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ROBUST NULL CONTROLLABILITY FOR HEAT EQUATIONS WITH UNKNOWN SWITCHING CONTROL MODE

Qi Lü

(1)School of Mathematics, Sichuan University,
Chengdu 610064, China.

Enrique Zuazua

(1)BCAM - Basque Center for Applied Mathematics, Mazarredo, 14,
E-48009 Bilbao-Basque Country-Spain.
(2) Ikerbasque. Bizkaia Plaza. E-48011 Bilbao-Basque Country-Spain.

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Abstract. We analyze the null controllability for heat equations in the presence of switching controls. The switching pattern is a priori unknown so that the control has to be designed in a robust manner, based only on the past dynamics, so to fulfill the final control requirement, regardless of what the future dynamics is. We prove that such a robust control strategy actually exists when the switching controllers are located on two non trivial open subsets of the domain where the heat process evolves. Our strategy to construct these robust controls is based on earlier works by Lebeau and Robbiano on the null controllability of the heat equation. It is relevant to emphasize that our result is specific to the heat equation as an extension of a property of finite-dimensional systems that we fully characterize but that it may not hold for wave-like equations.

1. **Introduction.** Let $T>0$ be a given finite time horizon, and $G\subset\mathbb{R}^d$ $(d\in\mathbb{N})$ a given bounded domain with a C^2 boundary ∂G. Let G_1 and G_2 be two nonempty open subsets of G such that $G_1\cap G_2=\emptyset$. Let $\gamma(\cdot):\mathbb{R}\to\{0,1\}$ be a measurable function.

Consider the following controlled heat equation:

$$\left\{\begin{array}{ll} y_t-\Delta y=\big[\gamma\chi_{G_1}+(1-\gamma)\chi_{G_2}\big]u & \text{in } G\times(0,T),\\ y=0 & \text{on } \partial G\times(0,T),\\ y(0)=y_0 & \text{in } G. \end{array}\right. \tag{1}$$

This paper is devoted to study the property of null controllability of system (1) which consists in driving the solution to rest,

$$y(x,T)\equiv 0, \tag{2}$$

by means of a suitable control u.

Here the initial state $y_0\in L^2(G)$. The value of the measurable function γ is assumed to switch between 0 and 1 so that the control activates, alternating, the control sets G_1 or G_2 in a manner so that, in each time t, only one actuator is active. The control is denoted by $u(\cdot)$ and the corresponding solution of the controlled system (1) by $y(\cdot;y_0,\gamma,u)$

We assume that $G_1\cap G_2=\emptyset$ here. In fact, if $G_1\cap G_2\neq\emptyset$, we can choose controls which are only supported on $G_1\cap G_2$. In this case, the switching law $\gamma(\cdot)$ does not affect the mode of control on $G_1\cap G_2$ since, the heat equation being null controllable in any time ([5]) and from any open

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subset of the domain where the equation evolves, and in particular from $G_1 \cap G_2$, the existence of the robust controller we look for is obvious.

Control systems with switching controllers arise in many fields of applications. Most of the existing works focus on designing smart switching control laws (see [15, 18] and the references therein). When the switching strategy γ is predetermined, the controls of minimal L^2-norm, for instance, can be characterized by classical duality arguments. But this leads to controls that depend globally on the switching function γ, i.e., in all its dynamics on the time interval $[0, T]$.

In this paper we address a different issue, relevant in applications: that of building possible strategies of robust control so that the control u at every time instant t is guaranteed to fulfill the control requirement at the final time $t = T$ and this regardless of the possible future evolution of the switching law γ in the time interval $[t, T]$. This is relevant indeed since, in many applications, the future dynamics of the switching law is unknown a priori and, as a result of that, it may depend on uncertain phenomena, external to the system, and therefore the control has to be implemented in a robust manner. Of course this is hard to achieve in practice.

As we shall prove below, this robust control strategy can be built for the heat equation under consideration, while it can not exist for wave like equations. The result is therefore intrinsic to parabolic dynamics. As we shall see, this parabolic result is an extension of a property of robust controllability with respect to the switching function that we can fully characterize in the finite-dimensional setting. In that case the control u can be chosen to be totally independent of the switching function both its past and future values, provided the two control operators have a sufficiently large overlapping in a fashion that we shall explain in Section 2.

In the parabolic setting, to some extent, the control u needs to depend on the switching function γ, since possible variations of γ modify the dynamics of the system. Our first result below shows that the control can not be completely independent of γ. But this fact is compatible with the main result of the paper showing that, at any time instant τ, $u(\tau)$ can be computed, independently of the variations of γ for $t \geq \tau$.

Proposition 1. *If for some $y_0 \in L^2(G)$ and time $T > 0$, we can find a $u(\cdot) \in L^2(0, T; L^2(G_1 \cup G_2))$ which is independent of $\gamma(\cdot)$, such that the corresponding solution $y(\cdot; y_0, \gamma, u)$ fulfills (15) for all γ, then $y_0 = 0$ in $L^2(G)$ and $u = 0$ in $L^2(0, T; L^2(G_1 \cup G_2))$.*

Remark 1. The statement in the proposition is in agreement with intuition. In fact, one may conjecture that for a given y_0, different $\gamma(\cdot)$ should lead to different controls $u(\cdot)$. This is so, for instance, if we are looking for the control u with some weighted (according to the switching function γ) minimal L^2-norm [1] since, then, it is characterized uniquely by an optimality system that depends on γ. But the conclusion is more subtle if we consider all possible controls. For example, let

$$\gamma_1(t) = \begin{cases} 1, & \text{if } t \in \left[0, \frac{T}{2}\right], \\ 0, & \text{if } t \in \left(\frac{T}{2}, T\right], \end{cases} \qquad \gamma_2(t) = \begin{cases} 0, & \text{if } t \in \left[0, \frac{T}{2}\right], \\ 1, & \text{if } t \in \left(\frac{T}{2}, T\right]. \end{cases}$$

Consider the following two controlled heat equations:

$$\begin{cases} z_{1,t} - \Delta z_1 = \gamma_1 \chi_{G_1} u_1 & \text{in } G \times (0, T), \\ z_1 = 0 & \text{on } \partial G \times (0, T), \\ z_1(0) = y_0 & \text{in } G, \end{cases} \tag{3}$$

and

$$\begin{cases} z_{2,t} - \Delta z_2 = \gamma_2 \chi_{G_1} u_2 & \text{in } G \times (0, T), \\ z_2 = 0 & \text{on } \partial G \times (0, T), \\ z_2(0) = y_0 & \text{in } G. \end{cases} \tag{4}$$

Note that in both systems the control is localized in the same set G_1 but is activated in time according to the two different laws γ_1 and γ_2.

Let $u_1 \in L^2(0, \frac{T}{2}; L^2(G_1))$ (*resp.* $u_2 \in L^2(\frac{T}{2}, T; L^2(G_1))$) such that the corresponding solution of system (3) (*resp.* system (4)) satisfies that $z_1(T; y_0, u_1) = 0$(*resp.* $z_2(T; y_0, u_2) = 0$). Let

$$u(\cdot) = \chi_{(0, \frac{T}{2})} \chi_{G_1} u_1(\cdot) + \chi_{(\frac{T}{2}, T)} \chi_{G_1} u_2(\cdot) \tag{5}$$

[1] More precisely, minimal with respect to the norm in $L^2(\frac{1}{\gamma} dt; L^2(G_1)) \cap L^2(\frac{1}{1-\gamma} dt; L^2(G_2))$. Note that, in view of the fact that $\gamma \in \{0, 1\}$, this norm is equivalent to the one in $L^2(I_1; L^2(G_1)) \cap L^2(I_2, L^2(G_2))$, where I_1 and I_2 are respectively the intervals in which $\gamma = 1$ and $\gamma = 0$.

in system (1). In this case, system (1) reads as follows, since the control is fully contained in G_1,

$$\begin{cases} y_t - \Delta y = \gamma \chi_{G_1}\big(\chi_{(0,\frac{T}{2})}u_1(\cdot) + \chi_{(\frac{T}{2},T)}u_2(\cdot)\big) & \text{in } G\times(0,T),\\ y = 0 & \text{on } \partial G\times(0,T),\\ y(0)=y_0 & \text{in } G, \end{cases} \tag{6}$$

since $\chi_{G_2}u \equiv 0$.

If $\gamma(\cdot) = \gamma_1(\cdot)$(*resp.* $\gamma(\cdot) = \gamma_2(\cdot)$), then system (6) is system (3)(*resp.* system (4)). Thus, from the construction of $u_1(\cdot)$ and $u_2(\cdot)$, we know that the solution $y(T; y_0, \gamma, u) = 0$, both for $\gamma(\cdot) \equiv \gamma_1(\cdot)$ and $\gamma(\cdot) \equiv \gamma_2(\cdot)$ with the choice of the control u as in (5).

This shows that, sometimes, the same control, u here, can be valid for two different switching patterns, $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ in this case.

The same construction can be made more complex by dividing the interval $[0,T]$ in an arbitrary finite number of disjoint intervals. In this way one can prove that there are controls that are simultaneously valid for an arbitrarily large finite number of switching patterns. But, according to Proposition 1, the control can never be valid for possible choices of γ since, otherwise, necessarily, $u \equiv 0$, which corresponds to $y_0 \equiv 0$.

Remark 2. We can look upon Proposition 1 from another point of view. Assume we have a control u that fulfills the control requirement $y(T) \equiv 0$ for a given $\gamma_0(\cdot)$. Assume the same control is valid for another switching function γ. Consider the adjoint system

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } G\times(0,T),\\ \varphi = 0 & \text{on } \partial G\times(0,T),\\ \varphi(T) = \varphi_T & \text{in } G, \end{cases} \tag{7}$$

where $\varphi_T \in L^2(G)$.

Multiplying (1) by φ and then (7) by y, integrating by parts and taking into account that $y(T) \equiv 0$ and

$$-\int_G y_0(x)\varphi(x,0)dx = \int_0^T\int_{G_1\cup G_2}(\gamma_0(t)\chi_{G_1} + (1-\gamma_0(t))\chi_{G_2})u(x,t)\varphi(x,t)dxdt,$$

we find that, if the control u successfully controls the system both for γ and γ_0, then for any φ solving (7),

$$\int_0^T\int_{G_1\cup G_2}\big[\big(\gamma(t)-\gamma_0(t)\big)\chi_{G_1} + \big(\gamma_0(t)-\gamma(t)\big)\chi_{G_2}\big]u(x,t)\varphi(x,t)dxdt \equiv 0.$$

In other words, if u is successful for γ_0, the set of γ's that are admissible are those such that $[(\gamma-\gamma_0)\chi_{G_1} + (\gamma_0-\gamma)\chi_{G_2}]\varphi$ is orthogonal to u, for all solution φ of the adjoint system. If the control u is aimed to be valid for all possible values of the switching function γ, then u has to be orthogonal to all functions of the form $[(\gamma-\gamma_0)\chi_{G_1} + (\gamma_0-\gamma)\chi_{G_2}]\varphi$ for all possible switching patterns γ and all solutions φ of the adjoint system. Since, as shown in Proposition 1, the only possible control fulfilling such condition is $u \equiv 0$, this implies that the set $[(\gamma-\gamma_0)\chi_{G_1} + (\gamma_0-\gamma)\chi_{G_2}]\varphi$ is a total subset of $L^2(0,T;L^2(G_1\cup G_2))$. When $G_1 \cap G_2 = \emptyset$, we can conclude that the set of functions of the form $(\gamma-\gamma_0)\chi_{G_1}\varphi(x,t)$ (resp. $(\gamma-\gamma_0)\chi_{G_2}\varphi(x,t)$) where γ is an arbitrary switching function and φ a solution of the adjoint system, is a total subset in $L^2(0,T;L^2(G_1))$ (resp. $L^2(0,T;L^2(G_2))$). Of course, these conclusions have a much more direct and simple proof. We give the details in Subsection 4.1. On the other hand, once we fix γ, the set constituted by all the elements of the form $(\gamma-\gamma_0)\chi_{G_1}\varphi(x,t)$ never be a total subset of $L^2(0,T;L^2(G_1))$, which, in this case, is a subspace. This means such subset will not be dense in $L^2(0,T;L^2(G_1))$. In fact, in this case, let us consider the following controlled heat equation:

$$\begin{cases} y - \Delta y = (\gamma-\gamma_0)\chi_{G_1}v & \text{in } G\times(0,T),\\ y = 0 & \text{on } \partial G\times(0,T),\\ y(0)=y_0 & \text{in } G. \end{cases} \tag{8}$$

Let us define the set V as

$$V \stackrel{\triangle}{=} \big\{v : \text{ The solution to (8) with } y_0 = 0 \text{ satisfies that } y(T) = 0\big\}.$$

Then, it is easy to see that for any $v \in V$, we have that

$$\int_0^T\int_{G_1}(\gamma(t)-\gamma_0(t))v(x,t)\varphi(x,t)dxdt = 0, \text{ for every } \varphi \text{ solves (7)}.$$

Further, it is obvious that V contains infinitely many linearly independent elements.

