Control of a hysteresis model subject to random failures

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Abstract. The note presents conditions to assure weak stability in the mean for a hysteresis Bouc-Wen model controlled by a proportional-integral controller subject to random failures. When a failure happens, the controller turns off and remains off for a while. After that the controller turns on and keeps on until the occurrence of the next failure. The failures occur according to a Poisson distributed process. A numerical example illustrates the result.

Keywords: Hysteresis, Bouc-Wen model, stochastic system, stochastic control

1 Introduction

Hysteresis is a nonlinear phenomenon encountered in a wide variety of processes in which the input-output dynamic relation between variables involve memory effects. An important model able to account hysteresis is called the Bouc-Wen model, a topic of intensive investigation in the recent years [1], [2], [3], [4], [5], [6], [7]. The survey paper [8] and the monograph [9] present in a unified and detailed way the most important results dedicated for such hysteresis model.

The normalized version of the Bouc-Wen model introduced in [1] (see also [4] and [9]) relates the single-output $\phi(x)(t)$ to the single-input $x(t)$ in the following way:

$$\Phi_{BW}(x)(t) = k_x x(t) + k_w w(t), \quad \forall t \geq 0, \tag{1}$$

and

$$\dot{w}(t) = \rho(\dot{x}(t) - \sigma|\dot{x}(t)||w(t)|^{n-1}w(t) - (\sigma - 1)\dot{x}(t)|w(t)|^n), \tag{2}$$

where the parameters are $k_x > 0$, $k_w > 0$, $\rho > 0$, $\sigma \geq 1/2$, and $n \geq 1$ In this single-input single-output relation, it is assumed that both $x(t)$ and $\phi(x)(t)$ are accessible to measurements but the internal nonlinear state, $w(t)$, is hidden and can not be measured.

Controlling Bouc-Wen models is a topic of interest [10], [11], [12], and the Proportional Integrative (PI) controller shows to be appropriate to handle such models as shown in [6]. A limitation of the results derived in [6] is that they are not appropriate to deal with the Bouc-Wen model subject to random elements. In contrary, our approach considers the PI controller subject to random failures. This approach is useful, for instance, to represent the case in which the source of energy supplying the controller fails. In this random context, our contribution is to derive conditions to assure stability of the Bouc-Wen model with PI controller subject to random failures.

The main contribution of this paper is to present conditions for a weak stability in the mean concept for the Bouc-Wen model. We assume that the PI controller is subject to random failures. The failures follow a stochastic process with Poisson distribution. At the instant of occurrence of a failure, the PI controller turns off and keeps in this situation for a while. After that, it is allowed to turn on. Under this random on-off behavior, we show that the resulting Bouc-Wen model system is stable in a weak sense. This sets the main contribution of this paper.

2 Basic definitions and main result

The Frobenius norm is denoted by $\|\cdot\|$, and the absolute value is denoted by $|\cdot|$. The symbol $\mathbb{1}_{\{\cdot\}}$ stands for the Dirac measure. We use $\mathsf{Re}(\cdot)$ to represent the real part of a complex number. When A is a square matrix, we let

$$\mathsf{Re}(A) := \{\max(\mathsf{Re}(\lambda_i)), i = 1, \ldots, n : \lambda_i \text{ is an eigenvalue of } A\}.$$

The scheme of the PI controller associated with the Bouc-Wen model is depicted in Fig. 1. The instant times

$$0 < t_1 < t_2 < \cdots < t_k < \cdots$$

denote the time of occurrence of a random failure. When a failure happens, a command to the controller to turn off is emitted and the two constants k_P and k_I vanish to zero. The system remains with the PI controller off during a certain period of time, say $\mu > 0$, in order to guarantee a certain degree of stability for the system. In applications, μ can be a variable chosen for safety requirements and its role in this investigation will be exploited in the sequence. After that off waiting time, the PI controller turns on and this happens precisely at $s_k = t_k + \mu$, see Fig. 2 for an illustration. Notice that s_k and t_k mark the instants for which the PI controller turns its status to 'on' and 'off', respectively. The time interval for which the PI controller keeps on is random, i.e., $t_{k+1} - s_k$ is a random variable. The next assumption sets this property.

