Reconstruction from boundary measurements for less regular conductivities *

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Abstract

In this paper, following Nachman's idea [14] and Haberman and Tataru's idea [9], we reconstruct C^1 conductivity γ or Lipchitz conductivity γ with small enough value of $|\nabla log\gamma|$ in a Lipschitz domain Ω from the Dirichlet-to-Neumann map Λ_γ. In the appendix the authors and R. M. Brown recover the gradient of a C^1-conductivity at the boundary of a Lipschitz domain from the Dirichlet-to-Neumann map Λ_γ.

MR Subject Classification: 35R30.

Keywords: Inverse conductivity problem, Dirichlet-to-Neumann map, Calderón problem, Boundary integral equation, Bourgain's space

Contents

1 Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be an open, bounded domain and let γ be a strictly positive real valued function defined on Ω which gives the conductivity at a given point. Given a voltage potential f on the boundary, the equation for the potential in the interior, under the assumption of no sinks or sources of current in Ω, is

$$\operatorname{div}(\gamma \nabla u) = 0, \quad \text{in } \Omega, \quad u|_{\partial\Omega} = f. \tag{1.1}$$

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The Dirichlet-to-Neumann map is defined in this case as follows:

$$\Lambda_\gamma(f) = \gamma \tfrac{\partial u}{\partial \nu}|_{\partial\Omega},$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative at the boundary. For $\gamma \in L^\infty(\Omega)$ and being the boundary of Ω Lipschitz, then Λ_γ is a well defined map from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$.

The Calderón problem concerns the inversion of the map $\gamma \to \Lambda_\gamma$, i.e., whether Λ_γ determines γ uniquely and in that case how to reconstruct γ from Λ_γ.

For the boundary determination: if $\partial\Omega$ is C^∞ and $\gamma \in C^\infty(\bar{\Omega})$, Kohn and Vogelius [11] showed that Λ_γ determines $\frac{\partial^k \gamma}{\partial \nu^k}|_{\partial\Omega}$ for all $k \geq 0$; if $\partial\Omega$ is Lipschitz and $\gamma \in Lip(\Omega)$, Brown [3] showed that $\gamma|_{\partial\Omega}$ can be recovered from the knowledge of Λ_γ; if $\partial\Omega$ is C^1 and $\gamma \in C^1(\bar{\Omega})$, Nakamura and Tanuma [17] showed that $\frac{\partial \gamma}{\partial \nu}|_{\partial\Omega}$ can be recovered from Λ_γ. Now, being $\gamma \in C^1(\bar{\Omega})$ and for Ω Lipschitz domain, the authors and Brown together show the recovering of the gradient of the conductivity at the boundary (see Appendix).

For the higher dimensional problem ($n \geq 3$), uniqueness was first proven for piecewise analytic conductivities by Kohn and Vogelius [12]. Later, Sylvester and Uhlmann showed that the uniqueness holds for smooth conductivities in their fundamental paper [21] which opened the door for studying the Calderón problem. Generalization to less regular conductivities had been obtained by several authors. Brown [2] obtained uniqueness under the assumption of $\frac{3}{2} + \epsilon$ derivatives. Päivärinta, Panchenko and Uhlmann [18] proved uniqueness under the assumption of $\frac{3}{2}$ bounded derivatives. Brown and Torres [4] obtained uniqueness under the assumption of $\frac{3}{2}$ derivatives being in $L^p, p > 2n$. Later, Uhlmann [23] proposed a conjecture whether uniqueness holds in dimension $n \geq 3$ for Lipschitz or less regular conductivities.

Recently, Haberman and Tataru [9] gave a partial answer to this conjecture. They used a totally new idea to construct CGO solutions in Bourgain's space and showed uniqueness for C^1 conductivity γ or Lipchitz conductivity γ with small enough value of $|\nabla log\gamma|$. The ideas and techniques coming from this work have been widely used as can be seen in the papers by Zhang [25], Caro, García and Reyes [5] or Caro and Zhou [7]. Some progress on the uniqueness has been done recently, as shown in Caro and Rogers [6] and Haberman [8].

If uniqueness holds, one whether or not construct the conductivity in the domain Ω by the boundary measurements. Nachman [14] and Novikov [16] independently solved this problem for C^2 conductivity.

We will briefly describe Nachman's idea as follows. For $\gamma \in C^2(\bar{\Omega})$, if u is a solution to (1.1), Sylvester and Uhlmann reduced $v = \gamma^{\frac{1}{2}} u$ to satisfy

$$-\triangle v + qv = 0 \qquad \text{in } \Omega, \tag{1.2}$$

where $q = \frac{\triangle \gamma^{1/2}}{\gamma^{1/2}}$, and the corresponding Dirichlet-to-Neumann map Λ_q to (1.2) is determined by Λ_γ. For $\rho_1^{(n)} \cdot \rho_1^{(n)} = 0, \rho_1^{(n)} \in \mathbb{C}^n$ and $|\rho_1^{(n)}| \to \infty,$ as $n \to \infty$, Sylvester and Uhlmann constructed a family of CGO solutions $v^{(n)} = e^{x\cdot\rho_1^{(n)}}(1 + \Phi_1^{(n)}(x))$ to

$$-\triangle v + qv = 0 \qquad \text{in } \mathbb{R}^n, \tag{1.3}$$

with the remainder term $\Phi_1^{(n)}(x)$ decaying to zero in some sense as $|\rho_1^{(n)}| \to \infty$, where $q = \frac{\triangle \gamma^{1/2}}{\gamma^{1/2}}$ in Ω and $q = 0$ outside Ω.

For the appropriate chosen $\rho_2^{(n)} \cdot \rho_2^{(n)} = 0, \rho_2^{(n)} \in \mathbb{C}^n$ and $\rho_1^{(n)} + \rho_2^{(n)} = ik$, Green's formula gives us

$$\int_{\mathbb{R}^n} q(x) e^{ix\cdot k}(1 + \Phi_1^{(n)}(x))dx = \int_{\partial\Omega} (\Lambda_q e^{x\cdot\rho_2^{(n)}} - \frac{\partial e^{x\cdot\rho_2^{(n)}}}{\partial \nu}) v^{(n)} dS, \tag{1.4}$$

where

$$\Lambda_q = \left(\gamma^{-1/2}\Lambda_\gamma\gamma^{-1/2} + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu}\right)|_{\partial\Omega}. \tag{1.5}$$

Letting $n \to \infty$ in (1.4) we conclude from the decay property of $\Phi_1^{(n)}(x)$

$$\hat{q}(k) = \lim_{n\to\infty}\int_{\partial\Omega}(\Lambda_q e^{x\cdot\rho_2^{(n)}} - \frac{\partial e^{x\cdot\rho_2^{(n)}}}{\partial\nu})v^{(n)}dS, \tag{1.6}$$

so the problem is then to recover the boundary values $v^{(n)}|_{\partial\Omega} = f_{\rho_1^{(n)}}$.

On the othe hand, $v^{(n)}$ is a solution to the exterior problem:

$$\begin{cases} -\triangle v^{(n)} = 0 & \text{in } \mathbb{R}^n\backslash\bar{\Omega}, \\ v^{(n)}|_{\partial\Omega} = f_{\rho_1^{(n)}}, & \frac{\partial v^{(n)}}{\partial\nu}|_{\partial\Omega} = \Lambda_q f_{\rho_1^{(n)}}. \end{cases} \tag{1.7}$$

If $v^{(n)}$ satisfies the radiation condition

$$\lim_{R\to\infty}\int_{|y|=R} G_{\rho_1^{(n)}}(x,y)\frac{\partial(v^{(n)} - e^{x\cdot\rho_1^{(n)}})}{\partial\nu(y)} - \frac{\partial G_{\rho_1^{(n)}}(x,y)}{\partial\nu(y)}(v^{(n)} - e^{x\cdot\rho_1^{(n)}})dS(y) = 0, \tag{1.8}$$

using Green's formula in (1.7) we can reduce $f_{\rho_1^{(n)}}$ to satisfy a boundary integral equation

$$f_{\rho_1^{(n)}} = e^{x\cdot\rho_1^{(n)}} - (S_{\rho_1^{(n)}}\Lambda_q - B_{\rho_1^{(n)}} - \frac{1}{2}I)f_{\rho_1^{(n)}} \tag{1.9}$$

and (1.9) is uniquely solvable by Fredholm alternative theory, where $G_{\rho_1^{(n)}}, S_{\rho_1^{(n)}}$ and $B_{\rho_1^{(n)}}$ are defined in Section 3.

Unfortunately, it is not easy to directly check that $v^{(n)}$ satisfy (1.8). To move around this obstacle, Nachman [14] dealt with this problem in an inverse direction: He first constructed CGO solutions to (1.3) automatically satisfying the radial condition (1.8) from boundary integral equation (1.9) and then showed that these solutions are the same as the solutions obtained by Sylvester and Uhlmann[21].

For $\gamma \in C^1(\bar{\Omega})$ or $\gamma \in Lip(\Omega)$ with the small enough value of $|\nabla log\gamma|$ we will follow Nachman's idea to construct CGO solutions to the equation

$$\text{div}(\gamma\nabla u) = 0, \quad \text{in } \mathbb{R}^n \tag{1.10}$$

from the boundary integral equation (1.9). In view of the less smooth regular γ, we need to do some changes in the above steps. First we reduce conductivity γ to the case $\gamma \equiv 1$ near the boundary of $B_R(0) \supset \Omega$ and show that the new Dirichlet-to-Neumann map $\tilde{\Lambda}_\gamma$ corresponding to γ in $B_R(0)$ is determined by Λ_γ. Next we construct CGO solutions to (1.10) from the boundary integral equation (1.9) on the boundary $\partial B_R(0)$ and show that these solutions are the same as the solutions obtained by Haberman and Tataru [9].

We state the theorem as follows.

Theorem 1.1. *The conductivity γ can be recovered in the domain Ω by the knowledge of Λ_γ under one of the following assumptions:*
(a) Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with Lipschitz boundary and let $\gamma(x) \in C^1(\bar{\Omega})$ be a real valued function with $\gamma(x) \geq C_0 > 0$.
(b) Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with Lipschitz boundary and let $\gamma(x) \in Lip(\Omega)$ be a real valued function with $\gamma(x) \geq C_0 > 0$ and such that there exists a constant $\varepsilon_{n,\Omega}$ satisfying $|\nabla log\gamma(x)| < \varepsilon_{n,\Omega}$.

