ON THE STRUCTURE OF ∞-HARMONIC MAPS

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Abstract. Let $H \in C^2(\mathbb{R}^N \otimes \mathbb{R}^n)$ be a Hamiltonian. Here we study *Aronsson Maps*, that is solutions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the system

$$\mathrm{A}_\infty u \; := \; \Big(H_P \otimes H_P \, + \, H[H_P]^\perp H_{PP}\Big)(Du) : D^2 u \; = \; 0 \tag{1}$$

with emphasis to the special case of 2-d ∞-*Harmonic maps* for $n = 2 \leq N$ with $H(P) = \frac{1}{2}|P|^2$ the Euclidean norm. We also consider the 1-d case of Aronsson ODEs when $H(x, u(x), u'(x))$ depends on all arguments. (1) was first derived in the author's recent work [K3]. By establishing a general Rigidity Theorem for Rank-One Maps of independent interest, we analyse the phase separation of 2-d ∞-Harmonic maps and their interfaces whereon the coefficients of the system become discontinuous. As a corollary, we extend the Aronsson-Evans-Yu theorem on the non-existence of zeros of $|Du|$ for solutions to all $N \geq 2$ and establish a Maximum Principle (Convex Hull Property) for $N = 2$. We further classify all H for which (1) is elliptic: they are the "geometric" ones, which depend on Du via the Riemannian metric $Du^\top Du$. We also study existence, uniqueness and regularity of the initial value problem for Aronsson ODEs.

1. Introduction

Let H be a Hamiltonian in $C^2(\mathbb{R}^N \otimes \mathbb{R}^n)$. In this paper we primarily study *Aronsson maps*, that is solutions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to system

$$\mathrm{A}_\infty u \; := \; \Big(H_\mathrm{P} \otimes H_\mathrm{P} + H[H_\mathrm{P}]^\perp H_\mathrm{PP}\Big)(Du) : D^2 u \; = \; 0. \tag{1.1}$$

In (1.1) $[H_\mathrm{P}(P)]^\perp$ is the projection on the nullspace of $H_\mathrm{P}(P)^\top : \mathbb{R}^N \longrightarrow \mathbb{R}^n$ and $H_P(P) \equiv D_P H(P)$ (for details and notation see Preliminaries 1.1). Particular emphasis is given in the important special case of 2-dimensional ∞-*Harmonic maps* for $n = 2 \leq N$ and $H(P) = \frac{1}{2}|P|^2$, where $|P| = (\mathrm{tr}(P^\top P))^{1/2}$ is the Euclidean norm on $\mathbb{R}^N \otimes \mathbb{R}^n$. In this case (1.1) simplifies to

$$\Delta_\infty u \; := \; \Big(Du \otimes Du + |Du|^2 [Du]^\perp \otimes I\Big) : D^2 u \; = \; 0. \tag{1.2}$$

We also consider the 1-dimensional Aronsson ODE system when $H\big(x, u(x), u'(x)\big)$ depends on all arguments. (1.1) is a quasilinear system in non-divergence form which arises in the limit of the Euler-Lagrange equations of the L^p functionals $\int_\Omega H(Du)^p$ as $p \to \infty$ and was first derived in the author's recent work [K3]. Let us briefly recall the derivation in the simpler case of (1.2). After expansion and normalization of the p-Laplace system $\Delta_p u = \mathrm{Div}\big(|Du|^{p-2} Du\big) = 0$, we have

$$Du \otimes Du : D^2 u \, + \, \frac{|Du|^2}{p-2} \Delta u \; = \; 0. \tag{1.3}$$

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Let $[Du]^\top$ and $[Du]^\perp$ denote the projections on range of Du and nullspace of $Du^\top$ respectively. Since $[Du]^\top + [Du]^\perp = I$, by expanding Δu with respect to these orthogonal projections, we get

$$Du \otimes Du : D^2u \,+\, \frac{|Du|^2}{p-2}[Du]^\top \Delta u \;=\; -\frac{|Du|^2}{p-2}[Du]^\perp \Delta u. \tag{1.4}$$

By perpendicularity, right and left hand side of (1.4) are normal to each other. Hence, they both vanish and (1.4) actually decouples to 2 systems. By renormalizing the right hand side of (1.4) and rearranging, we get

$$Du \otimes Du : D^2u + |Du|^2[Du]^\perp \Delta u \;=\; -\frac{|Du|^2}{p-2}[Du]^\top \Delta u. \tag{1.5}$$

As $p \to \infty$, (1.5) formally leads to (1.2). In the scalar case of $N = 1$, the normal coefficient $|Du|^2[Du]^\perp$ vanishes identically and the same holds for submersions in general. The scalar ∞-Laplacian $\Delta_\infty u = D_i u D_j u D^2_{ij} u = 0$ for $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ has a long history. It was first derived and studied by Aronsson in the '60s in [A3, A4] and has been extensively studied ever since (see for example Crandall [C], Barron, Evans, Jensen [BEJ] and references therein). A major difficulty in its study is its degeneracy and the emergence of singular solutions (see e.g. [A6, A7, K1]). In the last 20 years the scalar PDE has been studied in the context of Viscosity Solutions.

In the vector case even more intriguing phenomena occur, highlighted in [K3]. Firstly, there emerge highly singular solutions to (1.2). For example, for any unit speed curve $f \in C^1(\mathbb{R})^2$, the planar map $u(x, y) := f(x) + if(y)$ is a singular "∞-Harmonic" local diffeomorphism on a neighborhood of the diagonal. Secondly, a further difficulty not present in the scalar case is that (1.1) *has discontinuous coefficients* even for C^∞ solutions. There exist smooth ∞-Harmonic maps whose rank of the gradient is not constant: such an example is given by

$$u(x, y) \,:=\, e^{ix} - e^{iy}\ , \quad u \;:\; \{|x \pm y| < \pi\} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2. \tag{1.6}$$

Indeed, (1.6) is ∞-Harmonic on the rhombus and has $\mathrm{rk}(Du) = 1$ on the diagonal $\{x = y\}$, but it has $\mathrm{rk}(Du) = 2$ otherwise and the projection $[Du]^\perp$ is discontinuous. In general, *∞-Harmonic maps present a phase separation*, with a certain hierarchy. On each phase the dimension of the tangent space is constant and these phases are separated by *interfaces* whereon the rank of Du "jumps" and $[Du]^\perp$ gets discontinuous. On a phase, we interpret (1.2) as decoupling to the *tangential* system $DuD\big(\frac{1}{2}|Du|^2\big) = 0$ in the tangent bundle $[Du]^\top$ and the *normal* system $|Du|^2[Du]^\perp \Delta u = 0$ in the normal bundle $[Du]^\perp$.

In order to compensate the startling property of discontinuous coefficients, we proposed the following natural modification of (1.2):

$$\Gamma_\infty u \;:=\; \Big(Du \otimes Du \,+\, (Ju)^2\,[Du]^\perp \!\otimes I\Big) : D^2u \;=\; 0 \tag{1.7}$$

where J is the *Jacobian* of $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$Ju \;:=\; \left\{ \begin{array}{ll} \det\big(Du^\top Du\big)^{\frac{1}{2}}\,, & n \leq N, \\ \det\big(DuDu^\top\big)^{\frac{1}{2}}\,, & n \geq N. \end{array} \right. \tag{1.8}$$

We call solutions of (1.7) *∞-Harmonic varifolds.* System (1.7) has continuous coefficients: $(Ju)^2\,[Du]^\perp$ is continuous and vanishes at critical points of u. Interestingly, Γ_∞ and Δ_∞ are equivalent when either $n = 1$ or $N = 1$, but not for general maps. In particular, when $n = 1$ all ∞-Harmonic curves are *affine* and for $u : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$, (1.2) reduces to

$$\Delta_\infty u \;=\; (u' \otimes u')u'' \;+\; |u'|^2\Big(I - \frac{u'}{|u'|} \otimes \frac{u'}{|u'|}\Big)u'' \;=\; |u'|^2 u''. \tag{1.9}$$

Hence, the coefficients become *automatically continuous* since nornal and tangential part "match".

System (1.1) arises as a sort of Euler-Lagrange PDE related to L^∞ variational problems for the supremal functional

$$E_\infty(u,\Omega) \;:=\; \underset{\Omega}{\text{ess sup}}\, H(Du)\ , \quad u \in W^{1,\infty}_{loc}(\mathbb{R}^n)^N. \tag{1.10}$$

In the very recent paper [K4] we identify the variational principle govering ∞-Harmonic maps for the model functional $\|Du\|_{L^\infty(\Omega)}$ of vector-valued Calculus of Variations in L^∞. Recently, special cases of (1.1) and (1.10) in the vector case have attracted substantial interest. Ou, Troutman and Wilhelm in [OTW] and Wang and Ou in [WO] studied Riemannian variants of tangentially ∞-Harmonic maps which solve only the tangential part of (1.2). Sheffield and Smart in [SS] used as Hamiltonian the nonsmooth operator norm on $\mathbb{R}^N \otimes \mathbb{R}^n$

$$\|P\| \;:=\; \max_{w \in \mathbb{S}^{n-1}} \big(P^\top P : w \otimes w\big)^{\frac{1}{2}} \tag{1.11}$$

and derived a very singular variant of (1.2) for a norm different than the Euclidean, which governs *optimal Lipschitz extensions of maps*. The choice of (1.11) owes to that they employ an L^∞ variational approach and they need the coincidence

$$\underset{\Omega}{\text{ess sup}}\, \|Du\| \;=\; \text{Lip}(u,\Omega)\ , \quad u \in C^1(\mathbb{R}^n)^N, \tag{1.12}$$

which fails for the Euclidean norm $|Du|$ on $\mathbb{R}^N \otimes \mathbb{R}^n$. Capogna and Raich in [CR] used the Hamiltonian

$$H(\text{P}) \;:=\; \frac{|P|^n}{\det(P)} \tag{1.13}$$

defined on $GL(n,\mathbb{R}) \subseteq \mathbb{R}^n \otimes \mathbb{R}^n$ for local diffeomorphisms $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and developed an L^∞ variational approach to extremal *Quasiconformal maps*. They derived and studied a special important case of (1.1). Their results have very recently been advanced by the author in [K5].

Herein we are mostly interested to understand the structure of system (1.1) and of its regular solutions, with emphasis to those of (1.2). To this end, we first develop a Geometric Analysis tool of independent interest, and then employ it to our PDE analysis. Section 2 is devoted to this analytical result.

To begin with, consider a map $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ given as composition of a scalar function $f \in C^2(\Omega)$ with a unit speed curve $\nu : \mathbb{R} \longrightarrow \mathbb{R}^N$, that is $u = \nu \circ f$.

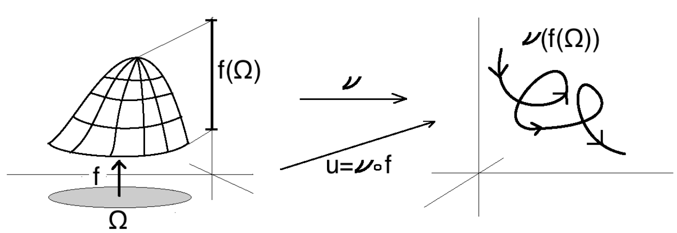

Figure 1.

Then, $Du = (\dot{\nu} \circ f) \otimes Df$ and hence u is a *Rank-One map*, that is $\mathrm{rk}(Du) \leq 1$ on Ω and Du can be written as $Du = \xi \otimes w$ for two vector fields ξ and w. Interestingly, the class of Rank-One maps is *rigid* since a certain converse is true as well: all maps which satisfy $\mathrm{rk}(Du) \leq 1$ arise as compositions of unit speed curves with scalar functions. More precisely, in Theorem 2.1 of Section 2 we prove that for any Rank-One map $u \in C^2(\Omega)^N$ over a contractible domain $\Omega \subseteq \mathbb{R}^N$, there exists an $f \in C^2(\Omega)$ and a partition of Ω to connected Borel sets B_i such that on each B_i, u can be represented as composition of f with a twice differentiable unit speed curve: $u = \nu^i \circ f$.

Theorem 2.1 is optimal. Without an extra assumption which reduces the complexity of $\mathrm{sgn}(Du)$, $u(\Omega)$ may bifurcate. The latter is an 1-rectifiable subset of $\mathbb{R}^N$, but there may not exist a single-valued curve ν such that $u = \nu \circ f$. (Corollary 2.2, Example 2.3) Moreover, if Ω is not homotopically trivial, then there exists no globally defined f either.

Theorem 2.1 is a result of rigidity type and has been motivated by the rigidity results of Rindler in [R1, R2]. Actually, we extend a part of his result from constant rank-one tensors $\xi \otimes w$ to variable rank-one $\xi(x) \otimes w(x)$ tensor fields. When compared to the rigidity results known in the literature which relate to Gromov's Convex Integration (see e.g. Kirchheim [Ki]), Theorem 2.1 is somewhat surprising in that most rigidity phenomena appear for rank greater than 2.

The idea of the proof of Theorem 2.1 has two main steps. Suppose that $Du = \xi \otimes w$. By using that $\mathrm{Curl}(Du) \equiv 0$, we first show that $\mathrm{rk}(D\xi) \leq 1$ as well and also invoke Poincaré's lemma of De Rham Cohomology to represent w by the gradient Df of a scalar function. Then, we employ geodesic flows, exponential maps of Riemannian Geometry and a curvilinear extension of "De Giorgi-type" arguments to show that ξ and f locally have the same level sets and hence ξ can be written as $\xi = \dot{\nu} \circ f$. Hence, locally we deduce $Du = (\dot{\nu} \circ f) \otimes Df = D(\nu \circ f)$.

It seems that the natural setting for the validity of the Rigidity Theorem 2.1 is that of Lipschitz maps $u \in W^{1,\infty}(\Omega)^N$. Indeed, we provide such an extension in Theorem 2.4. Yet, this does not follow by a direct approximation argument and substantial complications arise. The problem is that unless at least one of ξ, w is *constant*, all standard mollification schemes seem to fail when one tries to approximate Lipschitz Rank-One maps by smooth *and* Rank-One maps. If both ξ and w vary, the image of the mollification may "fatten" and its Hausdorff dimension increases (Remark 2.5). We remedy this problem by imposing a rather reasonable functional-analytic assumption on Du.

Using the analytical machinery developed in Section 2, in Section 3 we move to (1.2) and analyse the structure of ∞-Harmonic maps. In Theorem 3.1 we establish that if $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ is an ∞-Harmonic map in $C^2(\Omega)^N$, then u naturally separates to (at most) two different phases, a 2-dimensional Ω_2 whereon u has the structure of an Eikonal immersion and an 1-dimensional Ω_1 whereon u has the structure of a scalar ∞-Harmonic function along a fixed direction. The two phases are separated by an interface $\mathcal{S}$, whereon u has the structure of a scalar Eikonal function. Theorem 3.1 relates directly to the phase separation of tight maps (vectorial optimal Lipschitz extensions) observed by Sheffield and Smart in [SS]. For their non-smooth Hamiltonian (1.11), the 2-dimensional behavior of tight maps is Conformal and the 1-dimensional is that of "fans", that is, behavior of essentially scalar ∞-Harmonic functions but with variable direction.

By employing Aronsson's theorem in [A4], Theorem 3.1 implies an extension of the Aronsson-Evans-Yu theorem ([A4, E, Y]) on the non-existence of zeros of $|Du|$ for solutions to (1), for all $N \geq 2$. That is, if $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ is an ∞-Harmonic map, either $\mathrm{rk}(Du) > 0$ or $Du \equiv 0$. In view of our earlier observations, solutions of (1.2) have positive rank but generally non-constant rank. As a corollary, for $N = 2$ we establish a vectorial version of the Maximum Principle for solutions $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, known as *Convex Hull Property* (Corollary 3.3), which states that the range is contained in the closed convex hull of the boundary values:

$$u(\Omega) \ \subseteq \ \overline{\mathrm{co}}\big(u(\partial\Omega)\big). \tag{1.14}$$

Inclusion (1.14) is nothing but an elegant restatement of the Maximum Principle for all projections of u. It is well known in the context of Minimal Surfaces (see Colding-Minicozzi II [CM], Osserman [O]) and more generally in Calculus of Variations (see the Author's paper [K2] and references therein).

Motivated by Aronsson's paper [A6] and also from [SS], in Proposition 3.5 we investigate solutions to (1.2) of the radial form $u = \rho^k f(k\theta)$ for $k > 0$ and f a curve in $\mathbb{R}^N$. Interestingly, such solutions are very rigid, since they always have affine range and they moreover are essentially scalar if $k \neq 1$.