(A.1) Assumption: The process $\{t_k\}$ governing the failures of the PI controller follows the arrival times of a homogeneous Poisson process with rate $\lambda > 0$.

Remark 1. After the occurrence of a failure at t_k, the PI controller turns off and keeps in this situation in the time interval $[t_k, s_k)$. No other failure is allowed to happen when the controller is off. Since μ is a deterministic value chosen a

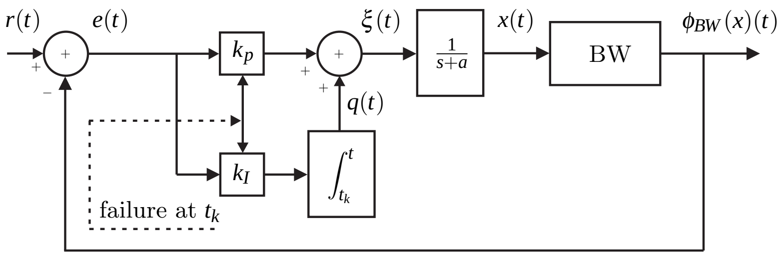

Fig. 1. Scheme diagram representing the Bouc-Wen model with Proportional Integrative controller subject to random failures.

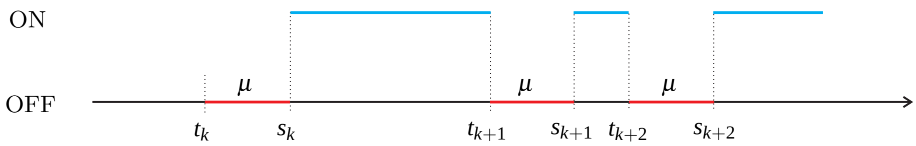

Fig. 2. Status of the PI controller. The controller remains 'off' from t_k to s_k and 'on' from s_k to t_{k+1}.

priori and the controller turns on at s_k, with $s_k = t_k + \mu$, the next failure t_{k+1} may happen at any random instant after s_k. Assumption (A.1) then implies that the inter-arrival times $\delta_k := t_{k+1} - s_k$, $k \geq 0$, are i.i.d. with exponential probability distribution [13, p. 202]

$$\Pr[\delta_k = t] = \lambda e^{-\lambda t}, \quad \forall t \geq 0, \, \forall k \geq 0.$$

With the Dirac function $\mathbb{1}_{t \in [s_k, t_{k+1})}$ indicating that the controller is on (one) when t lies within the interval $[s_k, t_{k+1})$ and off (zero) otherwise, we can define the PI parameters as

$$k_P(t) = \mathbb{1}_{t \in [s_k, t_{k+1})} k_P \quad \text{and} \quad k_I(t) = \mathbb{1}_{t \in [s_k, t_{k+1})} k_I, \quad \forall t \geq 0,$$

where k_P and k_I are fixed constants.

According to the scheme shown in Fig. 1, we can write the PI controller equations as

$$\xi(t) = k_P(t) e(t) + q(t), \tag{3}$$

$$q(t) = k_I(t) \int_{t_k}^{t} e(\tau) d\tau, \tag{4}$$

$$e(t) = r(t) - \phi_{BW}(x)(t), \quad \forall t \geq 0. \tag{5}$$

Notice that $x(t)$ satisfies the relation

$$\dot{x}(t) + a x(t) = \xi(t), \quad \forall t \geq 0, \tag{6}$$

where $a > 0$ is a given constant.

Remark 2. Even though the signal $\xi(t)$ presents infinitely many discontinuities when $\lambda > 0$, the solution $x(t)$ from the differential equation (6) is continuous.

At this point, after characterizing the control setup, we present the stability concept investigated in this paper.