Our paper is organized as follows. In Section 2 we reduce the conductivity γ to be $\gamma\equiv 1$ near the boundary. In Section 3 we construct CGO solutions from the boundary integral equation. In Section 4 we state the reconstruction of the conductivity γ from Λ_γ. In the appendix the authors and R. M. Brown give the proof of the recovering of the gradient of a C^1-conductivity at the boundary of a Lipschitz domain.

2 Reduction to The Case $\gamma\equiv 1$ near The Boundary

For a bounded domain Ω with Lipschitz boundary and $\gamma\in Lip(\Omega)$, if Λ_γ is known, we can recover the values of γ at the boundary $\partial\Omega$ (see [3]). From this knowledge we can extend γ to be a Lipchitz function in $\mathbb{R}^n$ with $\gamma\equiv 1$ outside a large ball B_{R_0} and $\gamma(x)\geq C_0>0$. Furthermore, If $|\nabla log\gamma(x)|<\varepsilon_{n,\Omega}$ we can keep this property holding in $\mathbb{R}^n$. For a bounded Lipschitz domain Ω and $\gamma\in C^1(\bar{\Omega})$, if Λ_γ is known, we can recover the values of γ and the gradient of γ at the boundary $\partial\Omega$ (see the appendix). Next, Whitney's extension allows us from the information of $\partial^\alpha\gamma(x)$ for all $x\in\partial\Omega$ and all $|\alpha|\leq 1$ to extend γ to be C^1 in $\mathbb{R}^n$ with $\gamma\equiv 1$ outside a large ball B_{R_0} and $\gamma(x)\geq C_0>0$(The readers are referred to see Chapter VI §4.7 of [20]). In the above two cases obviously γ is known in $\mathbb{R}^n\backslash\Omega$.

Now for fixed $R>R_0$ such that $\Omega\subset B_R(0)$, we define the new Dirichlet-to-Neumann map as follows:

$$\tilde{\Lambda}_\gamma:\qquad f|_{\partial B_R(0)}\longrightarrow \frac{\partial u_f}{\partial\nu}|_{\partial B_R(0)},$$

where u_f is a solution to

$$\begin{cases}\operatorname{div}(\gamma\nabla u)=0, & \text{in } B_R(0),\\ u|_{\partial B_R(0)}=f.\end{cases}$$

In the following, we will show $\tilde{\Lambda}_\gamma$ is determined by Λ_γ and the value of γ in $\bar{B}_R(0)\backslash\Omega$. Here we need to mention that Nachman already used this idea in [15] and obtained an exact formula in the plane with the the conductivity $\gamma\in W^{2,p}, p>1$ (see Proposition 6.1 of [15]). Following Nachman's idea, we will generalize this formula in a bounded Lipchitz domain $\Omega\subset\mathbb{R}^n, n\geq 2$, for a Lipchitz conductivity γ.

In fact, if we denote by Ω_1, respectively Ω_2, the domains Ω and $B_R(0)$, and by $\partial\Omega_1$, respectively $\partial\Omega_2$, the boundary of Ω and the boundary of $B_R(0)$, the Dirichlet-to-Neumann map corresponding to γ in the domain $\Omega_2\backslash\bar{\Omega}_1$ can be viewed as 2×2 matrix of operators $\Lambda^{ij}, H^1(\partial\Omega_i)\rightarrow H^{\frac{1}{2}}(\partial\Omega_j), i,j=1,2.$, defined as follows. Given $f_j\in H^1(\partial\Omega_j)$ for $j=1,2.$, considering the Dirichlet problem

$$\begin{cases}\operatorname{div}(\gamma\nabla w)=0, & \text{in } \Omega_2\backslash\bar{\Omega}_1,\\ w|_{\partial\Omega_1}=f_1, \quad w|_{\partial\Omega_2}=f_2,\end{cases}$$

we have the following Dirichlet-to Neumann map:$H^1(\partial\Omega_j)\rightarrow L^2(\partial\Omega_j)$ as follows:

$$\begin{pmatrix}-\gamma\frac{\partial w}{\partial\nu_+}|_{\partial\Omega_1}\\ \gamma\frac{\partial w}{\partial\nu_-}|_{\partial\Omega_2}\end{pmatrix}=\begin{pmatrix}\Lambda^{11} & \Lambda^{12}\\ \Lambda^{21} & \Lambda^{22}\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix},\tag{2.1}$$

where $\frac{\partial w}{\partial\nu_+}|_{\partial\Omega_1}$ denotes the nontangential limit of $\frac{\partial w}{\partial\nu}$ from outside of $\bar{\Omega}_1$ and $\frac{\partial w}{\partial\nu_-}|_{\partial\Omega_2}$ denotes the nontangential limit of $\frac{\partial w}{\partial\nu}$ from inside of Ω_2. Since γ is known in $\bar{\Omega}_2\backslash\Omega_1$, $\Lambda^{ij}, i,j=1,2.$, are determined. Now for any $f\in H^1(\partial\Omega_2)$ we have a solution $u\in H^{\frac{3}{2}}(\Omega_2)\bigcap C^s(\Omega_2)$, where $0\leq s<2$, satisfying

$$\begin{cases}\operatorname{div}(\gamma\nabla u)=0, & \text{in } \Omega_2,\\ u|_{\partial\Omega_2}=f.\end{cases}$$

Let $g = u|_{\partial\Omega_1} \in H^1(\partial\Omega_1)$. We have that (2.1) implies

$$\tilde{\Lambda}_\gamma = \gamma \frac{\partial w}{\partial \nu_-}|_{\partial\Omega_2} = \Lambda^{21} g + \Lambda^{22} f. \tag{2.2}$$

If g can be recovered from $\Lambda^{ij}, \Lambda_\gamma$ and f, then $\tilde{\Lambda}_\gamma$ is known. On the other hand, in view of $u \in H^{\frac{3}{2}}(\Omega_2) \bigcap C^s(\Omega_2)$, we can deduce from (2.1)

$$\Lambda^{11} g + \Lambda^{12} f = -\gamma \frac{\partial u}{\partial \nu_+}|_{\partial\Omega_1} = -\Lambda_\gamma g.$$

Hence, it follows that

$$(-\Lambda_\gamma - \Lambda^{11}) g = \Lambda^{12} f.$$

If we can show that $(-\Lambda_\gamma - \Lambda^{11})$ is an invertible operator: $H^1(\partial\Omega_1) \to L^2(\partial\Omega_1)$, then we have

$$g = (-\Lambda_\gamma - \Lambda^{11})^{-1} \Lambda^{12} f. \tag{2.3}$$

Finally (2.2) and (2.3) imply

$$\tilde{\Lambda}_\gamma = \Lambda^{21}(-\Lambda_\gamma - \Lambda^{11})^{-1} \Lambda^{12} f + \Lambda^{22} f. \tag{2.4}$$

Now we state the generalized result of Nachman [15] as follows.

Theorem 2.1. *Let $\bar{\Omega}_1 \subset \Omega_2$ be a bounded Lipschitz domain in $\mathbb{R}^n, n \geq 2$ and let $\gamma(x) \in Lip(\Omega_2)$ with $\gamma(x) \geq C_0 > 0$. Then $(-\Lambda_\gamma - \Lambda^{11})$ is an invertible operator: $H^1(\partial\Omega_1) \to L^2(\partial\Omega_1)$ and $\tilde{\Lambda}_\gamma = \Lambda^{21}(-\Lambda_\gamma - \Lambda^{11})^{-1}\Lambda^{12} + \Lambda^{22}$.*

Before proving Theorem 2.1, we first introduce some known results. Let $G(x, y)$ be the Green function such that, for $x \in \Omega_2$,

$$\begin{cases} \operatorname{div}(\gamma \nabla G(x,y)) = \delta_x, & \text{in } \Omega_2, \\ G(x,y)|_{\partial\Omega_2} = 0. \end{cases} \tag{2.5}$$

Now we define the single layer potential and double layer potential as

$$Sf(x) = \int_{\partial\Omega_1} G(x,y) f(y) dS(y), \quad x \in \bar{\Omega}_2, \tag{2.6}$$

$$Df(x) = \int_{\partial\Omega_1} \frac{\partial G(x,y)}{\partial \nu(y)} f(y) dS(y), \quad x \in \bar{\Omega}_2. \tag{2.7}$$

For the single potential $Sf(x)$ and the double layer potential $Df(x)$, we collect the following results from Mitrea and Taylor's paper [13] (see Proposition 1.6, Theorem 3.7, Proposition 3.8 and Proposition 8.2 of [13]).

Proposition 2.2. *If Ω_1 is a bounded Lipschitz domain in $\mathbb{R}^n, n \geq 2$ and $f \in L^2(\partial\Omega_1)$, then the functions $Sf(x)$ and $Df(x)$ have the following properties:*

(a) The following estimates hold

$$\|Df(x)\|_{L^2(\partial\Omega_1)} \leq C\|f\|_{L^2(\partial\Omega_1)},$$

$$\|Sf(x)\|_{H^1(\partial\Omega_1)} \leq C\|f\|_{L^2(\partial\Omega_1)}.$$

(b) $\operatorname{div}(\gamma\nabla Sf(x)) = 0, x \in \Omega_2 \backslash \partial\Omega_1$.

(c) $Sf(x) \in H^1(\Omega_2 - \bar{\Omega}_1)$ and $Sf(x) \in H^1(\Omega_1)$. The boundary value $Sf_+(x)(Sf_-(x))$ of $Sf(x)$ from outside (respectively inside) Ω_1 are identical as elements of $H^1(\partial\Omega_1)$ and agree with $Sf(x)|_{\partial\Omega_1}$.