In this paper, in order to build controls that are robust with respect to the switching pattern γ we adopt the strategy of proof of the null controllability of the heat equation developed in [5] which is based on a spectral decomposition strategy and therefore works only for time-independent heat equations. The control is built in an iterative manner, decomposing the time interval $[0,T]$ into an infinite sequence of subintervals in which an increasing number of Fourier components is controlled so, at the final time $t = T$, to control the whole dynamics and this with a control of finite L^2-norm. This strategy has been further developed in [16] to show that the system can be controlled by acting only of a set of time instants of positive measure, and not necessarily on the whole time interval $[0,T]$ or a non-trivial subinterval of it. This allows to show that, whatever the switching measurable function γ is, the null control u exists. But, we emphasize, the existing results lead to controls that, at each time instant t, depend on the whole trajectory of γ in $[0,T]$. We are however interested on control strategies that are independent of future possible fluctuations of γ.

Proposition 1 shows that the control u needs to depend on the switching function γ. But this does not exclude completely the possibility of making the control to be independent of the future unpredictability of $\gamma(\cdot)$. As we shall see, there is actually a control u such that, at any $t \in [0,T]$, $u(t)$ only depends on $\gamma(s)$ with $s < t$, for a.e. $t \in [0,T]$, that is, we only utilize the past information observed to determine the value of the control at time t. We need no prediction for the future of $\gamma(\cdot)$.

The following result answers positively to this question but the admissible delay vanishes as time approaches the final control time T.

Theorem 1.1. *System (1) is null controllable at any time $T > 0$ with a control strategy that is independent of future fluctuations or uncertainties on the switching control strategy γ. Furthermore, the control can be taken to be piecewise constant in time and allow some degree of delay that vanishes when approaching the final time T.*

More precisely, there is a sequence $\{t_i\}_{i=1}^{\infty}$ with $0 = t_1 < t_2 < \cdots$ and $\lim_{i\to\infty} t_i = T$ so that for every $y_0 \in L^2(G)$, we can find a control $u(\cdot) \in L^\infty(0,T; L^2(G_1 \cup G_2))$, such that $u(t)$ for $t \in (t_{i+1}, t_{i+2})$ $(i \in \mathbb{N})$ only depends on y_0 and $\gamma(s)$ for $s \in [0, t_i]$. Moreover,

$$u(t) = \begin{cases} \text{a function which is independent of } t, & \text{if } t \in (t_{2k-1}, t_{2k}),\ k \in \mathbb{N}, \\ 0, & \text{if } t \in (t_{2k}, t_{2k+1}),\ k \in \mathbb{N}. \end{cases}$$

Furthermore, there exists a constant $L > 0$ such that

$$|u|^2_{L^\infty(0,T;L^2(G_1\cup G_2))} \le L|y_0|^2_{L^2(G)} \tag{9}$$

for all measurable switching functions $\gamma(\cdot) : \mathbb{R} \to \{0,1\}$ and all y_0 in $L^2(G)$.

Remark 3. Note that the constant L in the upper bound on the control is independent of the switching function γ. This is natural since γ is uniformly bounded.

Remark 4. The proof of Theorem 1.1 allows to obtain a similar result of null controllability for a more general class of heat equations with variable but time-independent coefficients and homogeneous Robin boundary conditions. This can be done by combining the techniques of the present paper with the results in [8] and [13], that yield the needed generalizations of the spectral observability inequalities that we recall in Lemma 3.1 and Lemma 3.2.

Remark 5. Actually, in practice it is also relevant to be able to handle some delay on the observed information at hand when determining the value of the control u at a given t. This is in practice often the case because of the technological limitations of the sensor and actuator devices. Thus, not only it is worth to build control strategies so that $u(t)$ only depends on $\gamma(s)$ with $s < t$, but actually on $\gamma(s)$ for $s < t - \tau$ and some $\tau > 0$. Our control serves this aim to a certain extent. Indeed, observe that the control we have obtained at time t depends only on the past information of $\gamma(\cdot)$ on $[0, t_i]$ if $t \in [t_{i+1}, t_{i+2}]$. This implies that our control is robust with respect to the unknown possible future fluctuations of $\gamma(\cdot)$. It also allows some weak time delay between the observed values of the switching function γ and the actuator, of the order of $t_{i+1} - t_i$.

In particular, the control given by Theorem 1.1 can be computed in real time, based only on past measurements of γ. It is thus a control strategy in the spirit of a feedback control.

Remark 6. In this paper, for simplicity, we only consider the case where the control $u(\cdot)$ switches between two open subsets G_1 and G_2. One can also study the case where the control switches within an arbitrary finite number of controllers:

$$\begin{cases} y_t - \Delta y = u \sum_{k=1}^{N} \gamma_k \chi_{G_k} & \text{in } G \times (0,T), \\ y = 0 & \text{on } \partial G \times (0,T), \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{10}$$

Here $G_1, \cdots, G_N$ are a nonempty open subsets of G such that $G_k \cap G_j = \emptyset$ for $1 \le k < j \le N$, and $\gamma_k(\cdot) : \mathbb{R} \to \{0,1\} (k = 1, \cdots, N)$ are measurable functions such that $\sum_{k=1}^{N} \gamma_k(\cdot) = 1$.

One can easily generalize the result in Theorem 1.1 to this more complex case, with a similar proof.

In this paper, in order to present the key idea of our construction in the simplest way, we do not pursue the full technical generality but rather we focus on the particular case of Theorem 1.1.

The null controllability problem for heat equations has been studied extensively in the literature. In [2], by means of the moment method, the boundary null controllability of the heat equation was established in one space dimension. In [14], with the aid of a transmutation method, it was proved that if the wave equation is exactly controllable for some $T > 0$, then the heat equation is null controllable for all time with the same space-support of the control, provided that the coefficients of both equations are independent of time. A similar result was proved in [11] with the aid of Kannai's transform. The internal null controllability for the heat equation in several space dimensions was first proved in [5] using the Fourier decomposition of solutions and an observability estimate for the finite linear combinations of eigenfunctions of the homogeneous Dirichlet Laplacian. The method of proof in [5] was based on a time iterative method, allowing to drive all the Fourier coefficients of the solution to zero at time T. A simplified presentation was given in [6], in which the linear system of thermoelasticity was studied. The null controllability of the multi-dimensional heat equation was proved independently in [4] for a much larger class of heat equations with lower order terms by utilizing global parabolic Carleman inequalities. The result was generalized in [3] for weakly blowing up semilinear heat equations. We refer to [17] for a survey on this topic.

Although the method in [3, 4] can deal with general linear and semilinear heat equations, the method in [5, 6] has its advantages and it is actually the one we employ in the present paper. First, it can be used to establish the null-controllability for the internally controlled heat equation in $G \times [0,T]$ with the control restricted to a product set of an open nonempty subset of G and a subset of positive measure in the interval $[0,T]$(see [16] for example). Second, it can be applied to obtain the null controllability for some fractional order parabolic equations (see [7, 12] for example). Third, it can help us constructing piecewise constant controls with respect to the time variable (see [9] for example).

In this paper, we adopt the method in [5, 6], with some improvements, so to ensure robustness with respect to possible future variations of the switching function, to prove Theorem 1.1.

As a consequence of the previous controllability result, by duality, we get the following observability inequality.

Corollary 1. *There is a constant $C > 0$ such that*

$$\begin{aligned} & |\varphi(0)|_{L^2(G)} \\ \le\ & C\Big[\int_0^T \gamma(t)\Big(\int_{G_1} |\varphi(x,s)|^2 dx\Big)^{\frac{1}{2}} dt + \int_0^T [1-\gamma(t)]\Big(\int_{G_2} |\varphi(x,s)|^2 dx\Big)^{\frac{1}{2}} dt\Big], \end{aligned} \tag{11}$$

for every measurable function $\gamma(\cdot) : [0,T] \to \{0,1\}$ and solution $\varphi(\cdot)$ of the adjoint system (7).

Remark 7. As seen in Section 2, the analogue of (11) holds true for finite-dimensional systems, provided observability is guaranteed for each of the observation operators. However, this inequality may fail to be true for the wave equation. In the later case, roughly, the wave equation is observable if all characteristic rays pass by the observation region and, even in $1-d$, it is easy to see that even when (11) holds with $\gamma \equiv 0$ and $\gamma \equiv 1$, the result may fail for some values of γ. Similar issues are discussed in [10].

Remark 8. The statement (11) is a standard and well-known observability inequality in the case where γ or $1-\gamma$ are strictly positive and bounded below on a subinterval of some minimal length.

The novelty in (11) is that the result holds for all measurable function $\gamma(\cdot) : [0,T] \to \{0,1\}$ and with an observability constant that is independent of γ.

Note however that this uniform observability estimate is a direct consequence of the fact that the control u fulfills the uniform bound (9) but does not reflect the fact that the control is insensitive to possible future variations of the switching function γ.

Getting a complete characterization, by duality, of the fact that control can be taken to be insensitive to future fluctuations of the switching function γ is an interesting open problem.

The proof of Corollary 1 is standard, by means of Theorem 1.1. However, for the sake of completeness, we give it here.

Proof of Corollary 1. We multiply system (1) by φ and integrate the product on $G \times (0,T)$. Integrating by parts, we find that

$$\begin{aligned}
&\int_0^T \int_{G_1} \gamma(t)u(x,t)\varphi(x,t)dxdt + \int_0^T \int_{G_2} \big(1-\gamma(t)\big)u(x,t)\varphi(x,t)dxdt \\
&= \int_0^T \int_G \big(y_t(x,t) - \Delta y(x,t)\big)\varphi(x,t)dxdt \\
&= \int_G y(x,T)\varphi_T(x)dx - \int_G y_0(x)\varphi(x,0)dx - \int_0^T \int_G [\varphi_t(x,t) + \Delta\varphi(x,t)]\varphi(x,t)dxdt \\
&= \int_G y(x,T)\varphi_T(x)dx - \int_G y_0(x)\varphi(x,0)dx.
\end{aligned} \tag{12}$$

Let $y_0 = \varphi(0)$ and u be the control driving the solution $y(\cdot, y_0, \gamma, u)$ to 0 at time $t = T$ and such that $|u|^2_{L^\infty(0,T;L^2(G))} \le L|y_0|^2_{L^2(G)}$. From (12), we find that

$$\begin{aligned}
&\big(|\gamma\varphi|_{L^1(0,T;L^2(G_1))} + |(1-\gamma)\varphi|_{L^1(0,T;L^2(G_2))}\big)\sqrt{L}|\varphi(0)|_{L^2(G)} \\
&\ge \big(|\gamma\varphi|_{L^1(0,T;L^2(G_1))} + |(1-\gamma)\varphi|_{L^1(0,T;L^2(G_2))}\big)|u|_{L^\infty(0,T;L^2(G))} \\
&\ge -\int_0^T \int_{G_1} \gamma(t)u(x,t)\varphi(x,t)dxdt - \int_0^T \int_{G_2} \big(1-\gamma(t)\big)u(x,t)\varphi(x,t)dxdt \\
&= \int_G |\varphi(x,0)|^2 dx,
\end{aligned} \tag{13}$$

which implies inequality (11) immediately. □

The rest of this paper is organized as follows. In Section 2, we discuss the finite-dimensional case giving a necessary condition for the robust controllability with controls independent of the switching function. In Section 3 we show some preliminary results. Section 4 is devoted to the proof of Proposition 1 and Theorem 1.1 and Section 5 to discuss some closely related issues and open problems.

