

INTERNAL CONTROL FOR NON-LOCAL SCHRÖDINGER AND WAVE EQUATIONS INVOLVING THE FRACTIONAL LAPLACE OPERATOR*

UMBERTO BICCARI †

Abstract. We analyse the interior controllability problem for a non-local Schrödinger equation involving the fractional Laplace operator $(-\Delta)^s$, $s \in (0, 1)$, on a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$. The controllability from a neighbourhood of the boundary of the domain is obtained for exponents s in the interval $[1/2, 1)$, while for $s < 1/2$ the equation is shown to be not controllable. As a consequence of that, we obtain the controllability for a non-local wave equation involving the higher order fractional Laplace operator $(-\Delta)^{2s} = (-\Delta)^s(-\Delta)^s$, $s \in [1/2, 1)$. The results follow from a new Pohozaev-type identity for the fractional laplacian recently proved by X. Ros-Oton and J. Serra and from an explicit computation of the spectrum of the operator in the one dimensional case.

Key words. Fractional Laplacian, Schrödinger equation, wave equation, controllability, Pohozaev

AMS subject classifications. 35R11, 35S05, 35S11, 93B05, 93B07

1. Introduction and main results. This work is devoted to the analysis of a non-local Schrödinger equation, involving the fractional Laplace operator, defined on a bounded $C^{1,1}$ domain Ω of the euclidean space \mathbb{R}^n . The main purpose of this paper will be to address the interior controllability problem with a single control located in a neighbourhood of the boundary of the domain.

In the last years many attention has been given to the analysis of non-local operators and many interesting results have been proved. Indeed, this operators have shown to be particular appropriate for the study of a huge spectrum of phenomena, arising in several areas of physics, finance, biology, ecology, geophysics, and many others, such as anomalous transport and diffusion ([18]), hereditary phenomena with long memory, wave propagation in heterogeneous high contrast media ([26]), dislocation dynamics in crystals ([9]), non-local electrostatics. In this frame, the study of controllability properties for non-local evolution problem becomes a very interesting issue, both from a purely theoretical and from an applied point of view.

The complete problem we are considering for our fractional Schrödinger equation is

$$(1.1) \quad \begin{cases} iu_t + (-\Delta)^s u = 0 & \text{in } \Omega \times [0, T] := Q \\ u \equiv 0 & \text{in } \Omega^c \times [0, T] \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

in which the fractional Laplacian $(-\Delta)^s$ is the operator defined as ([21], [22], [23])

$$(1.2) \quad (-\Delta)^s u(x) := c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad s \in (0, 1),$$

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†BCAM - Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009, Bilbao, Basque Country, Spain. (ubiccari@bcamath.org).

with $c_{n,s}$ a normalization constant given by ([22])

$$c_{n,s} := \frac{s2^{2s}\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\Gamma(1-s)}.$$

A first important aspect we want to underline, is the particular formulation for the boundary conditions which, due to the non-local nature of the operator, are imposed not only on the boundary but everywhere outside of the domain Ω . Moreover, in this work we are considering boundary conditions of Dirichlet type, meaning that we are asking the solution u to vanish in Ω^c ; however, others possible choices can be done ([25]), as we are going to present in the last section of this paper, dedicated to open problems and perspectives.

Let us now formulate precisely the interior controllability problem for the fractional evolution equation we are considering. Let Ω be a bounded $C^{1,1}$ domain of \mathbb{R}^n with boundary Γ ; we consider a partition (Γ_0, Γ_1) of Γ given by

$$(1.3) \quad \Gamma_0 = \{x \in \Gamma \mid (x \cdot \nu) > 0\}, \quad \Gamma_1 = \{x \in \Gamma \mid (x \cdot \nu) \leq 0\},$$

where ν is the unit normal vector to Γ at x pointing towards the exterior of Ω . The main result of this work will be

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain with boundary Γ , $s \in [1/2, 1)$ and Γ_0 as defined in (1.3). Moreover, let $\omega = \mathcal{O}_\varepsilon \cap \Omega$, where \mathcal{O}_ε is a neighbourhood of Γ_0 in \mathbb{R}^n . For $u_0 \in L^2(\Omega)$ and $h \in L^2(\omega \times [0, T])$, let $u = u(x, t)$ be the solution of*

$$(1.4) \quad \begin{cases} iu_t + (-\Delta)^s u = h\chi_{\{\omega \times [0, T]\}} & \text{in } Q \\ u \equiv 0 & \text{in } \Omega^c \times [0, T] \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

- (i) *If $s \in (1/2, 1)$, for any $T > 0$ the control function h is such that the solution u of (1.4) satisfies $u(x, T) = 0$;*
- (ii) *if $s = 1/2$, the same controllability result as in (i) holds for any $T > 2Pd(\Omega) := T_0$, where P is the Poincaré constant associated to the domain Ω .*

The range of the exponent of the fractional Laplace operator is fundamental for the positivity of the controllability result; indeed, although the fractional Laplacian is well defined for any s in the interval $(0, 1)$, we can show that the sharp power when dealing with the control problem for our fractional Schrödinger equation is $s = 1/2$, meaning that below this critical value the equation becomes not controllable. This fact is proved in one space dimension by developing a Fourier analysis for our equation based on the results contained in [13] and [14], where the authors compute an explicit approximation of the eigenvalues of the fractional Laplacian in dimension one on the half-line $(0, +\infty)$ and on the interval $(-1, 1)$.

For proving the controllability theorem stated above, we are going to apply the very classical technique combining the multiplier method ([12]) and the Hilbert Uniqueness Method (HUM, [15]). Thus, as it is shown in many very classical works on control theory (see, for instance, [2], [15], [17]), we are reduced to prove what is called an ‘‘observability inequality’’ associated to our problem and then argue by duality. In our case we are going to prove:

$$(1.5) \quad \|u_0\|_{L^2(\Omega)}^2 \leq C \int_0^T \|u\|_{L^2(\omega)}^2 dt.$$

This inequality will be, in turn, a consequence of a Pohozaev-type identity for the solution of the equation considered, derived by applying the multiplier method and a new Pohozaev identity for the fractional Laplacian ([22]), which has been recently proved by X. Ros-Oton and J. Serra and which extends to the fractional case the by now well known identity proved by S.I. Pohozaev in [19]. However, the identity by Ros-Oton and Serra holds under very strict regularity assumptions for the functions involved (see [22, Prop. 1.6]), which are not automatically guaranteed for the solution of our fractional Schrödinger equation. Thus, in order to apply it, we are going to consider firstly solutions of (1.1) involving a finite number of eigenfunctions of the fractional Laplacian on Ω and, in a second step, we are going to recover the result for general solutions by employing a density argument. We are allowed to follow this path because, being a positive and self-adjoint operator, the fractional Laplacian possess a basis of eigenfunctions which forms a dense subspace of $L^2(\Omega)$ and, as we are going to show in the appendix to this work, these eigenfunctions are bounded on Ω , and this is enough to recover the regularity we need.

The paper is organised as follows. In section 2 we introduce the fractional Laplace operator; we present some very classical result ([10]) and the recent ones of Ros-Oton and Serra ([21], [22]) concerning the regularity of the fractional Dirichlet problem and the Pohozaev-type identity. In section 3 we analyse the fractional Schrödinger equation (1.1). We first check its well posedness applying Hille-Yosida theorem. Then, we derive the Pohozaev identity and we apply it for proving the observability inequality. Lastly, we prove our main result, Theorem 1.1. In section 4 we present a spectral analysis for our equation which will allow us to identify the sharp exponent needed for the fractional Laplace operator in order to get a positive control result. In section 5 we briefly present an abstract argument, due to Tucsnak and Weiss ([24]), which will allow us to employ the observability results for our fractional Schrödinger equation in order to obtain the observability for a fractional wave equation involving the higher order operator $(-\Delta)^{2s} := (-\Delta)^s(-\Delta)^s$. Finally, section 6 is devoted to some open problem and perspective related to our work.

2. Fractional Laplace operator: definition, Dirichlet problem and Pohozaev-type identity. We introduce here some preliminary results about the fractional Laplacian that we are going to use throughout this paper.

We start by introducing the fractional order Sobolev space $H^s(\Omega)$. Since we are dealing with smooth domains, say of class $C^{1,1}$, we introduce this space by assuming that our open set $\Omega \subset \mathbb{R}^n$ is smooth. For $s \in (0, 1)$, we denote by

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \mid \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx < \infty \right\}$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{H^s(\Omega)} = \left(\int_{\Omega} |u|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}}.$$

We denote with $H_0^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ (the space of the test functions) in $H^s(\Omega)$. The following characterization of $H_0^s(\Omega)$ is well known

$$H_0^s(\Omega) = \{ u \in H^s(\mathbb{R}^n) \mid u = 0 \text{ in } \Omega^c := \mathbb{R}^n \setminus \Omega \}.$$

We mention that $H_0^s(\Omega)$ is a Hilbert space and we denote its dual by $H^{-s}(\Omega)$. Throughout the remainder of the article, if we say that a function u belongs to $H_0^s(\Omega)$, we mean that $u \in H^s(\mathbb{R}^n)$

and $u = 0$ in Ω^c .

Let $u \in H^s(\mathbb{R}^n)$, $s \in (0, 1)$; let consider the fractional Laplace operator $(-\Delta)^s$ as defined in (1.2). The following result, (see e.g. [10, Prop. 3.3]), tells us that the fractional Laplacian is, in fact, the pseudo-differential operator associated to the symbol $|\xi|^{2s}$.

PROPOSITION 2.1. *Let $s \in (0, 1)$ and let $(-\Delta)^s$ be the fractional Laplace operator defined in (1.2). Then, for any $u \in H^s(\mathbb{R}^n)$*

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u) \quad \forall \xi \in \mathbb{R}^n.$$

Proposition 2.1 can be used, joint with the Plancherel theorem, to prove many other results such as the following.

PROPOSITION 2.2. *Let u, v be two $H^s(\mathbb{R}^n)$ functions such that $u \equiv v \equiv 0$ in Ω^c ; then, it holds the following integration formula*

$$(2.1) \quad \int_{\Omega} v(-\Delta)^s u dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx = \int_{\Omega} u (-\Delta)^s v dx.$$

Proof. The proof follows simply by using Fourier transform and Plancherel theorem and the fact that the functions u and v vanish out of Ω . Indeed, we have

$$\begin{aligned} \int_{\Omega} v(x)(-\Delta)^s u(x) dx &= \int_{\mathbb{R}^n} v(x)(-\Delta)^s u(x) dx = \int_{\mathbb{R}^n} \mathcal{F}v(\xi) \mathcal{F}[(-\Delta)^s u](\xi) d\xi \\ &= \int_{\mathbb{R}^n} \hat{v}(\xi) |\xi|^{2s} \hat{u}(\xi) d\xi = \int_{\mathbb{R}^n} |\xi|^s \hat{v}(\xi) |\xi|^s \hat{u}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \mathcal{F}(-\Delta)^{\frac{s}{2}} v(\xi) \mathcal{F}(-\Delta)^{\frac{s}{2}} u(\xi) d\xi = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} v(x) (-\Delta)^{\frac{s}{2}} u(x) dx. \end{aligned}$$

Finally, it is now trivial to show that also the second equality in our statement holds; this concludes our proof. \square

The fractional Laplacian is surely one of the most known non-local operators; in particular in the last years, it has been produced a huge literature on its properties and its employment in a very large spectrum of different applications.

Our work principally uses the results by Ros-Oton and Serra contained in [21] and [22]; we present here the most important ones. Let us consider the Dirichlet problem associated to the fractional Laplace operator

$$(2.2) \quad \begin{cases} (-\Delta)^s u = g & \text{in } \Omega \\ u \equiv 0 & \text{in } \Omega^c \end{cases}$$

In [21, Prop. 1.1] and in [22, Prop. 1.6] respectively, the following results have been proved.