(d) The (nontangential) limits $\frac{\partial Sf(x)}{\partial\nu_+}(\frac{\partial Sf(x)}{\partial\nu_-})$ as the boundary is approached from outside (respectively inside) Ω_1 are given by the formula

$$\frac{\partial Sf(x)}{\partial\nu_-^+} = {}^{-}_{+}\frac{1}{2}\frac{1}{\gamma(x)}f(x) + B^*f(x), \qquad \textit{almost everywhere } x \in \partial\Omega_1,$$

where

$$B^*f(x) = p.v. \int_{\partial\Omega_1} \frac{\partial G(x,y)}{\partial\nu(x)} f(y) dS(y).$$

In particular

$$\frac{\partial Sf(x)}{\partial\nu_-} - \frac{\partial Sf(x)}{\partial\nu_+} = \frac{1}{\gamma(x)} f(x).$$

(e) The nontangential limits $Df_+(x)(Df_-(x))$ of $Df(x)$ as we approach the boundary from outside (respectively inside) Ω_1 exist and satisfy

$$Df_{\substack{+\\-}} = {}^{+}_{-}\frac{1}{2}\frac{1}{\gamma(x)}f(x) + Bf(x), \qquad \textit{almost everywhere } x \in \partial\Omega_1,$$

where

$$Bf(x) = p.v. \int_{\partial\Omega_1} \frac{\partial G(x,y)}{\partial\nu(y)} f(y) dS(y).$$

Remark 2.3. Our case is a little bit different from the case of the paper [13]. In [13] they dealt with the operator $L = -\triangle + V(x)(V(x) \in L^\infty(M), V \geq 0,$ not identical zero.) in a compact, connected smooth M with C^1 metric tensor. But if we just want to get the above Proposition 2.2, we can reproduce the steps of the proof of Theorem 3.7 and Proposition 3.8 to show that the above Proposition 2.2 still holds for the operator $L = \operatorname{div}(\gamma\nabla u)$ in a bounded domain in Ω_2 with Lipchitz conductivity γ.

We know that the Dirichlet-to Neumann map Λ_γ is a bounded operator:$H^{\frac{1}{2}}(\partial\Omega_1) \to H^{-\frac{1}{2}}(\partial\Omega_1)$. In fact, when Ω_1 is a bounded Lipschitz domain and $\gamma(x) \in Lip(\Omega_1)$, the Dirichlet-to Neumann map Λ_γ is a bounded operator :$H^1(\partial\Omega_1) \to L^2(\partial\Omega_1)$(the same argument for $\tilde{\Lambda}_\gamma, \Lambda^{i,j}, i,j = 1,2.$). This result follows from the following Proposition 2.4 and Proposition 2.4 can be deduced from Proposition 7.4 and Proposition 8.2 of the paper [13].

Proposition 2.4. *Suppose Ω is a bounded domain in $\mathbb{R}^n, n \geq 2$ with Lipchitz boundary and $\gamma(x) \in Lip(\Omega)$ with $\gamma(x) \geq C_0 > 0$. Let $u(x) \in H^1(\Omega)$ be a solution to*

$$\begin{cases} \operatorname{div}(\gamma\nabla u) = 0, & \textit{in } \Omega, \\ u|_{\partial\Omega} = g \in H^1(\partial\Omega). \end{cases}$$

Then we have

$$\|\frac{\partial u}{\partial\nu}\|_{L^2(\partial\Omega)} \leq C\|g\|_{H^1(\partial\Omega)}$$

and

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \leq C\|g\|_{H^1(\partial\Omega)},$$

where $(\nabla u)^*$ *is the nontangential maximal fuction of* ∇u.

With the above preliminary works in hand we are now in the position to show an identity about $\Lambda_\gamma + \Lambda_{11}$.

Lemma 2.5. *For any* $g \in H^1(\partial\Omega_1)$*, we have the identity*

$$S(\Lambda_\gamma + \Lambda_{11})g|_{\partial\Omega_1} = g|_{\partial\Omega_1}.$$

Proof. For any $g \in H^1(\partial\Omega_1)$ we can find a solution $w \in H^1(\Omega_2\backslash\bar{\Omega}_1)$ satisfying

$$\begin{cases} \operatorname{div}(\gamma\nabla w) = 0, & \text{in } \Omega_2\backslash\bar{\Omega}_1, \\ w|_{\partial\Omega_1} = g, \quad w|_{\partial\Omega_2} = 0. \end{cases}$$

Since $\gamma(x) \in Lip(\Omega_2)$ the local regularity of the uniformly elliptic equation implies that $w \in H^2_{loc}(\Omega_2\backslash\bar{\Omega}_1)$ and Proposition 2.4 implies that $(\nabla w)^* \in L^2(\partial(\Omega_2\backslash\bar{\Omega}_1))$. Therefore we can still use Green's formula for $x \in \Omega_2\backslash\bar{\Omega}_1$,

$$\begin{aligned} w(x) &= \int_{\Omega_2\backslash\bar{\Omega}_1} \operatorname{div}(\gamma\nabla w)G(x,y)dy - \int_{\Omega_2\backslash\bar{\Omega}_1} w\operatorname{div}(\gamma\nabla G(x,y))dy \\ &= \int_{\partial\Omega_2} G(x,y)\gamma\frac{\partial w}{\partial\nu}dS(y) - \int_{\partial\Omega_2}\frac{\partial G(x,y)}{\partial\nu(y)}\gamma w dS(y) \\ &+ \int_{\partial\Omega_1} G(x,y)\Lambda^{11}g dS(y) + \int_{\partial\Omega_1}\frac{\partial G(x,y)}{\partial\nu(y)}\gamma w dS(y). \end{aligned} \tag{2.8}$$

Since $G(x,y) = 0, y \in \partial\Omega_2$ and $w|_{\partial\Omega_2} = 0$, (2.8) implies

$$w(x) = \int_{\partial\Omega_1} G(x,y)\Lambda^{11}g dS(y) + \int_{\partial\Omega_1}\frac{\partial G(x,y)}{\partial\nu(y)}\gamma w dS(y). \tag{2.9}$$

Let $x \to \partial\Omega_1$ in (2.9) and Proposition 2.2 implies

$$g(x) = \int_{\partial\Omega_1} G(x,y)\Lambda^{11}g dS(y) + \frac{1}{2}g(x) + \int_{\partial\Omega_1}\frac{\partial G(x,y)}{\partial\nu(y)}\gamma g dS(y), a.e., x \in \partial\Omega_1. \tag{2.10}$$

On the other hand we can find a solution u to

$$\begin{cases} \operatorname{div}(\gamma\nabla u) = 0, & \text{in } \Omega_1, \\ u|_{\partial\Omega_1} = g. \end{cases}$$

Using Green's formula again we know for $x \in \Omega_1$

$$\begin{aligned} u(x) &= \int_{\Omega_1} \operatorname{div}(\gamma\nabla u)G(x,y)dy - \int_{\Omega_1} u\operatorname{div}(\gamma\nabla G(x,y))dy \\ &= \int_{\partial\Omega_1} G(x,y)\Lambda_\gamma g dS(y) - \int_{\partial\Omega_1}\frac{\partial G(x,y)}{\partial\nu(y)}\gamma u dS(y). \end{aligned} \tag{2.11}$$

Let $x \to \partial\Omega_1$ in (2.11) and Proposition 2.2 gives

$$(2.12) \qquad g(x)=\int_{\partial\Omega_1} G(x,y)\Lambda_\gamma g dS(y)+\frac{1}{2}g(x)-\int_{\partial\Omega_1}\frac{\partial G(x,y)}{\partial\nu(y)}\gamma g dS(y), a.e., x\in\partial\Omega_1.$$

Finally (2.10) + (2.12) imply

$$g(x)=\int_{\partial\Omega_1} G(x,y)(\Lambda^{11}+\Lambda_\gamma)g dS(y),$$

which means

$$S(\Lambda_\gamma+\Lambda_{11})g|_{\partial\Omega_1}=g|_{\partial\Omega_1}.$$

At last we need to point out that the above proof is just a formal deduction due to the singularity of $\mathrm{div}(\gamma\nabla G(x,y))$. To overcome this obstacle we can deal with the integrations in a domain in which we remove a small ball $B_\varepsilon(x)$ centered at the singular point x. Finally letting $\varepsilon\to 0$ we get the above identity. □

Now we will show that $\Lambda_\gamma+\Lambda_{11}$ is an invertible operator: $H^1(\partial\Omega_1)\to L^2(\partial\Omega_1)$. From this Theorem 2.1 follows.

Lemma 2.6. $\Lambda_\gamma+\Lambda_{11}$ *is an invertible operator:* $H^1(\partial\Omega_1)\to L^2(\partial\Omega_1)$ *and the invertible operator is* $Sf|_{\partial\Omega_1}$ *for any* $f\in L^2(\partial\Omega_1)$.

Proof. We first show that $S|_{\partial\Omega_1}$ is an isomorphism from $L^2(\partial\Omega_1)$ to $H^1(\partial\Omega_1)$. In fact Lemma 2.5 implies $S|_{\partial\Omega_1}$ is surjective. To show that $S|_{\partial\Omega_1}$ is injective we assume that $\int_{\partial\Omega_1} G(x,y)f(y)dS(y)=0$ on $\partial\Omega_1$ for $f\in L^2(\partial\Omega_1)$. By Proposition 2.2 we know $Sf(x)$ is a solution to $\mathrm{div}(\gamma\nabla u)=0$ in Ω_1 and $Sf_-(x)=0$ on the boundary $\partial\Omega_1$. Hence the maximum principle of the uniformly elliptic equation implies $Sf(x)\equiv 0$ in $\bar\Omega_1$. On the other hand $Sf(x)$ is also a solution to $\mathrm{div}(\gamma\nabla u)=0$ in $\Omega_2\backslash\bar\Omega_1$. In view of the choice $G(x,y)=0, x\in\partial\Omega_2$, we know $Sf(x)|_{\partial\Omega_2}=0$ and Proposition 2.2 implies $Sf_+(x)=0$ on the boundary $\partial\Omega_1$, so the maximum principle of the uniformly elliptic equation also implies $Sf(x)\equiv 0$ in $\bar\Omega_2\backslash\Omega_1$.

From the analysis above we know that $Sf(x)\equiv 0$ in Ω_2. By Proposition 2.2, we have

$$\frac{1}{\gamma(x)}f(x)=\frac{\partial Sf(x)}{\partial\nu_-}-\frac{\partial Sf(x)}{\partial\nu_+}=0,$$

which implies $f\equiv 0$ on the boundary $\partial\Omega_1$. Thus $S|_{\partial\Omega_1}$ is bijective and Proposition 2.2 implies $S|_{\partial\Omega_1}$ is a bounded operator:$L^2(\partial\Omega_1)\to H^1(\partial\Omega_1)$. Finally Banach inverse mapping theorem gives us $S|_{\partial\Omega_1}$ is an isomorphism which $L^2(\partial\Omega_1)$ to $H^1(\partial\Omega_1)$.

