; see, e.g., [13], [32], [33], [36], [24] and [25]. More recently, X-elliptic operators were widely studied in [15], where a maximum principle, a non-homogeneous Harnack inequality and a Liouville theorem were obtained, and in [21], where a one-sided Liouville-type property was proved, which extends the previous result by Gutierrez and Lanconelli in [15] and a celebrated Liouville-type theorem by Colding and Minicozzi in [11].

We analyze degenerate parabolic problems

$$\begin{aligned} \partial_t u(x,t) &= \mathcal{L}u(x,t) + f(u(x,t)) && x \in \Omega, t > 0, \\ u(x,t) &= 0 && x \in \partial\Omega, t \geq 0, \\ u(x,0) &= u_0(x) && x \in \Omega, \end{aligned} \tag{1.1}$$

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and damped hyperbolic problems of the form

$$\begin{aligned} \partial_{tt}u(x,t)+\beta u_t(x,t) &= \mathcal{L}u(x,t)+f(u(x,t)) && x\in\Omega, t>0,\\ u(x,t) &= 0 && x\in\partial\Omega, t\geq 0,\\ u(x,0)=u_0(x),\ u_t(x,0) &= u_1(x) && x\in\Omega, \end{aligned} \tag{1.2}$$

in a bounded domain $\Omega\subset\mathbb{R}^N$, where $\partial\Omega$ denotes the boundary of Ω and the constant β is positive.

We study the local and global well-posedness of Problem (1.1) and Problem (1.2) and characterize the longtime behavior of solutions. In particular, we show that the operator $\mathcal{L}$ generates an analytic semigroup in $L^2(\Omega)$ and that the local well-posedness of solutions can be obtained under appropriate growth conditions on the nonlinearity. The growth conditions are determined by the geometry and functional setting naturally associated to the family of vector fields X. If we additionally assume certain sign conditions for the nonlinearity, the global existence of solutions follows similarly to the classical cases of the semilinear heat and damped wave equation. Finally, we show existence and finite fractal dimension of the global attractor for the generated semigroup and prove convergence of solutions to an equilibrium solution as time tends to infinity. We formulate proofs, that are valid for a large class of operators and immediately yield estimates for the fractal dimension of the global attractor.

1.1. **Hypotheses and functional setting.**

We consider the operator

$$\mathcal{L}u := \sum_{i,j=1}^{N}\partial_{x_i}(a_{ij}\partial_{x_j}u)$$

where the functions a_{ij} are measurable in $\mathbb{R}^N$ and $a_{ij}=a_{ji}$.

We assume that there exists a family $X:=\{X_1,\dots,X_m\}$ of vector fields in $\mathbb{R}^N$, $X_j=(\alpha_{j1},\dots,\alpha_{jN})$, $j=1,\dots,m$, where the functions α_{jk} are locally Lipschitz continuous in $\mathbb{R}^N$. As usual, we identify the vector-valued function X_j with the linear first order partial differential operator

$$X_j=\sum_{k=1}^{N}\alpha_{jk}\partial_{x_k}, \qquad j=1,\dots,m.$$

The operator $\mathcal{L}$ is *uniformly X-elliptic* in $\mathbb{R}^N$ if there exists a constant $C>0$ such that

$$\frac{1}{C}\sum_{j=1}^{m}\langle X_j(x),\xi\rangle^2 \leq \sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j \leq C\sum_{j=1}^{m}\langle X_j(x),\xi\rangle^2 \qquad \forall x,\xi\in\mathbb{R}^N, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$ and

$$\langle X_j(x),\xi\rangle=\sum_{k=1}^{N}\alpha_{jk}(x)\xi_k, \qquad j=1,\dots,m.$$

We define the Hilbert space H as the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_H := \Big(\sum_{j=1}^m \|X_j u\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}, \qquad u \in C_0^1(\Omega),$$

and the bilinear form

$$a(u,v) := \int_\Omega \sum_{i,j=1}^N a_{ij}(x)\partial_{x_i}u(x)\partial_{x_j}v(x)\, dx, \qquad u,v \in H.$$

We observe that the operator $\mathcal{L}$ is self-adjoint in $L^2(\Omega)$ with domain

$$\mathcal{D}(\mathcal{L}) = \big\{u \in H \;:\; \exists\, c \geq 0 \text{ such that } |a(u,v)| \leq c\|v\|_{L^2(\Omega)}\ \forall v \in H\big\},$$
$$\langle -\mathcal{L}u, v\rangle_{L^2(\Omega)} = a(u,v) \qquad \forall u \in \mathcal{D}(\mathcal{L}),\, v \in H.$$

Assuming that the following *Poincaré-type inequality* is valid,

$$\text{(P)} \qquad \|u\|^2_{L^2(\Omega)} \leq c\, a(u,u) \qquad \forall u \in H,$$

for some constant $c \geq 0$, the operator $-\mathcal{L}$ is positive sectorial and generates an analytic semigroup in $L^2(\Omega)$, which we denote by $e^{\mathcal{L}t}, t \geq 0$ (see Subsection 3.1).

To solve the semilinear problems we require a *Sobolev-type embedding* result: We assume that there exists $q > 2$ such that the embedding

$$\text{(S)} \qquad H \hookrightarrow L^p(\Omega)$$

is continuous for $p \in [1,q]$ and compact for every $p \in [1,q)$.

Remark 1. The X-ellipticity of the operator $\mathcal{L}$ implies that the Poincaré inequality (P) is in fact a particular case of the Sobolev embeddings (S).

The local well-posedness of Problem (1.1) and Problem (1.2) follows by classical techniques from the theory of analytic semigroups, if the nonlinearity is locally Lipschitz continuous and satisfies the sub-critical growth restrictions

$$|f(u) - f(v)| \leq c|u-v|(1 + |u|^\gamma + |v|^\gamma), \qquad u,v \in \mathbb{R},$$

for some constant $c \geq 0$, where we assume that

$$\text{(F1)} \qquad 0 \leq \gamma < q-2 \qquad \text{for Problem (1.1)},$$

$$\text{(F1')} \qquad 0 \leq \gamma < \frac{q-2}{2} \qquad \text{for Problem (1.2)}$$

(see Subsections 3.2, 4.1 and 5.1).

Furthermore, the following sign conditions ensure the global existence of solutions and allow to characterize their longtime behavior:

$$\text{(F2)} \qquad \limsup_{|u|\to\infty} \frac{f(u)}{u} < \mu_1, \qquad u \in \mathbb{R},$$

where $\mu_1 > 0$ denotes the first eigenvalue of the operator $-\mathcal{L}$ with homogeneous Dirichlet boundary conditions.

1.2. **Main results.**
To formulate our main results we recall some notions from the theory of infinite dimensional dynamical systems (e.g., see [5], [34] or [31]):

Let $S(t) : V \to V, t \geq 0$, be continuous operators in a Banach space $(V, \|\cdot\|_V)$. We call the family $S(t), t \geq 0$, a *semigroup* if it satisfies the properties

$$\begin{aligned} S(t) \circ S(s) = S(t+s) &\qquad \forall t, s \geq 0, \\ S(0) = \mathrm{Id}, & \\ (t, v) \mapsto S(t)v &\quad \text{is continuous from } [0, \infty) \times V \to V, \end{aligned}$$

where $\circ$ denotes the composition, and Id the identity operator in V.

A non-empty, compact subset $\mathcal{A} \subset V$ is called the *global attractor* of the semigroup $S(t), t \geq 0$, if $\mathcal{A}$ is invariant,

$$S(t)\mathcal{A} = \mathcal{A} \qquad \forall t \geq 0,$$

and $\mathcal{A}$ attracts every bounded subset $B \subset V$ under the action of the semigroup, i.e.,

$$\lim_{t \to \infty} \mathrm{dist}_H(S(t)B, \mathcal{A}) = 0.$$

Here, $\mathrm{dist}_H(\cdot, \cdot)$ denotes the Hausdorff semi-distance in V,

$$\mathrm{dist}_H(B, A) := \sup_{b \in B} \inf_{a \in A} \|a - b\|_V \qquad \text{for subsets } A, B \subset V.$$

We denote the set of equilibrium points of the semigroup $S(t), t \geq 0$, by

$$\mathcal{E} := \big\{ v \in V : S(t)v = v \ \forall t \geq 0 \big\},$$

and the *unstable set* of $\mathcal{E}$ by

$$\mathcal{W}^u(\mathcal{E}) = \big\{ v \in V : \ S(t)v \text{ is defined } \forall\, t \in \mathbb{R}, \mathrm{dist}_H(S(-t)v, \mathcal{E}) \to 0 \text{ as } t \to \infty \big\}.$$

Furthermore, the *ω-limit set* of an element $v \in V$ is

$$\omega(v) = \big\{ y \in V : \ \exists\, (t_n)_{n \in \mathbb{N}},\, t_n \geq 0,\, \lim_{n \to \infty} t_n = \infty, \text{ such that } \lim_{n \to \infty} S(t_n)v \to y \big\}.$$

Under our hypotheses we obtain the following result, which states the global existence of solutions of Problem (1.1) and characterizes their longtime behavior:

Theorem 1. *We assume the operator $\mathcal{L}$ is X-elliptic with respect to the family of vector fields $X = \{X_1, \ldots, X_m\}$, and the properties* (S), (F1) *and* (F2) *are satisfied. Then, for every initial data $u_0 \in H$ there exists a unique global solution of Problem* (1.1) *and*

$$u \in C([0, \infty); H) \cap C^1((0, \infty); H).$$

The semigroup $S(t), t \geq 0$, in H generated by Problem (1.1) *possesses a global attractor $\mathcal{A}$ of finite fractal dimension, which is connected and*

$$\mathcal{A} = \mathcal{W}^u(\mathcal{E}),$$

where $\mathcal{E} = \{u \in H \,|\, \mathcal{L}u + f(u) = 0\}$. Furthermore, for every initial data $u_0 \in H$ we have $\omega(u_0) \subset \mathcal{E}$ and, in particular,

$$\lim_{t \to \infty} \mathrm{dist}_H(S(t)u_0, \mathcal{E}) = 0.$$

This generalizes our previous result in [23], where we studied Problem (1.1) for the particular class of Δ_λ-Laplacians (see Subsection 2.1.2).

Similar results can be obtained for Problem (1.2). Setting $v := \partial_t u$ and $w := (u, v)$ we reformulate it as the first order system

$$
\begin{aligned}
(1.4) \qquad & \partial_t w^T = \hat{A} w^T + \hat{f}(w), \qquad w^T = \begin{pmatrix} u \\ v \end{pmatrix}, \\
& w|_{t=0} = w_0,
\end{aligned}
$$

where the initial data $w_0 = (u_0, u_1) \in V$, and $V := H \times L^2(\Omega)$ with

$$\|w\|_V := \left(a(u,u) + \|v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \qquad w = (u, v) \in V.$$

Furthermore, $\hat{A}$ and $\hat{f}$ are defined by

$$\hat{A} := \begin{pmatrix} 0 & Id \\ -A & -\beta \end{pmatrix}, \qquad \hat{f}(w) := \begin{pmatrix} 0 \\ f(u) \end{pmatrix},$$

where A denotes the operator $-\mathcal{L}$ in $L^2(\Omega)$ with homogeneous Dirichlet boundary conditions.

