Convergence to equilibrium of a linearized quantum Boltzmann equation for bosons at very low temperature

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December 2, 2014

Abstract

We consider an approximation of the linearised equation of the homogeneous Boltzmann equation that describes the distribution of quasiparticles in a dilute gas of bosons at low temperature. The corresponding collision frequency is neither bounded from below nor from above. We prove the existence and uniqueness of solutions satisfying the conservation of energy. We show that these solutions converge to the corresponding stationary state, at an algebraic rate as time tends to infinity.

Keyword: quantum Boltzmann equation, rate of convergence to equilibrium, algebraic decay.
MSC: 35Q20, 45A05, 47G10, 82B40, 82B40.

1 Introduction

A kinetic equation that describes the evolution of a non equilibrium spatially homogeneous distribution $n(t,p)$ of quasiparticles in a dilute Bose gas below the Bose Einstein transition temperature T_c has been obtained by several authors (see for example [13], [14], [15]) and reads as follows:

$$\frac{\partial n}{\partial t}(t,p) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dp_1 dp_2 \, [R(p,p_1,p_2) - R(p_1,p,p_2) - R(p_2,p_1,p)] \quad (1.1)$$

$$R(p,p_1,p_2) = |\mathcal{M}(p,p_1,p_2)|^2 \, [\delta(\omega(p)-\omega(p_1)-\omega(p_2))\delta(p-p_1-p_2)] \times$$
$$\times [n(p_1)n(p_2)(1+n(p)) - (1+n(p_1)(1+n(p_2))n(p)] \quad (1.2)$$

where $\mathcal{M}(p,p_1,p_2)$ is the transition probability, $\omega(p)$ is the so called Bogoliubov dispersion law:

$$\omega(p) = \left[\frac{gn_c}{m}|p|^2 + \left(\frac{|p|^2}{2m}\right)^2\right]^{1/2} \quad (1.3)$$

m is the mass of the particles, g is the interaction coupling constant and n_c is the density of particles in the superfluid. It is well known that the equation (1.1)–(1.3) has a family of equilibria:

$$n_0(p) = \frac{1}{e^{\frac{\omega(p)}{k_B T}} - 1}, \quad \beta > 0. \quad (1.4)$$

where k_B is the Boltzmann's constant and T the temperature of the quasiparticles whose distribution is n_0.

The relaxation of n towards its corresponding equilibrium is a question that has deserved some interest by several authors (cf. [3], [10], [11], [13]). In the more strictly mathematical literature, the convergence to equilibrium of Boltzmann equation has been extensively studied and still is. Since the works by T. Carleman [5] and H. Grad [12], then by L. Arkeryd [1], S. Ukai and K. Asano [19], G. Toscani [18] and L. Desvillettes [8] until those by L. Desvillettes and C. Villani [9] and later by Y. Guo and R. Strain [17] (cf. the review article [20] for more detailed references). However, we do not consider in this work the nonlinear problem (1.1)–(1.3). We only study, instead, the relaxation process of the equation linearised around one equilibrium. Let us then write:

$$n(t,p) = n_0(p) + n_0(p)[1+n_0(p)]\Omega(t,p) \quad (1.5)$$
$$= n_0(p) + \frac{\Omega(t,p)}{4\sinh^2\left(\frac{\omega(p)}{2k_BT}\right)} \quad (1.6)$$

Plugging this expression in the equation and keeping only the linear terms in Ω we obtain:

$$n_0(p)[1+n_0(p)]\frac{\partial \Omega}{\partial t}(t,p) = \mathcal{L}(\Omega)(t,p) \quad (1.7)$$

$$\mathcal{L}(\Omega)(t,p) = -M(p)\,\Omega(t,p) + \mathcal{T}(\Omega)(t,p) \tag{1.8}$$

$$\mathcal{T}(\Omega)(t,p) = \int_{\mathbb{R}^3} \mathcal{U}(p,p')\Omega(t,p')dp' \tag{1.9}$$

where the measure $\mathcal{U}(p,p')$ and the function $M(p)$ have been calculated in [11] and whose explicit expressions are recalled in formulas (6.7) and (6.8) of the Appendix.

The structure of the equation (1.7)–(1.9) is the same as in other linearised Boltzmann equations, as they may be seen for example in [4], [6], [11], [12], [16].

The relaxation to equilibrium of the solutions of (1.30)–(1.35) has been considered in [3], [7], [10], [11], [13].

As it is well known, the properties of the operator $\mathcal{L}$ crucially depend on the range of the function $M(p)$ and compactness properties of the integral operator $\mathcal{T}$. For the classical Boltzmann equation with hard potential the corresponding function M is such that, for some constant $M_0 > 0$, $M(p) \to M_0$ as $|p| \to 0$, $M(p) \to +\infty$ as $|p| \to \infty$, and its range is $[M_0, +\infty)$. For soft potentials, $M(p) \to M_0 > 0$ as $|p| \to 0$ but $M(p) \to 0$ as $|p| \to \infty$ and the range is $[0, M_0]$. In both cases the integral operator $\mathcal{T}$ is compact in some suitable functional space. It was shown in [3] that the values of the function $M(p)$ in (1.8) range from zero to ∞ as the variable $|p|$ goes from zero to ∞ (see Lemma 6.1 in the Appendix below). From this point of view, the situation for equation (1.7)–(1.9) is then similar to the case of the soft potentials for classical particles.

In the case of the spatially homogeneous linearized Boltzman equation for classical particles with soft potential it was observed in [12] (see also [4] and [19]) that the spectrum of the corresponding linearised operator $\mathcal{L}$ goes down until the origin and no exponential rate of convergence can be expected for the solutions. It is shown in [4] that for soft potentials and spatially homogeneous initial data $f(0,p)$ decaying exponentially fast as $|p| \to \infty$, the part of the solution f in the range of $\mathcal{L}$ decays in $L^2(\mathbb{R}^3)$ like $e^{-\lambda t^\theta}$ for some $\lambda > 0$ and $\theta \in (0,1)$. On the other hand, for non homogeneous initial data, the authors of [19] proved algebraic rates of decay in Lebesgue–Sobolev mixed type spaces.

1.1 Approximation of the linearised equation.

Since the functions $\omega(p)$ and $\mathcal{M}(p,p_1,p_2)$ appearing in equation (1.7)–(1.9) are complicated functions of their arguments, we restrict the range of our analysis. Following [3] we consider the situation where the equilibria n_0 in

(1.7)–(1.9) is at a quite low temperature T. More precisely, we suppose that the temperature T, the density n_c of superfluid and the interaction coupling constant g are such that $k_B T$ is much smaller than gn_c. That range has been widely considered in the physics literature, where the functions $\omega(p)$ and $\mathcal{M}(p, p_1, p_2)$ are then approximated as follows:

$$\omega(p) = c|p|, \;\; c = \sqrt{\frac{gn_c}{m}} \tag{1.10}$$

$$|\mathcal{M}(p, p_1, p_2)|^2 = \frac{9c}{64\pi^2 m n_c} |p||p_1||p_2|, \tag{1.11}$$

(cf. [3], [10], [13], [2]). This approximation has an important consequence. Indeed, if $\omega(p) = c|p|$, then the condition $\omega(p) = \omega(p') + \omega(p - p')$ reads $|p| = |p'| + |p - p'|$. This implies that p and p' must be parallel vectors of $\mathbb{R}^3$. The domain of integration in the integral at the right hand side of equation (1.7)–(1.9) is then reduced to the set $\mathcal{C}_p = \{\lambda\, p;\ \lambda \in \mathbb{R}\}$. More precisely, we are approximating the equation (1.7)–(1.9) by

$$n_0(p)[1 + n_0(p)] \frac{\partial \Omega}{\partial t}(t, p) = -M(p)\Omega(t, p) + \int_{\mathbb{R}^3} \Omega(t, p')\, W(p, p') dp' \tag{1.12}$$

where $W(p, p')$ and $M(p)$ are defined by (6.9) and (6.10) in the Appendix. Our goal is to study the solutions of the Cauchy problem associated to equation (1.12), their existence, uniqueness and relaxation towards equilibrium.

Due to the formulas (1.5), (1.6), and for the sake of notation we shall use the following convention all along this article. Given $p \in \mathbb{R}^3$, we shall denote:

$$k \equiv k(p) = \frac{c|p|}{2k_B T}. \tag{1.13}$$

Since, we will also denote $|p| = r$, we shall use sometimes

$$k = \frac{cr}{2k_B T}. \tag{1.14}$$

With some abuse of notations we will also write $n_0(p) = n_0(|p|) = n_0(r)$ and also, by (6.11), $M(p) = M(|p|) = M(r)$.

Proposition 1.1 *Let $\{Y_{\ell m}\}_{\ell,m}$ be the spherical harmonics on $\mathbb{S}^2$. For any sequence $\{c_{\ell m}\}$ of real numbers such that:*

$$\sum_{\ell=0}^{\infty} \sum_{n=-\ell}^{\ell} c_{\ell m}^2 < \infty$$

define

$$\Theta(p)=\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}c_{\ell\, m}Y_{\ell\, m}\left(\frac{p}{|p|}\right)\right)|p|.$$

Then:

$$(i)\quad \Theta\in L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right),$$

$$(ii)\quad -M(p)\Theta(p)+\int_{\mathbb{R}^3}\Theta(p')\,W(p,p')dp'=0.$$

Theorem 1.1 *Suppose that* $\Omega_0\in L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right)$. *Then, there exists a unique function* $\Omega(t,p)$ *such that*

$$\Omega\in L^\infty\left(0,\infty;L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right)\right)\cap C\left([0,\infty);L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right)\right),\tag{1.15}$$

$$\Omega-\Theta\in L^2\left(0,\infty;L^2\left(\mathbb{R}^3,M(p)dp\right)\right),\tag{1.16}$$

$$\frac{\partial\Omega}{\partial t}\in L^2\left(0,\infty;L^2\left(\mathbb{R}^3,\frac{dp}{M(p)\sinh^4 k}\right)\right),\tag{1.17}$$

satisfying the equation (1.12) in $L^2\left(0,\infty;L^2\left(\mathbb{R}^3,\frac{dp}{M(p)\sinh^4 k}\right)\right)$ *and taking the initial data* Ω_0 *in the following sense:*

$$\lim_{t\to 0}\left(||\Omega(t)-\Omega_0||_{L^2\left(\mathbb{R}^3,\frac{dp}{M(p)\sinh^4 k}\right)}+||\Omega(t)-\Omega_0||_{L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right)}\right)=0.\tag{1.18}$$

This solution also satisfies the following conservation property:

$$\frac{d}{dt}\int_{\mathbb{R}^3}n_0(p)(1+n_0(p))\Omega(t,p)|p|dp=0.\tag{1.19}$$

If Ω_0 *satisfies also:*

$$\int_{|p|<1}\frac{|\Omega_0(p)|^2}{|p|\sinh^2 k}dp<\infty\tag{1.20}$$

then

$$||\Omega(t)-\Theta||_{L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right)}\le\frac{C}{(1+t)^{1/2}}||\Omega_0-\Theta||_{L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right)},\tag{1.21}$$

where

$$\Theta(p) = \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell\, m} Y_{\ell\, m} \left(\frac{p}{|p|} \right) \right) |p| \tag{1.22}$$

$$c_{\ell\, m} = \left(\frac{\pi c}{2\sqrt{15} k_B T} \right)^4 \int_{\mathbb{R}^3} \Omega_0(p) n_0(p)(1 + n_0(p)) Y_{\ell\, m} \left(\frac{p}{|p|} \right) dp. \tag{1.23}$$

Remark 1.1 *The algebraic decay rate in (1.21) is not sufficient to have the integrability in time of* $||\Omega(t) - \Theta||^2_{L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2 k}\right)}$ *at infinity although, by (1.16), this integrability property is true for* $||\Omega(t) - \Theta||^2_{L^2(\mathbb{R}^3,\, M(p)dp)}$.

Remark 1.2 *The behaviors of the function* $M(p)$ *as* $|p| \to 0$ *and* $|p| \to \infty$ *are given in Proposition 6.1 of the Appendix.*

Remark 1.3 *The system of quasiparticles described by (1.1)–(1.2) satisfies the physical property of energy conservation. That property is expressed, in terms of the function* $n(t, p)$ *as:*

$$\frac{d}{dt} \int_{\mathbb{R}^3} n(t, p) \omega(p) dp = 0.$$

The identity (1.19) shows that this conservation of energy still holds for the equation (1.12).

Another natural quantity for the set of quasiparticles described by (1.1)-(1.2) is $N(t) = \int_{\mathbb{R}^3} n(t, p) dp$ *that represents the total number of particles. That physical quantity is not conserved by the system of particles described by (1.1)–(1.2), and the function* $N(t)$ *is not preserved, even formally, by equation (1.1)-(1.2). Nevertheless, the corresponding quantity for the linearised equation, namely* $M(t) = \int_{\mathbb{R}^3} n_0(p)(1 + n_0(p))\Omega(t, p) dp$ *is well defined for the solutions obtained in Theorem 1.1. See also Remark 5.1 below.*

The proof of theorem (1.1) is based on the following argument. Decompose first $\Omega(t, p)$ in spherical harmonics:

$$\Omega(t, p) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Omega_{\ell\, m}(t, |p|) Y_{\ell\, m} \left(\frac{p}{|p|} \right). \tag{1.24}$$

Using the decomposition of the measure W in Legendre's polynomial (recalled in the Appendix) we obtain for each ℓ and m:

$$n_0[1 + n_0] \frac{\partial \Omega_{\ell\, m}}{\partial t}(t, r) = -M(r) \Omega_{\ell\, m}(t, r) +$$

$$+\frac{1}{2\ell+1}\int_0^\infty W_\ell(r,r')\Omega_{\ell\, m}(t,r')dr' \tag{1.25}$$

where $r=|p|$, $r'=|p'|$ and:

$$\frac{1}{2\ell+1}W_\ell(r,r')=\frac{1}{2}\int_{-1}^{1}W(p,p')P_\ell(u)du,\ \ \ell=0,1,\cdots \tag{1.26}$$

It follows from the expression of $G(k,k')$ and $W(p,p')$ in (6.10) and (6.9) that

$$\begin{aligned}\frac{1}{2\ell+1}W_\ell(r,r') &= \frac{1}{2}\int_{-1}^{1}W(p,p')P_\ell(u)du\\ &= \frac{1}{2}\int_{-1}^{1}W(p,p')du=G(r,r'),\ \ \ell=1,2\cdots\end{aligned}$$

and all the coefficients $W_\ell(r,r')$ are equal. Therefore all the modes $\Omega_{\ell\, m}(t,r)$ satisfy the same equation:

$$n_0(r)[1+n_0(r)]\frac{\partial\Omega_{\ell\, m}}{\partial t}(t,r)=L(\Omega_{\ell\, m})(t,r) \tag{1.27}$$

$$L(\Omega_{\ell m})(t,r)=-M(r)\Omega_{\ell\, m}(t,r)+\int_0^\infty W_0(r,r')\Omega_{\ell\, m}(t,r')dr' \tag{1.28}$$

for all $\ell=0,1,2,\cdots$ and $m\in\{-\ell,\cdots,\ell\}$, where with some abuse of notation we denote:

$$n_0(p)=n_0(r)$$

and $W_0(r,r')$ is given by formula (6.12) in the Appendix.

Let us then consider an initial data $\Omega_0\in L^2(\mathbb{R}^3)$ and write its decomposition in spherical harmonics:

$$\Omega_0(p)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Omega_{0,\ell\, m}(|p|)Y_{\ell\, m}\left(\frac{p}{|p|}\right).$$

The solution to the equation (1.12) with the initial condition $\Omega(0,p)=\Omega_0(p)$ is then given by the function defined by the series (1.24) where every function $\Omega_{\ell\, m}(t,r)$ solves the equation (1.27), (1.28), with initial data $\Omega_{0,\ell\, m}$, for $\ell=0,1,2,\cdots$ and $m\in\{-\ell,\cdots,\ell\}$.