On the other hand, from Lemma 2.5's identity $S|_{\partial\Omega_1}(\Lambda_\gamma+\Lambda_{11})=I$, bijection of $S|_{\partial\Omega_1}$ implies $\Lambda_\gamma+\Lambda_{11}$ is also bijective and from Proposition 2.4 we have that $\Lambda_\gamma+\Lambda_{11}$ is a bounded operator: $H^1(\partial\Omega_1)\to L^2(\partial\Omega_1)$. Banach inverse mapping theorem implies $(\Lambda_\gamma+\Lambda_{11})^{-1}=S|_{\partial\Omega_1}$. □

3 Construction of CGO Solutions from the Boundary Integral Equation

In this section we will follow Nachman's idea to construct Complex Geometrical Optic solutions to the equation $\mathrm{div}(\gamma\nabla u)=0$ from the boundary integral equation. We begin with some basic properties of Green's function. For $\rho\in\mathbb{C}^n$ with $\rho\cdot\rho=0$, $(-\xi^2+2i\rho\cdot\xi)^{-1}$ is easily checked to be locally integrable as a function of ξ in $\mathbb{R}^n, n\geq 3$. Its inverse Fourier transform formally written as

$$g_\rho(x)=g(x,\rho)=\frac{1}{(2\pi)^n}\int\frac{e^{ix\cdot\xi}}{-\xi^2+2i\rho\cdot\xi}d\xi$$

is a tempered distribution satisfying

$$(\triangle_x + 2\rho \cdot \nabla_x) g(x,\rho) = \delta_0(x).$$

The distribution $G_\rho(x) = -e^{x\cdot\rho} g(x,\rho)$ is then a fundamental solution for the Laplace operator in $\mathbb{R}^n$:

$$-\triangle G_\rho(x) = \delta_0(x).$$

It follows that $G_\rho(x)$ differs from the Green's function of classical potential theory by a global harmonic function in $\mathbb{R}^n$:

$$G_\rho(x) = G_0(x) + H_\rho(x), \quad \triangle H_\rho = 0 \text{ in } \mathbb{R}^n,$$

where

$$G_0(x) = g(x,0) = \frac{1}{(n-2)\omega_n}|x|^{2-n}(\omega_n = \frac{(2\pi)^{n/2}}{\Gamma(n/2)}).$$

Note that $G_\rho(x)$ is a smooth function for x away from the origin and has the same singularity near $x = 0$ as that of $G_0(x)$.

Using the family G_ρ of Green's function defined above we now consider analogues of the classical single and double potentials. Let

$$S_\rho f(x) = \int_{\partial\Omega} G_\rho(x,y) f(y) dS(y),$$

$$Df_\rho(x) = \int_{\partial\Omega} \frac{\partial G_\rho(x,y)}{\partial\nu(y)} f(y) dS(y),$$

where $G_\rho(x,y) =: G_\rho(x-y)$ and to begin with, f is continuous on $\partial\Omega$ and $x \in \mathbb{R}^n \backslash \partial\Omega$. Define the boundary, or trace , double layer potential by

$$B_\rho f(x) = p.v. \int_{\partial\Omega} \frac{\partial G_\rho(x,y)}{\partial\nu(y)} f(y) dS(y), \text{for } x \in \partial\Omega.$$

Now we collect some properties about single and double layer potentials from Nachman's paper [14] as follows (see Lemma 2.4 and Lemma 2.5 of [14]).

Proposition 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$ with $C^{1,1}$ boundary and suppose $f \in H^{\frac{1}{2}}(\partial\Omega)$. Then the function $u = S_\rho f$ have the following properties:*

(a) $\triangle u = 0$ in $\mathbb{R}^n \backslash \partial\Omega$.

(b) u is in $H^2(\Omega)$ and in $H^2(\Omega'_\theta)$ for $\theta > \theta_0$, where θ_0 is a large enough number so that $\bar\Omega \subset \{x : \ |x| < \theta_0\}$, and for any $\theta > \theta_0$, $\Omega'_\theta = \{x : \ x \notin \bar\Omega, |x| < \theta\}$.

(c)The boundary values $u_+(u_-)$ of u from outside (respectively inside) Ω are identical as elements of $H^{\frac{3}{2}}(\partial\Omega)$ and agree with the trace of single layer potential $S_\rho f$.

(d) u satisfies the following decay properties:

$$\lim_{\theta\to\infty} \int_{|y|=\theta} G_\rho(x,y) \frac{\partial u}{\partial\nu(y)} - u(y) \frac{\partial G_\rho(x,y)}{\partial\nu(y)} dS(y) = 0, a.e., x \in \mathbb{R}^n. \tag{3.1}$$

(e) Let $g \in H^{\frac{3}{2}}(\partial\Omega)$ and $v = D_\rho g$ defined in $\mathbb{R}^n \backslash \partial\Omega$. Then the nontangential limits $v_+(v_-)$ of v as we approach the boundary from outside (respectively inside) Ω exist and satisfy

$$v_{\pm} = \pm\frac{1}{2} g(x) + B_\rho f(x) \ a.e., x \in \partial\Omega.$$

With the above preliminary works in hand we first deduce a boundary integral equation for the CGO solutions.

From the analysis we performed in Section 2 we can assume $\Omega = B_R(0)$ and $\gamma(x) \equiv 1$ near the boundary $\partial B_R(0)$ and for $\gamma \in Lip(B_R(0))$(or $\gamma \in C^1(\bar{B}_R(0))$) define the Dirichlet-to-Neumann map as follows

$$\Lambda_\gamma : \ f \in H^{\frac{3}{2}}(\partial B_R(0)) \to \frac{\partial u_f}{\partial \nu} \in H^{\frac{1}{2}}(\partial B_R(0)),$$

where u_f is the solution to

$$\begin{cases} \mathrm{div}(\gamma \nabla u) = 0, & \text{in } B_R(0), \\ u|_{\partial B_R(0)} = f. \end{cases}$$

Clearly Λ_γ is a bounded operator: $H^{\frac{3}{2}}(\partial B_R(0)) \to H^{\frac{1}{2}}(\partial B_R(0))$.

We are now ready to establish a one to one correspondence between the solutions of the boundary integral equation and those of the following exterior problem:

$$(3.2) \qquad \begin{array}{ll} (i)\ \triangle \psi = 0 & \text{in } \mathbb{R}^n \backslash \bar{B}_R(0), \\ (ii)\ \psi \in H^2(B_{R'} \backslash \bar{B}_R(0)), & \text{for } R' > R, \\ (iii)\ \psi(x,\rho) - e^{x\cdot\rho} & \text{satisfies (3.1)}, \\ (iv)\ \frac{\partial \psi}{\partial \nu_+} = \Lambda_\gamma \psi & \text{on } \partial B_R(0). \end{array}$$

We state Lemma 2.7 of Nachman's paper [14] as follows.

Proposition 3.2. *(a) Suppose ψ solves $((3.2)(i)-(iv))$. Then its trace on the boundary $f_\rho = \psi_+ = \psi|_{\partial B_R(0))}$ solves*

$$(3.3) \qquad f_\rho = e^{x\cdot\rho} - (S_\rho \Lambda_\gamma - B_\rho - \frac{1}{2} I) f_\rho.$$

(b) Conversely, suppose $f_\rho \in H^{\frac{3}{2}}(\partial B_R(0)))$ solves (3.3). Then the function $\psi(x,\rho)$ defined for x in $\mathbb{R}^n \backslash \bar{B}_R(0)$ by

$$(3.4) \qquad \psi(x,\rho) = e^{x\cdot\rho} - (S_\rho \Lambda_\gamma - D_\rho) f_\rho$$

solves the exterior problem $(3.2)(i),(ii),(iii),(iv)$. Furthermore $\psi|_{\partial B_R(0))} = f_\rho$.

In the following we will use the boundary integral equation (3.3) to construct CGO solutions to $\mathrm{div}(\gamma \nabla u) = 0$ in $\mathbb{R}^n$.

Lemma 3.3. *Suppose $f_\rho \in H^{\frac{3}{2}}(\partial B_R(0)))$ solves (3.3). Then*

(a) There exists a unique solution $u \in H^2_{loc}(\mathbb{R}^n)$ to

$$(3.5) \qquad \mathrm{div}(\gamma \nabla u) = 0$$

in $\mathbb{R}^n$ such that $u = \psi(x,\rho)$ in $\mathbb{R}^n \backslash B_R(0)$.

(b) Let $v = \gamma^{\frac{1}{2}} u$ and $v \in H^1_{loc}(\mathbb{R}^n)$ is a weak solution to the following Schrödinger equation

$$-\triangle v + qv = 0, in\ \mathbb{R}^n,$$

where $q = \dfrac{\triangle \gamma^{1/2}}{\gamma^{1/2}}$. Furthermore the following identity holds

$$(3.6) \qquad v(x) = e^{x\cdot\rho} + \int_{\mathbb{R}^n} \nabla \gamma^{1/2} \cdot \nabla (\frac{1}{\gamma^{1/2}} G_\rho(x,y) v) dy, \ a.e., x \in \mathbb{R}^n.$$

(c) Let $v(x) = e^{x\cdot\rho}(1+\Phi(x,\rho))$. Then $\Phi(x,\rho) \in H^1_{loc}(\mathbb{R}^n)$ satisfies

$$\Phi(x,\rho) = \int_{\mathbb{R}^n} g_\rho(x,y)q(y)dy + \int_{\mathbb{R}^n} g_\rho(x,y)q(y)\Phi(y,\rho)dy \tag{3.7}$$

as a tempered distribution.

