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Mixed weak type estimates: Examples and counterexamples related to a problem of E. Sawyer

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Abstract

We study mixed weighted weak-type inequalities for families of functions, which can be applied to study classical operators in harmonic analysis Our main theorem extends the key result from [CMP2].

1. Introduction and main results

In this work we consider mixed weighted weak-type inequalities of the form

$$uv\left(\left\{x\in\mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t\right\}\right) \le \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|\, Mu(x)v(x)\, dx, \tag{1.1}$$

where T is either the Hardy-Littlewood maximal operator or any Calderón-Zygmund operator. Similar inequalities were studied by Sawyer in [Sa] motivated by the work of Muckenhoupt and Wheeden [MW] (see also the works [AM] and [MOS]).

E. Sawyer proved that inequality (1.1) holds in $\mathbb{R}$ when $T = M$ is the Hardy-Littlewood maximal operator assuming that the weights u and v belong to the class A_1. This result can be seen as a very delicate extension of the classical weak type $(1,1)$ estimate. However, the reason why E. Sawyer considered (1.1) is due to the following interesting observation. Indeed, inequality (1.1) yields a new proof of the classical Muckenhoupt's theorem for M assuming that the A_p weights can be factored (P. Jones's theorem). This means that if $w \in A_p$ then $w = uv^{1-p}$ for some $u, v \in A_1$. Now, define the operator $f \to \frac{M(fv)}{v}$ which is bounded on $L^\infty(uv)$ and it is of weak type $(1,1)$ with respect to the measure $uvdx$ by (1.1). Hence by the Marcinkiewicz interpolation theorem we recover Muckenhoupt's theorem.

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In the same paper, Sawyer conjectured that if T is instead the Hilbert transform the inequality also holds with the same hypotheses on the weights u and v. This conjecture was proved in [CMP2]. In fact, it is proved in this paper that the inequality (1.1) holds for both the Hardy-Littlewood maximal operator and for any Calderón-Zygmund Operator in any dimension if either the weights u and v both belong to A_1 or u belongs to A_1 and $uv \in A_\infty$. The method of proof is quite different from that in [Sa] (also from [MW]) and it is based on certain ideas from extrapolation that go back to the work of Rubio de Francia (see [CMP2] and also the expository paper [CMP3]). Applications of these results can be found in [LOPTT]. The authors conjectured in [CMP2] that their results may hold under weaker hypotheses on the weights. To be more precise, they proposed that inequality (1.1) is true if $u \in A_1$ and $v \in A_\infty$. Very recently, some quantitative estimates in terms of the relevant constants of the weights have been obtained in [OPR] and some new conjectures have been formulated.

Inequalities like (1.1), when T is the Hardy-Littlewood maximal operator, can also be seen as generalizations of the classical Fefferman-Stein inequality

$$\|M(f)\|_{L^{1,\infty}(u)} \leq c\, \|f\|_{L^1(Mu)},$$

where c is a dimensional constant. However, in Section 3, we will see that (1.1) does not hold in general even for weights satisfying strong conditions like $v \in RH_\infty \subset A_\infty$.

In this work we generalize the extrapolation result in [CMP3] for a larger class of weights (see Theorem 1.5 below). This method of extrapolation is flexible enough with scope reaching beyond the classical linear operators. Indeed, it can be applied to square functions, vector valued operators as well as multilinear singular integral operators. See Section 2 for some of these applications. In fact, the best way to state the extrapolation theorem is without considering operators and the result can be seen as a property of families of functions. Hereafter, $\mathcal{F}$ will denote a family of ordered pairs of non-negative, measurable functions (f, g). Also we are going to assume that this family $\mathcal{F}$ of functions, satisfies the following property: for **some** p_0, $0 < p_0 < \infty$, and every $w \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x)\, dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w(x)\, dx, \tag{1.2}$$

for all $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, and where C depends only on the A_∞ constant of w. By the main theorem in [CMP1], this assumption turns out to be true for **any** exponent $p \in (0, \infty)$ and **every** $w \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^{p} w(x)\, dx \leq C \int_{\mathbb{R}^n} g(x)^{p} w(x)\, dx, \tag{1.3}$$

for all $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, and where C depends only on the A_∞ constant of w. See the papers [CMP1], [CGMP] and [CMP3] for more information and applications and the book [CMP4] for a general account. It is also interesting that both (1.2) and (1.3) are equivalent to the following vector-valued version: for all $0 < p, q < \infty$ and for all $w \in A_\infty$ we have

$$\Big\| \Big(\sum_j (f_j)^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)} \quad \leq \quad C \Big\| \Big(\sum_j (g_j)^q \Big)^{\frac{1}{q}} \Big\|_{L^p(w)}, \tag{1.4}$$

for any $\{(f_j,g_j)\}_j \subset \mathcal{F}$, where these estimates hold whenever the left-hand sides are finite.

