# A note on the off-diagonal Muckenhoupt-Wheeden conjecture

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We obtain the off-diagonal Muckenhoupt-Wheeden conjecture for Calderón-Zygmund operators. Namely, given $1 < p < q < \infty$ and a pair of weights $(u, v)$, if the Hardy-Littlewood maximal function satisfies the following two weight inequalities:

$$M : L^p(v) \to L^q(u) \quad \text{and} \quad M : L^{q'}(u^{1-q'}) \to L^{p'}(v^{1-p'}),$$

then any Calderón-Zygmund operator $T$ and its associated truncated maximal operator $T_\star$ are bounded from $L^p(v)$ to $L^q(u)$. Additionally, assuming only the second estimate for $M$ then $T$ and $T_\star$ map continuously $L^p(v)$ into $L^{q,\infty}(u)$. We also consider the case of generalized Haar shift operators and show that their off-diagonal two weight estimates are governed by the corresponding estimates for the dyadic Hardy-Littlewood maximal function.

*Keywords*: Haar shift operators; Calderón-Zygmund operators; two-weight inequalities; testing conditions.

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## 1. Introduction and Main results

In the 1970s, Muckenhoupt and Wheeden conjectured that given $p$, $1 < p < \infty$, a sufficient condition for the Hilbert transform to satisfy the two weight norm inequality

$$H : L^p(v) \to L^p(u)$$

is that the Hardy-Littlewood maximal operator satisfy the pair of norm inequalities

$$M : L^p(v) \to L^p(u),$$
$$M : L^{p'}(u^{1-p'}) \to L^{p'}(v^{1-p'}).$$

Moreover, they conjectured that the Hilbert transform satisfies the weak-type inequality

$$H : L^p(v) \to L^{p,\infty}(u)$$

provided that the maximal operator satisfies the second "dual" inequality. Both of these conjectures readily extend to all Calderón-Zygmund operators (see the definition below). Very recently, both conjectures were disproved: the strong-type inequality by Reguera and Scurry [1] and the weak-type inequality by the first author, Reznikov and Volberg [2].

**Remark 1.1.** A special case of these conjectures, involving the $A_p$ bump conditions, has been considered by several authors: see [2–7].

In this note we prove the somewhat surprising fact that the Muckenhoupt-Wheeden conjectures are true for off-diagonal inequalities. Our main result is Theorem 1.1 below. We also prove an analogous result for the Haar shift operators (the so-called dyadic Calderón-Zygmund operators) with the Hardy-Littlewood maximal operator replaced by the dyadic maximal operator: see Theorem 1.2 below.

To state our results we first give some preliminary definitions. By weights we will always mean non-negative, measurable functions. Given a pair of weights $(u, v)$, hereafter we will assume that $u > 0$ on a set of positive measure and $u < \infty$ a.e., and $v > 0$ a.e. and $v < \infty$ on a set of positive measure. We will also use the standard notation $0 \cdot \infty = 0$.

### *Calderón-Zygmund operators*

A Calderón-Zygmund operator $T$ is a linear operator that is bounded on $L^2(\mathbb{R}^n)$ and

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \qquad f \in L^\infty_c(\mathbb{R}^n), \quad x \notin \operatorname{supp} f,$$

where the kernel $K$ satisfies the size and smoothness estimates

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \qquad x \neq y,$$

and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}},$$

for all $|x-y| > 2|x-x'|$.