Proof. For $f_\rho \in H^{\frac{3}{2}}(\partial B_R(0)))$ we can find a solution $w \in H^2(B_R(0))$ such that

$$\begin{cases} \operatorname{div}(\gamma\nabla w) = 0 & \text{in } B_R(0), \\ w|_{\partial B_R(0)} = f_\rho. \end{cases}$$

Proposition 3.2 implies $w|_{\partial B_R(0)} = \psi(x,\rho)|_{\partial B_R(0)} = f_\rho$ and $\frac{\partial w}{\partial \nu_-} = \frac{\partial \psi(x,\rho)}{\partial \nu_+}$. Hence defining

$$u = \begin{cases} w & \text{in } B_R(0), \\ \psi(x,\rho) & \text{in } \mathbb{R}^n \backslash B_R(0), \end{cases}$$

we know $u \in H^2_{loc}(\mathbb{R}^n)$ is a solution to

$$\operatorname{div}(\gamma\nabla u) = 0 \quad \text{in } \mathbb{R}^n.$$

The uniqueness follows from the maximum principle of the uniformly elliptic equation. Then (a) follows.

Letting $v = \gamma^{\frac{1}{2}}u$ in view of $\gamma \equiv 1$ in $\mathbb{R}^n\backslash B_{R_0}(0)(R_0 < R)$, we can deduce from (3.5) that $v \in H^1_{loc}(\mathbb{R}^n)\bigcap H^2_{loc}(\mathbb{R}^n\backslash \bar{B}_{R_0}(0))$ is a weak solution to

$$-\triangle v + qv = 0, \text{in } \mathbb{R}^n, \tag{3.8}$$

where $q = \dfrac{\triangle\gamma^{1/2}}{\gamma^{1/2}}$.

For any $R^{'} > R$ and $\delta > 0$ small enough, we can consider a Lipschitz function $h_\delta(t)$ given by

$$h_\delta(t) = \begin{cases} 1 & t \in [0,R^{'}]\backslash[R-\delta, R+\delta], \\ \frac{R-t}{\delta} & t \in [R-\delta, R], \\ \frac{t-R}{\delta} & t \in [R, R+\delta], \end{cases}$$

and then define the function $H_\delta(x) = h_\delta(|x|) \in Lip(B_{R'}(0)), x \in \bar{B}_{R'}(0)$. Clearly, we have

$$(i)\ |H_\delta(x)| \le 1 \textit{ and } H_\delta(x)|_{\partial B_R(0)} = 0,$$
$$(ii)\ spt|\nabla H_\delta(x)| \subset T_\delta =: \bar{B}_{R+\delta}(0)\backslash B_{R-\delta}(0).$$

Given $\varepsilon > 0$ we define the following regularized version of $G_\rho(x,y)$ by

$$G^\varepsilon_\rho(x,y) = \frac{1}{(n-2)\omega_n}(|x-y|^2+\varepsilon^2)^{\frac{2-n}{2}} + H_\rho(x-y). \tag{3.9}$$

Taking the test function $H_\delta(y)G^\varepsilon_\rho(x,y) \in H^1_0(B_R(0))$ in (3.8), we have

$$\int_{B_R(0)} \nabla v \cdot \nabla(H_\delta(y)G^\varepsilon_\rho(x,y))dy = \int_{B_R(0)} \nabla\gamma^{1/2} \cdot \nabla(\frac{1}{\gamma^{1/2}} vH_\delta(y)G^\varepsilon_\rho(x,y))dy. \tag{3.10}$$

On the other hand since $v \in H^2(B_{R'}(0)\backslash\bar{B}_R(0))$ and $\gamma \equiv 1$ in $B_{R'}(0)\backslash\bar{B}_R(0)$, Green's formula gives us

$$
\begin{aligned}
&\int_{T_{R'}} \nabla\gamma^{1/2}\cdot\nabla(\frac{1}{\gamma^{1/2}}vH_\delta(y)G^\varepsilon_\rho(x,y))dy = \int_{T_{R'}} -\triangle vH_\delta(y)G^\varepsilon_\rho(x,y)dy \\
= &\int_{T_{R'}} \nabla v\cdot\nabla(H_\delta(y)G^\varepsilon_\rho(x,y))dy - \int_{\partial B_{R'}(0)}\frac{\partial v}{\partial\nu}G^\varepsilon_\rho(x,y)dS(y),
\end{aligned}
\tag{3.11}
$$

where $T_{R'} = B_{R'}(0)\backslash\bar{B}_R(0)$.

Now from (3.10)+(3.11) we have

$$
\begin{aligned}
&\int_{B_{R'}(0)} \nabla\gamma^{1/2}\cdot\nabla(\frac{1}{\gamma^{1/2}}vH_\delta(y)G^\varepsilon_\rho(x,y))dy \\
= &\int_{B_{R'}(0)} \nabla v\cdot\nabla(H_\delta(y)G^\varepsilon_\rho(x,y))dy - \int_{\partial B_{R'}(0)}\frac{\partial v}{\partial\nu}G^\varepsilon_\rho(x,y)dS(y) =: I + II.
\end{aligned}
\tag{3.12}
$$

Note that the left hand side of (3.12) can be rewritten as follows

$$
\begin{aligned}
&\int_{B_{R'}(0)} \nabla\gamma^{1/2}\cdot\nabla(\frac{1}{\gamma^{1/2}}vG^\varepsilon_\rho(x,y))H_\delta(y)dy \\
&+\int_{B_{R'}(0)} \nabla\gamma^{1/2}\cdot\nabla H_\delta(y)\frac{1}{\gamma^{1/2}}vG^\varepsilon_\rho(x,y)dy =: III + IV.
\end{aligned}
\tag{3.13}
$$

Letting $\delta\to 0$ Lebesgue dominated convergence theorem implies

$$
\lim_{\delta\to 0} III = \int_{B_{R'}(0)} \nabla\gamma^{1/2}\cdot\nabla(\frac{1}{\gamma^{1/2}}vG^\varepsilon_\rho(x,y))dy.
\tag{3.14}
$$

On the other hand in view of $spt|\nabla H_\delta(x)| \subset T_\delta =: \bar{B}_{R+\delta}(0)\backslash B_{R-\delta}(0)$ and $\gamma\equiv 1$ in $\mathbb{R}^n\backslash\bar{B}_{R_0}(0)$, when $\delta < R - R_0$ we have

$$
IV = 0.
\tag{3.15}
$$

Combining (3.12), (3.13), (3.14) and (3.15), we deduce that

$$
\lim_{\delta\to 0}\int_{B_{R'}(0)} \nabla\gamma^{1/2}\cdot\nabla(\frac{1}{\gamma^{1/2}}vH_\delta(y)G^\varepsilon_\rho(x,y))dy = \int_{B_{R'}(0)} \nabla\gamma^{1/2}\cdot\nabla(\frac{1}{\gamma^{1/2}}vG^\varepsilon_\rho(x,y))dy.
\tag{3.16}
$$

For I we have

$$
I = \int_{B_{R'}(0)} \nabla v\cdot\nabla G^\varepsilon_\rho(x,y)H_\delta(y)dy + \int_{B_{R'}(0)} \nabla v\cdot\nabla H_\delta(y)G^\varepsilon_\rho(x,y)dy =: V + VI.
\tag{3.17}
$$

For V by letting $\delta\to 0$ Lebesgue dominated convergence theorem implies

$$
\lim_{\delta\to 0} V = \int_{B_{R'}(0)} \nabla v\cdot\nabla G^\varepsilon_\rho(x,y)dy.
\tag{3.18}
$$

In view of $spt|\nabla H_\delta| \subset T_\delta$ and $v\in H^2_{loc}(\mathbb{R}^n\backslash\bar{B}_{R_0})$ integration by parts gives us

$$
\begin{aligned}
VI &= \int_{T_\delta}\nabla v\cdot\nabla H_\delta(y)G^\varepsilon_\rho(x,y)dy \\
&= -\int_{T_\delta}\triangle vH_\delta(y)G^\varepsilon_\rho(x,y)dy - \int_{T_\delta}\nabla v\cdot\nabla G^\varepsilon_\rho(x,y)H_\delta(y)dy \\
&+\int_{\partial B_{R+\delta}(0)}\frac{\partial v}{\partial\nu}G^\varepsilon_\rho(x,y)dS(y) - \int_{\partial B_{R-\delta}(0)}\frac{\partial v}{\partial\nu}G^\varepsilon_\rho(x,y)dS(y)
\end{aligned}
\tag{3.19}
$$

Since $v \in H^2_{loc}(\mathbb{R}^n \backslash \bar{B}_{R_0}) \bigcap C^\infty(\mathbb{R}^n \backslash \bar{B}_{R_0})$ and $G^\varepsilon_\rho(x,y)$ is smooth, letting $\delta \to 0$ in (3.19) we have

$$\lim_{\delta \to 0} VI = 0. \tag{3.20}$$

Combining (3.12), (3.17), (3.18) and (3.20), we have

$$\lim_{\delta \to 0} \int_{B_{R'}(0)} \nabla v \cdot \nabla (H_\delta(y) G^\varepsilon_\rho(x,y)) dy = \int_{B_{R'}(0)} \nabla v \cdot \nabla G^\varepsilon_\rho(x,y) dy, \tag{3.21}$$

and from (3.12), (3.16) and (3.21) we derive

$$\begin{aligned} & \int_{B_{R'}(0)} \nabla \gamma^{1/2} \cdot \nabla (\frac{1}{\gamma^{1/2}} v G^\varepsilon_\rho(x,y)) dy \\ = & \int_{B_{R'}(0)} \nabla v \cdot \nabla G^\varepsilon_\rho(x,y) dy - \int_{\partial B_{R'}(0)} \frac{\partial v}{\partial \nu} G^\varepsilon_\rho(x,y) dS(y). \end{aligned} \tag{3.22}$$

On the other hand, Green's formula gives us

$$\begin{aligned} & - \int_{B_{R'}(0)} v \triangle G^\varepsilon_\rho(x,y) dy \\ = & \int_{B_{R'}(0)} \nabla v \cdot \nabla G^\varepsilon_\rho(x,y) dy - \int_{\partial B_{R'}(0)} \frac{\partial G^\varepsilon_\rho(x,y)}{\partial \nu(y)} v dS(y). \end{aligned} \tag{3.23}$$

From (3.22) and (3.23), we deduce

$$\begin{aligned} - \int_{B_{R'}(0)} v \triangle G^\varepsilon_\rho(x,y) & = \int_{\partial B_{R'}(0)} \frac{\partial v}{\partial \nu} G^\varepsilon_\rho(x,y) dS(y) - \int_{\partial B_{R'}(0)} \frac{\partial G^\varepsilon_\rho(x,y)}{\partial \nu(y)} v dS(y) \\ & + \int_{B_{R'}(0)} \nabla \gamma^{1/2} \cdot \nabla (\frac{1}{\gamma^{1/2}} v G^\varepsilon_\rho(x,y)) dy. \end{aligned} \tag{3.24}$$

Due to $\gamma \equiv 1$ in $\mathbb{R}^n \backslash B_{R'}(0)$ and $v = \gamma^{1/2} u$, letting $\varepsilon \to 0$ in (3.24) we have for almost every $x \in B_{R'}(0)$

$$\begin{aligned} v(x) & = \int_{\partial B_{R'}(0)} \frac{\partial u}{\partial \nu} G_\rho(x,y) dS(y) - \int_{\partial B_{R'}(0)} \frac{\partial G_\rho(x,y)}{\partial \nu(y)} u dS(y) \\ & + \int_{\mathbb{R}^n} \nabla \gamma^{1/2} \cdot \nabla (\frac{1}{\gamma^{1/2}} v G_\rho(x,y)) dy, \end{aligned} \tag{3.25}$$

where we have used that $-\triangle G^\varepsilon_\rho(x,y)$ is an approximation of identity and L^p estimate of the convolution type integration about the kernels of $G_0(x-y)$ and $\nabla G_0(x-y)$.