Definition 1. *We say the stochastic nonlinear Bouc-Wen model in* (1)–(6) *is weakly stable in the mean if there exists a sequence of time instants* $\{t_k\}$ *and a constant* $c > 0$ *(which does not depend on* $\{t_k\}$*) such that*

$$|\mathrm{E}[\Phi_{BW}(x)(t_k)]| \leq c, \quad \forall k > 0.$$

The next matrix is useful to characterize the main result of this paper:

$$A = \begin{bmatrix} -(a+k_P k_x) & 1 \\ -k_I k_x & 0 \end{bmatrix}. \tag{7}$$

Now we are in position to present the main result of this paper.

Theorem 1. *Let* $\{t_k\}$ *be a stochastic process representing the occurrence of failures. Then the stochastic nonlinear Bouc-Wen model in* (1)–(6) *is weakly stable in the mean if and only if* $\mathrm{Re}(A) - \lambda < 0$.

The proof of Theorem 1 is shown in the sequence.

Remark 3. The stability condition $\mathrm{Re}(A) - \lambda < 0$ in Theorem 1 can be determined through the analysis of the characteristic equation

$$p(s) = s^2 + (a + k_I k_x + 2\lambda)s + \lambda(a + k_p k_x) + k_I k_x = 0.$$

According to the Routh-Hurwitz condition [14], the roots of $p(s)$ have negative real parts if and only if

$$a + k_I k_x + 2\lambda > 0 \quad \text{and} \quad \lambda(a + k_p k_x) + k_I k_x > 0 \tag{8}$$

are satisfied. Since the roots of $p(s)$ are also roots of $A - \lambda I$, the condition in (8) is necessary and sufficient for $\mathrm{Re}(A) - \lambda < 0$.

2.1 Proof of Theorem 1

To prove Theorem 1, some preliminary results are necessary.

Proposition 1. *([15, p. 84]). Let* A *be a matrix of dimension* $n \times n$*. Then there exists a similarity transformation matrix* Z *such that*

$$A = Z J_A Z^{-1},$$

where J_A *is the corresponding Jordan form. Moreover, for any scalar* c*, there holds*

$$\exp(cA) = Z \exp(cJ_A) Z^{-1}.$$

Lemma 1. *Let A be a square matrix of dimension two.*

(i) The square matrix $M_1 = \mathrm{E}[\exp(A\delta_k)]$, $\forall k \geq 0$, exists if and only if $\mathrm{Re}(A) - \lambda < 0$.

(ii) Let h be a two dimensional bounded continuous-time signal. The square matrix

$$M_2 = \mathrm{E}\left[\int_0^{\delta_k} \exp(A(\delta_k - \tau))h(\tau)d\tau\right] \tag{9}$$

exists only if $\mathrm{Re}(A) - \lambda < 0$.

Proof. The proof of Lemma 1 is in Appendix.

Proposition 2. *([1]). If the input signal $x(t)$ is bounded and continuous, then*

$$\sup_{t\geq 0} |w(t)| \leq \max\{w(0), 1\}. \tag{10}$$

Proof of Theorem 1 (continued)

The proof follows by an induction argument on k and is divided into two parts.

Case 1. Controller is on:
In this case t belongs to the interval $[s_k, t_{k+1})$, which results in $k_P(t) = k_P$ and $k_I(t) = k_I$. Substituting (3) into (6), we have

$$\dot{x}(t) = -ax(t) + q(t) + k_P e(t), \quad \forall t \in [s_k, t_{k+1}). \tag{11}$$

It follows from (4) that $\dot{q}(t) = k_I e(t)$ when $t \in [s_k, t_{k+1})$, which together with (1) and (5) allows us to write

$$\dot{q}(t) = k_I(r(t) - k_x x(t) - k_w w(t)), \quad \forall t \in [s_k, t_{k+1}). \tag{12}$$

On the other hand, by applying (1) and (5) in (11), we get

$$\dot{x}(t) = -ax(t) + q(t) + k_P(r(t) - k_x x(t) - k_w w(t)), \quad \forall t \in [s_k, t_{k+1}). \tag{13}$$