It is then enough to study the solutions of the Cauchy problem for the equation (1.27), (1.28). To this end we perform the following change of variables:

$$f(t,k)=\frac{c|p|}{2k_BT}\frac{\Omega(t,p)}{\sinh\left(\frac{c|p|}{2k_BT}\right)},\ \ k=\frac{c|p|}{2k_BT} \tag{1.29}$$

and obtain (cf. [3] and [21]):

$$\frac{\partial f}{\partial t}(t,k) = E(f) \equiv -\Gamma(k)\, f(t,k) + T_2[f] \equiv T_1[f] + T_2[f] \quad (1.30)$$

$$T_2[f] = 2\int_0^\infty K(k,k')f(t,k')dk' \quad (1.31)$$

$$\Gamma(k) = \sinh k \int_0^\infty \big(\phi(|k-k'|)\phi(k') + \phi(k+k')\phi(k')\big)\, dk' \quad (1.32)$$

$$K(k,k') = \big(\phi(|k-k'|) - \phi(k+k')\big)\, k\, k' \quad (1.33)$$

$$\phi(k) = \frac{k^2}{\sinh k} \quad (1.34)$$

$$f(0) = f_0. \quad (1.35)$$

The function $\Gamma(k)$ defined by (1.32) is such that:

$$\Gamma(k) \sim \frac{k^5}{15}, \text{ as } k \to +\infty \quad (1.36)$$

$$\Gamma(k) \sim \frac{\pi^4 k}{15}, \text{ as } k \to 0, \quad (1.37)$$

(cf. [3] and Appendix below), and then, its range is $[0,+\infty)$.

We introduce the following auxiliary function that will be needed in all the sequel:

$$\varphi_0 = \frac{\phi}{||\phi||_2} = \frac{\sqrt{30}}{\pi^2}\varphi. \quad (1.38)$$

Then, Theorem (1.1) is a consequence of the following result.

Theorem 1.2 *Suppose that $f_0 \in L^2(\mathbb{R}_+)$ and denote*

$$c_0 = \int_0^\infty f_0(k)\varphi_0(k)dk, \quad (1.39)$$

where φ_0 is defined in (1.38). Then,

(i) there exists a unique function f such that

$$(f - c_0\varphi_0) \in L^2((0,\infty), L^2(\Gamma)), \quad (1.40)$$

$$f \in L^\infty((0,\infty), L^2(\mathbb{R}_+)) \cap C([0,\infty), L^2(\mathbb{R}_+)), \quad (1.41)$$

$$\frac{\partial f}{\partial t} \in L^2(0,\infty; L^2(\Gamma^{-1})), \quad (1.42)$$

that satisfies the equation (1.30) in $L^2((0,\infty), L^2(\Gamma^{-1}))$ and takes the initial data f_0 in the following sense:

$$\lim_{t\to 0}\big(||f(t) - f_0||_{L^2(\Gamma^{-1})} + ||f(t) - f_0||_2\big) = 0. \quad (1.43)$$

This solution also satisfies

$$||f(t)||_2^2 + 2C_* \int_0^\infty ||f(t) - c_0\varphi_0||^2_{L^2(\Gamma)} dt \le 2||f_0||_2^2 \tag{1.44}$$

$$\left|\left|\frac{\partial f}{\partial t}\right|\right|_{L^2(0,\infty;L^2(\Gamma^{-1}))} \le (1+2C_0)||f||_{L^2(0,\infty;L^2(\Gamma))}, \tag{1.45}$$

for some constant $C_0 > 0$, and the conservation of energy:

$$\forall t > 0: \quad \frac{d}{dt}\int_0^\infty f(t,k)\frac{k^2\,dk}{\sinh(k)} = 0. \tag{1.46}$$

If $f_0 \ge 0$, then $f(t,k) \ge 0$ for all $t > 0$ and a. e. $k > 0$.

(ii) If f_0 also satisfies one of the two following conditions:

$$I = \int_0^1 \frac{|f_0(k)|^2}{k} dk < \infty \tag{1.47}$$

$$a = \lim_{k\to 0} f_0(k) \;\; \textit{exists}. \tag{1.48}$$

there exists a positive constant C, depending on I or a respectively, such that, for all $t > 0$:

$$\|f(t) - c_0\varphi_0\|_2 \le C\frac{||f_0 - c_0\varphi_0||_2}{(1+t)^{1/2}}. \tag{1.49}$$

where φ_0 is defined in (1.38) and c_0 is given by (1.39).

The algebraic rate of convergence in $L^2(\mathbb{R}_+)$ norm is proved using classical arguments. We first establish a coercivity property of the operator E in a suitable functional space. Then, this coercivity is used to obtain an upper estimate of the convergence rate. This last step uses the detailed behavior of the kernel K and the function Γ near $k = 0$.

The plan of the paper is the following. We prove in Section 2 two important properties of the operator E. Section 3 is devoted to the proof of an existence and uniqueness result for the solution of Cauchy problem (1.30)-(1.35). In Section 4 we prove the convergence rate of the solutions of the problem (1.30)–(1.35). In Section 5 we prove Proposition 1.1 and Theorem 1.1. We give in a final Appendix some auxiliary results, in particular the detailed behaviors of the functions Γ and K.

2 Properties of the operator E

In this Section we prove several important properties of the operator E. We will be using the following spaces.

$$L^2(\Gamma) = \big\{u : (0,\infty) \to \mathbb{R}; \text{measurable, such that } ||u||_{L^2(\Gamma)} < \infty\big\}$$
$$L^2(\Gamma^{-1}) = \big\{u : (0,\infty) \to \mathbb{R}; \text{measurable, such that } ||u||_{L^2(\Gamma^{-1})} < \infty\big\}$$

where

$$||u||_{L^2(\Gamma)} = \left(\int_0^\infty |u(k)|^2 \Gamma(k) dk\right)^{1/2}$$
$$||u||_{L^2(\Gamma^{-1})} = \left(\int_0^\infty \frac{|u(k)|^2}{\Gamma(k)} dk\right)^{1/2}.$$

We shall also use the classical $L^2(\mathbb{R}_+)$ of functions of integrable square in $(0,\infty)$, with its norm $||\cdot||_2$.

Since several Hilbert spaces will be used all along this work, we want to be careful with the notation. We denote by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\mathbb{R}_+)$:

$$\langle\varphi,\psi\rangle = \int_0^\infty \varphi(k)\psi(k)dk$$

whenever this integral is well defined. We will also use the notation $\perp$ to denote the orthogonality with respect to the scalar product of $L^2(\mathbb{R}_+)$:

$$\varphi \perp \psi \Longleftrightarrow \int_0^\infty \varphi(k)\psi(k)dk = 0$$

and similarly, if A is a set of measurable functions,

$$\varphi \in A^\perp \Longleftrightarrow \int_0^\infty \varphi(k)\psi(k)dk = 0, \ \ \forall \psi \in A.$$

We may then have $\varphi \perp \psi$ even if neither φ nor ψ belong to $L^2(\mathbb{R}_+)$, as long as the integral on the right hand side is well defined and equal to zero.

Lemma 2.1 *The operator E defined by (1.30)–(1.34) is linear and continuous from $L^2(\Gamma)$ into $L^2(\Gamma^{-1})$ and, for every $u \in L^2(\Gamma)$:*

$$||E(u)||_{L^2(\Gamma^{-1})} \leq (1 + 2C_0)||u||_{L^2(\Gamma)} \tag{2.1}$$

where

$$C_0 = \left(\int_0^\infty \left|\frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}}\right|^2 dk'dk\right)^{1/2} < \infty. \tag{2.2}$$

Proof The proof that the integral defining C_0 in (2.2) converges is given in detail in the Appendix. On the other hand, for all $u \in L^2(\Gamma)$ and $v \in L^2(\Gamma)$:

$$\langle E(u), v\rangle = -\int_0^\infty \Gamma(k)u(k)v(k)dk + 2\int_0^\infty \int_0^\infty K(k,k')u(k')v(k)dk'dk$$

$$\left|\int_0^\infty \Gamma(k)u(k)v(k)dk\right| \leq \left(\int_0^\infty \Gamma(k)|u(k)|^2 dk\right)^{1/2} \left(\int_0^\infty \Gamma(k)|v(k)|^2 dk\right)^{1/2}$$

$$\left|\int_0^\infty \int_0^\infty K(k,k')u(k')v(k)dk'dk\right| =$$

$$= \left|\int_0^\infty \int_0^\infty \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}}\sqrt{\Gamma(k)\Gamma(k')}u(k')v(k)dk'dk\right|$$

$$= \int_0^\infty \left|\sqrt{\Gamma(k)}v(k)\int_0^\infty \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}}\sqrt{\Gamma(k')}u(k')dk'\right| dk$$

$$\leq \left(\int_0^\infty \Gamma(k)|v(k)|^2 dk\right)^{1/2} \left(\int_0^\infty \left|\frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}}\sqrt{\Gamma(k')}u(k')dk'\right| dk\right)^{1/2}$$

$$\leq \left(\int_0^\infty \Gamma(k)|v(k)|^2\right)^{1/2} \left(\int_0^\infty \Gamma(k')|u(k')|^2 dk'\right)^{1/2} \times$$

$$\times \left(\int_0^\infty \int_0^\infty \left|\frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}}\right|^2 dk'dk\right)^{1/2}.$$

We have then for all $u \in L^2(\Gamma)$ and $v \in L^2(\Gamma)$:

$$|\langle E(u), v\rangle| \leq (1+2C_0)||u||_{L^2(\Gamma)}||v||_{L^2(\Gamma)}.$$

from where $E(u) \in (L^2(\Gamma))' = L^2(\Gamma^{-1})$ and (2.1) follows. ■

It was already shown in [3] that the operator E is non negative. The precise property and its proof are given in the following Lemma for the sake of completeness.

Lemma 2.2 *For all $f \in L^2(\Gamma)$ and $g \in L^2(\Gamma)$:*

$$\langle -E(f), g\rangle = \int_0^\infty \int_0^\infty \phi(k+k')\phi(k')\phi(k) \times \tag{2.3}$$
$$\times \left[\frac{\sinh(k)f(k)}{k} + \frac{\sinh(k')f(k')}{k'} - \frac{\sinh(k+k')f(k+k')}{k+k'}\right] \times$$
$$\times \left[\frac{\sinh(k)g(k)}{k} + \frac{\sinh(k')g(k')}{k'} - \frac{\sinh(k+k')g(k+k')}{k+k'}\right] dkdk'.$$

Proof We first notice that by definition:

$$\begin{aligned}
\langle -Ef, g\rangle &= \int_0^\infty\int_0^\infty \sinh k \left(\phi(|k-k'|)\phi(k') + \phi(k+k')\phi(k')\right) f(k)g(k)dk'dk \\
&\quad -2\int_0^\infty\int_0^\infty \left(\phi(|k-k'|) - \phi(k+k')\right) k\, k' f(k')g(k)dk'dk \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We now write the integrals I_1, I_2, I_3 and I_4 using the definitions and symmetries of the two functions $\Gamma(k)$ and $K(k,k')$.

$$\begin{aligned}
I_1 &= \int_0^\infty\int_0^\infty \sinh k \left(\frac{|k-k'|^2}{\sinh(|k-k'|)}\frac{|k'|^2}{\sinh(k')}\right) f(k)g(k)dk'dk \\
I_2 &= \int_0^\infty\int_0^\infty \sinh k \left(\frac{|k+k'|^2}{\sinh(|k+k'|)}\frac{|k'|^2}{\sinh(k')}\right) f(k)g(k)dk'dk \\
I_3 &= -2\int_0^\infty\int_0^\infty \left(\frac{|k-k'|^2}{\sinh(|k-k'|)}\right) k\, k' f(k')g(k)dk'dk \\
I_4 &= 2\int_0^\infty\int_0^\infty \left(\frac{|k+k'|^2}{\sinh(|k+k'|)}\right) k\, k' f(k')g(k)dk'dk.
\end{aligned}$$

Let us denote for the remaining of this calculation $Q[g](k) = \frac{\sinh(k)g(k)}{k}$

$$\begin{aligned}
I_1 &= \int_0^\infty\int_0^\infty \frac{|k-k'|^2}{\sinh(|k-k'|)}\frac{|k'|^2}{\sinh(k')}\frac{|k|^2}{\sinh(k)} Q[f](k)Q[g](k)dk'dk \\
&= \int_{\{k>k'\}} \frac{|k-k'|^2}{\sinh(|k-k'|)}\frac{|k'|^2}{\sinh(k')}\frac{|k|^2}{\sinh(k)} Q[f](k)Q[g](k)dk'dk \\
&+ \int_{\{k<k'\}} \frac{|k-k'|^2}{\sinh(|k-k'|)}\frac{|k'|^2}{\sinh(k')}\frac{|k|^2}{\sinh(k)} Q[f](k)Q[g](k)dk'dk \\
&= \int_0^\infty\int_0^\infty \phi(k)\phi(k')\phi(k+k')Q[f](k+k')Q[g](k+k') + \\
&+ \int_0^\infty\int_0^\infty \phi(k)\phi(k')\phi(k+k')Q[f](k)Q[g](k)dk'dk. \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^\infty\int_0^\infty \left(\frac{|k+k'|^2}{\sinh(|k+k'|)}\frac{|k'|^2}{\sinh(k')}\frac{|k|^2}{\sinh(k)}\right)\frac{\sinh(k)^2}{|k|^2} f(k)g(k)dk'dk \\
&= \int_0^\infty\int_0^\infty \phi(k+k')\phi(k')\phi(k)Q[f](k)Q[g](k)dk'dk \\
&= \int_0^\infty\int_0^\infty \phi(k+k')\phi(k')\phi(k)Q[f](k')Q[g](k')dk'dk. \qquad (2.5)
\end{aligned}$$

$$
\begin{aligned}
I_3 = &-2\int_{\{k>k'\}}\left(\frac{|k-k'|^2}{\sinh(|k-k'|)}\right)k\,k'f(k')g(k)dk'dk\\
&-2\int_{\{k>k'\}}\left(\frac{|k-k'|^2}{\sinh(|k-k'|)}\right)k\,k'f(k)g(k')dk'dk\\
&= -2\int_0^\infty\int_0^\infty\left(\frac{|k|^2}{\sinh(|k|)}\right)(k+k')\,k'f(k')g(k+k')dk'dk\\
&-2\int_0^\infty\int_0^\infty\left(\frac{|k|^2}{\sinh(|k|)}\right)(k+k')\,k'g(k')f(k+k')dk'dk\\
&= -\int_0^\infty\int_0^\infty\phi(k+k')\phi(k')\phi(k)\left(Q[f](k)+Q[f](k')\right)Q[g](k+k')dkdk'\\
&-\int_0^\infty\int_0^\infty\phi(k+k')\phi(k')\phi(k)\left(Q[g](k)+Q[g](k')\right)Q[f](k+k')dkdk'.
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
I_4 = &\int_0^\infty\int_0^\infty\phi(k+k')\phi(k')\phi(k)Q[f](k)Q[g](k')dk'dk+\\
&+\int_0^\infty\int_0^\infty\phi(k+k')\phi(k')\phi(k)Q[f](k')Q[g](k)dk'dk.
\end{aligned}
\tag{2.7}
$$

Identity (2.3) follows by combining (2.4)–(2.7).

■

Corollary 2.1 *Let ϕ be the function defined in (1.34). Then*

$$
E(\phi)=0. \tag{2.8}
$$

Conversely, if $f\in L^2(\Gamma)$ is such that $E(f)=0$, then $f=C\phi$ for some constant C.

Proof By (2.3), $\langle E(\phi),g\rangle=0$ for all $g\in L^2(\Gamma)$ and (2.1) follows. On the other hand, if $E(f)=0$ for some $f\in L^2(\Gamma)$, then $\langle E(f),f\rangle=0$ and by (2.3):

$$
\left[\frac{\sinh(k)f(k)}{k}+\frac{\sinh(k')f(k')}{k'}-\frac{\sinh(k+k')f(k+k')}{k+k'}\right]^2=0
$$

for a. e. $k>0$, $k'>0$. The function $\frac{\sinh(k)f(k)}{k}$ must then be linear, and we must then have $f=C\phi$ for some positive constant C. ■

Corollary 2.2 *For all $f \in L^2(\Gamma)$ and $g \in L^2(\Gamma)$:*

$$|\langle -E(f), g\rangle| \leq \frac{1}{2}\langle -E(f), f\rangle + \frac{1}{2}\langle -E(g), g\rangle. \tag{2.9}$$

Proof By (2.3) in Lemma 2.2:

$$|\langle -E(f), g\rangle| \leq \int_0^\infty \int_0^\infty |q(f)(k,k')q(g)(k,k')| d\mu(k,k').$$

where

$$q(h)(k,k') = \frac{\sinh(k)h(k)}{k} + \frac{\sinh(k')h(k')}{k'} - \frac{\sinh(k+k')h(k+k')}{k+k'}$$

and

$$d\mu = \phi(k+k')\phi(k')\phi(k)dkdk'$$

is a non negative measure. We deduce by Holder's inequality

$$\begin{aligned} |\langle -E(f), g\rangle| &\leq \frac{1}{2}\int_0^\infty \int_0^\infty |q(f)(k,k')|^2 d\mu + \\ &\quad + \frac{1}{2}\int_0^\infty \int_0^\infty |q(g)(k,k')|^2 d\mu \\ &= \frac{1}{2}\langle -E(f), f\rangle + \frac{1}{2}\langle -E(g), g\rangle. \end{aligned}$$

■

As we have seen, the operator E is continuous from $L^2(\Gamma)$ into $L^2(\Gamma^{-1})$. By the Corollary 2.1, its kernel, $N(E)$ is a one dimensional vector space generated by the function ϕ.