Associated with $T$ is the truncated maximal operator

$$T_\star f(x) = \sup_{0<\epsilon<\epsilon'<\infty} \Big| \int_{\epsilon<|x-y|<\epsilon'} K(x,y)f(y)dy \Big|.$$

Let $M$ denote the Hardy-Littlewood maximal operator, that is,

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)|dy = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

**Theorem 1.1.** *Given a Calderón-Zygmund operator $T$, let $1 < p < q < \infty$ and let $(u,v)$ be a pair of weights. If the maximal operator satisfies*

$$M : L^p(v) \to L^q(u) \quad \textit{and} \quad M : L^{q'}(u^{1-q'}) \to L^{p'}(v^{1-p'}), \tag{1}$$

*then*

$$\|Tf\|_{L^q(u)} \leq C\|f\|_{L^p(v)} \quad \textit{and} \quad \|T_\star f\|_{L^q(u)} \leq C\|f\|_{L^p(v)}. \tag{2}$$

*Analogously, if the maximal operator satisfies*

$$M : L^{q'}(u^{1-q'}) \to L^{p'}(v^{1-p'}), \tag{3}$$

*then*

$$\|Tf\|_{L^{q,\infty}(u)} \leq C\|f\|_{L^p(v)} \quad \textit{and} \quad \|T_\star f\|_{L^{q,\infty}(u)} \leq C\|f\|_{L^p(v)}. \tag{4}$$

If the pairs of weights $(u, v)$ satisfy any of the conditions in (1), then the weights $u$ and $v^{1-p'}$ are locally integrable. This is a consequence of a characterization of the two weight norm inequalities for the maximal operator due to Sawyer[8]. He proved that the $L^p - L^q$ inequality holds if and only if for every cube $Q$,

$$\left(\int_Q M(v^{1-p'}\chi_Q)(x)^q u(x)\,dx\right)^{1/q} \leq C\left(\int_Q v(x)^{1-p'}\,dx\right)^{1/p} < \infty,$$

and the $L^{q'} - L^{p'}$ inequality holds if and only if

$$\left(\int_Q M(u\chi_Q)(x)^{p'} v(x)^{1-p'}\,dx\right)^{1/p'} \leq C\left(\int_Q u(x)\,dx\right)^{1/q'} < \infty.$$

It is straightforward to construct pairs of weights that satisfy these conditions. For instance, in $\mathbb{R}$ both of these conditions follow easily for every $1 < p \leq q < \infty$ and the pair of weights $(u, v)$ with $u = \chi_{[0,1]}$ and $v^{-1} = \chi_{[2,3]}$ (i.e., $v = 1$ in $[2, 3]$ and $v = \infty$ elsewhere). Indeed, we only need to check Sawyer's inequalities for cubes Q that intersect both $[0, 1]$ and $[2, 3]$, in which case we have $M(\chi_{[2,3]\cap Q})(x) \leq |[2, 3] \cap Q|$ for every $x \in [0, 1] \cap Q$, and $M(\chi_{[0,1]\cap Q})(x) \leq |[0, 1] \cap Q|$ for every $x \in [2, 3] \cap Q$. These readily imply the desired estimates.

***Dyadic Calderón-Zygmund operators***

A generalized dyadic grid $\mathscr{D}$ in $\mathbb{R}^n$ is a set of generalized dyadic cubes with the following properties: if $Q \in \mathscr{D}$ then $\ell(Q) = 2^k$, $k \in \mathbb{Z}$; if $Q, R \in \mathscr{D}$ and $Q \cap R \neq \emptyset$ then $Q \subset R$ or $R \subset Q$; the cubes in $\mathscr{D}$ with $\ell(Q) = 2^{-k}$ form a disjoint partition of $\mathbb{R}^n$ (see[7] and[9] for more details).

We say that $g_Q$ is a generalized Haar function associated with $Q \in \mathscr{D}$ if

($a$) $\operatorname{supp}(g_Q) \subset Q$;

($b$) if $Q' \in \mathscr{D}$ and $Q' \subsetneq Q$, then $g_Q$ is constant on $Q'$;

($c$) $\|g_Q\|_\infty \leq 1$.