In Section 4 we focus to the general Aronsson PDE system (1.1). We motivate our results by observing that (1.2) is quasilinear and degenerate elliptic, that is, for

$$\mathrm{A}(P) \ := \ P \otimes P \ + \ |P|^2 [P]^\perp \otimes I \tag{1.15}$$

we can rewrite the ∞-Laplacian (1.2) as

$$\mathrm{A}(Du) : D^2 u \ = \ 0 \tag{1.16}$$

and A satisfies the Legendre-Hadamard ellipticity condition (4.4) and also the symmetry condition (4.5). However, (1.1) is *not* degenerate elliptic, not even when the Hamiltonian $H \in C^2(\mathbb{R}^N \otimes \mathbb{R}^n)$ is strictly convex. The problem is that in the normal Aronsson system

$$H(Du)[H_P(Du)]^\perp H_{PP}(Du) : D^2 u \ = \ 0 \tag{1.17}$$

both tensors $[H_P(Du)]^\perp$ and $H_{PP}(Du)$ are symmetric, but when $N \geq 2$ their product may *not commute*. Which are the Hamiltonians which lead to elliptic Aronsson systems? This question has empty content for $N = 1$, since Aronsson's equation $H_{P_i}(Du)H_{P_j}(Du)D^2_{ij}u = 0$ is always degenerate elliptic. In Theorem 4.1 we give a complete answer, by classifying the Hamiltonians which lead to elliptic Aronsson systems. Every "geometric" Hamiltonian which depends on Du via the induced Riemannian metric $Du^\top Du$ on the range $u(\Omega) \subseteq \mathbb{R}^N$, that is when

$$H(P) \ = \ h\Big(\frac{1}{2}P^\top P\Big) \ , \quad h = h(p), \tag{1.18}$$

gives rise to a degenerate elliptic PDE system which takes the form

$$\mathrm{A}_\infty u \ = \ \Big(Duh_p \otimes Duh_p \ + \ h[Du]^\perp \otimes h_p\Big) : D^2 u \ = \ 0. \tag{1.19}$$

with $h = h\big(\frac{1}{2}Du^\top Du\big)$. In the case of Δ_∞, we have $h(p) = \mathrm{tr}(p) = p_{11} + ... + p_{nn}$. In dimensions $n \leq 3$, the converse is true as well and this is the only way that degenerate elliptic Aronsson systems can arise. However, if $n \geq 4$ complicated structures in the minors of the higher order derivative tensors $H_{P...P}(0)$ appear and a necessary extra assumption is required for the converse to be true. Without it, H can be written in the form (1.18) up to an $O(|P|^4)$ correction. In the case $n = 1$, then (1.18) requires that $H(P)$ is radially symmetric: $H(P) = h\big(\frac{1}{2}|P|^2\big)$. This is very restrictive, but should be compared with the rigidity of Lipschitz extensions for maps in Kirszbraun's theorem (see e.g. Federer [F], p. 201), in contrast to the flexibility of scalar Lipschitz extensions.

In the light of our general Theorem 4.1, it is not a coincidence that all Hamiltonian known in the literature have the form (1.18). For, the Euclidean norm trivially gives $H(P) = \frac{1}{2}\mathrm{tr}(P^\top P)$. The Hamiltonian (1.11) of Sheffield and Smart evidently has this form and the Hamiltonian (1.13) of Capogna and Raich can be written as

$$H(P) \ = \ \left(\frac{\mathrm{tr}(P^\top P)^n}{\det(P^\top P)}\right)^{\frac{1}{2}}. \tag{1.20}$$

Moreover, by comparing the elliptic version (1.19) of (1.1) with (1.2), it follows that by studying ∞-Harmonic maps we gain a great deal of intuition regarding the behavior of general Aronsson maps.

Finally, in Section 5 we focus on the 1-dimensional case of Aronsson ODEs. We first formally derive Aronsson's system in the limit as $p \to \infty$ of the Euler-Lagrange equations of an L^p-functional with Hamiltonian $H \in C^2(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N)$ depending on all arguments: $H = H\big(x, u(x), u'(x)\big)$ (equation (5.8)). Then, by arguing as in Theorem 4.1, we derive the *degenerate elliptic system of Aronsson ODEs with dependence on all the arguments*:

$$A_\infty u \ = \ |u'|^2\Big(h_p u'' \ - \ \mathbf{R}_{u'} h_\eta\Big) \ + \ h_x u' \ = \ 0. \tag{1.21}$$

Here $h = h\big(x, u(x), \frac{1}{2}|u'|^2\big)$ and $\mathbf{R}_{u'}$ is the *reflection* operator with respect to the normal hyperplane bundle $[u']^\perp$. As in the case of (1.9), the coefficients of (1.21) are continuous, although $\mathbf{R}_{u'}$ becomes discontinuous at critical points on $\{u' = 0\}$. In Theorem 5.2 we study existence, uniqueness and $W^{2,\infty}_{loc}(\mathbb{R})^N$ regularity of solutions to the initial value problem for (1.21).

1.1. **Preliminaries.** Throughout this paper we reserve $n, N \in \mathbb{N}$ for the dimensions of Euclidean spaces and $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$. Greek indices $\alpha, \beta, \gamma, ...$ run from 1 to N and Latin $i, j, k, ...$ form 1 to n. The summation convention will always be employed in repeated indices in a product. Vectors are always viewed as columns and we differentiatiate along lines. Hence, for $a, b \in \mathbb{R}^n$, $a^\top b$ is their inner product and $ab^\top$ equals $a \otimes b$. If $u = u_\alpha e_\alpha : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is a map, the gradient matrix Du is viewed as $D_i u_\alpha e_\alpha \otimes e_i : \Omega \longrightarrow \mathbb{R}^N \otimes \mathbb{R}^n$ and the Hessian tensor D^2u as $D^2_{ij} u_\alpha e_\alpha \otimes e_i \otimes e_j : \Omega \longrightarrow \mathbb{R}^N \otimes \mathbb{S}(\mathbb{R}^n)$. If V is a vector space, then $\mathbb{S}(V)$ denotes the symmetric linear operators $T : V \longrightarrow V$ for which $T = T^\top$. The Euclidean (Frobenious) norm on $\mathbb{R}^N \otimes \mathbb{R}^n$ is $|P| = (P_{\alpha i}P_{\alpha i})^{\frac{1}{2}} = (\mathrm{tr}(P^\top P))^{\frac{1}{2}}$. If $F \in C^q(\mathbb{R}^N \otimes \mathbb{R}^n)$ is a function and we denote the standard basis elements of $\mathbb{R}^N \otimes \mathbb{R}^n$ by $e_{\alpha i} := e_\alpha \otimes e_i$, then its q-th order derivative tensor $F_{P...P}$ at P_0

$$F_{P...P}(P_0) = F_{P_{\alpha_1 i_1}...P_{\alpha_q i_q}}(P_0) e_{\alpha_1 i_1} \otimes ... \otimes e_{\alpha_q i_q} \tag{1.22}$$

is viewed as a multilinear map $\otimes^{(q)}\mathbb{R}^n \longrightarrow \otimes^{(q)}\mathbb{R}^N$, or equivalenly as an element of $\otimes^{(q)}(\mathbb{R}^n \otimes \mathbb{R}^n)$. Here "$\otimes^{(q)}$" is the q-fold tensor product. Hence, $F_{P...P}$ is a map $\mathbb{R}^N \otimes \mathbb{R}^n \longrightarrow \otimes^{(q)}(\mathbb{R}^N \otimes \mathbb{R}^n)$. We will say that a q-th order tensor $C \in \otimes^{(q)}(\mathbb{R}^N \otimes \mathbb{R}^n)$ is fully symmetric in all its arguments when

$$C_{...\alpha i...\beta j...} \ = \ C_{...\alpha j...\beta i...} \ = \ C_{...\beta j...\alpha i...}. \tag{1.23}$$

We also introduce the following *contraction operation* for tensors which extends the trace inner product $P : Q = \mathrm{tr}(P^\top Q) = P_{\alpha i}Q_{\alpha i}$ of $\mathbb{R}^N \otimes \mathbb{R}^n$. For, if $C \in \otimes^{(q)}(\mathbb{R}^N \otimes \mathbb{R}^n)$ is a q-th order tensor and $A \in \otimes^{(p)}(\mathbb{R}^N \otimes \mathbb{R}^n)$ is a p-th order tensor with $p \leq q$, we define

$$C : A \ := \ \big(C_{\alpha_q i_q ... \alpha_1 i_1} A_{\alpha_p i_p ... \alpha_1 i_1}\big) e_{\alpha_q i_q} \otimes ... \otimes e_{\alpha_{p+1} i_{p+1}} \ \in \ \otimes^{(q-p)}(\mathbb{R}^N \otimes \mathbb{R}^n). \tag{1.24}$$

Let now $P : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be linear map. Upon identifying linear subspaces with projections on them, we have the split $\mathbb{R}^N = [P]^\top \oplus [P]^\perp$ where $[P]^\top$ and $[P]^\perp$ denote range of P and nullspace of $P^\top$ respectively. Hence, if $\xi \in \mathbb{S}^{N-1}$, then $[\xi]^\perp$ is (the projection on) the normal hyperplane $I - \xi \otimes \xi$. Consequently, the ∞-Laplacian (1.2) in index form reads

$$D_i u_\alpha D_j u_\beta D^2_{ij} u_\beta \ + \ |Du|^2 [Du]^\perp_{\alpha\beta} D^2_{ii} u_\beta \ = \ 0 \tag{1.25}$$

and the Aronsson system (1.1) becomes

$$\Big(H_{P_{\alpha i}} H_{P_{\beta j}} \ + \ H[H_P]^\perp_{\alpha\gamma} H_{P_{\gamma i} P_{\beta j}}\Big)(Du) D^2_{ij} u_\beta \ = \ 0. \tag{1.26}$$

For convenience we use a different scaling in (1.25) and (1.26) and we multiply the normal term of (1.25) by a factor 2 which is plausible since (1.26) consists of two systems normal to each other. Finally, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure and for measure theoretic notions we use herein we refer to Simon [S].

2. Rigidity of Rank-One Maps.

2.1. **The case of smooth Rank-One Maps.** In this subsection we establish our Geometric Analysis rigidity result in the case of C^2 maps.

Theorem 2.1 (Rigidity of Rank-One Maps)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and contractible and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is in $C^2(\Omega)^N$. Then, the following are equivalent:*

(i) u is a Rank-One map, that is $rk(Du) \leq 1$ on Ω or equivalently there exist $\xi : \Omega \longrightarrow \mathbb{R}^N$ and $w : \Omega \longrightarrow \mathbb{R}^n$ such that $Du = \xi \otimes w$.

(ii) There exists $f \in C^2(\Omega)$, a partition $\{B_i\}_{i\in\mathbb{N}}$ of Ω to Borel sets where each B_i equals a connected open set with a boundary portion and Lipschitz curves $\{\nu^i\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{loc}(\mathbb{R})^N$ such that on each B_i u equals the composition of the curve ν^i with the scalar function f:

$$u \ = \ \nu^i \circ f \ , \quad \text{on } B_i \subseteq \Omega. \tag{2.1}$$

Moreover, $|\dot\nu^i| \equiv 1$ on $f(B_i)$, $\dot\nu^i \equiv 0$ on $\mathbb{R} \setminus f(B_i)$ and there exists $\ddot\nu^i$ on $f(B_i)$, interpreted as 1-sided on $\partial f(B_i)$, if any. Also,

$$Du \ = \ (\dot\nu^i \circ f) \otimes Df \ , \quad \text{on } B_i \subseteq \Omega, \tag{2.2}$$

and the image $u(\Omega)$ is an 1-rectifiable subset of $\mathbb{R}^N$:

$$u(\Omega) \ = \ \bigcup_{i=1}^\infty \nu^i\big(f(B_i)\big) \ \subseteq \ \mathbb{R}^N. \tag{2.3}$$

As we have already mentioned in the Introduction, an extra assumptions is required in order to deduce that a rank-one map u has the form $u = \nu \circ f$ for a unique single-valued unit speed curve ν. This assumption guarrantees "low complexity" for the direction field ξ.

Corollary 2.2 (Strong Rigidity of Rank-One Maps)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and contractible and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is in $C^2(\Omega)^N$. Consider the following statements:*

(i) u is a strictly Rank-One map, that is $rk(Du) = 1$ on Ω or equivalently there exist $\xi : \Omega \longrightarrow \mathbb{R}^N \setminus \{0\}$ and $w : \Omega \longrightarrow \mathbb{R}^n \setminus \{0\}$ such that $Du = \xi \otimes w$. Moreover, the following condition holds

$$E \;:=\; \Omega \cap \left(\bigcup_{\alpha=1}^{N} \partial\big\{|D\xi_\alpha| > 0\big\} \right) \;=\; \emptyset. \tag{2.4}$$

(ii) u equals the composition of a single curve $\nu \in W^{1,\infty}_{loc}(\mathbb{R})^N$ with a scalar function $f \in C^2(\Omega)$, without critical points that is $u = \nu \circ f$ with $|\dot{\nu}| \equiv 1$ on $f(\Omega)$, $\dot{\nu} \equiv 0$ on $\mathbb{R} \setminus f(\Omega)$. Moreover, $Du = (\dot{\nu} \circ f) \otimes Df$ on Ω and $u(\Omega)$ is 1-rectifiable, equal to $\nu(f(\Omega))$.

Then, (i) implies (ii) and also (ii) implies that u is a strictly rank-one map, that is assertion (i) without (2.4)*.*

Example 2.3. *The additional assumption* (2.4) *of Corollary* 2.2 *is necessary in order to obtain $u = \nu \circ f$. It reduces the complexity of ξ and leads to the avoidance of bifurcations in the curve ν. For, let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by*

$$u(x) \;:=\; \begin{cases} \big(+ f^4(x), f(x)\big)^\top, & \text{on } \{f > 0\} \cap \{x_1 > 0\}, \\ \big(- f^4(x), f(x)\big)^\top, & \text{on } \{f > 0\} \cap \{x_1 < 0\}, \\ (0, f(x))^\top, & \text{on } \{f \leq 0\}, \end{cases}$$

where

$$f(x) \;:=\; 1 \,-\, |x - e_1|^2 |x + e_1|^2.$$

Then, u can not be written as $u = \nu \circ f$ for a single-valued curve ν since the unique ν bifurcates and has two braunches: $\nu^{\pm}(t) = \big(\pm\, t^4 \chi_{(0,\infty)}(t), t\big)^\top$.

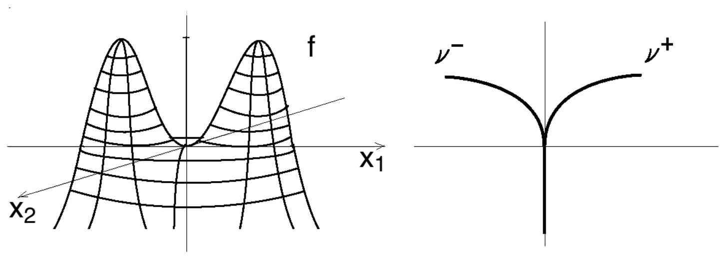

Figure 2.