In view of $-\triangle_y G_\rho(x,y) = \delta_x$ and $\triangle e^{x\cdot\rho} = 0$ whenever $\rho \cdot \rho = 0$, we deduce from Green's formula

$$e^{x\cdot\rho} = \int_{\partial B_{R'}(0)} \frac{\partial e^{x\cdot\rho}}{\partial \nu} G_\rho(x,y) dS(y) - \int_{\partial B_{R'}(0)} \frac{\partial G_\rho(x,y)}{\partial \nu(y)} e^{x\cdot\rho} dS(y). \tag{3.26}$$

Since $u = \psi(x,\rho)$ in $\mathbb{R}^n \backslash B_R(0)$ and Proposition 3.2 implies that $u - e^{x\cdot\rho}$ satisfies (3.1), we can deduce from (3.26)

$$\lim_{R' \to \infty} \Big(\int_{\partial B_{R'}(0)} \frac{\partial u}{\partial \nu} G_\rho(x,y) dS(y) - \int_{\partial B_{R'}(0)} \frac{\partial G_\rho(x,y)}{\partial \nu(y)} u dS(y) \Big) = e^{x\cdot\rho}. \tag{3.27}$$

Then (3.25) and (3.27) imply

$$v(x) = e^{x\cdot\rho} + \int_{\mathbb{R}^n} \nabla\gamma^{1/2}\cdot\nabla(\frac{1}{\gamma^{1/2}}G_\rho(x,y)v)dy,\ a.e., x\in\mathbb{R}^n. \tag{3.28}$$

Then (b) follows.

Let $\gamma_\varepsilon = J_\varepsilon * \gamma$, where J_ε is the standard mollifier. Since $\gamma\in Lip(\mathbb{R}^n)$ and $\gamma\equiv 1$ in $\mathbb{R}^n\backslash B_R(0)$, by integration by parts we can deduce from (3.28)

$$v(x) = e^{x\cdot\rho} - \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \frac{\triangle\gamma_\varepsilon^{1/2}}{\gamma_\varepsilon^{1/2}}G_\rho(x,y)vdy. \tag{3.29}$$

Recalling $v(x) = e^{x\cdot\rho}(1+\Phi(x,\rho))$ we can deduce from (3.29)

$$\Phi(x,\rho) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} g_\rho(x,y)\frac{\triangle\gamma_\varepsilon^{1/2}}{\gamma_\varepsilon^{1/2}}dy + \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} g_\rho(x,y)\frac{\triangle\gamma_\varepsilon^{1/2}}{\gamma_\varepsilon^{1/2}}\Phi(y,\rho)dy, \tag{3.30}$$

where we have used the relations $G_\rho(x,y) = G_\rho(x-y)$, $g_\rho(x,y) = g_\rho(x-y)$ and $g_\rho(x) = -e^{x\cdot\rho}G_\rho(x)$.

Hence, we conclude from (3.30) that

$$\Phi(x,\rho) = \int_{\mathbb{R}^n} g_\rho(x,y)\frac{\triangle\gamma^{1/2}}{\gamma^{1/2}}dy + \int_{\mathbb{R}^n} g_\rho(x,y)\frac{\triangle\gamma^{1/2}}{\gamma^{1/2}}\Phi(y,\rho)dy \tag{3.31}$$

as a tempered distribution.

In fact,

$$\Big(\int_{\mathbb{R}^n} g_\rho(x,y)q(y)dy\Big)\hat{}(\xi) = \frac{1}{-|\xi|^2+2i\rho\cdot\xi}\Big(\frac{\triangle\gamma^{1/2}}{\gamma^{1/2}}\Big)\hat{}(\xi), \tag{3.32}$$

and

$$\Big(\int_{\mathbb{R}^n} g_\rho(x,y)q(y)\Phi(y,\rho)dy\Big)\hat{}(\xi) = \frac{1}{-|\xi|^2+2i\rho\cdot\xi}\Big(\frac{\triangle\gamma^{1/2}}{\gamma^{1/2}}\Phi(y,\rho)\Big)\hat{}(\xi). \tag{3.33}$$

For $\gamma\in Lip(\mathbb{R}^n)$ and $\gamma\equiv 1$ in $\mathbb{R}^n\backslash B_R(0)$ and $\Phi(x,\rho)\in H^1_{loc}(\mathbb{R}^n)$ from the results of the paper of [9], we know (3.31), (3.32) and (3.33) all make sense. Then (c) follows. □

In Lemma 3.3, we proved the identity (3.7). We mentioned that $\int_{\mathbb{R}^n} g_\rho(x,y)q(y)dy$ and $\int_{\mathbb{R}^n} g_\rho(x,y)q(y)\Phi(y,\rho)dy$ are tempered distributions. In fact in the paper [9] by Habermann and Tataru, they showed $\int_{\mathbb{R}^n} g_\rho(x,y)q(y)dy \in \dot{X}_\rho^{\frac{1}{2}}$ and $\int_{\mathbb{R}^n} g_\rho(x,y)q(y)\ \Phi(y,\rho)dy \in \dot{X}_\rho^{\frac{1}{2}}$. We are now in the position to introduce the definition of Bourgain's space $\dot{X}_\rho^b$ (see [1]). Following the idea of Haberman and Tataru we define the space $\dot{X}_\rho^b$ by the norm

$$\|u\|_{\dot{X}_\rho^b} = \||p_\rho(\xi)|^b\hat{u}(\xi)\|_{L^2},$$

where $p_\rho(\xi) = -|\xi|^2 + 2i\rho\cdot\xi$ is the symbol of $\triangle_\rho := \triangle + 2\rho\cdot\nabla$.

Now given $k\in\mathbb{R}^n$, consider P a 2-dimensional linear subpace orthogonal to k and set

$$\rho_1 = s\eta_1 + i\left(\frac{k}{2} + r\eta_2\right), \tag{3.34}$$

$$\rho_2 = -s\eta_1 + i\left(\frac{k}{2} - r\eta_2\right), \tag{3.35}$$

where $\eta_1, \eta_2 \in S^{n-1}$ satisfy $(k, \eta_1) = (k, \eta_2) = (\eta_1, \eta_2) = 0$ and $\frac{|k|^2}{4} + r^2 = s^2$. We have that η_1 can be chosen to be $\eta_1 \in P \cap S^{n-1}$ (for later references set $S := P \cap S^{n-1}$) and η_2 is the unique vector making $\{\eta_1, \eta_2\}$ a positively oriented orthonormal basis of P. The vectors are chosen so that $\rho_i \cdot \rho_i = 0, i = 1, 2.$ and $\rho_1 + \rho_2 = ik$.

We can prove the following result.

Theorem 3.4. *Let $\gamma(x) \in Lip(B_R(0))$ be a real valued function and assume that $\gamma(x) \geq C_0 > 0$ with $\gamma(x) \equiv 1$ near the boundary $\partial B_R(0)$. Then there exists a constant $\varepsilon_{n,R}$ such that if γ satisfies either $\|\nabla log\gamma\| \leq \varepsilon_{n,R}$ or $\gamma \in C^1(\bar{B}_R(0))$, then the following properties hold*

(a) $S_\rho \Lambda_\gamma - B_\rho - \frac{1}{2}I$ is a compact operator on $H^{\frac{3}{2}}(\partial B_R(0))$.

(b) For any $\rho \cdot \rho = 0, \rho \in \mathbb{C}^n$, when $|\rho|$ is large enough, then there exists a unique $f_\rho \in H^{\frac{3}{2}}(\partial B_R(0))$ such that

$$f_\rho = e^{x\cdot\rho} - (S_\rho \Lambda_\gamma - B_\rho - \frac{1}{2}I)f_\rho.$$

(c) Let us consider $\rho_i = \rho_i(s, \eta_1)$ as above. We have that $u = \gamma^{-1/2} e^{x\cdot\rho_i}(1 + \Phi(x, \rho_i))$ are solutions to

$$\mathrm{div}(\gamma \nabla u) = 0, \textit{ in } \mathbb{R}^n.$$

Moreover, for a fixed $k \in \mathbb{R}^n$ and P as above we have that for sufficiently large λ,

$$\frac{1}{\lambda}\int_S \int_\lambda^{2\lambda} \|\Phi(x, \rho_i)\|_{\dot{X}^{\frac{1}{2}}_{\rho_i}} ds d\eta_1 \to 0,$$

$$\frac{1}{\lambda}\int_S \int_\lambda^{2\lambda} \|\eta_B q\|_{\dot{X}^{-\frac{1}{2}}_{\rho_i}} ds d\eta_1 \to 0,$$

where η_B is a smooth function with compact support.