Next theorem improves the corresponding Theorem from [CMP2].

Theorem 1.5. *Let $\mathcal{F}$ be a family of functions satisfying (1.2) and let $\theta \geq 1$. Suppose that $u \in A_1$ and that v is a weight such that for some $\delta > 0$, $v^\delta \in A_\infty$.*

Then, there is a constant C such that

$$\Big\| \frac{f}{v^\theta} \Big\|_{L^{1/\theta,\infty}(uv)} \leq C \Big\| \frac{g}{v^\theta} \Big\|_{L^{1/\theta,\infty}(uv)}, \qquad (f,g) \in \mathcal{F}. \tag{1.6}$$

Similarly, the following vector-valued extension holds: if $0 < q < \infty$,

$$\Big\| \frac{\Big(\sum_j (f_j)^q\Big)^{\frac{1}{q}}}{v^\theta} \Big\|_{L^{1/\theta,\infty}(uv)} \leq C \Big\| \frac{\Big(\sum_j (g_j)^q\Big)^{\frac{1}{q}}}{v^\theta} \Big\|_{L^{1/\theta,\infty}(uv)}, \tag{1.7}$$

for any $\{(f_j,g_j)\}_j \subset \mathcal{F}$.

Observe that the singular class of weights $v(x) = |x|^{-nr}$, $r \geq 1$, is covered by the hypothesis of the Theorem but not in the corresponding Theorem from [CMP2].

The proof of (1.7) is immediate since we can extrapolate using as initial hypothesis (1.4) and then applying (1.6).

Corollary 1.8. *Let $\mathcal{F}$, u and $\theta \geq 1$ as in the Theorem. Suppose now that v_i, $i = 1, \cdots, m$, are weights such that for some $\delta_i > 0$, $v_i^{\delta_i} \in A_\infty$, $i = 1, \cdots, m$.*

Then, if we denote $v = \prod_{i=1}^m v_i$

$$\Big\| \frac{f}{v^\theta} \Big\|_{L^{1/\theta,\infty}(uv)} \leq C \Big\| \frac{g}{v^\theta} \Big\|_{L^{1/\theta,\infty}(uv)}, \qquad (f,g) \in \mathcal{F}.$$

and similarly for $0 < q < \infty$,

$$\Big\| \frac{\Big(\sum_j (f_j)^q\Big)^{\frac{1}{q}}}{v^\theta} \Big\|_{L^{1/\theta,\infty}(uv)} \leq C \Big\| \frac{\Big(\sum_j (g_j)^q\Big)^{\frac{1}{q}}}{v^\theta} \Big\|_{L^{1/\theta,\infty}(uv)},$$

for any $\{(f_j,g_j)\}_j \subset \mathcal{F}$.

The proof reduces to the Theorem by choosing $\delta > 0$ small enough such that $v^\delta = \prod_{i=1}^m v_i^\delta \in A_\infty$ which follows by convexity since $v_i^{\delta_i} \in A_\infty$, $i = 1, \cdots, m$.

To apply Theorem 1.5 above to some of the classical operators we need a mixed weak type estimate for the Hardy-Littlewood maximal operator. This is the content of next Theorem which was obtained in dimension one by Andersen and Muckenhoupt in [AM] and by Martín-Reyes, Ortega Salvador and Sarrión Gavián [MOS] in higher dimensions. In each case the proof follows as a consequence of a more general result with the additional hypothesis that $u \in A_1$. For completeness we will give an independent and direct proof with the advantage that no condition on the weight u is assumed.

Theorem 1.9. *Let $u \geq 0$ and $v(x) = |x|^{-nr}$ for some $r > 1$. Then there is a constant C such that for all $t > 0$,*

$$uv\bigg(\bigg\{x \in \mathbb{R}^n : \frac{M(fv)(x)}{v(x)} > t\bigg\}\bigg) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|\, Mu(x) v(x)\, dx. \tag{1.10}$$

Remark 1.11. We remark that the theorem could be false when $r = 1$ even in the case $u = 1$, see [AM]. However, we already mentioned that the singular weight $v(x) = |x|^{-n}$ is included in the extrapolation Theorem 1.5.

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2. Some applications

In this section we show the flexibility of the method by giving two applications.

2.1. The vector-valued case

Let T be any singular integral operator with standard kernel and let M be the Hardy-Littlewood maximal function. We are going to show that starting from the following inequality due to Coifman [Coi]: for $0 < p < \infty$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) \, dx \leq C \int_{\mathbb{R}^n} Mf(x)^p \, w(x) \, dx, \tag{2.1}$$

combined with the extrapolation Theorem 1.5 together with Theorem 1.9 yields the following corollary.