Given a dyadic grid $\mathscr{D}$ and a pair $(m, k) \in \mathbb{Z}^2_+$, a linear operator $\mathcal{S}$ is a generalized Haar shift operator (that is, a dyadic Calderón-Zygmund operator) of complexity type $(m, k)$ if it is bounded on $L^2(\mathbb{R}^n)$ and

$$\mathcal{S}f(x) = \sum_{Q\in\mathscr{D}} \mathcal{S}_Q f(x) = \sum_{Q\in\mathscr{D}} \sum_{\substack{Q'\in\mathscr{D}_m(Q)\\ Q''\in\mathscr{D}_k(Q)}} \frac{\langle f, g_{Q'}^{Q''}\rangle}{|Q|} g_{Q''}^{Q'}(x),$$

where $\mathscr{D}_j(Q)$ stands for the dyadic subcubes of $Q$ with side length $2^{-j}\ell(Q)$, $g_{Q'}^{Q''}$ is a generalized a Haar function associated with $Q'$ and $g_{Q''}^{Q'}$ is a generalized a Haar function associated with $Q''$. We say that the complexity of $\mathcal{S}$ is $\kappa = \max(m,k)$. We also define the truncated Haar shift operator

$$\mathcal{S}_\star f(x) = \sup_{0<\epsilon<\epsilon'<\infty} |\mathcal{S}_{\epsilon,\epsilon'} f(x)| = \sup_{0<\epsilon<\epsilon'<\infty} \Big| \sum_{\substack{Q\in\mathscr{D}\\ \epsilon\le\ell(Q)\le\epsilon'}} \mathcal{S}_Q f(x) \Big|.$$

An important example of a Haar shift operator on the real line is the Haar shift (also known as the dyadic Hilbert transform) $H^d$, defined by

$$H^d f(x) = \sum_{I\in\Delta} \langle f, h_I\rangle \big(h_{I_-}(x) - h_{I_+}(x)\big),$$

where, given a dyadic interval $I$, $I_+$ and $I_-$ are its right and left halves, and

$$h_I(x) = |I|^{-1/2}\big(\chi_{I_-}(x) - \chi_{I_+}(x)\big).$$

After renormalizing, $h_I$ is a Haar function on $I$ and one can write $H^d$ as a generalized Haar shift operator of complexity 1. These operators have played a very important role in the proof of the $A_2$ conjecture: see[6,10,11] and the references they contain for more information.

Associated with the dyadic grid $\mathscr{D}$ is the dyadic maximal function

$$M_{\mathscr{D}} f(x) = \sup_{x\in Q\in\mathscr{D}} \fint_Q |f(y)|dy.$$

Note that $M_{\mathscr{D}}$ is dominated pointwise by the Hardy-Littlewood maximal operator.

We can now state our result for dyadic Calderón-Zygmund operators.

**Theorem 1.2.** *Let $\mathcal{S}$ be a generalized Haar shift operator of complexity $\kappa$. Given $1<p<q<\infty$ and a pair of weights $(u,v)$, if the dyadic maximal operator satisfies*

$$M_{\mathscr{D}} : L^p(v) \to L^q(u) \quad \text{and} \quad M_{\mathscr{D}} : L^{q'}(u^{1-q'}) \to L^{p'}(v^{1-p'}), \tag{5}$$

*then*

$$\|\mathcal{S}f\|_{L^q(u)} \le C\kappa^2 \|f\|_{L^p(v)} \quad \text{and} \quad \|\mathcal{S}_\star f\|_{L^q(u)} \le C\kappa^2\|f\|_{L^p(v)}. \tag{6}$$

*Analogously, if the dyadic maximal operator satisfies*

$$M_{\mathscr{D}} : L^{q'}(u^{1-q'}) \to L^{p'}(v^{1-p'}) \tag{7}$$

*then*

$$\|\mathcal{S}f\|_{L^{q,\infty}(u)} \le C\kappa^2\|f\|_{L^p(v)} \quad \text{and} \quad \|S_\star f\|_{L^{q,\infty}(u)} \le C\kappa^2\|f\|_{L^p(v)}. \tag{8}$$

## 2. Proofs of the Main results

### *Proof of Theorem 1.1*

We will prove our estimates for $T_\star$; the ones for $T$ are completely analogous.