Theorem 2. *We assume the operator $\mathcal{L}$ is X-elliptic with respect to the family of vector fields $X = \{X_1, \ldots, X_m\}$, and the properties* (S), (F1') *and* (F2) *are satisfied. Then, for every initial data $w_0 \in V$ there exists a unique global solution of Problem* (1.4) *and*

$$w \in C([0, \infty); V).$$

The semigroup $U(t), t \geq 0$, in V generated by Problem (1.4) *possesses a global attractor $\mathcal{A}$ of finite fractal dimension, which is connected and*

$$\mathcal{A} = \mathcal{W}^u(\mathcal{E}),$$

where $\mathcal{E} = \{(u, 0) \in V \mid \mathcal{L}u + f(u) = 0\}$. Furthermore, for every initial data $w_0 \in V$ we have $\omega(u_0) \subset \mathcal{E}$ and, in particular,

$$\lim_{t \to \infty} \operatorname{dist}_V(U(t)w_0, \mathcal{E}) = 0.$$

The aim of this article is to show that our previous result in [23] extends to a large class of parabolic equations involving degenerate elliptic operators and that similar results can be obtained for damped hyperbolic equations. Moreover, we formulate for the different classes of X-elliptic operators the admissible growth restrictions on the nonlinearity, which are determined by Sobolev type embedding theorems.

In Section 2 we give an overview of the different classes of operators to which our results apply. In Section 3 we collect some notions and results from the theory of semigroups and infinite dimensional dynamical systems that are used in the subsequent sections. Theorem 1 can be shown by following the arguments in [23]. For convenience of the reader we give a sketch of the proof in Section 4 and indicate the main ideas. In Section 5 we consider the degenerate damped hyperbolic problem (1.4), show its well-posedness and prove Theorem 2. In the Appendix we derive a Poincaré type inequality and an auxiliary result, which yields estimates for the fractal dimension of global attractors.

2. Classes of operators satisfying our hypotheses

2.1. X-elliptic operators in homogeneous metric spaces.

We recall the definition of *control distance* or *Carnot-Caratheodory distance* $d = d_X$ related to the family of vector fields X. A piecewise regular path $\gamma : [0,1] \longrightarrow \mathbb{R}^N$ is called X*-subunit curve* in $[0,T]$, if there exist measurable functions $c_1, \dots, c_N : [0,T] \longrightarrow \mathbb{R}$ such that

$$\dot\gamma(t) = \sum_{j=1}^m c_j(t) X_j(\gamma(t)), \qquad \sum_{j=1}^m c_j^2 \le 1 \quad \text{a.e. in } [0,1].$$

We call $\mathbb{R}^N$ X*-connected* if for every $x, y \in \mathbb{R}^N$ there exists a X-subunit curve such that $\gamma(0) = x$ and $\gamma(T) = y$ and define

$$d(x,y) := \inf\{T > 0 \ : \ \exists\, \gamma \text{ subunit curve s.t. } \gamma(0) = x, \gamma(T) = y\}.$$

We further assume that the following properties are satisfied:

- $\mathbb{R}^N$ is X-connected, the control distance is continuous with respect to the Euclidean topology and there exists $A > 1$ such that the *doubling condition* holds

$$0 < |B_{2R}| \le A|B_R|, \tag{D}$$

  for every d-ball B_R of radius R, such that $B_R \subset B_{2R_0}$, where $\Omega \subset B_{R_0}$. Here, $|E|$ denotes the Lebesgue measure of a set $E \subseteq \mathbb{R}^N$.
- There exist positive constants C and $\nu \ge 1$ such that the following Poincaré inequality holds

$$\fint_{B_R} |u(x) - u_R|\, dx \le CR \fint_{B_{\nu R}} |Xu(x)|\, dx \tag{$\widetilde{P}$}$$

  for every Lipschitz continuous function u in $B_{\nu R}$ and any d-ball B_R, such that $B_R \subset B_{2R_0}$. Here, we denote by $u_R = \fint_{B_R} u := \frac{1}{|B_R|} \int_{B_R} u$ the mean value and by Xu the X*-gradient* of u, i.e.,

$$Xu = (X_1 u, \dots, X_m u).$$

These properties imply the Sobolev embeddings (S) and have been verified for wide classes of operators that we discuss in Subsections 2.1.1 and 2.1.2.

The real number $Q := \log_2(A)$ is called the *homogeneous dimension* of $(\mathbb{R}^N, d)$.

Remark 2. The homogeneous dimension is not unique, but taking

$$A_0 := \inf\{A \ : \ \text{property } (D) \text{ holds}\}$$

leads to the largest exponent in the Sobolev type embeddings (S) and the weakest growth restrictions on the nonlinearities in Problem (1.1) and Problem (1.2).

In the particular case that the vector fields X_j are homogeneous of degree one with respect a group of dilations $(\delta_r)_{r>0}$ in $\mathbb{R}^N$,

$$\delta_r : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad \delta_r(x) = \delta_r(x_1, \dots, x_N) = (r^{\sigma_1} x_1, \dots, r^{\sigma_N} x_N),$$

$\sigma_1 = 1 \le \sigma_2 \le \dots \le \sigma_N$, the doubling property can be easily verified. We have $0 < |B_{2R}| = 2^Q |B_R|$ and $A = 2^Q$ is clearly the optimal constant in (D). In this case the homogeneous dimension of $\mathbb{R}^N$ is defined as $Q := \sigma_1 + \dots + \sigma_N$.

Under the assumptions (D) and $(\widetilde{P})$ the Poincaré-Sobolev inequality

$$\|u\|_{L^q(\Omega)} \leq c\|Xu\|_{L^2(\Omega)} \qquad \forall\, u \in C_0^1(\Omega),$$

for some constant $c \geq 0$, with $q = \frac{2Q}{Q-2}$ was proved in Remark 9 in [21]. Since the domain Ω is bounded, the inequalities are satisfied if $L^q(\Omega)$ is replaced by $L^p(\Omega)$ for any $p \in [1, q]$, and the Poincaré inequality (P) follows by the X-ellipticity of the operator $\mathcal{L}$.

Furthermore, $(\mathbb{R}^N, d)$ is a Carnot-Carathéodory space in the sense of [17], Section 11, and by Corollary 9.5 in [17] every ball in $\mathbb{R}^N$ is a John's domain. Theorem 1.28 in [16][1], states that the embedding

$$H^1_X(\Omega) \hookrightarrow L^p(\Omega)$$

is compact if $1 \leq p < \frac{2Q}{Q-2}$, where

$$H^1_X(\Omega) = \{u \in L^2(\Omega) \,:\, X_j u \in L^2(\Omega), j = 1, \ldots, m\}$$

with inner product

$$\langle u, v\rangle_{H^1_X(\Omega)} = \int_\Omega (uv + Xu \cdot Xv), \qquad u, v \in H^1_X(\Omega).$$

This shows that the assumptions (D) and $(\widetilde{P})$ imply our hypotheses (P) and (S), where H is the closure of $C_0^1(\Omega)$ with respect to the norm $\|\cdot\|_{H^1_X(\Omega)}$. For our problems (1.1) and (1.2) this leads to the subcritical growth restrictions (F1) with $0 < \rho < \frac{4}{Q-2}$ and (F1') with $0 < \rho < \frac{2}{Q-2}$.

We observe that the homogeneous dimension Q plays the same role as the dimension N in the classical Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [1, q^*]$ and $q^* = \frac{2N}{N-2}$, $N \geq 3$, which determines the critical exponents in the growth restrictions on the nonlinearity $\frac{4}{N-2}$ and $\frac{2}{N-2}$ for the classical semilinear heat and damped wave equation.

The properties (D) and $(\widetilde{P})$ are well known for two wide classes of operators, which we describe in Subsections 2.1.1 and 2.1.2: Operators that are X-elliptic with respect to a family of smooth vector fields satisfying the Hörmander rank condition, and Lipschitz continuous vector fields of diagonal type.

2.1.1. *Vector fields satisfying the Hörmander rank condition.*
Let $X = \{X_1, \ldots, X_m\}$ be a family of smooth vector fields satisfying the Hörmander rank condition

$$\operatorname{rank}\left(\operatorname{Lie}\{X_1, \ldots, X_m\}\right)(x) = N \qquad \forall x \in \mathbb{R}^N,$$

where $\operatorname{Lie}\{X_1, \ldots, X_m\}$ denotes the Lie-algebra generated by the family of vector fields X. Then, the doubling condition (D) was proved by Nagel, Stein and Weinger in [29] and the Poincaré inequality $(\widetilde{P})$ by Jerison in [19].

[1]The hypotheses of the theorem are satisfied since every ball B_R is a John's domain and John's domains are X-PS domains in the sense of [16] (see p. 1093). Moreover, using the doubling property (D) and the inequality

$$\|u\|_{L^1(\Omega)} \geq \lambda|\{x \in B_R \,:\, |u(x) - u_{B_R}| \geq \lambda\}| \qquad \forall\, \lambda > 0,$$

we can deduce Hypothesis (H.2) in [16] from our assumption $(\widetilde{P})$.

Our results apply to operators $\mathcal{L}$ which are X-elliptic with respect to the family $X = \{X_1, \ldots, X_m\}$ and, in particular, to *sub-Laplacians on Carnot groups*:

Let $(\mathbb{R}^N, \circ)$ be a Lie group in $\mathbb{R}^N$. We assume that $\mathbb{R}^N$ can be split as follows

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_n},$$

and that there exists a group of dilations $\delta_r : \mathbb{R}^N \longrightarrow \mathbb{R}^N$,

$$\delta_r(x) = \delta_r(x^{(N_1)}, \ldots, x^{(N_n)}) := (rx^{(N_1)}, \ldots, r^n x^{(N_n)}), \quad r > 0,$$
$$x^{(N_i)} \in \mathbb{R}^{N_i}, \qquad i = 1, \ldots, n,$$

which are automorphisms of $(\mathbb{R}^N, \circ)$.
We assume that

$$\text{rank } \left(\text{Lie}\{X_1, \ldots, X_{N_1}\}\right)(x) = N \qquad \forall x \in \mathbb{R}^N,$$

where the vector fields X_j are left invariant on $(\mathbb{R}^N, \circ)$ and

$$X_j(0) = \frac{\partial}{\partial x_j^{(N_1)}}, \qquad j = 1, \ldots, N_1.$$

Then, $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a Carnot group, and the homogeneous dimension is $Q = N_1 + 2N_2 + \cdots + nN_n$. Our results apply to the sub-Laplacian

$$\mathcal{L} = \Delta_{\mathbb{G}} = \sum_{j=1}^{N_1} X_j^2,$$

where the vector fields $X_1, \ldots, X_{N_1}$ are the *generators* of $\mathbb{G}$. We remark that every sub-Laplacian can be written in divergence form (see p. 64 in [6]).