By stacking the differential equations (12) and (13), we have

$$\dot{Y}(t) = AY(t) + h(t), \quad \forall t \in [s_k, t_{k+1}) \tag{14}$$

where

$$Y(t) = \begin{bmatrix} x(t) \\ q(t) \end{bmatrix}, \quad A = \begin{bmatrix} -(a + k_P k_x) & 1 \\ -k_I k_x & 0 \end{bmatrix},$$

and

$$h(t) = \begin{bmatrix} k_P \\ k_I \end{bmatrix} r(t) + \begin{bmatrix} -k_P k_w \\ -k_I k_w \end{bmatrix} w(t).$$

The solution of (14) is given by

$$Y(t) = \exp(A(t-s_k))Y(s_k) + \int_{s_k}^{t} \exp(A(t-\tau))h(\tau)d\tau. \tag{15}$$

Taking $t \uparrow t_{k+1}$ in (15), and recalling that $\delta_k = t_{k+1} - s_k$, we have

$$Y(t_{k+1}) = \exp(A\delta_k)Y(s_k) + \int_{0}^{\delta_k} \exp(A(\delta_k-\tau))h(\tau - s_k)d\tau. \tag{16}$$

Passing the expected value operator on both sides of (16) yields

$$\begin{aligned} \mathrm{E}[Y(t_{k+1})] &= \mathrm{E}[\exp(A\delta_k)Y(s_k)] + \mathrm{E}\left[\int_{0}^{\delta_k} \exp(A(\delta_k-\tau))h(\tau - s_k)d\tau\right] \\ &= \mathrm{E}[\exp(A\delta_k)]\mathrm{E}[Y(s_k)] + \mathrm{E}\left[\int_{0}^{\delta_k} \exp(A(\delta_k-\tau))h(\tau - s_k)d\tau\right] \end{aligned} \tag{17}$$

where the last equality follows from the i.i.d property of the Poisson process. Lemma 1 allows us to get from (17) that

$$|\mathrm{E}[Y(t_{k+1})]| \leq \|M_1\| |\mathrm{E}[Y(s_k)]| + \|M_2\|, \quad \forall k \geq 0, \tag{18}$$

where M_1 and M_2 are matrices satisfying Lemma 1.

Notice from the definition of the vector $Y(t)$ that

$$\mathrm{E}[Y(s_k)] = \lim_{t\uparrow s_k} \mathrm{E}\begin{bmatrix} x(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \mathrm{E}[x(s_k)] \\ 0 \end{bmatrix}. \tag{19}$$

We now introduce the induction argument. Let $k = n = 0$ in (18) and (19) to get that

$$|\mathrm{E}[Y(t_1)]| \leq \|M_1\| + \|M_2\|,$$

since $|x(s_0)| = |x(0)| = 0$.

Case 2. Controller is off:

We now show that

$$|\mathrm{E}[x(s_k)]| \leq 1, \quad \forall k > 0. \tag{20}$$

When the controller is off, t belongs to the interval $[t_k, s_k)$ and this results in $k_P(t) = k_I(t) = 0$. The equations (3)-(6) guarantee that $q(t) = \xi(t) = 0$ and so $\dot{x}(t) + ax(t) = 0$ whenever $t \in [t_k, s_k)$. The solution of this autonomous system is given by

$$x(t) = x(t_k)\exp(-a(t-t_k)), \quad \forall t \in [t_k, s_k). \tag{21}$$

Let us assume for the moment that $|\mathrm{E}[x(t_k)]|$ is bounded above by a constant $c_1 := \max(1, \|M_1\| + \|M_2\|)$ which does not depend on k, where M_1 and M_2 are the matrices as defined in Lemma 1. This assumption applied in (21) yields

$$|\mathrm{E}[x(t)]| = |\mathrm{E}[x(t_k)]| \exp(-a(t-t_k)) \leq c_1 \exp(-a(t-t_k)), \tag{22}$$

when $t \in [t_k, s_k)$.

Recalling that $s_k = t_k + \mu$ for all $k > 0$, we now choose the value of $\mu > 0$ to guarantee the result. Indeed, if we let $\mu > 0$ be large enough such that

$$\mu > \frac{\log(c_1)}{a}, \tag{23}$$

which is equivalent to

$$c_1 \exp(-a\mu) < 1,$$

then one can take $t \uparrow s_k$ in (22) to produce

$$\lim_{t \uparrow s_k} |\mathrm{E}[x(t)]| \leq c_1 \exp(-a\mu) < 1. \tag{24}$$

A direct consequence of (24) is that $|\mathrm{E}[x(s_1)]| \leq 1$, since the assumption $|\mathrm{E}[x(t_k)]| \leq c_1$ is considered valid for $k = n = 1$ according to (19).