2. **The finite-dimensional case.** Let us analyze the following linear finite-dimensional system with switching control:

$$\begin{cases} \dfrac{dx}{dt} = Ax + \gamma B_1 u_1 + (1-\gamma)B_2 u_2 & \text{in } [0,T], \\ x(0) = x_0. \end{cases} \tag{14}$$

Here A is a $n \times n$ matrix.

To begin with we consider the simplest case in which B_1 and B_2 are $n \times n$ invertible matrices.

The initial state $x_0 \in \mathbb{R}^n$ and the switching function γ belongs to the set of all measurable functions from $[0,T]$ to $\{0,1\}$. The controls u_1 and u_2 belong to $L^2(0,T;\mathbb{R}^m)$.

Under suitable rank conditions on the matrices A, B_1 and B_2, the system (14) is null controllable provided the pair $(A, \tilde{B})$ fulfills the classical rank condition with $\tilde{B}$ being $\tilde{B} = (B_1, B_2)$(see [18]). As we shall see, under further sharp rank conditions, the controls u_1 and u_2 in the system (14) can be chosen to be independent of γ. In fact, in this case, the control can be chosen to be totally independent of γ, both its past and future values, and not only of future values of the switching function as it occurs on parabolic problems.

Let us consider the auxiliary control problem

$$\begin{cases} \dfrac{dx}{dt} = Ax + Cv & \text{in } [0,T], \\ x(0) = x_0. \end{cases} \tag{15}$$

Assume the system (15) is controllable, i. e. that the pair (A, C) fulfills the rank condition. Given the initial datum x_0 then let $v = v(t)$ be a control steering the system (15) to the rest, the target being $x_1 = 0$ just to fix ideas, in time $t = T$. We can for instance consider the control v of minimal $L^2(0,T)$-norm that can be characterized through the adjoint system.

Let us then set, for all $t \in (0,T)$,

$$\begin{cases} u_1(t) \equiv (B_1)^{-1}Cv \\ u_2(t) \equiv (B_2)^{-1}Cv. \end{cases}$$

Of course for this definition to make sense, we need to make sure that the range of C is included both in the range of B_1 and B_2, i. e. that the following condition holds:

$$\text{Range}(C) \subset \text{Range}(B_1) \cap \text{Range}(B_2). \tag{16}$$

Of course, in the particular case under consideration, (16) holds according to the fact that B_1 and B_2 are $n \times n$ invertible matrices.

Then, obviously, the system (14) is also under control with these controls u_1 and u_2 since, actually, their action coincides with that of Cv simply because

$$Cv = \gamma B_1 u_1 + (1-\gamma)B_2 u_2,$$

for this choice of u_1 and u_2.

Note that the controls u_1 and u_2 are completely independent of γ, not only of its future values but also of their past.

The above strategy can be generalized to more general cases where neither B_1 nor B_2 need to be invertible.

Indeed, the following result holds.

Theorem 2.1. *Let B_1 be a $n \times m_1$ matrix and B_2 a $n \times m_2$ matrix, with $m_1, m_2 \in \mathbb{N}$. Then the system (14) is null controllable with controls u_1 and u_2 which are completely independent of γ if and only if the condition (16) holds with a matrix C such that (A, C) fulfills the Kalman rank condition.*

Remark 9. If Range(B_1) and Range(B_2) are orthogonal, then we know that Rank$(C) = 0$, which implies that the only solution to the system (15), which can be driven to the rest is the one with null initial datum. This is the analogous result of Proposition 1 for finite-dimensional system. In [1], the authors analyze a similar issue from the viewpoint of stabilization.

Proof. The "if" part. Let us assume that Rank$(C) = m$. In this case, we can find a $v \in L^2(0,T;\mathbb{R}^m)$ such that the solution to the system (15) enjoys that $x(T) = 0$. Let us define two operators $\widetilde{B}_i (i = 1,2)$ as follows:

$$\begin{cases} \widetilde{B}_1 : \mathbb{R}^{m_i}/\text{Ker}(B_i) \to \text{Range}(C), \\ \widetilde{B}_i \tilde{v} = B_i \tilde{v}. \end{cases}$$

Since Range$(C) \subset$ Range(B_1), we know both $\widetilde{B}_1$ and $\widetilde{B}_2$ are isomorphisms. Put $\tilde{u}_i = (\widetilde{B}_i)^{-1}v (i = 1,2)$, then we know that $\tilde{u}_i \in L^2(0,T;\mathbb{R}^{m_i}/\text{Ker}(B_i)) (i = 1,2)$. For $i = 1,2$, denote by 0_i the zero element in Ker(B_i) and let $u_i(t) = \tilde{u}_i(t) \oplus 0_i$. Then we know that $u_i \in L^2(0,T;\mathbb{R}^{m_i}) (i = 1,2)$. It is easy to check that the solution to the system (1) with these u_1 and u_2 satisfies that $x(T) = 0$. It is also obvious that neither u_1 nor u_2 depends on γ.

The "only if" part. Let u_1 and u_2 be the controls which are completely independent of γ and drive the solution to the system (1) to the rest. Then we have that for any measurable $\gamma : [0,T] \to \{0,1\}$, it holds that

$$0 = e^{AT}x_0 + \int_0^T e^{A(T-t)}\big[\gamma B_1 u_1 + (1-\gamma)B_2 u_2\big]\,dt. \tag{17}$$

For arbitrary measurable set $E \subset [0,T]$, we set $\gamma_1 \equiv 1$ and

$$\gamma_2 = \begin{cases} 0, & t \in E, \\ 1 & t \in [0,T] \setminus E. \end{cases}$$

Then, from (17), we see

$$0 = e^{AT}x_0 + \int_0^T e^{A(T-t)}\big[\gamma_1 B_1 u_1 + (1-\gamma_1)B_2 u_2\big]\,dt$$

and

$$0 = e^{AT}x_0 + \int_0^T e^{A(T-t)}\big[\gamma_2 B_1 u_1 + (1-\gamma_2)B_2 u_2\big]dt.$$

From the above two equalities, we obtain that

$$0 = \int_0^T \chi_E e^{A(T-t)}\big[B_1 u_1 - B_2 u_2\big]dt.$$

Thanks to the fact that E can be any measurable set contains in $[0,T]$, we find that $B_1u_1(t) = B_2u_2(t)$, a.e. $t \in [0,T]$. This means that the system (14) is null controllable with controls that are in $\mathrm{Range}(B_1)\cap\mathrm{Range}(B_2)$. Assume that the dimension of $\mathrm{Range}(B_1)\cap\mathrm{Range}(B_2)$ is m. Then, there is an isomorphism $\widetilde{C} : \mathbb{R}^m \to \mathrm{Range}(B_1)\cap\mathrm{Range}(B_2)$. Denote by C the corresponding matrix of $\widetilde{C}$. Then we know that C is a $n\times m$ matrix. We claim that this C is what we are looking for. Indeed, thanks to that, for any $x_0 \in \mathbb{R}^n$, we can find a $\tilde{v} \in L^2(0,T;\mathrm{Range}(B_1)\cap\mathrm{Range}(B_2))$ such that the solution to the system (14) satisfies that $x(T) = 0$. Hence, we know that there is a $v \in L^2(0,T;\mathbb{R}^m)$ such that $Cv = \tilde{v}$. Therefore, we can conclude that the solution to the system (15) with control v and initial datum x_0 enjoys that $x(T) = 0$. This, together with the fact that x_0 is an arbitrary element in $\mathbb{R}^n$, implies that (A,C) satisfies the Kalman rank condition. □

3. Some preliminaries.

In this section, we present some preliminary results which will be used later.

Let A be an unbounded linear operator on $L^2(G)$ defined as follows:

$$D(A) = H^2(G)\cap H_0^1(G), \quad Az = \Delta z, \quad \forall z \in D(A).$$

Denote by $\{\lambda_i\}_{i=1}^\infty$ the eigenvalues of $-A$ repeated according to their multiplicity, organized as an increasing sequence, and by $\{e_i\}_{i=1}^\infty$ the corresponding eigenfunctions satisfying $|e_i|_{L^2(G)} = 1$ for $i = 1, 2, \cdots$ It is well known that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \cdots$ and that $\{e_i\}_{i=1}^\infty$ constitutes an orthonormal basis of $L^2(G)$.

Let us recall the following result on the observability of eigenfunction clusters.

Lemma 3.1. ([6], [8, Theorem 1.2]) *For all open non-empty subset ω of G there exists a positive constant C such that*

$$\sum_{\lambda_i\le r}|a_i|^2 \le Ce^{C\sqrt{r}}\int_\omega\Big|\sum_{\lambda_i\le r}a_ie_i(x)\Big|^2dx \tag{18}$$

for each finite $r > 0$ and any choice of the coefficients $\{a_i\}_{\lambda_i\le r}$ with $a_i \in \mathbb{C}$.

As an immediate consequence of this, for the sets G_1 and G_2 under consideration, there exist two positive constants C_1 and C_2 such that

$$\sum_{\lambda_i\le r}|a_i|^2 \le C_1e^{C_1\sqrt{r}}\int_{G_1}\Big|\sum_{\lambda_i\le r}a_ie_i(x)\Big|^2dx \tag{19}$$

and that

$$\sum_{\lambda_i\le r}|a_i|^2 \le C_2e^{C_2\sqrt{r}}\int_{G_2}\Big|\sum_{\lambda_i\le r}a_ie_i(x)\Big|^2dx \tag{20}$$

for each finite $r > 0$ and any choice of the coefficients $\{a_i\}_{\lambda_i\le r}$ with $a_i \in \mathbb{C}$.

Remark 10. Lemma 3.1 was first established in [6] for C^∞ domains G and later in [8] for domains of class C^2.

This result can be recast as a quantitative result on the positive definiteness of the mass matrices

$$B_m^{(1)} = \Big(\int_{G_1}e_ie_jdx\Big)_{1\le i,j\le m} \text{ and } B_m^{(2)} = \Big(\int_{G_2}e_ie_jdx\Big)_{1\le i,j\le m}, \quad \text{for } m \in \mathbb{N},$$

with explicit lower bounds on the coercivity constants as $m \to \infty$.

Proposition 2. *Both $B_m^{(1)}$ and $B_m^{(2)}$ are positive definite. Furthermore, there exist two positive constants $C_1, C_2 > 0$ such that for any $b \in \mathbb{R}^m$, it holds that*

$$|(B_m^{(1)})^{-1}b|^2_{\mathbb{R}^m} \le C_1^2e^{2C_1\sqrt{\lambda_m}}|b|^2_{\mathbb{R}^m}$$

and that

$$|(B_m^{(2)})^{-1}b|^2_{\mathbb{R}^m} \le C_2^2e^{2C_2\sqrt{\lambda_m}}|b|^2_{\mathbb{R}^m},$$

for all $b \in \mathbb{R}^m$ and $m \ge 1$.

Remark 11. Proposition 2 was first established in [9] for C^∞ domains G. Although the proof is very similar, we give it here for the sake for completeness.

Proof. From Lemma 3.1, for every $g=(g_1,g_2,\dots,g_m)^T\in\mathbb{R}^m$, it holds that

$$|g|^2_{\mathbb{R}^m}=\sum_{i=1}^m g_i^2\le C_1e^{C_1\sqrt{\lambda_m}}\int_{G_1}\Big|\sum_{i=1}^m g_ie_i\Big|^2dx=C_1e^{C_1\sqrt{\lambda_m}}g^TB_m^{(1)}g.$$

This shows that $B_m^{(1)}$ is a positive definite matrix and

$$|g|^2_{\mathbb{R}^m}\le C_1e^{C_1\sqrt{\lambda_m}}\Big|\sqrt{B_m^{(1)}}g\Big|^2_{\mathbb{R}^m},\quad\text{for all } g\in\mathbb{R}^m.$$

This yields the upper bound for $(B_m^{(1)})^{-1}b$. The same argument applies to $B_m^{(2)}$.
This completes the proof. □

Now we recall the following classical result on the asymptotic behavior of $\{\lambda_i\}_{i=1}^\infty$.