Lemma 2.3 *There exists a constant $C_* > 0$ such that, for all $h \in L^2(\Gamma)$:*

$$\langle -Eh, h\rangle \geq C_* \|h - \mathbb{P}h\|^2_{L^2(\Gamma)}, \tag{2.10}$$

where

$$\mathbb{P}h = c_0(h)\varphi_0, \;\; c_0(h) = \int_0^\infty h(k)\varphi_0(k)dk.$$

Remark 2.1 *The map $\mathbb{P}$ is the orthogonal projection on the kernel $N(E)$ for the scalar product of $L^2(\mathbb{R}_+)$. Since $\varphi_0 \in L^2(\Gamma^{-1})$ it is well defined for all $h \in L^2(\Gamma)$.*

Proof For all $h \in L^2(\Gamma)$, we denote

$$h = c_0(h)\varphi_0 + g, \;\; c_0(h)\varphi_0 = \mathbb{P}h \in N(E).$$

Notice that,

$$\int_0^\infty g(k)\varphi_0(k)dk = \int_0^\infty (h(k) - c_0(h)\varphi_0(k))\varphi_0(k)dk = 0$$

and so $g \in N(E)^\perp$. Moreover, by Lemma 2.2, we deduce that

$$\langle E(g), \mathbb{P}h\rangle = c_0(h)\langle E(g), \varphi_0\rangle = 0$$

and then,

$$\langle Eh, h\rangle = \langle E(g), \mathbb{P}h + g\rangle = \langle E(g), g\rangle.$$

Therefore, property (2.10) is equivalent to

$$\forall g \in L^2(\Gamma), \, \mathbb{P}g = 0 : \quad \langle -Eg, g\rangle \geq C_* \|g\|^2_{L^2(\Gamma)}. \tag{2.11}$$

In order to prove (2.11), we show that for all $h \in L^2(\Gamma)$:

$$\langle -Eh, h\rangle + c_0^2(h) \geq C_* \|h\|^2_{L^2(\Gamma)}. \tag{2.12}$$

To this end we make a change of unknown variable and define $g = \alpha h$, with $\alpha = \sqrt{\Gamma}$. The problem is now equivalent to prove that for all $g \in L^2(\mathbb{R}_+)$:

$$\begin{aligned} &\int_0^\infty |g(k)|^2 dk - 2\int_0^\infty \int_0^\infty \frac{K(k,k')}{\alpha(k)\alpha(k')} g(k')g(k)dk'dk \\ &+ \int_0^\infty \int_0^\infty \frac{\varphi_0(k)\varphi_0(k')}{\alpha(k)\alpha(k')} g(k')g(k)dk'dk \geq C_* \|g\|^2_{L^2}. \end{aligned} \tag{2.13}$$

This follows from simple spectral properties of the operator $\widetilde{E} = -I + T$ with

$$T : g \to \int_0^\infty \frac{2K(k,k')}{\alpha(k)\alpha(k')} g(k')dk' - \int_0^\infty \frac{\varphi_0(k)\varphi_0(k')}{\alpha(k)\alpha(k')} g(k')dk'.$$

Since the two functions $\frac{2K(k,k')}{\alpha(k)\alpha(k')}$ and $\frac{\varphi_0(k)\varphi_0(k')}{\alpha(k)\alpha(k')}$ belong to $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$, (for the first function this is proved in detail in Lemma 6.2 of the Appendix), the operator T is a Hilbert Schmidt, and then a compact, operator from $L^2(\mathbb{R}_+)$ into itself. Its spectrum is then reduced to a sequence $(\mu_j)_{j\in\mathbb{N}}$ of eigenvalues satisfying $\mu_j \to 0$ as $j \to \infty$. The spectrum of $-\widetilde{E}$ is then also reduced to a sequence $(\lambda_j)_{j\in\mathbb{N}}$ of eigenvalues such that $\lambda_j \to 1$ as $j \to \infty$. Since the operator $-E$ is non negative on $L^2(\Gamma)$ it is easy to deduce that $-\widetilde{E}$ is non negative on $L^2(\mathbb{R}_+)$, and then $\lambda_j \geq 0$ for all $j \in \mathbb{N}$. In order to prove (2.13) we then only need to show that zero is not an eigenvalue of $-\widetilde{E}$. If that was the case, any associated eigenfunction $g \in L^2(\mathbb{R}_+)$ would satisfy $\widetilde{E}(g) = 0$ and then, multiplying by g and integrating

$$-\left\langle E\left(\frac{g}{\alpha}\right), \frac{g}{\alpha}\right\rangle + \int_0^\infty \int_0^\infty \frac{\varphi_0(k)\varphi_0(k')}{\alpha(k)\alpha(k')} g(k')g(k)dk'dk = 0.$$

But this would imply that, for the function $h = \frac{g}{\alpha} \in L^2(\Gamma)$, we have

$$\langle -Eh, h\rangle + c_0(h)^2 = 0.$$

Since $\langle -Eh, h\rangle \geq 0$ this implies that $\langle -Eh, h\rangle = 0$ and $c_0(h) = 0$. By Corollary 2.1, the first condition implies that $h \in N(E)$. Then we deduce from the second that $h = 0$ and then $g = 0$. This proves that zero is not an eigenvalue of $\widetilde{E}$ and we deduce that

$$C_* = \min_{j\in\mathbb{N}} \lambda_j > 0.$$

Property (2.13) follows, and then also (2.12) for all $g \in L^2(\Gamma)$ and (2.11) for all $g \in L^2(\Gamma)$ such that $\mathbb{P}g = 0$. This concludes the proof of (2.10). ■

3 Existence and uniqueness of global solution.

In this Section we prove that the Cauchy problem (1.30)-(1.35) is well posed in $L^2(\mathbb{R}_+)$. More precisely, we have the following proposition that is the first part of Theorem 1.2.

Proposition 3.1 *Suppose that $f_0 \in L^2(\mathbb{R}_+)$. Then, the problem (1.30)–(1.35) has a unique solution f such that*

$$(f - \mathbb{P}(f_0)) \in L^2((0,\infty), L^2(\Gamma)), \tag{3.1}$$

$$f \in L^\infty((0,\infty), L^2(\mathbb{R}_+)) \cap C([0,\infty), L^2(\mathbb{R}_+)), \tag{3.2}$$

$$\partial_t f \in L^2((0,\infty), L^2(\Gamma^{-1})), \tag{3.3}$$

that satisfies the equation (1.30) in $L^2((0,T); L^2(\Gamma^{-1}))$ for all $T > 0$ and takes the initial data in the following sense:

$$\lim_{t\to 0} \left(||f(t) - f_0||_{L^2(\Gamma^{-1})} + ||f(t) - f_0||_2\right) = 0. \tag{3.4}$$

This solution is such that, for all $\varphi \in L^2(\Gamma)$:

$$\frac{d}{dt}\int_0^\infty f(t,k)\varphi(k)dk = \int_0^\infty\int_0^\infty \phi(k+k')\phi(k')\phi(k)\times \tag{3.5}$$
$$\times\left[\frac{\sinh(k)f(k)}{k}+\frac{\sinh(k')f(k')}{k'}-\frac{\sinh(k+k')f(k+k')}{k+k'}\right]\times$$
$$\times\left[\frac{\sinh(k)\varphi(k)}{k}+\frac{\sinh(k')\varphi(k')}{k'}-\frac{\sinh(k+k')\varphi(k+k')}{k+k'}\right]dkdk'.$$

In particular, for all $t > 0$:

$$\frac{d}{dt}\int_0^\infty f(t,k)\frac{k^2dk}{\sinh(k)} = 0. \tag{3.6}$$

Moreover, for all $t > 0$:

$$||f(t)||_2^2 + 2C_*\int_0^\infty ||f(t)-\mathbb{P}(f_0)||^2_{L^2(\Gamma)}dt \le 2||f_0||_2^2. \tag{3.7}$$

and

$$\left|\left|\frac{\partial f}{\partial t}\right|\right|_{L^2(0,\infty;L^2(\Gamma^{-1}))} \le (1+2C_0)||f-\mathbb{P}(f_0)||_{L^2(0,\infty;L^2(\Gamma))}, \tag{3.8}$$

where the constant C_0 is defined in (2.2).
If $f_0 \ge 0$ then for all $t > 0$, $f(t,k) \ge 0$ for a.e. $k > 0$.

Proof *Step 1: Uniqueness.* We first prove that if there is a solution of (1.30)-(1.35) satisfying (3.2)-(3.3), then it is unique. Since the equation is linear it is sufficient to prove that the only solution of (1.30)-(1.35) satisfying (3.2)-(3.3) with initial data $f_0 = 0$ is the function such that $f(t) = 0$ for all $t > 0$. To this end, we multiply the equation (1.30) by f and integrate on $k > 0$ to obtain:

$$\frac{1}{2}\frac{d}{dt}||f(t)||_2^2 = \langle E(f), f\rangle.$$

Since $c_0 = 0$ by hypothesis, we deduce using (2.12):

$$\frac{1}{2}\frac{d}{dt}||f(t)||_2^2 \le -C_*||f(t)||^2_{L^2(\Gamma)}.$$

If we now integrate this in time:

$$||f(t)||_2^2 + 2C_*\int_0^t ||f(s)||^2_{L^2(\Gamma)}ds \le 0$$

since $||f(0)||_2^2 = ||f_0||_2^2$ by the continuity of the application $t \mapsto ||f(t)||_2$ and uniqueness then follows.

Step 2. We define the following truncation of the operator E and the initial data f_0:

$$E_n[h] \equiv -\Gamma_n(k)\, h(k) + T_{n2}[h], \tag{3.9}$$
$$\Gamma_n(k) = \Gamma(k)\chi_n(k), \tag{3.10}$$
$$T_{n2}[h] = 2\int_0^\infty K_n(k,k')h(k')dk', \tag{3.11}$$
$$K_n(k,k') = \chi_n(k)\chi_n(k')K(k,k'), \tag{3.12}$$
$$f_{0,n}(k) = \chi_n(k)f_0(k). \tag{3.13}$$
$$\chi_n(k) = \chi_{\{1/n<|k|<n\}},$$

where χ_A is the characteristic function of the set A.

For every $n \in \mathbb{N}$, E_n is now a linear and bounded operator from $L^2(\mathbb{R}_+)$ into itself. Therefore, the linear problem

$$\frac{\partial f}{\partial t}(t,k) = E_n(f)(t,k), \;\; t > 0, k > 0, \tag{3.14}$$
$$f(0,k) = f_{0,n}(k), \;\; k > 0, \tag{3.15}$$

has a solution:

$$f_n(t,k) = e^{tE_n}(f_{0,n})$$

satisfying

$$f_n \in C([0,\infty); L^2(\mathbb{R}_+)) \cap C^\infty(0,\infty; L^2(\mathbb{R}_+)). \tag{3.16}$$

The same argument as in Step 1 shows that f_n is unique. If moreover $f_0 \geq 0$, then $f_{0,n} \geq 0$ and then $f_n(t) \geq 0$ for all $t > 0$.

Since $supp(f_{0,n}) \subset (1/n, n)$, $supp(\Gamma_n) \subset (1/n, n)$ and $supp(K_n) \subset (1/n, n) \times (1/n, n)$, we have $supp(f_n(t)) \subset (1/n, n)$ for all $t > 0$, and therefore:

$$E_n f_n = (Ef_n)\chi_n(k),$$

where $\chi_{(1/n,n)}$ is the characteristic function of $(1/n, n)$. The function f_n solves then:

$$\frac{\partial f_n}{\partial t} = (Ef_n)\chi_n(k) \tag{3.17}$$
$$f_n(0) = f_{0,n} \tag{3.18}$$

Multiplying (3.17) by f_n we obtain, after integration on $(0,t)\times\mathbb{R}_+$:

$$\|f_n(t)\|_2^2 - \|f_{0,n}\|_2^2 = 2\int_0^t \langle (Ef_n(s))\chi_n, f_n(s)\rangle dt. \tag{3.19}$$

Since for all $s>0$ $supp(f_n(s))\subset(1/n,n)$ we notice first that:

$$\langle (Ef_n(s))\chi_n, f_n(s)\rangle = \langle Ef_n(s), f_n(s)\rangle$$

and second, that $f_n(t)\in L^2(\Gamma)$. Then, by (2.12) in the proof of Lemma 2.3, we deduce, for all $T>0$:

$$||f_n(T)||_2^2 + 2C_*\int_0^T ||f_n(t)-\mathbb{P}f_n(t)||^2_{L^2(\Gamma)}dt \quad\leq\quad \|f_{0,n}\|_2^2. \tag{3.20}$$

It first follows from (3.20) that for all $t\geq 0$

$$||f_n(t)||_2^2 \leq \|f_0\|_2^2. \tag{3.21}$$

Using (3.21) we obtain that for all $t>0$:

$$\begin{aligned}||\mathbb{P}f_n(t)||^2_{L^2(\Gamma)} &= \left(\int_0^\infty f_n(t,k)\varphi_0(k)dk\right)^2 ||\varphi_0||^2_{L^2(\Gamma)}\\ &\leq ||f_n(t)||_2^2||\varphi_0||^2_{L^2(\Gamma)} \leq \|f_0\|_2^2||\varphi_0||^2_{L^2(\Gamma)}\end{aligned}$$

and then, using this in (3.20):

$$||f_n(T)||_2^2 + C_*\int_0^T ||f_n(t)||^2_{L^2(\Gamma)}dt \leq \|f_0\|_2^2\left(1+2C_*T||\varphi_0||^2_{L^2(\Gamma)}\right)$$

$$\int_0^T ||f_n(t)||^2_{L^2(\Gamma)}dt \leq \frac{\|f_0\|_2^2}{C_*}\left(1+2C_*T||\varphi_0||^2_{L^2(\Gamma)}\right). \tag{3.22}$$

By (3.22), the sequence $(f_n)_{n\in\mathbb{N}}$ is then bounded in $L^2(0,T;L^2(\Gamma))$ for all $T>0$. We prove now that it is also a Cauchy sequence in that space.