PROPOSITION 2.3. *Let Ω be a bounded $C^{1,1}$ domain of \mathbb{R}^n , $s \in (0, 1)$, $g \in L^\infty(\Omega)$; let $u \in H^s(\mathbb{R}^n)$ satisfying (2.2). Then $u \in C^s(\mathbb{R}^n)$ and $\|u\|_{C^s(\mathbb{R}^n)} \leq C(s, \Omega) \|g\|_{L^\infty(\Omega)}$, where C is a constant depending only on Ω and s .*

PROPOSITION 2.4. *Let Ω be a bounded $C^{1,1}$ domain of \mathbb{R}^n , $s \in (0, 1)$ and $\delta(x) = \text{dist}(x, \partial\Omega)$, with $x \in \Omega$, be the distance of a point x from $\partial\Omega$. Assume that $u \in H^s(\mathbb{R}^n)$ vanishes in Ω^c and satisfies the following:*

(i) $u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1 + 2s)$, u is of class $C^\beta(\Omega)$ and

$$[u]_{C^\beta(\{x \in \Omega | \delta(x) \geq \rho\})} \leq C\rho^{s-\beta} \quad \text{for all } \rho \in (0, 1);$$

(ii) The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\gamma \in (0, 1)$ such that $u/\delta^s \in C^\gamma(\overline{\Omega})$. In addition, for all $\beta \in [\gamma, s + \gamma]$ it holds the estimate

$$[u/\delta^s]_{C^\beta(\{x \in \Omega | \delta(x) \geq \rho\})} \leq C\rho^{\gamma-\beta} \quad \text{for all } \rho \in (0, 1);$$

(iii) $(-\Delta)^s u$ is pointwise bounded in Ω .

Then, the following identity holds

$$(2.3) \quad \int_{\Omega} (x \cdot \nabla u)(-\Delta)^s u dx = \frac{2s-n}{2} \int_{\Omega} u(-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma$$

where ν is the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function.

In the two propositions above, following the notation introduced by Ros-Oton and Serra in [21] and [22], $C^\beta(\Omega)$ with $\beta > 0$ indicates the space $C^{k, \beta'}(\Omega)$, where k is the greatest integer such that $k < \beta$ and $\beta' = \beta - k$.

Identity (2.3) is the Pohozaev identity for the fractional Laplacian and it will be the starting point for our control problem. In it, $u/\delta^s|_{\partial\Omega}$ plays the role that the normal derivative $\partial_\nu u$ plays in the classical Pohozaev identity. Moreover, as the authors already remark in [22], it is surprising that, starting from a non-local problem, they obtained a completely local term in their identity. This means that, although the function u has to be defined in all \mathbb{R}^n in order to compute its fractional Laplacian at a given point, knowing u only in a neighbourhood of the boundary we can already compute $\int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma$.

3. Fractional Schrödinger equation. We analyse here the fractional Schrödinger equation (1.1). As already written before, our principal aim will be to show that the problem is exactly controllable from a neighbourhood of the boundary of the domain. However, the first issue we have to deal with is, of course, the one of the well posedness.

3.1. Well posedness. We check here the well posedness of Problem (1.1) using the Hille-Yosida theorem.

Let $X = L^2(\Omega)$, and $A : \mathcal{D}(A) \rightarrow X$ the operator defined as

$$\mathcal{D}(A) = \left\{ u \mid u \in L^2(\mathbb{R}^n); u|_{\Omega^c} = 0, (-\Delta)^s u \in L^2(\Omega) \right\} = \left\{ u \in H_0^1(\Omega) \mid (-\Delta)^s u \in L^2(\Omega) \right\}$$

$$Au := -i(-\Delta)^s u.$$

Then, Problem (1.1) becomes

$$(3.1) \quad \begin{cases} \frac{du}{dt} + Au = 0 \\ u(0) = u_0 \end{cases}$$

We can easily check that the operator A is monotone, since

$$(Au, u)_X = \Re \int_{\Omega} -i(-\Delta)^s u \bar{u} dx = \Im \int_{\Omega} (-\Delta)^s u \bar{u} dx = \Im \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \bar{u} dx = 0,$$

and that it is maximal since $(A + I)u = f$ has always a solution for any function $f \in X$; indeed, by taking the Fourier transform, we find $\hat{u}(\xi) = \hat{f}(\xi)/(1 + i|\xi|^{2s})$, which implies that u is defined over all the domain. Thus, from Hille-Yosida theorem follows that (3.1) admits a unique solution

$$u \in C(0, T; \mathcal{D}(A)) \cap C^1(0, T; L^2(\Omega)).$$

3.2. Pohozaev-type identity and observability. In this section we present the tools we need in order to prove the controllability Theorem 1.1. Firstly, we are going to employ the results of [22] in order to obtain a Pohozaev-type identity for the solution u of (1.1); from there, we derive two estimates for the $H^s(\Omega)$ norm of the initial datum, from above and from below, with respect to the boundary term appearing in the identity. These boundary inequalities will finally be used for the proof of the observability inequality (1.5).

3.2.1. Pohozaev-type identity. We introduce here the Pohozaev-type identity for our fractional Schrödinger equation, obtained by the classical method of multipliers ([12]) and applying the identity proved by Ros-Oton and Serra in [22]. However, as it is written in the statement of Proposition 2.4, and as we already said before, this identity requires the functions involved to satisfy some very strict regularity assumptions, and we do not know if the solution u of (3.2) possess this regularity. Thus, we are going to proceed in two steps. In the next proposition, we are going to prove our identity for solutions of the equation corresponding to an initial datum $u_{k,0}$ given as a linear combination of a finite number of eigenfunctions; indeed, in the appendix to this work we will show that the eigenfunctions of the fractional Laplacian are bounded on Ω , and that this is enough to guarantee the regularity we need to apply (2.3). Once we will have the identity for this class of solutions of our equation, the same result in the general case will be obtained through a density argument.

PROPOSITION 3.1. *Let Ω be a bounded $C^{1,1}$ domain of \mathbb{R}^n , $s \in (0, 1)$ and $\delta(x)$ be the distance of a point x from $\partial\Omega$. Given $f \in C(0, T; L^\infty(\Omega))$, let consider the following non-homogeneous problem for the fractional Schrödinger equation*

$$(3.2) \quad \begin{cases} iu_t + (-\Delta)^s u = f & \text{in } Q \\ u \equiv 0 & \text{in } \Omega^c \times [0, T] \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

For any initial datum $u_{k,0} \in \text{span}(\phi_1, \dots, \phi_k)$, where ϕ_1, \dots, ϕ_k are the first k eigenfunctions of the fractional Laplacian on Ω , let

$$u_k(x, t) = \sum_{j=1}^k a_j(t) \phi_j(x)$$

be the corresponding solution of (3.2). Then, the following identity holds

$$(3.3) \quad \begin{aligned} \Gamma(1+s)^2 \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt &= 2s \int_0^T \|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2(\Omega)}^2 dt + \Im \int_{\Omega} \bar{u}_k(x \cdot \nabla u_k) dx \Big|_0^T \\ &- \Re \int_Q u_k (n \bar{f}_k + 2x \cdot \nabla \bar{f}_k) dx dt, \end{aligned}$$

where ν is the unit outward normal to $\partial\Omega$ at x , Γ is the Gamma function and $\Sigma := \partial\Omega \times [0, T]$.

Proof. We are going to apply the multipliers method, joint with the Pohozaev identity for the fractional Laplacian; in particular, we are employing the multiplier

$$x \cdot \nabla \bar{u}_k + \frac{n}{2} \bar{u}_k.$$

Since the eigenfunctions of the fractional Laplacian are bounded, so are $u_{k,0}$ and u_k ; this assures that we have enough regularity in order to apply the result of Ros-Oton and Serra. Indeed, we have

$$(-\Delta)^s u_k = \sum_{j=1}^k a_j (-\Delta)^s \phi_j = \sum_{j=1}^k a_j \lambda_j \phi_j$$

and

$$x \cdot \nabla u_k = \sum_{l=1}^n x_l \partial_{x_l} u_k = \sum_{l=1}^n x_l \sum_{j=1}^k a_j \partial_{x_l} \phi_j = \sum_{j=1}^k a_j (x \cdot \nabla \phi_j).$$

Thus,

$$\begin{aligned} (-\Delta)^s u_k (x \cdot \nabla u_k) &= \left[\sum_{l=1}^k \lambda_l a_l \phi_l \right] \cdot \left[\sum_{j=1}^k a_j (x \cdot \nabla \phi_j) \right] = \sum_{j=1}^k a_j \left[\sum_{l=1}^k \lambda_l a_l \phi_l \right] (x \cdot \nabla \phi_j) \\ &= \sum_{j=1}^k \sum_{l=1}^k \lambda_l a_l a_j \phi_l (x \cdot \nabla \phi_j) \end{aligned}$$

and

$$\int_{\Omega} (-\Delta)^s u_k (x \cdot \nabla u_k) dx = \int_{\Omega} \sum_{j,l=1}^k \lambda_l a_l a_j \phi_l (x \cdot \nabla \phi_j) dx = \sum_{j,l=1}^k a_l a_j \int_{\Omega} (-\Delta)^s \phi_l (x \cdot \nabla \phi_j) dx.$$

Since in the previous equality we have to deal also with cross terms, appearing each time that $j \neq l$, we use the identity

$$\begin{aligned} &\int_{\Omega} (-\Delta)^s \phi_l (x \cdot \nabla \phi_j) dx + \int_{\Omega} (-\Delta)^s \phi_j (x \cdot \nabla \phi_l) dx \\ &= \frac{2s-n}{2} \int_{\Omega} \phi_l (-\Delta)^s \phi_j dx + \frac{2s-n}{2} \int_{\Omega} \phi_j (-\Delta)^s \phi_l dx - \Gamma(1+s)^2 \int_{\partial\Omega} \frac{\phi_l}{\delta^s} \frac{\phi_j}{\delta^s} (x \cdot \nu) d\sigma, \end{aligned}$$

which follows from [22, Lemma 5.1, 5.2] and holds for functions satisfying the same hypothesis of Proposition 2.4; after some simple technical computation we get

$$\begin{aligned} &\sum_{j,l=1}^k a_j a_l \int_{\Omega} (-\Delta)^s \phi_l (x \cdot \nabla \phi_j) dx \\ &= (2s-n) \int_{\Omega} u_k (-\Delta)^s u_k dx - \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u_k}{\delta^s} \right)^2 (x \cdot \nu) d\sigma - \int_{\Omega} (-\Delta)^s u_k (x \cdot \nabla u_k) dx. \end{aligned}$$

Summarizing,

$$\begin{aligned} & \int_{\Omega} (-\Delta)^s u_k (x \cdot \nabla u_k) dx \\ &= (2s - n) \int_{\Omega} u_k (-\Delta)^s u_k dx - \Gamma(1 + s)^2 \int_{\partial\Omega} \left(\frac{u_k}{\delta^s} \right)^2 (x \cdot \nu) d\sigma - \int_{\Omega} (-\Delta)^s u_k (x \cdot \nabla u_k) dx, \end{aligned}$$

and from here we finally recover the Pohozaev identity for the fractional Laplacian (2.3) applied to the function u_k . We can now use it in order to prove the identity (3.3). Hence, we multiply our equation by $x \cdot \nabla \bar{u}_k + (n/2)\bar{u}_k$, we take the real part and we integrate over Q .