Example 1. The Kohn-Laplacian on the Heisenberg group.

The Heisenberg group $\mathbb{H}^N$, whose elements we denote by $\zeta = (x, y, z)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the composition law

$$\zeta \circ \zeta' = (x + x', y + y', z + z' + 2(\langle x', y\rangle - \langle x, y'\rangle)),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. The Kohn Laplacian is the operator $\Delta_{\mathbb{H}^N} = \sum_{j=1}^N (X_j^2 + Y_j^2)$, where

$$X_j = \partial_{x_j} + 2y_j\partial_z, \quad Y_j = \partial_{y_j} - 2x_j\partial_z.$$

A natural group of dilations is given by

$$\delta_r(\zeta) = \delta_r(x, y, z) = (rx, ry, r^2 z), \quad r > 0,$$

and the homogeneous dimension is $Q = 2N + 2$.

For further examples we refer to [6].

2.1.2. *The Δ_λ-Laplacian.*
As in [22], we consider operators of the form

$$\Delta_\lambda := \sum_{i=1}^N \partial_{x_i}(\lambda_i^2 \partial_{x_i}),$$

where $\partial_{x_i} = \frac{\partial}{\partial_{x_i}}, i = 1, \ldots, N$. The functions $\lambda_i : \mathbb{R}^N \to \mathbb{R}$ are continuous, strictly positive and of class C^1 outside the coordinate hyperplanes[2] and satisfy the following properties:

[2] $\lambda_i > 0$ in $\mathbb{R}^N \backslash \Pi$, where $\Pi = \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N \ : \prod_{i=1}^N x_i = 0\right\}$

(*i*) $\lambda_1(x) \equiv 1,\ \lambda_i(x) = \lambda_i(x_1, \dots, x_{i-1}),\ i = 2, \dots, N.$

(*ii*) For every $x \in \mathbb{R}^N$ the function $\lambda_i(x) = \lambda_i(x^*)$, $i = 1, \dots, N$, where

$$x^* = (|x_1|, \dots, |x_N|) \quad \text{if } x = (x_1, \dots, x_N).$$

(*iii*) There exists a constant $\rho \geq 0$ such that

$$0 \leq x_k \partial_{x_k} \lambda_i(x) \leq \rho \lambda_i(x) \qquad \forall k \in \{1, \dots, i-1\}, i = 2, \dots, N,$$

for every $x \in \mathbb{R}^N_+ := \left\{(x_1, \dots, x_N) \in \mathbb{R}^N \ : x_i \geq 0\ \forall i = 1, \dots, N\right\}$.

(*iv*) There exists a group of dilations $(\delta_r)_{r>0}$,

$$\delta_r : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_r(x) = \delta_r(x_1, \dots, x_N) = (r^{\sigma_1} x_1, \dots, r^{\sigma_N} x_N),$$

where $1 \leq \sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_N$, such that λ_i is δ_r-*homogeneous of degree* $\sigma_i - 1$, i.e.,

$$\lambda_i(\delta_r(x)) = r^{\sigma_i - 1} \lambda_i(x), \qquad \forall x \in \mathbb{R}^N,\ r > 0,\ i = 1, \dots, N.$$

This implies that the operator Δ_λ is δ_r-homogeneous of degree two, i.e.,

$$\Delta_\lambda(u(\delta_r(x))) = r^2 (\Delta_\lambda u)(\delta_r(x)) \qquad \forall u \in C^\infty(\mathbb{R}^N).$$

We assumed in our definition of X-elliptic operators that the coefficient functions are Lipschitz continuous. The assumptions (*i*)-(*iv*) do not necessarily imply the Lipschitz continuity of the functions λ_i, but the hypotheses we need to prove our main results can still be verified for Δ_λ-Laplacians.

The doubling property (D) is certainly satisfied, the homogeneous dimension is $Q = \sigma_1 + \dots + \sigma_N$, and the Poincaré inequality $(\widetilde{P})$ was obtained in [14]. However, for the Δ_λ-Laplacian the Poincaré inequality (P) can also be directly verified as in A, and the Sobolev embeddings (S) were proved in [22] and [14]. If we apply Theorem 1 to the Δ_λ-Laplacian we recover our previous result in [23].

Example 2. We split $\mathbb{R}^N$ into $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$ and write

$$x = (x^{(1)}, x^{(2)}, x^{(3)}), \qquad x^{(i)} \in \mathbb{R}^{N_i},\ i = 1, 2, 3.$$

Let α, β and γ be nonnegative real constants. For the operator

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} \Delta_{x^{(2)}} + |x^{(1)}|^{2\beta} |x^{(2)}|^{2\gamma} \Delta_{x^{(3)}},$$

where $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ with

$$\begin{aligned} &\lambda_j^{(1)}(x) \equiv 1, && j = 1, \dots, N_1, \\ &\lambda_j^{(2)}(x) = |x^{(1)}|^\alpha, && j = 1, \dots, N_2, \\ &\lambda_j^{(3)}(x) = |x^{(1)}|^\beta |x^{(2)}|^\gamma, && j = 1, \dots, N_3, \end{aligned}$$

we find the group of dilations

$$\delta_r \left(x^{(1)}, x^{(2)}, x^{(3)}\right) = \left(r x^{(1)}, r^{\alpha+1} x^{(2)}, r^{\beta + (\alpha+1)\gamma + 1} x^{(3)}\right).$$

Similarly, for operators of the form

$$\begin{aligned} \Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha_{1,1}} \Delta_{x^{(2)}} + |x^{(1)}|^{2\alpha_{2,1}} |x^{(2)}|^{2\alpha_{2,2}} \Delta_{x^{(3)}} + \cdots \\ + \left(\prod_{i=1}^{k-1} |x^{(i)}|^{2\alpha_{k-1,i}}\right) \Delta_{x^{(k)}}, \end{aligned}$$

where $\alpha_{i,j} \geq 0$, $i = 1, \dots, k-1, j = 1, \dots, i$, are real constants, the group of dilations is given by

$$\delta_r \left(x^{(1)}, \dots, x^{(k)}\right) = \left(r^{\sigma_1} x^{(1)}, \dots, r^{\sigma_k} x^{(k)}\right)$$

with $\sigma_1 = 1$ and $\sigma_j = 1 + \sum_{i=1}^{j-1} \alpha_{j-1,i}\sigma_i$, for $i = 2, \dots, k$.

In particular, if $\alpha_{1,1} = \dots = \alpha_{k-1,k-1} = \alpha$, the dilations become

$$\delta_r \left(x^{(1)}, \dots, x^{(k)}\right) = \left(r x^{(1)}, r^{\alpha+1} x^{(2)}, \dots, r^{(\alpha+1)^{k-1}} x^{(k)}\right).$$

A particular case of Δ_λ-Laplacians are operators that are commonly called of Grushin type,

$$\Delta_x + |x|^{2\alpha} \Delta_y, \qquad (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},\ \alpha > 0.$$

The global well-posedness and existence of the global attractor for semilinear parabolic problems involving Grushin-type operators was proved in [4] and the result slightly extended in [35].

2.1.3. *Further examples.*
We close this subsection with examples of X-elliptic operators that do not fall into the classes of operators discussed in Subsections 2.1.1 and 2.1.2.

Example 3. We consider in $\mathbb{R}^3$ the family of vector fields $X = \{X_1, X_2\}$ with

$$X_1 = \partial_x \quad \text{and} \quad X_2 = |x|^m \partial_y + \partial_z,$$

where m is a real constant, $m \geq 1$.

As observed in [15], Section 6.1, the doubling condition (D) and the Poincaré inequality $(\widetilde{P})$ are satisfied. Consequently, our results apply to the operator

$$\mathcal{L} = X_1^2 + X_2^2 = \partial_x^2 + |x|^{2m} \partial_y^2 + \partial_z^2 + 2|x|^m \partial_{yz}.$$

In this case, it can be easily verified that the fields X_1 and X_2 are invariant with respect to the group of dilations

$$\delta_r(x, y, z) = (rx, r^{m+1} y, rz), \qquad r > 0,$$

and the homogeneous dimension is $Q = 3 + m$.

Example 4. Another operator for which Properties (D) and $(\widetilde{P})$ hold was considered in Example 4.6 in [26]:

In $\mathbb{R}^3$ we consider the family of vector fields $X = \{X_1, X_2\}$, where

$$X_1 = \partial_x + a\partial_z \quad \text{and} \quad X_2 = \partial_y + b\partial_z.$$

The functions $a, b \in C^1(\mathbb{R}^3, \mathbb{R})$ and $X_1 a - X_2 b > 0$ at any point in $\mathbb{R}^3$.

We consider the operator

$$\mathcal{L}u := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij} \partial_{x_j} u),$$

where the matrix $A = (a_{ij})_{1 \leq i,j \leq 3}$ is given by

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ a & b & a^2 + b^2 \end{pmatrix} (X_1\ X_2) \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}.$$

Since

$$\langle A\xi, \xi\rangle = \langle X_1, \xi\rangle^2 + \langle X_2, \xi\rangle^2$$

the operator $\mathcal{L}$ is X-elliptic. It takes the form

$$\begin{aligned}\mathcal{L} = \partial_x^2 + \partial_y^2 + (a^2+b^2)\partial_z^2 + \partial_x(a\partial_z) + \partial_y(b\partial_z)\\ + \partial_z(a\partial_x + b\partial_y) + \partial_z(a^2+b^2)\partial_z.\end{aligned}$$

Such operators arise when studying the Levi-curvature equation in $\mathbb{C}^2$(see [10]). The doubling property (D) and Poincaré inequality $(\widetilde{P})$ were proved in [27], the homogeneous dimension is $Q = 4$ and can be computed as in [28].

2.2. **A more general class of X-elliptic operators.**
The doubling property (D) and the Poincaré inequality $(\widetilde{P})$ are generally difficult to verify. They are known for the classes of operators in Subsection 2.1. In certain cases, we can verify the Poincaré inequality (P) and the Sobolev embeddings (S) directly under weaker assumptions and without using the fact that hypotheses (D) and $(\widetilde{P})$ imply properties (P) and (S).