Lemma 3.2. [13, Corollary 1.9]*(Weyl's asymptotic formula) There exists a constant $C_3>0$ such that for every $r>0$, it holds that*

$$\max\{i:\ \lambda_i\le r,\ i\in\mathbb{N}\}\le C_3r^{\frac{d}{2}}. \tag{21}$$

Throughout this paper, C_1 and C_2 stand for the positive constants given by Lemma 3.1, and C_3 represents the positive constant given by Lemma 3.2. Without loss of generality, we assume that $C_1\ge C_2\ge 1$.

Let $0\le t_1<t_2<+\infty$. Consider the following system of controlled ordinary differential equations:

$$\begin{cases} z_t=A_mz+\gamma B_m^{(1)}f_1+(1-\gamma)B_m^{(2)}f_2 & \text{in } [t_1,t_2],\\ z(t_1)=z_0. \end{cases} \tag{22}$$

Here,

$$A_m=\begin{pmatrix} -\lambda_1 & 0 & \dots & 0\\ 0 & -\lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & -\lambda_m\end{pmatrix},$$

$f_1(\cdot)$ and $f_2(\cdot)$ are controls taken from $L^\infty(t_1,t_2;\mathbb{R}^m)$, and $z_0\in\mathbb{R}^m$. We denote by $z(\cdot;z_0,\gamma,f_1,f_2)$ the solution of the equation (22) corresponding to controls $f_1(\cdot)$ and $f_2(\cdot)$ and the switching function γ.

We have the following controllability result for system (22), which plays a key role in the proof of Theorem 1.1.

Proposition 3. *Let $m\in\mathbb{N}$. Then for each $z_0\in\mathbb{R}^m$, the controls $f_1(\cdot)$ and $f_2(\cdot)$ defined by*

$$\begin{cases} f_1(t)\equiv-(B_m^{(1)})^{-1}\Big(\displaystyle\int_{t_1}^{t_2}e^{A_m(t_1-s)}ds\Big)^{-1}z_0, & \textit{for all } t\in(t_1,t_2),\\ f_2(t)\equiv-(B_m^{(2)})^{-1}\Big(\displaystyle\int_{t_1}^{t_2}e^{A_m(t_1-s)}ds\Big)^{-1}z_0, & \textit{for all } t\in(t_1,t_2),\end{cases}$$

which are independent of γ, drive the solution $z(\cdot;z_0,\gamma,f_1,f_2)$ from z_0 at time t_1 to the origin at time t_2. Furthermore, these controls satisfy the estimate that

$$\begin{cases} |f_1|^2_{L^\infty(t_1,t_2;\mathbb{R}^m)}\le C_1^2e^{2C_1\sqrt{\lambda_m}}\Big|\Big(\displaystyle\int_{t_1}^{t_2}e^{A_m(t_1-s)}ds\Big)^{-1}z_0\Big|^2_{\mathbb{R}^m},\\ |f_2|^2_{L^\infty(t_1,t_2;\mathbb{R}^m)}\le C_2^2e^{2C_2\sqrt{\lambda_m}}\Big|\Big(\displaystyle\int_{t_1}^{t_2}e^{A_m(t_1-s)}ds\Big)^{-1}z_0\Big|^2_{\mathbb{R}^m}.\end{cases}$$

Proof. Note, first of all, that, A_m being diagonal, f_1 and f_2 are well defined.

On one hand, we can easily check that

$$\begin{aligned}
&z(t_2; z_0, \gamma, f_1, f_2)\\
&= e^{A_m(t_2-t_1)}z_0 + \int_{t_1}^{t_2} e^{A_m(t_2-s)}\gamma(s)B_m^{(1)}f_1(s)ds\\
&\quad + \int_{t_1}^{t_2} e^{A_m(t_2-s)}(1-\gamma(s))B_m^{(2)}f_2(s)ds\\
&= e^{A_m(t_2-t_1)}z_0 + \int_{t_1}^{t_2} e^{A_m(t_2-s)}\gamma(s)B_m^{(1)}\Big[-(B_m^{(1)})^{-1}\Big(\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1}\Big]z_0 ds\\
&\quad + \int_{t_1}^{t_2} e^{A_m(t_2-s)}(1-\gamma(s))B_m^{(2)}\Big[-(B_m^{(2)})^{-1}\Big(\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1}\Big]z_0 ds\\
&= e^{A_m(t_2-t_1)}z_0 - e^{A_m(t_2-t_1)}\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big(\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1}z_0\\
&= 0.
\end{aligned}$$

On the other hand, by means of Proposition 2, we find that

$$\begin{aligned}
&|f_1|^2_{L^\infty(t_1,t_2;\mathbb{R}^m)}\\
&= \Big|-(B_m^{(1)})^{-1}\Big(\int_0^T e^{A_m(s-t_1)}ds\Big)^{-1}z_0\Big|^2_{\mathbb{R}^m}\\
&\le C_1^2 e^{2C_1\sqrt{\lambda_m}}\Big|\Big(\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1}z_0\Big|^2_{\mathbb{R}^m}.
\end{aligned}$$

By a similar argument, we get that

$$|f_2|^2_{L^\infty(t_1,t_2;\mathbb{R}^m)} \le C_2^2 e^{2C_2\sqrt{\lambda_m}}\Big|\Big(\int_{t_1}^{t_2} e^{A_m(t_1-s)}ds\Big)^{-1}z_0\Big|^2_{\mathbb{R}^m}.$$

Hence, we complete the proof. □

4. Proof of Proposition 1 and Theorem 1.1.

4.1. Proof of Proposition 1.

Proof. We give two alternative proofs.

Proof 1. Assume that y_0 is an element in $L^2(G)$ and $u_T(\cdot) \in L^2(0,T;L^2(G_1 \cup G_2))$ is the corresponding control which is independent of $\gamma(\cdot)$ and drives the solution of system (1) to 0 at time $t = T$. Then we know that

$$\begin{aligned}
0 &= y(T)\\
&= e^{AT}y_0 + \int_0^T e^{A(T-s)}\gamma(s)\chi_{G_1}u_T(s)ds + \int_0^T e^{A(T-s)}\big[1-\gamma(s)\big]\chi_{G_2}u_T(s)ds.
\end{aligned}\tag{23}$$

For each $t \in [0,T]$, let us choose $\gamma^t(\cdot) \in L^\infty(0,T;\mathbb{R})$ such that

$$\gamma^t(s) = \begin{cases} 1, & \text{if } s \in [0,t],\\ 0, & \text{if } s \in (t,T]. \end{cases}$$

Since $u_T(\cdot)$ is independent of $\gamma(\cdot)$, according to equality (23), we find that for every $t \in [0,T]$, it holds that

$$0 = e^{AT}y_0 + \int_0^t e^{A(T-s)}\chi_{G_1}u_T(s)ds + \int_t^T e^{A(T-s)}\chi_{G_2}u_T(s)ds. \tag{24}$$

Combining this identity with the analogue one obtains for $\gamma \equiv 1$, we have

$$\int_t^T e^{A(T-s)}\chi_{G_1}u_T(s)ds = \int_t^T e^{A(T-s)}\chi_{G_2}u_T(s)ds, \quad \text{for all } t \in [0,T].$$

Taking derivatives with respect to t we deduce that

$$e^{A(T-s)}\big[\chi_{G_1}u_T(s) - \chi_{G_2}u_T(s)\big] = 0 \text{ in } L^2(G), \quad \text{for a.e. } s \in [0,T]. \tag{25}$$

The well-known backward uniqueness property of the heat equation yields

$$\chi_{G_1}u_T(s) - \chi_{G_2}u_T(s) = 0 \text{ in } L^2(G), \text{ for a.e. } s \in [0,T].$$

This, together with the fact that $G_1 \cap G_2 = \emptyset$, implies that

$$\chi_{G_1 \cup G_2} u_T(s) = 0 \text{ in } L^2(G), \quad \text{for a.e. } s \in [0,T]. \tag{26}$$

Combining (23) and (26), we get that $e^{AT} y_0 = 0$ in $L^2(G)$, which implies that $y_0 = 0$ in $L^2(G)$. Hence, we get that $u = 0$ in $L^2(0,T; L^2(G_1 \cup G_2))$. This completes the first proof.

Proof 2. It is sufficient to prove this result for the case $\gamma_0 \equiv 0$. For this, let us denote by U the subset of $L^2(0,T; L^2(G_1 \cup G_2))$ such that for each $u \in U$, it holds that

$$\int_0^T \int_{G_1} \gamma u(x,s) \varphi(x,s) dx ds - \int_0^T \int_{G_2} \gamma u(x,s)\varphi(x,s) dx ds = 0,$$

for all φ solves the equation (7) with some final datum $\varphi_T \in L^2(G)$ and every $\gamma : [0,T] \to \{0,1\}$. Now, let us choose

$$\gamma^t(t) = \begin{cases} 1, & \text{if } s \in [0,t], \\ 0, & \text{if } s \in (t,T]. \end{cases}$$

Then, we know that

$$\int_0^t \int_{G_1} u(x,s)\varphi(x,s) dx ds - \int_0^t \int_{G_2} u(x,s) \varphi(x,s) dx ds = 0,$$

which implies that

$$\int_{G_1} u(x,t)\varphi(x,t) dx - \int_{G_2} u(x,t)\varphi(x,t) dx = 0 \text{ for a.e.} t \in [0,T]. \tag{27}$$

On the other hand, by means of the fact that the set

$$\{\varphi(t,\cdot) : \varphi \text{ solves the equation (7) with some initial datum } \varphi_T \in L^2(G)\}$$

is dense in $L^2(G)$, we see that

$$\{\chi_{G_1 \cup G_2} \varphi(s,\cdot) : \varphi \text{ solves the equation (7) with some initial datum} \varphi_T \in L^2(G)\}$$

is dense in $L^2(G_1 \cup G_2)$. This, together with the equality (27), implies that $u(s,\cdot) = 0$ in $L^2(G_1 \cup G_2)$ for a.e. $s \in [0,T]$. Hence, we conclude that $u = 0$ in $L^2(0,T; L^2(G_1 \cup G_2))$. Then, we have proved the desired result.

□

4.2. **Proof of Theorem 1.1.** Before giving the detailed proof, we introduce the main idea briefly. Without loss of generality, we assume that $T \leq 1$.

Let

$$T_k = \begin{cases} 0, & \text{if } k = 1, \\ T \displaystyle\sum_{i=1}^{k-1} 2^{-i}, & \text{if } k > 1, \end{cases} \tag{28}$$

and

$$\widetilde{T}_k = \begin{cases} \dfrac{T}{4}, & \text{if } k = 1, \\ T\Big(\displaystyle\sum_{i=1}^{k-1} 2^{-i} + 2^{-k-1} \Big), & \text{if } k > 1. \end{cases} \tag{29}$$

We define the following sequences of time intervals:

$$I_k = [T_k, \widetilde{T}_k) \tag{30}$$

and

$$J_k = [\widetilde{T}_k, T_{k+1}). \tag{31}$$

We put

$$r_k = \frac{16 C_1^2}{(T_{k+1} - \widetilde{T}_k)^4}, \quad \text{for } k = 1, 2, \cdots \tag{32}$$

Then we know that

$$r_k \to \infty \text{ as } k \to \infty. \tag{33}$$

For each $k \in \mathbb{N}$, let us denote by P_k the orthogonal projection from $L^2(G)$ to $\text{Span}_{\lambda_i \leq r_k}\{e_i\}$. On each interval I_k, we control the heat equation with a control switching from G_1 to G_2 in an unknown mode. By Proposition 3, we can find a control $u^{(k)}(\cdot) \in L^\infty(I_k; L^2(G_1 \cup G_2))$ such that the corresponding solution $y^{(k)}(\cdot)$ to the equation on I_k satisfies

$$P_k\big(y^{(k)}(\widetilde{T}_k)\big) = 0.$$

On every interval J_k, we let the heat equation freely evolve. We start by having the initial datum for the equation on I_1 to be y_0. For the initial datum on I_k, $k = 2, 3, \cdots$, we define it to be the ending value of the solution to the equation on J_{k-1}. The initial datum of the equation on J_k, $k = 1, 2, \cdots$, is given by the ending value of the solution for the equation on I_k. If there is no eigenvalue of $-A$ in $(r_k, r_{k+1}]$, we simply set $u^{(k)}(\cdot) = 0$ on I_k.

Notice that for each $k \in \mathbb{N}$, by Proposition 3, the control $u^{(k)}(\cdot)$ is independent of time t and the value of $\gamma(\cdot)$ in I_k. Further, Proposition 3 provides an estimate for the control $u^{(k)}(\cdot)$. On the other hand, thanks to the energy decay of the heat equation, we can get a suitable $L^2(G)$-norm estimate for the ending value of the solution to the equation on J_k. These two estimates yield that the control

$$u(\cdot) = \sum_{k=1}^{\infty} \chi_{I_k}(\cdot) u^{(k)} \in L^\infty(0, T; L^2(G_1 \cap G_2)),$$

drives the solution of system (1) to 0 at time T.

In order to adopt the above strategy, we need to know the ending values of the solution on every $I_k (k \in \mathbb{N})$. These values cannot be obtained by the initial datum of the solution on every I_k, $k = 1, 2, \cdots$, if we do not know the value of $\gamma(s)$ for $s \in I_k$. Hence, we have to observe them. This is reasonable and necessary according to Proposition 1. Moreover, this is operable since we only need the previous and present state of the system rather than the future of $\gamma(\cdot)$.

Now we give the details of the proof.

Proof of Theorem 1.1. Let

$$m_k = \max\{i : \lambda_i \le r_k,\ i \in \mathbb{N}\}.$$

It follows from Lemma 3.2 that there exists a constant $C_3 > 0$, such that

$$m_k \le C_3 r_k^{\frac{d}{2}}, \quad \text{for } k \in \mathbb{N}. \tag{34}$$

Denote by

$$\begin{cases} \alpha_k = 3e^{-2r_k(\widetilde{T}_k - T_k)} + 3C_1^2 e^{2C_1\sqrt{r_k}} m_k + 3C_2^2 e^{2C_2\sqrt{r_k}} m_k, \\ \beta_k = e^{-2r_k(T_{k+1} - \widetilde{T}_k)}, \\ \theta_{1,k} = C_1^2 e^{2C_1\sqrt{r_k}} \displaystyle\sum_{i=1}^{m_k} \Big(\frac{\lambda_i}{e^{\lambda_i(\widetilde{T}_k - T_k)} - 1}\Big)^2, \\ \theta_{2,k} = C_2^2 e^{2C_2\sqrt{r_k}} \displaystyle\sum_{i=1}^{m_k} \Big(\frac{\lambda_i}{e^{\lambda_i(\widetilde{T}_k - T_k)} - 1}\Big)^2. \end{cases} \tag{35}$$

Consider the following equations:

$$\begin{cases} y_t^{(1)} - \Delta y^{(1)} = \big[\gamma\chi_{G_1} + (1-\gamma)\chi_{G_2}\big] u^{(1)} & \text{in } G \times I_1, \\ y^{(1)} = 0 & \text{on } \partial G \times I_1, \\ y^{(1)}(T_1) = y_0 & \text{in } G, \end{cases} \tag{36}$$

$$\begin{cases} z_t^{(k)} - \Delta z^{(k)} = 0 & \text{in } G \times J_k, \\ z^{(k)} = 0 & \text{on } \partial G \times J_k, \\ z^{(k)}(\widetilde{T}_k) = y^{(k)}(\widetilde{T}_k) & \text{in } G, \end{cases} \tag{37}$$

for $k = 1, 2, \cdots$, and

$$\begin{cases} y_t^{(k)} - \Delta y^{(k)} = \big[\gamma\chi_{G_1} + (1-\gamma)\chi_{G_2}\big] u^{(k)} & \text{in } G \times I_k, \\ y^{(k)} = 0 & \text{on } \partial G \times I_k, \\ y^{(k)}(T_k) = z^{(k-1)}(T_k) & \text{in } G, \end{cases} \tag{38}$$

for $k = 2, 3, \cdots$ We are going to prove that for each $k \ge 1$, there exists a control

$$u^{(k)} = \chi_{G_1} \sum_{i=1}^{m_k} u_{1i}^{(k)} e_i + \chi_{G_2} \sum_{i=1}^{m_k} u_{2i}^{(k)} e_i \in L^\infty(I_k; L^2(G_1 \cap G_2)),$$

where $u_{1i}^{(k)}$ and $u_{2i}^{(k)}$, $i = 1, \cdots, m_k$, are real numbers, such that

$$P_{m_k}\big(y^{(k)}(\widetilde{T}_k)\big) = 0, \tag{39}$$

$$\big|y^{(k)}(\widetilde{T}_k)\big|_{L^2(G)} \le \Big[\alpha_k \prod_{i=1}^{k-1}(\beta_i\alpha_i)\Big]|y_0|^2_{L^2(G)}, \tag{40}$$

and that

$$|u^{(k)}|^2_{L^\infty(I_k;L^2(G))} \le 2\big(\theta_{1,k} + \theta_{2,k}\big)\prod_{i=1}^{k-1}(\beta_i\alpha_i)|y_0|^2_{L^2(G)}. \tag{41}$$

We first do that for $k = 1$. Write $\sum_{i=1}^{\infty} y_{0i}e_i$ for the Fourier expansion of y_0. Put

$$\begin{cases} (u_{11}^{(1)}, \cdots, u_{1m_1}^{(1)})^T = -\big(B_{m_1}^{(1)}\big)^{-1}\Big(\displaystyle\int_{I_1} e^{A_{m_1}(T_1-s)}ds\Big)^{-1}(y_{01}, \cdots, y_{0m_1})^T, \\ (u_{21}^{(1)}, \cdots, u_{2m_1}^{(1)})^T = -\big(B_{m_1}^{(2)}\big)^{-1}\Big(\displaystyle\int_{I_1} e^{A_{m_1}(T_1-s)}ds\Big)^{-1}(y_{01}, \cdots, y_{0m_1})^T. \end{cases} \tag{42}$$

Define a control $u^{(1)}(\cdot) = \chi_{G_1}u_1^{(1)}(\cdot) + \chi_{G_2}u_2^{(1)}(\cdot)$ by setting

$$u_1^{(1)}(\cdot) = \chi_{G_1}\sum_{i=1}^{m_1} u_{1i}^{(1)}e_i \ \text{ and } \ u_2^{(1)}(\cdot) = \chi_{G_2}\sum_{i=1}^{m_1} u_{2i}^{(1)}e_i. \tag{43}$$

Write

$$y^{(1)}(x,t) = \sum_{j=1}^{\infty} y_j^{(1)}(t)e_j, \ \text{ for all } \ t \in I_1.$$

Then, by (43) and (36), we find that $\big(y_1^{(1)}(\cdot), \cdots, y_{m_1}^{(1)}(\cdot)\big)$ is the solution to the following system of ordinary differential equations:

$$\begin{cases} (y_j^{(1)})_t + \lambda_j y_j^{(1)} = \gamma\displaystyle\sum_{i=1}^{m_1} u_{1i}^{(1)}\int_{G_1} e_ie_jdx + (1-\gamma)\sum_{i=1}^{m_1} u_{2i}^{(1)}\int_{G_2} e_ie_jdx \ \text{ over } \ I_1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \ j = 1, 2, \cdots, m_1, \\ y_j^{(1)}(0) = y_{0j}, \ \text{ for all } \ j = 1, 2, \cdots, m_1. \end{cases}$$

Now, by means of Proposition 3, we get that

$$\big(y_1^{(1)}(\widetilde{T}_1), \cdots, y_{m_1}^{(1)}(\widetilde{T}_1)\big) = 0,$$

which implies that

$$P_{m_1}\big(y^{(1)}(x, \widetilde{T}_1)\big) = \sum_{i=1}^{m_1} y_i^{(1)}(\widetilde{T}_1)e_i = 0. \tag{44}$$

Thus, we obtain (39) for $k = 1$. Moreover, it holds that

$$\begin{aligned} \sum_{i=1}^{m_1}(u_{1i}^{(1)})^2 &\le C_1^2 e^{2C_1\sqrt{r_1}}\Big|\Big(\int_{I_1} e^{A_{m_1}(T_1-s)}ds\Big)^{-1}(y_{01}, \cdots, y_{0m_1})^T\Big|^2_{\mathbb{R}^{m_1}} \\ &= C_1^2 e^{2C_1\sqrt{r_1}}\Big|\sum_{i=1}^{m_1}\frac{\lambda_i}{e^{\lambda_i\widetilde{T}_1} - 1}y_{0i}\Big|^2_{\mathbb{R}^{m_1}} \\ &\le \theta_{1,1}\sum_{i=1}^{m_1} y_{0i}^2. \end{aligned} \tag{45}$$

By a similar argument, we can show that

$$\sum_{i=1}^{m_1}(u_{2i}^{(1)})^2 \le \theta_{2,1}\sum_{i=1}^{m_1} y_{0i}^2.$$

Hence, we find that

$$|u^{(1)}|^2_{L^\infty(I_1;L^2(G))} \le 2\big(\theta_{1,1} + \theta_{2,1}\big)|y_0|^2_{L^2(G)},$$

which verifies (41) for $k = 1$.

Denote by $\mathcal{L}(L^2(G);L^2(G))$ the Banach space of all bounded linear operators from $L^2(G)$ to $L^2(G)$. Write

$$\Lambda^{(1)} \triangleq \Big|\Big(I-P_{m_1}\Big)e^{A\widetilde{T}_1}y_0\Big|^2_{L^2(G)},$$

$$\Gamma_1^{(1)} \triangleq \Big|\Big(I-P_{m_1}\Big)\int_{I_1}e^{A(\widetilde{T}_1-s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))}\Big|\sum_{i=1}^{m_1}u_{1i}^{(1)}e_i\Big|^2_{L^2(G)}$$

and

$$\Gamma_2^{(1)} \triangleq \Big|\Big(I-P_{m_1}\Big)\int_{I_1}e^{A(\widetilde{T}_1-s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))}\Big|\sum_{i=1}^{m_1}u_{2i}^{(1)}e_i\Big|^2_{L^2(G)}.$$

From the energy decay of equation (36), we find that

$$\Lambda^{(1)} \le e^{-2\lambda_{m_1+1}\widetilde{T}_1}|y_0|^2_{L^2(G)}. \tag{46}$$

Since

$$\begin{aligned}\Big|\Big(I-P_{m_1}\Big)\int_{I_1}e^{A(\widetilde{T}_1-s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))} &\le \Big(\int_{I_1}e^{-\lambda_{m_1+1}(\widetilde{T}_1-s)}ds\Big)^2\\ &= \frac{1}{(\lambda_{m_1+1})^2}\Big(1-e^{-\lambda_{m_1+1}\widetilde{T}_1}\Big)^2,\end{aligned}$$

it follows from (45), the third inequality in (35) and the fact that $r_1 \le \lambda_{m_1+1}$ that

$$\begin{aligned}\Gamma_1^{(1)} &\le \frac{1}{(\lambda_{m_1+1})^2}\Big(1-e^{-\lambda_{m_1+1}\widetilde{T}_1}\Big)^2\theta_{1,1}\sum_{i=1}^{m_1}y_{0i}^2\\ &\le \frac{1}{r_1^2}C_1^2e^{2C_1\sqrt{r_1}}\frac{m_1}{(\widetilde{T}_1-T_1)^2}\sum_{i=1}^{m_1}y_{0i}^2.\end{aligned}$$