$$\begin{aligned} (3.4) \quad \Re \int_Q f_k \left(x \cdot \nabla \bar{u}_k + \frac{n}{2} \bar{u}_k \right) dx dt &= \underbrace{\Re \int_Q (-\Delta)^s u_k (x \cdot \nabla \bar{u}_k) dx dt}_{A_1} + \underbrace{\Re \int_Q \frac{n}{2} \bar{u}_k (-\Delta)^s u_k dx dt}_{A_2} \\ &+ \underbrace{\Re \int_Q i(u_k)_t \left(\frac{n}{2} \bar{u}_k + x \cdot \nabla \bar{u}_k \right) dx dt}_{A_3} \end{aligned}$$

For what concerns the right hand side, we compute the three contributes separately. For the first integral, we have

$$\begin{aligned} A_1 &= \int_Q \left\{ [(-\Delta)^s \Re(u_k)] (x \cdot \nabla \Re(u_k)) + [(-\Delta)^s \Im(u_k)] (x \cdot \nabla \Im(u_k)) \right\} dx dt \\ &= \frac{2s - n}{2} \int_Q \left\{ \Re(u_k) \Re[(-\Delta)^s u_k] + \Im(u_k) \Im[(-\Delta)^s u_k] \right\} dx dt \\ &\quad - \frac{\Gamma(1 + s)^2}{2} \int_{\Sigma} \left[\left(\frac{\Re(u_k)}{\delta^s} \right)^2 + \left(\frac{\Im(u_k)}{\delta^s} \right)^2 \right] (x \cdot \nu) d\sigma dt \\ &= \frac{2s - n}{2} \int_Q u_k (-\Delta)^s \bar{u}_k dx dt - \frac{\Gamma(1 + s)^2}{2} \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt \\ &= \frac{2s - n}{2} \int_0^T \|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2(\Omega)}^2 dt - \frac{\Gamma(1 + s)^2}{2} \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt, \end{aligned}$$

while, for the second one,

$$A_2 = \frac{n}{2} \Re \int_Q \bar{u}_k (-\Delta)^s u_k dx dt = \frac{n}{2} \int_0^T \|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2(\Omega)}^2 dt;$$

thus,

$$A_1 + A_2 = s \int_0^T \|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2(\Omega)}^2 dt - \frac{\Gamma(1 + s)^2}{2} \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt.$$

Finally, let compute the integral A_3 ; we observe that, by considering the function $\psi(x) := |x|^2/4$ we have

$$\nabla\psi = \frac{x}{2}, \quad \Delta\psi = \frac{n}{2}.$$

Thus

$$\begin{aligned} A_3 &= \Re \int_Q i(u_k)_t (\bar{u}_k \Delta\psi + 2\nabla\psi \cdot \nabla\bar{u}_k) dxdt = -\Im \int_Q (u_k)_t (\bar{u}_k \Delta\psi + 2\nabla\psi \cdot \nabla\bar{u}_k) dxdt \\ &= -\Im \int_Q \left\{ -\nabla[(u_k)_t \bar{u}_k] \cdot \nabla\psi + 2(u_k)_t \nabla\bar{u}_k \cdot \nabla\psi \right\} dxdt \\ &= -\Im \int_Q \left\{ -\bar{u}_k \nabla(u_k)_t \cdot \nabla\psi - (u_k)_t \nabla\bar{u}_k \cdot \nabla\psi + 2(u_k)_t \nabla\bar{u}_k \cdot \nabla\psi \right\} dxdt \\ &= \Im \int_Q [\bar{u}_k \nabla(u_k)_t \cdot \nabla\psi - (u_k)_t \nabla\bar{u}_k \cdot \nabla\psi] dxdt = \Im \int_Q \partial_t [\bar{u}_k \nabla u_k \cdot \nabla\psi] dxdt \\ &= \Im \int_Q \partial_t \left[\frac{\bar{u}_k}{2} (x \cdot \nabla u_k) \right] dxdt = \Im \int_\Omega \frac{\bar{u}_k}{2} (x \cdot \nabla u_k) dx \Big|_0^T. \end{aligned}$$

Concerning the left hand side of (3.4), we simply apply classical integration by parts. Thus, by adding the components just obtained we finally get (3.3). \square

Starting from the Pohozaev identity for u_k , it is now possible to recover the same result for the solution $u = \lim_{k \rightarrow +\infty} u_k$, by applying a density argument. When taking the limit for $k \rightarrow +\infty$ in (3.3), on the right hand side we do not have any problem and we get

$$2s \int_0^T \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)}^2 dt + \Im \int_\Omega \bar{u} (x \cdot \nabla u) dx \Big|_0^T - \Re \int_Q u (n\bar{f} + 2x \cdot \nabla \bar{f}) dxdt.$$

However, we cannot affirm immediately that the boundary term in the limit goes to the corresponding one we would get for u ; nevertheless, this fact will be a direct consequence of the following result.

LEMMA 3.2. *Let u_k be the solution of (3.2) corresponding to the initial datum $u_{k,0}$ we introduced before. Then, $|u_k|/\delta^s \rightarrow |u|/\delta^s$ in $L^2(\Sigma)$ as $k \rightarrow +\infty$.*

Proof. First of all, we have that $\{|u_k|/\delta^s\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\Sigma)$. This is simply a consequence of [21, Thm. 1.2] which tells us that, for any $k \in \mathbb{N}$, $|u_k|/\delta^s$ is Hölder regular up to the boundary of the domain. Now, since $L^2(\Sigma)$ is, of course, a complete space, we have

$$\lim_{k \rightarrow +\infty} \frac{|u_k|}{\delta^s} = h$$

in $L^2(\Sigma)$. We claim that $h = |u|/\delta^s$, with

$$u \in C(0, T; \mathcal{D}[(-\Delta)^s]) \cap C^1(0, T; L^2(\Omega))$$

solution of (3.2). Indeed, using the fact that u_k solves the equation, we have

$$(-\Delta)^s u_k = f_k - i(u_k)_t \in L^2(0, T; L^\infty(\Omega))$$

which implies

$$|u_k|/\delta^s \in L^2(0, T; C^\alpha(\bar{\Omega}))$$

with $\alpha < 1 - s$.

Thus, $|u_k|/\delta^s \rightarrow |u|/\delta^s$ in the sense of distribution; however, since we already know that $|u_k|/\delta^s \rightarrow h$ in $L^2(\Sigma)$ and since the limit in the sense of distribution is unique, we can finally conclude

$$\lim_{k \rightarrow +\infty} \frac{|u_k|}{\delta^s} = \frac{|u|}{\delta^s}$$

in $L^2(\Sigma)$. \square

Now that we know that Lemma 3.2 holds, we can pass to the limit also in the boundary term of the identity for u_k and we finally recover

$$(3.5) \quad \begin{aligned} \Gamma(1+s) \int_{\Sigma} \left(\frac{|u|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt &= 2s \int_0^T \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)}^2 dt + \Im \int_{\Omega} \bar{u}(x \cdot \nabla u) dx \Big|_0^T \\ &\quad - \Re \int_Q u(n\bar{f} + 2x \cdot \nabla \bar{f}) dx dt. \end{aligned}$$

3.2.2. Boundary observability. We now use (3.5), with $f = 0$, to obtain upper and lower estimates for the H^s norm of the initial datum u_0 on Ω with respect to the boundary term appearing in the identity. In order to do that, we will firstly need some preliminary tool.

PROPOSITION 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $g \in H_0^s(\Omega)$. For all $s \in (0, 1)$ there exist a constant $M > 0$, depending only on n, s , and Ω , such that*

$$(3.6) \quad \|g\|_{H^s(\Omega)} \leq M \|g\|_{\dot{H}^s(\Omega)},$$

where

$$(3.7) \quad \dot{H}^s(\Omega) := \left\{ u \mid \|\xi|^s \hat{u}(\xi)\|_{L^2(\Omega)} \leq +\infty \right\}.$$

Proof. The proof is made by interpolation and it is very trivial. We have

$$\begin{aligned} \|g\|_{H^0(\Omega)} \|g\|_{L^2(\Omega)} &= \|\hat{g}\|_{L^2(\Omega)} = \|g\|_{\dot{H}^0(\Omega)} \\ \|g\|_{H^1(\Omega)} &= \|g\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega)} \leq P \|\nabla g\|_{L^2(\Omega)} = \|g\|_{\dot{H}^1(\Omega)}, \end{aligned}$$

where P is the Poincaré constant associated to the domain Ω . Thus, (3.6) immediately follows for any $s \in (0, 1)$. \square

PROPOSITION 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $g, h \in H_0^s(\Omega)$. Moreover, let*

$$(3.8) \quad T(g, h) := \int_{\Omega} \bar{g}(x \cdot \nabla h) dx.$$

Then, for all $s \in [1/2, 1)$ there exist a constant $N > 0$, depending only on n, s and Ω , such that

$$(3.9) \quad |T(g, h)| \leq N \|g\|_{H^s(\Omega)} \|h\|_{H^s(\Omega)}.$$

Proof. Since Ω is bounded, we have

$$\int_{\Omega} \bar{g}(x \cdot \nabla h) dx = \int_{\Omega} \nabla h \cdot \bar{x} \bar{g} dx \leq Pd(\Omega) \|h\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)},$$

where P is the Poincaré constant associated to the domain Ω . Moreover, integrating by parts

$$\begin{aligned} \int_{\Omega} \bar{g}(x \cdot \nabla h) dx &= \int_{\Omega} (\bar{\nabla} g \cdot x h + n \bar{g} h) dx \leq Pd(\Omega) \|g\|_{H^1(\Omega)} \|h\|_{L^2(\Omega)} + n \|g\|_{L^2(\Omega)} \|h\|_{L^2(\Omega)} \\ &\leq P(d(\Omega) + n) \|g\|_{H^1(\Omega)} \|h\|_{L^2(\Omega)}. \end{aligned}$$

Thus, by interpolation, for every $s \in (0, 1)$ we have

$$|T(g, h)| \leq [Pd(\Omega)]^s [P(d(\Omega) + n)]^{1-s} \|h\|_{H^s(\Omega)} \|g\|_{H^{1-s}(\Omega)}.$$

Since $s \geq 1/2$, the thesis follows from the continuous embedding $H^s(\Omega) \hookrightarrow H^{1-s}(\Omega)$ ([10]). \square

We now have all we need in order to prove the following result.

THEOREM 3.5. *Let $u_0 \in H_0^s(\Omega)$ and let $u = u(x, t)$ be the solution of (1.1), with initial datum u_0 . Moreover, assume that u satisfies the hypothesis of Proposition 3.1. Then, there exist two positive constants A_1 and A_2 , depending only on s, T, n and Ω , such that*

(i) *if $s \in (1/2, 1)$, then for any $T > 0$ it holds*

$$(3.10) \quad A_1 \|u_0\|_{H^s(\Omega)}^2 \leq \int_{\Sigma} \left(\frac{|u|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt \leq A_2 \|u_0\|_{H^s(\Omega)}^2;$$

(ii) *if $s = 1/2$, then (3.10) holds for any $T > 2Pd(\Omega) := T_0$, where P is the Poincaré constant associated to the domain Ω .*

Proof. First of all, we observe that, since $i(-\Delta)^s$ is a skew-adjoint operator,

$$(3.11) \quad \|D^\alpha u(x, t)\|_{L^2(\Omega)} = \|D^\alpha u(x, 0)\|_{L^2(\Omega)} \quad \forall t \in [0, T], \quad |\alpha| \geq 0.$$

Now, once again we split the proof in the following two steps: we firstly prove the inequality for solutions of the equation with an initial datum which is combination of a finite number of eigenfunctions of the fractional Laplacian and then we recover the same result for general solutions arguing by density.

