A note on the off-diagonal Muckenhoupt-Wheeden conjecture

David Cruz-Uribe, SFO
Dept. of Mathematics, Trinity College,
Hartford, CT 06106-3100, USA
E-mail: david.cruzuribe@trincoll.edu

José María Martell
Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM
Consejo Superior de Investigaciones Científicas
C/ Nicolás Cabrera, 13-15, E-28049 Madrid, Spain,
E-mail: chema.martell@icmat.es

Carlos Pérez
Departamento de Análisis Matemático,
Facultad de Matemáticas, Universidad de Sevilla,
41012 Sevilla, Spain
E-mail: carlosperez@us.es

We obtain the off-diagonal Muckenhoupt-Wheeden conjecture for Calderón-Zygmund operators. Namely, given $1 < p < q < \infty$ and a pair of weights $(u, v)$, if the Hardy-Littlewood maximal function satisfies the following two weight inequalities:

$$M: L^p(v) \rightarrow L^q(u) \quad \text{and} \quad M: L^{p'}(u^{1-p'}) \rightarrow L^{q'}(v^{1-q'})$$

then any Calderón-Zygmund operator $T$ and its associated truncated maximal operator $T_*$ are bounded from $L^p(v)$ to $L^q(u)$. Additionally, assuming only the second estimate for $M$ then $T$ and $T_*$ map continuously $L^p(v)$ into $L^{q, \infty}(u)$.

We also consider the case of generalized Haar shift operators and show that their off-diagonal two weight estimates are governed by the corresponding estimates for the dyadic Hardy-Littlewood maximal function.

Keywords: Haar shift operators; Calderón-Zygmund operators; two-weight inequalities; testing conditions.
1. Introduction and Main results

In the 1970s, Muckenhoupt and Wheeden conjectured that given \( p, 1 < p < \infty \), a sufficient condition for the Hilbert transform to satisfy the two weight norm inequality

\[ H : L^p(v) \to L^p(u) \]

is that the Hardy-Littlewood maximal operator satisfy the pair of norm inequalities

\[ M : L^p(v) \to L^p(u), \]
\[ M : L^{p'}(u^{1-p'}) \to L^{p'}(v^{1-p'}). \]

Moreover, they conjectured that the Hilbert transform satisfies the weak-type inequality

\[ H : L^p(v) \to L^{p,\infty}(u) \]

provided that the maximal operator satisfies the second “dual” inequality. Both of these conjectures readily extend to all Calderón-Zygmund operators (see the definition below). Very recently, both conjectures were disproved: the strong-type inequality by Reguera and Scurry\(^1\) and the weak-type inequality by the first author, Reznikov and Volberg\(^2\).

Remark 1.1. A special case of these conjectures, involving the \( A_p \) bump conditions, has been considered by several authors: see\(^2\)\(^-\)\(^7\).

In this note we prove the somewhat surprising fact that the Muckenhoupt-Wheeden conjectures are true for off-diagonal inequalities. Our main result is Theorem 1.1 below. We also prove an analogous result for the Haar shift operators (the so-called dyadic Calderón-Zygmund operators) with the Hardy-Littlewood maximal operator replaced by the dyadic maximal operator: see Theorem 1.2 below.

To state our results we first give some preliminary definitions. By weights we will always mean non-negative, measurable functions. Given a pair of weights \((u, v)\), hereafter we will assume that \( u > 0 \) on a set of positive measure and \( u < \infty \) a.e., and \( v > 0 \) a.e. and \( v < \infty \) on a set of positive measure. We will also use the standard notation \( 0 \cdot \infty = 0 \).
Calderón-Zygmund operators

A Calderón-Zygmund operator $T$ is a linear operator that is bounded on $L^2(\mathbb{R}^n)$ and
\[ Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy, \quad f \in L^\infty_c(\mathbb{R}^n), \quad x \notin \text{supp} f, \]
where the kernel $K$ satisfies the size and smoothness estimates
\[ |K(x,y)| \leq \frac{C}{|x-y|^a}, \quad x \neq y, \]
and
\[ |K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \frac{C|y-x'|^b}{|x-y|^{a+b}}, \]
for all $|x-y| > 2|x-x'|$.