To this end, let n,m be two positive integers such that for example $m>n$. By (3.14):

$$\frac{\partial}{\partial t}(f_n - f_m) \quad=\quad E_nf_n - E_mf_m \tag{3.23}$$

$$f_n(0)-f_m(0) \quad=\quad f_{0,n}-f_{0,m} \tag{3.24}$$

After multiplication by $f_n - f_m$ and integration over $(0,\infty)$ we deduce as usual

$$\|f_n(t) - f_m(t)\|_2^2 - \|f_{0,n} - f_{0,m}\|_2^2 = 2\int_0^t (\langle E_n f_n, f_n - f_m\rangle - \langle E_m f_m, f_n - f_m\rangle)\, dt. \tag{3.25}$$

We decompose the function f_m as follows:

$$f_m(t,k) = f_{m,n}(t,k) + \varphi_{m,n}(t,k) \tag{3.26}$$

$$f_{m,n}(t,k) = f_m(t,k)\chi_n(k) \tag{3.27}$$

$$\varphi_{m,n}(t,k) = f_m(t,k)\,(\chi_m(k) - \chi_n(k)) \tag{3.28}$$

and use this to rewrite the two right hand side terms of (3.25). We have first:

$$\langle E_n f_n, f_n - f_m\rangle = -\int_0^\infty \Gamma_n(k) f_n(k)(f_n(k) - f_m(k))dk + \tag{3.29}$$

$$+\int_{\mathbb{R}_+^2} K_n(k,k') f_n(k')(f_n(k) - f_m(k))(k)dk'dk = J_1 + J_2.$$

Since the supports of f_n and $\varphi_{m,n}$ are disjoint we have:

$$\begin{aligned} J_1 &= -\int_0^\infty \Gamma_n(k) f_n(k)(f_n(k) - f_m(k))dk \\ &= -\int_0^\infty \Gamma_n(k) f_n(k)(f_n(k) - f_{m,n}(k))dk \\ &= -\int_0^\infty \Gamma(k) f_n(k)(f_n(k) - f_{m,n}(k))dk \end{aligned} \tag{3.30}$$

Using that for any $k' > 0$ the supports of $K_n(\cdot, k')$ and $\varphi_{m,n}$ are also disjoints we obtain:

$$\begin{aligned} J_2 &= \int_{\mathbb{R}_+^2} K_n(k,k') f_n(k')(f_n(k) - f_m(k))(k)dk'dk \\ &= \int_{\mathbb{R}_+^2} K_n(k,k') f_n(k')(f_n(k) - f_{m,n}(k))(k)dk'dk \\ &= \int_{\mathbb{R}_+^2} K(k,k') f_n(k')(f_n(k) - f_{m,n}(k))(k)dk'dk \end{aligned} \tag{3.31}$$

By (3.30) and (3.31), we deduce from (3.29) that

$$\langle E_n f_n, f_n - f_m\rangle = \langle E f_n, f_n - f_{m,n}\rangle\,. \tag{3.32}$$

On the other hand,

$$\langle E_m f_m, f_n - f_m\rangle = -\int_{\mathbb{R}_+} \Gamma_m(k) f_m(k)(f_n(k) - f_m(k))dk +$$
$$+\int_{\mathbb{R}_+^2} K_m(k,k')f_m(k')(f_n(k) - f_m(k))(k)dk'dk = L_1 + L_2. \quad (3.33)$$

We have now

$$\begin{aligned} L_1 &= -\int_0^\infty \Gamma_m(k)(f_{m,n}(k)+\varphi_{m,n}(k))(f_n(k)-f_{m,n}(k)-\varphi_{m,n}(k))dk \\ &= -\int_0^\infty \Gamma_m(k) f_{m,n}(k)(f_n(k)-f_{m,n}(k))dk + \\ &\qquad + \int_0^\infty \Gamma_m(k) f_{m,n}(k)\varphi_{m,n}(k)dk \\ &\quad -\int_0^\infty \Gamma_m(k)\varphi_{m,n}(k)(f_n(k)-f_{m,n}(k))dk + \\ &\qquad + \int_0^\infty \Gamma_m(k)\varphi_{m,n}(k)\varphi_{m,n}(k)dk. \end{aligned} \quad (3.34)$$

Using the properties of the support of the functions f_n, $f_{m,n}$, Γ_n and $\varphi_{m,n}$ we deduce as above that the second and third terms in the right hand side of (3.34) are zero, from where:

$$\begin{aligned} L_1 &= -\int_0^\infty \Gamma(k) f_{m,n}(k)(f_n(k)-f_{m,n}(k))dk + \\ &\qquad + \int_0^\infty \Gamma(k)\varphi_{m,n}(k)\varphi_{m,n}(k)dk \end{aligned} \quad (3.35)$$

Consider now L_2, that may be written as follows:

$$\begin{aligned} L_2 &= \int_{\mathbb{R}_+^2} K_m(k,k')(f_{m,n}(k')+\varphi_{m,n}(k'))(f_n(k)-f_{m,n}(k)-\varphi_{m,n}(k))dk'dk \\ &= \int_{\mathbb{R}_+^2} K_m(k,k') f_{m,n}(k')(f_n(k)-f_{m,n}(k))dk'dk \\ &\qquad - \int_{\mathbb{R}_+^2} K_m(k,k') f_{m,n}(k')\varphi_{m,n}(k)dk'dk + \\ &\quad + \int_{\mathbb{R}_+^2} K_m(k,k')\varphi_{m,n}(k')(f_n(k)-f_{m,n}(k))dk'dk \\ &\qquad - \int_{\mathbb{R}_+^2} K_m(k,k')\varphi_{m,n}(k')\varphi_{m,n}(k)dk'dk. \end{aligned} \quad (3.36)$$

We rewrite L_2 as follows:

$$\begin{aligned} L_2 &= \int_{\mathbb{R}^2_+} K(k,k')f_{m,n}(k')(f_n(k)-f_{m,n}(k))dk'dk \\ &- \int_{\mathbb{R}^2_+} K(k,k')\varphi_{m,n}(k')\varphi_{m,n}(k)dk'dk + R_{m,n}(t), \quad (3.37)\\ R_{m,n}(t) &= \int_{\mathbb{R}^2_+} K_m(k,k')\varphi_{m,n}(k')(f_n(k)-f_{m,n}(k))dk'dk \\ &\qquad - \int_{\mathbb{R}^2_+} K_m(k,k')f_{m,n}(k')\varphi_{m,n}(k)dk'dk. \quad (3.38)\end{aligned}$$

It follows from (3.33), (3.35) and (3.37) that:

$$\begin{aligned}\langle E_m f_m, f_n - f_m\rangle = &- \int_0^\infty \Gamma(k) f_{m,n}(k)(f_n(k)-f_{m,n}(k))dk + \\ &+ \int_0^\infty \Gamma(k)\varphi_{m,n}(k)\varphi_{m,n}(k)dk + \\ &+ \int_{\mathbb{R}^2_+} K(k,k')f_{m,n}(k')(f_n(k)-f_{m,n}(k))dk'dk \\ &- \int_{\mathbb{R}^2_+} K(k,k')\varphi_{m,n}(k')\varphi_{m,n}(k)dk'dk + R_{m,n}(t,k)\end{aligned}$$

and then

$$\langle E_m f_m, f_n - f_m\rangle = \langle E f_{m,n}, f_n - f_m\rangle - \langle E\varphi_{m,n}, \varphi_{m,n}\rangle + R_{m,n}(t). \quad (3.39)$$

We deduce, using (3.32) and (3.39) that

$$\begin{aligned}\langle E_n f_n, f_n - f_m\rangle - \langle E_m f_m, f_n - f_m\rangle = \langle E(f_n - f_{m,n}), f_n - f_m\rangle + \\ + \langle E\varphi_{m,n}, \varphi_{m,n}\rangle + R_{m,n}(t). \quad (3.40)\end{aligned}$$

By (2.10) in Lemma 2.3 we deduce

$$\begin{aligned}\langle E_n f_n, f_n - f_m\rangle - \langle E_m f_m, f_n - f_m\rangle \le -C_* \|(\mathbb{I}-\mathbb{P})(f_n - f_{m,n})\|^2_{L^2(\Gamma)} \\ -C_*\|(\mathbb{I}-\mathbb{P})\varphi_{m,n}\|^2_{L^2(\Gamma)} + |R_{m,n}(t)|, \quad (3.41)\end{aligned}$$

where $\mathbb{I}$ is the identity operator.

On the other hand, since

$$\|(\mathbb{I}-\mathbb{P})(f_n - f_m)\|^2_{L^2(\Gamma)} \le \|(\mathbb{I}-\mathbb{P})(f_n - f_{m,n})\|^2_{L^2(\Gamma)} + \|(\mathbb{I}-\mathbb{P})\varphi_{m,n}\|^2_{L^2(\Gamma)}$$

it follows that

$$\langle E_n f_n, f_n - f_m\rangle - \langle E_m f_m, f_n - f_m\rangle \le -C_* \|(\mathbb{I}-\mathbb{P})(f_n - f_m)\|^2_{L^2(\Gamma)} + |R_{m,n}(t)|. \quad (3.42)$$

We now estimate $R_{m,n}$ given by (3.38). Since $K(k,k') = K(k',k)$ for all $k>0$, $k'>0$ it is easy to check that this term may be written as follows

$$R_{m,n}(t) = \int_{\mathbb{R}^2_+} K(k,k')\chi_m(k')(\chi_m(k') - \chi_n(k'))f_m(k')f_n(k)dk'dk$$
$$-2\int_{\mathbb{R}^2_+} K(k,k')\chi_n(k)(\chi_m(k') - \chi_n(k'))f_m(k')f_m(k)dk'dk. \quad (3.43)$$

from where we deduce the following estimate:

$$\begin{aligned} |R_{m,n}(t)| &\le \int_{\mathbb{R}^2_+} K(k,k')\chi_m(k')(\chi_m(k') - \chi_n(k'))|f_m(k')||f_n(k)|dk'dk \\ &\quad +2\int_{\mathbb{R}^2_+} K(k,k')\chi_m(k')(\chi_n(k') - \chi_m(k'))|f_m(k')||f_m(k)|dk'dk \\ &\le \rho_{n,m}\left(||f_n||_{L^2(\Gamma)}||f_m||_{L^2(\Gamma)} + 2||f_m||^2_{L^2(\Gamma)}\right) \quad (3.44) \\ \rho_{n,m} &= \left\|\frac{K(k,k')(\chi_m(k') - \chi_n(k'))}{\sqrt{\Gamma}(k)\sqrt{\Gamma}(k')}\right\|_{L^2(\mathbb{R}^2_+)} \quad (3.45) \end{aligned}$$

Using now that $\frac{K(k,k')}{\sqrt{\Gamma}(k)\sqrt{\Gamma}(k')} \in L^2(\mathbb{R}^2_+)$ and the dominated convergence Theorem, it is easy to check that

$$\lim_{n\to\infty, m>n} \rho(n,m) = 0 \quad (3.46)$$

Combining now (3.25) and (3.42):

$$\|f_n(t) - f_m(t)\|_2^2 + 2C_* \int_0^t \|(\mathbb{I}-\mathbb{P})(f_n - f_m)\|^2_{L^2(\Gamma)} ds \le$$
$$\le \|f_{0,n} - f_{0,m}\|_2^2 + \int_0^t |R_{m,n}(s)|ds. \quad (3.47)$$

On the other hand, since

$$||\mathbb{P}(f_n(t) - f_m(t))||^2_{L^2(\Gamma)} \le ||f_n(t) - f_m(t)||_2^2 ||\varphi_0||^2_{L^2(\Gamma)},$$

we have by (3.47):

$$||\mathbb{P}(f_n(t)-f_m(t))||^2_{L^2(\Gamma)} \le \left(\|f_{0,n}-f_{0,m}\|_2^2 + \int_0^t |R_{m,n}(s)|ds\right) ||\varphi_0||^2_{L^2(\Gamma)}.$$

Integrating both sides of this inequality with respect to t, we deduce

$$\int_0^t ||\mathbb{P}(f_n(t)-f_m(t))||^2_{L^2(\Gamma)}ds \le t\left(\|f_{0,n}-f_{0,m}\|_2^2 + \right.$$
$$\left. + \int_0^t |R_{m,n}(s)|ds\right) ||\varphi_0||^2_{L^2(\Gamma)},$$

and then,

$$\|f_n(t)-f_m(t)\|_2^2 + 2C_*\int_0^t \|(f_n-f_m)\|^2_{L^2(\Gamma)}ds \le (1+2C_*t||\varphi_0||_{L^2(\Gamma)}) \times$$
$$\times \left(\|f_{0,n}-f_{0,m}\|_2^2 + \int_0^t |R_{m,n}(s)|ds\right). \quad (3.48)$$

By (3.44),

$$\int_0^t |R_{m,n}(s)|ds \le 3\rho_{n,m}\int_0^t \left(||f_n||^2_{L^2(\Gamma)} + ||f_m||^2_{L^2(\Gamma)}\right). \quad (3.49)$$

Since the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(\Gamma))$ for all $T>0$ and $\rho_{n,m}$ satisfies (3.46), we deduce that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0,T;L^2(\Gamma))$ for all $T>0$.

Then, there exists $f\in L^2(0,T;L^2(\Gamma))$ for all $T>0$, and a subsequence, that we still denote f_n, satisfying

$$\lim_{n\to\infty} ||f_n - f||_{L^2(0,T;L^2(\Gamma))} = 0, \ \ \forall T>0, \quad (3.50)$$
$$\lim_{n\to\infty} f_n(t,k) = f(t,k), \ \ a.e.\ t>0, k>0. \quad (3.51)$$

On the other hand, it also follows from (3.46), (3.48) and (3.49) that $(f_n)_{n\in\mathbb{N}}$ is now a Cauchy sequence in $C([0,T);L^2(\mathbb{R}_+))$. We then deduce that, for all $T>0$:

$$f\in L^\infty((0,T);L^2(\mathbb{R}_+))\cap C([0,T);L^2(\mathbb{R}_+)), \quad (3.52)$$
$$\lim_{n\to\infty} ||f_n - f||_{L^\infty(0,T;L^2(\mathbb{R}_+))} = 0. \quad (3.53)$$

We now take the limit in (3.20) as $n\to\infty$ to obtain:

$$||f(T)||_2^2 + 2C_*\int_0^T ||f(t)-\mathbb{P}f(t)||^2_{L^2(\Gamma)}dt \le \|f_0\|_2^2, \ \ \forall T>0,$$

and then,

$$||f(t)||_2^2 + 2C_* \int_0^\infty ||f(t) - \mathbb{P}f(t)||^2_{L^2(\Gamma)} dt \le 2\|f_0\|_2^2 \tag{3.54}$$

Let us show now that $\partial_t f \in L^2(0,T; L^2(\Gamma^{-1}))$ and f satisfies the equation (1.30) in $L^2(0,T; L^2(\Gamma^{-1}))$, for all $T > 0$. To this end we notice that for all $u \in L^2(0,T; L^2(\Gamma))$ and $v \in L^2(0,T; L^2(\Gamma))$:

$$\left| \int_0^T \int_0^\infty E(u)(s,k) v(s,k) dk ds \right| \le (1 + 2C_0) ||u||_{L^2(0,T;L^2(\Gamma))} ||v||_{L^2(0,T;L^2(\Gamma))}$$

Then, the linear operator:

$$\mathcal{T}: \ \ v \to \int_0^T \int_0^\infty E(u)(s,k) v(s,k) dk ds$$

is linear and bounded from $L^2(0,T; L^2(\Gamma))$ to $\mathbb{R}$. It belongs then to $(L^2(0,T; L^2(\Gamma)))'$. We deduce the existence of $\omega \in L^2(0,T; L^2(\Gamma))$ such that, for all $v \in L^2(0,T; L^2(\Gamma))$:

$$T(v) = \int_0^T \int_0^\infty E(u)(t,k) v(t,k) dk dt = \int_0^T \int_0^\infty \omega(t,k) v(t,k) \Gamma(k) dk dt.$$

Then,

$$E(u)(t,k) = \omega(t,k)\Gamma(k), \text{ for } \ a.e. \ t \in (0,T), \text{ and } \ a.e. \ k > 0.$$

This implies that $E(u) \in L^2(0,T; L^2(\Gamma^{-1}))$ and we have:

$$||E(u)||_{L^2(0,T;L^2(\Gamma^{-1}))} \le (1 + 2C_0) ||u||_{L^2(0,T;L^2(\Gamma))}. \tag{3.55}$$

On the other hand, we know by (3.7) that $f - \mathbb{P}(f) \in L^2(0,\infty; L^2(\Gamma))$. But we also have $\mathbb{P}(f)(t) \in L^\infty(0,\infty; L^2(\Gamma))$ since , for all $t > 0$:

$$||\mathbb{P}(f)(t)||_{L^2(\Gamma)} = |\langle f(t), \varphi_0 \rangle| ||\varphi_0||_{L^2(\Gamma)} \le ||f_0||_2 ||\varphi_0||_{L^2(\Gamma)}$$

we deduce, that $f \in L^2(0,T; L^2(\Gamma))$, then $E(f) \in L^2(0,T; L^2(\Gamma^{-1}))$ and by (3.50), for a new subsequence still denoted (f_n):

$$||E(f_n) - E(f)||_{L^2(0,T;L^2(\Gamma^{-1}))} \le (1 + 2C_0) ||f_n - f||_{L^2(0,T;L^2(\Gamma))} \to 0 \tag{3.56}$$

as $n \to \infty$ and

$$\lim_{n \to \infty} E(f_n)(t,k) = E(f)(t,k), \ \ a.e. \ t \in (0,T), k > 0. \tag{3.57}$$

We then deduce, passing to the limit in (3.17), that $\partial_t f \in L^2(0,T;L^2(\Gamma^{-1}))$ and f satisfies the equation (1.30) in $L^2(0,T;L^2(\Gamma^{-1}))$, for all $T>0$. Moreover, by (3.55):

$$\left|\left|\frac{\partial f}{\partial t}\right|\right|_{L^2((0,T),L^2(\Gamma^{-1}))} \le (1+2C_0)||f||_{L^2((0,T),L^2(\Gamma))}, \;\; \forall T>0. \tag{3.58}$$

We leave the proof of (3.8) until the end of the proof of Proposition 3.1.