Step 1: inequality for u_k . Let $u_{k,0} \in \text{span}(\phi_1, \dots, \phi_k)$ and let u_k be the corresponding solution of (1.1). Considering (3.5) with $f = 0$ we have

$$(3.12) \quad \Gamma(1+s)^2 \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt = 2s \int_0^T \|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2(\Omega)}^2 dt + \Im \int_{\Omega} \bar{u}_k (x \cdot \nabla u_k) dx \Big|_0^T.$$

We now have to distinguish two cases. Indeed, for $s > 1/2$ on the right hand side of the previous identity we have two terms of different orders and we can deal with the lower order one by applying a compactness-uniqueness argument. However for $s = 1/2$ this two terms have the

same order and we have to proceed differently.

Let assume firstly $s > 1/2$; concerning the upper estimate, we have

$$\Gamma(1+s)^2 \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt \leq 2sT \|u_{k,0}\|_{\dot{H}^s(\Omega)}^2 + 2 \left| \int_{\Omega} \bar{u}_k (x \cdot \nabla u_k) dx \right| \leq A_2 \|u_{k,0}\|_{H^s(\Omega)}^2,$$

where we used (3.9) with $g = h := u_k$ and the fact that

$$\|(-\Delta)^{\frac{s}{2}} u_k\|_{L^2(\Omega)} = \|u_k\|_{\dot{H}^s(\Omega)}.$$

Let us prove, now, the other estimate. By using Young inequality, we have

$$\left| \int_{\Omega} \bar{u}_k (x \cdot \nabla u_k) dx \right| \leq c_{\varepsilon} \|u_{k,0}\|_{L^2(\Omega)}^2 + \varepsilon \|u_{k,0}\|_{H_0^1(\Omega)}^2.$$

Thus, choosing $\varepsilon < 2sT$, since $\|u_k\|_{H^s(\Omega)} \leq \|u_k\|_{H_0^1(\Omega)}$,

$$(3.13) \quad (2sT - \varepsilon) \|u_{k,0}\|_{H^s(\Omega)}^2 dt \leq \Gamma(1+s)^2 \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt + c_{\varepsilon} \|u_{k,0}\|_{L^2(\Omega)}^2.$$

We conclude now by observing that, thanks to a compactness-uniqueness argument we can immediately prove

$$\|u_{k,0}\|_{L^2(\Omega)}^2 \leq M \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt.$$

Indeed, let assume that the previous inequality does not hold; thus, there exists a sequence $\{u_k^j\}_{j \in \mathbb{N}}$ of solutions of our equation such that

$$(3.14) \quad \left\| u_k^j(0) \right\|_{L^2(\Omega)} = 1 \quad \forall j \in \mathbb{N}$$

and

$$(3.15) \quad \lim_{j \rightarrow +\infty} \int_{\Sigma} \left(\frac{|u_k^j|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt = 0.$$

From (3.13) we deduce that $\{u_k^j\}_{j \in \mathbb{N}}$ is bounded in $H^s(\Omega)$ and then $\{u_k^j\}_{j \in \mathbb{N}}$ is bounded in $L^\infty(0, T; H^s(\Omega)) \cap W^{1,\infty}(0, T, H^{-s}(\Omega))$. Thus, by extracting a subsequence, that we will still note by $\{u_k^j\}$, we have

$$\begin{cases} u_k^j \rightharpoonup u & \text{in } L^\infty(0, T; H^s(\Omega)) \\ (u_k^j)_t \rightharpoonup u_t & \text{in } L^\infty(0, T; H^{-s}(\Omega)). \end{cases}$$

From (3.14) we deduce that $\|u_{k,0}\|_{L^2(\Omega)} = 1$; on the other hand, (3.15) implies $|u_k|/\delta^s = 0$ on Σ . However, thanks to (1.1) and to Holmgren's uniqueness theorem [15, Ch.1, Thm. 8.2], this last fact implies $u_k \equiv 0$ which, of course, is a contradiction.

Let assume, now, $s = 1/2$; as we already said, in this case we cannot apply compactness-uniqueness and we have to estimate the reminder term in a different way. Simply by applying Cauchy-Schwartz inequality we have

$$\left| \int_{\Omega} \bar{u}_k(x \cdot \nabla u_k) dx \right| \leq d(\Omega) \|u_k\|_{L^2(\Omega)} \|\nabla u_k\|_{L^2(\Omega)} \leq Pd(\Omega) \|u_{k,0}\|_{H_0^1(\Omega)}^2,$$

where P is the Poincaré constant associated to the domain Ω . Hence,

$$\frac{4T - 8Pd(\Omega)}{\pi} \|u_{k,0}\|_{H^{1/2}(\Omega)}^2 \leq \int_{\Sigma} \left(\frac{|u_k|}{\delta^{1/2}} \right)^2 (x \cdot \nu) d\sigma dt.$$

Thus, finally, if $T > 2Pd(\Omega) := T_0$,

$$(3.16) \quad A_1 \|u_{k,0}\|_{H^{1/2}(\Omega)}^2 \leq \int_{\Sigma} \left(\frac{|u_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt \leq A_2 \|u_{k,0}\|_{H^{1/2}(\Omega)}^2$$

holds with $A_1 > 0$.

Step 2: inequality for general solutions. We conclude the proof of the theorem by applying a density argument. Indeed, we know that the eigenfunctions ϕ_k form a basis of $L^2(\Omega)$ and that the space generated by them is dense in $L^2(\Omega)$; thus, taking the limit for $k \rightarrow +\infty$ in (3.16), since, clearly, the norms are preserved, and since we have already shown before that

$$\lim_{k \rightarrow +\infty} \frac{|u_k|}{\delta^s} = \frac{|u|}{\delta^s},$$

we finally have (3.17). \square

3.2.3. Observability from a neighbourhood of the boundary. We introduce here the observability inequality for the fractional Schrödinger equation; this inequality will then be the starting point for the proof of the controllability result.

PROPOSITION 3.6. *Let $s \in [1/2, 1)$ and let Ω and Γ_0 be as in the statement of Theorem 1.1. Moreover, let $\omega = \mathcal{O}_\varepsilon \cap \Omega$, where \mathcal{O}_ε is a neighbourhood of Γ_0 in \mathbb{R}^n . For any $u_0 \in L^2(\Omega)$, let $u = u(x, t)$ be the solution of (1.4) with initial datum u_0 .*

(i) *If $s \in (1/2, 1)$, then for every $T > 0$ there exists a positive constant C , depending only on s, T, n and Ω , such that*

$$(3.17) \quad \|u_0\|_{L^2(\Omega)}^2 \leq C \int_0^T \|u\|_{L^2(\omega)}^2 dt.$$

(i) *If $s = 1/2$, then (3.17) holds for any $T > 2Pd(\Omega)$, where P is the Poincaré constant associated to the domain Ω .*

Proof. First of all we notice that in the proposition we distinguish two cases: $s = 1/2$ and $s \in (1/2, 1)$. This is simply a natural consequence of Theorem 3.5. In this proof, there is no need to consider this distinction anymore since, in both cases, the path is the same.

Once again, we firstly prove the inequality for solutions of the equation with an initial datum which is combination of a finite number of eigenfunctions of the fractional Laplacian and then we recover the same result for general solutions arguing by density. Thus, let $u_{k,0} \in \text{span}(\phi_1, \dots, \phi_k)$ and let u_k be the corresponding solution of (1.1). We proceed in several steps

Step 1. We firstly establish

LEMMA 3.7. *For every $u_k = u_k(x, t)$ solution of (1.1) with initial data $u_{k,0} \in \text{span}(\phi_1, \dots, \phi_k)$ it holds*

$$(3.18) \quad \|u_{k,0}\|_{H^s(\Omega)}^2 \leq C_1 \int_0^T \|u_k\|_{H^s(\hat{\omega})}^2 dt,$$

where $\hat{\omega} \subset \Omega$ is a neighbourhood of the boundary such that $(\Omega \cap \hat{\omega}) \subset \omega$.

Proof. Let consider the cut-off function $\eta \in C^\infty(\mathbb{R}^n)$ defined as follows

$$(3.19) \quad \begin{cases} \eta(x) \equiv 1 & \text{in } \hat{\omega} \\ 0 < \eta(x) < 1 & \text{in } \omega \\ \eta(x) \equiv 0 & \text{in } \Omega \setminus \omega \end{cases}$$

and $v_k(x, t) := \eta(x)u_k(x, t)$; thus, v_k satisfies the equation

$$i(v_k)_t + (-\Delta)^s v_k = u_k(-\Delta)^s \eta + I_s(u_k, \eta) := g,$$

where (see [22])

$$I_s(u_k, \eta) = c_{n,s} \text{ P.V. } \int_{\mathbb{R}^n} \frac{(u_k(x) - u_k(y))(\eta(x) - \eta(y))}{|x - y|^{n+2s}} dy.$$

Now, starting from (3.5) applied to v_k , we have

$$\underbrace{\Gamma(1+s)^2 \int_{\Sigma} \left(\frac{|v_k|}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt}_J = 2s \int_0^T \|(-\Delta)^{\frac{s}{2}} v_k\|_{L^2(\omega)}^2 dt + \Im \int_{\omega} \bar{v}_k (x \cdot \nabla v_k) dx \Big|_0^T \\ + \Re \int_0^T \int_{\omega} v_k \left(\frac{n}{2} \bar{g} - ng - x \cdot \nabla g \right) dx dt.$$

Thus,

$$J \leq \alpha_1 \int_0^T \|v_k\|_{H^s(\omega)}^2 dt + \Re \int_0^T \int_{\omega} n v_k \left(\frac{\bar{g}}{2} - g \right) dx dt - \Re \int_0^T \int_{\omega} v_k (x \cdot \nabla g) dx dt \\ \leq \alpha_1 \int_0^T \|v_k\|_{H^s(\omega)}^2 dt + \alpha_2 \int_0^T \|v_k\|_{L^2(\omega)}^2 \|g\|_{H^1(\omega)}^2 dt + \Re \int_0^T \int_{\omega} n v_k \left(\frac{\bar{g}}{2} - g \right) dx dt \\ \leq \left(\alpha_1 + \alpha_2 \|g\|_{H^1(\omega)}^2 \right) \int_0^T \|v_k\|_{H^s(\omega)}^2 dt + \alpha_3 \int_0^T \|v_k\|_{L^2(\omega)}^2 dt \leq A \int_0^T \|v_k\|_{H^s(\omega)}^2 dt.$$

Now

$$\|v_k\|_{H^s(\omega)}^2 = \|\eta u_k\|_{H^s(\omega)}^2 \leq \beta_1 \|\eta u_k\|_{\dot{H}^s(\omega)}^2 = \beta_1 \|(-\Delta)^s(\eta u_k)\|_{L^2(\omega)}^2 \\ \leq \beta_1 \left(\|\eta(-\Delta)^s u_k\|_{L^2(\omega)}^2 + \|u_k(-\Delta)^s \eta\|_{L^2(\omega)}^2 + \|I_s(u_k, \eta)\|_{L^2(\omega)}^2 \right) \\ \leq \beta_2 \|u_k\|_{H^s(\omega)}^2 + \frac{\beta_1}{2} \left(\|u_k\|_{L^2(\omega)}^2 + \|(-\Delta)^s \eta\|_{L^2(\omega)}^2 \right) + \beta_3 \|u_k\|_{H^s(\omega)}^2 \\ \leq \beta_4 \|u_k\|_{H^s(\omega)}^2.$$

Thus, since $0 < \eta < 1$, applying (3.10) we finally get (3.18). \square

Step 2. We now prove the following

$$(3.20) \quad \|u_{k,0}\|_{H^s(\omega)}^2 \leq C_2 \int_0^T \left(\|(u_k)_t\|_{H^{-s}(\omega)}^2 + \|u_k\|_{L^2(\omega)}^2 \right) dt.$$

This inequality follows immediately from (3.18) and the following result.