Associated with $T$ is the truncated maximal operator
\[ T_* f(x) = \sup_{0 < \epsilon < c' < \infty} \left| \int_{|x-y| < c'} K(x,y) f(y) dy \right|. \]

Let $M$ denote the Hardy-Littlewood maximal operator, that is,
\[ Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_{Q} |f(y)| dy = \sup_{Q \ni x} \frac{1}{|Q|} \int_{Q} |f(y)| dy, \]
where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

**Theorem 1.1.** Given a Calderón-Zygmund operator $T$, let $1 < p < q < \infty$ and let $(u, v)$ be a pair of weights. If the maximal operator satisfies
\[ M : L^p(v) \to L^q(u) \quad \text{and} \quad M : L^{q'}(u^{1-q'}) \to L^{p'}(u^{1-p'}), \]
then
\[ \|Tf\|_{L^q(u)} \leq C\|f\|_{L^p(v)} \quad \text{and} \quad \|T_*f\|_{L^{q'}(u^{1-q'})} \leq C\|f\|_{L^{p'}(u^{1-p'})}. \]

Analogously, if the maximal operator satisfies
\[ M : L^{q'}(u^{1-q'}) \to L^{p'}(u^{1-p'}), \]
then
\[ \|Tf\|_{L^{q',\infty}(u)} \leq C\|f\|_{L^p(v)} \quad \text{and} \quad \|T_*f\|_{L^{q',\infty}(u)} \leq C\|f\|_{L^{p'}(v)}. \]
If the pairs of weights \((u, v)\) satisfy any of the conditions in (1), then the weights \(u\) and \(v^{1-p'}\) are locally integrable. This is a consequence of a characterization of the two weight norm inequalities for the maximal operator due to Sawyer\(^8\). He proved that the \(L^p - L^q\) inequality holds if and only if for every cube \(Q\),

\[
\left( \int_Q M(u^{1-p'} \chi_Q)(x) u(x) \, dx \right)^{1/q} \leq C \left( \int_Q v(x)^{1-p'} \, dx \right)^{1/p} < \infty,
\]

and the \(L^{q'} - L^{p'}\) inequality holds if and only if

\[
\left( \int_Q M(u \chi_Q)(x)^{p'} v(x)^{1-p'} \, dx \right)^{1/p'} \leq C \left( \int_Q u(x) \, dx \right)^{1/q'} < \infty.
\]

It is straightforward to construct pairs of weights that satisfy these conditions. For instance, in \(\mathbb{R}\) both of these conditions follow easily for every \(1 < p \leq q < \infty\) and the pair of weights \((u, v)\) with \(u = \chi_{[0,1]}\) and \(v^{-1} = \chi_{[2,3]}\) (i.e., \(v = 1\) in \([2,3]\) and \(v = \infty\) elsewhere). Indeed, we only need to check Sawyer’s inequalities for cubes \(Q\) that intersect both \([0,1]\) and \([2,3]\), in which case we have \(M(\chi_{[2,3]} \cap \cdot)(x) \leq |[2,3] \cap \cdot|\) for every \(x \in [0,1] \cap \cdot\), and \(M(\chi_{[0,1]} \cap \cdot)(x) \leq |[0,1] \cap \cdot|\) for every \(x \in [2,3] \cap \cdot\). These readily imply the desired estimates.

**Dyadic Calderón-Zygmund operators**

A generalized dyadic grid \(\mathcal{D}\) in \(\mathbb{R}^n\) is a set of generalized dyadic cubes with the following properties: if \(Q \in \mathcal{D}\) then \(\ell(Q) = 2^k\), \(k \in \mathbb{Z}\); if \(Q, R \in \mathcal{D}\) and \(Q \cap R \neq \emptyset\) then \(Q \subset R\) or \(R \subset Q\); the cubes in \(\mathcal{D}\) with \(\ell(Q) = 2^{-k}\) form a disjoint partition of \(\mathbb{R}^n\) (see\(^7\) and\(^9\) for more details).