Given a dyadic grid $\mathscr{D}$ we say that $\{Q_j^k\}_{j,k}$ is a *sparse family* of dyadic cubes if for any $k$ the cubes $\{Q_j^k\}_j$ are pairwise disjoint; if $\Omega_k := \cup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$; and $|\Omega_{k+1} \cap Q_{j,k}| \leq \frac{1}{2}|Q_j^k|$. Given $\mathscr{D}$ and a sparse family $\mathscr{S} = \{Q_j^k\}_{j,k} \subset \mathscr{D}$, define the positive dyadic operator $\mathscr{A}$ by

$$\mathscr{A} f(x) = \mathscr{A}_{\mathscr{D},\mathscr{S}} f(x) = \sum_{j,k} f_{Q_j^k} \chi_{Q_j^k}(x)$$

where $f_Q = \fint_Q f(y)dy$.

For our proof we will use the main result in[7,9]. Given a Banach function space $X$ and a non-negative function $f$,

$$\|T_\star f\|_X \leq C(T,n) \sup_{\mathscr{D},\mathscr{S}} \|\mathscr{A}_{\mathscr{D},\mathscr{S}} f\|_X, \tag{9}$$

where the supremum is taken over all dyadic grids $\mathscr{D}$ and sparse families $\mathscr{S} \subset \mathscr{D}$. To prove Theorem 1.1 we apply this result with $X = L^q(u)$ or $X = L^{q,\infty}(u)$; it will then suffice to show that our assumptions on $M$ guarantee that $\mathscr{A}_{\mathscr{D},\mathscr{S}}$ satisfies the corresponding two weight inequalities.

To prove this fact we will use a result by Lacey, Sawyer and Uriate-Tuero[12]. Given a sequence of non-negative constants $\alpha = \{\alpha_Q\}_{Q\in\mathscr{D}}$, define the positive operator

$$T_\alpha f(x) = \sum_{Q\in\mathscr{D}} \alpha_Q f_Q \chi_Q(x).$$

Further, given $R \in \mathscr{D}$ we define the "outer truncated" operator

$$T_\alpha^R f(x) = \sum_{\substack{Q\in\mathscr{D}\\ mathcalQ\supset R}} \alpha_Q f_Q \chi_Q(x).$$

In[12] it was shown that for all $1 < p < q < \infty$, $T_\alpha : L^p(v) \to L^q(u)$ if and only if there exist constants $C_1$ and $C_2$ such that for every $R \in \mathscr{D}$

$$\left(\int_{\mathbb{R}^n} T_\alpha^R (v^{1-p'}\chi_R)(x)^q u(x)dx\right)^{\frac{1}{q}} \leq C_1 \left(\int_R v(x)^{1-p'}dx\right)^{\frac{1}{p}}, \tag{10}$$

and

$$\left(\int_{\mathbb{R}^n} T_\alpha^R (u\chi_R)(x)^{p'} v(x)^{1-p'}dx\right)^{\frac{1}{p'}} \leq C_2 \left(\int_R u(x)dx\right)^{\frac{1}{q'}}. \tag{11}$$

Furthermore, for $1 < p < q < \infty$, $T_\alpha : L^p(v) \to L^{q,\infty}(u)$ holds if and only if there exists a constant $C_2$ such that for every $R \in \mathscr{D}$, (11) holds.