Corollary 2.2. *Let $u \in A_1$ and $v(x) = |x|^{-nr}$ for some $r > 1$. Also let $1 < q < \infty$. Then, there is a constant C such that for all $t > 0$,*

$$uv\Bigg(\Bigg\{x \in \mathbb{R}^n : \frac{\Big(\sum_j M(f_j v)(x)^q\Big)^{\frac{1}{q}}}{v(x)} > t\Bigg\}\Bigg) \leq \frac{C}{t} \int_{\mathbb{R}^n} \Big(\sum_j |f_j(x)|^q\Big)^{\frac{1}{q}} u(x)v(x) \, dx,$$

$$uv\Bigg(\Bigg\{x \in \mathbb{R}^n : \frac{\Big(\sum_j |T(f_j v)(x)|^q\Big)^{\frac{1}{q}}}{v(x)} > t\Bigg\}\Bigg) \leq \frac{C}{t} \int_{\mathbb{R}^n} \Big(\sum_j |f_j(x)|^q\Big)^{\frac{1}{q}} u(x)v(x) \, dx.$$

Observe that in particular we have the following scalar version,

$$uv\Bigg(\Bigg\{x \in \mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t\Bigg\}\Bigg) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| \, u(x)v(x) \, dx.$$

This scalar version was proved in [MOS].

The second inequality of the corollary follows from the first one by applying inequality (1.7) in Theorem 1.5 with initial hypothesis (2.1):

$$\sup_{t>0} tuv\Bigg(\Bigg\{x \in \mathbb{R}^n : \frac{\Big(\sum_j |T(f_j)(x)|^q\Big)^{\frac{1}{q}}}{v(x)} > t\Bigg\}\Bigg) \leq$$

$$C \sup_{t>0} tuv\Bigg(\Bigg\{x \in \mathbb{R}^n : \frac{\Big(\sum_j M(f_j)(x)^q\Big)^{\frac{1}{q}}}{v(x)} > t\Bigg\}\Bigg).$$

To prove the first inequality in Corollary 2.2 we first note that in [CGMP] it was shown for $1 < q < \infty$ and for all $0 < p < \infty$ and $w \in A_\infty$,

$$\Big\|\Big(\sum_j (M(f_j))^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w)} \leq C\,\Big\|M\Big(\big(\sum_j |f_j|^q\big)^{\frac{1}{q}}\Big)\Big\|_{L^p(w)}.$$

To conclude we apply Theorem 1.5 combined with Theorem 1.9.

2.2. Multilinear Calderón-Zygmund operators:

We now apply our main results to multilinear Calderón-Zygmund operator. We follow here the theory developed by Grafakos and Torres in [GT1], that is, T is an m-linear operator such that $T : L^{q_1} \times \cdots \times L^{q_m} \longrightarrow L^q$, where $1 < q_1, \ldots, q_m < \infty$, $0 < q < \infty$ and

$$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}. \tag{2.3}$$

The operator T is associated with a Calderón-Zygmund kernel K in the usual way:

$$T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \ldots, y_m)\, f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m,$$

whenever $f_1, \ldots, f_m$ are in C_0^∞ and $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$. We assume that K satisfies the appropriate decay and smoothness conditions (see [GT1] for complete details). Such an operator T is bounded on any product of Lebesgue spaces with exponents $1 < q_1, \ldots, q_m < \infty$, $0 < q < \infty$ satisfying (2.3). Further, it also satisfies weak endpoint estimates when some of the q_i's are equal to one. There are also weighted norm inequalities for multilinear Calderón-Zygmund operators; these were first proved in [GT2] using a good-λ inequality and fully characterized in [LOPTT] using the sharp maximal function $\mathcal{M}$ and a new maximal type function which plays a central role in the theory,

$$\mathcal{M}(f_1, \ldots, f_m)(x) = \sup_{\substack{Q \ni x \\ Q \text{ cube}}} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(z)|\, dz,$$

where the supremum is taken over cubes with sides parallel to the axes. Indeed, one of the main results from [LOPTT] is that for any $0 < p < \infty$ and for any $w \in A_\infty$,

$$\Big\|T(f_1, \ldots, f_m)\Big\|_{L^p(w)} \leq C\,\Big\|\mathcal{M}(f_1, \ldots, f_m)\Big\|_{L^p(w)}.$$

Beginning with these inequalities, we can apply Theorem 1.5 to the family $\mathcal{F}\Big(T(f_1, \ldots, f_m), \mathcal{M}(f_1, \ldots, f_m)\Big)$. Hence, if $u \in A_1$ and $v(x) = |x|^{-nr}$ for some $r > 1$.

$$\Big\|\frac{T(f_1, \ldots, f_m)}{v^m}\Big\|_{L^{1/m,\infty}(uv)} \leq C\,\Big\|\frac{\mathcal{M}(f_1, \ldots, f_m)}{v^m}\Big\|_{L^{1/m,\infty}(uv)} \tag{2.4}$$