LEMMA 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a bonded regular domain, $f \in H^{-s}(\Omega)$ and let $u \in H_0^s(\Omega)$ be the solution of*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u \equiv 0 & \text{in } \Omega^c. \end{cases}$$

Then, there exists a constant $\gamma > 0$ such that

$$(3.21) \quad \|u\|_{H^s(\hat{\omega})}^2 \leq \gamma \left[\|f\|_{H^{-s}(\omega)}^2 + \|u\|_{L^2(\omega)}^2 \right].$$

Proof. Let consider again the function $\eta(x)$ as defined in (3.19) and let $v(x, t) = \eta(x)u(x, t)$. Thus, v satisfies

$$\begin{cases} (-\Delta)^s v = \eta f + u(-\Delta)^s \eta + I_s(u, \eta) & \text{in } \omega \\ v \in H^s(\omega) \end{cases}$$

Now,

$$\begin{aligned} \|v\|_{H^s(\omega)} &\leq \gamma_1 \|(-\Delta)^s v\|_{L^2(\omega)} \\ &\leq \gamma_1 \left(\|u(-\Delta)^s \eta\|_{L^2(\omega)} + \|\eta(-\Delta)^s u\|_{L^2(\omega)} + \|I_s(u, \eta)\|_{L^2(\omega)} \right) \\ &\leq \gamma_2 \|u\|_{L^2(\omega)} + \gamma_3 \|(-\Delta)^s u\|_{L^2(\omega)} = \gamma_2 \|u\|_{L^2(\omega)} + \gamma_3 \|f\|_{H^{-s}(\omega)}. \end{aligned}$$

Hence, since

$$\|u\|_{H^s(\hat{\omega})} = \|v\|_{H^s(\hat{\omega})} \leq \|v\|_{H^s(\omega)},$$

we finally obtain the estimate (3.21). \square

By combining (1.1), (3.18) and (3.21) applied to u_k , we obtain

$$(3.22) \quad \|u_{k,0}\|_{H^s(\Omega)}^2 \leq B \int_0^T \left(\|(u_k)_t\|_{H^{-s}(\omega)}^2 + \|u_k\|_{L^2(\omega)}^2 \right) dt$$

and, from this, immediately follows (3.20).

Step 3. From (3.22), proceeding by contradiction and using a compactness-uniqueness argument, we obtain

$$(3.23) \quad \|u_{k,0}\|_{H^s(\Omega)}^2 \leq C_3 \int_0^T \|(u_k)_t\|_{H^{-s}(\omega)}^2 dt.$$

On the other hand, since $(-\Delta)^s$ is an isomorphism from $H^s(\Omega)$ to $H^{-s}(\Omega)$, from (3.23) we have

$$(3.24) \quad \|(u_k)_t(0)\|_{H^{-s}(\Omega)}^2 \leq C_3 \int_0^T \|(u_k)_t\|_{H^{-s}(\omega)}^2 dt.$$

Step 4. Let now define

$$\phi(x, t) := \int_0^t u_k(x, s) ds + \Theta(x),$$

where

$$\begin{cases} (-\Delta)^s \Theta = -i u_{k,0} & \text{in } \Omega \\ \Theta \in H^s(\mathbb{R}^n) \end{cases}$$

Thus, ϕ is a solution of (1.1) with initial datum $\phi(x, 0) = \Theta(x)$ and $\phi_t = u_k$. Applying (3.24) to ϕ we have

$$(3.25) \quad \|u_{k,0}\|_{H^{-s}(\Omega)}^2 \leq C_3 \int_0^T \|u_k\|_{H^{-s}(\omega)}^2 dt.$$

Step 5. From (3.18) and (3.25) we have

$$(3.26) \quad \|u_{k,0}\|_{H^s(\Omega)}^2 \leq C_1 \int_0^T \|u_k\|_{H^s(\omega)}^2 dt = C_1 \|u_k\|_{L^2(0,T;H^s(\omega))}^2$$

$$(3.27) \quad \|u_{k,0}\|_{H^{-s}(\Omega)}^2 \leq C_3 \int_0^T \|u_k\|_{H^{-s}(\omega)}^2 dt = C_3 \|u_k\|_{L^2(0,T;H^{-s}(\omega))}^2.$$

We are finally going to prove (3.17) by interpolation. Let consider the linear operator

$$\Lambda : H^{-s}(\Omega) \rightarrow L^2(0, T; H^{-s}(\omega))$$

defined by

$$\Lambda u_k(t) := \left(e^{it(-\Delta)^s} u_k \right) \Big|_{\omega}.$$

Clearly,

$$\|\Lambda u_k\|_{L^2(0,T;H^{-s}(\omega))} \leq \lambda_1 \|u_k\|_{H^{-s}(\Omega)}.$$

Furthermore, from (3.25) it follows that

$$\|\Lambda u_k\|_{L^2(0,T;H^{-s}(\omega))} \geq \lambda_2 \|u_k\|_{H^{-s}(\Omega)}.$$

Therefore, we can consider the closed subspace $X_0 := \Lambda(H^{-s}(\Omega))$ of $L^2(0,T;H^{-s}(\omega))$ and the linear operator $\Pi := \Lambda^{-1}$ (since Λ is an isomorphism between $H^{-s}(\Omega)$ and X_0). Thus,

$$(3.28) \quad \Pi \in \mathcal{L}(X_0, Y_0),$$

with $Y_0 := H^{-s}(\Omega)$. If now we set $X_1 := X_0 \cap L^2(0,T;H^s(\omega))$, it follows from (3.26) that

$$(3.29) \quad \Pi \in \mathcal{L}(X_1, Y_1),$$

with $Y_1 := H^s(\Omega)$. From (3.28), (3.29) and [16, Thm. 5.1], we have

$$\Pi \in \mathcal{L}([X_0, X_1]_{1/2}, [Y_0, Y_1]_{1/2}).$$

Moreover, from [16, Lemma 12.1] we have $[Y_0, Y_1]_{1/2} = L^2(\Omega)$ and from [4, Thm. 5.1.2] we have that

$$[L^2(0,T;H^s(\omega)), L^2(0,T;H^{-s}(\omega))]_{1/2} = L^2(0,T;[H^s(\omega); H^{-s}(\omega)]_{1/2}) = L^2(0,T;L^2(\omega)).$$

Hence, since X_0 and X_1 are closed subspaces of $L^2(0,T;H^{-s}(\omega))$ and $L^2(0,T;H^s(\omega))$ respectively, using [16, Thm. 15.1] we can verify that the norm of the space $[X_0, X_1]_{1/2}$ is equivalent to the norm of $L^2(0,T;L^2(\omega))$ and, since $\Pi \in \mathcal{L}([X_0, X_1]_{1/2}; L^2(\Omega))$, we finally have

$$(3.30) \quad \|u_{k,0}\|_{L^2(\Omega)}^2 \leq \int_0^T \|u_k\|_{L^2(\omega)}^2 dt.$$

Step 6. We now conclude our proof by applying a density argument. Indeed, we know that the eigenfunctions ϕ_k form a basis of $L^2(\Omega)$ and that the space generated by them is dense in $L^2(\Omega)$. Thus, taking the limit for $k \rightarrow +\infty$ in both sides of (3.30), since, clearly, the L^2 norm is preserved, we finally have (3.17). \square

3.3. Controllability from a neighbourhood of the boundary. The aim of this section is to obtain a result of exact controllability for the fractional Schrödinger equation (1.1), with one control acting only in a neighbourhood of the boundary.

Let consider the domain Ω , which we assumed to be bounded and $C^{1,1}$ with boundary Γ . For a fixed $\varepsilon > 0$ we define

$$(3.31) \quad \mathcal{O}_\varepsilon = \bigcup_{x \in \Gamma_0} B(x, \varepsilon), \quad \omega = \mathcal{O}_\varepsilon \cap \Omega$$

where, we remind, $\Gamma_0 = \{x \in \Gamma \mid (x \cdot \nu) > 0\}$.

Proof of Theorem 1.1. We apply HUM. Let consider the problem

$$(3.32) \quad \begin{cases} iy_t + (-\Delta)^s y = 0 & \text{in } Q \\ y \equiv 0 & \text{in } \Omega^c \times [0, T] \\ y(x, 0) = y_0(x) & \text{in } \Omega. \end{cases}$$

Let also consider the backward system

$$(3.33) \quad \begin{cases} i\phi_t + (-\Delta)^s \phi = y\chi_{\{\omega \times [0, T]\}} & \text{in } Q \\ \phi \equiv 0 & \text{in } \Omega^c \times [0, T] \\ \phi(x, T) = 0 & \text{in } \Omega \end{cases}$$

Problem (3.33) admits at least a weak solution, defined by transposition (see e.g. [16]); the function ϕ is a solution of the problem if it holds

$$(3.34) \quad \Re \int_Q \phi \bar{g} dx dt - \Re \int_{\Omega} i\phi(x, 0) \bar{\theta}(x, 0) dx = \Re \int_0^T \int_{\omega} y \bar{\theta} dx dt,$$

where θ is the solution of

$$\begin{cases} i\theta_t + (-\Delta)^s \theta = g & \text{in } Q \\ \theta \equiv 0 & \text{in } \Omega^c \times [0, T] \\ \theta(x, 0) = \theta_0 & \text{in } \Omega. \end{cases}$$

We then introduce the linear operator

$$\Lambda : L^2(\Omega) \rightarrow L^2(\Omega)$$

defined by

$$\Lambda y_0 := -i\phi(0).$$

By considering (3.34) with $\theta = y$ we immediately get

$$(3.35) \quad \langle \Lambda y_0, y_0 \rangle = \int_0^T \int_{\omega} |y|^2 dx dt.$$

From the observability inequality (3.17) and the identity (3.35) we deduce that Λ is an isomorphism from $L^2(\Omega)$ to $L^2(\Omega)$. Hence, given $u_0 \in L^2(\Omega)$ in (1.1) we can choose the control function $h = y|_{\omega}$, where y is the solution of (3.32) with initial datum $y_0 = \Lambda^{-1}(-iu_0) \in L^2(\Omega)$, such that $u(x, T) = 0$.

4. Fourier analysis for the one dimensional problems. We want here to show that, if we want to prove a positive control result, we need to consider a Schrödinger equation with a fractional Laplacian of order $s \geq 1/2$. In order to do that, we analyse our evolution problem in one space dimension and we show that, when the exponent of the fractional Laplace operator is below the critical value written above, we are not able to prove the observability inequality. In this way we immediately obtain the sharpness of the exponents $s = 1/2$. Thus, the main result of this section will be the following Theorem.

THEOREM 4.1. *Let us consider the following one dimensional problem for the fractional Schrödinger equation on the interval $(-1, 1)$*

$$(4.1) \quad \begin{cases} iu_t + (-d_x^2)^\beta u = 0 & \text{in } (-1, 1) \times [0, T] \\ u(-1, t) = u(1, t) = 0 & t \in [0, T] \\ u(x, 0) = u_0(x) & x \in (-1, 1) \end{cases}$$

with $\beta \in (0, 1)$. Then, (4.1) is controllable if and only if $\beta \geq 1/2$.

For the proof Theorem 4.1, we will use the results contained in [13] and [14]. In this two works, the authors have studied the eigenvalue problem for the fractional Laplacian both on the half line $(0, +\infty)$ and on the interval $(-1, 1)$. In particular, [13] is devoted only to the analysis of the square root of the Laplacian. The main result we will apply is the following, taken from [14, Thm. 1].