In order to prove (3.5) we first notice that, using $\partial_t f \in L^2(0,T;L^2(\Gamma^{-1}))$ and Lemma 2.1, we can multiply the equation (1.30) by any function $\varphi \in L^2(\Gamma)$ to obtain:

$$\frac{d}{dt}\langle f,\varphi\rangle = \langle E(f),\varphi\rangle.$$

By Lemma 2.2, identity (3.5), and then (3.6) follows.

From (3.6) we now deduce that,

$$\mathbb{P}(f)(t) = \langle f(t),\varphi_0\rangle\varphi_0 = \langle f_0,\varphi_0\rangle\varphi_0 = \mathbb{P}(f_0) \;\; \forall t>0,$$

and by (3.54), (3.7) immediately follows. We then easily deduce (3.1), (3.2).

We prove now (3.4). Since f_n satisfies (3.16), (3.17) and (3.18), we obtain after integration on $(0,t)$:

$$f_n(t,k) - f_{0,n}(k) = \int_0^t E(f_n)(s,k)ds, \;\; \forall n>0, \forall t>0, \forall k>0. \tag{3.59}$$

Using now (3.56) we notice that, for all $t>0$:

$$\left\|\int_0^t \left(E(f_n)(s) - E(f)(s)\right) ds\right\|_{L^2(\Gamma^{-1})} \le C_0\sqrt{t}\,\|f_n - f\|_{L^2(0,t;L^2(\Gamma))}.$$

We deduce that

$$\lim_{n\to 0}\left\|\int_0^t E(f_n)(s)ds - \int_0^t E(f)(s)ds\right\|_{L^2(\Gamma^{-1})} = 0$$

and then, up to a new subsequence still denoted (f_n):

$$\lim_{n\to 0}\int_0^t E(f_n)(s)ds = \int_0^t E(f)(s)ds = 0, \;\; a.e.\ k>0 \tag{3.60}$$

Using now (3.51), (3.57) and (3.60) we first pass to the limit in (3.59) as $n\to\infty$ for almost every $t\in(0,T)$ and $k>0$ and deduce that:

$$f(t,k) = f_0(k) + \int_0^t E(f)(s,k)ds, \;\; a.e.\ t\in(0,T), k>0.$$

Therefore,

$$\lim_{t\to 0} ||f(t)-f_0||_{L^2(0,t;L^2(\Gamma^{-1}))} \leq C_0\int_0^t ||f(s)||_{L^2(\Gamma)} \leq C_0\sqrt{t}\,||f||_{L^2(0,t;L^2(\Gamma))}.$$

Since, on the other hand, $f\in C([0,T);L^2(\mathbb{R}_+))$, (3.4) follows.

If we assume that $f_0\geq 0$, we have seen that, for every n, $f_n(t)\geq 0$ for all $t>0$. We deduce by (3.51) that $f(t,k)\geq 0$ for all $t>0$ and a. e. $k>0$.

Finally, in order to prove the estimate we argue as follows. Consider the function $g(t,k)=f(t,k)-\mathbb{P}(f_0)$. By (3.6), g satisfies all the properties that have been already proved for the function f. Moreover, by construction $\mathbb{P}(g)(t)=0$ for all $T\geq 0$. Therefore, using (3.58):

$$\left|\left|\frac{\partial g}{\partial t}\right|\right|^2_{L^2((0,T),L^2(\Gamma^{-1}))} \leq (1+2C_0)^2||g||^2_{L^2((0,T),L^2(\Gamma))}$$

and then,

$$\left|\left|\frac{\partial f}{\partial t}\right|\right|^2_{L^2((0,T),L^2(\Gamma^{-1}))} \leq (1+2C_0)^2||f(t)-\mathbb{P}(f_0)||^2_{L^2((0,T),L^2(\Gamma))}, \forall T>0 \tag{3.61}$$

from where (3.8) follows. ■

4 Rate of decay

In this Section we prove the algebraic rate of convergence of the solutions obtained in Section 3 towards the corresponding equilibrium. To this end we first need the following Lemma.

Lemma 4.1 *Let $f_0\in L^2(\mathbb{R}_+)$ such that $\int_0^\infty f_0(k)\varphi_0(k)dk=0$ and satisfies (1.47) or (1.48). Suppose that there exist $C^*>0$, $\omega>0$ and $\tau>0$ such that, the solution f of (1.30)–(1.35) obtained in Proposition 3.1 satisfies:*

$$\|f(t)\|_2\leq C^*\|f_0\|_2(t+1)^{-\omega}\quad \forall t\geq\tau. \tag{4.1}$$

Then, there exist $\theta_1>0$, $\kappa_1>0$ and $\kappa_2>0$, where κ_1 and κ_2 are independent on θ_1, such that, for all $0<\theta<\theta_1$ and for all $t>\max\{1,\tau\}$

$$\int_0^\infty |f(t,k)|^2\Gamma(k)dk\geq\kappa_1\theta\int_0^\infty|f(t,k)|^2dk-\kappa_2\left(\frac{\theta^2}{(t+1)^{2\omega}}+\frac{\theta}{(t+1)}\right). \tag{4.2}$$

Proof By hypothesis:

$$\frac{\partial f}{\partial t} = -\Gamma(k) f(t,k) + \int_0^\infty K(k,k') f(t,k') dk'.$$

Multiply both sides of the above equation by $2f$, we get

$$\frac{\partial f^2}{\partial t} = -2\Gamma(k) f^2(t,k) + 2\int_0^\infty K(k,k') f(t,k') dk' f(t,k).$$

Using (6.1) and (6.4) in the Appendix we deduce, that there exist two positive constants $\theta_0 < 1$ and C_K such that, for all $k \in (0, \theta_0)$:

$$(i) \quad \Gamma(k) \geq \frac{k}{2},$$
$$(ii) \quad \int_0^\infty K(k,k') f(t,k') dk' \leq ||f(t)||_2 ||K(k,\cdot)||_2 \leq \frac{C_K}{2} \ k ||f(t)||_2.$$

Therefore, for $\theta \in (0, \theta_0)$ and all $t > 0$:

$$\frac{\partial f^2}{\partial t}(t,k) \leq -k f^2(t,k) + C_K k ||f(t)||_2 |f(t,k)| \quad a.e. k \in (0,\theta).$$

Using now (4.1) we deduce, for $\theta \in (0,\theta_0)$ and all $t > \tau$:

$$\frac{\partial f^2}{\partial t}(t,k) + k f^2(t,k) \leq C_K k ||f(t)||_2 |f(t,k)| \leq C_K C^* k (t+1)^{-\omega} |f(t,k)| ||f_0||_2$$
$$\frac{\partial}{\partial t}\left(f^2(t,k) e^{kt}\right) \leq C_K C^* k (t+1)^{-\omega} e^{kt} |f(t,k)| ||f_0||_2.$$

Since

$$\begin{aligned} \frac{\partial}{\partial t}\left(f^2(t,k)e^{kt}\right) &= \frac{\partial}{\partial t}\left(\left(f(t,k)e^{\frac{k}{2}t}\right)^2\right) \\ &= 2\left|f(t,k)e^{\frac{k}{2}t}\right| \frac{\partial}{\partial t}\left|f(t,k)e^{\frac{k}{2}t}\right| \text{ for a. e. } k, \end{aligned}$$

then

$$\frac{\partial}{\partial t}\left(|f(t,k)| e^{\frac{k}{2}t}\right) \leq \frac{C_K C^*}{2} k ||f_0||_2 (t+1)^{-\omega} e^{\frac{k}{2}t}$$
$$|f(t,k)| e^{\frac{k}{2}t} \leq |f_0(k)| + \frac{C_K C^*}{2} ||f_0||_2 k \int_0^t (s+1)^{-\omega} e^{\frac{k}{2}s} ds.$$

By lemma 6.3 with $\rho = k/2$ and $\theta = \omega$:

$$\int_0^t (s+1)^{-\omega} e^{\frac{ks}{2}} ds \quad \leq \quad C_\omega [(t+1)^{-\omega} + e^{-\frac{kt}{6}}] \frac{e^{\frac{k}{2}t}}{k}$$

for all $\omega > 0$, and $t > 0$, where we can take $C_\omega = 6 \times 2^\omega$. Then, for all $t > \tau$ and $\theta \in (0, \theta_0)$:

$$
\begin{aligned}
&|f(t,k)|e^{\frac{k}{2}t} \le |f_0(k)| + \frac{C_K C^* C_\omega}{2}||f_0||_2 \left[(t+1)^{-\omega} + e^{-\frac{kt}{6}}\right] e^{\frac{k}{2}t} \\
&|f(t,k)| \le |f_0(k)|e^{-\frac{kt}{2}} + \frac{C_K C^* C_\omega}{2}||f_0||_2 \left[(t+1)^{-\omega} + e^{-\frac{kt}{6}}\right] \\
&|f(t,k)|^2 \le 2|f_0(k)|^2 e^{-kt} + A||f_0||_2^2 \left[(t+1)^{-2\omega} + e^{-\frac{kt}{3}}\right] \\
&A = (C_K C^* C_\omega)^2.
\end{aligned}
$$

As a consequence, if $0 < \theta \le \theta_0$:

$$
\int_0^\theta |f(t,k)|^2 dk \le 2\int_0^\theta f_0^2(k)e^{-kt} + A||f_0||_2^2 \left(\frac{\theta}{(1+t)^{2\omega}} + \frac{3}{t}\right). \tag{4.3}
$$

If we now assume that f_0 satisfies (1.47):

$$
I = \int_0^1 \frac{|f_0(k)|^2}{k} dk < \infty,
$$

then we obtain, for all $t \ge \max\{1, \tau\}$:

$$
\int_0^\theta |f(t,k)|^2 dk \le \frac{2I}{(t+1)} + A||f_0||_2^2 \left[\frac{\theta}{(t+1)^{2\omega}} + \frac{3}{1+t}\right]. \tag{4.4}
$$

On the other hand, by (6.1) and (6.2) it easily follows that there exists a positive constant $\kappa > 0$ such that for all $k > 0$ we have $\Gamma(k) \ge \kappa\, k$. We then have:

$$
\begin{aligned}
\int_0^\infty |f(t,k)|^2 \Gamma(k) dk &= \int_0^\theta |f(t,k)|^2 \Gamma(k) dk + \int_\theta^\infty |f(t,k)|^2 \Gamma(k) dk \\
&\ge \kappa\, \theta \int_\theta^\infty |f(t,k)|^2 dk \\
&= -\kappa\, \theta \int_0^\theta |f(t,k)|^2 dk + \kappa\, \theta \int_0^\infty |f(t,k)|^2 dk \\
&\ge -\kappa\theta \left(\frac{2I}{(t+1)} + (C_K C^* C_\omega ||f_0||_2)^2 \left[\frac{\theta}{(t+1)^{2\omega}} + \frac{3}{1+t}\right]\right) + \\
&\quad +\kappa\theta \int_0^\infty |f(t,k)|^2 dk.
\end{aligned}
$$

Then, condition (4.2) is satisfied with

$$
\kappa_1 = \kappa, \tag{4.5}
$$

$$\kappa_2 = \kappa \left(2I + 4A||f_0||_2^2\right). \tag{4.6}$$

for all $t \geq \max\{1, \tau\}$.

If, on the other hand, the initial data f_0 satisfies (1.48) then, by Lebesgue convergence Theorem:

$$\lim_{t\to\infty} t \int_0^\theta f_0^2(k) e^{-kt} dk = \lim_{t\to\infty} \int_0^{\theta t} f_0^2\left(\frac{x}{t}\right) e^{-x} dx = a^2$$

Notice that if the limit a exists, then the function f_0 is bounded in a neighborhood of the origin, from where, for all $x \in (0, t\theta)$, $x/t \in (0, \theta)$ and $f(x/t)$ is bounded if θ_0 is sufficiently small. We then deduce by (4.3) that

$$\int_0^\theta |f(t,k)|^2 dk \leq \frac{2a^2}{(t+1)} + A||f_0||_2^2 \left[\frac{\theta}{(t+1)^{2\omega}} + \frac{3}{1+t}\right]. \tag{4.7}$$

Arguing as above we deduce that condition (4.2) is now satisfied with

$$\kappa_1 = \kappa, \tag{4.8}$$

$$\kappa_2 = \kappa \left(2a^2 + 4A||f_0||_2^2\right). \tag{4.9}$$

■

Remark 4.1 *The constants θ_0 and C_K are determined by the behavior of $\Gamma(k)$ and $||K(k,\cdot)||_2$ respectively as $k \to 0$. The value of κ is determined by the global behavior of the function Γ. The constants κ_1 and κ_2 given by (4.5) and (4.6) or (4.8) and (4.9) depend on the global behavior of the function Γ, but also on the quantities $\int_0^1 \frac{|f_0(k)|^2}{k} dk$ or a respectively.*

The algebraic convergence rate of the solution of problem (1.30)–(1.35) follows as a consequence of Lemma 4.1, using the following result.

Lemma 4.2 *Suppose that $f_0 \in L^2(\mathbb{R}_+)$ is such that $\mathbb{P}(f_0) = 0$ and satisfies (1.47) or (1.48). Then, there exists a positive constant C, that does not depend on $\|f_0\|_2$ such that for all $t > 0$:*

$$\|f(t)\|_2 \leq C\|f_0\|_2(1+t)^{-1/2}. \tag{4.10}$$

Proof Since equation (1.30) is linear, we may suppose without any loss of generality that $||f_0||_2 = 1$. We divide the proof into two steps.

Step 1. We first apply Lemma 4.1 with $\omega = 0$. To this end we multiply the equation (1.30) by f and integrate over $\mathbb{R}_+$ and obtain, using Lemma 2.3:

$$\frac{d}{dt}\|f\|_2^2 = \langle E(f), f\rangle \leq -C_* \int_0^\infty |\sqrt{\Gamma}(k) f(k)|^2 dk.$$

Since the solution that we have obtained is such that $||f(t)||_2 \leq ||f_0||_2$ for all $t > 0$, condition (4.1) holds with $\omega = 0$, $\tau = 0$ and $C^* = 1$. Then, by Lemma 4.1, there exist three positive constants θ_0, κ_1 and κ_2, with κ_1 and κ_2 independent of θ_0, such that for all $\theta \in (0, \theta_0)$ and for all $t > 1$:

$$\frac{d}{dt}\|f\|_2^2 \leq -C_*\kappa_1\theta\|f\|_2^2 + C_*\kappa_2\left(\theta^2 + \frac{\theta}{(t+1)}\right).$$

This leads to

$$\frac{d}{dt}\left(\|f\|_2^2 \exp(C_1\theta t)\right) \leq C_2\left(\theta^2 + \frac{\theta}{(t+1)}\right)\exp(C_1\theta t), \tag{4.11}$$

$$\text{with: } C_1 = \max\{1, C_*\kappa_1\}, \;\; C_2 = C_*\kappa_2. \tag{4.12}$$

Thus, for all $t > 1$:

$$\|f(t)\|_2^2 \leq \exp(-C_1\theta t) + C_2\int_0^t \left(\theta^2 + \frac{\theta}{(s+1)}\right)\exp(-C_1\theta(t-s))ds.$$

and, by (6.6) in Lemma 6.3:

$$\|f(t)\|_2^2 \leq \exp(-C_1\theta t) + C_2\theta^2 t + C_2\left[\frac{2}{1+t} + 3e^{-\frac{C_1\theta t}{3}}\right] \tag{4.13}$$

for all $\theta \in (0, \theta_0)$ and $t \geq 1$.