Proof of Theorem 2.1. The implication $(ii) \Rightarrow (i)$ is trivial and the whole proof is devoted to establish the reverse implication $(i) \Rightarrow (ii)$. For, suppose there exist $\xi : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ and $w : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $Du = \xi \otimes w$. By replacing ξ by $\xi/|\xi|$ on $\{|\xi| > 0\}$ and w by $|\xi| w$ on $\{|\xi| > 0\}$, we may pass all the zeros of Du to w and assume that $|\xi| \equiv 1$ on

$$\Omega_0 \;:=\; \{|Du| > 0\} \;=\; \{|w| > 0\}. \tag{2.5}$$

By differentiating $D_k u_\alpha = \xi_\alpha w_k$, we have

$$D^2_{ij} u_\alpha \;=\; (D_j \xi_\alpha)\, w_i \;+\; \xi_\alpha (D_j w_i), \tag{2.6}$$

$$D^2_{ji}u_\alpha \;=\; (D_i\xi_\alpha)\,w_j \;+\; \xi_\alpha(D_iw_j). \tag{2.7}$$

Since $u \in C^2(\Omega)^N$, the curl of Du vanishes and we have

$$D^2_{ij}u_\alpha \;=\; D^2_{ji}u_\alpha. \tag{2.8}$$

Hence, by (2.6), (2.7), (2.8),

$$(D_j\xi)\,w_i \;-\; (D_i\xi)\,w_j \;=\; \xi(D_iw_j \;-\; D_jw_i). \tag{2.9}$$

Since $|\xi|^2 = 1$ on Ω_0, we have $D_k\xi^\top\xi = 0$ thereon. Hence, the two sides of (2.9) are normal to each other. By applying the projections $\xi\otimes\xi$ and $[\xi]^\perp = I - \xi\otimes\xi$, (2.9) decouples on Ω_0 to

$$\mathrm{Curl}(w)_{ij} \;=\; D_iw_j \;-\; D_jw_i \;\equiv\; 0, \tag{2.10}$$

$$(D_j\xi)\,w_i \;-\; (D_i\xi)\,w_j \;\equiv\; 0. \tag{2.11}$$

By (2.10), the curl of $w : \Omega_0 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ vanishes and by $w \equiv 0$ on $\Omega\setminus\Omega_0$. Hence, since Ω is contractible, by Poincaré 's Lemma w can be represented by the gradient of a scalar function $f \in C^2(\Omega)$: $w = Df$. By (2.11), for all $i, j \in \{1, ..., n\}$ for which $\{w_i \neq 0\} \cap \{w_j \neq 0\} \neq \emptyset$, we have

$$\frac{D_j\xi_\alpha}{w_j} \;=\; \frac{D_i\xi_\alpha}{w_i}. \tag{2.12}$$

By (2.12), the quotient $D_k\xi_\alpha/w_k$ is independent of k. Hence, we may define

$$\eta \;:=\; \frac{D_k\xi}{w_k} \;\;:\;\; \{w_k \neq 0\} \subseteq \Omega_0 \;\longrightarrow\; \mathbb{R}^N. \tag{2.13}$$

By (2.12), η is well defined on all of Ω_0 since $\cup_1^n\{w_k \neq 0\}$ is an open cover of $\Omega_0 = \{|w| > 0\}$ and on the overlaps the different expressions coincide. By (2.13), we have $D_k\xi_\alpha = \eta_\alpha w_k$ on $\{w_k \neq 0\}$. Actually, this extends to the whole of Ω_0 since by (2.11) we get $D_k\xi = 0$ whenever $w_k = 0$. Thus,

$$D\xi \;=\; \eta\otimes Df\;, \quad \text{on } \Omega_0, \tag{2.14}$$

and also η is normal to ξ, since $\eta^\top\xi = \frac{1}{w_k}D_k(\frac{1}{2}|\xi|^2) = 0$, on $\{w_k \neq 0\}$. We now employ (2.14) to show that in a certain local sence ξ and f have the same level sets.

Fix $\alpha \in \{1, ..., N\}$ and set

$$A \;:=\; \Omega_0 \cap \{|\eta_\alpha| > 0\}, \tag{2.15}$$

$$g \;:=\; \xi_\alpha, \tag{2.16}$$

$$\lambda \;:=\; \eta_\alpha. \tag{2.17}$$

We then obtain

$$Dg \;=\; \lambda\, Df\;, \quad \text{on } A, \tag{2.18}$$

while $|Dg| > 0$ and $|\lambda| > 0$ on A. (2.18) says that the level hypersurfaces $\{f = f(x)\}$ and $\{g = g(x)\}$ passing through x have, for all $x \in A$ the same tangent spaces given by

$$[Dg]^\perp \;=\; [Df]^\perp \;=\; I - \frac{Df}{|Df|}\otimes\frac{Df}{|Df|}. \tag{2.19}$$

Consider the level hypersurfaces of f, g as Riemannian submanifolds of A with the induced metrics from $\mathbb{R}^n$. Since covariant derivatives coincide with tangential projections of derivatives in $\mathbb{R}^n$, the geodesic equations for χ, ψ with initial conditions $\chi(0) = \psi(0) = x \in A$ and $\dot{\chi}(0) = \dot{\psi}(0) = e \in [Df(x)]^\perp = [Dg(x)]^\perp$ are

$$\begin{cases} [Df(\chi(t))]^\perp \ddot{\chi}(t) \ = \ 0, \quad t > 0, \\ \chi(0) = x, \quad \dot{\chi}(0) = e, \end{cases} \tag{2.20}$$

and

$$\begin{cases} [Dg(\psi(t))]^\perp \ddot{\psi}(t) \ = \ 0, \quad t > 0, \\ \psi(0) = x, \quad \dot{\psi}(0) = e. \end{cases} \tag{2.21}$$

Since $[Dg]^\perp \equiv [Df]^\perp$, χ and ψ satisfiy the same ODEs with the same initial conditions. Hence, by uniqueness, $\chi \equiv \psi$. Consequently, the esponential maps $\exp^f_x$ and $\exp^g_x$ of $\{f = f(x)\}$ and $\{g = g(x)\}$ coincide and hence $(\exp^g_x)^{-1} \circ \exp^f_x$ equals the identity their common geodesically convex neighborhood centered at x.

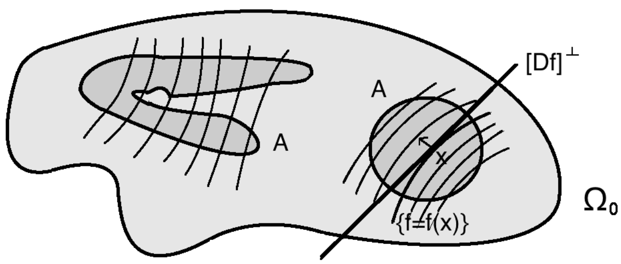

Figure 3.

Hence, the level hypersurfaces of f, g within A coincide, but perhaps they are at different hights. Cover A by countably many balls whose radii are small enough to guarrantee that the intersections of the level sets of f, g with each ball are connected. Using this cover, we decompose A to a partition of connected Borel sets by writting $A = \cup_1^\infty A_i$, where each A_i equals an open subset of the ball of the cover with possibly some boundary portion. Then, for each $t \in \mathbb{R}$ and each $i \in \mathbb{N}$ there is a unique $\rho^i(t) \in \mathbb{R}$ such that $\{f = t\}$ equals $\{g = \rho^i(t)\}$ locally within A_i. Hence, there exists a unique bijection $\rho^i : f(A_i) \subseteq \mathbb{R} \longrightarrow g(A_i) \subseteq \mathbb{R}$ such that

$$\{g = \rho^i(t)\} \ = \ \{f = t\} \ = \ \{\rho^i \circ f = \rho^i(t)\}, \tag{2.22}$$

within $A_i \subseteq \Omega_0$. Equivalently,

$$g \ = \ \rho^i \circ f \ , \quad \text{on } A_i, \quad i \in \mathbb{N}. \tag{2.23}$$

We extend ρ^i from $f(A_i)$ to $\mathbb{R}$ by zero.

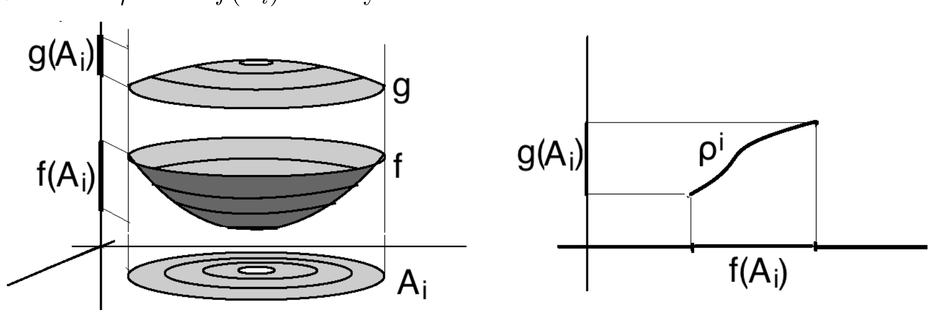

Figure 4.

On $\Omega_0 \setminus A = \Omega_0 \setminus \cup_1^\infty A_i$, we have $Dg \equiv 0$. Hence, there exists a constant function $\rho^0 : f(\Omega_0 \setminus A) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$g \;=\; \rho^0 \circ f\;, \quad \text{on } \Omega_0 \setminus \cup_1^\infty A_i. \tag{2.24}$$

We extend ρ^0 by zero on $\mathbb{R}$ as well. By recalling (2.15), (2.16), (2.17), we have shown that for any ξ_α, $1 \leq \alpha \leq N$, there exists a partition of Ω_0 to disjoint connected Borel sets A_i^α where each A_i^α equals an open set with possibly some boundary portion and also their complement $A_0^\alpha := \Omega_0 \setminus \cup_1^\infty A_i^\alpha$. There also exist functions $\rho_\alpha^i : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\xi_\alpha \;=\; \rho_\alpha^i \circ f\;, \quad \text{on } A_i^\alpha, \quad i = 0, 1, 2, \dots\,. \tag{2.25}$$

Hence, by recalling that $|\xi| \equiv 1$ on Ω_0, there exists a partitition of Ω_0 to connected Borel sets $\{B_i\}_{i\in\mathbb{N}}$ which are intersections of the A_i's and respective bounded curves $\mu^i : \mathbb{R} \longrightarrow \{0\} \cup \mathbb{S}^{N-1} \subseteq \mathbb{R}^N$ which satisfy

$$|\mu^i| \equiv 1 \text{ on } f(B_i)\;, \quad \mu^i \equiv 0 \text{ on } \mathbb{R} \setminus f(B_i), \tag{2.26}$$

and are such that

$$\xi \;=\; \mu^i \circ f\;, \quad \text{on } B_i, \tag{2.27}$$

for all $i \in \mathbb{N}$. We set

$$\nu^i(t) \;:=\; \int_0^t \mu^i(s)\, ds\;, \quad i \in \mathbb{N}. \tag{2.28}$$

Then, by (2.26) we have that $\nu^i \in W^{1,\infty}_{loc}(\mathbb{R})^N$, while $|\dot{\nu}^i| \equiv 1$ on the interval $f(B_i)$ and also $\dot{\nu}^i \equiv 0$ on $\mathbb{R} \setminus f(B_i)$. By (2.27) we have

$$\xi \;=\; \dot{\nu}^i \circ f\;, \quad \text{on } B_i. \tag{2.29}$$

Hence, since $Du = \xi \otimes w$, (2.29) implies

$$\begin{aligned} Du \;&=\; \xi \otimes w \\ &=\; (\dot{\nu}^i \circ f) \otimes Df \\ &=\; D(\nu^i \circ f), \end{aligned} \tag{2.30}$$

on B_i. Thus, $u = \nu^i \circ f$ on each $B_i \subseteq \Omega_0$, up to an additive constant. By differencing (2.29), comparing with (2.14) and passing to limits, we obtain

$$D\xi \;=\; (\ddot{\nu}^i \circ f) \otimes Df, \tag{2.31}$$

and hence $\ddot{\nu}^i_\alpha \circ f = D_k \xi_\alpha\, D_k f$, on B_i. Thus, $\ddot{\nu}^i$ exists on $f(B_i) \subseteq \mathbb{R}$ and is interpreted as 1-sided at the endpoints of this interval in case it is not open. Since $Du = 0$ and $Df = 0$ on $\partial(\Omega_0) \cap \Omega$, we can extend the partition $\cup_1^\infty B_i$ of Ω_0 to $\overline{\Omega_0} \cap \Omega$ and further extend the families $\{B_i\}_{i\in\mathbb{N}}$ and $\{\nu^i\}_{i\in\mathbb{N}}$ by attaching the limit values and setting

$$B_0 \;:=\; \Omega \setminus \overline{\Omega_0}, \tag{2.32}$$

$$\nu^0 \;:=\; u\big|_{\Omega \setminus \overline{\Omega_0}} \;=\; \text{const}. \tag{2.33}$$

Hence, since $u = \nu^i \circ f$ on each B_i of the partition $\cup_0^\infty B_i = \Omega$, we conclude that u is 1-rectifiable and the image $u(\Omega)$ equals a union of images of Lipschitz curves:

$$u(\Omega) \;=\; \bigcup_{i=1}^{\infty} \nu^i\big(f(B_i)\big). \tag{2.34}$$

The theorem follows. □

Proof of Corollary 2.2. In the setting of the proof of Theorem 3.1, if in addition the set E given by (2.4) is empty and moreover $\mathrm{rk}(Du) > 0$ on Ω, then for all $\alpha \in \{1, ..., N\}$, either $D\xi_\alpha$ does not vanish anywhere inside $\Omega_0 = \Omega$ or it is identically constant. In both cases, the previous set A is connected and coincides with Ω. Hence, the curve ν constructed is unique and consequently $u = \nu \circ f$ with $|\dot{\nu}| \equiv 1$ on $f(\Omega)$ and $\dot{\nu} \equiv 0$ on $\mathbb{R} \setminus f(\Omega)$. The reverse implication is obvious. □

2.2. **Extension to Lipschitz Rank-One Maps.**

In this subsection we extend Theorem 2.1 to the Lipschitz setting. As we have already explained, this does not follow by a straightforward mollification argument and an additional assumption is required.

Theorem 2.4 (Rigidity of Lipschitz Rank-One Maps)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open, bounded and contractible and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is in $W^{1,\infty}(\Omega)^N$.*

We moreover assume that there exists a family $\{V^\varepsilon\}_{\varepsilon>0}$ of rank-one smooth tensor fields in $C^\infty(\Omega)^{Nn}$ where each V^ε is curl-free (that is $rk(V^\varepsilon) \leq 1$ and also $D_j V^\varepsilon_{\alpha i} - D_i V^\varepsilon_{\alpha j} = 0$) such that

$$V^\varepsilon \overset{*}{\longrightarrow} Du \ \text{ in } L^\infty(\Omega)^{Nn} \text{ and } \ V^\varepsilon \longrightarrow Du \text{ a.e. on } \Omega, \quad \text{as } \ \varepsilon \to 0. \tag{2.35}$$

Then, the following are equivalent:

(i) u is a Rank-One map, that is $rk(Du) \leq 1$ a.e. on Ω or equivalently there exist $\xi : \Omega \longrightarrow \mathbb{R}^N$ and $w : \Omega \longrightarrow \mathbb{R}^n$ both L^∞ vector fields such that $Du = \xi \otimes w$ a.e. on Ω.

(ii) There exists $f \in W^{1,\infty}(\Omega)$, a partition $\{B_i\}_{i\in\mathbb{N}}$ of Ω to measurable sets which covers it a.e., that is $\big|\Omega \setminus (\cup_1^\infty B_i)\big| = 0$ and Lipschitz curves $\{\nu^i\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{loc}(\mathbb{R})^N$ such that on each B_i u equals the composition of the curve ν^i with the scalar function f:

$$u \ = \ \nu^i \circ f \ , \quad \text{on } B_i \subseteq \Omega. \tag{2.36}$$

Moreover, $\|\dot{\nu}^i\|_{L^\infty(\mathbb{R})} \leq 1$ and $\dot{\nu}^i = 0$ a.e. on $\mathbb{R} \setminus f(B_i)$. Also,

$$Du \ = \ (\dot{\nu}^i \circ f) \otimes Df \ , \quad \text{a.e. on } B_i \subseteq \Omega, \tag{2.37}$$

and the image $u(\Omega)$ is an 1-rectifiable subset of $\mathbb{R}^N$:

$$\mathcal{H}^1\left(u(\Omega) \ \setminus \ \bigcup_{i=1}^{\infty} \nu^i\big(f(B_i)\big)\right) \ = \ 0. \tag{2.38}$$

Remark 2.5. The extra approximation assumption (2.35) of Theorem 2.4 requires that Du is in the intersection of the weak* and the pointwise closures in $L^\infty(\Omega)^{Nn}$ of the cone which consists of smooth rank-one curl-free tensor fields. Such an assumption is superfuous if either ξ or w is identically constant, since mollification of $Du = \xi \otimes w$ produces the desired approximations V^ε.

Generally, however, all standard mollification methods average at each point contributions from nearby points. As a result, if such a "partial affineness" of u fails to hold and both ξ *and* w vary, its easy to see that the range $u(\Omega)$ generally "fattens" and the mollification of u is not be rank-one any more.

Unfortunately, we have not been able neither to verify the necessity of the assumption nor to construct an appropriate mollification scheme which allows to drop it.

Proof of Theorem 2.4. Is suffices to demonstrate the implication $(i) \Rightarrow (ii)$. Suppose $Du = \xi \otimes w$ a.e. on Ω. By a rescaling of the form $Du = (\frac{1}{|\xi|}\xi) \otimes (|\xi| w)$ on $\{|\xi| > 0\}$, we may assume that $\xi : \Omega_0 \longrightarrow \mathbb{S}^{N-1}$, where $\Omega_0 := \{|Du| > 0\} \subseteq \Omega$ and also that $\xi = 0$ a.e. on $\Omega \setminus \Omega_0$.