Proof. For (a) let $g \in H^{\frac{3}{2}}(\partial B_R(0))$ and consider $u \in H^2(B_R(0))$ solution to

$$\begin{cases} \mathrm{div}(\gamma \nabla u) = 0, & \text{in } B_R(0), \\ u|_{\partial B_R(0)} = g. \end{cases}$$

Let $x \in \Omega$ and apply Green's formula to $G^\varepsilon_\rho(x, y)$ in Ω. After taking the limit as ε tends to zero we obtain

$$\begin{aligned} \int_{B_R(0)} G_\rho(x, y) \triangle u dy + u(x) &= \int_{\partial B_R(0)} G_\rho(x, y)\frac{\partial u}{\partial \nu} dS(y) - \int_{\partial B_R(0)} G_\rho(x, y)\frac{\partial G_\rho(x, y)}{\partial \nu} g dS(y) \\ &= (S_\rho \Lambda_\gamma - D_\rho)g(x). \end{aligned}$$

Letting $x \to \partial B_R(0)$ (nontangential from inside $B_R(0)$), we deduce from Proposition 3.1

$$(S_\rho \Lambda_\gamma - B_\rho - \frac{1}{2}I)g = T\int_{B_R(0)} G_\rho(x, y)\nabla log\gamma \cdot \nabla u dy, \tag{3.36}$$

where T denotes the trace operator.

Since $\gamma \in Lip(B_R(0))$ from the estimates of the uniformly elliptic equation we know that the operator $P_\gamma : g \in H^{\frac{3}{2}}(\partial B_R(0)) \to u \in H^2(B_R(0))$ is bounded. Furthermore Rellich's compact embedding theorem implies that the operator $\Xi : u \in H^2(B_R(0)) \to \nabla log\gamma \cdot \nabla u$ is compact and

Calderón-Zygmund estimates imply that the operator $G_\rho : f \in L^2(B_R(0) \to G_\rho f \in H^2(B_R(0))$ is bounded. On the other hand, the trace operator $T : H^2(B_R(0)) \to H^{\frac{3}{2}}(\partial B_R(0))$ is bounded. Summing up the above analysis we can deduce from (3.36) that $S_\rho\Lambda_\gamma - B_\rho - \frac{1}{2}I = TG_\rho \Xi P_\gamma$ is a compact operator on $H^{\frac{3}{2}}(\partial B_R(0))$. Then (a) follows.

To prove (b) by Fredholm alternative theorem we just need to show that the homogeneous equation

$$f_\rho = (-S_\rho\Lambda_\gamma + B_\rho + \frac{1}{2}I)f_\rho \tag{3.37}$$

only has the zero solution. For any $g \in H^{\frac{3}{2}}(\partial B_R(0))$ satisfying (3.37) repeating the steps of the proof of Lemma 3.3 we can find a solution $u(x,\rho) = \gamma^{-1/2}e^{x\cdot\rho}\tilde{\Phi}(x,\rho) \in H^2_{loc}(\mathbb{R}^n)$ to

$$\mathrm{div}(\gamma\nabla u) = 0, \quad \text{in } \mathbb{R}^n,$$

with $u|_{\partial B_R(0)} = g$ and

$$\tilde{\Phi}(x,\rho) = \int_{\mathbb{R}^n} g_\rho(x,y)q(y)\tilde{\Phi}(y,\rho)dy. \tag{3.38}$$

Noting $\tilde{\Phi}(x,\rho) \in H^1_{loc}(\mathbb{R}^n)$ and $q = \frac{\triangle\gamma^{1/2}}{\gamma^{1/2}}$ with compact support, by dual method we can show

$$q(y)\tilde{\Phi}(y,\rho) \in \dot{X}_\rho^{-\frac{1}{2}}. \tag{3.39}$$

Then (3.38) and (3.39) imply $\tilde{\Phi}(x,\rho) \in \dot{X}_\rho^{\frac{1}{2}}$.

Under our assumptions: $\gamma \in Lip(B_R(0))$ and $\|\nabla log\gamma\| \leq \varepsilon_{n,R}$ or $\gamma \in C^1(\bar{B}_R(0))$, when ρ is large enough Haberman and Tataru [9] showed the operator :

$$f(x) \in \dot{X}_\rho^{\frac{1}{2}} \longrightarrow \int_{\mathbb{R}^n} g_\rho(x,y)q(y)f(y)dy \in \dot{X}_\rho^{\frac{1}{2}}$$

is a contraction. Therefore it follows from (3.38) that $\tilde{\Phi}(x,\rho) \equiv 0$. Hence $u \equiv 0$ and $g \equiv 0$. Then (b) follows.

In view of (b), for any $\rho\cdot\rho = 0, \rho \in \mathbb{C}^n$, when $|\rho|$ is large enough, then there exists a unique $f_\rho \in H^{\frac{3}{2}}(\partial B_R(0))$ such that

$$f_\rho = e^{x\cdot\rho} - (S_\rho\Lambda_\gamma - B_\rho - \frac{1}{2}I)f_\rho.$$

Now Lemma 3.3 implies that $u(x) = \gamma^{-1/2}e^{x\cdot\rho}(1 + \Phi(x,\rho))$ is a solution to

$$\mathrm{div}(\gamma\nabla u) = 0, \text{ in } \mathbb{R}^n,$$

and

$$\Phi(x,\rho) = \int_{\mathbb{R}^n} g_\rho(x,y)q(y)dy + \int_{\mathbb{R}^n} g_\rho(x,y)q(y)\Phi(y,\rho)dy. \tag{3.40}$$

Finally from (3.40) under our assumptions about γ, Lemma 3.1 and Theorem 4.1 in [9] imply that (c) follows. □

4 Reconstruction of the Conductivity γ

Theorem 4.1. *Under the assumptions of Theorem 3.4 γ can be recovered from the knowledge of Λ_γ at the boundary $\partial B_R(0)$.*

Before proving Theorem 4.1 we first recall an integral identity appearing in Lemma 4.1 in [10].

Proposition 4.2. *Suppose $\gamma_i \in C^1(\bar{\Omega}), i = 1, 2.$ and $u_1, u_2 \in H^1(\Omega)$ satisfy $\nabla \cdot (\gamma_i \nabla u_i) = 0$ in Ω. Suppose further that $\tilde{u}_1 \in H^1(\Omega)$ satisfies $\nabla \cdot (\gamma_1 \nabla \tilde{u}_1) = 0$ with $\tilde{u}_1 = u_2$ on $\partial\Omega$. Then*

$$\int_\Omega (\gamma_1^{\frac{1}{2}} \nabla \gamma_2^{\frac{1}{2}} - \gamma_2^{\frac{1}{2}} \nabla \gamma_1^{\frac{1}{2}}) \cdot \nabla(u_1 u_2) dx = \int_{\partial\Omega} \gamma_1 \partial_\nu (\tilde{u}_1 - u_2) u_1 ds,$$

where the integral on the boundary is understood in the sense of the dual pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$.

Remark 4.3. Proposition 4.2 also holds for $\gamma_i \in Lip(\Omega)$.

Proof of Theorem 4.1

Proof. Taking the conductivities $\gamma_1 = \gamma$ and $\gamma_2 = 1$, ρ_1 and ρ_2 as in (3.34) and (3.35) respectively and the solutions $u_1 = \gamma^{-1/2} e^{x\cdot\rho_1}(1 + \Phi(x, \rho_1))$ and $u_2 = e^{x\cdot\rho_2}$ in Proposition 4.2, we have

$$\int_{B_R(0)} -\nabla\gamma^{1/2} \cdot \nabla(\gamma^{-1/2} e^{x\cdot\rho_1} e^{x\cdot\rho_2}(1 + \Phi(x, \rho_1)) dx = \int_{\partial B_R(0)} (\Lambda_\gamma e^{x\cdot\rho_2} - \gamma \frac{\partial(e^{x\cdot\rho_2})}{\partial\nu}) u_1 dS. \tag{4.1}$$

We have that $u_1|_{\mathbb{R}^n \setminus \bar{B}_R(0)}$ is a solution of the exterior problem (3.2) such that $u_1|_{\partial B_R(0)} = f_{\rho_1}$. Since $e^{x\cdot\rho_1}|_{\partial B_R(0)}$, S_{ρ_1}, B_{ρ_1} are known and Λ_γ and γ at $\partial B_R(0)$ can be determined, Proposition 3.2 and (b) of Theorem 3.4 imply that the right hand side of (4.1) is known. Since we have that $\rho_1 + \rho_2 = ik$ we deduce from (4.1)

$$\int_{B_R(0)} -\nabla\gamma^{1/2} \cdot \nabla(\gamma^{-1/2} e^{ix\cdot k}(1 + \Phi(x, \rho_1))) dx = \int_{\partial B_R(0)} (\Lambda_\gamma e^{x\cdot\rho_2} - \gamma \frac{\partial(e^{x\cdot\rho_2})}{\partial\nu}) f_{\rho_1} dS. \tag{4.2}$$

Let γ_ε be as in Lemma 3.3. Now integration by parts in the left hand side of (4.2) gives us

$$\lim_{\varepsilon\to 0} \int_{B_R(0)} \frac{\triangle\gamma_\varepsilon^{1/2}}{\gamma_\varepsilon^{1/2}} e^{ix\cdot k}(1 + \Phi(x, \rho_1)) dx = \int_{\partial B_R(0)} (\Lambda_\gamma e^{x\cdot\rho_2} - \gamma \frac{\partial(e^{x\cdot\rho_2})}{\partial\nu}) f_{\rho_1} dS. \tag{4.3}$$

Clearly we have

$$\lim_{\varepsilon\to 0} \int_{B_R(0)} \frac{\triangle\gamma_\varepsilon^{1/2}}{\gamma_\varepsilon^{1/2}} e^{ix\cdot k} = \int_{B_R(0)} \frac{\triangle\gamma^{1/2}}{\gamma^{1/2}} e^{ix\cdot k} = \hat{q}(k) \tag{4.4}$$

in the sense of the tempered distribution.