We fix now a constant δ such that

$$\frac{2}{3} < \delta < 1, \tag{4.14}$$

and define

$$T_0 = \left(\frac{1}{C_1\theta_0}\right)^\delta. \tag{4.15}$$

Then for all $t \geq T_0$, we have $t^{-\delta}C_1^{-1} \leq T_0^{-\delta}C_1^{-1} = \theta_0$. We may therefore choose $\theta = (t+1)^{-\delta}C_1^{-1}$ in (4.13) to obtain that, for all $t \geq \max\{1, T_0\}$:

$$
\begin{aligned}
\|f(t)\|_2^2 &\leq \exp(-C_1 t(1+t)^{-\delta}) + \frac{C_1^{-2}C_2 t}{(1+t)^{2\delta}} + \\
&\qquad\qquad\qquad + C_2\left[\frac{2}{1+t} + 3e^{-\frac{t(1+t)^{-\delta}}{3}}\right] \\
&\leq \exp(-C_1 t(1+t)^{-\delta}) + C_1^{-2}C_2(1+t)^{1-2\delta} + \\
&\qquad\qquad\qquad + C_2\left[\frac{2}{1+t} + 3e^{-\frac{t(1+t)^{-\delta}}{3}}\right] \quad (4.16) \\
&\leq (1+3C_2)e^{-\frac{t(1+t)^{-\delta}}{3}} + C_1^{-2}C_2(1+t)^{1-2\delta} + \frac{2C_2}{1+t}. \quad (4.17)
\end{aligned}
$$

Since $\delta < 1$, there is a unique positive number T_1 such that

$$
(1+3C_2)e^{-\frac{T_1(1+T_1)^{-\delta}}{3}} = C_1^{-2}C_2(1+T_1)^{1-2\delta}. \quad (4.18)
$$

Then, if $t \geq T_2 = \max\{1, T_0, T_1\}$,

$$
(1+3C_2)e^{-\frac{t(1+t)^{-\delta}}{3}} \leq C_1^{-2}C_2(1+t)^{1-2\delta}
$$

and

$$
\|f(t)\|_2^2 \leq 2C_1^{-2}C_2(1+t)^{1-2\delta} + \frac{2C_2}{1+t}.
$$

Since $\delta \in (2/3, 1)$, if we call $\omega_0 = \frac{2\delta-1}{2}$we have $\omega_0 \in (1/6, 1/2)$ and then

$$
\|f(t)\|_2^2 \leq 2C_2(1+C_1^{-2})(1+t)^{-2\omega_0} \quad \forall t \geq T_2. \quad (4.19)
$$

Step 2. Using the estimate (4.19) we may apply now Lemma 4.1 with $\omega = \omega_0$, $\tau = T_2$ and $2C_2(1+C_1^{-2})$ in the role of C^*. Let us call $2C_2(1+C_1^{-2}) = C^{**}$. Arguing as above we first write that, by Lemma 4.1, there exists three positive constants θ_0', κ_1' and κ_2' with κ_1' and κ_2' independent of θ_0', such that for all $\theta \in (0, \theta_0')$ and for all $t > T_2$:

$$
\frac{d}{dt}\|f(t)\|_2^2 \leq -C_*\kappa_1'\theta\|f(t)\|_2^2 + C_*\kappa_2'\left(\frac{\theta^2}{(1+t)^{2\omega_0}} + \frac{\theta}{(t+1)}\right).
$$

Then, for all $t \geq T_2$:

$$
\|f(t)\|_2^2 \leq \|f_0\|_2^2 e^{-C_1'\theta t} + C_2'\int_0^t \left(\frac{\theta^2}{(s+1)^{2\omega_0}} + \frac{\theta}{(s+1)}\right)e^{-C_1'\theta(t-s)}ds.
$$

where

$$C_1' = C_* C^{**} \kappa_1', \;\; C_2' = C_* \kappa_2'.$$

Using (6.6):

$$\begin{aligned}\int_0^t \left(\frac{\theta^2}{(s+1)^{2\omega_0}} + \frac{\theta}{(s+1)}\right) e^{C_1'\theta s} ds \leq \\ \leq \theta \left(4^{\omega_0}(t+1)^{-2\omega_0} + 3e^{-C_1'\theta t/3}\right) \frac{e^{C_1'\theta t}}{C_1'} + \\ + \left(2(t+1)^{-1} + 3e^{-C_1'\theta t/3}\right) \frac{e^{C_1'\theta t}}{C_1'}\end{aligned}$$

from where we deduce that for all $\theta \in (0, \theta_0')$ and $t \geq T_2$:

$$\begin{aligned}&||f(t)||_2^2 \leq e^{-C_1'\theta t} + \\ &+\frac{C_2'}{C_1'}\left(\frac{4^{\omega_0}\theta}{(t+1)^{2\omega_0}} + 3\theta e^{-C_1'\theta t/3} + \frac{2}{(t+1)} + 3e^{-C_1'\theta t/3}\right) \quad (4.20)\\ \leq \;\; &\left(1 + 6\frac{C_2'}{C_1'}\right) e^{-C_1'\theta t/3} + \frac{C_2'}{C_1'}\frac{2^{2\omega_0}\theta}{(t+1)^{2\omega_0}} + \frac{C_2'}{C_1'}\frac{2}{(t+1)}. \quad (4.21)\end{aligned}$$

(where we have used that $||f_0||_2 \leq 1$). We define now

$$T_3 = \left(\frac{1}{C_1'\theta_0'}\right)^{\delta}. \quad (4.22)$$

Then, if $t > \max\{T_2, T_3\}$, $t^{-\delta} {C_1'}^{-1} \leq T_3^{-\delta} {C_1'}^{-1} = \theta_0'$. We may therefore choose $\theta = (t+1)^{-\delta} {C_1'}^{-1}$ in (4.21) and obtain

$$||f(t)||_2^2 \leq \left(1 + 6\frac{C_2'}{C_1'}\right) e^{-\frac{t(1+t)^{-\delta}}{3}} + \frac{C_2'}{{C_1'}^2}\frac{4^{\omega_0}}{(t+1)^{2\omega_0+\delta}} + \frac{C_2'}{C_1'}\frac{2}{(t+1)}$$

for all $t \geq \max\{T_2, T_3\}$. We now call T_4 the positive number such that

$$\left(1 + 6\frac{C_2'}{C_1'}\right) e^{-\frac{T_4(1+T_4)^{-\delta}}{3}} = \frac{C_2'}{{C_1'}^2}\frac{4^{\omega_0}}{(T_4+1)^{2\omega_0+\delta}}$$

then, for all $t \geq \max\{T_2, T_3, T_4\}$,

$$||f(t)||_2^2 \leq 2\frac{C_2'}{{C_1'}^2}\frac{4^{\omega_0}}{(t+1)^{2\omega_0+\delta}} + \frac{C_2'}{C_1'}\frac{2}{(t+1)}.$$

Since $\delta > 2/3$ and $2\omega_0 > 1/3$, $2\omega_0 + \delta > 1$ and for all $t \geq \max\{T_2, T_3, T_4\}$:

$$||f(t)||_2^2 \leq 2\left(\frac{C_2' 4^{\omega_0}}{{C_1'}^2} + \frac{C_2'}{C_1'}\right)\frac{1}{(t+1)}.$$

Since on the other hand, $||f(t)||_2^2 \leq ||f_0||_2^2 = 1$ for all $t \geq 0$ we deduce (4.10) for some positive constant C and for all $t > 0$. If the initial data is such that $||f_0||_2 \geq 1$, we apply the previous argument to the function $f(t)/||f_0||_2$ and (4.10) by the linearity of the equation (1.30)–(1.34). ■

We may state now the following Corollary that follows from Lemma 4.2 and Lemma 4.1.

Corollary 4.1 *For any solution f of (1.30)–(1.35) given by Proposition 3.1 such that the initial data f_0 satisfies (1.47) or (1.48), there exists a positive constant C, depending the behavior of $\Gamma(k)$ on $[0, \infty)$, of $||K(k, \cdot)||_2$ as $k \to 0$ and on $\int_0^1 \frac{|f_0(k)|^2}{k} dk$ or a respectively, such that, for all $t > 0$:*

$$||f(t) - c_0\varphi_0||_2 \leq C\frac{||f_0 - \mathbb{P}(f_0)||_2}{(1+t)^{1/2}}. \tag{4.23}$$

Proof If $c_0 = \int_0^\infty f_0(k)\varphi_0(k)dk = 0$, the conclusion follows from Lemma 4.1. Suppose that $c_0 \neq 0$. Consider then the initial data

$$g_0 = f_0 - \mathbb{P}(f_0).$$

By the properties of φ_0 and the hypothesis on f_0, it easily follows that g_0 satisfies all the hypothesis of Lemma 4.2 and Lemma 4.1. The solution g of the problem (1.30)–(1.35) with initial data g_0 satisfies then

$$\|g(t)\|_2 \leq C(1+t)^{-\frac{1}{2}}||g_0||_2. \tag{4.24}$$

Notice on the other hand that the function

$$G(t, k) = f(t, k) - \mathbb{P}(f_0)$$

is also a solution of (1.30)–(1.35) with initial data g_0 satisfying properties (3.1)–(3.3). Then, by the uniqueness of solution to (1.30)–(1.35) proved in Proposition 3.1, $g = f - \mathbb{P}(f_0)$ and (4.23) follows from (4.24). ■

Proof of Theorem 1.2. The point (i) follows from Proposition 3.1. The point (ii) follows from Corollary 4.1. ■

We do not know if the rate of convergence obtained in Theorem 1.2 is optimal. One may also wonder whether it is necessary to impose one of the conditions (1.47), (1.48) in order to have the algebraic decay (1.49). We do not know neither if these conditions are optimal in any sense. But we show in the next Lemma that it is not possible to have any convergence rate uniform for all the functions in $L^2(\mathbb{R}_+)\cap L^2(\Gamma)$, without any other restriction. More precisely, we have the following.

Lemma 4.3 *There is no function $\rho(t)\geq 0$ satisfying $\overline{\lim}_{t\to\infty}\rho(t)<1$ and such that, for all data $f_0\in L^2(\mathbb{R}_+)\cap L^2(\Gamma)$, the solution of (1.30)–(1.35) given by Proposition 3.1 satisfies:*

$$\|f(t)-\mathbb{P}(f_0)\|_2\leq \rho(t)\|f_0-\mathbb{P}(f_0)\|_2,\ \ \forall t>0. \tag{4.25}$$

Proof Suppose by contradiction that such a function ρ do exists. Let us call, $g(t,k)=f(t,k)-\mathbb{P}(f_0)(k)$. From (4.25) we deduce that, for any $T>0$:

$$\|g_0\|_2^2-C\|g_0\|_2^2\,\rho(T)\leq\|g_0\|_2^2-\|g(T)\|_2^2=-\int_0^T\langle E(g),g\rangle dt. \tag{4.26}$$

By (4.26), there exists $\delta>0$ and $T_0>0$ such that if $T>T_0$,

$$\delta\|g_0\|_2^2\leq\|g_0\|_2^2-\|g(T)\|_2^2\leq-\int_0^T\langle E(g),g\rangle dt. \tag{4.27}$$

In order to estimate the right hand side of (4.27) we consider the norm of $\|g(T)-g_0\|_2^2$:

$$\begin{aligned}\|g(T)-g_0\|_2^2 &= 2\int_0^T\langle\partial_t g,g-g_0\rangle\,dt\\ &= \int_0^T 2\,\langle E(g),g-g_0\rangle\,dt\\ &= 2\int_0^T\langle E(g),g\rangle dt-2\int_0^T\langle E(g),g_0\rangle\,dt\\ &\leq \int_0^T\langle E(g),g\rangle dt-\int_0^T\langle E(g_0),g_0\rangle dt.\end{aligned}$$

where, in the last step, we have used (2.9) in Corollary 2.2.
We then have:

$$-\int_0^T\langle E(g),g\rangle dt\ \ \leq\ \ -T\langle E(g_0),g_0\rangle, \tag{4.28}$$

Since $g_0 \in L^2(\Gamma)$, by (2.2):

$$-\langle E(g_0), g_0\rangle \leq C_0\|g_0\|^2_{L^2(\Gamma)} = C_0\|\sqrt{\Gamma}g_0\|_2^2. \tag{4.29}$$

We deduce from (4.27), (4.28) and (4.29) that, for all $g_0 \in L^2(\mathbb{R}_+) \cap L^2(\Gamma)$:

$$\|g_0\|_2^2 \leq \frac{TC_0}{\delta}\|\sqrt{\Gamma}g_0\|_2^2 \tag{4.30}$$

By property (6.1) of the function Γ this is not possible if $g_0 \in L^2(\mathbb{R}_+)\cap L^2(\Gamma)$ with support in an interval (k_1, k_2), with $0 < k_1 < k_2$ sufficiently small. ∎

Remark 4.2 *The results in the Appendix say that*

$$\Gamma(k) \sim \frac{\pi k}{15}, \;\; k \to 0,$$
$$||K(k,\cdot)||_2 \leq \frac{2\pi^3 k}{\sqrt{21}}, \;\; 0 < k << 1.$$

This suggest that a very rough approximation of the equation (1.30) near $k = 0$ *could be given by*

$$\frac{d}{dt}f(t,k) = -Ckf(t,k), \;\; for\ t > 0, \;\; k\ small$$
$$f(0,k) = f_0(k)\ for\ k\ small,$$

for some constant C*. By the positivity of the operator* E *it seems reasonable to have* $C > 0$*. Since the solution* f *of that simple equation is*

$$f(t,k) = e^{-Ckt}f_0(k), \;\; \forall t > 0,$$

we have

$$\int_0^{k_0} |f(t,k)|^2 dk = \int_0^{k_0} |f_0(k)|^2 e^{-2Ckt}dk, \;\; \forall t > 0.$$

Therefore, if f_0 *satisfies (1.47),*

$$\int_0^{k_0} |f(t,k)|^2 dk \leq \frac{1}{2Ct}\int_0^{k_0} \frac{|f_0(k)|^2}{k}dk, \;\; \forall t > 0.$$

If on the other hand, f_0 *is continuous at* $k = 0$*,*

$$t\int_0^{k_0} |f(t,k)|^2 dk = \frac{1}{2C}\int_0^{2Ck_0t} \left|f_0\left(\frac{x}{2Ct}\right)\right|^2 e^{-x}dx.$$

Since, by (1.48),

$$\lim_{t\to\infty} t\int_0^{k_0} |f(t,k)|^2 dk = \frac{a^2}{2C}$$

we deduce

$$\int_0^{k_0} |f(t,k)|^2 dk = \frac{a^2}{2Ct} + o\left(\frac{1}{t}\right), \text{ as } t \to \infty.$$

The convergence rate (1.49) *seems then in some sense optimal.*

5 Proofs of Proposition 1.1 and Theorem 1.1.

We give in this Section the proofs of Proposition 1.1 and of Theorem 1.1. These follow easily from the results that have been proved in Sections 2, 3 and 4. We begin with the proof of the Proposition.

Proof of Proposition 1.1. Point (i) follows immediately from the orthogonality property of the spherical harmonic functions and the fact that $|p| \in L^2\left(\mathbb{R}^+, \frac{dp}{\sinh^2(k)}\right)$. In order to prove point (ii) let us notice first of all that, if $f(k)$ is such that $f \in L^2(\mathbb{R}^+)$, respectively $f \in L^2(\Gamma)$, and we consider the function g defined by the change of variables (1.29):

$$g(p) \equiv g(|p|) = \frac{\sinh(k)}{k} f(k), \;\; k = \frac{c|p|}{2k_B T}$$

then $g \in L^2\left(\mathbb{R}^+, \frac{k^2}{\sinh^2(k)} dr\right)$, respectively $g \in L^2\left(\mathbb{R}^+, \frac{k^2\Gamma(k)}{\sinh^2(k)} dr\right)$. Moreover, by definition

$$L(g)(|p|) = (k \sinh k)\, E(f)(k),$$

where L is defined in (1.28). Then, if $f \in L^2(\Gamma)$, we have $E(f) \in L^2(\Gamma^{-1})$ by Lemma (2.1), and therefore $L(g) \in L^2\left(\mathbb{R}^+, \frac{k^2 \sinh^2(k) dr}{\Gamma(k)}\right)$.
We then deduce that $L(|p|) \in L^2\left(\mathbb{R}^+, \frac{k^2 \sinh^2(k) dr}{\Gamma(k)}\right)$ and therefore

$$\Lambda(p) = -M(p)\Theta(p) + \int_{\mathbb{R}^3} \Theta(p')\, W(p,p') dp' \in L^2\left(\frac{\sinh^2(k) dp}{\Gamma(k)}\right)$$

It is then enough to check that all the components $\Lambda_{\ell m}$ of the function Λ in the spherical harmonic basis are zero. Using the orthonormality properties of the spherical harmonic functions $Y_{\ell m}$ and the definitions of the Legendre's polynomial we readily check that these components are, up to a constant factor:

$$\Lambda_{\ell m}(|p|) = -M(|p|)\Theta_{\ell m}(|p|) + \frac{1}{2\ell+1}\int_0^\infty \Theta_{\ell m}(r')W_\ell(|p|,r')dr'$$

Since, by Corollary 2.1, the function $\phi(k)$ satisfies $E(\phi) = 0$ and the function $\Theta_{\ell m}(r) = c_{\ell m} r$ is obtained from $\phi(k)$ through the change of variables (1.29), it follows that $\Lambda_{\ell m}(|p|) = 0$ for all ℓ and m. ■

Proof of Theorem 1.1. We decompose the initial data Ω_0 that by hypothesis belongs to $L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2(k)}\right)$ using the basis of $L^2(\mathbb{S}^2)$ of spherical harmonics:

$$\Omega_0(p) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Omega_{0,\ell m}(|p|)Y_{\ell m}\left(\frac{p}{|p|}\right).$$

Using the orthonormality of the basis $\{Y_{\ell m}\}$ we deduce

$$\begin{aligned}
||\Omega_0||^2_{L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2(k)}\right)} &= \int_{\mathbb{R}^3}\left|\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Omega_{0,\ell m}(|p|)Y_{\ell m}\left(\frac{p}{|p|}\right)\right|^2 \frac{dp}{\sinh^2(k)} \\
&= \int_{\mathbb{S}^2} d\sigma \int_0^\infty \left|\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Omega_{0,\ell m}(|p|)Y_{\ell m}(\sigma)\right|^2 \frac{|p|^2 d|p|}{\sinh^2(k)} \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^\infty |\Omega_{0,\ell m}(|p|)|^2 \frac{|p|^2 d|p|}{\sinh^2(k)},
\end{aligned}$$

and then:

$$\Omega_{\ell m} \in L^2\left(\mathbb{R}^+; \frac{|p|^2 d|p|}{\sinh^2(k)}\right), \quad \forall \ell\in\mathbb{N}, m\in\{-\ell,-\ell+1,\cdots,\ell-1,\ell\}$$

Therefore, if we define:

$$f_{0,\ell,m}(k) = k\frac{\Omega_{0,\ell m}(|p|)}{\sinh k}, \quad k = \frac{c|p|}{2k_BT} \tag{5.1}$$

it follows that $f_{0,\ell,m} \in L^2(\mathbb{R}^+)$. Let then be $f_{\ell,m}$ the solution of the equation (1.30) with initial data $f_{0,\ell m}$ given by Theorem 1.2 and define:

$$\Omega_{\ell m}(t,r) = f_{\ell,m}(t,k)\frac{\sinh k}{k}, \quad k = \frac{cr}{2k_BT}. \tag{5.2}$$

It follows from (1.44) that:

$$||\Omega_{\ell\, m}(t)||^2_{L^2\left(\mathbb{R}_+;\frac{r^2}{\sinh^2 k}\right)} \le 2||\Omega_{0,\ell\, m}||^2_{L^2\left(\mathbb{R}_+;\frac{r^2}{\sinh^2 k}\right)}\quad \forall t>0. \tag{5.3}$$

We deduce that

$$\begin{aligned}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}||\Omega_{\ell\, m}(t)||^2_{L^2\left(\mathbb{R}_+;\frac{r^2}{\sinh^2 k}\right)} &\le 2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}||\Omega_{0,\ell\, m}||^2_{L^2\left(\mathbb{R}_+;\frac{r^2}{\sinh^2 k}\right)}\\ &= 2\,||\Omega_0||^2_{L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right)} \end{aligned}\tag{5.4}$$

and the following function is then well defined in $L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2 k}\right)$ for all $t>0$:

$$\Omega(t,p)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Omega_{\ell\, m}(|p|)Y_{\ell\, m}\left(\frac{p}{|p|}\right).$$

It follows from (5.3), (5.4) and (3.2) that Ω satisfies (1.15).

Similarly, by (1.44) and (1.45):

$$\left|\left|\frac{\partial f_{\ell\, m}}{\partial t}\right|\right|_{L^2(0,\infty;L^2(\Gamma^{-1}(k)dk))}\le\frac{1+2C_0}{\sqrt{C_*}}\,||f_{0,\ell\, m}||_{L^2}$$

and then

$$\left|\left|\frac{\partial \Omega_{\ell\, m}}{\partial t}\right|\right|_{L^2\left(0,\infty;L^2\left(\frac{r^2}{\Gamma(k)\,\sinh^2 k}\right)\right)}\le\frac{1+2C_0}{\sqrt{C_*}}||\Omega_{0,\ell\, m}||^2_{L^2\left(\mathbb{R}_+;\frac{r^2}{\sinh^2 k}\right)}$$

Using that $M(p)\equiv M(r)=\Gamma(k)n_0(p)(1+n_0(p))$ and $n_0(p)(1+n_0(p))=1/(4\sinh^2 k)$ we have:

$$\left|\left|\frac{\partial \Omega_{\ell\, m}}{\partial t}\right|\right|_{L^2\left(0,\infty;L^2\left(\frac{r^2}{M(r)\sinh^4 k}\right)\right)}\le\frac{1+2C_0}{\sqrt{C_*}}||\Omega_{0,\ell\, m}||^2_{L^2\left(\mathbb{R}_+;\frac{r^2}{\sinh^2 k}\right)},$$

and

$$\begin{aligned}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left|\left|\frac{\partial \Omega_{\ell\, m}}{\partial t}\right|\right|^2_{L^2\left(0,\infty;L^2\left(\frac{r^2}{M(r)\sinh^4 k}\right)\right)} &\le \frac{(1+2C_0)^2}{C_*}\times\\ \times\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}||\Omega_{0,\ell\, m}||^2_{L^2\left(\mathbb{R}_+;\frac{r^2}{\sinh^2 k}\right)} &= \frac{(1+2C_0)^2}{C_*}||\Omega_0||^2_{L^2\left(\mathbb{R}^3,\frac{dp}{\sinh^2(k)}\right)}.\end{aligned}$$

The following function:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\partial\Omega_{\ell\, m}}{\partial t}(|p|)Y_{\ell\, m}\left(\frac{p}{|p|}\right)$$

is then well defined in $L^2\left(\mathbb{R}^3, \frac{dp}{M(|p|)\sinh^4 k}\right)$ for all $t>0$ and

$$\frac{\partial\Omega}{\partial t}(t,p)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\partial\Omega_{\ell\, m}}{\partial t}(|p|)Y_{\ell\, m}\left(\frac{p}{|p|}\right).$$

Since $f_{\ell\, m}(t,k)$ satisfies the equation (1.30)–(1.34) in $L^2((0,\infty),L^2(\Gamma^{-1}))$, and $M(p)\equiv M(r)=\Gamma(k)n_0(p)(1+n_0(p),\ n_0(p)(1+n_0(p))=1/(4\sinh^2 k)$, the function $\Omega_{\ell\, m}$ satisfies equation (1.27), (1.28) in $L^2\left(\mathbb{R}^+, \frac{r^2 dr}{M(r)\sinh^4 k}\right)$. One easily deduces that Ω satisfies equation (1.12) in $L^2\left(\mathbb{R}^3, \frac{dp}{M(p)\sinh^4 k}\right)$. The two properties in (1.18) are deduced from those in (3.4) using similar arguments.

We wish to prove the uniqueness of solutions of (1.12) in the sense of $L^2\left(0,\infty; L^2\left(\mathbb{R}^3, \frac{dp}{M(p)\sinh^4 k}\right)\right)$, satisfying (1.15)–(1.17) and such that

$$\lim_{t\to 0}||\Omega(t)-\Omega_0||_{L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2 k}\right)}=0. \tag{5.5}$$

To this end we suppose that Ω_1 and Ω_2 are two such solutions and call $\widetilde{\Omega}=\Omega_1-\Omega_2$. It is then also a solution of (1.12) in $L^2\left(0,\infty; L^2\left(\mathbb{R}^3, \frac{dp}{M(p)\sinh^4 k}\right)\right)$, satisfying (1.15)–(1.17) and (5.6) with $\Omega_0=0$. It then follows that the modes $\widetilde{\Omega}_{\ell\, m}$ of $\widetilde{\Omega}$ satisfy equation (1.27)-(1.28) with initial data $\widetilde{\Omega}_{\ell\, m}(0)=0$. By the uniqueness part of Theorem (1.2) it follows that $\widetilde{\Omega}_{\ell\, m}=0$ for each ℓ and m and then $\widetilde{\Omega}\equiv 0$.

Suppose now that $\Omega_0(p)$ also satisfies (1.20). Then, for every ℓ and m, the function $f_{0,\ell\, m}(k)$, defined in (5.2), satisfies (1.47). By Theorem 1.2 we then have:

$$||f_{\ell\, m}(t)-c_{0,\ell\, m}\varphi_0||_2\le C\frac{||f_{0,\ell\, m}-c_{0,\ell\, m}\varphi_0||_2}{(1+t)^{1/2}} \tag{5.6}$$

where

$$c_{0,\ell\, m}=\int_0^{\infty}f_{0,\ell\, m}(k)\varphi_0(k)dk. \tag{5.7}$$

Therefore, using (5.2) we deduce

$$\int_0^\infty |\Omega_{\ell m}(t,r) - c_{\ell m} r|^2 \frac{r^2 dr}{\sinh^2 k} \le \frac{C}{1+t}\int_0^\infty |\Omega_{0,\ell m}(r) - c_{\ell m} r|^2 \frac{r^2 dr}{\sinh^2 k}$$

where

$$c_{\ell m} = \frac{c}{2k_B T ||\phi||_2} c_{0,\ell m}.$$

If we sum now with respect to ℓ and m we obtain

$$\begin{aligned}
||\Omega(t) - \Theta||^2_{L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2 k}\right)} &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{\mathbb{R}^3} |\Omega_{\ell m}(t,|p|) - c_{\ell m}|p||^2 \frac{dp}{\sinh^2 k} \\
&\le \frac{C}{1+t}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{\mathbb{R}^3} |\Omega_{0,\ell m}(|p|) - c_{\ell m}|p||^2 \frac{dp}{\sinh^2 k} \\
&= \frac{C}{1+t}||\Omega(0) - \Theta||^2_{L^2\left(\mathbb{R}^3, \frac{dp}{\sinh^2 k}\right)}.
\end{aligned}$$

Since

$$\begin{aligned}
c_{\ell m} &= \frac{c}{2k_B T||\phi||_2}\int_0^\infty f_{0,\ell m}(k)\varphi_0(k)dk \\
&= \left(\frac{c}{2k_B T}\right)^4 \frac{1}{||\phi||_2^2}\int_0^\infty \frac{\Omega_{0,\ell m}(r)}{\sinh^2 k} r^2 dr \\
&= \left(\frac{c}{2k_B T}\right)^4 \frac{4}{||\phi||_2^2}\int_{\mathbb{R}^3} \Omega_0(p) Y_{\ell m}\left(\frac{p}{|p|}\right) n_0(p)(1+n_0(p))dp
\end{aligned}$$

and $||\phi||_2^2 = \pi^4/30$, this concludes the proof of (1.21)-(1.23). ∎

Remark 5.1 *The total number of particles in the physical system described by equation (1.1)-(1.2) is given by*

$$N(t) = \int_{\mathbb{R}^3} n(t,p)dp.$$

The corresponding quantity in the linear approximation that we consider in this work is:

$$M(t) = \int_{\mathbb{R}^3} n_0(p)dp + \int_{\mathbb{R}^3} n_0(p)(1+n_0(p))\Omega(t,p)dp,$$

It follows from Theorem 1.1 that, if the initial data $\Omega_0 \in L^2(\mathbb{R}^3)$ satisfies (1.20) then:

$$\lim_{t\to\infty} M(t) = \int_{\mathbb{R}^3} n_0(p)dp + \int_{\mathbb{R}^3} n_0(p)(1+n_0(p))\Theta(p)dp \equiv M_\infty.$$

where Θ is defined by (1.22) (1.23). It is easy to see that M_∞ may be greater or smaller than $M(0)$. If we choose the initial data $\Omega_0 = \Theta + g_0$ then,

$$\begin{aligned} M(0) &= \int_{\mathbb{R}^3} n_0(p)dp + \int_{\mathbb{R}^3} n_0(p)(1+n_0(p))\Omega_0(p)dp \\ &= \int_{\mathbb{R}^3} n_0(p)dp + \int_{\mathbb{R}^3} n_0(p)(1+n_0(p))\Omega_0(p)dp + \int_{\mathbb{R}^3} n_0(p)(1+n_0(p))g_0(p)dp \\ &= M_\infty + \int_{\mathbb{R}^3} n_0(p)(1+n_0(p))g_0(p)dp. \end{aligned}$$

The sign of $M(0)-M_\infty$ is then given by the sign of $\int_{\mathbb{R}^3} n_0(p)(1+n_0(p))g_0(p)dp$ and may be positive or negative.

6 Appendix

In this Appendix we recall the definition of Legendre's polynomials, we describe the formal approximation argument leading to the simplified equation (1.27)-(1.28) and present some auxiliary results on the functions Γ and K that appear in the operator E defined in (1.30).

6.1 The functions Γ and K.

We present in this Appendix some auxiliary results, in particular several properties of the functions Γ and K that are needed in the proof of our main results. They have already been obtained in [3] and we state and prove them here just for the sake of completeness.

Lemma 6.1 *The function Γ defined in (1.32), (1.34) satisfies $\Gamma \in C(0,\infty)$ and $\Gamma(k) > 0$ for all $k > 0$. Moreover,*

$$\lim_{k\to 0} \frac{\Gamma(k)}{k} = \frac{\pi^4}{15} \tag{6.1}$$

$$\lim_{k\to\infty} \frac{\Gamma(k)}{k^5} = \frac{1}{15}. \tag{6.2}$$

Proof The continuity of Γ follows immediately from the integrability properties of the integrand in (1.32). The strict positivity of $\Gamma(k)$ for $k > 0$ is deduced from the fact that the integrand in (1.32) is non negative. In order to prove (6.1) and (6.2) we first notice that, by a simple change of variables, the function Γ may be written as:

$$\Gamma(k) = \sinh k \int_0^k \phi(k-k')\phi(k')dk' + 2\sinh k \int_0^\infty \phi(k+k')\phi(k')dk' \quad (6.3)$$

By Lebesgue's convergence Theorem it follows that

$$\lim_{k\to 0} \int_0^k \phi(k-k')\phi(k')dk' = 0$$

and

$$\lim_{k\to 0} \int_0^\infty \phi(k+k')\phi(k')dk' = \int_0^\infty \frac{x^4}{\sinh^2 x}dx = \frac{\pi^4}{30}$$

from where (6.1) follows.