THEOREM 4.2. *Let $\beta \in (0, 1)$. For the eigenvalues associated to the problem*

$$(-d_x^2)^\beta \phi_k(x) = \lambda_k \phi_k(x), \quad k \geq 1$$

it holds

$$(4.2) \quad \lambda_k = \left(\frac{k\pi}{2} - \frac{(2-2\beta)\pi}{8} \right)^{2\beta} + O\left(\frac{1}{k}\right) \quad \text{as } k \rightarrow +\infty.$$

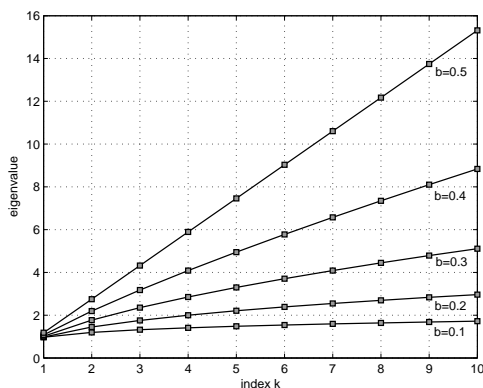


FIGURE 1. First 10 eigenvalues of $(-d_x^2)^\beta$ on $(-1, 1)$ for $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$.

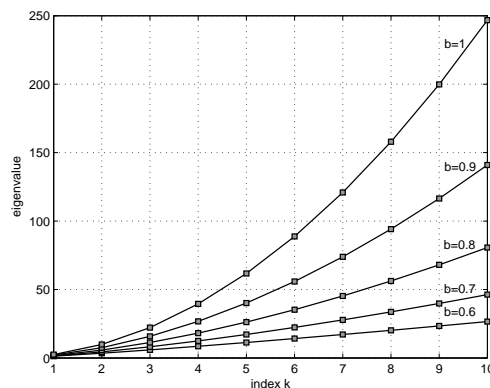


FIGURE 2. First 10 eigenvalues of $(-d_x^2)^\beta$ on $(-1, 1)$ for $\beta = 0.6, 0.7, 0.8, 0.9, 1$.

Proof of Theorem 4.1. We are interested in getting a control result by applying HUM. It is well known that this is equivalent to the proof of an observability inequality, which follows from the Pohozaev identity for the fractional Schrödinger equation (3.5) and from (3.10). In our case, $s = \beta$ and $n = 1$, the boundary integral in (3.10) simply corresponds to computing the value of the integrand in the extremal points of the interval considered, $x = \pm 1$; thus, the inequality we get is

$$(4.3) \quad C \|u_0\|_{H^\beta(-1,1)}^2 \leq \int_0^T \left(\frac{|u|}{(1-|x|)^\beta} \right)^2 \Big|_{x=-1}^{x=1} dt.$$

Since (4.3) involves the H^β norm of the initial datum, the natural space in which to analyse the problem is $H^\beta(-1, 1)$; we remind that this is an Hilbert space, naturally endowed with the inner product

$$(4.4) \quad (u, v)_{H^\beta(-1,1)} = \int_{-1}^1 uvdx + \int_{-1}^1 (-d_x^2)^{\beta/2} u (-d_x^2)^{\beta/2} v dx.$$

The solution of (4.1) will be given spectrally, i.e in terms of the eigenvalues and eigenfunctions of the operator $(-d_x^2)^\beta$, namely $\{\lambda_k, \phi_k(x)\}_{k \geq 1}$.

First of all, it is possible to show that $\phi_k(1) = \phi_k(-1) = 0$. Indeed, since clearly every ϕ_k solves the problem

$$(4.5) \quad (-d_x^2)^\beta \phi_k(x) = \lambda_k \phi_k(x) \quad x \in (-1, 1),$$

from [21, Cor. 1.6] we have that, for every $\alpha \in [\beta, 1 + 2\beta)$, $\phi_k \in C^\alpha(-1, 1)$ and

$$[\phi_k]_{C^\alpha(\{x \in (-1,1): 1-|x| \geq \rho\})} \leq C\rho^{\beta-\alpha} \quad \text{for all } \rho \in (0, 1)$$

and that the function $\phi_k(x)/(1-|x|)^\beta$ is continuous up to the boundary of the domain which, in this case, consists only in the set $\{-1, 1\}$. Thus, each eigenfunction $\phi_k(x)$ has to vanish in $x = \pm 1$.

Now, since the ϕ_k are eigenfunctions, they form an orthonormal basis of $L^2(-1, 1)$, i.e.

$$(\phi_k, \phi_j)_{L^2(-1,1)} = \delta_{kj}.$$

If, instead, we compute $(\phi_k, \phi_j)_{H^\beta(-1,1)}$ we have

$$\begin{aligned} (\phi_k, \phi_j)_{H^\beta(-1,1)} &= \int_{-1}^1 \phi_k(x)\phi_j(x)dx + \int_{-1}^1 (-d_x^2)^{\beta/2} \phi_k(x)(-d_x^2)^{\beta/2} \phi_j(x)dx \\ &= (\phi_k, \phi_j)_{L^2(-1,1)} + \int_{-1}^1 \phi_k(x)(-d_x^2)^\beta \phi_j(x)dx \\ &= \delta_{kj} + \int_{-1}^1 \lambda_j \phi_k(x)\phi_j(x)dx = \delta_{kj} + \lambda_j (\phi_k, \phi_j)_{L^2(-1,1)} = (1 + \lambda_j)\delta_{kj}. \end{aligned}$$

This fact tells us that if we introduce the following normalization for the eigenfunctions ϕ_k

$$\{\theta_k\}_{k \geq 1} = \left\{ \frac{\phi_k}{\sqrt{1 + \lambda_k}} \right\}_{k \geq 1}$$

we get an orthonormal basis for the space $H^\beta(-1, 1)$; this is the basis we are going to use for the representation of the solution of the problem; we remark here that for the $\{\theta_k\}_{k \geq 1}$ clearly holds

$$(-d_x^2)^\beta \theta_k(x) = \lambda_k \theta_k(x).$$

Formally, (4.1) has a solution of the form

$$u(x, t) = \sum_{k \geq 1} a_k \theta_k(x) e^{i\lambda_k t},$$

where the Dirichlet boundary conditions are satisfied since $\theta_k(\pm 1) = 0$. The coefficients a_k are the Fourier coefficients of the function $u_0(x)$ with respect to the basis of the eigenfunctions and are the ones which guarantee that the solution u satisfies the initial condition. Since $\{\theta_k\}_{k \geq 1}$ is an orthonormal basis, they are given by

$$(4.6) \quad a_k = \frac{1}{2} \int_{-1}^1 u_0(x) \theta_k(x) dx.$$

Now, coming back to (4.3), we have

$$\|u_0\|_{H^\beta(-1,1)}^2 = \left(\sum_{k \geq 1} a_k \theta_k, \sum_{k \geq 1} a_k \theta_k \right)_{H^\beta(-1,1)} = \sum_{k \geq 1} |a_k|^2 (\theta_k, \theta_k)_{H^\beta(-1,1)} = \sum_{k \geq 1} |a_k|^2;$$

thus, the observability inequality becomes

$$(4.7) \quad C_1 \sum_{k \geq 1} |a_k|^2 \leq \int_0^T \left(\sum_{k \geq 1} a_k \frac{\theta_k(x)}{(1-|x|)^\beta} e^{i\lambda_k t} \right)^2 \Big|_{x=-1}^{x=1} dt.$$

As we already stated before, and as it is proved in [21], the function $\theta_k(x)/(1-|x|)^\beta$ is continuous up to the boundary. In our case, this means that, in the limit for $x \rightarrow \pm 1$, even if either the numerator and the denominator separately goes to zero, we get a constant value; moreover, for any $k \geq 1$, each eigenvalue λ_k is positive. Hence, (4.7) holds only if the vectors $\{e^{i\lambda_k t}\}_{k \geq 1}$ are linear independent; if not, there will exist at least a coefficient $a_k \neq 0$ such that

$$C_1 \sum_{k \geq 1} |a_k|^2 \leq 0$$

and the constant C_1 could only be zero.

For having the linear independence of the $e^{i\lambda_k t}$, for any $k \geq 1$ the eigenvalues λ_k must be all different; since we can order them as to be an increasing sequence, we get that it has to hold

$$(4.8) \quad \lambda_{k+1} - \lambda_k \geq \gamma > 0.$$

Now, we know from (4.2) the behaviour of the eigenvalues of $(-d_x^2)^\beta$. Thus, we can immediately check that (4.8) holds only for $\beta \geq 1/2$ while for $\beta < 1/2$ we have

$$\liminf_{k \rightarrow +\infty} (\lambda_{k+1} - \lambda_k) = 0.$$

This means that we are able to prove the observability inequality, i.e. we can control the equation (4.1), only for $\beta \geq 1/2$.

5. Application to the observability of a fractional wave equation. As an immediate consequence of the observability result for the fractional Schrödinger equation (1.1), we derive here an observability inequality for the fractional wave equation

$$(5.1) \quad \begin{cases} u_{tt} + (-\Delta)^{2s} u = 0 & \text{in } \Omega \times [0, T] := Q \\ u \equiv (-\Delta)^s u \equiv 0 & \text{in } \Omega^c \times [0, T] \\ u(x, 0) = u_0(x) & \text{in } \Omega. \\ u_t(x, 0) = u_1(x) & \text{in } \Omega. \end{cases}$$

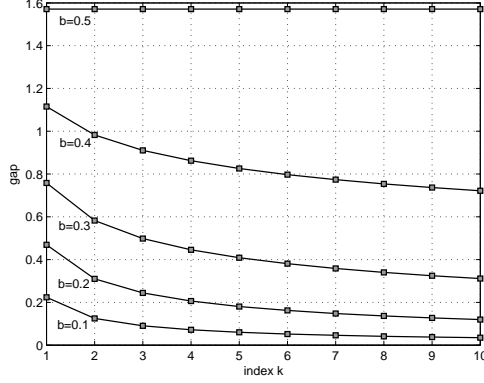


FIGURE 3. Gap between the first 10 eigenvalues of $(-d_x^2)^\beta$ on $(-1, 1)$ for $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$. At any index k corresponds the gap $\lambda_{k+1} - \lambda_k$.

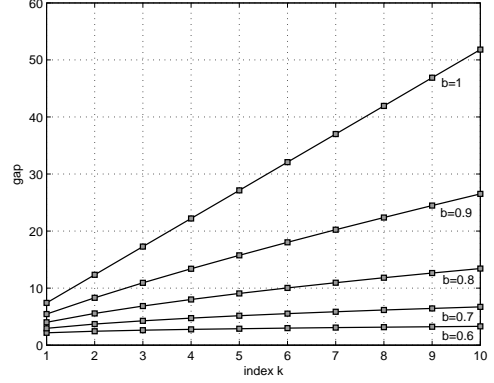


FIGURE 4. Gap between the first 10 eigenvalues of $(-d_x^2)^\beta$ on $(-1, 1)$ for $\beta = 0.6, 0.7, 0.8, 0.9, 1$. At any index k corresponds the gap $\lambda_{k+1} - \lambda_k$.

This inequality, of course, will imply a controllability result analogous to the one we already proved for the Schrödinger equation.