This, together with the choice of r_1(recalling (32) for the definition of r_1) and the fact that $C_1 \ge 1$ and $T \le 1$, implies that

$$\Gamma_1^{(1)} \le \frac{\big(T_2-\widetilde{T}_1\big)^8}{256C_1^4}C_1^2e^{2C_1\sqrt{r_1}}\frac{m_1}{\big(\widetilde{T}_1-T_1\big)^2}\sum_{i=1}^{m_1}y_{0i}^2 \le C_1^2e^{2C_1\sqrt{r_1}}m_1|y_0|^2_{L^2(G)}. \tag{47}$$

Similarly, we see

$$\Gamma_2^{(1)} \le C_2^2e^{2C_2\sqrt{r_1}}m_1\sum_{i=1}^{m_1}y_{0i}^2 \le C_2^2e^{2C_2\sqrt{r_1}}m_1|y_0|^2_{L^2(G)}. \tag{48}$$

It follows from (46)–(48) that

$$\begin{aligned}\big|y^{(1)}(\widetilde{T}_1)\big|^2_{L^2(\Omega)} &= \Big|\Big(I-P_{m_1}\Big)e^{\widetilde{T}_1A}y_0+\Big(I-P_{m_1}\Big)\int_{I_1}e^{A(\widetilde{T}_1-s)}\gamma(s)\chi_{G_1}u_1(s)ds\\ &\quad+\Big(I-P_{m_1}\Big)\int_{I_1}e^{A(\widetilde{T}_1-s)}(1-\gamma(s))\chi_{G_2}u_2(s)ds\Big|^2_{L^2(G)}\\ &\le 3\Lambda^{(1)}+3\Gamma_1^{(1)}+3\Gamma_2^{(1)}\\ &\le 3\Big(e^{-2\lambda_{m_1+1}\widetilde{T}_1}+C_1^2e^{2C_1\sqrt{r_1}}m_1+C_2^2e^{2C_2\sqrt{r_1}}m_1\Big)|y_0|^2_{L^2(G)}\\ &= \alpha_1|y_0|^2_{L^2(G)}.\end{aligned} \tag{49}$$

Therefore, we have (40) for $k=1$.

The next step should be to prove that (39)–(41) hold for $k=n+1$, provided that (39)–(41) hold for $k=n$. However, in order the give a more readable proof, here we also prove that (39)–(41) are true for $k=2$.

Let $z^{(1)}(\cdot)$ be the solution to the freely evolved heat equation:

$$\begin{cases} z_t^{(1)}-\Delta z^{(1)}=0 & \text{in } G\times J_1,\\ z^{(1)}=0 & \text{on } \partial G\times J_1,\\ z^{(1)}(\widetilde{T}_1)=y^{(1)}(\widetilde{T}_1) & \text{in } G.\end{cases} \tag{50}$$

Due to the energy decay of solutions of (50), we have that

$$\begin{aligned}\big|z^{(1)}(T_2)\big|^2_{L^2(G)} &\le e^{-2\lambda_{m_1+1}(T_2-\widetilde{T}_1)}\big|y^{(1)}(\widetilde{T}_1)\big|^2_{L^2(G)}\\ &\le \beta_1\alpha_1|y_0|^2_{L^2(G)}.\end{aligned} \tag{51}$$

Let $y^{(2)}(\cdot)$ be the solution to the following controlled equation:

$$\begin{cases} y^{(2)}_t - \Delta y^{(2)} = \big[\gamma\chi_{G_1} + (1-\gamma)\chi_{G_2}\big]u^{(2)} & \text{in } G\times I_2, \\ y^{(2)} = 0 & \text{on } \partial G\times I_2, \\ y^{(2)}(T_2) = z^{(1)}(T_2) & \text{in } G. \end{cases} \tag{52}$$

Write

$$P_{m_2}\big(z^{(1)}(x,T_2)\big) = \sum_{i=1}^{m_2} z^{(1)}_i e_i, \tag{53}$$

where the coefficients $z^{(1)}_i$, $i=1,\cdots,m_2$, are real numbers and $z^{(1)}_i = 0$ for $i=1,\cdots,m_1$. Put

$$\begin{cases} (u^{(2)}_{11},\cdots,u^{(2)}_{1m_2})^T = -\big(B^{(1)}_{m_2}\big)^{-1}\Big(\displaystyle\int_{I_2} e^{A_{m_2}(T_2-s)}ds\Big)^{-1}(z^{(1)}_1,\cdots,z^{(1)}_{m_2})^T, \\ (u^{(2)}_{11},\cdots,u^{(2)}_{1m_2})^T = -\big(B^{(2)}_{m_2}\big)^{-1}\Big(\displaystyle\int_{I_2} e^{A_{m_2}(T_2-s)}ds\Big)^{-1}(z^{(1)}_1,\cdots,z^{(1)}_{m_2})^T. \end{cases} \tag{54}$$

Define a control $u^{(2)}(\cdot) = \chi_{G_1}u^{(2)}_1(\cdot) + \chi_{G_2}u^{(2)}_2(\cdot)$ by setting

$$u^{(2)}_1(\cdot) = \chi_{G_1}\sum_{i=1}^{m_2} u^{(2)}_{1i}e_i \text{ and } u^{(2)}_2(\cdot) = \chi_{G_2}\sum_{i=1}^{m_2} u^{(2)}_{2i}e_i. \tag{55}$$

Write

$$y^{(2)}(x,t) = \sum_{j=1}^{\infty} y^{(2)}_j(t)e_j, \quad \text{for all } \ t\in I_2.$$

Then, by (52), (54) and (55), we find that $\big(y^{(2)}_1(\cdot),\cdots,y^{(2)}_{m_2}(\cdot)\big)$ solves the following system of ordinary differential equations:

$$\begin{cases} (y^{(2)}_j)_t + \lambda_j y^{(2)}_j = \gamma\displaystyle\sum_{i=1}^{m_2} u^{(2)}_{1i}\int_{G_1} e_ie_j + (1-\gamma)\sum_{i=1}^{m_2} u^{(2)}_{2i}\int_{G_2} e_ie_j \ \text{ over } \ I_2, \\ \hfill \text{for all } \ j=1,2,\cdots,m_2, \\ y^{(2)}_j(T_2) = z^{(1)}_j, \ \text{ for all } \ j=1,2,\cdots,m_2. \end{cases}$$

By Proposition 3, again, we can get that

$$P_{m_2}\big(y^{(2)}(x,\widetilde{T}_2)\big) = \sum_{i=1}^{m_2} y^{(2)}_i(\widetilde{T}_2)e_i = 0, \tag{56}$$

which implies that (39) is true for $k=2$. Moreover, it holds that

$$\begin{aligned} \sum_{i=1}^{m_2}(u^{(2)}_{1i})^2 &\le C_1^2 e^{2C_1\sqrt{r_2}}\Big|\Big(\int_{I_2} e^{A_{m_2}(T_2-s)}ds\Big)^{-1}(z^{(1)}_1,\cdots,z^{(1)}_{m_2})^T\Big|^2_{\mathbb{R}^{m_2}} \\ &= C_1^2 e^{2C_1\sqrt{r_2}}\Big|\sum_{i=1}^{m_2}\frac{\lambda_i}{e^{\lambda_i(\widetilde{T}_2-T_2)}-1}z^{(1)}_i\Big|^2_{\mathbb{R}^{m_2}} \\ &\le \theta_{1,2}\sum_{i=1}^{m_2}(z^{(1)}_i)^2. \end{aligned} \tag{57}$$

This, together with (51), indicates that

$$\sum_{i=1}^{m_2}(u^{(2)}_{1i})^2 \le \theta_{1,2}\alpha_1\beta_1|y_0|^2_{L^2(G)}. \tag{58}$$

By a similar argument, we can see

$$\sum_{i=1}^{m_2}(u^{(2)}_{2i})^2 \le \theta_{2,2}\alpha_1\beta_1|y_0|^2_{L^2(G)}.$$

Hence, we get that

$$|u^{(2)}|^2_{L^\infty(I_2;L^2(G))} \le 2\big(\theta_{1,2}+\theta_{2,2}\big)\alpha_1\beta_1|y_0|^2_{L^2(G)}.$$

This verifies (41) for $k=2$.

Write

$$\Lambda^{(2)} \triangleq \Big|\Big(I-P_{m_2}\Big)e^{A(\widetilde{T}_2-T_2)}y^{(1)}(T_2)\Big|^2_{L^2(G)},$$

$$\Gamma_1^{(2)} = \Big|\Big(I - P_{m_2}\Big)\int_{I_2} e^{A(\widetilde{T}_2 - s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))}\Big|\sum_{i=1}^{m_2} u_{1i}^{(2)} e_i\Big|^2_{L^2(G)}$$

and

$$\Gamma_2^{(2)} = \Big|\Big(I - P_{m_2}\Big)\int_{I_2} e^{A(\widetilde{T}_2 - s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))}\Big|\sum_{i=1}^{m_2} u_{2i}^{(2)} e_i\Big|^2_{L^2(G)}.$$

By (51), we find that

$$\Lambda^{(2)} \leq e^{-2(\widetilde{T}_2 - T_2)\lambda_{m_2+1}}\alpha_1\beta_1|y_0|^2_{L^2(G)}. \tag{59}$$

Since

$$\begin{aligned}\Big|\Big(I - P_{m_2}\Big)\int_{I_2} e^{A(\widetilde{T}_2 - s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))} &\leq \Big(\int_{I_2} e^{-\lambda_{m_2+1}(\widetilde{T}_2 - s)}ds\Big)^2\\ &\leq \frac{1}{(\lambda_{m_2+1})^2}\big(1 - e^{-\lambda_{m_2+1}(\widetilde{T}_2 - T_2)}\big)^2,\end{aligned}$$

it follows from (58) that

$$\Gamma_1^{(2)} \leq C_1^2 e^{2C_1\sqrt{r_2}} m_2\alpha_1\beta_1|y_0|^2_{L^2(G)}. \tag{60}$$

Similarly, we find that

$$\Gamma_2^{(2)} \leq C_2^2 e^{2C_2\sqrt{r_2}} m_2\alpha_1\beta_1|y_0|^2_{L^2(G)}. \tag{61}$$

By (59)–(61), we obtain that

$$\begin{aligned}\big|y^{(2)}(\widetilde{T}_2)\big|^2_{L^2(G)} =& \Big|\Big(I - P_{m_2}\Big)\Big(e^{A(\widetilde{T}_2 - T_2)}y^{(2)}(T_2) + \int_{I_2} e^{A(\widetilde{T}_2 - s)}\gamma(s)\chi_{G_1}\sum_{i=1}^{m_2} u_{1i}^{(2)} e_i ds\\ &+ \int_{I_2} e^{A(\widetilde{T}_2 - s)}\big(1 - \gamma(s)\big)\chi_{G_1}\sum_{i=1}^{m_2} u_{1i}^{(2)} e_i ds\Big)\Big|^2_{L^2(G)}\\ \leq& 3\Lambda^{(2)} + 3\Gamma_1^{(2)} + 3\Gamma_2^{(2)}\\ \leq& \alpha_2\beta_1\alpha_1|y_0|^2_{L^2(G)}.\end{aligned} \tag{62}$$

Thus, we get (40) for $k = 2$.

We next prove that (39)–(41) are true for $k = n + 1$, on the condition that they are true for $k = n$. Here is the argument: Since there are $y^{(n)}(\cdot)$ and $u^{(n)}(\cdot)$ which satisfy (38)–(40) for $k = n$, then equation (37) for $k = n$, has a unique solution $z^{(n)}(\cdot)$ satisfying that

$$\big|z^{(n)}(T_{n+1})\big|^2_{L^2(G)} \leq e^{-\lambda_{m_n+1}(T_{n+1} - \widetilde{T}_n)}\big|y^n(\widetilde{T}_n)\big|^2_{L^2(G)} \leq \Big[\prod_{i=1}^{n}(\beta_i\alpha_i)\Big]|y_0|^2_{L^2(G)}. \tag{63}$$

Write

$$P_{m_{n+1}}\big(z^{(n)}(x, T_{n+1})\big) = \sum_{i=1}^{m_{n+1}} z_i^{(n)} e_i, \tag{64}$$

where the coefficients $z_i^{(n)}$, $i = 1, \cdots, m_{n+1}$, are real numbers and $z_i^{(n)} = 0$ for $i = 1, \cdots, m_n$. Set

$$\begin{aligned}&\Big(u_{11}^{(n+1)}, \cdots, u_{1m_{n+1}}^{(n+1)}\Big)^T\\ &= -\big(B_{m_{n+1}}^{(1)}\big)^{-1}\Big(\int_{I_{n+1}} e^{A_{n+1}(T_{n+1} - s)}ds\Big)^{-1}\Big(z_1^{(n+1)}, \cdots, z_{m_{n+1}}^{(n+1)}\Big)^T,\end{aligned} \tag{65}$$

$$\begin{aligned}&\Big(u_{21}^{(n+1)}, \cdots, u_{2m_{n+1}}^{(n+1)}\Big)^T\\ &= -\big(B_{m_{n+1}}^{(2)}\big)^{-1}\Big(\int_{I_{n+1}} e^{A_{n+1}(T_{n+1} - s)}ds\Big)^{-1}\Big(z_1^{(n+1)}, \cdots, z_{m_{n+1}}^{(n+1)}\Big)^T,\end{aligned} \tag{66}$$

and

$$\left\{\begin{aligned}&u_1^{(n+1)}(\cdot) = \chi_{G_1}\sum_{i=1}^{m_{n+1}} u_{1i}^{(n+1)} e_i,\\ &u_2^{(n+1)}(\cdot) = \chi_{G_2}\sum_{i=1}^{m_{n+1}} u_{2i}^{(n+1)} e_i,\\ &u^{(n+1)}(\cdot) = u_1^{(n+1)}(\cdot) + u_2^{(n+1)}(\cdot).\end{aligned}\right. \tag{67}$$

Let $y^{(n+1)}(\cdot)$ be the solution to equation (38), where $k=n+1$ and $u^{(k)}(\cdot)$ is replaced by $u^{(n+1)}(\cdot)$ which is given by (67). Write

$$y^{(n+1)}(x,t)=\sum_{j=1}^{\infty} y_j^{(n+1)}(t)e_j, \quad \text{for all } \ t\in I_{n+1}.$$

Then, from (67) and equation (38), we observe that $\big(y_1^{(n+1)}(\cdot),\cdots,y_{m_{n+1}}^{(n+1)}(\cdot)\big)$ solves the following system:

$$\begin{cases} \Big(y_j^{(n+1)}\Big)_t+\lambda_j y_j^{(n+1)}=\gamma\displaystyle\sum_{i=1}^{m_{n+1}} u_{1i}^{(n+1)}\int_{G_1} e_ie_j dx \\ \qquad\qquad +(1-\gamma)\displaystyle\sum_{i=1}^{m_{n+1}} u_{2i}^{(n+1)}\int_{G_2} e_ie_j dx \text{ on } I_{n+1}, \\ \qquad\qquad\qquad \text{for all } \ j=1,\cdots,m_{n+1}, \\ y_j^{(n+1)}\Big(\widetilde{T}_{n+1}\Big)=z_j^{(n+1)}(\widetilde{T}_{n+1}), \ \text{ for all } \ j=1,2,\cdots,m_{n+1}. \end{cases}$$

Taking into account (65) and (66), we can apply Proposition 3 to the above system to derive that

$$P_{m_{n+1}}\big(y^{(n+1)}(x,\widetilde{T}_{n+1})\big)=\sum_{i=1}^{m_{n+1}} y_i^{(n+1)}(\widetilde{T}_{n+1})e_i=0. \tag{68}$$

Namely, (39) holds for $k=n+1$. Moreover, it holds that

$$\begin{aligned} &\sum_{i=1}^{m_{n+1}}\Big(u_{1i}^{(n+1)}\Big)^2 \\ &\le C_1^2e^{2C_2\sqrt{r_{n+1}}}\Big|\Big(\int_{I_{n+1}} e^{A_{m_{n+1}}(T_{n+1}-s)}ds\Big)^{-1}(z_1^{(n)},\cdots,z_{m_{n+1}}^{(n)})^T\Big|^2_{\mathbb{R}^{m_{n+1}}} \\ &= C_1^2e^{2C_1\sqrt{r_{n+1}}}\Big|\sum_{i=1}^{m_{n+1}}\frac{\lambda_i}{e^{\lambda_iT_{n+1}}-1}z_i^{(n+1)}\Big|^2_{\mathbb{R}^{m_{n+1}}} \\ &\le \theta_{1,n+1}\sum_{i=1}^{m_{n+1}}(z_i^{(n+1)})^2. \end{aligned}$$

Hence, we see that

$$\big|u_1^{(n+1)}\big|^2_{L^\infty(I_{n+1};L^2(G))}\le\theta_{1,n+1}\big|z^{(n)}(T_{n+1})\big|^2_{L^2(G)}\le\theta_{1,n+1}\prod_{i=1}^{n}(\beta_i\alpha_i)|y_0|^2_{L^2(G)}. \tag{69}$$

By a similar argument, we find that

$$\big|u_2^{(n+1)}\big|^2_{L^\infty(I_{n+1};L^2(G))}\le\theta_{2,n+1}\big|z^{(n)}(T_{n+1})\big|^2_{L^2(G)}\le\theta_{2,n+1}\prod_{i=1}^{n}(\beta_i\alpha_i)|y_0|^2_{L^2(G)}. \tag{70}$$

From (69) and (70), we get that

$$\big|u^{(n+1)}\big|^2_{L^\infty(I_{n+1};L^2(G))}\le 2\big(\theta_{1,n+1}+\theta_{2,n+1}\big)\prod_{i=1}^{n}(\beta_i\alpha_i)|y_0|^2_{L^2(G)}.$$

Hence, we find that (41) holds for $k=n+1$. Write

$$\Lambda^{(n+1)}\triangleq\Big|\Big(I-P_{m_{n+1}}\Big)e^{A(\widetilde{T}_{n+1}-T_{n+1})}y^{(n+1)}(T_{n+1})\Big|^2_{L^2(G)},$$

$$\Gamma_1^{(n+1)}=\Big|\Big(I-P_{m_{n+1}}\Big)\int_{I_2} e^{A(\widetilde{T}_{n+1}-s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))}\Big|\sum_{i=1}^{m_{n+1}}u_{1i}^{(n+1)}e_i\Big|^2_{L^2(G)}$$

and

$$\Gamma_2^{(n+1)}=\Big|\Big(I-P_{m_{n+1}}\Big)\int_{I_2} e^{A(\widetilde{T}_{n+1}-s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))}\Big|\sum_{i=1}^{m_{n+1}}u_{2i}^{(n+1)}e_i\Big|^2_{L^2(G)}.$$

From (68), we see that

$$\Lambda^{(n+1)}\le e^{-2\lambda_{m_{n+1}+1}(\widetilde{T}_{n+1}-T_{n+1})}\prod_{i=1}^{n}(\alpha_i\beta_i)|y_0|^2_{L^2(G)}. \tag{71}$$

Since

$$\begin{aligned}
&\Big|\Big(I-P_{m_{n+1}}\Big)\int_{I_{n+1}} e^{A(\widetilde T_{n+1}-s)}ds\Big|^2_{\mathcal{L}(L^2(G);L^2(G))}\\
&\leq \Big(\int_{I_2} e^{-\lambda_{m_{n+1}+1}(\widetilde T_{n+1}-s)}ds\Big)^2\\
&\leq \frac{1}{(\lambda_{m_{n+1}+1})^2}\big(1-e^{-\lambda_{m_{n+1}+1}(\widetilde T_{n+1}-T_{n+1})}\big)^2,
\end{aligned}$$

by virtue of (69), we get that

$$\Gamma_1^{(n+1)} \leq C_1^2 e^{2C_1\sqrt{r_{n+1}}} m_{n+1}\prod_{i=1}^{n}(\beta_i\alpha_i)|y_0|^2_{L^2(G)}. \tag{72}$$

Similarly, we find that

$$\Gamma_2^{(n+1)} \leq C_2^2 e^{2C_2\sqrt{r_{n+1}}} m_{n+1}\prod_{i=1}^{n}(\beta_i\alpha_i)|y_0|^2_{L^2(G)}. \tag{73}$$

It follows from (71)–(73) that

$$\begin{aligned}
&\big|y^{(n+1)}(\widetilde T_{n+1})\big|^2_{L^2(G)}\\
&=\Big|\Big(I-P_{m_{n+1}}\Big)\Big(e^{A(\widetilde T_{n+1}-T_{n+1})}y^{(n+1)}(T_{n+1})\\
&\quad+\int_{I_{n+1}} e^{A(\widetilde T_{n+1}-s)}\gamma(s)\chi_{G_1}\sum_{i=1}^{m_{n+1}} u_{1i}^{(n+1)}e_i ds\\
&\quad+\int_{I_{n+1}} e^{A(\widetilde T_{n+1}-s)}\big(1-\gamma(s)\big)\chi_{G_1}\sum_{i=1}^{m_{n+1}} u_{1i}^{(n+1)}e_i ds\Big)\Big|^2_{L^2(G)}\\
&\leq 3\Lambda^{(n+1)}+3\Gamma_1^{(n+1)}+3\Gamma_2^{(n+1)}\\
&\leq \Big[\alpha_{n+1}\prod_{i=1}^{n}(\beta_i\alpha_i)\Big]|y_0|^2_{L^2(G)}.
\end{aligned} \tag{74}$$

This verifies (40) for $k=n+1$. Therefore, we know that (39)–(41) hold for all $k\in\mathbb{N}$.

Now we show that there exists a constant $L>0$ such that

$$\theta_{1,k+1}\prod_{i=1}^{k}(\beta_i\alpha_i)\leq L \text{ and } \theta_{2,k+1}\prod_{i=1}^{k}(\beta_i\alpha_i)\leq L,\quad \text{for all } k\in\mathbb{N}. \tag{75}$$

We first estimate $\theta_{1,k+1}\beta_k\alpha_k$. It follows from Lemma 3.2 that

$$m_k\leq C_3 r_k^{\frac d2}\leq C_3\big(16C_1^2 2^{4(k+1)}\big)^{\frac d2}T^{2d}.$$

Noting that

$$\frac{\lambda_i}{e^{\lambda_i(\widetilde T_k-T_k)}-1}\leq\frac{\lambda_i}{\lambda_i(\widetilde T_k-T_k)}\leq\frac{1}{(\widetilde T_k-T_k)}=\frac{2^{k+1}}{T},$$

from (35) and the definition of r_k, and utilizing the fact that $T\leq 1$, we get that

$$\begin{aligned}
&\theta_{1,k+1}\beta_k\alpha_k\\
&=C_1^2e^{2C_1\sqrt{r_{k+1}}}\sum_{i=1}^{m_{k+1}}\Big(\frac{\lambda_i}{e^{\lambda_i(\widetilde T_k-T_k)}-1}\Big)^2e^{-2r_k(T_{k+1}-\widetilde T_k)}\\
&\quad\times 3\Big(e^{-2r_k(\widetilde T_k-T_k)}+C_1^2e^{2C_1\sqrt{r_k}}m_k+C_2^2e^{2C_2\sqrt{r_k}}m_k\Big)\\
&\leq C_1^2e^{8C_1^22^{2(k+2)}T^{-2}}m_k2^{k+2}e^{-32C_1^22^{3(k+1)}T^{-3}}6\big(1+C_1^2e^{8C_1^22^{2(k+1)}T^{-2}}m_k\big)T^{-2}\\
&\leq 12C_1^2C_3^2\big(16C_1^22^{4(k+1)}\big)^d2^{k+2}e^{-16C_1^22^{3(k+1)}T^{-3}}\big(1+T^{-2(d+1)}\big).
\end{aligned}$$

Then, we know that for given $T\leq 1$, there exists a $N_1>0$ such that for all $k\geq N_1$, it holds that $\theta_{1,k+1}\beta_k\alpha_k\leq 1$.