As in [20] we assume that the operator $\mathcal{L}$ is X-elliptic with respect to the family of vector fields

$$X = \{X_1, \dots, X_N\} = \{\eta_1\partial_{x_1}, \dots, \eta_N\partial_{x_N}\},$$

where the η_j's are non-negative functions on Ω such that

$$\eta_1(x) \geq 1 \quad \text{and} \quad \eta_j(x) \geq c|x_1|^{\alpha_1}\cdots|x_{j-1}|^{\alpha_{j-1}} \qquad \forall x \in \Omega,$$

for suitable positive constants c and $\alpha_1, \dots, \alpha_{j-1}$.

The Poincaré inequality (P) can be verified by slightly modifying the classical proof in [1] (see A). Furthermore, from Proposition 3.1 in [20] we obtain that there exists $q \in (2,\infty)$ such that the embedding $H \hookrightarrow L^p(\Omega)$ is continuous for all $p \in [1, q]$ and compact for $p \in [1, q)$, which shows that (S) is satisfied. Here, the space H is the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_H := \Big(\int_\Omega |Xu|^2\Big)^{\frac{1}{2}}, \qquad u \in C_0^1(\Omega).$$

Consequently, our results also apply to this class of X-elliptic operators, which were studied in [20].

Remark 3. For simplicity and to avoid introducing new notation we have discussed in this subsection a particular class of the X-elliptic operators in [20]. However, the results can be verified for a larger class of operators studied in the cited article.

3. Semigroups and infinite dimensional dynamical systems: some well-known properties

For the convenience of the reader we collect in this section well-known notions and results from the theory of semigroups and infinite dimensional dynamical systems that we apply in the subsequent sections to prove our main results. For details we refer to [5], [9], [18], [30], [31] and [34].

3.1. **Analytic semigroups and fractional power spaces.** Let $\mathcal{L}$ be a uniformly X-elliptic operator as defined in Section 1 and H be the corresponding Hilbert space. We consider $\mathcal{L}$ in $L^2(\Omega)$ with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ u \in H \mid \exists\, c \geq 0 \text{ such that } |a(u,v)| \leq c\|v\|_{L^2(\Omega)}\ \forall v \in H \right\},$$
$$\langle -\mathcal{L}u, v\rangle_{L^2(\Omega)} = a(u,v) \qquad \forall u \in \mathcal{D}(\mathcal{L}),\ v \in H.$$

Proposition 3. *The operator $A := -\mathcal{L}$ generates an analytic semigroup $e^{-At}, t \geq 0$, in $L^2(\Omega)$.*

Proof. We observe that $\mathcal{L}$ is densely defined and self-adjoint in $L^2(\Omega)$. Furthermore, the Poincaré type inequality (P) implies

$$-\int_\Omega \mathcal{L}u(x)u(x)dx = a(u,u) \geq \frac{1}{c}\|u\|^2_{L^2(\Omega)} \qquad \forall u \in \mathcal{D}(\mathcal{L}),$$

which shows that $-\mathcal{L}$ is bounded from below by a positive constant. We conclude that $A = -\mathcal{L}$ is sectorial (see [18], p.19) and generates an analytic semigroup $e^{-At}, t \geq 0$, in $L^2(\Omega)$ by Theorem 1.3.4 in [18]. □

The operator A is positive,

$$\langle -\mathcal{L}u, u\rangle_{L^2(\Omega)} = a(u,u) \geq 0 \qquad \forall u \in \mathcal{D}(\mathcal{L}),$$

and self-adjoint in $L^2(\Omega)$, and the Sobolev type embedding (S) implies that A has compact inverse. Consequently, there exists an orthonormal basis of $L^2(\Omega)$ of eigenfunctions $\psi_j \in H, j \in \mathbb{N}$, of A with eigenvalues

$$0 < \mu_1 \leq \mu_2 \leq \dots, \qquad \mu_j \to \infty \text{ as } j \to \infty.$$

We denote the fractional power spaces associated to A by $X^\alpha = \big(\mathcal{D}(A^\alpha), \langle\cdot,\cdot\rangle_{X^\alpha}\big)$, $\alpha \in \mathbb{R}$. The inner product in X^α is given by $\langle u, v\rangle_{X^\alpha} = \langle A^\alpha u, A^\alpha v\rangle_{X^0}$, $u, v \in \mathcal{D}(A^\alpha)$, where

$$\mathcal{D}(A^\alpha) = \Big\{ \psi = \sum_{j\in\mathbb{N}} c_j\psi_j,\ c_j \in \mathbb{R} \ \Big|\ \sum_{j\in\mathbb{N}} \mu_j^{2\alpha} c_j^2 < \infty \Big\}$$

and

$$A^\alpha\psi = A^\alpha \sum_{j\in\mathbb{N}} c_j\psi_j = \sum_{j\in\mathbb{N}} \mu_j^\alpha c_j \psi_j.$$

In this notation,

$$X^1 = \mathcal{D}(A), \quad X^{\frac{1}{2}} = H, \quad X^0 = L^2(\Omega), \quad X^{-\frac{1}{2}} = H',$$

where H' denotes the dual space of H. Moreover, for $\alpha > \beta$ the embedding $X^\alpha \hookrightarrow X^\beta$ is compact.

The operator A in X^0 can be extended to a positive sectorial operator in X^α with domain $X^{\alpha+1}$ for $\alpha \in [-1, 0]$ and restricted to a positive sectorial operator in X^α with domain $X^{\alpha+1}$ for $\alpha \in [0,\infty)$ (see [3] and Section 6 in [2]). The corresponding semigroups $e^{-At}, t \geq 0$, in X^α and X^β for $-1 \leq \beta < \alpha < \infty$ are obtained from each other by natural restrictions and extensions. Moreover, if $\beta \leq \alpha$ we have $e^{-At}(X^\beta) \subset X^\alpha$ and

$$\left\|e^{-At}\right\|_{L(X^\beta;X^\alpha)} \leq \frac{C_{\alpha,\beta}}{t^{\alpha-\beta}}, \qquad t > 0, \tag{3.5}$$

for some constant $C_{\alpha,\beta} \geq 0$, where $\|\cdot\|_{L(V;W)}$ denotes the norm of a linear operator between the normed spaces V and W (e.g., see Theorem 2.4 in [3]).

3.2. **Local well-posedness of semi-linear problems.** Let V be a Banach space. We consider the abstract semi-linear problem

$$\begin{aligned} &\frac{d}{dt}u(t) = Bu(t) + F(u(t)) \qquad\qquad t > 0, \\ &u|_{t=0} = u_0, \qquad u_0 \in V, \end{aligned} \tag{3.6}$$

where B is the infinitesimal generator of a strongly continuous semigroup $S(t), t \geq 0$, and the nonlinearity $F : V \to V$ is *Lipschitz continuous on bounded subsets* of V, i.e., there exists a non-decreasing function $L : [0, \infty) \to [0, \infty)$ such that

$$\|F(u) - F(v)\|_V \leq L(r)\|u - v\|_V \qquad \forall u, v \in B_r(0),$$

where $B_r(0)$ denotes the ball of radius $r > 0$ and center 0 in V.

Definition 4. We call u a *local mild solution* of the initial value problem (3.6), if there exists $T > 0$ such that $u \in C([0, T); V)$, $u(0) = 0$ and u satisfies the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t - s)F(u(s))ds, \qquad t \in [0, T).$$

For the proof of the following result we refer to Theorem 1.4, Chapter 6, in [30].

Theorem 5. *Under the hypotheses above, for every initial data $u_0 \in V$ there exists a unique local mild solution of the initial value problem* (3.6)*, which is defined on the maximal interval of existence* $[0, T), T > 0$*, and either $T = \infty$, or, if $T < \infty$, then*

$$\lim_{t\to T} \|u(t)\|_V = \infty.$$

If the operator B is positive sectorial, i.e., the generated semigroup is analytic, the solutions have stronger regularity properties (see Theorem 2.1.1 and Corollary 2.3.1 in [9]).

Theorem 6. *Let the operator B be positive sectorial and the nonlinearity $F : X^\alpha \to X$ be Lipschitz continuous on bounded subsets of X^α, for some $\alpha \in [0, 1)$. Then, for every $u_0 \in X^\alpha$ there exists a unique local mild solution $u \in C([0, T); X^\alpha) \cap C^1((0, T); X)$ of* (3.6) *defined on the maximal interval of existence* $[0, T), T > 0$. *Moreover, either $T = \infty$, or, if $T < \infty$, then*

$$\lim_{t\to T} \|u(t)\|_{X^\alpha} = \infty,$$

and u satisfies

$$u \in C((0, T); X^1), \qquad \dot{u} \in C((0, T); X^\gamma) \quad \forall \gamma \in [0, 1).$$

3.3. **Global attractors of infinite dimensional dynamical systems.** Let $S(t), t \geq 0$, be a semigroup in the Banach space $(V, \|\cdot\|_V)$. For a subset $B \subset V$ we define the positive orbit of B by

$$\gamma^+(B) := \bigcup_{t\geq 0} S(t)B,$$

and more generally, for $\tau \geq 0$ we define the orbit of B after time τ by

$$\gamma^+_\tau(B) := \gamma^+(S(\tau)B).$$

The semigroup $S(t), t \geq 0$, is *asymptotically compact* if for every bounded subset $B \subset V$ such that $\gamma^+_\tau(B)$ is bounded for some $\tau \geq 0$, the set

$$\{S(t_n + \tau)v_n,\ n \in \mathbb{N}\}$$

is relatively compact for all sequences v_n in B and $t_n \geq 0$ such that $t_n \to \infty$ as $n \to \infty$.

A *Lyapunov functional* for the semigroup $S(t), t \geq 0$, is a continuous function $\Phi : V \to \mathbb{R}$ such that

$$\Phi(S(t)v) \leq \Phi(v) \qquad \forall t \geq 0, \forall v \in V,$$
$$\Phi(S(t)v) = \Phi(v) \qquad \forall t \geq 0 \text{ implies that } v \text{ is an equilibrium point.}$$

If $S(t), t \geq 0$, possesses a Lyapunov functional we call it a *gradient semigroup.*

For the proof of the following theorem about the existence of global attractors for gradient semigroups we refer to Theorem 4.6 and Proposition 2.19, [31].

Theorem 7. *Let $S(t), t \geq 0$, be an asymptotically compact gradient semigroup such that for every bounded subset $B \subset V$ there exists $\tau \geq 0$ such that the orbit $\gamma_\tau^+(B)$ is bounded. If the set of equilibrium points $\mathcal{E}$ is bounded, then the global attractor exists, is connected and $\mathcal{A} = \mathcal{W}^u(\mathcal{E})$.*

The invariance principle of LaSalle (see Proposition 4.2, [31]) characterizes the longtime behavior of trajectories.