In (5.1), the operator $(-\Delta)^{2s}$ is an higher order fractional Laplacian, which is defined by composition between two lower order operators as follows.

$$(5.2) \quad \begin{aligned} (-\Delta)^{2s}u(x) &:= (-\Delta)^s(-\Delta)^s u(x), \quad s \in [1/2, 1), \\ \mathcal{D}((-\Delta)^{2s}) &= \left\{ u \in H_0^1(\Omega) \mid (-\Delta)^s u|_{\Omega^c} \equiv 0, (-\Delta)^{2s}u \in L^2(\Omega) \right\}. \end{aligned}$$

The reason why we are introducing it is that, with an analysis similar to the one presented in section 4, we can show that a wave equation involving the fractional Laplacian is controllable if and only if we consider an operator of order $s \geq 1$; otherwise, we are not able to prove any observability inequality. Moreover, we are defining the operator as in (5.2) because this choice allows us to preserve the regularity properties that $(-\Delta)^s$ possess. In particular, $(-\Delta)^{2s}$ is symmetric, positive and self-adjoint on the domain Ω , simply because it is defined applying twice the same symmetric, positive and self-adjoint operator, namely $(-\Delta)^s$. Of course, we can admit other definition of an higher order fractional Laplacian on a regular domain by composition, but not always we obtain a suitable operator; for instance

$$(-\Delta)^{s+1}u(x) := (-\Delta)^s(-\Delta u)(x) = c_{n,s} \text{ P.V. } \int_{\mathbb{R}^n} \frac{-\Delta u(x) + \Delta u(y)}{|x-y|^{n+2s}} dy, \quad s \in (0, 1)$$

is a well defined higher order fractional Laplacian, meaning that we can identify its domain and the way it operates but, in this case, it is easy to see through the definition that the operator is not self-adjoint.

For the observability of (5.1), we are going to apply an abstract argument introduced by Tucsnak and Weiss in [24]. Let A_0 be a linear, self-adjoint operator such that A_0^{-1} is compact, H be an Hilbert space and $H_1 := \mathcal{D}(A_0)$; moreover, let us denote $X := H_1 \times H$, which is an Hilbert

space with the inner product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \langle A_0 f_1, A_0 f_2 \rangle_H + \langle g_1, g_2 \rangle_H = \int_{\Omega} A_0 f_1 A_0 f_2 dx + \int_{\Omega} g_1 g_2 dx.$$

We define $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow X$ by $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A_0^2) \times H$ and

$$\mathcal{A} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix}.$$

Now, let Y be another Hilbert space and let $C_0 \in \mathcal{L}(H_1, Y)$ be such that the pair (iA_0, C_0) is exactly observable in some time T_0 . From [24, Prop. 6.8.2] we have that, if the eigenvalues of the operator A_0 satisfy

$$(5.3) \quad \sum_{k \in \mathbb{N}} \lambda_k^{-d} < +\infty$$

for some $d \in \mathbb{N}$, then the pair (\mathcal{A}, C) , with $C = [0 \ C_0]$, is exactly observable in any time $T > T_0$.

In our case, we have $A_0 := (-\Delta)^s$, $A_0^2 := (-\Delta)^{2s}$, $H = Y := L^2(\Omega)$ and

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \int_{\Omega} (-\Delta)^s f_1 (-\Delta)^s f_2 dx + \int_{\Omega} g_1 g_2 dx;$$

moreover, the eigenvalues condition (5.3) is satisfied with $d = n$ (see e.g. [5], [11]).

Thus, we can apply [24, Prop. 6.8.2] and, from the observability of the fractional Schrödinger equation we immediately get the observability of the fractional wave equation for any $T > 0$, when $s \in (1/2, 1)$, or for $T > T_0$, when $s = 1/2$; in particular, we get

$$(5.4) \quad \|u_0\|_{H^{2s}(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \leq C \int_0^T \|u_t\|_{L^2(\omega)}^2 dt.$$

Of course, the same result could have been obtained by applying the multipliers method, in the same way we did for the Schrödinger equation. With this procedure, we are led to a different observability inequality, namely

$$(5.5) \quad \|u_0\|_{H^{2s}(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \leq C \int_0^T \|u\|_{H^{2s}(\omega)}^2 dt$$

which, however, is totally equivalent to (5.4). Indeed, starting from (5.4) we can easily recover (5.5) proceeding as follows; first of all we introduce the change of variables $\phi(x, t) := (-\Delta)^s u(x, t)$; thus ϕ satisfies

$$\begin{cases} \phi_{tt} + (-\Delta)^{2s} \phi = 0 & \text{in } \Omega \times [0, T] := Q \\ \phi \equiv (-\Delta)^s \phi \equiv 0 & \text{in } \Omega^c \times [0, T] \\ \phi(x, 0) = (-\Delta)^s u_0(x) & \text{in } \Omega \\ \phi_t(x, 0) = (-\Delta)^s u_1(x) & \text{in } \Omega \end{cases}$$

and we get the inequality

$$(5.6) \quad \|u_0\|_{H^{4s}(\Omega)}^2 + \|u_1\|_{H^{2s}(\Omega)}^2 \leq C \int_0^T \|u_t\|_{H^{2s}(\omega)}^2 dt.$$

Then, we define

$$\psi(x, t) := \int_0^t u(x, \tau) d\tau - h(x),$$

with $h(x)$ satisfying $(-\Delta)^{2s}h(x) = u_1(x)$; thus, the function ψ satisfies

$$\begin{cases} \psi_{tt} + (-\Delta)^{2s}\psi = 0 & \text{in } \Omega \times [0, T] := Q \\ \psi \equiv (-\Delta)^s\psi \equiv 0 & \text{in } \Omega^c \times [0, T] \\ \psi(x, 0) = -h(x) & \text{in } \Omega \\ \psi_t(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

and, by applying to it (5.6), we finally obtain (5.5).

6. Open problems and perspectives. We conclude this work by briefly presenting some open problems and future perspectives.

As we have already mentioned in the introduction, an interesting issue, and a natural extension of the work presented in this paper, could be the study of a variant of our original problem including different boundary conditions, for instance of Neumann or Robin type. In particular, we would like to address the problem of the boundary controllability of the fractional Schrödinger equation, i.e. the possibility of getting a controllability result for the system

$$(6.1) \quad \begin{cases} iu_t + (-\Delta)^s u = 0 & \text{in } \Omega \times [0, T] \\ u = \begin{cases} v & \text{on } \Gamma_0 \times [0, T] \\ 0 & \text{on } \Gamma_1 \times [0, T] \end{cases} \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases}$$

In this frame, we should refer to a recent work of M. Warma ([25]), which generalize the Pohozaev identity for the fractional Laplacian presented in [22].

Another possible extension of our work could be the study of a Schrödinger equation involving some more general non-local operators. By definition, the fractional Laplace operator belongs to a class of operators of the type

$$(6.2) \quad \mathcal{L}u(x) = \int_{\mathbb{R}^n} (u(x) - u(x+y))K(y)dy$$

where $K(y)$ is a suitable kernel. In the case of the fractional Laplacian, the kernel is simply $1/|y|^{n+2s}$; by choosing $K(y)$ differently we could define other non-local operators and, consequently, other non-local evolution problems. In particular, by choosing a kernel $K(x, y)$, i.e. depending also on the x variable, could be addressed problems with variable coefficients.

Finally, in section 4 we showed that Theorem 1.1 holds only in the case of a fractional Laplacian of order $s \in [1/2, 1)$. We would like to extended this result, either in positive or in negative, also for real exponents $s > 1$. This, however, is still a subject under study; one path which seem promising to follow consists in analysing the WKB (Wentzel–Kramers–Brillouin) asymptotic expansion in Geometric Optics [20] for the solutions of our fractional Schrödinger equation. Anyhow this work is at the very beginning and we still do not have clear ideas about it.

Appendix A. Justification of the use of Pohozaev for the solution of the fractional Schrödinger equation.

In order to bypass the regularity issue for the solution of our fractional Schrödinger equation and to be allowed to apply the Pohozaev identity for the fractional Laplacian in the proof of Proposition 3.1, we firstly dealt with solutions given as a linear combination of a finite number of eigenfunctions and, in a second moment, we recovered the result we needed for general finite energy solutions by density. To justify this procedure, we show here that the eigenfunctions of the fractional Laplacian on a bounded, regular domain Ω possess the regularity required in the hypothesis of Proposition 2.4. We are going to proceed in two steps. First of all, we show L^p regularity for the eigenfunctions for any $p \in [2, +\infty)$; then, we show that we can reach L^∞ regularity and, according to [22, Thm. 1.4], this will imply enough regularity to apply the Pohozaev identity.

Step 1: L^p -regularity of the eigenfunctions. Let consider the eigenvalues problem for the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ u \equiv 0 & \text{in } \Omega^c. \end{cases}$$

We multiply the equation for $\phi := |u|^{p+1} \text{sgn}(u)$ and we integrate over Ω . First of all, we notice that the function ϕ vanishes outside the domain, thus we can consider the integrals over Ω as integrals over the whole space \mathbb{R}^n .

$$\begin{aligned} & \lambda \int_{\mathbb{R}^n} u(x) |u(x)|^{p+1} \text{sgn}(u(x)) dx \\ &= \lambda \int_{\mathbb{R}^n} |u(x)|^{p+2} dx = \int_{\mathbb{R}^n} |u(x)|^{p+1} \text{sgn}(u(x)) (-\Delta)^s u(x) dx \\ &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} |u(x)|^{p+1} \text{sgn}(u(x)) (-\Delta)^{\frac{s}{2}} u(x) dx \\ &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \left[|u(x)|^{p+1} \text{sgn}(u(x)) - |u(y)|^{p+1} \text{sgn}(u(y)) \right] dx dy \\ &\geq c_{n,s} \frac{2(p+1)}{(p+2)^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left| |u(x)|^{\frac{p+2}{2}} - |u(y)|^{\frac{p+2}{2}} \right|^2}{|x-y|^{n+2s}} dx dy. \end{aligned}$$

In the previous computations, we used the inequality

$$\left| |\alpha|^{\frac{p}{2}} - |\beta|^{\frac{p}{2}} \right|^2 \leq \frac{p^2}{4(p-1)} (\alpha - \beta) (|\alpha|^{p-1} \text{sgn}(\alpha) - |\beta|^{p-1} \text{sgn}(\beta)) \quad \forall \alpha, \beta \in \mathbb{R}, \quad \forall p \geq 2$$

presented in chapter 4 of [3]. Thus, at the end we have

$$\lambda \int_{\Omega} |u(x)|^{p+2} dx \geq c_{n,s} \frac{2(p+1)}{(p+2)^2} \int_{\Omega} \int_{\Omega} \frac{\left| |u(x)|^{\frac{p+2}{2}} - |u(y)|^{\frac{p+2}{2}} \right|^2}{|x-y|^{n+2s}} dx dy.$$

Using the embedding theorems for the fractional Sobolev spaces (see e.g. [10, Thm. 6.5]), we finally get

$$\lambda \int_{\Omega} |u|^{p+2} dx \geq Ac_{n,s} \frac{2(p+1)}{(p+2)^2} \left\| |u|^{\frac{p+2}{2}} \right\|_{L^{\frac{2n}{n-2s}}(\Omega)}^2,$$

which is, of course, the same as

$$\lambda \left\| |u|^{\frac{p+2}{2}} \right\|_{L^2(\Omega)}^2 \geq Ac_{n,s} \frac{2(p+1)}{(p+2)^2} \left\| |u|^{\frac{p+2}{2}} \right\|_{L^{\frac{2n}{n-2s}}(\Omega)}^2.$$

Since $n/(n-2s) > 1$, this argument allows us to gain regularity for the function u as follows

$$p+2 \mapsto (p+2) \frac{n}{n-2s}.$$

Coming back now to our original problem, since u is an eigenfunction for the fractional Laplacian, we know that it is, at least, L^2 regular. Thus, by applying the procedure above for $p=0$ we can increase its regularity up to $L^{\frac{2n}{n-2s}}$.

If now we iterate the same argument we see that, in a finite number of steps, we can get L^p regularity for any $p \in [2, +\infty)$.

Step 3: L^∞ -regularity of the eigenfunctions. We prove here the L^∞ -regularity for the eigenfunctions of the fractional Laplacian, as an immediate consequence of the following result.