We say that \(g_Q\) is a generalized Haar function associated with \(Q \in \mathcal{D}\) if

\begin{enumerate}[\(\text{(a)}\)]
\item \(\text{supp}(g_Q) \subset Q\);
\item if \(Q' \in \mathcal{D}\) and \(Q' \subset Q\), then \(g_Q\) is constant on \(Q'\);
\item \(\|g_Q\|_{L^\infty} \leq 1\).
\end{enumerate}

Given a dyadic grid \(\mathcal{D}\) and a pair \((m, k) \in \mathbb{Z}_+^2\), a linear operator \(S\) is a generalized Haar shift operator (that is, a dyadic Calderón-Zygmund operator) of complexity type \((m, k)\) if it is bounded on \(L^2(\mathbb{R}^n)\) and

\[
Sf(x) = \sum_{Q \in \mathcal{D}} S_Q f(x) = \sum_{Q \in \mathcal{D}} \sum_{Q' \in \mathcal{D}_m(Q)} \sum_{Q'' \in \mathcal{D}_k(Q)} \frac{\langle f, g_Q'' \rangle}{|Q|} g_{Q'}(x),
\]
where $\mathcal{D}_j(Q)$ stands for the dyadic subcubes of $Q$ with side length $2^{-j}\ell(Q)$, $g_{Q'}^{Q''}$ is a generalized a Haar function associated with $Q'$ and $g_{Q''}^{Q''}$ is a generalized a Haar function associated with $Q''$. We say that the complexity of $S$ is $\kappa = \max(m,k)$. We also define the truncated Haar shift operator

$$S_\star f(x) = \sup_{0 < \epsilon < \epsilon' < \infty} |S_{\epsilon,\epsilon'} f(x)| = \sup_{0 < \epsilon < \epsilon' < \infty} \left| \sum_{Q \in \mathcal{D}_{\epsilon} \leq \ell(Q) \leq \epsilon'} S_Q f(x) \right|.$$  

An important example of a Haar shift operator on the real line is the Haar shift (also known as the dyadic Hilbert transform) $H^d$, defined by

$$H^d f(x) = \sum_{I \in \Delta} \langle f, h_I \rangle (h_I - (x) - h_I (x)),$$

where, given a dyadic interval $I$, $I_+$ and $I_-$ are its right and left halves, and

$$h_I = |I|^{-1/2} (\chi_{I_-} (x) - \chi_{I_+}(x)).$$

After renormalizing, $h_I$ is a Haar function on $I$ and one can write $H^d$ as a generalized Haar shift operator of complexity 1. These operators have played a very important role in the proof of the $A_2$ conjecture: see\(^6,10,11\) and the references they contain for more information.

Associated with the dyadic grid $\mathcal{D}$ is the dyadic maximal function

$$M_{\mathcal{D}} f(x) = \sup_{x \in Q \in \mathcal{D}} \int_Q |f(y)|dy.$$

Note that $M_{\mathcal{D}}$ is dominated pointwise by the Hardy-Littlewood maximal operator.

We can now state our result for dyadic Calderón-Zygmund operators.

**Theorem 1.2.** Let $S$ be a generalized Haar shift operator of complexity $\kappa$. Given $1 < p < q < \infty$ and a pair of weights $(u,v)$, if the dyadic maximal operator satisfies

$$M_{\mathcal{D}} : L^p(v) \to L^p(u) \quad \text{and} \quad M_{\mathcal{D}} : L^p(u^{1-\frac{1}{q}}) \to L^{p'}(u^{1-\frac{1}{p'}}), \quad (5)$$

then

$$\|Sf\|_{L^p(u)} \leq C\kappa^2 \|f\|_{L^p(v)} \quad \text{and} \quad \|S_\star f\|_{L^p(u)} \leq C\kappa^2 \|f\|_{L^p(v)}. \quad (6)$$

Analogously, if the dyadic maximal operator satisfies

$$M_{\mathcal{D}} : L^p(u^{1-\frac{1}{q}}) \to L^{p'}(u^{1-\frac{1}{p'}}) \quad (7)$$

then

$$\|Sf\|_{L^p(u^{1-\frac{1}{q}})} \leq C\kappa^2 \|f\|_{L^p(v)} \quad \text{and} \quad \|S_\star f\|_{L^p(u^{1-\frac{1}{q}})} \leq C\kappa^2 \|f\|_{L^p(v)}. \quad (8)$$
2. Proofs of the Main results

Proof of Theorem 1.1

We will prove our estimates for $T^\star$; the ones for $T$ are completely analogous.