Corollary 2.5. *Let T be a multilinear Calderón-Zygmund operator as above. Let $u \in A_1$ and $v(x) = |x|^{-nr}$ for some $r > 1$. Then*

$$\Big\|\frac{T(f_1, \ldots, f_m)}{v^m}\Big\|_{L^{1/m,\infty}(uv)} \leq C \prod_{j=1}^m \int_{\mathbb{R}^n} |f_j|\, u\, dx, .$$

To prove this corollary we will use the following version of the generalized Holder's inequality: for $1 \leq q_1, \ldots, q_m < \infty$ with

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q},$$

there is a constant C such that

$$\| \prod_{j=1}^{m} h_j \|_{L^{q,\infty}(w)} \leq C \prod_{j=1}^{m} \|h_j\|_{L^{q_j,\infty}(w)}.$$

The proof of this inequality follows in a similar way that the proof of the classic generalized Holder's inequality in L^p theory.

Now, if we combine this with (2.4) and with the trivial observation that

$$\mathcal{M}(f_1, \ldots, f_m)(x) \leq \prod_{i=1}^{m} M(f_i)\,,$$

we have

$$\Big\| \frac{T(f_1, \ldots, f_m)}{v^m} \Big\|_{L^{1/m,\infty}(uv)} \leq C \prod_{j=1}^{m} \Big\| \frac{Mf_j}{v} \Big\|_{L^{1,\infty}(uv)},$$

Finally, an application of Theorem 1.9 concludes the proof of the corollary.

3. counterexamples

An interesting point from Theorem 1.9 is that if $v(x) = |x|^{-nr}$, $r > 1$, the estimate

$$uv\bigg(\bigg\{x \in \mathbb{R}^n : \frac{M(fv)(x)}{v(x)} > t\bigg\}\bigg) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|\, Mu(x)v(x)\, dx, \tag{3.1}$$

holds for any $u \geq 0$. On the other hand we have already mentioned that the same inequality holds if $u \in A_1$ and $v \in A_1$ or $uv \in A_\infty$ [CMP2]. In particular, this is the case if $u \in A_1$ and $v \in RH_\infty$. Assuming that $v \in RH_\infty$, a natural question is whether inequality (3.1) holds with **no** assumption on u. This would improve the classical Fefferman-Stein inequality. However, we will show in the next example that this is **false** in general.

Example 3.2. On the real line we let $v(x) = \sum_{k \in Z} |x - k|\, \chi_{I_k}(x)$, where I_k denotes the interval $|x - k| \leq 1/2$. It is not difficult to see that $v \in RH_\infty$. If we choose

$$u(x) = \sum_{\substack{k \in N \\ k > 10}} \frac{k}{\log(k)} \chi_{J_k}(x)\,,$$

where $J_k = \left[k + \frac{1}{4k}, k + \frac{1}{k}\right]$, and $f = \chi_{[-1,1]}$, then there is no finite constant C such that the inequality

$$uv(\{x : Mf(x) > v(x)\}) \leq C \int |f|\, M^2 u \tag{3.3}$$

holds. To prove this we will make use of the following observation:

There is a geometric constant such that

$$M^2w(x) \approx M_{L\log L}w(x) \qquad x \in \mathbb{R}^n$$

where

$$M_{L\log L}f(x) = \sup_{Q \ni x} \|f\|_{L\log L,Q}$$

and

$$\|f\|_{L\log L,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi(\frac{|f|}{\lambda})\,dx \le 1\}.$$

with $\Phi(t) = t\log(e+t)$, see [PW] or [G]. Now, it is a computation to see that if $x \in [-1,1]$, $M^2u(x) \approx M_{L\log L}u(x) \le C$ then the right hand side of (3.3) is finite, while the left hand side is infinite. Let us check that. For $|x| > 2$ we have that $Mf\,(x) \ge \frac{1}{|x|}$ and if $x \in J_k \subset I_k$ for $k > 10$ $\frac{1}{|x|} > \frac{1}{2k}$, then it is easy to see that $(k+\frac{1}{4k}, k+\frac{1}{2k}) \subset \{x \in J_k : Mf(x) > v(x)\}$ and therefore we obtain that

$$\begin{aligned} uv(\{x : Mf\,(x) > v(x)\}) \quad &> \quad \sum_{\substack{k\in N\\ k>10}} \frac{k}{\log(k)} \int_{k+\frac{1}{4k}}^{k+\frac{1}{2k}} (x-k)\,dx > \\ &> \quad \sum_{\substack{k\in N\\ k>10}} \frac{1}{8k\log(k)} = \infty. \end{aligned}$$

4. Proof of Theorem 1.5

The following Lemmas will be useful:

Lemma 4.1. *If $u \in A_1$, $w \in A_1$, then there exists $0 < \epsilon_0 < 1$ depending only on $[u]_{A_1}$ such that $uw^\epsilon \in A_1$ for all $0 < \epsilon < \epsilon_0$.*

Proof. Since $u \in A_1$, $u \in RH_{s_0}$ for some $s_0 > 1$ depending on $[u]_{A_1}$. Let $\epsilon_0 = 1/s_0{}'$ and $0 < \epsilon < \epsilon_0$. This implies that $u \in RH_s$ with $s = (1/\epsilon)'$.