Proposition 8. *Let $S(t), t \geq 0$, be a gradient semigroup in V with Lyapunov functional $\Phi : V \to \mathbb{R}$ and let $u \in V$. If the orbit $\gamma_\tau^+(u)$ is relatively compact in V for some $\tau \geq 0$, then the limit $\lim_{t\to\infty} \Phi(S(t)u) = a$ exists and $\Phi(v) = a$ for all $v \in \omega(u)$. Moreover, $\omega(u) \subset \mathcal{E}$, $\mathcal{E} \neq \emptyset$ and*

$$\lim_{t\to\infty} \operatorname{dist}_H\big(S(t)u, \mathcal{E}\big) = 0.$$

4. Semilinear degenerate parabolic problems

The local well-posedness of Problem (1.1) and Theorem 1 can be shown by slightly extending the arguments in [23]; for convenience of the reader we present a sketch of the proof and summarize the main ideas.

4.1. **Local well-posedness.**

Definition 9. We call u a *local weak solution* of (1.1) if there exists $T > 0$ such that

$$u \in C\big([0,T); H\big), \quad u(0) = u_0, \quad u \in C^1\big((0,T); H'\big),$$

and u satisfies the equation

$$\frac{d}{dt}\langle u(t), v\rangle_{L^2(\Omega)} = a(u(t), v) + \langle f(u), v\rangle_{L^2(\Omega)} \qquad \forall v \in H, t \in (0,T).$$

Theorem 10. *We assume the operator $\mathcal{L}$ is X-elliptic with respect to the family of vector fields $X = \{X_1, \dots, X_m\}$, and the properties* (S) *and* (F1) *are satisfied. Then, for every initial data $u_0 \in H$ there exists a unique local solution defined on the maximal interval of existence $[0,T)$, $T > 0$, and*

$$u \in C([0,T); H) \cap C^1((0,T); H).$$

Furthermore, either $T = \infty$, or if $T < \infty$, then

$$\lim_{t\to T} \|u(t)\|_H = \infty,$$

and the solution satisfies the variation of constants formula,

$$u(t) = e^{\mathcal{L}t}u_0 + \int_0^t e^{\mathcal{L}(t-s)} f(u(s))ds \qquad t \in [0,T),$$

where $u(t) := u(\cdot, t; u_0)$ *denotes the solution of* (1.1) *corresponding to initial data* $u_0 \in H$.

Sketch of the proof. Applying Young's inequality if necessary, we can assume that f satisfies Hypothesis (F1) with exponent $\frac{q-2}{2} < \rho < q-2$. It then follows from complex interpolation, as in Lemma 1 in [23], that there exists $\alpha \in (0, \frac{1}{2})$ such that the mapping f is locally Lipschitz continuous on bounded subsets from $H = X^{\frac{1}{2}}$ to $X^{-\alpha}$. The operator $A = -\mathcal{L}$ can be extended to a positive sectorial operator in $X^{-\alpha}$ with domain $X^{1-\alpha}$ (see Subsection 3.2), and we observe that $X^{1-\alpha} \hookrightarrow X^{\frac{1}{2}} \hookrightarrow X^{-\alpha}$. The local existence, uniqueness and regularity of solutions then follows by considering the problem

$$\begin{aligned} \partial_t u &= -Au + f(u) \qquad\qquad t > 0, \\ u|_{t=0} &= u_0, \quad u_0 \in X^{\frac{1}{2}}, \end{aligned}$$

in $X^{-\alpha}$ and applying Theorem 6 (see also the proof of Theorem 2 in [23]). □

Remark 4. If the non-linearity satisfies the growth restriction (F1) with exponent $0 \le \rho < \frac{q-2}{2}$, the proof of Theorem 10 simplifies. It follows from Hölder's inequality and the Sobolev embeddings (S) that f is Lipschitz continuous on bounded subsets from $X^{\frac{1}{2}}$ to X^0. In this case, the local well-posedness of Problem (1.1) and the regularity of solutions is an immediate consequence of Theorem 6.

4.2. **Global existence and longtime behavior of solutions.** The global existence of solutions is a consequence of the hypothesis (F2) and can be shown by considering the Lyapunov functional $\Phi : X^{\frac{1}{2}} \to \mathbb{R}$,

$$\Phi(u) := \int_\Omega \left(\frac{1}{2} |\nabla u(x)|^2 - F(u(x)) \right) dx,$$

where $F(u) := \int_0^u f(s)ds$ denotes the primitive of f. If u is a weak solution of (1.1), then $\Phi(u(\cdot)) \in C\big([0,T);\mathbb{R}\big) \cap C^1\big((0,T);\mathbb{R}\big)$ by Theorem 10, and

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|^2_{L^2(\Omega)} < \infty, \qquad t \in (0,T).$$

Using the growth restriction (F1), the sign condition (F2) and Young's inequality we obtain an estimate of the form

$$C_1 \left(1 + \|u(t)\|^2_{X^{\frac{1}{2}}} \right) \le \Phi(u(t)) \le \Phi(u_0) \le C_2 \left(1 + \|u_0\|^2_{X^{\frac{1}{2}}} + \|u_0\|^{\rho+2}_{L^{\rho+2}(\Omega)} \right),$$

for some constants $C_1, C_2 \ge 0$ (see [23]). The Sobolev embeddings (S) now imply that solutions are uniformly bounded in $H = X^{\frac{1}{2}}$ for $t > 0$ and therefore exist globally.

We denote by $S(t), t \ge 0$, the semigroup in H generated by Problem (1.1),

$$S(t)u_0 := u(t; u_0), \qquad t \ge 0,$$

where $u \in C\big([0,\infty);H\big) \cap C^1\big((0,\infty);H\big)$ denotes the global weak solution corresponding to initial data $u_0 \in H$.

Sketch of the proof of Theorem 1. We only give a sketch of the proof, for all details we refer to the proof of Theorem 3 in [23].

We observed that solutions of Problem (1.1) exist globally, that the generated semigroup $S(t), t \geq 0$, in H possesses a Lyapunov functional and orbits of bounded sets are bounded. The sign condition (F2) implies that the set of equilibrium points

$$\mathcal{E} = \{u \in H : \mathcal{L}u + f(u) = 0\}$$

is bounded in H. Moreover, it can be shown as in Lemma 2 in [23], that the semigroup $S(t), t \geq 0$, satisfies the smoothing property in bounded subsets $D \subset H$: For every $t^* > 0$ there exists a constant $\kappa^* > 0$ such that

$$\|S(t^*)u_0 - S(t^*)v_0\|_H \leq \kappa^* \|u_0 - v_0\|_{L^2(\Omega)} \qquad \forall\, u_0, v_0 \in D. \tag{4.7}$$

This estimate implies that the semigroup is asymptotically compact in H. The existence of the global attractor, its connectedness and structure, $\mathcal{A} = \mathcal{W}^u(\mathcal{E})$, now follow by Theorem 7.

The finite fractal dimension of the global attractor is a consequence of the smoothing property (4.7) and Lemma 3 in [23], which is a special case of the result we prove in B. In particular, for every $\nu \in (0, \frac{1}{2})$ the fractal dimension is bounded by

$$\dim_f(\mathcal{A}) \leq \log_{\frac{1}{2\nu}} \left(N^{L^2(\Omega)}_{\frac{\nu}{\kappa^*}} \left(B^H_1(0) \right) \right).$$

The convergence of trajectories to equilibrium solutions can be deduced from the smoothing property and by applying the invariance principle of LaSalle (Proposition 8).

□

5. Semilinear degenerate hyperbolic problem

5.1. Local well-posedness.

Definition 11. We call u a *local weak solution* of (1.2) if there exists $T > 0$ such that

$$u \in C\big([0,T); H\big) \cap C^1((0,T); L^2(\Omega)) \cap C^2((0,T); H'),$$
$$u(0) = u_0, \ u_t(0) = u_1,$$

and u satisfies the equation

$$\frac{d^2}{dt^2}\langle u(t), v\rangle_{L^2(\Omega)} + \beta \frac{d}{dt}\langle u(t), v\rangle_{L^2(\Omega)} = a(u(t), v) + \langle f(u), v\rangle_{L^2(\Omega)},$$

for all $v \in H$, $t \in (0,T)$.

Problem (1.2) is equivalent to the first order system (1.4), which is locally well-posed in $V = H \times L^2(\Omega)$ if f satisfies the growth restrictions (F1').

Theorem 12. *We assume the operator $\mathcal{L}$ is X-elliptic with respect to the family of vector fields $X = \{X_1, \dots, X_m\}$, and the properties* (S) *and* (F1') *are satisfied. Then, for every initial data $w_0 \in V$ there exists a unique local solution of Problem* (1.4) *defined on the maximal interval of existence* $[0,T)$, $T > 0$, *and $w \in C([0,T); V)$. Furthermore, either $T = \infty$, or if $T < \infty$, then*

$$\lim_{t \to T} \|w(t)\|_V = \infty,$$

and the solution satisfies the variation of constants formula,

$$w(t) = e^{\hat{A}t}w_0 + \int_0^t e^{\hat{A}(t-s)}(\hat{f}(w(s)))ds, \qquad t \in [0,T),$$

where $w(t) := w(\cdot, t; w_0)$ *denotes the solution of* (1.4) *corresponding to initial data* $w_0 \in V$.