By assumption, we have $\mathrm{rk}(V^\varepsilon) \leq 1$ and hence there exist smooth vector fields $\xi^\varepsilon : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ and $w^\varepsilon : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $V^\varepsilon = \xi^\varepsilon \otimes w^\varepsilon$. By an appropriate rescaling inside the products $(\frac{1}{|\xi^\varepsilon|}\xi^\varepsilon) \otimes (|\xi^\varepsilon| w^\varepsilon)$ on $\{|\xi^\varepsilon| > 0\}$, we may assume that $\xi^\varepsilon : \Omega_\varepsilon \longrightarrow \mathbb{S}^{N-1}$ where $\Omega_\varepsilon := \{|V^\varepsilon| > 0\} \subseteq \Omega$ and also that $\xi^\varepsilon \equiv 0$ on $\Omega \setminus \Omega_\varepsilon$.

We now claim that $\xi^\varepsilon \longrightarrow \xi$ and also that $w^\varepsilon \longrightarrow w$ as $\varepsilon \to 0$, both weakly* in $L^\infty(\Omega)$ and also a.e. on Ω; indeed, there exists η such that $\xi^\varepsilon \overset{*}{\longrightarrow} \eta$ and hence by the $L^1(\Omega)^{Nn}$ strong convergence of $\xi^\varepsilon \otimes w^\varepsilon$ which follows by the Dominated Convergence theorem, we have

$$w^\varepsilon = (\xi^\varepsilon)^\top (\xi^\varepsilon \otimes w^\varepsilon) \overset{*}{\longrightarrow} \eta^\top (\xi \otimes w) = (\eta^\top \xi) w, \tag{2.39}$$

as $\varepsilon \to 0$. Thus, by uniqueness of limits of $\xi^\varepsilon \otimes w^\varepsilon$ we have $[(\eta \otimes \eta)\xi] \otimes w = \xi \otimes w$ a.e. on Ω and hence $\xi = \eta$.

Since Ω is contractible, by Poincaré's lemma, for any $\varepsilon > 0$ there exists a smooth map $u^\varepsilon : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ such that V^ε can be represented as the gradient of u^ε: $Du^\varepsilon = \xi^\varepsilon \otimes w^\varepsilon$. Moreover, each u^ε is a smooth rank-one map: by Theorem 2.1, there exist scalar functions $f^\varepsilon \in C^\infty(\Omega)$, partitions of Ω to Borel sets $\{B_i^\varepsilon\}_{i\in\mathbb{N}}$ with $\Omega = \cup_1^\infty B_i^\varepsilon$, families of Lipschitz curves $\{\nu^{i\varepsilon}\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{loc}(\mathbb{R})^N$ with $\|\dot\nu^{i\varepsilon}\|_{L^\infty(\mathbb{R})} \leq 1$ and $\dot\nu^{i\varepsilon} \equiv 0$ on $\mathbb{R} \setminus f^\varepsilon(B_i^\varepsilon)$ such that $u^\varepsilon = \nu^{i\varepsilon} \circ f^\varepsilon$ on each $B_i^\varepsilon \subseteq \Omega$, while the images $u^\varepsilon(\Omega)$ are 1-rectifiable, equal to $\cup_1^\infty \nu^{i\varepsilon}(f^\varepsilon(B_i^\varepsilon))$.

We will now show that appropriate normalized shifts of the maps u^ε approximate u. Fix a point $\overline{x} \in \Omega$ and set $d := \mathrm{diam}(\Omega)$. Since $Du^\varepsilon \overset{*}{\longrightarrow} Du$ in $L^\infty(\Omega)^{Nn}$ as $\varepsilon \to 0$, for all $x, y \in \Omega$ and $\varepsilon > 0$ small we have

$$\begin{aligned} |u^\varepsilon(x) - u^\varepsilon(y)| \ &\leq\ \|Du^\varepsilon\|_{L^\infty(\Omega)}|x-y| \\ &\leq\ \big(\|Du\|_{L^\infty(\Omega)} + 1\big)|x-y|. \end{aligned} \tag{2.40}$$

We further normalize u^ε by considering appropriate shifts, denoted again by u^ε, such that $u^\varepsilon(\overline{x}) = u(\overline{x})$. By (2.40), we have

$$\begin{aligned} \|u^\varepsilon\|_{L^\infty(\Omega)} \ &\leq\ d\|Du^\varepsilon\|_{L^\infty(\Omega)} \ +\ |u^\varepsilon(\overline{x})| \\ &\leq\ d\big(\|Du\|_{L^\infty(\Omega)} + 1\big) \ +\ |u(\overline{x})|. \end{aligned} \tag{2.41}$$

Hence, there exists v such that $u^\varepsilon \overset{*}{\longrightarrow} v$ in $W^{1,\infty}(\Omega)^N$ as $\varepsilon \to 0$. We will now show that $u \equiv v$. Since $Du^\varepsilon \longrightarrow Du$ a.e. on Ω, for $\mathcal{H}^{n-1}$-a.e. direction $e \in \mathbb{S}^{n-1}$, we have that $Du^\varepsilon \longrightarrow Du$ $\mathcal{H}^1$-a.e. on the set $(\overline{x} + \mathrm{span}[e]) \cap \Omega =: I$. We fix such an e. By Egoroff's theorem, for any $\sigma \in (0,1)$, there is an $\mathcal{H}^1$-measurable set $E\sigma \subseteq I$ with $\mathcal{H}^1(E_\sigma) \leq \sigma$ such that $Du^\varepsilon \longrightarrow Du$ uniformly on $I \setminus E_\sigma$ as $\varepsilon \to 0$. Since

$u^\varepsilon(\overline{x}) = u(\overline{x})$, by the 1-dimensional Poincaré inequality, for $\varepsilon > 0$ small we have

$$
\begin{aligned}
\int_0^d \big|u^\varepsilon(\overline{x}+te) - u(\overline{x}+te)\big|\,dt \ &\leq\ d\int_0^d \big|Du^\varepsilon(\overline{x}+te)e - Du(\overline{x}+te)e\big|\,dt \\
&\leq\ d^2 \sup_{I\setminus E_\sigma} \big|Du^\varepsilon - Du\big| \\
&\quad +\ d\big(2\|Du\|_{L^\infty(\Omega)} + 1\big)\mathcal{H}^1(E_\sigma).
\end{aligned}
\tag{2.42}
$$

Since $u^\varepsilon \longrightarrow v$ in $C^0\big(\overline{\Omega}\big)^N$ and $Du^\varepsilon \longrightarrow Du$ in $C^0(I\setminus E_\sigma)^{Nn}$ as $\varepsilon \to 0$, by passing to the limit in (2.42) we obtain

$$
\int_0^d \big|v(\overline{x}+te) - u(\overline{x}+te)\big|\,dt \ \leq\ d\,\big(2\|Du\|_{L^\infty(\Omega_R)} + 1\big)\sigma. \tag{2.43}
$$

By letting $\sigma \to 0$, by (2.43) we get $u \equiv v$ on $I \subseteq \Omega$. Since this holds for $\mathcal{H}^{n-1}$-a.e. direction $e \in \mathbb{S}^{N-1}$, we get $u \equiv v$ on Ω. Hence, $u^\varepsilon \overset{*}{\longrightarrow} u$ in $W^{1,\infty}(\Omega)^N$ as $\varepsilon \to 0$. Since $Df^\varepsilon \overset{*}{\longrightarrow} w$ in $L^\infty(\Omega)^n$, for $\varepsilon > 0$ small we have

$$
\begin{aligned}
|f^\varepsilon(x) - f^\varepsilon(y)| \ &\leq\ \|Df^\varepsilon\|_{L^\infty(\Omega)}|x-y| \\
&\leq\ \big(\|w\|_{L^\infty(\Omega)} + 1\big)|x-y|.
\end{aligned}
\tag{2.44}
$$

We further normalize the family f^ε by considering appropriate shifts denoted again by f^ε such that $f^\varepsilon(\overline{x}) = f(\overline{x})$. By replacing also each $\nu^{\varepsilon i}$ with the translate $\nu^{\varepsilon i}\big(\_ - (f(\overline{x}) - f^\varepsilon(\overline{x}))\big)$, we do not affect the previous nornalization $u^\varepsilon(\overline{x}) = u(\overline{x})$. Consequently, (2.44) implies

$$
\begin{aligned}
\|f^\varepsilon\|_{L^\infty(\Omega)} \ &\leq\ d\|Df^\varepsilon\|_{L^\infty(\Omega)} \ +\ |f^e(\overline{x})| \\
&\leq\ d\big(\|w\|_{L^\infty(\Omega)} + 1\big) \ +\ |f(\overline{x})|.
\end{aligned}
\tag{2.45}
$$

As a result, there exists an f such that $f^\varepsilon \overset{*}{\longrightarrow} f$ in $W^{1,\infty}(\Omega)$ as $\varepsilon \to 0$.

Since $\dot{\nu}^{\varepsilon i} \circ f^\varepsilon = \xi^\varepsilon$ on B_i^ε and $\dot{\nu}^{\varepsilon i} \circ f^\varepsilon = 0$ on $\Omega \setminus B_i^\varepsilon$, for $\varepsilon, \delta > 0$ small we have

$$
\begin{aligned}
\big|B_i^\varepsilon \triangle B_i^\delta\big| \ &=\ \int_\Omega \big|\chi_{B_i^\varepsilon} - \chi_{B_i^\delta}\big| \\
&=\ \int_\Omega \big||\dot{\nu}^{\varepsilon i} \circ f^\varepsilon| - |\dot{\nu}^{\delta i} \circ f^\delta|\big| \\
&\leq\ \int_\Omega \big|\dot{\nu}^{\varepsilon i} \circ f^\varepsilon - \dot{\nu}^{\delta i} \circ f^\delta\big| \\
&=\ \int_\Omega \big|\xi^\varepsilon - \xi^\delta\big|.
\end{aligned}
\tag{2.46}
$$

Since $\xi^\varepsilon \longrightarrow \xi$ in $L^1(\Omega)^N$, for each $i \in \mathbb{N}$ the family $\{B_i^\varepsilon\}_{\varepsilon>0}$ is Cauchy in measure and hence has a measurable limit $B_i \subseteq \Omega$. Since for all $\varepsilon > 0$ we have $\Omega = \cup_1^\infty B_i^\varepsilon$ and $B_i^\varepsilon \cap B_j^\varepsilon = \emptyset$ for $i \neq j$, the limit family forms a cover of Ω except perhaps for a nullset: $\big|\Omega \setminus (\cup_1^\infty B_i)\big| = 0$.

We recall that we have $u^\varepsilon = \nu^{\varepsilon i} \circ f^\varepsilon$ on B_i^ε and also $\|\dot\nu^{\varepsilon i}\|_{L^\infty(\mathbb{R})} \leq 1$ and $\dot\nu^{\varepsilon i} \equiv 0$ on $\mathbb{R} \setminus f^\varepsilon(B_i^\varepsilon)$.

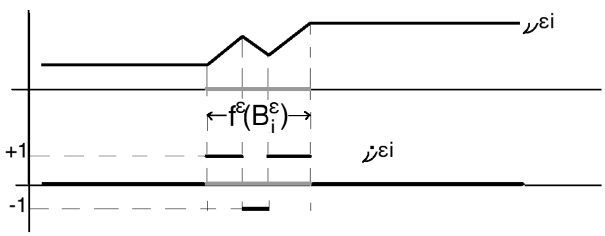

Figure 5.

Hence, if $\overline{x} \in B_i^\varepsilon$, for any $t \in \mathbb{R}$ we have

$$
\begin{aligned}
|\nu^{\varepsilon i}(t)| \ &\leq \ \|\dot\nu^{\varepsilon i}\|_{L^\infty(\mathbb{R})}|t - f^\varepsilon(\overline{x})| \ + \ |\nu^{\varepsilon i}(f^\varepsilon(\overline{x}))| \\
&= \ |t - f(\overline{x})| \ + \ |u^\varepsilon(\overline{x})| \\
&= \ |t - f(\overline{x})| \ + \ |u(\overline{x})|.
\end{aligned}
\tag{2.47}
$$

If $\overline{x} \notin B_i^\varepsilon$, then $f(\overline{x})$ is in the complement of the interval $f^\varepsilon(B_i^\varepsilon)$ and since $|\nu^{\varepsilon i}|$ is constant on $\mathbb{R} \setminus f^\varepsilon(B_i^\varepsilon)$, for any $t \in \mathbb{R}$ we have

$$
\begin{aligned}
|\nu^{\varepsilon i}(t)| \ &\leq \ \|\dot\nu^{\varepsilon i}\|_{L^\infty(\mathbb{R})}|t - f(\overline{x})| \ + \ |\nu^{\varepsilon i}(f(\overline{x}))| \\
&\leq \ |t - f(\overline{x})| \ + \ \max\left|\nu^{\varepsilon i}(\partial(f^\varepsilon(B_i^\varepsilon))\right| \\
&= \ |t - f(\overline{x})| \ + \ \sup_{B_i^\varepsilon}|\nu^{\varepsilon i} \circ f^\varepsilon| \\
&\leq \ |t - f(\overline{x})| \ + \ \|u^\varepsilon\|_{L^\infty(\Omega)}.
\end{aligned}
\tag{2.48}
$$

As a result, since the family u^ε is uniformly bounded on Ω, for each $i \in \mathbb{N}$ the family $\{\nu^{\varepsilon i}\}_{\varepsilon>0} \subseteq W^{1,\infty}_{loc}(\mathbb{R})^N$ has a weak* limit $\nu^i \in W^{1,\infty}_{loc}(\mathbb{R})^N$ which satisfies $\|\dot\nu^i\|_{L^\infty(\mathbb{R})} \leq 1$. By passing to the limit as $\varepsilon \to 0$ we get $u = \nu^i \circ f$ on $B_i \subseteq \Omega$ and $\nu^i = 0$ on $\mathbb{R} \setminus f(B_i)$.

Finally, the image $u(\Omega)$ is 1-rectifiable in $\mathbb{R}^N$ and up to an $\mathcal{H}^1$-nullset of $\mathbb{R}^N$, we have $u(\Omega) = \cup_1^\infty \nu^i(f(B_i))$. The theorem follows. □

3. The Structure of 2-Dimensional ∞-Harmonic Maps.

In this section we use the Rigidity Theorem 2.1 of Section 2 to analyse the phase separation of classical solutions to (1.2) when $n = 2$ and $N \geq 2$.

Theorem 3.1 (Structure of ∞-Harmonic Maps)**.** *Suppose $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^N$, where u is an ∞-Harmonic map in $C^2(\Omega)^N$, that is solution to*

$$
\Delta_\infty u \ = \ \Big(Du \otimes Du \ + \ |Du|^2[Du]^\perp \otimes I\Big) : D^2u \ = \ 0.
\tag{3.1}
$$

Then, there exists disjoint open sets Ω_1, $\Omega_2 \subseteq \Omega$ and a a closed nowhere dense set $\mathcal{S}$ such that $\Omega = \Omega_1 \cup \mathcal{S} \cup \Omega_2$ and:

(i) On Ω_2 we have rk$(Du) = 2$ and the map $u\big|_{\Omega_2} : \Omega_2 \longrightarrow \mathbb{R}^N$ is an Eikonal immersion with

$$
|Du|^2 \ = \ const. > 0,
\tag{3.2}
$$

on each connected component of Ω_2.

(ii) On Ω_1 we have rk$(Du) = 1$ and the map $u\big|_{\Omega_1} : \Omega_1 \longrightarrow \mathbb{R}^N$ is given by an essentially scalar ∞-Harmonic function $f \in C^2(\Omega_1)$:

$$u \ = \ a \ + \ \xi f \ , \ \ \Delta_\infty f \ = \ 0, \ a \in \mathbb{R}^N, \ \xi \in \mathbb{S}^{N-1}, \tag{3.3}$$

where ξ and a may vary on different connected components of Ω_1.

(iii) On $\mathcal{S}$, $|Du|^2$ is constant and also rk$(Du) = 1$. Moreover if $\mathcal{S} = \partial\Omega_1 \cap \partial\Omega_2$ (that is if both the $1-$ and $2-$dimensional phases exists, one on each side of the interface $\mathcal{S}$) then $u\big|_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathbb{R}^N$ is given by an essentially scalar Eikonal function:

$$u \ = \ a \ + \ \xi f \ , \ \ |Df| \ = \ const. > 0, \ a \in \mathbb{R}^N, \ \xi \in \mathbb{S}^{N-1}. \tag{3.4}$$

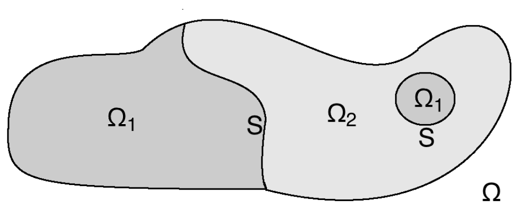

Figure 6.