Then since u, $v \in A_1$, for any cube Q and almost every $x \in Q$,

$$\begin{aligned} \frac{1}{|Q|}\int_Q u(y)w(y)^\epsilon\,dy &\le \left(\frac{1}{|Q|}\int_Q u(y)^s\,dy\right)^{1/s}\left(\frac{1}{|Q|}\int_Q w(y)\,dy\right)^{1/s'} \\ &\le \frac{[u]_{RH_s}}{|Q|}\int_Q u(y)\,dy\left(\frac{1}{|Q|}\int_Q w(y)\,dy\right)^{1/s'} \le [u]_{RH_s}[u]_{A_1}[w]_{A_1}^\epsilon u(x)w(x)^\epsilon. \end{aligned}$$

Hence $uw^\epsilon \in A_1$ with $[uw^\epsilon]_{A_1} \le [u]_{RH_s}[u]_{A_1}[w]_{A_1}^\epsilon$.

□

We also need the following version of the Marcinkiewicz interpolation theorem in the scale of Lorentz spaces. In fact we need a version of this theorem with precise constants. The proof can be found in [CMP2].

Proposition 4.2. *Given p_0, $1 < p_0 < \infty$, let T be a sublinear operator such that*

$$\|Tf\|_{L^{p_0,\infty}} \le C_0 \, \|f\|_{L^{p_0,1}} \qquad \text{and} \qquad \|Tf\|_{L^\infty} \le C_1 \, \|f\|_{L^\infty}.$$

Then for all $p_0 < p < \infty$,

$$\|Tf\|_{L^{p,1}} \le 2^{1/p} \left(C_0 \, (1/p_0 - 1/p)^{-1} + C_1 \right) \|f\|_{L^{p,1}}.$$

Fix $u \in A_1$ and v such that $v^\delta \in A_\infty$ for some $\delta > 0$. Then by the factorization theorem $v^\delta = v_1 v_2$ for some $v_1 \in A_1$ and $v_2 \in RH_\infty$. Define the operator S_λ by

$$S_\lambda f(x) = \frac{M(fuv_1^{1/\lambda\delta})}{uv_1^{1/\lambda\delta}}$$

for some large enough constant $\lambda > 1$ that will be chosen soon.

By Lemma 4.1, there exists $0 < \epsilon_0 < 1$ (that depends only on $[u]_{A_1}$) such that $u\,w^\epsilon \in A_1$ for all $w \in A_1$ and $0 < \epsilon < \epsilon_0$.

Choose $\lambda > \frac{1}{\delta\epsilon_0}$ such that $uv_1^{1/\lambda\delta} \in A_1$. Hence, S_λ is bounded on $L^\infty(uv)$ with constant $C_1 = [u]_{A_1}$. We will now show that for some larger λ, S_λ is bounded on $L^m(uv)$. Observe that

$$\int_{\mathbb{R}^n} Sf(x)^\lambda \, u(x)\, v(x)\, dx = \int_{\mathbb{R}^n} M(fuv_1^{1/\lambda\delta})(x)^\lambda \, u(x)^{1-\lambda}\, v_2(x)^{1/\delta}\, dx.$$

Since $v_2 = \tilde{v}_2^{1-t}$ for some $\tilde{v}_2 \in A_1$ and $t > 1$ we have

$$u^{1-\lambda}\, v_2^{1/\delta} = u^{1-\lambda}\, \tilde{v}_2^{\frac{1-t}{\delta}} = \left(u\, \tilde{v}_2^{\frac{t-1}{\delta(\lambda-1)}}\right)^{1-\lambda}.$$

By Lemma 4.1 there exists λ sufficiently large ($\lambda > 1 + \frac{t-1}{\delta\epsilon_0}$) such that $u\,\tilde{v}_2^{\frac{t-1}{\delta(\lambda-1)}} \in A_1$ and hence $u^{1-\lambda}\, v_2^{1/\delta} \in A_\lambda$. By Muckenhoupt's theorem, M is bounded on $L^\lambda(u^{1-\lambda} v_2^{1/\delta})$ and therefore S is bounded on $L^\lambda(uv)$ with some constant C_0. Observe that λ depends on the A_1 constant of u. We fix one such λ from now on.