On the other hand,

$$\sinh k \int_0^k \phi(k-k')\phi(k')dk' = k^5 \sinh k \int_0^1 \frac{z^2(1-z)^2}{\sinh(k(1-z))\sinh(kz)}dz$$

But,

$$\sinh k \frac{z^2(1-z)^2}{\sinh(k(1-z))\sinh(kz)} = \frac{1-e^{-2k}}{2} \frac{e^k z^2(1-z)^2}{\frac{e^{k-kz}-e^{kz-k}}{2}\frac{e^{kz}-e^{-kz}}{2}}$$

$$= \frac{2(1-e^{-2k})z^2(1-z)^2}{(e^{-kz}-e^{kz-2k})(e^{kz}-e^{-kz})} = \frac{2(1-e^{-2k})z^2(1-z)^2}{1-e^{-2kz}-e^{-2k(1-z)}+e^{-2k}}$$

$$= 2(1-e^{-2k})\frac{z^2}{(1-e^{-2kz})}\frac{(1-z)^2}{(1-e^{-2k(1-z)})}.$$

And we observe that,

$$\frac{x^2}{1-e^{-Ax}} \leq \frac{x^2}{1-e^{-1/2}},\ \forall x \in \left(\frac{1}{2A}, 1\right).$$

When $Ax \in (0, 1/2)$, $1 - e^{-Ax} \geq Ax/2$ so,

$$\frac{x^2}{1-e^{-Ax}} \leq \frac{2x}{A}\ \forall x \in \left(0, \frac{1}{2A}\right),$$

and then, for $A > 1$:

$$\frac{x^2}{1-e^{-Ax}} \leq 2x, \ \forall x \in (0,1)$$

since $2(1-e^{-1/2}) < 1$. This gives

$$\sinh k \frac{z^2(1-z)^2}{\sinh(k(1-z))\sinh(kz)} \leq 8(1-e^{-2k})z(1-z)$$

for all $z \in (0,1)$ and $k > 1$. The Lebesgues convergence Theorem gives then,

$$\lim_{k\to\infty} k^{-5}\left(\sinh k \int_0^k \phi(k-k')\phi(k')dk'\right) =$$

$$= \lim_{k\to\infty} \sinh k \int_0^1 \frac{z^2(1-z)^2}{\sinh(k(1-z))\sinh(kz)}dz$$

$$= 2\int_0^1 z^2(1-z)^2 dz = \frac{1}{15}.$$

It is not difficult to check, using similar arguments, that the second integral in the right hand side of (6.3) is of lower order when $k \to \infty$ and then (6.1) follows. ■

Lemma 6.2

$$\int_0^\infty |K(k,k')|^2 dk' < \frac{4}{15}\pi^4 k^4 + \frac{4}{21}\pi^6 k^2 \tag{6.4}$$

and

$$\int_0^\infty \int_0^\infty \left|\frac{K(k,k')}{\sqrt{\Gamma(k')\Gamma(k)}}\right|^2 dk dk' < +\infty. \tag{6.5}$$

Proof

$$\int_0^\infty |K(k,k')|^2 dk' \leq 2k^2 \int_0^\infty \left(\phi(k-k')^2 + \phi(k+k')^2\right) k'^2 dk'$$

$$\int_0^\infty \phi(k+k')^2 k'^2 dk' = \int_0^\infty \frac{(k+k')^4}{\sinh^2(k+k')} k'^2 dk'$$

$$= \int_k^\infty \frac{z^4}{\sinh^2 z}(z-k)^2 dz \leq \int_k^\infty \frac{z^4}{\sinh^2 z} z^2 dz$$
$$\leq \int_0^\infty \frac{z^4}{\sinh^2 z} z^2 dz = \frac{\pi^6}{42}$$

$$\int_0^\infty \phi(k-k')^2 k'^2 dk' = \int_0^\infty \frac{(k'-k)^4}{\sinh^2(k'-k)} k'^2 dk'$$
$$= \int_{-k}^\infty \frac{z^4}{\sinh^2 z}(z+k)^2 dz \leq \int_{-\infty}^\infty \frac{z^4}{\sinh^2 z}(z+k)^2 dz$$
$$\leq 2\int_0^\infty \frac{z^4}{\sinh^2 z}(z^2+k^2) dz = 2\frac{\pi^6}{42} + 2k^2\frac{\pi^4}{30}$$

In order to prove (6.5) we first write:

$$\int_0^\infty \int_0^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 dkdk' = I_1 + 2I_2 + I_4$$
$$I_1 = \int_0^1 \int_0^1 \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 dkdk'$$
$$I_2 = \int_0^1 \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 dkdk'$$
$$I_3 = \int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 dkdk'$$

We notice that,

$$I_3 = \int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 dkdk' \leq 2\left(\int_1^\infty \int_1^\infty \frac{|K(k,k')|^2}{|\Gamma(k)|^2} dkdk' \right)$$

since:

$$\int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 dkdk' = \int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 1_{\Gamma(k)>\Gamma(k')} dkdk' +$$
$$+ \int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 1_{\Gamma(k)<\Gamma(k')} dkdk'$$

$$= \int_1^\infty \int_1^\infty \left| \frac{K(k',k)}{\sqrt{\Gamma(k')\Gamma(k)}} \right|^2 1_{\Gamma(k')>\Gamma(k)} dk dk' +$$

$$+ \int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 1_{\Gamma(k)<\Gamma(k')} dk dk'$$

$$= 2 \int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 1_{\Gamma(k')>\Gamma(k)} dk dk'$$

$$\leq 2 \int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\Gamma(k)} \right|^2 1_{\Gamma(k')>\Gamma(k)} dk dk' \leq 2 \int_1^\infty \int_1^\infty \left| \frac{K(k,k')}{\Gamma(k)} \right|^2 dk dk'$$

and this integral I_3 converges by Lemma 6.1 and (6.4).
We have that

$$\lim_{k,k' \to 0} \frac{\sqrt{\Gamma(k)\Gamma(k')}}{\sqrt{kk'}} = \frac{\pi^4}{15},$$

and

$$\lim_{k,k' \to 0} \frac{K(k,k')}{\sqrt{kk'}} = 0,$$

we have

$$\lim_{k,k' \to 0} \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} = 0.$$

Therefore $\frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}}$ is continuous on $[0,1] \times [0,1]$ and the first integral is then convergent. Finally let us estimate I_2. We first notice that for all $k > 0$, $\Gamma(k) > 0$ and then, by the continuity of Γ on $[0, \infty)$ and (6.2), there exists a positive constant $C > 0$ such that

$$\Gamma(k) \geq C > 0, \;\; \forall k \geq 1.$$

Therefore

$$\begin{aligned} \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k')}} \right|^2 dk \;\; &\leq \;\; \frac{1}{C} \int_1^\infty \left| K(k,k') \right|^2 dk \\ &\leq \;\; \frac{1}{C} \left(8|B_4|\pi^4 k^4 + 8|B_6|\pi^6 k^2 \right) \end{aligned}$$

from where we deduce, for some positive constant C':

$$I_2 = \int_0^1 \int_1^\infty \left| \frac{K(k,k')}{\sqrt{\Gamma(k)\Gamma(k')}} \right|^2 dk dk' \leq C' \int_0^1 \frac{k^2}{\Gamma(k)} dk$$

and this integral converges by (6.1). ■

Finally, the following elementary estimate is used in the proof of Lemma 4.1 and Lemma 4.2.

Lemma 6.3 *For all $t > 0$, $\theta \geq 0$, $\rho > 0$, define*

$$Z(t, \theta, \rho) = \int_0^t (s+1)^{-\theta} e^{\rho s} ds.$$

Then, for all $\theta > 0$:

$$Z(t, \theta, \rho) \leq [2^\theta (t+1)^{-\theta} + 3e^{-\rho t/3}] \frac{e^{\rho t}}{\rho}. \tag{6.6}$$

Proof We define

$$S(t, \theta, \rho) = Z(t, \theta, \rho) \rho e^{-\rho t}.$$

and split $S(t)$ into two parts $S(t, \theta, \rho) = S_1(t, \theta, \rho) + S_2(t, \theta, \rho)$:

$$S_1(t, \theta, \rho) = \int_{t/2}^t \rho (s+1)^{-\theta} e^{\rho(s-t)} ds,$$

$$S_2(t, \theta, \rho) = \int_0^{t/2} \rho (s+1)^{-\theta} e^{\rho(s-t)} ds.$$

In the first integral, we have

$$\begin{aligned} S_1(t, \theta, \rho) &= \int_{t/2}^t \rho (s+1)^{-\theta} e^{\rho(s-t)} ds \leq \int_{t/2}^t \rho (t/2+1)^{-\theta} e^{\rho(s-t)} ds \\ &\leq (t/2+1)^{-\theta} \int_{t/2}^t \rho e^{\rho(s-t)} ds \leq (t/2+1)^{-\theta} (1 - e^{-\rho t/2}) \\ &\leq (t/2+1)^{-\theta} \leq 2^\theta (t+1)^{-\theta}. \end{aligned}$$

For the second integral we notice that, since $s \in (0, t/2)$ we have $s - t < -(s+t)/3$ and then

$$\begin{aligned} S_2(t, \theta, \rho) &= \int_0^{t/2} \rho (s+1)^{-\theta} e^{\rho(s-t)} ds \leq \int_0^{t/2} \rho (s+1)^{-\theta} e^{-\rho s/3} e^{-\rho t/3} ds \\ &\leq e^{-\rho t/3} \int_0^{t/2} \rho e^{-\rho s/3} ds \leq 3e^{-\rho t/3}. \end{aligned}$$

■

6.2 The measure $\mathcal{U}$ and the function M.

The following expressions for $\mathcal{U}(p,p')$ and $M(p)$ have been obtained in [11]:

$$
\begin{aligned}
\mathcal{U}(p,p') = 2\left|\mathcal{M}(p,p',p-p')\right|^2 \delta(\omega(p)-\omega(p')-\omega(p-p')) \times \\
\times n_0(\omega(p))(1+n_0(\omega(p')))(1+n_0(\omega(p)-\omega(p'))) + \\
+2\left|\mathcal{M}(p',p,p'-p)\right|^2 \delta(\omega(p')-\omega(p)-\omega(p'-p)) \times \\
\times n_0(\omega(p'))(1+n_0(\omega(p)))(1+n_0(\omega(p')-\omega(p))) - \\
-2\left|\mathcal{M}(p'+p,p,p')\right|^2 \delta(\omega(p)+\omega(p')-\omega(p+p')) \times \\
\times(1+n_0(\omega(p)))(1+n_0(\omega(p')))n_0(\omega(p)+\omega(p')).
\end{aligned}
\tag{6.7}
$$

$$
M(p) = \frac{1}{\omega(p)}\int_{\mathbb{R}^3} \mathcal{U}(p,p')\omega(p')dp' \tag{6.8}
$$

6.2.1 The formal approximation argument.

In the limit $|p|/4mgn_c \to 0$ we have:

$$
\begin{aligned}
&\omega(p) = \left[\frac{gn_c}{m}|p|^2 + \left(\frac{|p|^2}{2m}\right)^2\right]^{1/2} = c(|p|+\psi(|p|))^{1/2} \\
&0 < \psi(|p|) = o(|p|^3).
\end{aligned}
$$

In order to see how the equation (1.12) may be formally obtained from equation (1.7)–(1.9) we first express the delta measures in $\mathcal{U}(p,p')$ in terms of $r=|p|$, $r'=|p'|$, and the angle $u=\cos\theta_{p,p'}$. We notice first that, given $r>0$ and $r'>0$, we call $u_1(r,r')$ the positive solution u of the equation $\omega(p)-\omega(p')-\omega(p-p')=0$, or equivalently, of the equation:

$$
r = r' + (r^2 + r'^2 - 2rr'u)^{1/2}
$$

We may then express:

$$
\delta(\omega(p)-\omega(p')-\omega(p-p')) = F_1(r,r')\delta(u-u_1(r,r'))
$$

with

$$
\begin{aligned}
F_1(r,r') &= \frac{-1}{\frac{\partial h}{\partial u}(r,r',u_1(r,r'))} \\
h(r,r',u) &= \omega(p-p')
\end{aligned}
$$

An asymptotic expression may be obtained for $u_1(r, r')$ in the limit $r \to 0$ and $r' \to 0$ as follows (cf. [2]):

$$u_1(r, r') = 1 - \frac{r - r'}{rr'}(\psi(r) - \psi(r') - \psi(r - r')) + \\ +\mathcal{O}\left(\psi(r)^2 + \psi(r')^2 + 2\psi(r)\psi(r')\right) \quad \text{as } r \to 0, r' \to 0.$$

Similar arguments yield:

$$\delta(\omega(p') - \omega(p) - \omega(p' - p)) = F_2(r, r')\delta(u - u_2(r, r'))$$
$$\delta(\omega(p + p') - \omega(p) - \omega(p')) = F_3(r, r')\delta(u - u_3(r, r'))$$

In the limit considered in this article we are approximating $\omega(p)$ as $c|p|$. Therefore, the angles between the vectors p, p' and $p - p'$ involved in the collisions must all be equal to one. This corresponds to the approximation:

$$u_i(r, r') = 1 \quad i = 1, 2, 3.$$

The measure $\mathcal{U}(p, p')$ is then approximated as:

$$\mathcal{U}(p, p') \approx W(p, p') \equiv G(r, r')\delta(u - 1) \tag{6.9}$$
$$G(r, r') = \frac{9c}{32\pi^2 m n_c}\left[|rr'(r - r')|^2 F_1(r, r') \times \right. \\ \times n_0(r)(1 + n_0(r'))(1 + n_0(r - r')) + \\ +|rr'(r - r')|^2 F_2(r, r') \times \\ \times n_0(r')(1 + n_0(r))(1 + n_0(r' - r)) - \\ -|rr'(r + r')|^2 F_3(r, r') \times \\ \left. \times(1 + n_0(r))(1 + n_0(r'))n_0(r + r')\right]. \tag{6.10}$$

Using the rotational invariance of $W(p, p')$ we may write its expansion in terms of Legendre's polynomials:

$$W(p, p') = \sum_{\ell=0}^{\infty} W_\ell(r, r')P_\ell(\cos\theta(p, p'))$$

where $r = |p|$, $r' = |p'|$, P_ℓ is the Legendre polynomial of degree ℓ and $\theta(p, p')$ is the angle between p and p' and

$$W_\ell(r, r') = \frac{2\ell + 1}{2}\int_{-1}^{1} W P_\ell(u)du, \quad u = \cos\theta(p, p').$$

It follows that, with some abuse of notation:

$$M(p) \equiv M(|p|) = M(r) = \frac{1}{\omega(cr)} \int_0^\infty W_0(r, r')\omega(cr')r'^2 dr', \qquad (6.11)$$

where we recall that $r = |p|$. On the other hand the function $W_0(r, r')$ is given by

$$\begin{aligned} W_0(r, r') = & \frac{9(r-r')^2 H(r-r')}{32\pi^2 n} n_0(r)(1+n_0(r'))(1+n_0(r-r')) + \\ & + \frac{9(r'-r)^2 H(r'-r)}{32\pi^2 n} n_0(r')(1+n_0(r))(1+n_0(r'-r)) - \\ & - \frac{9(r+r')^2}{32\pi^2 n} n_0(r+r')(1+n_0(r))(1+n_0(r')). \qquad (6.12) \end{aligned}$$

where, we denote $n_0(r) = n_0(p)$, and $H(r)$ is the Heaviside function (see [11]).

Proposition 6.1 *Let $M(p)$ be the function defined in (6.11). Then,*

$$M(p) \equiv M(r) = \Gamma(k)\, n_0(r)(1+n_0(r)) \qquad (6.13)$$

where $r = |p|$ and $k = cr/2k_B T$. Moreover:

$$\lim_{r\to 0} \frac{M(r)\sinh^2 k}{k} = \frac{\pi^4}{60} \qquad (6.14)$$

$$\lim_{r\to\infty} \frac{M(r)\sinh^2 k}{k^5} = \frac{1}{60}. \qquad (6.15)$$

Proof The function f satisfies the equation (1.30). Using that, by (1.29), $f(t, k) = (k/\sinh k)\Omega(t, p)$ and that $\Omega(t, p)$ satisfies equation (1.12), identity (6.13) follows. Properties (6.14) and (6.15) are then consequence of Lemma 6.1. ■

6.3 Legendre's polynomials.

We recall that the Legendre's polynomial of degree $n \in \mathbb{N}$ is defined as:

$$P_n(x) = \frac{1}{2}^n n! \frac{d^n}{dx^n}\left[\left(x^2-1\right)^2\right], \quad n = 0, 1, 2, \cdots \qquad (6.16)$$

These polynomials form a complete orthogonal set of functions in $L^2(-1, 1)$ such that:

$$\int_{-1}^1 P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}. \qquad (6.17)$$

The following property of the Legendre's polynomials is useful to obtain fornula (1.25):

$$P_\ell(u \cdot u') = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(u) Y^*_{\ell m}(u') \tag{6.18}$$

for all $u \in \mathbb{S}^2, u' \in \mathbb{S}^2$.

Acknowledgements. The work of M. E. has been supported by Grant 2011-29306-C02-00, MICINN, Spain and Basque Government Grant IT641-13. The second author has been supported by Grant MTM2011-29306-C02-00, MICINN, Spain, ERC Advanced Grant FP7-246775 NUMERIWAVES, and Grant PI2010-04 of the Basque Government. Both authors are supported by SEV-2013-0323, MINECO, Spain. T. M. B. would like to thank his advisor, Professor Enrique Zuazua for fruitful discussions on this work. Both authors acknowledge enlightening conversations with Pr. M. A. Valle.

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