Repeating the arguments of Case 1, now with $k = n = 1$, one can show that

$$|\mathrm{E}[Y(t_2)]| \leq \|M_1\| + \|M_2\|.$$

And taking this inequality in Case 2 with $k = n = 2$ one gets that $|\mathrm{E}[x(s_2)]| \leq 1$. Proceeding similarly, one can conclude that

$$|\mathrm{E}[Y(t_k)]| \leq \|M_1\| + \|M_2\|, \quad \forall k > 0. \tag{25}$$

Taking the expected value operator on both sides of (1), we get

$$\mathrm{E}[\Phi_{BW}(x)(t)] = k_x \mathrm{E}[x(t)] + k_w \mathrm{E}[w(t)], \quad \forall t \geq 0. \tag{26}$$

Combining Proposition 2, which assures that $|w(t)| \leq 1$ for all $t > 0$, (25), and (26), we obtain

$$|\mathrm{E}[\Phi_{BW}(x)(t_k)]| \leq k_x(\|M_1\| + \|M_2\|) + k_w, \quad \forall k \geq 0,$$

which shows the result. □

3 Experimental evaluation

We simulated the stochastic Bouc-Wen model in (1)–(6) with $r(t) = sin(t)$, $t \geq 0$,

$$k_x = 2, \quad k_w = 2, \quad \rho = 2, \quad n = 1.5, \quad \sigma = 1$$

and

$$k_P = 0.9, \quad k_I = 0.9, \quad a = 1, \quad \mu = 1.$$

The exponential distribution to generate the failures was taken with $\lambda = 2$. Figure 3 shows a sample path for the output $\Phi_{BW}(t)$. It can be seen that after the occurrence of a failure, the system returns to track the desired reference signal.

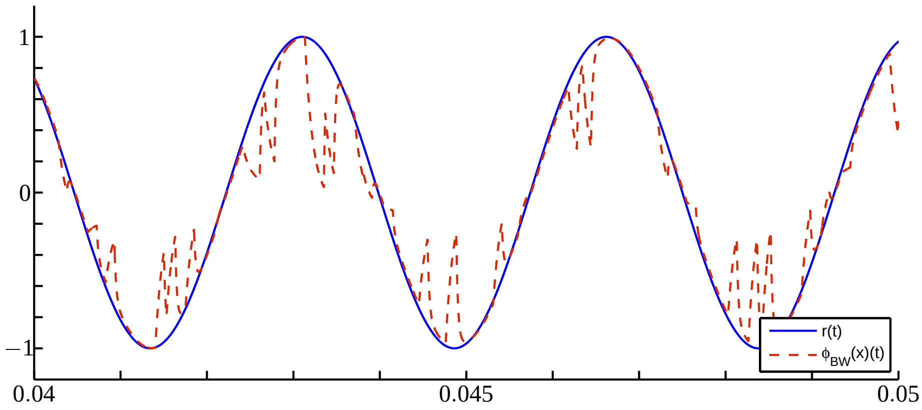

Fig. 3. Response of the Bouc-Wen model with PI controller for a sample path. The continuous line represents the reference $r(t)$ and the dotted one represents the output of the Bouc-Wen model $\Phi_{BW}(x)(t)$.

4 Concluding remarks

The paper presents necessary and sufficient conditions to guarantee weak stability in mean for the Bouc-Wen model. The conditions basically rely on the analysis of eigenvalues of a two dimensional matrix. Further investigation is under progress to convert the weak stability, valid for a time subsequence $\{t_k\}$ on the real line, into a strong one valid for all times $t > 0$.