By Proposition 4.2 above we have that S is bounded on $L^{q,1}(uv)$, $q > \lambda$. Hence,

$$\|Sf\|_{L^{q,1}(uv)} \le 2^{1/q} \left(C_0 \, (1/\lambda - 1/q)^{-1} + C_1 \right) \|f\|_{L^{q,1}(uv)}.$$

Thus, for all $q \ge 2\lambda$ we have that $\|Sf\|_{L^{q,1}(uv)} \le K_0 \, \|f\|_{L^{q,1}(uv)}$ with $K_0 = 4\lambda\,(C_0 + C_1)$. We emphasize that the constant K_0 is valid for every $q \ge 2\lambda$.

Fix $(f, g) \in \mathcal{F}$ such that the left-hand side of (1.6) is finite. We let r be such that $\theta < r < \theta(2\lambda)'$, to be chosen soon. Now, by the duality of $L^{r,\infty}$ and $L^{r',1}$,

$$\left\|f\, v^{-\theta}\right\|_{L^{1/\theta,\infty}(uv)}^{\frac{1}{r}} = \left\|(f\, v^{-\theta})^{\frac{1}{r}}\right\|_{L^{r/\theta,\infty}(uv)} = \sup \int_{\mathbb{R}^n} f(x)^{\frac{1}{r}}\, h(x)\, u(x)\, v(x)^{1-\theta/r}\, dx,$$

where the supremum is taken over all non-negative $h \in L^{(\frac{r}{\theta})',1}(uv)$ with $\|h\|_{L^{(\frac{r}{\theta})',1}(uv)} = 1$. Fix such a function h. We are going to build a larger function $\mathbb{R}h$ using the Rubio de Francia‘s method such $\mathbb{R}h\, uv^{1-\theta/r} \in A_\infty$. Hence we will use the hypothesis (1.3) with $p = \theta/r$ (recall that is equivalent to (1.2)) with the weight $\mathbb{R}h\, uv^{1-\theta/r} \in A_\infty$

We let r be such that $(\frac{r}{\theta})' > 2\lambda$ and hence $S_{(\frac{r}{\theta})'}$ is bounded on $L^{(\frac{r}{\theta})',1}(uv)$ with constant bounded by K_0. Now apply the Rubio de Francia algorithm (see [GCRdF]) to define the operator $\mathbb{R}$ on $h \in L^{(\frac{r}{\theta})',1}(uv)$, $h \geq 0$, by

$$\mathbb{R}h(x) = \sum_{j=0}^{\infty} \frac{S^j_{(\frac{r}{\theta})'} h(x)}{2^j\, K_0^j},$$

Recall that the operator $S_{(\frac{r}{\theta})'}$ is defined by

$$S_{(\frac{r}{\theta})'} f(x) = \frac{M(f uv_1^{1/(\frac{r}{\theta})'\delta})}{uv_1^{1/(\frac{r}{\theta})'\delta}}.$$

Also, recall that by the choice of r $uv_1^{1/(\frac{r}{\theta})'\delta} \in A_1$.

It follows immediately from this definition that:

(a) $h(x) \leq \mathbb{R}h(x)$;

(b) $\|\mathbb{R}h\|_{L^{(\frac{r}{\theta})',1}(uv)} \leq 2\,\|h\|_{L^{(\frac{r}{\theta})',1}(uv)}$;

(c) $S_{(\frac{r}{\theta})'}(\mathbb{R}h)(x) \leq 2\,K_0\,\mathbb{R}h(x)$.

In particular, it follows from (c) and the definition of S that $\mathbb{R}h\, uv_1^{1/(\frac{r}{\theta})'\delta} \in A_1$ and therefore $\mathbb{R}h\, uv^{1/(\frac{r}{\theta})'} = \mathbb{R}h\, uv_1^{1/\delta(\frac{r}{\theta})'} v_2^{1/\delta(\frac{r}{\theta})'} \in A_\infty$.

To apply the hypothesis (1.3) we must first check that the left-hand side is finite, but this follows at once from Hölder's inequality and (b):

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{\frac{1}{r}}\,\mathbb{R}h(x)\,u(x)\,v(x)^{1-\frac{\theta}{r}}\,dx &\leq \left\|(f\,v^{-\theta})^{\frac{1}{r}}\right\|_{L^{r/\theta,\infty}(uv)} \|\mathbb{R}h\|_{L^{(r/\theta)',1}(uv)} \\ &\leq 2\left\|f\,v^{-\theta}\right\|^{\frac{1}{r}}_{L^{1/\theta,\infty}(uv)} \|h\|_{L^{(\frac{r}{\theta})',1}(uv)} < \infty.\end{aligned}$$

Thus since $\mathbb{R}h\, uv^{1/(\frac{r}{\theta})'} \in A_\infty$ by (1.3)