Proof. The domain of the operator $\hat{A}$ in V is $\mathcal{D}(\hat{A}) = \mathcal{D}(\mathcal{L}) \times H$. We define A_1 and A_2 by

$$\hat{A} = A_1 + A_2 = \begin{pmatrix} 0 & Id \\ -A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\beta \end{pmatrix},$$

where the operator $A_2 : V \to V$ is linear and bounded. Since A is self-adjojnt, the operator A_1 is dissipative. Indeed, if $w = (u, v) \in \mathcal{D}(A_1) = \mathcal{D}(\hat{A})$, then

$$\langle w^T, A_1 w^T \rangle_V = \langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} v \\ -Au \end{pmatrix} \rangle_V = a(u,v) + \langle v, -Au \rangle_{L^2(\Omega)} = 0.$$

By Corollary 4.4, Chapter 1, in [30] the operator A_1 generates a strongly continuous semigroup of contractions in V. Moreover, Hölder's inequality and the embedding $H \hookrightarrow L^q(\Omega)$ imply that f is Lipschitz continuous on bounded subsets from H to $L^2(\Omega)$, and therefore, the mapping $\hat{f} : V \to V$ is Lipschitz continuous on bounded subsets. The local well-posedness of Problem (1.4) now follows from Theorem 5. □

5.2. **Global existence and longtime behavior.** We first show the exponential decay of solutions of the linear homogeneous problem

$$\partial_t w = \hat{A}w \qquad t > 0, \qquad w|_{t=0} = w_0, \quad w_0 \in V. \tag{5.8}$$

Lemma 13. *Let* $C(t) = e^{\hat{A}t}, t \geq 0$, *be the semigroup in* V *generated by Problem* (5.8)*. Then, there exist constants* $C \geq 0$ *and* $\omega > 0$ *such that*

$$\|C(t)\|_{L(V;V)} \leq Ce^{-\omega t} \qquad \text{for all } t \geq 0.$$

Proof. We define the functional $\mathcal{F} : V \to \mathbb{R}$ by

$$\mathcal{F}(\phi,\psi) := \frac{1}{2}a(\phi,\phi) + \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} + 2b\langle\phi,\psi\rangle_{L^2(\Omega)},$$

where the constant $b > 0$ will be chosen below. If $(\phi,\psi) \in V$ and if we take $b < \frac{1}{4}\min\{1,\mu_1\}$, Poincaré's inequality and the X-ellipticity of $\mathcal{L}$ imply that

$$\begin{aligned} \mathcal{F}(\phi,\psi) &\leq \frac{1}{2}a(\phi,\phi) + \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} + b(\|\phi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)}) \\ &\leq (\frac{1}{2} + \frac{b}{\mu_1})a(\phi,\phi) + (\frac{1}{2} + b)\|\psi\|^2_{L^2(\Omega)} \leq \frac{3}{4}\|(\phi,\psi)\|^2_V, \end{aligned} \tag{5.9}$$

and, on the other hand,

$$\begin{aligned} \mathcal{F}(\phi,\psi) &\geq \frac{1}{2}a(\phi,\phi) + \frac{1}{2}\|\psi\|^2_{L^2(\Omega)} - b(\|\phi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)}) \\ &\geq (\frac{1}{2} - \frac{b}{\mu_1})a(\phi,\phi) + (\frac{1}{2} - b)\|\psi\|^2_{L^2(\Omega)} \geq \frac{1}{4}\|(\phi,\psi)\|^2_V. \end{aligned} \tag{5.10}$$

Here, μ_1 denotes the first eigenvalue of the operator A and $\frac{1}{\mu_1}$ is the optimal constant in the Poincaré inequality (P). These estimates show that $\mathcal{F}$ defines an equivalent norm on V.

If $w(t) = (u(t), v(t))$ is a solution of (5.8) we observe that

$$0 = \langle v, v_t\rangle_{L^2(\Omega)} + a(u, v) + \beta\|v\|^2_{L^2(\Omega)},$$
$$0 = \langle u, v_t\rangle_{L^2(\Omega)} + a(u, u) + \beta\langle u, v\rangle_{L^2(\Omega)}.$$

Using these identities it follows by Poincaré's and Young's inequality that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(u, v) &= a(u, v) + \langle v, v_t\rangle_{L^2(\Omega)} + 2b\langle u_t, v\rangle_{L^2(\Omega)} + 2b\langle u, v_t\rangle_{L^2(\Omega)} \\
&= -2ba(u, u) - \big(\beta - 2b\big)\|v\|^2_{L^2(\Omega)} - 2b\beta\langle u, v\rangle_{L^2(\Omega)} \\
&\le -2ba(u, u) - \big(\beta - 2b\big)\|v\|^2_{L^2(\Omega)} + b\beta\big(\frac{\mu_1}{\beta}\|u\|^2_{L^2(\Omega)} + \frac{\beta}{\mu_1}\|v\|^2_{L^2(\Omega)}\big) \\
&\le -2ba(u, u) - \big(\beta - 2b\big)\|v\|^2_{L^2(\Omega)} + b\beta\big(\frac{1}{\beta}a(u, u) + \frac{\beta}{\mu_1}\|v\|^2_{L^2(\Omega)}\big) \\
&\le -ba(u, u) - (\beta - 2b - \frac{b\beta^2}{\mu_1})\|v\|^2_{L^2(\Omega)} \le -b\|(u, v)\|^2_V,
\end{aligned}$$

where we chose $b \le \frac{\mu_1\beta}{3\mu_1+\beta^2}$. Setting $\alpha = \min\{\frac{1}{4}, \frac{\mu_1}{4}, \frac{\mu_1\beta}{3\mu_1+\beta^2}\}$ we obtain

$$\frac{d}{dt}\mathcal{F}(u, v) \le -\alpha\|(u, v)\|^2_V \le -\frac{4}{3}\alpha\mathcal{F}(u, v).$$

Gronwall's Lemma now implies that

$$\mathcal{F}(u, v) \le \mathcal{F}(u_0, v_0)e^{-\alpha\frac{4}{3}t}, \qquad t \ge 0,$$

and the norm equivalence yields the exponential decay of solutions. □

The sign conditions (F2) ensure the global existence of solutions. We define the functional $\Phi : V \to \mathbb{R}$ by

$$\Phi(w) = \Phi((u, v)) = \frac{1}{2}a(u, u) + \int_\Omega \left(\frac{1}{2}(v(x))^2 - F(u(x))\right) dx,$$

where F denotes the primitive of f. If $w = (u, v) \in C([0, T); V)$ is a solution of Problem (1.4), then $\Phi(w(\cdot)) \in C^1((0, T); \mathbb{R})$ and

$$\frac{d}{dt}\Phi(w(t)) = -\beta\|v(t)\|^2_{L^2(\Omega)}, \qquad t > 0.$$

Using the growth restriction (F1'), the sign condition (F2) and Young's inequality we obtain an estimate of the form

$$\begin{aligned}
C_1\left(1 + \|(u(t), v(t))\|^2_V\right) &\le \Phi(w(t)) \\
&\le \Phi(w_0) \le C_2\left(1 + \|(u_0, v_0)\|^2_V + \|u_0\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}\right),
\end{aligned}$$

for some constants $C_1, C_2 \ge 0$ (see Section 4.2 in [23] and Section 5.2). The Sobolev embeddings (S) and the X-ellipticity of the operator $\mathcal{L}$ now imply that solutions are uniformly bounded in V for $t > 0$ and therefore exist globally.

We denote by $U(t), t \ge 0$, the semigroup in V generated by Problem (1.4),

$$U(t)w_0 := w(t; w_0), \qquad t \ge 0,$$

where $w \in C([0, \infty); V)$ denotes the unique global solution corresponding to initial data $w_0 \in V$. Using the variation of constants formula U can be represented as sum

$U = C + S$, where the semigroup C corresponds to the linear homogeneous problem (5.8) and decays exponentially, and the family of operators S is defined by

$$U(t)w_0 = C(t)w_0 + \int_0^t C(t-s)\hat{f}(U(s)w_0)ds = C(t)w_0 + S(t)w_0, \qquad t \geq 0.$$

To show the asymptotic compactness of U and to establish a bound for the fractal dimension of the global attractor we prove in several steps that S satisfies the smoothing property. The proof is based on the method applied in [8].

As in Subsection 3.1, we denote by X^α, $\alpha \in [-1, \infty)$, the fractional power spaces associated to the operator $A = -\mathcal{L}$ with domain $\mathcal{D}(A) = X^1$ in $L^2(\Omega) = X^0$. The solution theory for Problem (1.4) can be extended to the fractional power space $X^\alpha \times X^{\alpha-\frac{1}{2}}$, for some $\alpha \in (0, \frac{1}{2}]$.

Lemma 14. *There exists $\varepsilon \in (0,1)$ such that f is Lipschitz continuous from $X^{\frac{1-\varepsilon}{2}}$ to $L^2(\Omega)$ on bounded subsets of $D \subset H$:*

$$\|f(u) - f(v)\|_{L^2(\Omega)} \leq c_D \|u - v\|_{X^{\frac{1-\varepsilon}{2}}} \qquad \forall\ u, v \in D,$$

for some constant $c_D \geq 0$.

Proof. Let $u, v \in D$. The growth restrictions (F1') and Hölder's inequality with $p' = \frac{q}{2\gamma}$ and $q' = \frac{q}{q-2\gamma}$ imply

$$\begin{aligned}
&\|f(u) - f(v)\|_{L^2(\Omega)} \leq c\|(1 + |u|^\gamma + |v|^\gamma)(u - v)\|_{L^2(\Omega)} \\
&\leq c\big(\|u - v\|_{L^2(\Omega)} + (\|u\|^\gamma_{L^{2\gamma p'}(\Omega)} + \|v\|^\gamma_{L^{2\gamma p'}(\Omega)})\|u - v\|_{L^{2q'}(\Omega)}\big) \\
&\leq c\|u - v\|_{L^{2q'}(\Omega)},
\end{aligned}$$

where we used the embeddings $H \hookrightarrow L^{2\gamma p'}(\Omega) = L^q(\Omega)$ and $L^{2q'}(\Omega) \hookrightarrow L^2(\Omega)$ in the last inequality. Here, c denotes a non-negative constant that may vary from line to line.

If we define $\varepsilon := 1 - \frac{2\gamma}{q-2}$, then $\varepsilon \in (0, 1)$. Moreover, the identity E is a bounded linear operator from $X^0 = L^2(\Omega)$ to $L^2(\Omega)$ and from $X^{\frac{1}{2}}$ to $L^q(\Omega)$ by our assumption (S) and the X-ellipticity of the operator $\mathcal{L}$. Using complex interpolation we conclude that

$$E : [X^0, X^{\frac{1}{2}}]_{1-\varepsilon} = X^{\frac{1-\varepsilon}{2}} \to [L^2(\Omega), L^q(\Omega)]_{1-\varepsilon} = L^{2q'}(\Omega), \qquad \frac{1}{2q'} = \frac{\varepsilon}{2} + \frac{1-\varepsilon}{q},$$

is linear and bounded (see Section II.2.1. in [34], Example 7.56 in [1] and Proposition 1.3.9 in [9]). Using this embedding in the above inequality follows the statement of the lemma. □

Lemma 15. *Let $\varepsilon := 1 - \frac{2\gamma}{q-2}$ be as in Lemma 14 and $V^\varepsilon := X^{\frac{1-\varepsilon}{2}} \times X^{-\frac{\varepsilon}{2}}$. Then, for every initial data $w_0 = (u_0, v_0) \in V^\varepsilon$ there exists a unique local solution $w \in C([0, T); V^\varepsilon)$ of Problem (1.4).*

In particular, the semigroup C_ε generated by the linear homogeneous problem (5.8) in V^ε is the extension of the semigroup C and uniformly bounded in V^ε,

$$\|C_\varepsilon(t)\|_{L(V^\varepsilon;V^\varepsilon)} \leq d \qquad t \geq 0,$$

for some constant $d \geq 0$.