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Appendix: Proof of Lemma 1

[Proof of (i)]: Recalling that the inter-arrival process $\delta_k = t_{k+1} - s_k$, $k > 0$, obeys a Poisson process, we can write (see Remark 1)

$$\begin{aligned} M_1 = \mathrm{E}[\exp(A\delta_k)] &= \int_0^{+\infty} \exp(At)\,\mathrm{Pr}[\delta_k = t]dt \\ &= \lambda \int_0^{+\infty} \exp(At)\exp(-\lambda t)dt. \end{aligned} \tag{27}$$

Using the result of Proposition 1 in (27), we have

$$\begin{aligned} M_1 &= \lambda \int_0^{+\infty} Z\exp(J_A t)Z^{-1}\exp(-\lambda t)dt \\ &= \lambda \int_0^{+\infty} Z\exp\left((J_A - \lambda I)t\right)Z^{-1}dt. \end{aligned} \tag{28}$$

On the other hand, if we denote by σ_1 and σ_2 the two eigenvalues of A, we get that the matrix $J_A - \lambda I$ assumes either the form

$$\begin{bmatrix} \sigma_1 - \lambda & 0 \\ 0 & \sigma_2 - \lambda \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sigma_1 - \lambda & 1 \\ 0 & \sigma_1 - \lambda \end{bmatrix}. \tag{29}$$

It follows from (29) that the exponential matrix $\exp\left((J_A - \lambda I)t\right)$ is identical to either [14, Ch. 3.6]

$$\begin{bmatrix} \exp((\sigma_1 - \lambda)t) & 0 \\ 0 & \exp((\sigma_2 - \lambda)t) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \exp((\sigma_1 - \lambda)t) & t\exp((\sigma_1 - \lambda)t) \\ 0 & \exp((\sigma_1 - \lambda)t) \end{bmatrix}. \tag{30}$$

Notice from (30) that the integral $\int_0^{+\infty} \exp\left((J_A - \lambda I)t\right)dt$ exists if and only if both $\mathrm{Re}(\sigma_1) - \lambda < 0$ and $\mathrm{Re}(\sigma_2) - \lambda < 0$ hold true, which is equivalent to the condition $\mathrm{Re}(A) - \lambda < 0$. This argument completes the proof of (i).

[Proof of (ii)]: A direct evaluation of (9) yields

$$\begin{aligned} M_2 &= \mathrm{E}\left[\int_0^{\delta_k} \exp(A(\delta_k - \tau))h(\tau)d\tau\right] \\ &= \int_0^{+\infty} \left(\int_0^t \exp(A(t-\tau))h(\tau)d\tau\right)\lambda\exp(-\lambda t)dt \\ &= \lambda \int_0^{+\infty} \exp(At)\exp(-\lambda t)\left(\int_0^t \exp(-A\tau)h(\tau)d\tau\right)dt. \end{aligned}$$

Applying Proposition 1 in the above identity, we have

$$\begin{aligned} M_2 &= \lambda \int_0^{+\infty} Z\exp((J_A - \lambda I)t)Z^{-1}\left(\int_0^t Z\exp(-J_A\tau)Z^{-1}h(\tau)d\tau\right)dt \\ &= \lambda \int_0^{+\infty} Z\exp((J_A - \lambda I)t)\left(\int_0^t \exp(-J_A\tau)Z^{-1}h(\tau)d\tau\right)dt. \end{aligned} \tag{31}$$

After some algebraic manipulation, we get that (31) is identical to

$$M_2 = \int_0^{+\infty} \Gamma(t)\left(\int_0^t h(\tau)d\tau\right)dt, \tag{32}$$

where $\Gamma(t)$ is two-dimensional square matrix where all of its entries are bounded from above and below by a term in the form $t\exp(-\rho t)$, $\rho > 0$. Combining this and the assumption that $h(\cdot)$ is bounded, we can conclude from (32) that

$$\|M_2\| \leq \int_0^{+\infty} \|\Gamma(t)\| \left(t \sup_{s\geq 0} \|h(s)\| \right) dt$$
$$\leq \sup_{s\geq 0} \|h(s)\| \int_0^{+\infty} t^2 \exp(-\rho t) dt = \frac{2 \sup_{s\geq 0} \|h(s)\|}{\rho^3},$$

which shows the result. □

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