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{\frac{1}{r}}\,h(x)\,u(x)\,v(x)^{1-\frac{\theta}{r}}\,dx &\leq \int_{\mathbb{R}^n} f(x)^{\frac{1}{r}}\,\mathbb{R}h(x)\,u(x)\,v(x)^{1-\frac{\theta}{r}}\,dx \\ &\leq C\int_{\mathbb{R}^n} g(x)^{\frac{1}{r}}\,\mathbb{R}h(x)\,u(x)\,v(x)^{1-\frac{\theta}{r}}\,dx \\ &\leq C\left\|(g\,v^{-\theta})^{\frac{1}{r}}\right\|_{L^{r/\theta,\infty}(uv)} \|\mathbb{R}h\|_{L^{(\frac{r}{\theta})',1}(uv)} \\ &\leq 2\,C\left\|g\,v^{-\theta}\right\|^{\frac{1}{r}}_{L^{1/\theta,\infty}(uv)}.\end{aligned}$$

Since C is independent of h, inequality (1.6) follows finishing the proof of the theorem.

5. Proof of Theorem 1.9

5.1. Proof of (1.10)

The following lemma is important in the proof.

Lemma 5.1. *Let f be a positive and locally integrable function. Then for $r > 1$ there exists a positive real number a depending on f and λ such that the inequality*

$$\left(\int_{|y|\leq a^{\frac{1}{r-1}}} f(y)dy\right) a^n = \lambda$$

holds.

Proof. Consider the function

$$g(a) = \left(\int_{|y|\leq a^{\frac{1}{r-1}}} f(y)dy\right) a^n, \text{ for } a \geq 0,$$

then by the hypothesis we have that g is a continuous and non decreasing function. Furthermore, $g(0) = 0$, and $g(+\infty) = +\infty$, and therefore by the mean value theorem there exists a which satisfies the conditions of lemma. □

Let $u \geq 0$ and $v(x) = |x|^{-nr}$ with $r > 1$. By homogeneity we can assume that $\lambda = 1$. Also, for simplicity we denote $g = fv$. Now, for each integer k we denote $G_k = \left\{2^k < |x| \leq 2^{k+1}\right\}$, $I_k = \left\{2^{k-1} < |x| \leq 2^{k+2}\right\}$, $L_k = \left\{2^{k+2} < |x|\right\}$, $C_k = \left\{|x| \leq 2^{k-1}\right\}$.

It will be enough to prove the following estimates

$$\sum_{k\in Z} uv\left\{x \in G_k : M(g\chi_{I_k})(x) > \frac{1}{|x|^{nr}}\right\} \leq C_{r,n} \int g\, Mu, \tag{5.2}$$

$$\sum_{k\in Z} uv\left\{x \in G_k : M(g\chi_{L_k})(x) > \frac{1}{|x|^{nr}}\right\} \leq C_{r,n} \int g\, Mu, \tag{5.3}$$

$$\sum_{k\in Z} uv\left\{x \in G_k : M(g\chi_{C_k})(x) > \frac{1}{|x|^{nr}}\right\} \leq C_{r,n} \int g\, Mu. \tag{5.4}$$

Taking into account that in G_k, $v(x) = \frac{1}{|x|^{nr}} \sim 2^{-knr}$, using the $(1,1)$ weak type inequality of M with respect to the pair of weights (u, Mu) and since the subsets I_k overlap at most three times we obtain (5.2).

To prove inequality (5.3) we will estimate $M(g\chi_{L_k})(x)$. Observe that if x belongs to G_k and $y \in L_k = \left\{2^{k+2} < |y|\right\}$, and if $|y - x| \leq \rho$, we have that $\frac{|y|}{2} \leq \rho$,

$$\frac{1}{\rho^n}\int_{|y-x|\leq\rho} g(y)\chi_{L_k}(y)\, dy \leq C_n \int_{2^{k+2}<|y|} \frac{g(y)}{|y|^n}dy \leq C_n \int_{|x|<|y|} \frac{g(y)}{|y|^n}dy.$$

If we denote $F(x) = \int_{|x|<|y|} \frac{g(y)}{|y|^n}dy$ the left hand side of (5.3) is bounded by

$$\sum_{k\in Z} 2^{-krn} u\left\{x \in \mathbb{R}^n : F(x) > C\, 2^{-knr}\right\} \approx \int_0^\infty tu\left\{x \in \mathbb{R}^n : F(x) > t\right\} \frac{dt}{t}$$

$$= \int_{\mathbb{R}^n} F(x)\, u(x)dx = \int_{\mathbb{R}^n} \int_{|x|<|y|} \frac{g(y)}{|y|^n} dy\, u(x)dx$$

$$= \int_{\mathbb{R}^n} g(y) \frac{1}{|y|^n} \int_{|x|<|y|} u(x)dx\, dy \leq C \int_{\mathbb{R}^n} g(y)\, Mu(y)dy.$$