By employing Aronsson's result in [A4] on the non-existence of zeros for the gradient of scalar ∞-Harmonic functions on the plane, Theorem 3.1 readily implies the following

Corollary 3.2 (∞-Harmonic Maps have Positive Rank)**.** *Let $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ be an ∞-Harmonic map in $C^2(\Omega)^N$. Then, either $|Du| > 0$ on Ω or $|Du| \equiv 0$ on Ω. Hence, non-constant ∞-Harmonic maps have positive rank.*

Proof of Theorem 3.1. We begin by assuming $N \geq 2$ and setting

$$\Omega_1 \ := \ \text{int}\big\{\text{rk}(Du) \leq 1\big\}, \tag{3.5}$$

$$\Omega_2 \ := \ \big\{\text{rk}(Du) = 2\big\}, \tag{3.6}$$

and let also $\mathcal{S} := \Omega \setminus (\Omega_1 \cup \Omega_2)$. Our PDE system (3.1) decouples to

$$Du\, D\Big(\frac{1}{2}|Du|^2\Big) \ = \ 0, \tag{3.7}$$

$$|Du|^2[Du]^\perp \Delta u \ = \ 0. \tag{3.8}$$

On Ω_2, we have rk$(Du) = 2$ and hence $u\big|_{\Omega_2} : \Omega_2 \longrightarrow \mathbb{R}^N$ is an immersion. Thus, $Du(x)$ possesses a left inverse $(Du(x))^{-1}$ for all $x \in \Omega_2$. Hence, (3.7) implies

$$(Du)^{-1} Du\, D\Big(\frac{1}{2}|Du|^2\Big) \ = \ 0 \tag{3.9}$$

and hence $D\Big(\frac{1}{2}|Du|^2\Big) = 0$ on Ω_2, or equivalently

$$|Du|^2 \ = \ \text{const.}, \tag{3.10}$$

on each connected component of Ω_2. Moreover, (3.10) holds on $\mathcal{S}$ as well, the common boundary of Ω_2 and Ω_1.

On the other hand, on Ω_1 we have rk$(Du) \leq 1$. Hence, there exist vector fields $\xi : \Omega_1 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ and $w : \Omega_1 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ such that $Du = \xi \otimes w$. Suppose first that Ω_1 is contractible. Then, by the Rigidity Theorem 2.1, there exists a function $f \in C^2(\Omega_1)$, a partition of Ω_1 to Borel sets $\{B_i\}_{i\in\mathbb{N}}$ and Lipschitz curves $\{\nu^i\}_{i\in\mathbb{N}} \subseteq W^{1,\infty}_{loc}(\mathbb{R})^N$ with $|\dot{\nu}^i| \equiv 1$ on $f(B_i)$, $|\dot{\nu}^i| \equiv 0$ on $\mathbb{R}\setminus f(B_i)$ twice differentiable on $f(B_i)$, such that $u = \nu^i \circ f$ on each B_i and hence $Du = (\dot{\nu}^i\circ f)\otimes Df$ on B_i. By (3.7), we obtain

$$\begin{aligned}(3.11)\qquad &\big((\dot{\nu}^i\circ f)\otimes Df\big)\otimes\big((\dot{\nu}^i\circ f)\otimes Df\big) :\\ &: \Big[(\ddot{\nu}^i\circ f)\otimes Df\otimes Df \,+\, (\dot{\nu}^i\circ f)\otimes D^2 f\Big] \;=\; 0,\end{aligned}$$

on $B_i \subseteq \Omega_1$. Since $|\dot{\nu}^i| \equiv 1$ on B_i, we have that $\ddot{\nu}^i$ is normal to $\dot{\nu}^i$ and hence

$$(3.12)\qquad \big((\dot{\nu}^i\circ f)\otimes Df\big)\otimes\big((\dot{\nu}^i\circ f)\otimes Df\big) : \big((\dot{\nu}^i\circ f)\otimes D^2 f\big) \;=\; 0,$$

on $B_i \subseteq \Omega_1$. Hence, by using again that $|\dot{\nu}^i|^2 \equiv 1$ on B_i we get

$$(3.13)\qquad \big(Df\otimes Df : D^2 f\big)(\dot{\nu}^i\circ f) \;=\; 0,$$

on $B_i \subseteq \Omega_1$. Thus, $\Delta_\infty f = 0$ on B_i. By (3.8) and again since $|\dot{\nu}^i|^2 \equiv 1$ on B_i, we have $[Du]^\perp = [\dot{\nu}^i\circ f]^\perp$ and hence

$$(3.14)\qquad |Df|^2\,[\dot{\nu}^i\circ f]^\perp \mathrm{Div}\big((\dot{\nu}^i\circ f)\otimes Df\big) \;=\; 0,$$

on $B_i \subseteq \Omega_1$. Hence,

$$(3.15)\qquad |Df|^2\,[\dot{\nu}^i\circ f]^\perp\Big((\ddot{\nu}^i\circ f)|Df|^2 \,+\, (\dot{\nu}^i\circ f)\Delta f\Big) \;=\; 0,$$

on B_i, which by using once again $|\dot{\nu}^i|^2 \equiv 1$ gives

$$(3.16)\qquad |Df|^4(\ddot{\nu}^i\circ f) \;=\; 0,$$

on B_i. Since $\Delta_\infty f = 0$ on B_i and $\Omega_1 = \cup_1^\infty B_i$, f is ∞-Harmonic on Ω_1. Thus, by Aronsson's theorem in [A4], either $|Df| > 0$ or $|Df| \equiv 0$ on Ω_1.

If the first alternative holds, then by (3.16) we have $\ddot{\nu}^i \equiv 0$ on $f(B_i)$ for all i and hence ν^i is affine on $f(B_i)$, that is $\nu^i(t) = t\xi^i + a^i$ for some $|\xi^i| = 1$, $a^i \in \mathbb{R}^N$. Thus, since $u = \nu^i\circ f$ and $u \in C^2(\Omega_1)^N$, all ξ^i and all a^i coincide and consequently $u = \xi f + a$, $\xi \in \mathbb{S}^{N-1}$, where $a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1)$.

If the second alternative holds, then f is constant on Ω_1 and hence by the representation $u = \nu^i\circ f$, u is piecewise constant on each B_i. Since $u \in C^2(\Omega_1)^N$ and $\Omega_1 = \cup_1^\infty B_i$, necessarily u is constant on Ω_1. But then $|Du|_{\Omega_2}| = |Df|_{\mathcal{S}}| = 0$ and necessarily $\Omega_2 = \emptyset$. Hence, $|Du| \equiv 0$ on Ω, that is u is affine on each of the connected components of Ω.

If Ω_1 is not contractible, cover it with balls $\{\mathbb{B}_m\}_{m\in\mathbb{N}}$ and apply the previous argument. Hence, on each $\mathbb{B}_m$, we have $u = \xi^m f^m + a^m$, $\xi^m \in \mathbb{S}^{N-1}$, $a^m \in \mathbb{R}^N$ and $f^m \in C^2(\mathbb{B}_m)$ with $\Delta_\infty f^m = 0$ on $\mathbb{B}_m$ and hence either $|Df^m| > 0$ or $|Df^m| \equiv 0$. Since $u \in C^2(\Omega_1)^N$, on the overlaps of the balls the different expressions of u must coincide and hence we obtain $u = \xi f + a$ for $\xi \in \mathbb{S}^{N-1}$, $a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1)$ where ξ and a may vary on different connected components of Ω_1. The theorem follows. □

Theorem 3.1 implies a vectorial version of the Maximum Principle when $n = N = 2$, which we now prove. It seems that under a thorough analysis of the hierachy of the multiple m phases $\Omega_1, ..., \Omega_m$ of higher-dimensional ∞-Harmonic maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ with $m = \min\{n, N\}$, Corollary 3.3 below extends to the case of $n \geq N$, but generally not the case of $n < N$ for positive codimension.

Corollary 3.3 (Convex Hull Property)**.** *Suppose that $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an ∞-Harmonic map. Then, for all $\Omega' \subset\subset \Omega$, the image $u(\Omega')$ is contained in the closed convex hull of the boundary values:*

$$u(\Omega') \ \subseteq\ \overline{co}\big(u(\partial\Omega')\big). \tag{3.17}$$

Proof of Corollary 3.3. We begin by observing that (3.17) is an elegant restatement of the Maximum Principle for all projections $\eta^\top u$ of u, that is, when for all $\Omega' \subset\subset \Omega$ and all directions $\eta \in \mathbb{S}^{N-1}$ we have

$$\sup_{\Omega'} \eta^\top u \ \leq\ \max_{\partial\Omega'} \eta^\top u. \tag{3.18}$$

Indeed, (3.18) says that $u(\Omega')$ is contained in the intersection of all halfspaces containing $u(\partial\Omega')$.

To see (3.18), fix Ω' and $\eta \in \mathbb{S}^{N-1}$ and let Ω_1, Ω_2, $\mathcal{S}$ respectively be the constant rank domains and the interface of u, as in Theorem 3.1. Suppose that $u = \xi f + a$ on $\Omega_1 \cup \mathcal{S}$, where $\xi \in \mathbb{S}^{N-1}$, $a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1 \cup \mathcal{S})$. Then,

$$\begin{aligned} |D(\eta^\top u)| \ &= \ |\eta^\top Du|\chi_{\Omega_2} \ + \ |\eta^\top Du|\chi_{\mathcal{S}\cup\Omega_1} \\ &= \ |\eta^\top Du|\chi_{\Omega_2} \ + \ |\eta^\top \xi|\,|Df|\chi_{\mathcal{S}\cup\Omega_1}. \end{aligned} \tag{3.19}$$

If $|Df| \equiv 0$ on Ω_1, then $\Omega_2 = \emptyset$ and u is affine. Hence, (3.18) follows. Hence, suppose $|Df| > 0$ on Ω_1. Since $u|_{\Omega_2}$ is a local diffeomorphism, we have $|\eta^\top Du| > 0$ for all $\eta \in \mathbb{S}^{N-1}$.

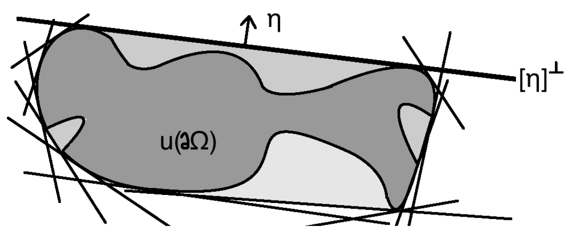

Figure 7.

Consequently, for all $\eta \in \mathbb{S}^{N-1} \setminus [\xi]^\perp$, in view of (3.19) we have $|D(\eta^\top u)| > 0$ on Ω. Hence, $\eta^\top u$ has no interior critical points inside Ω and consequently we have

$$\max_{\overline{\Omega'}} \eta^\top u \ = \ \max_{\partial\Omega'} \eta^\top u, \tag{3.20}$$

for all directions $\eta \not\perp \xi$. By letting $\mathrm{dist}(\eta, [\xi]^\perp) \longrightarrow 0$, (3.20) implies (3.18) for all $\eta \in \mathbb{S}^{N-1}$. □

As a topological consequence of the Convex Hull Property for planar ∞-Harmonic maps, we deduce the following:

Corollary 3.4 (Absence of Interfaces)**.** *Suppose that $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an ∞-Harmonic map. Then:*

(i) If $\Omega_2 = \{rk(Du) = 2\} \subset\subset \Omega$, then $\Omega_2 = \emptyset$ and $\mathcal{S} = \emptyset$. Hence, either the set whereon u is a local diffeomorphism has a common boundary portion with Ω or it is empty and u is everywhere essentially scalar without any interface $\mathcal{S}$.

(ii) If $\Omega \subset\subset \mathbb{R}^2$ and u extends on $\partial\Omega$ as an essentially scalar function, there is no interface $\mathcal{S}$ inside Ω and u is an essentially scalar ∞-Harmonic function throughout Ω.

3.1. **Rigidity of Radial 2-Dimensional Solutions.** In this subsection we study a class of special solutions of the ∞-Laplacian, that of smooth ∞-Harmonic maps $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^N$, $N \geq 2$ of the form $u = \rho^k f(k\theta)$ in polar coordinates (ρ, θ). Here $k > 0$ is a parameter and $f : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a curve in $\mathbb{R}^N$. It follows that such solutions are very rigid, because if $k \neq 1$ they are essentially scalar and if $k = 1$ they always have affine image. Accordingly, we have

Proposition 3.5 (Rigidity of Radial 2-D ∞-Harmonic maps)**.** *Suppose that $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^N$, $N \geq 2$, in an ∞-Harmonic map of the form $u = \rho^k f(k\theta)$ in polar coordinates $(\rho, \theta) \in \mathbb{R}^2$, $k > 0$, $f \in C^\infty(\mathbb{R})^N$. Then, f solves the ODE systems*

$$f' \otimes f'\big(f'' + f\big) \; + \; \frac{k-1}{k}\big(|f'|^2 + |f|^2\big) f \; = \; 0, \tag{3.21}$$

$$[(f', f)]^\perp f'' \; = \; 0. \tag{3.22}$$

Moreover:

(i) If $k \neq 1$, then all such solutions have constant rank one and the image $u(\mathbb{R}^2)$ is contained into a line passing through the origin and f can be represented as $f(\theta) = \xi g(\theta)$ for some $\xi \in \mathbb{S}^{N-1}$ and $g \in C^\infty(\mathbb{R})$.

If $k = 1$, then all such solutions have rank at most two and the image $u(\mathbb{R}^2)$ is contained into a 2-plane of $\mathbb{R}^N$ passing through the origin. On this plane f can be represented by

$$f(\theta) \; = \; c \cos B(\theta) \big(\cos A(\theta), \sin A(\theta)\big)^\top, \tag{3.23}$$

where $c \in \mathbb{R}$ and $A, B \in C^\infty(\mathbb{R})$ satisfy the differential equation

$$|B'|^2 + |A'|^2 \cot^2 B \; = \; 1 \tag{3.24}$$

and also $0 < B \leq \frac{\pi}{2}$.

Proof of Proposition 3.5. The derivation of the "tangential part" (3.21) of Δ_∞ is entirely analogous to Aronsson's derivation of its scalar counterpart in the paper [A6], p. 138. Hence, it suffices to outline the derivation of the "normal part" (3.22). Since for all $\alpha \in \{1, ..., N\}$ we have $u_\alpha = \rho^k f_\alpha(k\theta)$, we obtain

$$\begin{aligned} \begin{bmatrix} D_x u_\alpha \\ D_y u_\alpha \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} D_\rho u_\alpha \\ \frac{1}{\rho} D_\theta u_\alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} k\rho^{k-1} f_\alpha \\ k\rho^{k-1} f'_\alpha \end{bmatrix}. \end{aligned} \tag{3.25}$$

Let $O(\theta)$ denote the rotation-by-θ appearing in (3.25). Since (f, f') is a matrix-valued curve $\mathbb{R} \longrightarrow \mathbb{R}^N \otimes \mathbb{R}^2$, we have

$$Du \; = \; k\rho^{k-1}(f, f')\, O(\theta)^\top. \tag{3.26}$$

Hence, since $O(\theta)^\top = O(\theta)^{-1}$ we have

$$\begin{aligned} N(Du^\top) \; &= \; \big\{\eta \in \mathbb{R}^N \; : \; \eta^\top (f, f')\, O(\theta)^\top = 0\big\} \\ &= \; \big\{\eta \in \mathbb{R}^N \; : \; \eta^\top (f, f') = 0\big\} \\ &= \; N\big((f, f')^\top\big). \end{aligned} \tag{3.27}$$

and consequently $[Du]^\perp = [(f,f')]^\perp$. Moreover,

$$
\begin{aligned}
[Du]^\perp \Delta u\ &= [(f,f')]^\perp \left(\frac{1}{\rho}D_\rho u \,+\, D^2_{\rho\rho}u \,+\, \frac{1}{\rho^2}D^2_{\theta\theta}u\right)\\
&= [(f,f')]^\perp \Big(k\rho^{k-2}f \,+\, k(k-1)\rho^{k-2}f \,+\, k^2\rho^{k-2}f''\Big)\\
&= k^2\rho^{k-2}[(f,f')]^\perp f''.
\end{aligned}
\tag{3.28}
$$

By Corollary 3.2, we may require $|Du|>0$ and hence (3.22) follows by (3.8) and (3.28). Now, for (i) we have that if $k\neq 1$ then on $\{|f|>0\}$ (3.21) gives

$$
-\,\frac{k\big(f''\,+\,f\big)^\top f'}{(k-1)\big(|f'|^2\,+\,|f|^2\big)}f' \;=\; f. \tag{3.29}
$$

Consequently, f' is everywhere proportional to f and as a result $f(\mathbb{R})$ is contained into an 1-dimensional subspace of $\mathbb{R}^N$.