THEOREM A.1. *Let $u \in H_0^s(\Omega)$ be the solution of*

$$(A.1) \quad \begin{cases} (-\Delta)^s u - \lambda u = f & \text{in } \Omega \\ u \equiv 0 & \text{in } \Omega^c. \end{cases}$$

If $f \in L^p(\Omega) + L^\infty(\Omega)$ for some $p > 1$, $p > n/2s$, i.e. $f = f_1 + f_2$ with $f_1 \in L^p(\Omega)$ and $f_2 \in L^\infty(\Omega)$, then $u \in L^\infty(\Omega)$.

Proof. First of all we observe that, since $-u$ solves the same equation as u with f replaced by its opposite $-f$, which clearly satisfies the same assumptions, it is enough to estimate $\|u^+\|_{L^\infty(\Omega)}$, where

$$u^+ = \begin{cases} u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0. \end{cases}$$

Set $T := \|u^+\|_{L^\infty(\Omega)} \in [0, +\infty]$ and, for any $t \in (0, T)$, set $v(t) := (u - t)^+$. Let now define

$$\alpha(t) := |\{x \in \Omega \mid u(x) > t\}|$$

for all $t > 0$ (note that $\alpha(t)$ is always finite).

Since $v(t) \in L^2(\Omega)$ is supported in the set $\{x \in \Omega \mid u(x) > t\}$, we have $v(t) \in L^1(\Omega)$. Thus, it is well defined

$$\beta(t) := \int_{\Omega} v(t) dx.$$

Integrating the characteristic function $\chi_{\{u>s\}}$ on $(t, +\infty) \times \Omega$ and applying Fubini's theorem we obtain

$$\beta(t) := \int_t^{+\infty} \alpha(s) ds,$$

so that $\beta \in W_{loc}^{1,1}(0, +\infty)$ and $\beta'(t) = -\alpha(t)$ for a.e. $t > 0$. Now, from (A.1) we obtain

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} u dx - \lambda \int_{\mathbb{R}^n} u v dx = \int_{\mathbb{R}^n} f v dx,$$

which yields to

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 dx - \lambda \int_{\mathbb{R}^n} |v|^2 dx = \int_{\mathbb{R}^n} (f + \lambda t) v dx.$$

From this last inequality and from the fact that u vanishes outside Ω , it follows immediately

$$(A.2) \quad |1 - \lambda \|u\|_{H^s(\Omega)}^2 \leq \int_{\Omega} (f - \lambda t) dx \leq \int_{\Omega} (|f| + t|\lambda|) v dx.$$

We now observe that

$$\begin{aligned} \int_{\Omega} |f| v dx &\leq \int_{\Omega} (|f_1| + |f_2|) v dx \leq \|f_1\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)} + \|f_2\|_{L^\infty(\Omega)} \|v\|_{L^1(\Omega)} \\ &\leq C_1 \|v\|_{L^{p'}(\Omega)} + C_2 \|v\|_{L^1(\Omega)} \end{aligned}$$

and we deduce from (A.2) that

$$(A.3) \quad \|v\|_{H^s(\Omega)} \leq C_3(1+t) (\|v\|_{L^{p'}(\Omega)} + \|v\|_{L^1(\Omega)}).$$

Fix now $\rho > 2p'$ such that $\rho < 2n/(n-2s)$. From the embedding theorems for the fractional Sobolev spaces ([8], [10]) we have $H^s(\Omega) \hookrightarrow L^\rho(\Omega)$. Moreover, it follows from the Hölder inequality that

$$\|v\|_{L^1(\Omega)} \leq \alpha(t)^{1-\frac{1}{\rho}} \|v\|_{L^\rho(\Omega)}$$

and

$$\|v\|_{L^{p'}(\Omega)} \leq \alpha(t)^{\frac{1}{p'}-\frac{1}{\rho}} \|v\|_{L^\rho(\Omega)}.$$

Thus, we deduce from (A.3) that

$$\|v\|_{L^\rho(\Omega)}^2 \leq C_3(1+t) \left[\alpha(t)^{\frac{1}{p'}-\frac{1}{\rho}} + \alpha(t)^{1-\frac{1}{\rho}} \right] \|v\|_{L^\rho(\Omega)}.$$

Since $\beta(t) = \|v\|_{L^1(\Omega)} \leq \alpha(t)^{1-\frac{1}{\rho}} \|v\|_{L^\rho(\Omega)}$, we obtain

$$\beta(t) \leq C_3(1+t) \left[\alpha(t)^{1+\frac{1}{p'}-\frac{2}{\rho}} + \alpha(t)^{2-\frac{2}{\rho}} \right]$$

which can be written as

$$\beta(t) \leq C_3(1+t) F(\alpha(t)),$$

with $F(s) = s^{1+\frac{1}{p'}-\frac{2}{\rho}} + s^{2-\frac{2}{\rho}}$. It follows that

$$-\alpha(t) + F^{-1} \left(\frac{\beta(t)}{C_3(1+t)} \right) \leq 0.$$

Setting now $z(t) = \beta(t)/C_3(1+t)$, and remembering that $\beta'(t) = -\alpha(t)$, we deduce

$$z'(t) + \frac{\psi(z(t))}{C_3(1+t)} \leq 0$$

with $\psi(s) = F^{-1}(s) + C_3s$. Integrating the above differential inequality we get

$$\int_s^t \frac{d\sigma}{C_3(1+\sigma)} \leq \int_{z(t)}^{z(s)} \frac{d\sigma}{\psi(\sigma)}$$

for all $0 < s < t < T$. Now, if $T \leq 1$, then $\|u^+\|_{L^\infty} \leq 1$ by definition. Otherwise, we obtain

$$\int_1^t \frac{d\sigma}{C_3(1+\sigma)} \leq \int_{z(t)}^{z(1)} \frac{d\sigma}{\psi(\sigma)}$$

for all $1 < t < T$, which implies in particular that

$$\int_1^T \frac{d\sigma}{C_3(1+\sigma)} \leq \int_0^{z(1)} \frac{d\sigma}{\psi(\sigma)}.$$

Note now that $F(s) \approx s^{1+\frac{1}{p'}-\frac{2}{p}}$ as $s \downarrow 0$ and $1 + 1/p' - 2/p > 1$, so that $1/\psi$ is integrable near zero. Since, instead, the function $1/(1+\sigma)$ is not integrable at $+\infty$, this finally implies that $T = \|u^+\|_{L^\infty(\Omega)} < +\infty$. \square

Since, of course, the theorem we just proved can be applied to the function $f \equiv 0$, this automatically imply the L^∞ -regularity for the eigenfunctions of the fractional Laplacian. Now, this is enough to allow us to apply the Pohozaev identity for the fractional Laplacian to the solution u of our fractional Schrödinger equation. Indeed, Theorem 1.4 of [22] states that any bounded solution of

$$(A.4) \quad \begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u \equiv 0 & \text{in } \Omega^c \end{cases}$$

with $f \in C_{loc}^{0,1}(\overline{\Omega} \times \mathbb{R})$, i.e. Lipschitz, satisfies the hypothesis (i) and (ii) of Proposition 3.1. But this is exactly our case, since, by definition any eigenfunction of the fractional Laplacian satisfies the problem

$$\begin{cases} (-\Delta)^s \phi_k = \lambda_k \phi_k & \text{in } \Omega \\ \phi_k \equiv 0 & \text{in } \Omega^c \end{cases}$$

which is in the form of (A.4) with f clearly Lipschitz, and since we just showed that all the eigenfunctions are bounded. Moreover, we can conclude by observing that, always from the definition of eigenfunction, also hypothesis (iii) is clearly satisfied.

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REFERENCES

- [1] R.A. ADAMS, *Sobolev spaces*, Academic press, New York-London, 1975
- [2] F. ALABAU-BOUSSOUIRA, R. BROCKETT, O.GLASS, J. LE ROUSSEAU AND E. ZUAZUA, *Control of partial differential equations*, Springer Lecture Notes in Mathematics 2048, CIME Foundation Subseries, 2010
- [3] F. ANDREU-VAILLO, J.M. MAZÓN, J.D. ROSSI AND J. JULIÁN TOLEDO-MELERO, *Nonlocal diffusion problems*, Mathematical Surveys and Monographs, Volume 165, 2010
- [4] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces: an introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976
- [5] R.M. BLUMENTHAL AND R.K. GETTOO, *The asymptotic distribution of the eigenvalues for a class of Markov operators*, Pacific J. Math., Vol 9 (1959), pp. 399–408
- [6] H. BREZIS, *Analyse Fonctionnelle - théorie et applications*, Masson, 1983
- [7] T. CAZENAVE AND A. HARAUX, *An introduction to semilinear elliptic equations*
- [8] F. DEMENGEL AND J. DEMENGEL, *Functional spaces for the theory of elliptic PDEs*, Springer-Verlag London Limited, 2012
- [9] S. DIPIERRO, G. PALATUCCI AND E. VALDINOCI, *Dislocation dynamics in crystals: a macroscopic theory in a fractional Laplace setting*, Comm. Math. Phys. (2014)
- [10] E. DI NEZZA, G. PALATUCCI AND E. VALDINOCI, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math., Vol. 136 (2012), No. 5, pp. 521–573
- [11] R.L. FRANK AND L. GEISINGER, *Refined semiclassical asymptotics for fractional powers of the Laplace operator*, J. Reine Angew. Math., to appear
- [12] V. KOMORNINK, *Exact controllability and stabilization-the multiplier method*, Université Louis Pasteur, Strasbourg, 1993
- [13] T.KULCZYCKI, M. KWAŚNICKI, J. MAŁECKI AND A. STOS, *Spectral properties of the Cauchy process on half-line and interval*, Proc. Lond. Math. Soc., Vol. 101 (2010), No. 2, pp. 589-622
- [14] M. KWAŚNICKI, *Eigenvalues of the fractional Laplace operator int the interval*, J. Funct. Anal., Vol. 262 (2012), pp. 2379-2402
- [15] J.L. LIONS, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Tome 1 - contrôlabilité exacte*, Masson, 1988
- [16] J.L. LIONS AND E. MAGENES, *Problèmes aux limites non homogènes et applications*, Dunod, Paris, 1968
- [17] E. MACHTYNGIER, *Exact controllability for the Schrödinger equation*, SIAM J. Control Optim., Vol. 32 (1994), No. 1, pp. 24-34
- [18] M. M. MEERSCHAERT, *Fractional calculus, anomalous diffusion, and probability*, World Scientific Publishing Co. Pte. Ltd.
- [19] S.I. POHOZAEV, *On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$* , Dokl. Akad. Nauk, SSSR 165 (1965), pp. 1408-1411
- [20] J. RAUCH, X. ZHANG AND E. ZUAZUA, *Polynomial decay for a hyperbolic-parabolic coupled system*, J. Math. Pures Appl., Vol. 84 (2005), pp. 407-470
- [21] X. ROS-OTON AND J. SERRA, *The Dirichlet problem for the fractional Laplacian-Regularity up to the boundary*, J.Math. Pures Appl., Vol. 101, Issue 3 (2014), pp. 275-302
- [22] X. ROS-OTON AND J. SERRA, *The Pohozaev identity for the fractional Laplacian*, Arch. Rational Mech. Anal., Vol. 213 (2014), pp. 587-628
- [23] S.G. SAMKO, A.A. KILBAS AND O.I. MARICHEV, *Fractional integrals and derivatives - theory and applications*, Gordon and Breach Science Publishers S.A., 1993
- [24] M. TUCSNAK AND G. WEISS, *Observation and control for operator semigroups*, Birkhäuser Verlag AG, Basel-Boston-Berlin, 2009
- [25] M. WARMA, *The Pohozaev identity for the regional fractional Laplacian with general boundary conditions*
- [26] T. ZHU AND J.M. HARRIS, *Modeling acoustic wave propagation in heterogeneous attenuating media using decoupled fractional Laplacians*, GEOPHYSICS, vol. 79, issue 3, pp. T105-T116