To prove (5.4) we estimate $M(g\chi_{C_k})(x)$ for $x \in G_k$. Indeed, if $y \in C_k$, $2\,|y| < |x|$ and since $M(g\chi_{C_k})(x) \leq \frac{c_n}{|x|^n} \int_{C_k} g(y)dy$, we obtain

$$M(g\chi_{C_k})(x) \leq \frac{C}{|x|^n} \int_{C_k} g \leq \frac{C}{|x|^n} \int_{|y| \leq \frac{|x|}{2}} g,$$

Thus, since the subsets G_k are disjoint, the left hand side in (5.4) is bounded by

$$uv \left\{ x \in \mathbb{R}^n : \frac{C}{|x|^n} \int_{|y| \leq \frac{|x|}{2}} g > \frac{1}{|x|^{nr}} \right\}.$$

Now, if a denotes the positive real number that appears in Lemma 5.1 (i.e., a satisfies $1 = \left(\int_{|y| \leq a^{\frac{1}{r-1}}} g \right) a^n$, we express the last integral in the following way:

$$uv \left(\left\{ x : \frac{C}{|x|^n} \int_{|y| \leq \frac{|x|}{2}} g > \frac{1}{|x|^{nr}} \right\} \right) = uv \left(\left\{ |x| \leq a^{\frac{1}{r-1}} : \frac{C}{|x|^n} \int_{|y| \leq \frac{|x|}{2}} g > \frac{1}{|x|^{nr}} \right\} \right) + \tag{5.5}$$

$$+ \sum_{k=0}^{\infty} uv \left(\left\{ x : 2^k a^{\frac{1}{r-1}} < |x| \leq 2^{k+1} a^{\frac{1}{r-1}} \text{ and } \frac{C}{|x|^n} \int_{|y| \leq \frac{|x|}{2}} g > \frac{1}{|x|^{nr}} \right\} \right)$$

If $|x| \leq a^{\frac{1}{r-1}}$, since $|y| \leq \frac{|x|}{2}$ we have that $|y| \leq a^{\frac{1}{r-1}}$, thus the set

$$\left\{ |x| \leq a^{\frac{1}{r-1}} : \frac{C}{|x|^n} \int_{|y| \leq \frac{|x|}{2}} g > \frac{1}{|x|^{nr}} \right\} \subset \left\{ |x| \leq a^{\frac{1}{r-1}} : |x|^{n(r-1)} > C \left(\int_{|y| \leq a^{\frac{1}{r-1}}} g \right)^{-1} \right\}.$$

Taking into account the last inclusion and since $\left(\int_{|y| \leq a^{\frac{1}{r-1}}} g \right)^{-1} = a^n$, the first summand in the second term in (5.5) is bounded by

$$uv(\{|x|^{r-1} > Ca\}) = uv(\{|x| > ca^{r'-1}\}).$$

Using again Lemma 5.1, the last term can be estimated by

$$\int_{|x| > C\, a^{r'-1}} uv\, dx \leq C \sum_{k=1}^{\infty} \frac{1}{(2^k a^{r'-1})^{nr}} \int_{c2^{k-1} a^{r'-1} \leq |x| < c2^k a^{r'-1}} u(x)\, dx \leq$$

$$\leq C \sum_{k=1}^{\infty} \frac{1}{2^{k(r-1)n}} \frac{1}{a^n} \frac{1}{(c2^k a^{r'-1})^n} \int_{|x| \leq c2^k a^{r'-1}} u(x)\, dx$$

$$= C \sum_{k=1}^{\infty} \frac{1}{2^{k(r-1)n}} \int_{|y| \leq a^{r'-1}} g(y)\, dy \frac{1}{(c2^k a^{r'-1})^n} \int_{|x| \leq c2^k a^{r'-1}} u(x)\, dx,$$

and this is bounded by

$$\leq C\sum_{k=1}^{\infty}\frac{1}{2^{k(r-1)n}}\int_{|y|\leq a^{r'-1}}g(y)Mu(y)\,dy\leq C\int g\,Mu.$$

To finish, we must estimate the series in (5.5). It is clear that sum is bounded by

$$\sum_{k=0}^{\infty}uv\left(\left\{x\in 2^k a^{r'-1}<|x|\leq 2^{k+1}a^{r'-1}\right\}\right)\leq C\sum_{k=0}^{\infty}\frac{1}{(2^k a^{r'-1})^{nr}}\int_{2^{k-1}a^{r'-1}\leq|x|<2^k a^{r'-1}}u\,dx$$

and arguing as before we conclude the proof of (5.4).

Remark 5.6. We observe that the proof only uses the following conditions for a sublinear operator T: a) T is of weak type $(1,1)$ with respect to the pair of weights (u, Mu) and b) T is a convolution type operator such that the associated kernel satisfies the usual standard condition:

$$|K(x)|\leq\frac{c}{|x|^n}.$$

In particular if $u\in A_1$, this observation can be applied to the usual Calderón-Zygmund singular integral operators and moreover to the strongly singular integral operators (see [Ch] and [F]).

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