Proof. We consider the operator

$$\hat{A} = A_1 + A_2 = \begin{pmatrix} 0 & Id \\ -A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\beta \end{pmatrix},$$

in V^ε, where $A_2 : V^\varepsilon \to V^\varepsilon$ is linear and bounded and A is considered as an operator in $X^{-\frac{\varepsilon}{2}}$ with domain $X^{\frac{1-\varepsilon}{2}}$ (see Subsection 3.1). Since A is selfadjoint in $X^{-\frac{\varepsilon}{2}}$ the operator A_1 is dissipative in V^ε. Indeed, if $w = (u, v) \in \mathcal{D}(A_1) = X^{\frac{2-\varepsilon}{2}} \times X^{\frac{1-\varepsilon}{2}}$, then

$$\begin{aligned} \langle w^T, A_1 w^T \rangle_{V^\varepsilon} &= \langle (\begin{smallmatrix} u \\ v \end{smallmatrix}), (\begin{smallmatrix} v \\ -Au \end{smallmatrix}) \rangle_{V^\varepsilon} \\ &= \langle A^{\frac{1-\varepsilon}{2}} u, A^{\frac{1-\varepsilon}{2}} v \rangle_{X^0} + \langle A^{-\frac{\varepsilon}{2}} v, A^{-\frac{\varepsilon}{2}}(-Au) \rangle_{X^0} \\ &= \langle A^{\frac{1-\varepsilon}{2}} u, A^{\frac{1-\varepsilon}{2}} v \rangle_{X^0} - \langle A^{\frac{1-\varepsilon}{2}} v, A^{\frac{1-\varepsilon}{2}} u \rangle_{X^0} = 0. \end{aligned}$$

By Corollary 4.4, Chapter 1, in [30] the operator A_1 generates a strongly continuous semigroup of contractions in V^ε. Moreover, if $B \subset X^{\frac{1-\varepsilon}{2}}$ is bounded, Lemma 14 implies that

$$\|f(u) - f(v)\|_{X^{-\frac{\varepsilon}{2}}} \le C_1 \|f(u) - f(v)\|_{X^0} \le C_2 \|u - v\|_{X^{\frac{1-\varepsilon}{2}}}, \qquad \forall\, u, v \in B,$$

for some constants $C_1, C_2 \ge 0$. Consequently, $\hat{f}$ is Lipschitz continuous on bounded subsets in V^ε, and the lemma follows from Theorem 5. □

Lemma 16. *Let* $\varepsilon = 1 - \frac{2\gamma}{q-2}$ *and* $V^\varepsilon = X^{\frac{1-\varepsilon}{2}} \times X^{-\frac{\varepsilon}{2}}$. *Then, the embedding* $V \hookrightarrow V^\varepsilon$ *is compact and* S *satisfies the smoothing property in bounded subsets* $D \subset V$*: For every* $t_* > 0$ *there exists a constant* $\kappa_* > 0$ *such that*

$$\|S(t_*)w - S(t_*)z\|_V \le \kappa_* \|w - z\|_{V^\varepsilon} \qquad \forall w, z \in D.$$

Proof. Using Lemma 15 we first prove the Lipschitz continuity of the semigroup U in V^ε. For initial data $w, z \in D$ we denote the corresponding solutions of (1.4) by

$$U(t)w = \big(U_1(t)w, U_2(t)w\big),\ U(t)z = \big(U_1(t)z, U_2(t)z\big), \qquad t \ge 0.$$

The variation of constants formula implies

$$\begin{aligned} \|U(t)w - U(t)z\|_{V^\varepsilon} &\le \|C(t)\|_{L(V^\varepsilon;V^\varepsilon)} \|w - z\|_{V^\varepsilon} \\ &\quad + \int_0^t \|C(t-s)\|_{L(V^\varepsilon;V^\varepsilon)} \|\hat{f}(U(s)w) - \hat{f}(U(s)z)\|_{V^\varepsilon} ds \\ &\le d\Big(\|w - z\|_{V^\varepsilon} + \int_0^t \|f(U_1(s)w) - f(U_1(s)z)\|_{X^{-\frac{\varepsilon}{2}}} ds\Big) \\ &\le d\Big(\|w - z\|_{V^\varepsilon} + \int_0^t C\|U_1(s)w - U_1(s)z\|_{X^{\frac{1-\varepsilon}{2}}} ds\Big) \\ &\le d\Big(\|w - z\|_{V^\varepsilon} + \int_0^t C\|U(s)w - U(s)z\|_{V^\varepsilon} ds\Big), \end{aligned}$$

for some constant $C \ge 0$, and the Lipschitz continuity follows by Gronwall's Lemma,

$$\|U(t)w - U(t)z\|_{V^\varepsilon} \le d\|w - z\|_{V^\varepsilon} e^{dCt}, \qquad t > 0.$$

Let now $t_* > 0$. In the following, c will denote a non-negative constant that may vary from line to line. We obtain

$$\begin{aligned}\|S(t_*)w - S(t_*)z\|_V &\le \int_0^{t_*} \|C(t_* - s)\big(\hat{f}(U(s)w) - \hat{f}(U(s)z))\big)\|_V ds \\ &\le c\int_0^{t_*} e^{-\omega(t_*-s)}\|f(U_1(s)w - f(U_1(s)z)\|_{L^2(\Omega)} ds \\ &\le c\int_0^{t_*} \|U_1(s)w - U_1(s)z\|_{X^{\frac{1-\varepsilon}{2}}} ds \\ &\le c\int_0^{t_*} \|U(s)w - U(s)z\|_{V^\varepsilon} ds \\ &\le c\int_0^{t_*} de^{dCs}\|w - z\|_{V^\varepsilon} ds = \kappa_*\|w - z\|_{V^\varepsilon},\end{aligned}$$

for some constant $\kappa_* > 0$, where we used Lemma 14 in the third estimate. □

Finally, we formulate the proof of Theorem 2, where we essentially use the following observation. Let $B \subset V$ be a bounded subset. By Lemma 16 and Lemma 13 we conclude that there exists $T_* > 0$ and constants $\lambda \in [0, \frac{1}{2})$ and $\kappa \ge 0$ such that

$$\begin{aligned}\|U(T_*)w - U(T_*)z\|_V &\le \|S(T_*)w - S(T_*)z\|_V + \|C(T_*)w - C(T_*)z\|_V \\ &\le \kappa\|w - z\|_{V^\varepsilon} + \lambda\|w - z\|_V,\end{aligned} \tag{5.11}$$

for all $w, z \in B$.

Proof of Theorem 2. Existence of the global attractor: We proved that the semigroup $U(t), t \ge 0$, possesses a Lyapunov functional and orbits of bounded sets are bounded. It remains to show that the set of equilibria is bounded in V and that $U(t), t \ge 0$, is asymptotically compact (see Theorem 7).

We observe that the sign condition (F2) implies that there exist constants $0 \le c_0 < \mu_1$ and $c_1 \in \mathbb{R}$ such that

$$uf(u) \le c_1|u| + c_0u^2, \qquad u \in \mathbb{R}.$$

Let $u \in \mathcal{E} = \{u \in H : \mathcal{L}u + f(u) = 0\}$. Multiplying the equation by u and using Young's and Poincaré's inequality we obtain

$$\begin{aligned}0 = -a(u,u) + \int_\Omega f(u(x))u(x)dx &\le -a(u,u) + \int_\Omega \big(c_1|u(x)| + c_0|u(x)|^2\big)dx \\ \le -a(u,u)\left(1 - \frac{c_0 + \varepsilon}{\mu_1}\right) + C_\varepsilon,\end{aligned}$$

for $\varepsilon > 0$ and some constant $C_\varepsilon \ge 0$. Since $c_0 < \mu_1$ this estimate implies that the set $\mathcal{E}$ is bounded in V.

To prove the asymptotic compactness of $U(t), t \ge 0$, we assume that $B \subset V$ is a bounded subset. Let $(x_n)_{n\in\mathbb{N}} \subset B$ and $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ be sequences such that $t_n \to \infty$ as $n \to \infty$. Since orbits of bounded sets are bounded, the set $\{U(t_n)x_n | \, n \in \mathbb{N}\}$ is bounded in V and consequently, there exists a subsequence $(U(t_{n_k})x_{n_k})_{k\in\mathbb{N}}$ converging weakly in V and strongly in V^ε. Let $m \in \mathbb{N}$ and $T_* > 0$

be as in (5.11). Then, there exists $N_0 \in \mathbb{N}$ such that $t_{n_l}, t_{n_k} \geq mT_*$ for all $k, l \geq N_0$. Moreover, we obtain

$$\begin{aligned}
&\|U(t_{n_k})x_{n_k} - U(t_{n_l})x_{n_l}\|_V \\
&\leq \kappa\|U(t_{n_k} - T_*)x_{n_k} - U(t_{n_l} - T_*)x_{n_l}\|_{V^\varepsilon} \\
&\quad + \lambda\|U(t_{n_k} - T_*)x_{n_k} - U(t_{n_l} - T_*)x_{n_l}\|_V \\
&\leq \kappa C(1 + \lambda + \cdots + \lambda^{m-1})\|x_{n_k} - x_{n_l}\|_{V^\varepsilon} + C\lambda^m\|x_{n_k} - x_{n_l}\|_V,
\end{aligned}$$

for some constant $C \geq 0$, where we used (5.11) and the Lipschitz continuity of U in V and V^ε. This implies that $(U(t_{n_k})x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in V and shows the asymptotic compactness of the semigroup $U(t), t \geq 0$.

Fractal dimension of the global attractor: The global attractor $\mathcal{A}$ is compact and invariant, and the semigroup $U(t), t \geq 0$, can be decomposed as $U = S + C$, where the operators S satisfy the smoothing property in $\mathcal{A}$ and the semigroup C is a contraction in V. Proposition 18 applied to the semigroup $U(t), t \geq 0$, with V and $W = V^\varepsilon$ implies the finite fractal dimension of $\mathcal{A}$. In particular, for every $\nu \in (0, \frac{1}{2} - \lambda)$, the fractal dimension of the global attractor is bounded by

$$\dim_f(\mathcal{A}) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N_{\frac{\nu}{\kappa}}^{V^\varepsilon}(B_1^V(0))\right),$$

where $\kappa > 0$ and $\lambda \in [0, \frac{1}{2})$ denote the constants in (5.11).