We can apply these results to the operator $\mathscr{A} = \mathscr{A}_{\mathscr{D},\mathscr{S}}$ where $\mathscr{D}$ and $\mathscr{S}$ are fixed, since $\mathscr{A} = T_\alpha$ with $\alpha_Q = 1$ if $Q \in \mathscr{S}$ and $\alpha_Q = 0$ otherwise. Fix $R \in \mathscr{D}$; to estimate $\mathscr{A}^R$, take the increasing family of cubes $R = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq \dots$ with $R_k \in \mathscr{D}$ and $\ell(R_k) = 2^k \ell(R)$. Define $R_{-1} = \emptyset$. Note that $\operatorname{supp} \mathscr{A}^R \subset \cup_{k \geq 0} R_k$. Then for every non-negative function $f$ and for every $x \in R_k \setminus R_{k-1}$ with $k \geq 0$ we have that

$$
\begin{aligned}
0 \leq \mathscr{A}^R(f\chi_R)(x) \leq \sum_{j=0}^{\infty} (f\chi_R)_{R_j} \chi_{R_j}(x) = f_R \sum_{j=k}^{\infty} 2^{-j\,n} \\
\lesssim f_R 2^{-k\,n} = (f\chi_R)_{R_k} \leq M_{\mathscr{D}}(f\chi_R)(x).
\end{aligned}
$$

Consequently, for every $x \in \mathbb{R}^n$,

$$
0 \leq \mathscr{A}^R(f\chi_R)(x) \lesssim M_{\mathscr{D}}(f\chi_R)(x) \leq M(f\chi_R)(x). \tag{12}
$$

Inequality (12) together with our hypothesis (1) implies (10) and (11). Therefore, we have that $\mathscr{A} : L^p(v) \to L^q(u)$ with constants depending on the dimension, $p$, $q$ and the implicit constants in (1). Therefore, by Lerner's estimate (9) we get $T_\star : L^p(v) \to L^q(u)$ as desired.

For the weak-type estimates we proceed in the same manner, using the fact that (3) yields (11) and therefore $\mathscr{A} : L^p(v) \to L^{q,\infty}(u)$. This in turn implies, by Lerner's estimate (9) applied to $X = L^{q,\infty}(u)$, that $T_\star : L^p(v) \to L^{q,\infty}(u)$.

***Proof of Theorem 1.2***

Fix $\mathscr{D}$ and a generalized Haar shift operator of complexity $\kappa$. As before we can work with $\mathcal{S}_\star$. We can repeat the previous argument except that we want to keep the fixed dyadic structure $\mathscr{D}$. A careful examination of[7] Section 5 shows that, given a Banach function space $X$, we have

$$
\|\mathcal{S}_\star f\|_X \leq C_n \kappa^2 \sup_{\mathscr{S}} \|\mathscr{A}_{\mathscr{D},\mathscr{S}} f\|_X, \qquad f \geq 0, \tag{13}
$$

where the supremum is taken over all sparse families $\mathscr{S} \subset \mathscr{D}$. We emphasize that in[7] Section 5 there is an additional supremum over the dyadic grids $\mathscr{D}$. This is because at some places the dyadic maximal operator is majorized by the regular Hardy-Littlewood maximal operator and the latter is in turn controlled by a sum of $\mathscr{A}_{\mathscr{D}_\alpha,\mathscr{S}_\alpha}$ for $2^n$ dyadic grids $\mathscr{D}_\alpha$. However, keeping $M_{\mathscr{D}}$ one can easily show that (13) holds. Details are left to the interested reader.

Given (13), we fix a sparse family $\mathscr{S} \subset \mathscr{D}$ and write $\mathscr{A} = \mathscr{A}_{\mathscr{D},\mathscr{S}}$. Arguing exactly as before we obtain (12). Thus, (5) implies (10) and (11) and therefore the result from[12] yields $\mathscr{A} : L^p(v) \to L^q(u)$ with constants depending on the dimension, $p$, $q$ and the implicit constants in (5). Combining this with Lerner's estimate (13) applied to $X = L^q(u)$ we conclude as desired that $\mathcal{S}_\star : L^p(v) \to L^q(u)$. We get the weak-type estimate by adapting the above proof in exactly the same way.

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