For (ii), we have that if $k=1$ then (3.22) implies $f''=\lambda f+\mu f'$ for some $\lambda,\mu\in C^\infty(\mathbb{R})$. Hence, $f(\mathbb{R})$ is contained into a 2-dimensional subspace of $\mathbb{R}^N$. (3.21) gives the extra condition that $f'^\top(f''+f)=0$ which implies $|f'|^2+|f|^2=c^2$ for some $c\in\mathbb{R}$. Hence, if $c\neq 0$ then $\frac{1}{c}(|f'|,|f|)^\top$ is on the unit circle and as such $|f|=c\cos B$ and $|f'|=c\sin B$, for some B valued in $[0,\frac{\pi}{2}]$. Hence, $f=c\cos B(\cos A,\sin A)^\top$ for some A. The differential relation $|B'|^2+|A'|^2\cot^2 B=1$ follows easily. $\square$

4. Classification of Elliptic Aronsson PDE Systems.

In this section we focus to the more general Aronsson system (1.1). As already explained in the introduction, when $N\geq 2$ the normal coefficient $H[H_P]^\perp H_{PP}$ is not commutative and as a result the system generally is not degenerate elliptic, not even for strictly convex Hamiltonians.

In Theorem 4.1 below we establish that all "geometric" Hamiltonians which depend on Du via the induced Riemannian metric $Du^\top Du$ lead to elliptic systems. Moreover, in low dimensions $n\leq 3$ the converse is true as well for (normalized) analytic Hamiltonians with fully symmetric Hessian tensor. When $n\geq 4$, there appear complicated structures in the minors of forth and higher order derivatives and an additional assumption is required. The constructive method of proof reveals that it is necessary.

The main idea in the reverse direction is to impose the commutativity relation $[H_P]^\perp H_{PP}=H_{PP}[H_P]^\perp$ and use power-series expansions of H to derive the form (1.18) inductively, by a term-after-term blow-up argument along inverse images under H_P of rank-one directions.

Theorem 4.1 (Classification of Hamiltonians leading to elliptic Aronsson systems)**.** *Suppose that $H\in C^2(\mathbb{R}^N\otimes\mathbb{R}^n)$ is a non-negative Hamiltonian with $n\geq 1$, $N\geq 2$.*

Consider the following statements:

(i) There exists $h\in C^2(\mathbb{S}(\mathbb{R}^n))$, $h=h(p)$ with positive symmetric gradient $h_p=h_p^\top>0$ such that

$$
H(P)\;=\;h\left(\frac{1}{2}P^\top P\right). \tag{4.1}
$$

(ii) The Aronsson PDE system

$$A_\infty u \;:=\; \Big(H_P \otimes H_P \;+\; H\,[H_P]^\perp H_{PP}\Big)(Du) : D^2 u \;=\; 0 \tag{4.2}$$

is quasilinear and degenerate elliptic, that is, the tensor map

$$A_{\alpha i \beta j}(P) \;:=\; H_{P_{\alpha i}}(P) H_{P_{\beta j}}(P) + H(P)[H_P(P)]^\perp_{\alpha\gamma} H_{P_{\gamma i} P_{\beta j}}(P) \tag{4.3}$$

satisfies the strict Legendre-Hadamard condition and the symmetry condition

$$A(P) : (\eta\otimes w)\otimes(\eta\otimes w) \;>\; 0, \tag{4.4}$$

$$A(P) : (Q\otimes R - R\otimes Q) \;=\; 0, \tag{4.5}$$

for all $\eta \in \mathbb{R}^N \setminus \{0\}$, $w \in \mathbb{R}^n \setminus \{0\}$ *and* $P, Q, R \in \mathbb{R}^N \otimes \mathbb{R}^n \setminus \{0\}$.

Then, (i) implies (ii). If moreover H *is analytic at* 0 *and satisfies*

$$\{H_P = 0\} = \{H = 0\} = \{0\}\,, \quad H_{PP}(0) > 0 \quad \text{and} \quad H_{PP} : (v\otimes w - w \otimes v) = 0, \tag{4.6}$$

for $v, w \in \mathbb{R}^n$, *then, (ii) implies (i) when either*
a) $n \leq 3$,
or
b) $n \geq 4$ *and the q-th order derivative tensor* $H_{P\ldots P}(0) \in \otimes^{(q)}(\mathbb{R}^N \otimes \mathbb{R}^n)$ *is contained in the linear subspace* $\mathscr{L}^q$ *which consists of fully symmetric tensors* T *for which the only non-trivial components are of the form* $T_{\alpha_1 i \alpha_2 j \alpha_3 k \ldots \alpha_q k}$, *where* $\alpha_m \in \{1, ..., N\}$, $i, j, k \in \{1, ..., n\}$.

If $n \geq 4$ *but* $H_{P\ldots P}(0) \notin \mathscr{L}^q$, *then* H *has the form* (4.1) *up to a forth order correction:* $H(P) = h\big(\frac{1}{2}P^\top P\big) + O(|P|^4)$.

In the case where $H(P)$ *equals* $h\big(\frac{1}{2}P^\top P\big)$, *the elliptic Aronsson system takes the form*

$$A_\infty u \;=\; \Big(Du h_p \otimes Du h_p \;+\; h[Du]^\perp \otimes h_p\Big) : D^2 u \;=\; 0 \tag{4.7}$$

and then $h = h\big(\frac{1}{2} Du^\top Du\big)$.

The extra assumption $H_{P\ldots P}(0) \in \mathscr{L}^q$ is necessary only in higher dimensions $n \geq 4$. It requires that $H_{P\ldots P}(0)$ vanishes when more than 3 of its Latin indices are different to each other. The linear space $\mathscr{L}^q$ can be descrided as

$$\begin{aligned} \mathscr{L}^q \;:=\; \Big\{ T \in \otimes^{(q)}(\mathbb{R}^N \otimes \mathbb{R}^n) \;\big|\; & T = T_{\alpha_1 i_1 \ldots \alpha_q i_q} e_{\alpha_1 i_1} \otimes \ldots \otimes e_{\alpha_q i_q} : \\ & T_{\ldots \alpha i \ldots \beta j \ldots} \;=\; T_{\ldots \beta i \ldots \alpha j \ldots} \;=\; T_{\ldots \beta j \ldots \alpha i \ldots}\,, \\ & \{i_1, \ldots i_q\} \neq \{i, j, k, \ldots, k\} \implies T = 0 \Big\}. \end{aligned} \tag{4.8}$$

If $H_{P\ldots P}(0) \notin \mathscr{L}^q$, then Hamiltonians with a little more complicated fourth and higher order derivatives also give rise to elliptic systems.

Proof of Theorem 4.1. We first prove the implication $(i) \Rightarrow (ii)$. For, assume that the Hamiltonian H has the form (4.1). We begin by observing that the symmetry assumption $h_{p_{ij}} = h_{p_{ji}}$ implies that second derivatives of h are fully symmetric in all indices: obviously since h is in $C^2(\mathbb{S}(\mathbb{R}^n))$ we have $h_{p_{ij} p_{kl}} = h_{p_{kl} p_{ij}}$ and also

$$h_{p_{ij} p_{kl}} = (h_{p_{kl}})_{p_{ij}} = (h_{p_{lk}})_{p_{ij}} = h_{p_{ij} p_{lk}}. \tag{4.9}$$

Using that, we suppress the arguments in the notation of h and calculate

$$\begin{aligned} H_{P_{\alpha i}}(P) &= \frac{1}{2}h_{p_{kl}}\Big(\delta_{\alpha\beta}\delta_{ki}P_{\beta l} \ + \ \delta_{\alpha\beta}\delta_{il}P_{\beta k}\Big) \\ &= P_{\alpha k}h_{p_{ki}}, \end{aligned} \tag{4.10}$$

and also

$$\begin{aligned} H_{P_{\alpha i}P_{\beta j}}(P) &= \frac{1}{2}h_{p_{ik}p_{lm}}\Big(\delta_{\beta\gamma}\delta_{jl}P_{\gamma m} \ + \ \delta_{\beta\gamma}\delta_{jm}P_{\gamma l}\Big)P_{\alpha k} \\ &\quad + \ h_{p_{ik}}\delta_{\alpha\beta}\delta_{kj} \\ &= \delta_{\alpha\beta}h_{p_{ij}} \ + \ h_{p_{ik}p_{jm}}P_{\alpha k}P_{\beta m}. \end{aligned} \tag{4.11}$$

Also, since $h_p > 0$ in $\mathbb{S}(\mathbb{R}^n)$, the operators $H_P(P)^\top$, $P^\top : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ have the same nullspaces:

$$\begin{aligned} N(H_P(P)^\top) &= \{\eta \in \mathbb{R}^N \ : \ \eta^\top H_P(P) = 0\} \\ &= \{\eta \in \mathbb{R}^N \ : \ \eta^\top P h_p = 0\} \\ &= \{\eta \in \mathbb{R}^N \ : \ \eta^\top P = 0\} \\ &= N(P^\top). \end{aligned} \tag{4.12}$$

Hence, we obtain that $[H_P(P)]^\perp = [P]^\perp$. By (4.10), (4.11), (4.12), we have

$$\begin{aligned} \Big(H_P \otimes H_P + H[H_P]^\perp H_{PP}\Big)(P) &= Ph_p \otimes Ph_p \ + \ h[P]^\perp\Big(I \otimes h_p \ + \ Ph_{pp}P^\top\Big) \\ &= Ph_p \otimes Ph_p \ + \ h[P]^\perp \otimes h_p, \end{aligned} \tag{4.13}$$

where $h = h\big(\frac{1}{2}P^\top P\big)$. Hence, in view of (4.3), equation (4.7) follows. Also, since $h \geq 0$ and h_p, $[P]^\perp$ are positive symmetric, conditions (4.4) and (4.5) follow as well:

$$\begin{aligned} \mathrm{A}(P) : (\eta \otimes w) \otimes (\eta \otimes w) &= (P_{\alpha k}h_{p_{ki}}\eta_\alpha w_i)(P_{\beta l}h_{p_{lj}}\eta_\beta w_j) \\ &\quad + h([P]^\perp_{\alpha\beta}\eta_\alpha\eta_\beta)(h_{p_{ij}}w_i w_j) \\ &= (Ph_p : \eta \otimes w)^2 \\ &\quad + h([P]^\perp : \eta \otimes \eta)(h_p : w \otimes w) \\ &\geq \ 0, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \mathrm{A}(P) : (Q \otimes R - R \otimes Q) &= \pm(P_{\alpha k}h_{p_{ki}}Q_{\alpha i})(P_{\beta l}h_{p_{lj}}R_{\beta j}) \\ &\quad \pm \ h[P]^\perp_{\alpha\beta}Q_{\alpha i}h_{p_{ij}}R_{\beta j} \\ &= \pm(Ph_p : Q)(Ph_p : R) \ \pm \ h[P]^\perp : (Qh_pR^\top) \\ &= \ 0, \end{aligned} \tag{4.15}$$

for all $\eta \in \mathbb{R}^N$, $w \in \mathbb{R}^n$, $P, Q, R \in \mathbb{R}^N \otimes \mathbb{R}^n$. Hence, (ii) follows. Now we assume (ii) and prove the reverse implication. For, suppose H is analytic at 0 and assume (4.4) - (4.8). By (4.5), we have

$$\Big(H_P \otimes H_P + H[H_P]^\perp H_{PP}\Big)(P) : (Q \otimes R - R \otimes Q) \ = \ 0, \tag{4.16}$$

for all $P, Q, R \in \mathbb{R}^N \otimes \mathbb{R}^n$. By symmety of $H_P \otimes H_P$ and since by (4.6) we have $H > 0$ and $H_P \neq 0$ on $(\mathbb{R}^N \otimes \mathbb{R}^n) \setminus \{0\}$, (4.16) gives

$$[H_P]^\perp H_{PP} : (Q \otimes R - R \otimes Q) \ = \ 0. \tag{4.17}$$

By the identity $[H_P]^\perp = I - [H_P]^\top$ and since I, H_{PP} and $[H_P]^\perp$ are symmetric, for $Q = e_\alpha \otimes e_i$ and $R = e_\beta \otimes e_j$, (4.17) gives the commutativity relation

$$[H_P]^\perp_{\alpha\gamma} H_{P_{\gamma i}P_{\beta j}} \; = \; H_{P_{\alpha i}P_{\gamma j}} [H_P]^\perp_{\gamma\beta} \tag{4.18}$$

on $(\mathbb{R}^N \otimes \mathbb{R}^n) \setminus \{0\}$, that is

$$[H_P]^\perp H_{PP} \; = \; H_{PP}[H_P]^\perp. \tag{4.19}$$

We set $A_{\alpha i\beta j} := H_{P_{\alpha i}P_{\gamma j}}(0)$. By assumtpion (4.6), we have $A > 0$ in $\mathbb{S}(\mathbb{R}^N \otimes \mathbb{R}^n)$. By analyticity of H and since $H(0) = 0$ and $H_P(0) = 0$, we have

$$H(P) \; = \; \frac{1}{2} A : P \otimes P \; + \; O(|P|^3), \tag{4.20}$$

$$H_P(P) \; = \; A : P \; + \; O(|P|^2), \tag{4.21}$$

$$H_{PP}(P) \; = \; A \; + \; O(|P|), \tag{4.22}$$

as $|P| \to 0$. Since $A = H_{PP}(0) > 0$ and $H_P(0) = 0$, the map H_P is a diffeomorphism between open neighborhoods of zero in $\mathbb{R}^N \otimes \mathbb{R}^n$. Hence, there is an $r > 0$ such that

$$H_P \; : \; \mathbb{B}^{Nn}_r := \{Q \in \mathbb{R}^N \otimes \mathbb{R}^n : |Q| < r\} \longrightarrow H_P(\mathbb{B}^{Nn}_r) \subseteq \mathbb{R}^N \otimes \mathbb{R}^n \tag{4.23}$$

is a diffeomorphism. Hence, there is a $\rho > 0$ such that for $0 < t < \rho$, $\xi \in \mathbb{S}^{N-1}$ and $w \in \mathbb{S}^{n-1}$, there exists a unique $P(t) \in \mathbb{B}^{Nn}_r$ such that

$$t\,\xi \otimes w \; = \; H_P\big(P(t)\big). \tag{4.24}$$

Moreover, $|P(t)| \to 0$ as $t \to 0$. The path $P(\,.\,)$ is the inverse image through H_P of the rank-one line spanned by $\xi \otimes w$. By (4.24), we have

$$[H_P\big(P(t)\big)]^\top \; = \; [t\,\xi \otimes w]^\top \; = \; \xi \otimes \xi. \tag{4.25}$$

By evaluating (4.19) at $P(t)$ and using (4.25) and (4.22), we obtain

$$(\xi \otimes \xi)(A \; + \; o(1)) \; = \; (A \; + \; o(1))(\xi \otimes \xi), \tag{4.26}$$

at $t \to 0$. In the limit we get $(\xi \otimes \xi)A \; = \; A(\xi \otimes \xi)$, that is

$$\xi_\alpha \xi_\kappa A_{\kappa i\beta j} \; = \; A_{\alpha i\kappa j}\xi_\kappa \xi_\beta, \tag{4.27}$$

for all $i, j \in \{1, ..., n\}$, $\alpha, \beta \in \{1, ..., N\}$. By the symmetry condition in assumption (4.6), for all i, j fixed the matrix $A_{\alpha i\beta j}$ commutes with all 1-dimensional projections of $\mathbb{R}^N$. Hence, it is simultaneously diagonalizable with them and as such a multiple of the identity. Thus, there is a symmetric matrix $\hat{A}_{ij}$ such that

$$A_{\alpha i\beta j} \; = \; \hat{A}_{ij}\delta_{\alpha\beta}. \tag{4.28}$$

Consequently, $A : P \otimes P = \hat{A} : P^\top P$. We now set $B_{\alpha i\beta j\gamma k} := H_{P_{\alpha i}P_{\beta j}P_{\gamma k}}(0)$. Then, by (4.28), equations (4.21) and (4.22) become

$$H_P(P) \; = \; P\hat{A} \; + \; O(|P|^2), \tag{4.29}$$

$$H_{PP}(P) \; = \; I \otimes \hat{A} \; + \frac{1}{2} B : P \; + \; O(|P|^2), \tag{4.30}$$

and hence by (4.29) and (4.24) we get

$$t\,\xi \otimes w \; = \; P(t)\hat{A} \; + \; O(|P(t)|^2). \tag{4.31}$$

Since $A > 0$ in $\mathbb{S}(\mathbb{R}^N \otimes \mathbb{R}^n)$, we have $\hat{A} > 0$ in $\mathbb{S}(\mathbb{R}^n)$ as well. Thus, for $0 < t < \rho$, we have

$$\frac{P(t)}{|P(t)|} \ + \ O(|P(t)|) \ = \ \frac{t}{|P(t)|}\xi \otimes \big((\hat{A}^{-1})^\top w\big). \tag{4.32}$$

As $t \to 0$, we have $|P(t)| \to 0$ and by compactness along an infinitessimal sequence $t_m \to 0$ there exists a $\bar{P}$ with $|\bar{P}| = 1$ such that $P(t_m)/|P(t_m)| \to \bar{P}$. By passing to the limit in (4.32) as $m \to \infty$ along $\{t_m\}$, we obtain that the limit of $t_m/|P(t_m)|$ exists and

$$\lim_{m\to\infty} \frac{P(t_m)}{|P(t_m)|} \ = \ \bar{P} \ = \ \xi \otimes \left[\left(\lim_{m\to\infty} \frac{t_m}{|P(t_m)|}\right)(\hat{A}^{-1})^\top w\right]. \tag{4.33}$$

Since $\hat{A}^{-1} > 0$ and $|\bar{P}| = 1$, for any $v \in \mathbb{S}^{n-1}$, there is a $w \in \mathbb{S}^{n-1}$ such that (4.33) becomes

$$\lim_{m\to\infty} \frac{P(t_m)}{|P(t_m)|} \ = \ \bar{P} \ = \ \xi \otimes v. \tag{4.34}$$

By (4.19), (4.25), (4.29) (4.30), we have

$$\begin{aligned} &\xi \otimes \xi\Big(I \otimes \hat{A} + \frac{1}{2}B : P(t) \ + \ O(|P(t)|^2)\Big) \\ &\qquad = \ \Big(I \otimes \hat{A} \ + \ \frac{1}{2}B : P(t) \ + \ O(|P(t)|^2)\Big)\xi \otimes \xi. \end{aligned} \tag{4.35}$$

By cancelling the commutative term $\xi \otimes \xi(I \otimes \hat{A})$, (4.35) gives

$$\xi \otimes \xi \left(B : \frac{P(t)}{|P(t)|} \ + \ O(|P(t)|)\right) = \left(B : \frac{P(t)}{|P(t)|} \ + \ O(|P(t)|)\right)\xi \otimes \xi. \tag{4.36}$$

By passing to the limit in (4.36) along $t_m \to 0$, in view of (4.34) we obtain

$$\xi \otimes \xi(B : \xi \otimes v) \ = \ (B : \xi \otimes v)\xi \otimes \xi, \tag{4.37}$$

for all $\xi \in \mathbb{S}^{N-1}$, $v \in \mathbb{S}^{n-1}$. Hence, (4.37) says

$$\xi_\alpha \xi_\lambda (B_{\beta i \lambda j \mu k}\, \xi_\mu v_k) \ = \ (B_{\alpha i \lambda j \mu k}\, \xi_\mu v_k)\xi_\lambda \xi_\beta \tag{4.38}$$

and for $v = e_k$ we get

$$\xi_\alpha (B_{\beta i \lambda j \mu k}\, \xi_\mu \xi_\lambda) \ = \ (B_{\alpha i \lambda j \mu k}\, \xi_\mu \xi_\lambda)\xi_\beta, \tag{4.39}$$

or,

$$\xi \otimes (B : \xi \otimes \xi) \ = \ (B : \xi \otimes \xi) \otimes \xi. \tag{4.40}$$

By (4.40), $B : \xi \otimes \xi$ is proportional to ξ; hence, there exists a tensor map $\hat{B} : \mathbb{R}^N \longrightarrow \otimes^{(3)}\mathbb{R}^n$ such that $B : \xi \otimes \xi = \hat{B}(\xi) \otimes \xi$, or

$$B_{\alpha i \lambda j \mu k}\, \xi_\mu \xi_\lambda \ = \ \hat{B}_{ijk}(\xi)\, \xi_\alpha. \tag{4.41}$$

By assumption (4.6) and induction, all second and higher order derivatives are fully symmetric in all their indices. Hence, we may fix $i, j, k \in \{1, ..., n\}$ and suppress the dependence in them to obtain $B_{\alpha\kappa\lambda}\, \xi_\kappa \xi_\lambda \ = \ \hat{B}(\xi)\, \xi_\alpha$ with $\hat{B} \in C^\infty(\mathbb{R}^N \setminus \{0\})$. The idea now is to differentiate in order to cancel both ξ's contracted with B and then contract again with a vector which annihilates ξ from the right hand side. For, by differentiating we get

$$D_\beta \hat{B}(\xi)\xi_\alpha \ + \ \hat{B}(\xi)\delta_{\alpha\beta} \ = \ 2B_{\alpha\beta\gamma}\xi_\gamma, \tag{4.42}$$

or equivalently

$$D\hat{B}(\xi)\otimes\xi \;=\; -\hat{B}(\xi)\,I \;+\; 2B:\xi. \tag{4.43}$$

By (4.43), we obtain that $D\hat{B}(\xi)\otimes\xi\in\mathbb{S}(\mathbb{R}^N)$. Hence, we get that $D\hat{B}(\xi)\otimes\xi = \xi\otimes D\hat{B}(\xi)$ and hence there exists $\bar{B}\in C^\infty(\mathbb{R}^N\setminus\{0\})$, such that $D\hat{B}(\xi)=\bar{B}(\xi)\xi$. Thus, (4.43) gives

$$\bar{B}(\xi)\xi\otimes\xi \;+\hat{B}(\xi)\,I \;=\; 2B:\xi. \tag{4.44}$$

By differentiating $D\hat{B}(\xi)=\bar{B}(\xi)\xi$, we get

$$D\bar{B}(\xi)\otimes\xi \;=\; D^2\hat{B}(\xi) \;-\; \bar{B}(\xi)\,I. \tag{4.45}$$

By (4.45), we obtain $D\bar{B}(\xi)\otimes\xi\in\mathbb{S}(\mathbb{R}^N)$. Hence, there exists $\check{B}\in C^\infty(\mathbb{R}^N\setminus\{0\})$, such that $D\bar{B}(\xi)=\check{B}(\xi)\xi$ and hence (4.45) gives

$$D^2\hat{B}(\xi) \;=\; \check{B}(\xi)\xi\otimes\xi \;+\; \bar{B}(\xi)\,I. \tag{4.46}$$

By differentiating (4.42) again and inserting (4.46) we get

$$\begin{aligned} 2B_{\alpha\beta\gamma} \;&=\; D^2_{\beta\gamma}\hat{B}(\xi)\xi_\alpha \;+\; D_\beta\hat{B}(\xi)\delta_{\alpha\gamma} \;+\; D_\gamma\hat{B}(\xi)\delta_{\alpha\beta} \\ &=\; \check{B}(\xi)\xi_\alpha\xi_\beta\xi_\gamma \;+\; \bar{B}(\xi)\big(\xi_\alpha\delta_{\beta\gamma} \;+\; \xi_\beta\delta_{\alpha\gamma} \;+\; \xi_\gamma\delta_{\beta\alpha}\big), \end{aligned} \tag{4.47}$$

for all $\xi\in\mathbb{R}^N\setminus\{0\}$. Since $N\geq 2$, for each $\eta\in\mathbb{R}^N$ we can choose $\xi\in[\eta]^\perp\setminus\{0\}$. Hence, by triple contraction in (4.47) we obtain

$$\begin{aligned} B:\eta\otimes\eta\otimes\eta \;&=\; \frac{1}{2}\Big[\check{B}(\xi)(\xi^\top\eta)^2 \;+\; \bar{B}(\xi)|\eta|^2\Big](\xi^\top\eta) \\ &=\; 0. \end{aligned} \tag{4.48}$$

Hence, by full symmetry in all indices we obtain $H_{P_{\alpha i}P_{\beta j}P_{\gamma k}}(0)=B_{\alpha i\beta j\gamma k}=0$ and consequently third order derivatives vanish. We now set

$$C_{\alpha i\beta j\gamma k\delta l} \;:=\; H_{P_{\alpha i}P_{\beta j}P_{\gamma k}P_{\delta l}}(0) \tag{4.49}$$

and then for $0<t<\rho$, (4.30) and (4.36) become

$$H_{PP}(P(t)) \;= I\otimes\hat{A} \;+\; \frac{1}{3!}C:P(t)\otimes P(t) \;+\; O(|P(t)|^3), \tag{4.50}$$

$$\begin{aligned} \xi\otimes\xi\left(C:\frac{P(t)}{|P(t)|}\otimes\frac{P(t)}{|P(t)|} \;+\; O(|P(t)|)\right) \\ =\; \left(C:\frac{P(t)}{|P(t)|}\otimes\frac{P(t)}{|P(t)|} \;+\; O(|P(t)|)\right)\xi\otimes\xi. \end{aligned} \tag{4.51}$$

By setting $t=t_m$ and letting $m\to\infty$, in view of (4.34), we get

$$\xi\otimes\xi\Big[C:(\xi\otimes v)\otimes(\xi\otimes v)\Big] \;=\; \Big[C:(\xi\otimes v)\otimes(\xi\otimes v)\Big]\xi\otimes\xi, \tag{4.52}$$

for all $\xi\in\mathbb{R}^N$, $v\in\mathbb{R}^n$. Hence, for $v=e_k$,

$$\xi_\alpha\Big[C_{\beta i\kappa j\lambda k\mu k}\,\xi_\kappa\xi_\lambda\xi_\mu\Big] \;=\; \Big[C_{\alpha i\kappa j\lambda k\mu k}\,\xi_\kappa\xi_\lambda\xi_\mu\Big]\xi_\beta. \tag{4.53}$$

By (4.53), there exists a tensor $\hat{C}:\mathbb{R}^N\setminus\{0\}\longrightarrow\otimes^{(4)}\mathbb{R}^n$ with $\hat{C}_{ijkk}\in C^\infty(\mathbb{R}^N\setminus\{0\})$ such that

$$C_{\alpha i\kappa j\lambda k\mu k}\,\xi_\kappa\xi_\lambda\xi_\mu \;=\; \hat{C}_{ijkk}(\xi)\xi_\alpha. \tag{4.54}$$

By fixing again the indices i, j, k, dropping them and arguing exactly as we did before for $B_{\alpha\beta\gamma}$, there exist $\bar{C}, \check{C} \in C^\infty(\mathbb{R}^N \setminus \{0\})$ such that

$$3! C_{\alpha\beta\gamma\delta}\xi_\delta \;=\; \check{C}(\xi)\xi_\alpha\xi_\beta\xi_\gamma \;+\; \bar{C}(\xi)\big(\xi_\alpha\delta_{\beta\gamma} \;+\; \xi_\beta\delta_{\alpha\gamma} \;+\; \xi_\gamma\delta_{\beta\alpha}\big). \tag{4.55}$$

By differentiating (4.55), we get

$$\begin{aligned} 3!\, C_{\alpha\beta\gamma\delta} \,-\, \bar{C}(\xi)\big(\delta_{\alpha\beta}\delta_{\gamma\delta} \,+\, \delta_{\gamma\beta}\delta_{\alpha\delta} \,+\, \delta_{\delta\beta}\delta_{\gamma\alpha}\big) & \\ = \check{C}(\xi)\big(\xi_\alpha\xi_\beta\delta_{\gamma\delta} \,+\, \xi_\beta\xi_\gamma\delta_{\alpha\delta} \,+\, \xi_\gamma\xi_\alpha\delta_{\beta\delta}\big) & \\ + D_\delta\check{C}(\xi)\Big[\xi_\alpha\xi_\beta\xi_\gamma \,+\, \big(\xi_\alpha\delta_{\beta\gamma} \,+\, \xi_\beta\delta_{\alpha\gamma} \,+\, \xi_\gamma\delta_{\beta\alpha}\big)\Big]. & \end{aligned} \tag{4.56}$$

Fix $\eta \in \mathbb{R}^N$. Since $N \geq 2$, there exists $\xi \perp \eta$, $\xi \neq 0$. Then, (4.56) gives

$$\begin{aligned} \left[C_{\alpha\beta\gamma\delta} \;-\; \frac{\bar{C}(\xi)}{3!}\big(\delta_{\alpha\beta}\delta_{\gamma\delta} \,+\, \delta_{\gamma\beta}\delta_{\alpha\delta} \,+\, \delta_{\delta\beta}\delta_{\gamma\alpha}\big)\right]\eta_\alpha\eta_\beta\eta_\gamma\eta_\delta \;&=\; O(|\eta^\top\xi|) \\ &=\; 0. \end{aligned} \tag{4.57}$$

By (4.57), the function $\bar{C}$ is constant and moreover for all i, j, k,

$$C_{\alpha i\beta j\gamma k\delta k} \;=\; \frac{\bar{C}_{ijkk}}{3!}\big(\delta_{\alpha\beta}\delta_{\gamma\delta} \,+\, \delta_{\gamma\beta}\delta_{\alpha\delta} \,+\, \delta_{\delta\beta}\delta_{\gamma\alpha}\big). \tag{4.58}$$

If either $n \leq 3$ or $n \geq 4$ but $H_{PPPP}(0) \in \mathscr{L}^4$, where $\mathscr{L}^4$ is given by (4.8), then in view of (4.49), the tensor $C_{\alpha i\beta j\gamma k\delta l}$ has no more than 3 different indices i, j, k, l for which it is non-zero. Hence, by full symmetry in all indices, (4.58) completely determines $H_{PPPP}(0)$ and gives

$$\begin{aligned} H_{PPPP}(0) : \otimes^{(4)} P \;&=\; \frac{1}{2}\bar{C}_{ijkl}P_{\alpha i}P_{\alpha j}P_{\beta k}P_{\beta l} \\ &=\; \frac{\bar{C}}{2} : (P^\top P) \otimes (P^\top P). \end{aligned} \tag{4.59}$$

Now we iterate the above arguments. The analog of (4.52) after blowing up along t_m for q-th order derivatives is

$$\xi \otimes \xi\Big[H_{P\ldots P}(0) : \otimes^{(q-2)}(\xi \otimes v)\Big] \;=\; \Big[H_{P\ldots P}(0) : \otimes^{(q-2)}(\xi \otimes v)\Big]\xi \otimes \xi, \tag{4.60}$$

for all $\xi \in \mathbb{R}^N$, $v \in \mathbb{R}^n$. When $H_{P\ldots P}(0) \in \mathscr{L}^q$, the only components of the tensor $H_{P_{\alpha_1 i_1}\ldots P_{\alpha_q i_q}}(0)$ which may not vanish are of the form

$$H_{P_{\alpha_1 i}P_{\alpha_2 j}P_{\alpha_3 k}\ldots P_{\alpha_q k}}(0), \tag{4.61}$$

where $i, j, k \in \{1, ..., n\}$ and $\alpha_1, ..., \alpha_q \in \{1, ..., N\}$. Hence, (4.60), completely determines $H_{P\ldots P}(0)$. By induction, all odd order derivatives of H vanish and all even order derivatives depend on P via $P^\top P$: we have

$$\underbrace{H_{P\ldots P}(0)}_{\text{q-th order}} \;=\; \begin{cases} C_q : \otimes^{(q/2)} P^\top P, & q \in 2\mathbb{N}, \\ 0, & q \in 2\mathbb{N}+1, \end{cases} \tag{4.62}$$

for certain tensors $C_q \in \otimes^{(q/2)}\mathbb{R}^n$. Hence, by defining $h : \mathbb{R}^n \otimes \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$h(p) \;:=\; \sum_{m=1}^{\infty} 2^m C_{2m} : \otimes^{(m)} p, \tag{4.63}$$

we obtain

$$H(P) \;=\; h\Big(\frac{1}{2}P^\top P\Big). \tag{4.64}$$

Hence, $h \geq 0$ with $h \in C^\infty\big(\mathbb{S}(\mathbb{R}^n)\big)$ and also $h_p = h_p^\top$. Moreover by (4.3), (4.4) and (4.13), for $w \in \mathbb{S}^{n-1}$ and $\eta \in [P]^\perp \cap \mathbb{S}^{N-1}$, we have

$$h_p(p) : w \otimes w \ = \ \frac{\mathrm{A}(P) : (\eta \otimes w) \otimes (\eta \otimes w)}{h(p)[P]^\perp : \eta \otimes \eta} \ > \ 0, \tag{4.65}$$

where $p = \frac{1}{2}P^\top P$. Hence, $h_p > 0$. If finally $H_{P\dots P}(0) \notin \mathscr{L}^q$, then H has the form (4.64) up to a correction of order $O(|P|^4)$. This follows by decomposing each $H_{P\dots P}(0)$ to the sum of a term in $\mathscr{L}^q$ and a term in the orthogonal complement of $\mathscr{L}^q$. The $O(|P|^4)$ function arises from the series consisting of the forth and higher order parts of $H_{P\dots P}(0) : \otimes^{(q)} P$ in the orthogonal complements. The theorem follows. ☐

5. The 1-Dimensional Case: Aronsson's ODE System.

5.1. **Formal derivation of the general Aronsson ODE System.** Let H be a non-negative Hamiltonian in $C^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$, where $N \geq 2$ and we denote the arguments of H by $H(x, \eta, P)$. Consider the integral functional

$$E_m(u, I) \ := \ \int_I \big(H(x, u(x), u'(x))\big)^m dx, \tag{5.1}$$

where $m \geq 2$ and $u : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$. The Euler-Lagrange equation of functional (5.1) is the ODE system

$$\Big(H^{m-1}(\_, u, u')H_P(\_, u, u')\Big)' \ = \ H^{m-1}(\_, u, u')H_\eta(\_, u, u') \tag{5.2}$$

which after expansion and normalization gives

$$\big(H(\_, u, u')\big)' H_P(\_, u, u') \ + \ \frac{H(\_, u, u')}{m-1}\Big(\big(H_P(\_, u, u')\big)' - H_\eta(\_, u, u')\Big) \ = \ 0, \tag{5.3}$$

on $I \subseteq \mathbb{R}$. We define the following projections of $\mathbb{R}^N$:

$$[H(x, \eta, P)]^\top \ := \ \mathrm{sgn}\big(H_P(x, \eta, P)\big) \otimes \mathrm{sgn}\big(H_P(x, \eta, P)\big), \tag{5.4}$$

$$[H(x, \eta, P)]^\perp \ := \ I \ - \ [H(x, \eta, P)]^\top. \tag{5.5}$$

Then, by employing (5.4) and (5.5) to expand the term in bracket of (5.3), we obtain

(5.6)

$$\begin{aligned} \big(H(\_, u, u')\big)' H_P(\_, u, u') + \frac{H(\_, u, u')}{m-1}[H_P(\_, u, u')]^\top \Big(\big(H_P(\_, u, u')\big)' - H_\eta(\_, u, u')\Big) \\ = - \ \frac{H(\_, u, u')}{m-1}[H_P(\_, u, u')]^\perp \Big(\big(H_P(\_, u, u')\big)' - H_\eta(\_, u, u')\Big). \end{aligned}$$

By perpendicularity of the orthogonal projections (5.4) and (5.5), the left and right hand sides of (5.6) are normal to each other. Hence, they both vanish. By renormalizing the right hand side and rearranging, we get

(5.7)

$$\begin{aligned} \big(H(\_, u, u')\big)' H_P(\_, u, u') \ + \ H(\_, u, u')[H_P(\_, u, u')]^\perp \Big(\big(H_P(\_, u, u')\big)' - H_\eta(\_, u, u')\Big) \\ = - \ \frac{H(\_, u, u')}{m-1}[H_P(\_, u, u')]^\top \Big(\big(H_P(\_, u, u')\big)' - H_\eta(\_, u, u')\Big). \end{aligned}$$

As $m \to \infty$, we obtain the complete system of Aronsson ODEs for a general Hamiltonian with dependence on all the arguments

$$\begin{aligned} &\big(H(\_,u,u')\big)' H_P(\_,u,u') \ + \ H(\_,u,u')\cdot \\ &\qquad \cdot [H_P(\_,u,u')]^\perp \Big(\big(H_P(\_,u,u')\big)' - H_\eta(\_,u,u')\Big) = \ 0, \end{aligned} \tag{5.8}$$

whose solutions are curves $u : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^N$.

5.2. **Degenerate elliptic Aronsson ODE systems.** We begin by observing that the Ellipticity Classification Theorem 4.1 readily extends to the case of Hamiltonians $H(x, \eta, P)$ with dependence on all arguments; the form (4.1) of the Hamiltonian modifies to

$$H(x,\eta,P) \ = \ h\Big(x,\eta,\frac{1}{2}P^\top P\Big) \tag{5.9}$$

and the PDE systems (4.2) and (4.7) modify by the appearance of first and lower order tems. In the case of ODEs where $n = 1$, the "geometric" Hamiltonians of the form (5.9) become the *radially symmetric* ones:

$$H(x,\eta,P) \ = \ h\Big(x,\eta,\frac{1}{2}|P|^2\Big), \tag{5.10}$$

where $h \in C^2\big(\mathbb{R} \times \mathbb{R}^N \times [0,\infty)\big)$ and the degenerate elliptic Aronsson ODE system takes a particularly important and simple form. In the case of Δ_∞, we have $h(x, \eta, p) = p$. We now perform the derivation of the ODEs in the elliptic case.

Suppose $h \in C^2\big(\mathbb{R} \times \mathbb{R}^N \times [0,\infty)\big)$ with arguments denoted by $h(x, \eta, p)$ and define $H \in C^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$ by means of (5.10). We henceforth assume

$$\big\{h_p(x,\eta,\_) = 0\big\} \subseteq \{0\} \ = \ \big\{h(x,\eta,\_) = 0\big\}, \tag{5.11}$$

for all $(x, \eta) \in \mathbb{R}^{1+N}$. Assumption (5.11) is natural and will make the normal coefficient $H[H_P]^\perp$ of (5.8) continuous. By using (5.10) and supressing arguments, we calculate derivatives:

$$H_P \ = \ h_p P, \qquad H_{PP} \ = \ h_{pp} P\otimes P \ + \ h_p I, \tag{5.12}$$

$$H_{P\eta} \ = \ P \otimes h_{p\eta}, \quad H_{Px} \ = \ h_{px} P, \tag{5.13}$$

$$H_\eta \ = \ h_\eta, \qquad H_x \ = \ h_x. \tag{5.14}$$

By expanding derivatives in (5.8) and using (5.10) and (5.12)-(5.14), we get

$$\begin{aligned} &(h_p)^2 (u'\otimes u')u'' \ + \ h_p(u'\otimes h_\eta)u' \ + \ h_x h_p u' \\ &\quad + h[h_p u']^\perp \Big(h_{pp}(u'\otimes u')u'' \ + \ (u'\otimes h_{p\eta})u' \\ &\qquad + \ h_{px}u' \ + \ h_p u'' \ - \ h_\eta\Big) \ = \ 0, \end{aligned} \tag{5.15}$$

where $h = h\big(\_,u,\frac{1}{2}|u'|^2\big)$. By assumption, (5.11), we have $\{h_p u' = 0\} = \{u' = 0\} = \{h = 0\}$. Hence, we obtain that $[h_p u']^\perp = [u']^\perp$. On $\{u' \neq 0\}$, we multiply the normal term of (5.15) by $\frac{|u'|^2 h_p}{h}$ to obtain

$$\begin{aligned} &(h_p)^2 (u'\otimes u')u'' \ + \ h_p\Big((u'\otimes u')h_\eta \ + \ h_x u'\Big) \\ &\qquad + \ |u'|^2 h_p [u']^\perp \Big(h_p u'' \ - \ h_\eta\Big) \ = \ 0. \end{aligned} \tag{5.16}$$

Hence, by using the identity $|u'|^2 I = u' \otimes u' + |u'|^2[u']^\perp$, (5.16) gives

$$(h_p)^2|u'|^2u'' \;-\; h_p\left(|u'|^2\Big(I - 2\frac{u'}{|u'|}\otimes\frac{u'}{|u'|}\Big)h_\eta \;-\; h_x u'\right) \;=\; 0. \tag{5.17}$$

By introducing the *reflection operator* $\mathbf{R}_\xi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *with respect to the hyperplane* $[\xi]^\perp$, $\xi \in \mathbb{R}^N \setminus \{0\}$, given by

$$\mathbf{R}_\xi \;:=\; I \;-\; 2\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}, \tag{5.18}$$

the ODE system (5.17) becomes

$$A_\infty u \;:=\; |u'|^2\Big(h_p u'' \;-\; \mathbf{R}_{u'}h_\eta\Big) \;+\; h_x u' \;=\; 0, \tag{5.19}$$

where $h = h\big(\_, u, \frac{1}{2}|u'|^2\big)$.

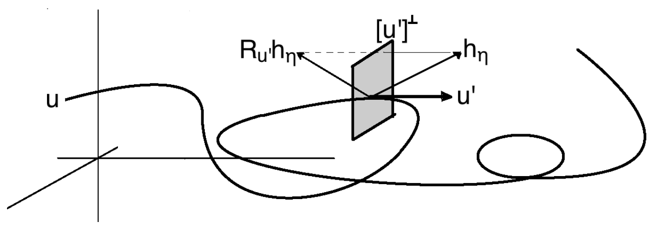

Figure 8.

In view of (5.11), the systems (5.19) and (5.8) are equivalent on $\{u' = 0\}$ as well. The system (5.19) comprises the *degenerate elliptic Aronsson ODE system.*

Remark 5.1. We observe that in the special case where $h = h\big(\frac{1}{2}|u'|^2\big)$ and $h_\eta \equiv 0$, $h_x \equiv 0$, solutions of (5.19) trivialize to *affine* and actually (5.19) is equivalent to Δ_∞. In the special case where $h = h\big(\_, \frac{1}{2}|u'|^2\big)$ and $h_\eta \equiv 0$, solutions of (5.19) become essentially scalar with *affine rank-one range*, that is $u(\mathbb{R})$ is contained in an affine line of $\mathbb{R}^N$ since u'' becomes proportional to u' and (5.19) becomes semi-monotone.

Consequently, (5.19) is most interesting when $h\big(x, u(x), \frac{1}{2}|u'(x)|^2\big)$ depends on $u(x)$ and hence $h_\eta \not\equiv 0$. In this case the reflection operator $\mathbf{R}_{u'}$ with respect to the normal hyperplane $[u']^\perp$ is *discontinuous on* $\{u' = 0\}$ *at critical points of* u, but the product $|u'|^2\mathbf{R}_{u'}$ is continuous. However, in any case the system is always degenerate.

5.3. **The initial value problem for elliptic Aronsson ODE systems.** In this subsection we solve the initial value problem for ODE system (5.19) and consider some regularity questions.

Theorem 5.2 (The initial value problem for elliptic Aronsson ODEs)**.** *Suppose that* $h \in C^2\big(\mathbb{R}\times\mathbb{R}^N\times[0,\infty)\big)$ *satisfies* $h, h_p \geq 0$ *and also* (5.11) *and consider the following problem for Aronsson ODEs*

$$\left\{\begin{array}{l} A_\infty u \;=\; |u'|^2\Big(h_p u'' \;-\; \mathbf{R}_{u'}h_\eta\Big) \;+\; h_x u' \;=\; 0, \\ \qquad u(x_0) = u_0\ , \quad u'(x_0) = v_0\ , \quad x_0 \in \mathbb{R}. \end{array}\right. \tag{5.20}$$

Then:
(i) For any non-critical *initial conditions* $(u_0, v_0) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$, *there exists a unique maximal smooth solution* $u : (x_0 - r, x_0 + r) \longrightarrow \mathbb{R}^N$ *for some* $r > 0$ *which solves* (5.20) *and satisfies* $|u'| > 0$.

(ii) For any critical *initial condition* $(u_0, 0) \in \mathbb{R}^N \times \{0\}$*, there exists at least one solution to* (5.20)*, one of them being the constant one* $u \equiv u_0$*.*

(iii) If

$$h_\eta(x, \eta, 0) \neq 0 \quad \text{and} \quad h_x(x, \eta, p) = O(p) \quad \text{as} \quad p \to 0, \tag{5.21}$$

then bounded maximal solutions of (5.20) *starting (in positive time) from non-critical data, either are defined on* $[x_0, \infty)$ *being smooth and satisfying* $|u'| > 0$*, or they reach a critical point* $u' = 0$ *and form a discontinuity in* u'' *in finite time.*

(iv) If

$$c \leq h_p \leq \frac{1}{c} \quad \text{for } c > 0, \quad \text{and} \quad h_x(x, \eta, p) = O(p) \quad \text{as} \quad p \to 0, \tag{5.22}$$

then bounded maximal solutions of (5.20) *either are globally smooth or can be extended past singularities as* $W^{2,\infty}_{loc}(\mathbb{R})^N$ *strong solutions which satisfy* (5.19) *everywhere and are eventually constant.*

The interpretation of $W^{2,\infty}_{loc}(\mathbb{R})^N$ solutions to (5.20) as strong everywhere solutions is the same as in Aronsson [A1, A2, A5]: at critical points of u whereon u'' may not exist but is essentially bounded in a neighborhood of $\{u' = 0\}$, the coefficient $|u'|^2$ vanishes.

Example 5.3. *The solution of problem* (5.20) *is generally* non-unique *for critical initial conditions. Choose* $h(x, \eta, p) := \frac{1}{2}|\eta|^2 + p$*. Then,* (5.19) *takes the form*

$$|u'|^2 \Big(u'' \; - \; \mathbf{R}_{u'} u \Big) \;=\; 0 \tag{5.23}$$

and the Hamiltonian is $H(u, u') = \frac{1}{2}(|u|^2 + |u'|^2)$*. In view of example* 3 *in Aronsson's paper* [A1]*, for essentially scalar solutions* $u = \xi v$ *where* $\xi \in \mathbb{S}^{N-1}$ *and* $v : \mathbb{R} \longrightarrow \mathbb{R}$*,* (5.23) *takes the form* $|v'|^2 \big(v'' + v \big) \xi = 0$*. Hence, for initial conditions* $u\big(-\frac{\pi}{2} \big) = -e_1$*,* $u'\big(-\frac{\pi}{2} \big) = 0$*,* (5.24) *admits the solutions* $u_1(x) = e_1 \sin x$ *and* $u_2(x) = -e_1$*.*

The non-uniqueness for critical data owes to the fact that (5.19) is an 1-dimensional degenerate elliptic system and the initial value problem generally is not well-posed for it.

Proof of Theorem 5.2. All assertions follow directly by considering the following dynamical formulation of the ODE (5.19). For, we write the N-dimensional second order degenerate implicit system (5.19) as a $2N$-dimensional first order explicit system for a vector field defined off an N-dimensional "slice" of $\mathbb{R}^{2N}$. For $U = (u, v)^\top \in \mathbb{R}^{2N}$, we set

$$U(x) \;:=\; (u(x), u'(x))^\top, \qquad U : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{2N}, \tag{5.24}$$

$$F(x, U) \;:=\; \left[\begin{array}{c} v \\ \frac{1}{h_p\left(x, u, \frac{1}{2}|v|^2\right)} \left(\mathbf{R}_v h_\eta\big(x, u, \tfrac{1}{2}|v|^2\big) \;-\; \frac{h_x\left(x, u, \frac{1}{2}|v|^2\right)}{|v|^2} v \right) \end{array} \right], \tag{5.25}$$

where

$$F \;:\; \mathbb{R} \times \mathbb{R}^N \times \big(\mathbb{R}^N \setminus \{0\} \big) \;\longrightarrow\; \mathbb{R}^{2N}. \tag{5.26}$$

Then, in view of (5.24) and (5.25), ODE system (5.19) can be written as

$$\dot{U}(x) \;=\; F\big(x, U(x)\big)\;, \qquad U : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{2N}. \tag{5.27}$$

We now merely observe that the equation

$$u'' = \frac{1}{h_p\big(\_, u, \frac{1}{2}|u'|^2\big)}\left(\mathbf{R}_{u'} h_\eta\Big(\_, u, \frac{1}{2}|u'|^2\Big) - \frac{h_x\big(\_, u, \frac{1}{2}|u'|^2\big)}{|u'|^2}u'\right) \tag{5.28}$$

which follows by (5.27), implies that under assumption (5.21) the first term in the bracket becomes discontinuous at critical points of u, while the second one vanishes. Solutions extend past critical points where u'' "jumps" by constant solutions. □

In the forthcoming work [K6] we present a theory of non-differentiable solutions which applies to fully nonlinear PDE systems and extends Viscosity Solutions to the general vector case. This approach is based on the existence of an extremality principle which applies to maps. In this context, we consider the existence of solution to the Dirichlet problem for (1.2).

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