Convergence to stationary states: We deduce the last statement of the theorem from the invariance principle of LaSalle (Proposition 8). Let $w_0 \in V$. It suffices to show that the orbit

$$\gamma^+(w_0) = \bigcup_{t\geq 0} U(t)w_0$$

is relatively compact in V. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\gamma^+(w_0)$, then without loss of generality $x_n = U(t_n)w_0$, where $t_n \to \infty$ as $n \to \infty$. Since the orbit $\gamma^+(w_0)$ is bounded in V there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that weakly converges in V and strongly in V^ε. We iteratively apply the decomposition (5.11) and conclude as above that the sequence $(U(t_{n_k})w_0)_{k\in\mathbb{N}}$ is Cauchy in V, which proves the precompactness of the orbit $\gamma^+(w_0)$ and concludes the proof of the theorem. □

Appendix A. Poincaré type inequality

In this appendix we prove the Poincaré inequality (P) for operators that are X-elliptic with respect to the family of vector fields

$$X = \{X_1, \ldots, X_N\} = \{\eta_1\partial_{x_1}, \ldots, \eta_N\partial_{x_N}\}$$

in Subsection 2.2. In particular, the inequality follows from the X-ellipticity of the operator $\mathcal{L}$ and the following proposition:

Proposition 17. *Let $\Omega \subset \mathbb{R}^N$ be a domain, which is bounded in the x_1-direction, and $p \in (1, \infty)$. Then, there exists a constant $c \geq 0$ such that*

$$\|u\|_{L^p(\Omega)}^p \leq c\int_\Omega |Xu|^p \qquad \forall u \in C_0^1(\Omega).$$

Proof. Let $u \in C_0^1(\Omega)$. Without loss of generality we can assume that the support of u is contained in $\left(-\frac{M}{2}, \frac{M}{2}\right) \times \mathbb{R}^{N-1}$, for some $M > 0$. Let $e_1 = (1, 0, \ldots, 0)$ and

$x \in \Omega$. Then,

$$u(x) = u(x) - u(x + Me_1) = -\int_0^1 \frac{d}{dt}\,(u(x + tMe_1))\,dt$$
$$= -\int_0^1 M\partial_{x_1} u(x + tMe_1)dt,$$

and Hölder's inequality yields

$$|u(x)| \le M \left(\int_0^1 |\partial_{x_1} u(x + tMe_1)|^p dt \right)^{\frac{1}{p}}.$$

Consequently, we obtain

$$\begin{aligned}
\|u\|^p_{L^p(\Omega)} &\le M^p \int_{\mathbb{R}^N} \left(\int_0^1 |\partial_{x_1} u(x + tMe_1)|^p dt \right) dx \\
&= M^p \int_0^1 \left(\int_{\mathbb{R}^N} |\partial_{x_1} u(x + tMe_1)|^p dx \right) dt \\
&= M^p \int_0^1 \left(\int_{\mathbb{R}^N} |\partial_{y_1} u(y)|^p dy \right) dt = M^p \int_0^1 \left(\int_{\Omega} |\partial_{y_1} u(y)|^p dy \right) dt \\
&\le M^p \int_0^1 \left(\int_{\Omega} |\eta_1 \partial_{y_1} u(y)|^p dy \right) dt \le M^p \int_{\Omega} |Xu|^p,
\end{aligned}$$

where we applied the change of variables $x \mapsto y = x + tMe_1$ and used that the function $\eta_1 \ge 1$. □

Appendix B. Fractal dimension of compact invariant sets

In this appendix we prove an auxiliary result that allows to estimate the fractal dimension of global attractors. It can be deduced from the method applied in [7] for the construction of exponential attractors, which is based on the article [12]. We recall that the fractal dimension of a compact subset A of a metric space V is defined as

$$\dim_f(A) := \lim_{\varepsilon \to 0} \frac{\ln(N^V_\varepsilon(A))}{-\ln(\varepsilon)},$$

where $N^V_\varepsilon(A)$ denotes the minimal number of balls in V with radius $\varepsilon > 0$ and centers in A needed to cover the set A.

We assume that $U(t), t \ge 0$, is a semigroup in the Banach space V, and V is dense and compactly embedded into an auxiliary normed space W. Moreover, it exists a compact invariant set $\mathcal{A} \subset V$, and the semigroup can be represented as $U = S + C$, where the families of operators $S(t), t \ge 0$, and $C(t), t \ge 0$, satisfy the following properties:

(H_1) There exists $T^* > 0$ and a constant $\kappa \ge 0$ such that

$$\|S(T^*)u - S(T^*)v\|_V \le \kappa \|u - v\|_W \qquad \forall u, v \in \mathcal{A}.$$

(H_2) There exists a constant $\lambda \in [0, \frac{1}{2})$ such that

$$\|C(T^*)u - C(T^*)v\|_V \le \lambda \|u - v\|_V \qquad \forall u, v \in \mathcal{A}.$$

In the sequel, we denote by $B^V_r(v)$ the ball in a Banach space V with radius $r > 0$ and center $v \in V$.

Proposition 18. *Let the semigroup U, the spaces V, W and the set $\mathcal{A}$ be as above and hypotheses (H_1) and (H_2) be satisfied. Then, for every $\nu \in (0, \frac{1}{2} - \lambda)$ the fractal dimension of $\mathcal{A}$ in V is bounded by*

$$\dim_f(\mathcal{A}) \leq \log_{\frac{1}{2(\nu+\lambda)}}(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))),$$

where $B^V_1(0)$ denotes the unit ball in V and $N^W_\varepsilon(A)$ the minimal number of ε-balls in W needed to cover the set $A \subset V$.

Proof. Without loss of generality we can assume $T^* = 1$. Let $\nu \in (0, \frac{1}{2} - \lambda)$, $R > 0$ and $a \in \mathcal{A}$ be such that $\mathcal{A} \subset B^V_R(a)$. Moreover, let $v_1, \ldots, v_M \in V$ be such that

$$B^V_1(0) \subset \bigcup_{i=1}^M B^W_{\frac{\nu}{\kappa}}(v_i),$$

where M denotes the minimal number of balls with radius $\frac{\nu}{\kappa}$ in W needed to cover the unit ball $B^V_1(0)$. First, we construct by induction the family of sets $V^n, n \in \mathbb{N}$, with the following properties:

$$V^n \subset U(n)\mathcal{A} = \mathcal{A}, \qquad \sharp V^n \leq M^n, \qquad U(n)\mathcal{A} = \mathcal{A} \subset \bigcup_{u \in V^n} B^V_{(2(\nu+\lambda))^n R}(u),$$

where $\sharp A$ denotes the cardinality of the set A.

We define $V^0 := \{v_0\}$ and assume the sets $V^l \subset \mathcal{A}$ have already been constructed for $l \leq n$, which yields the covering

$$\mathcal{A} = U(n)\mathcal{A} \subset \bigcup_{u \in V^n} B^V_{(2(\nu+\lambda))^n R}(u).$$

To construct a covering of the iterate

$$U(n+1)\mathcal{A} = U(1)U(n)\mathcal{A} = U(1) \bigcup_{u \in V^n} \left(B^V_{(2(\nu+\lambda))^n R}(u) \cap \mathcal{A} \right),$$

let $u \in V^n$. We use the covering of the unit ball $B^V_1(0)$ by balls with radius $\frac{\nu}{\kappa}$ in the space W and obtain

$$\mathcal{A} \cap B^V_{(2(\nu+\lambda))^n R}(u) \subset \bigcup_{i=1}^M \left(B^W_{(2(\nu+\lambda))^n R \frac{\nu}{\kappa}}\big((2(\nu+\lambda))^n R v_i + u\big) \cap \mathcal{A} \right) =: \bigcup_{i=1}^M A_i,$$

where we can assume that the sets $A_i, i = 1, \ldots, N$, are non-empty. For elements v, w in the set $\bigcup_{i=1}^M A_i \subset \mathcal{A}$ the smoothing property implies

$$\|S(1)v - S(1)w\|_V \leq \kappa \|v - w\|_W < 2\nu(2(\nu+\lambda))^n R,$$

and we obtain

$$S(1) \left(\mathcal{A} \cap B^V_{(2(\nu+\lambda))^n R}(u) \right) \subset S(1) \bigcup_{i=1}^M A_i \subset \bigcup_{i=1}^M B^V_{2\nu(2(\nu+\lambda))^n R}(y_i),$$

for some $y_1, \ldots, y_M \in S(1)\big(\mathcal{A} \cap B^V_{(2(\nu+\lambda))^n R}(u)\big)$. In particular, there exist $z_1, \ldots, z_M \in \mathcal{A}$ such that $y_i = S(1)z_i, i = 1, \ldots, M$. The contraction property (H_2) implies

$$C(1) \left(\mathcal{A} \cap B^V_{(2(\nu+\lambda))^n R}(u) \right) \subset B^V_{2\lambda(2(\nu+\lambda))^n R}(C(1)z_i) \qquad \forall i = 1, \ldots, M,$$

and we obtain the covering

$$U(1)\left(\mathcal{A}\cap B^V_{(2(\nu+\lambda))^nR}(u)\right) = (S(1)+C(1))\left(\mathcal{A}\cap B^V_{(2(\nu+\lambda))^nR}(u)\right) \subset \bigcup_{i=1}^{M} B^V_{(2(\nu+\lambda))^{n+1}R}(U(1)z_i),$$

with centers $U(1)z_i \in \mathcal{A}$.

Constructing in this way for every $u \in V^n$ such a covering of the set

$$U(1)\left(\mathcal{A}\cap B^V_{(2(\nu+\lambda))^nR}(u)\right)$$

by balls of radius $(2(\nu+\lambda))^{n+1}R$ in V and centers in $\mathcal{A}$, yields a covering of the image $U(n+1)\mathcal{A}=\mathcal{A}$. We denote the new set of centers by V^{n+1} and observe that

$$\sharp V^{n+1} \le M\sharp V^n \le M^{n+1}.$$

Moreover, by construction, the set of centers satisfies $V^{n+1}\subset U(n+1)\mathcal{A}=\mathcal{A}$, and

$$\mathcal{A} = U(n+1)\mathcal{A} \subset \bigcup_{u\in V^{n+1}} B^V_{(2(\nu+\lambda))^{n+1}R}(u).$$

Finally, to prove the finite fractal dimension of $\mathcal{A}$ let $\varepsilon>0$. If we choose m sufficiently large such that

$$(2(\nu+\lambda))^mR \le \varepsilon < (2(\nu+\lambda))^{m-1}R$$

holds, we can estimate the number of ε-balls needed to cover the set $\mathcal{A}$ by

$$N^V_\varepsilon(\mathcal{A}) \le \sharp V^m \le M^m.$$

Furthermore, we have

$$m < \frac{\ln\frac{1}{\varepsilon}}{\ln\frac{1}{2(\nu+\lambda)}} + C,$$

for some constant $C\ge 0$ depending on R and ν, and we obtain for the fractal dimension of $\mathcal{A}$

$$\dim_f(\mathcal{A}) = \limsup_{\varepsilon\to 0}\frac{\ln(N^V_\varepsilon(\mathcal{A}))}{-\ln\varepsilon} \le \limsup_{\varepsilon\to 0}\frac{m\ln(M)}{-\ln\varepsilon} \le \limsup_{\varepsilon\to 0}\frac{\left(\frac{-\ln\varepsilon}{\ln\frac{1}{2(\nu+\lambda)}}+C\right)\ln(M)}{-\ln\varepsilon} \le \log_{\frac{1}{2(\nu+\lambda)}}(M).$$

□

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