Given a dyadic grid $\mathcal{D}$ we say that $\{Q^k_{j,k}\}$ is a sparse family of dyadic cubes if for any $k$ the cubes $\{Q^k_{j}\}$ are pairwise disjoint; if $\Omega_k := \bigcup_j Q^k_{j}$, then $\Omega_{k+1} \subset \Omega_k$; and $|\Omega_{k+1} \cap Q^k_{j,k}| \leq \frac{1}{2}|Q^k_{j}|$. Given $\mathcal{D}$ and a sparse family $\mathcal{F} = \{Q^k_{j,k}\} \subset \mathcal{D}$, define the positive dyadic operator $A$ by

$$A_f(x) = \sum_{j,k} f_{Q^k_{j,k}} \chi_{Q^k_{j,k}}(x)$$

where $f_Q = f(y)dy$.

For our proof we will use the main result in 7, 9. Given a Banach function space $X$ and a non-negative function $f$,

$$\|T^\star f\|_X \leq C(T, n) \sup _{\mathcal{D}, \mathcal{F}} \|\mathcal{A}\|_X,$$

where the supremum is taken over all dyadic grids $\mathcal{D}$ and sparse families $\mathcal{F} \subset \mathcal{D}$. To prove Theorem 1.1 we apply this result with $X = L^q(u)$ or $X = L^{q, \infty}(u)$; it will then suffice to show that our assumptions on $M$ guarantee that $\mathcal{A}$ satisfies the corresponding two weight inequalities.

To prove this fact we will use a result by Lacey, Sawyer and Uriate-Tuero 12. Given a sequence of non-negative constants $\alpha = \{\alpha_Q\}_{Q \in \mathcal{D}}$, define the positive operator

$$T^\alpha f(x) = \sum_{Q \in \mathcal{D}} \alpha_Q f_Q \chi_Q(x).$$

Further, given $R \in \mathcal{D}$ we define the “outer truncated” operator

$$T^\alpha_R f(x) = \sum_{Q \in \mathcal{D}} \alpha_Q f_Q \chi_Q(x).$$

In 12 it was shown that for all $1 < p < q < \infty$, $T^\alpha : L^p(v) \rightarrow L^q(u)$ if and only if there exist constants $C_1$ and $C_2$ such that for every $R \in \mathcal{D}$

$$\left( \int_{R^n} T^\alpha_R (v^{1-p'}(\chi_R)(x)^q u(x)dx \right)^{\frac{1}{q}} \leq C_1 \left( \int_{R^n} v(x)^{1-p'}dx \right)^{\frac{1}{p}} ,$$

and

$$\left( \int_{R^n} T^\alpha_R (u\chi_R)(x)^p v(x)^{1-p'}dx \right)^{\frac{1}{p'}} \leq C_2 \left( \int_{R^n} u(x)dx \right)^{\frac{1}{p}} .$$
Furthermore, for $1 < p < q < \infty$, $T_\alpha : L^p(v) \to L^{q,\infty}(u)$ holds if and only if there exists a constant $C_2$ such that for every $R \in \mathcal{D}$, (11) holds.