Since $\|\gamma_\varepsilon - \gamma\|_{H^1(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$ it holds

$$\lim_{\varepsilon\to 0} \int_{B_R(0)} \frac{\triangle\gamma_\varepsilon^{1/2}}{\gamma_\varepsilon^{1/2}} e^{ix\cdot k} \Phi(x, \rho_1) dx = \langle \frac{\triangle\gamma^{1/2}}{\gamma^{1/2}} e^{ix\cdot k} \varphi_B,\ \Phi(x, \rho_1) \rangle \tag{4.5}$$

where $\langle \ \ \rangle$ means the dual pairing between $\dot{X}_{\rho_1}^{\frac{1}{2}}$ and $\dot{X}_{\rho_1}^{-\frac{1}{2}}$ and $\varphi_B \in C_0^\infty(\mathbb{R}^n)$ with $\varphi_B \equiv 1$ in $B_R(0)$. Therefore, we have from (4.3), (4.4) and (4.5) that the following holds

$$\hat{q}(k) + \langle \frac{\triangle\gamma^{1/2}}{\gamma^{1/2}} e^{ix\cdot k} \varphi_B,\ \Phi(x, \rho_1) \rangle = \int_{\partial B_R(0)} (\Lambda_\gamma e^{x\cdot\rho_2} - \gamma \frac{\partial(e^{x\cdot\rho_2})}{\partial\nu}) f_{\rho_1} dS. \tag{4.6}$$

For a fixed $k \in \mathbb{R}^n$ and $\lambda \geq |k|$, we next take average of the last identity in $(s, \eta_1) \in [\lambda, 2\lambda] \times S$. We get that

$$
(4.7) \qquad \frac{1}{\lambda}\int_S \int_\lambda^{2\lambda} \hat{q}(k) ds d\eta_1 + \frac{1}{\lambda}\int_S \int_\lambda^{2\lambda} \langle \frac{\triangle \gamma^{1/2}}{\gamma^{1/2}} e^{ix\cdot k}\varphi_B,\ \Phi(x,\rho_1)\rangle ds d\eta_1 = \frac{1}{\lambda}\int_S \int_\lambda^{2\lambda} \int_{\partial B_R(0)} (\Lambda_\gamma e^{x\cdot\rho_2} - \gamma \frac{\partial (e^{x\cdot \rho_2})}{\partial \nu}) f_{\rho_1} dS ds d\eta_1
$$

By (c) of Theorem 3.4 in Section 3 we have that

$$
(4.8) \qquad \frac{1}{\lambda}\int_S \int_\lambda^{2\lambda} \langle \frac{\triangle \gamma^{1/2}}{\gamma^{1/2}} e^{ix\cdot k}\varphi_B,\ \Phi(x,\rho_1)\rangle ds d\eta_1 \to 0,
$$

as $\lambda \to \infty$. Since the right hand side of (4.7) is still known and $\hat{q}(k)$ does not depend neither on s nor on η_1, we have that $\hat{q}(k)$ is known as a tempered distribution. By inverting the Fourier transform the potential $q(x)$ can be recovered in $\mathbb{R}^n$. From the definition of $q(x) = \frac{\triangle\gamma^{1/2}}{\gamma^{1/2}}$ we know $w = log\sqrt{\gamma} \in H_0^1(B_R(0))$ is a weak solution to

$$
(4.9) \qquad \begin{cases} \triangle w + |\nabla w|^2 = q, & \text{in } B_R(0), \\ w|_{\partial B_R(0)} = 0. \end{cases}
$$

The nonlinear Dirichlet problem (4.9) has a unique solution by the maximum principle of uniform elliptic equations and since we have already recovered $q(x)$ we may construct γ in $B_R(0)$ by solving the equation (4.9). In other words γ can be recovered by the knowledge of Λ_γ at the boundary $\partial B_R(0)$. Then Theorem 4.1 follows.

□

5 Appendix.

The goal of this appendix is to give a method for recovering the gradient of a coefficient from the Dirichlet to Neumann map. The argument is an extension of a method from earlier work of Brown [3]. More precisely, in a collaboration with Professor R. M. Brown, we recover the gradient of a C^1-conductivity at the boundary of the domain.

Throughout this appendix, we let Ω be a Lipschitz domain as defined in [24], for example, and we let γ denote a function on $\bar{\Omega}$ that is continuous and satisfies the ellipticity condition for some $\lambda > 0$,

$$
(5.1) \qquad \lambda \leq \gamma \leq \lambda^{-1}.
$$

We will use H^s to denote the standard scale of L^2 Sobolev spaces. Given $f \in H^{1/2}(\partial\Omega)$, we may solve the Dirichlet problem

$$
\begin{cases} \operatorname{div} \gamma \nabla u = 0 & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases}
$$

We define $\gamma \partial u/\partial \nu$, the co-normal derivative of u, as an element of $H^{-1/2}(\partial\Omega)$ by

$$
\langle \gamma \frac{\partial u}{\partial \nu}, \phi \rangle = \int_\Omega \gamma \nabla u \cdot \nabla \phi \, dy.
$$

Here, $\langle \cdot, \cdot \rangle : H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \to \mathbb{C}$ is the bilinear pairing of duality and we use ϕ to denote both a function in $H^{1/2}(\partial\Omega)$ and an extension into Ω which lies in $H^1(\Omega)$. Because u is a solution, the right-hand side depends only on the boundary values and not on the particular extension chosen for ϕ.

We give a reformulation of a result of the author Brown for recovering the conductivity at the boundary [3].

Theorem 5.1. *Let Ω be a Lipschitz domain. For almost every $x \in \partial\Omega$, we may find a sequence of functions f_N which are Lipschitz on $\partial\Omega$, supported in $\{y : |x-y| < N^{-1/2}\}$, satisfy*

$$\|f_N\|_{H^s(\partial\Omega)} \leq CN^{s-1/2}, \qquad 0 \leq s \leq 1,$$

and so that if γ is a continuous function satisfying (5.1), and u_N is the solution of the Dirichlet problem for $\operatorname{div}\gamma\nabla$ *with data f_N, and ψ is a continuous function in $\bar{\Omega}$, we have*

$$\psi(x) = \lim_{N\to\infty} \int_\Omega \psi(y)|\nabla u_N(y)|^2\, dy.$$

As an immediate consequence of the above theorem, we obtain recovery at the boundary and a stability result. The stability result for the recovery of the boundary values of a continuous coefficient in smooth domains was proved by Sylvester and Uhlmann [22, Theorem 0.2].

Theorem 5.2. *Let γ be continuous function in $\bar{\Omega}$ and let $x \in \partial\Omega$ and f_N be as in Theorem 5.1, then we have*

$$\gamma(x) = \lim_{N\to\infty} \langle \Lambda_\gamma f_N, \bar{f}_N\rangle.$$

As a consequence, we obtain the stability result: If γ and $\tilde{\gamma}$ are continuous on $\bar{\Omega}$ and elliptic, then

$$\|\gamma - \tilde{\gamma}\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_\gamma - \Lambda_{\tilde{\gamma}}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}.$$

Proof. This follows immediately from Theorem 5.1. □

If γ is continuously differentiable in the closure of Ω, we have additional regularity of the solution u_N. Since the boundary values f_N lie in $H^1(\partial\Omega)$, we have that the full gradient of u_N lies in $L^2(\partial\Omega)$ and we obtain estimates for the non-tangential maximal function of the gradient $(\nabla u_N)^*$

$$\|(\nabla u_N)^*\|_{L^2(\partial\Omega)} \leq C\|f_N\|_{H^1(\partial\Omega)} \leq CN^{1/2}. \tag{5.2}$$

This result holds for elliptic operators in Lipschitz domains with C^1 coefficients and follows from the work of Mitrea and Taylor on manifolds with C^1-metrics [13, Section 7]. We will use the estimate (5.2) to justify the use of the divergence theorem for expressions involving the gradient of a solution on the boundary. In our next result, we let $\nabla_t u = \nabla u - \nu\frac{\partial u}{\partial \nu}$ denote the tangential component of the gradient of u. The following theorem gives a recipe for recovering $\nabla\gamma$ from the gradient of f_N on the boundary, the boundary values of γ, and the Dirichlet to Neumann map acting on f_N.

Theorem 5.3. *Let Ω be a Lipschitz domain and let x and f_N be as in Theorem 5.1. Let α be a constant vector. We have*

$$\alpha\cdot\nabla\gamma(x) = \lim_{N\to\infty}\int_{\partial\Omega} (\gamma|\nabla_t f_N|^2 + \frac{1}{\gamma}|\Lambda_\gamma f_N|^2)\alpha\cdot\nu - 2\Re((\Lambda_\gamma f_N)\,\alpha\cdot\nabla\bar{u}_N)\, d\sigma$$

Proof. The rather mysterious expression on the right is obtain by rewriting the Rellich identity [19] using $\frac{1}{\gamma}\Lambda_\gamma f_N$ for the normal derivative. The following identity holds because the integrands are equal as functions

$$\int_{\partial\Omega}(\gamma|\nabla_t f_N|^2+\frac{1}{\gamma}|\Lambda_\gamma f_N|^2)\alpha\cdot\nu-2\Re(\Lambda_\gamma f_N\,\alpha\cdot\nabla\bar{u}_N)\,d\sigma$$
$$=\int_{\partial\Omega}\gamma|\nabla u_N|^2\alpha\cdot\nu-2\Re(\gamma\frac{\partial u}{\partial\nu}\,\alpha\cdot\nabla\bar{u}_N)\,d\sigma.$$

Recalling that u_N is a solution, an application of the divergence theorem gives

$$\int_{\partial\Omega}\gamma|\nabla u_N|^2\alpha\cdot\nu-2\Re(\gamma\frac{\partial u}{\partial\nu}\,\alpha\cdot\nabla\bar{u}_N)\,d\sigma=\int_{\Omega}\alpha\cdot\nabla\gamma\,|\nabla u_N|^2\,dy.$$

We use the non-tangential maximal function estimate (5.2) to justify the application of the divergence theorem. With this, the theorem follows from the properties of u_N given in Theorem 5.1. □

Acknowledgements. The authors want to express their gratitude to Professor R M Brown, from University of Kentucky, USA, for pointing out the result appearing in the appendix. He deserves to be considered as a co-author. We also want to thank Professor Mikko Salo and Dr Pedro Caro for many useful suggestions and deep comments. The first author is supported by the Basque Government through the BERC 20142017 program and by the Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa accreditation SEV-2013-0323 and the project MTM2014-53145-P. The second author is supported by the National Natural Science Foundation of China (Grant Nos: 11401238).

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