Further,

$$\begin{aligned}
\beta_k\alpha_k&=e^{-2(T_{k+1}-\widetilde T_k)r_k}\times 3\Big(e^{-2(\widetilde T_k-T_k)r_k}+C_1^2e^{2C_1\sqrt{r_k}}m_k+C_2^2e^{2C_2\sqrt{r_k}}m_k\Big)\\
&\leq e^{-32C_1^22^{3(k+1)}T^{-3}}6\big(1+C_1^2e^{8C_1^22^{2(k+1)}T^{-2}}m_k\big)\\
&\leq 12C_1^2C_3\big(16C_1^22^{4(k+1)}\big)^{\frac d2}e^{-16C_1^22^{3(k+1)}T^{-3}}\big(1+T^{-2(d+1)}\big).
\end{aligned}$$

Hence, we know that for given $T \le 1$, there exists a $N_2 > 0$ so that for all $k \ge N_2$, it holds that $\beta_k \alpha_k \le 1$. Thus, we know that

$$\theta_{1,k+1} \prod_{i=1}^{k} (\beta_i \alpha_i) \le 1 \quad \text{for all } k \ge \max\{N_1, N_1\}.$$

By a similar argument, we can show that there exists a $N_3 > 0$ so that

$$\theta_{2,k+1} \prod_{i=1}^{k} (\beta_i \alpha_i) \le 1 \quad \text{for all } k \ge N_3.$$

Let

$$L = 4 \max_{k \le \max\{N_1, N_2, N_3\}} \Big\{ \theta_{1,k+1} \prod_{i=1}^{k} (\beta_i \alpha_i),\ \theta_{2,k+1} \prod_{i=1}^{k} (\beta_i \alpha_i), 1 \Big\}.$$

Then we have that for all $k \in \mathbb{N}$,

$$|u^{(k)}|^2_{L^\infty(I_k;L^2(G))} \le 2|u_1^{(k)}|^2_{L^\infty(I_k;L^2(G))} + 2|u_2^{(k)}|^2_{L^\infty(I_k;L^2(G))} \le L|y_0|^2_{L^2(G)}.$$

Finally, we are going to construct the control which drives the solution of system (1) to 0 at time $t = T$. To achieve such a goal, we let

$$u(t) = \begin{cases} u^{(k)}(t), & \text{if } t \in I_k, \\ 0, & \text{if } t \in J_k. \end{cases} \tag{76}$$

Then we find that

$$|u|^2_{L^\infty(0,T;L^2(G))} \le L|y_0|^2_{L^2(G)}.$$

Now we only need to prove that

$$y(T; y_0, u) = 0. \tag{77}$$

Indeed, since $y^{(k)}(\cdot)$, $u^{(k)}(\cdot)$ and $z^{(k)}(\cdot)$, $k \in \mathbb{N}$, satisfy (38) and (37), we can make use of (76) to obtain that

$$y\big(\widetilde{T}_k; y_0, u\big) = y^{(k)}\big(\widetilde{T}_k\big), \quad \text{for each } k \in \mathbb{N}.$$

This, together with (39), yields that

$$P_{m_k}\big(y\big(\widetilde{T}_k; u\big)\big) = 0, \quad \text{for all } k \in \mathbb{N}. \tag{78}$$

We arbitrarily fix a $n \in \mathbb{N}$. Then it follows from (78) that

$$P_{m_n}\big(y\big(\widetilde{T}_k; u\big)\big) = 0, \quad \text{for all } k \ge n.$$

This implies that

$$0 = \lim_{k \to +\infty} P_{m_n}\big(y\big(\widetilde{T}_k; u\big)\big) = P_{m_n}\big(y\big(T; u\big)\big). \tag{79}$$

By means of (33), we can pass to the limit for $n \to +\infty$ in (79) to get (77).

Now we choose

$$t_i = \begin{cases} T_k, & \text{if } i = 2k-1,\ k \in \mathbb{N}, \\ \widetilde{T}_k, & \text{if } i = 2k,\ k \in \mathbb{N}. \end{cases}$$

This completes the proof. □

5. Further comments and open problems.

- **Observability inequalities.** As indicated in Corollary 1 our results yield the observability inequality

$$\begin{aligned} |\varphi(0)|^2_{L^2(G)} \ &\le C\Big[\int_0^T \int_{G_1} \gamma(t)|\varphi(x,s)|^2 dxdt \\ &\qquad + \int_0^T \int_{G_2} (1-\gamma(t))|\varphi(x,s)|^2 dxdt\Big], \end{aligned} \tag{80}$$

  with a constant $C > 0$, independent of the measurable function $\gamma(\cdot) : [0,T] \to \{0,1\}$ and the solution $\varphi(\cdot)$ of the adjoint system (7). Clearly, the inequality (80) is equivalent to the following one

$$|\varphi(0)|^2_{L^2(G)} \le C \int_0^T \min\Big\{ \int_{G_1} |\varphi(x,t)|^2 dx, \int_{G_2} |\varphi(x,t)|^2 dx \Big\} dt, \tag{81}$$

  for every $\varphi(\cdot)$ solution of (7).

As far as we know this observability inequality is new and can not be proved by the methods based on the use of Carleman inequalities as in [3, 4].

- **Minimal norm controls.** Note however that the inequality (80) or (11) is not sufficient to deduce Theorem 1.1. For example, by inequality (80), we can only get the null controllability of the system (1) with a control depending on $\gamma(\cdot)$.
For instance, to find the control with the minimal $L^2(\frac{1}{\gamma}dt; L^2(G_1))\cap L^2(\frac{1}{1-\gamma}dt; L^2(G_2))$-norm, it is sufficient to minimize the functional
$$\begin{aligned} J(\varphi_T) &= \int_0^T\int_{G_1}\gamma(t)|\varphi(x,t)|^2dxdt + \int_0^T\int_{G_2}(1-\gamma(t))|\varphi(x,t)|^2dxdt \\ &\quad + \int_G \varphi(x,0)y_0(x)dx \end{aligned}$$
on the Hilbert space H, which is the completion of $C_0^\infty(G)$ with respect to the following norm:
$$|\varphi_T|_H^2 \triangleq \int_0^T\int_{G_1}\gamma(t)|\varphi(x,t)|^2dxdt + \int_0^T\int_{G_2}(1-\gamma(t))|\varphi(x,t)|^2dxdt.$$
Clearly, $J(\cdot)$ is convex and continuous in H. By virtue of the inequality (80), we find the coercivity of $J(\cdot)$. Hence, there is a minimizer of $J(\cdot)$. This minimizer provides the minimal $L^2(\frac{1}{\gamma}dt; L^2(G_1))\cap L^2(\frac{1}{1-\gamma}dt; L^2(G_2))$-norm control we are looking for but it depends on γ. Note that our control given by Theorem 1.1, which is designed to deal with the uncertainty of the future variations of γ, does not enjoy any minimal norm condition.
Recall also that Theorem 1.1 cannot be obtained by the observability estimate (80) or (11). Characterizing the robust control result in Theorem 1.1 by a suitable observability estimate for the adjoint system (7) is an interesting open problem.
In view of the observability inequality (81) which is independent of γ and following the methods in [18], by minimizing a suitable functional in the spirit of $J(\varphi_T)$ above in which the quadratic term is replaced by the right hand side term of (81), one can obtain a different switching strategy.

- **More complex switching laws.** One can also consider the problem that the control switches among a countable number of subsets $\{G_k\}_{k=1}^\infty (G_i\cap G_j=\emptyset)$ of G:
$$\begin{cases} y_t-\Delta y = \sum_{k=1}^\infty\Big[\gamma_k\prod_{j=1,j\neq k}^\infty(1-\gamma_j)\chi_{G_k}\Big]u & \text{in } G\times(0,T),\\ y=0 & \text{on } \partial G\times(0,T),\\ y(0)=y_0 & \text{in } G.\end{cases} \tag{82}$$
In this case, we should assume an uniform lower bound on the Lebesgue measure of the sets $G_k (k\in\mathbb{N})$. Otherwise, if there were a subsequence $\{G_{k_l}\}_{l=1}^\infty$ of $\{G_k\}_{k=1}^\infty$ with the Lebesgue measure of G_{k_l} tending to zero as $l\to\infty$, choosing $\{\gamma_{k_l}(\cdot)\}_{l=1}^\infty$ such that $\gamma_{k_l}(\cdot)\equiv 1$ in $[0,T]$, then one could see that the $L^\infty(0,T;L^2(G_{k_l}))$-norm of the controls $u_{k_l}(\cdot)$ driving the solution to the rest for $\gamma(\cdot)=\gamma_{k_l}(\cdot)$, would diverge as $l\to\infty$. But if the Lebesgue measure of the sets $G_k(k\in\mathbb{N})$ are bounded from below and $G_i\cap G_j=\emptyset$ for $i\neq j$ the domain would be unbounded. Then either one should work on unbounded domains or drop the assumption of empty intersection of these sets. This issues will be considered elsewhere.

- **Switching boundary controls.** In this paper we have only analyzed the problem of internal null controllability. It is also natural to consider the corresponding boundary control problem. If there is no switching in the control, then the boundary control result is an easy consequence of the internal control result. But this is not the case when the control switches. For example, let us consider the null boundary controllability problem with switching boundary controls localized on two arbitrary open subsets Γ_1 and Γ_2 of the boundary. If we follow the standard argument to obtain the boundary null controllability by null internal controllability, we should extend the domain G by two small open subsets, G_1 and G_2, attached to Γ_1 and Γ_2, respectively. Theorem 1.1 allows us to control the system in the large domain by means of an internal switching control supported in these small added domains. The restriction of the solution to the original domain satisfies all the requirements and its restriction to Γ_1 and Γ_2 gives the boundary control which drives the solution to rest at $t=T$. However, it is not the control we are looking for, since it does not follow the switching

mode given by $\gamma(\cdot)$. Indeed, the restrictions on B_1 and B_2 will be supported on the whole time-duration $[0, T]$. They will not be the switching controls any more.
Thus, the extension of the results of this paper to the case of boundary switching controls is open.

- **More general heat equations.** We only study heat equation with constant coefficients in this article. As we have mentioned in Remark 4, Theorem 1.1 can be generalized to a more general class of heat equations with variable but time-independent coefficients. It is natural to ask whether Theorem 1.1 still holds for heat equations with time-dependent coefficients or semilinear heat equations.
- **Wave equations.** As indicated in Remark 7, the results of this paper do not hold for the wave equation. This is the case even in one space dimension. Indeed, assuming that G_1 and G_2 are two open non-empty subintervals of the interval G where the wave equation is posed, the exact controllability property of the wave equation is ensured when the time of control is sufficiently large. But this does not suffice to guarantee the exact controllability for all possible switching functions γ. Indeed, it is easy to build a switching function γ such that there exists a broken characteristic line reflected on the boundary but that never meets the control sets G_1 and G_2 when they are active. In this situation the wave equation is not controllable.
However, using the arguments we developed in the finite-dimensional case and the estimates on spectral clusters, we can show a finite-dimensional controllability result. Indeed, consider the following controlled wave equation:

$$\begin{cases} y_{tt} - \Delta y = \big[\gamma\chi_{G_1} + (1-\gamma)\chi_{G_2}\big]u & \text{in } G \times (0, T), \\ y = 0 & \text{on } \partial G \times (0, T), \\ y(0) = y_0,\ y_t(0) = y_1 & \text{in } G. \end{cases} \tag{83}$$

Here $(y_0, y_1) \in H_0^1(G) \cap L^2(G)$. By an argument similar to the proof of Theorem 2.1, we can show that there is a control $u \in L^2(0, T; L^2(G_1 \cup G_2))$, which is independent of γ, such that $(P_k y(T), P_k y_t(T)) = (0, 0)$ in span $_{\lambda_i \le r_k}\{e_i\}$.

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E-mail address: luqi59@163.com
E-mail address: zuazua@bcamath.org