We can apply these results to the operator $A = A_D,S$ where $D$ and $S$ are fixed, since $A = T_\alpha$ with $\alpha_Q = 1$ if $Q \in S$ and $\alpha_Q = 0$ otherwise. Fix $R \in \mathcal{D}$; to estimate $A_R$, take the increasing family of cubes $R = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq \ldots$ with $R_k \in \mathcal{D}$ and $\ell(R_k) = 2^k \ell(R)$. Define $R_{-1} = \emptyset$. Note that $\text{supp } A_R \subset \bigcup_{k \geq 0} R_k$. Then for every non-negative function $f$ and for every $x \in R_k \setminus R_{k-1}$ with $k \geq 0$ we have that

$$0 \leq A^R(f\chi_{R_k})(x) \leq \sum_{j=0}^{\infty} (f\chi_{R_j})_{R_j}(x) = f_R \sum_{j=0}^{\infty} 2^{-j n} \leq f_R 2^{-k n} = (f\chi_{R_k})_{R_k} \leq M_{\mathcal{D}}(f\chi_{R_k})(x).$$

Consequently, for every $x \in \mathbb{R}^n$,

$$0 \leq A^R(f\chi_{R})(x) \leq M_{\mathcal{D}}(f\chi_{R})(x) \leq M(f\chi_{R})(x).$$

Inequality (12) together with our hypothesis (1) implies (10) and (11). Therefore, we have that $A : L^p(v) \to L^q(u)$ with constants depending on the dimension, $p$, $q$ and the implicit constants in (1). Therefore, by Lerner’s estimate (9) we get

$$T^* : L^p(v) \to L^{q,\infty}(u)$$

as desired.

For the weak-type estimates we proceed in the same manner, using the fact that (3) yields (11) and therefore $A^R(f\chi_{R_k})(x) \leq M_{\mathcal{D}}(f\chi_{R_k})(x)$ for $k \geq 0$. Then for every non-negative function $f$ and for every $x \in R_k \setminus R_{k-1}$ with $k \geq 0$ we have that

$$0 \leq A^R(f\chi_{R_k})(x) \leq \sum_{j=0}^{\infty} (f\chi_{R_j})_{R_j}(x) = f_R \sum_{j=0}^{\infty} 2^{-j n} \leq f_R 2^{-k n} = (f\chi_{R_k})_{R_k} \leq M_{\mathcal{D}}(f\chi_{R_k})(x).$$

Inequality (12) together with our hypothesis (1) implies (10) and (11). Therefore, we have that $A : L^p(v) \to L^q(u)$ with constants depending on the dimension, $p$, $q$ and the implicit constants in (1). Therefore, by Lerner’s estimate (9) we get $T^* : L^p(v) \to L^{q,\infty}(u)$ as desired.

**Proof of Theorem 1.2**

Fix $\mathcal{D}$ and a generalized Haar shift operator of complexity $\kappa$. As before we can work with $S_\alpha$. We can repeat the previous argument except that we want to keep the fixed dyadic structure $\mathcal{D}$. A careful examination of Section 5 shows that, given a Banach function space $X$, we have

$$\|S_\alpha f\|_X \leq C_n \kappa^2 \max_{\mathcal{D}} \|A_{\mathcal{D},\alpha} f\|_X, \quad f \geq 0,$$

where the supremum is taken over all sparse families $\mathcal{F} \subset \mathcal{D}$. We emphasize that in Section 5 there is an additional supremum over the dyadic grids $\mathcal{D}$. This is because at some places the dyadic maximal operator is majorized by the regular Hardy-Littlewood maximal operator and the latter is in turn controlled by a sum of $A_{\alpha_n,\mathcal{F}_n}$ for $2^n$ dyadic grids $\mathcal{F}_n$. However, keeping $M_{\mathcal{D}}$ one can easily show that (13) holds. Details are left to the interested reader.
Given (13), we fix a sparse family $\mathcal{S} \subset \mathcal{D}$ and write $A = A_{\mathcal{D}, \mathcal{S}}$. Arguing exactly as before we obtain (12). Thus, (5) implies (10) and (11) and therefore the result from (5) yields $\mathcal{S}_*: L^p(v) \rightarrow L^q(u)$ with constants depending on the dimension, $p$, $q$ and the implicit constants in (5). Combining this with Lerner’s estimate (13) applied to $X = L^q(u)$ we conclude as desired that $\mathcal{S}_*: L^p(v) \rightarrow L^q(u)$. We get the weak-type estimate by adapting the above proof in exactly the same way.

References

12. M. T. Lacey, E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero,