

HARDY UNCERTAINTY PRINCIPLE, CONVEXITY AND PARABOLIC EVOLUTIONS

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ABSTRACT. We give a new proof of the L^2 version of Hardy's uncertainty principle based on calculus and on its dynamical version for the heat equation. The reasonings rely on new log-convexity properties and the derivation of optimal Gaussian decay bounds for solutions to the heat equation with Gaussian decay at a future time. We extend the result to heat equations with lower order variable coefficient.

1. INTRODUCTION

In this paper we continue the study in [18, 6, 8, 9, 10, 11] related to the Hardy uncertainty principle and its relation to unique continuation properties for some evolutions.

One of our motivations came from a well known result due to G. H. Hardy ([14], [21, pp. 131]), which concerns the decay of a function f and its Fourier transform,

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx.$$

If $f(x) = O(e^{-|x|^2/\beta^2})$, $\widehat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$ and $1/\alpha\beta > 1/4$, then $f \equiv 0$. Also, if $1/\alpha\beta = 1/4$, f is a constant multiple of $e^{-|x|^2/\beta^2}$.

As far as we know, the known proofs for this result and its variants - before the one in [18, 6, 9, 10, 11] - use complex analysis (the Phragmén-Lindelöf principle). There has also been considerable interest in a better understanding of this result and on extensions of it to other settings: [3], [15], [20], [1] and [2].

The result can be rewritten in terms of the free solution of the Schrödinger equation

$$i\partial_t u + \Delta u = 0, \text{ in } \mathbb{R}^n \times (0, +\infty),$$

with initial data f ,

$$u(x, t) = (4\pi it)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} f(y) dy = (2\pi it)^{-\frac{n}{2}} e^{\frac{i|x|^2}{4t}} \widehat{e^{\frac{i|\cdot|^2}{4t}} f} \left(\frac{x}{2t} \right),$$

in the following way:

If $u(x, 0) = O(e^{-|x|^2/\beta^2})$, $u(x, T) = O(e^{-|x|^2/\alpha^2})$ and $T/\alpha\beta > 1/4$, then $u \equiv 0$. Also, if $T/\alpha\beta = 1/4$, u has as initial data a constant multiple of $e^{-(1/\beta^2 + i/4T)|y|^2}$.

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The corresponding results in terms of L^2 -norms, established in [4], are the following:

If $e^{|x|^2/\beta^2} f, e^{4|\xi|^2/\alpha^2} \widehat{f}$ are in $L^2(\mathbb{R}^n)$ and $1/\alpha\beta \geq 1/4$, then $f \equiv 0$.

If $e^{|x|^2/\beta^2} u(x, 0), e^{|x|^2/\alpha^2} u(x, T)$ are in $L^2(\mathbb{R}^n)$ and $T/\alpha\beta \geq 1/4$, then $u \equiv 0$.

In [10] we proved a uniqueness result in this direction for variable coefficients Schrödinger evolutions

$$(1.1) \quad \partial_t u = i(\Delta u + V(x, t)u), \text{ in } \mathbb{R}^n \times [0, T].$$

with bounded potentials V verifying, $V(x, t) = V_1(x) + V_2(x, t)$, with V_1 real-valued and

$$\sup_{[0, T]} \|e^{T^2|x|^2/(\alpha t + \beta(T-t))^2} V_2(t)\|_{L^\infty(\mathbb{R}^n)} < +\infty$$

or

$$\lim_{R \rightarrow +\infty} \int_0^T \|V(t)\|_{L^\infty(\mathbb{R}^n \setminus B_R)} dt = 0.$$

More precisely, we showed that the only solution u to (1.1) in $C([0, T], L^2(\mathbb{R}^n))$, which verifies

$$\|e^{|x|^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty$$

is the zero solution, when $T/\alpha\beta > 1/4$. When $T/\alpha\beta = 1/4$, we found a complex valued potential V with

$$|V(x, t)| \lesssim \frac{1}{1 + |x|^2}, \text{ in } \mathbb{R}^n \times [0, T]$$

and a nonzero smooth solution u in $C^\infty([0, T], \mathcal{S}(\mathbb{R}^n))$ of (1.1) with

$$\|e^{|x|^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty.$$

Thus, we established in [10] that the optimal version of Hardy's Uncertainty Principle in terms of L^2 -norms holds for solutions to (1.1) holds when $T/\alpha\beta > 1/4$ for many general bounded potentials, while it can fail for some complex-valued potentials in the end-point case, $T/\alpha\beta = 1/4$. Finally, in [11] we showed that the reasonings in [18, 6, 8, 9, 10, 11] provide the first proof (up to the end-point case) that we know of Hardy's uncertainty principle for the Fourier transform without the use of holomorphic functions.

The Hardy uncertainty principle also has a dynamical version associated to the heat equation,

$$\partial_t u - \Delta u = 0, \text{ in } \mathbb{R}^n \times (0, +\infty),$$

with initial data f ,

$$u(x, t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y) dy, \quad \widehat{u}(\xi, t) = e^{-t|\xi|^2} \widehat{f}(\xi), \quad x, \xi \in \mathbb{R}^n, \quad t > 0.$$

In particular, its L^∞ and L^2 versions yield the following statements:

If $u(0)$ is a finite measure in \mathbb{R}^n , $u(x, T) = O(e^{-|x|^2/\delta^2})$ and $\delta < \sqrt{4T}$, then $f \equiv 0$. Also, if $\delta = \sqrt{4T}$, then $u(0)$ is a multiple of the Dirac delta function.

If $u(0)$ is in $L^2(\mathbb{R}^n)$, $\|e^{|x|^2/\delta^2} u(T)\|_{L^2(\mathbb{R}^n)}$ is finite and $\delta \leq \sqrt{4T}$, then $u \equiv 0$.

In [9, Theorem 4] we proved that a dynamical L^2 -version of Hardy uncertainty principle holds for solutions u in $C([0, T], L^2(\mathbb{R}^n)) \cap L^2([0, T], H^1(\mathbb{R}^n))$ to

$$(1.2) \quad \partial_t u = \Delta u + V(x, t)u, \text{ in } \mathbb{R}^n \times [0, T],$$

when V is any bounded complex potential in $\mathbb{R}^n \times [0, T]$ and $\delta < \sqrt{T}$. Here, we find the optimal interior Gaussian decay over $[0, 1]$ for solutions to (1.2) with

$$\|e^{|x|^2/\delta^2} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty,$$

when $\delta > \sqrt{4T}$ and derive from it the full dynamical L^2 version of the Hardy uncertainty principle for solutions to (1.2), reaching the end-point case, $\delta = \sqrt{4T}$.

Theorem 1. *Assume that u in $C([0, T], L^2(\mathbb{R}^n)) \cap L^2([0, T], H^1(\mathbb{R}^n))$ verifies (1.2) with V in $L^\infty(\mathbb{R}^n \times [0, T])$. Assume that*

$$(1.3) \quad \|e^{T|x|^2/4(T^2+R^2)} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty$$

for some $R > 0$. Then, there is a universal constant N such that

$$(1.4) \quad \sup_{[0, T]} \|e^{t|x|^2/4(t^2+R^2)} u(t)\|_{L^2(\mathbb{R}^n)} \leq e^{N(1+T^2\|V\|_{L^\infty(\mathbb{R}^n \times [0, T])}^2)} \left[\|u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{T|x|^2/4(T^2+R^2)} u(T)\|_{L^2(\mathbb{R}^n)} \right].$$

Moreover, u must be identically zero when $\|e^{|x|^2/4T} u(T)\|_{L^2(\mathbb{R}^n)}$ is finite.

Theorem 1 is optimal because

$$(1.5) \quad u_R(x, t) = (t - iR)^{-\frac{n}{2}} e^{-|x|^2/4(t-iR)} = (t - iR)^{-\frac{n}{2}} e^{-(t+iR)|x|^2/4(t^2+R^2)},$$

is a solution to the heat equation and for each fixed $t > 0$, $t/4(t^2 + R^2)$ is decreasing in the R -variable for $R > 0$. Also, observe that $t/4(t^2 + R^2)$ attains its maximum value in the interior of $[0, T]$, when $R \neq T$,

Notice that the finiteness condition on condition on $\|e^{|x|^2/4T} u(T)\|_{L^2(\mathbb{R}^n)}$ is independent of the size of the potential or the dimension and that we do not assume any regularity or strong decay of the potentials.

This improvement of our results in [9, Theorem 4] on the relation between Hardy uncertainty principle and its dynamical version for parabolic evolutions comes from a better understanding of the solutions to (1.2) which have Gaussian decay and of the adaptation to the parabolic context of the same kind of log-convexity arguments that we used in [10] to derive the dynamical version of the Hardy uncertainty principle for Schrödinger evolutions.

We have not tried to extend the results in Theorems 1 to parabolic evolutions with nonzero drift terms

$$(1.6) \quad \partial_t u = \Delta u + W(x, t) \cdot \nabla u + V(x, t)u.$$

We expect that similar methods will yield analogue results for solutions to (1.6) (See [5] for initial results following the approach initiated in [18] and [6] for the case of Schrödinger evolutions).

In what follows, N denotes a universal constant depending at most on the dimension, $N_{a, \xi, \dots}$ a constant depending on the parameters a, ξ, \dots . In section 2 we give three Lemmas which are necessary for our proof in section 3 of Theorem 1.

2. A FEW LEMMAS

In the sequel

$$(f, g) = \int_{\mathbb{R}^n} f \bar{g} dx, \quad \|f\|^2 = (f, f) \quad \text{and} \quad \|V\|_\infty = \|V\|_{L^\infty(\mathbb{R}^n \times [0, 1])}.$$

In Lemma 1, \mathcal{S} and \mathcal{A} denote respectively a symmetric and skew-symmetric bounded linear operators on $\mathcal{S}(\mathbb{R}^n)$. Both are allowed to depend smoothly on the time-variable, $\mathcal{S}_t = \partial_t \mathcal{S}$ and $[\mathcal{S}, \mathcal{A}]$ is the space commutator of \mathcal{S} and \mathcal{A} . The reader can find a proof of Lemma 1 in [10, Lemma 2].

Lemma 1. *Let \mathcal{S} and \mathcal{A} be as above, f lie in $C^\infty([c, d], \mathcal{S}(\mathbb{R}^n))$ and $\gamma : [c, d] \rightarrow (0, +\infty)$ be a smooth function such that*

$$(\gamma \mathcal{S}_t f(t) + \gamma [\mathcal{S}, \mathcal{A}] f(t) + \dot{\gamma} \mathcal{S} f(t), f(t)) \geq 0, \text{ when } c \leq t \leq d.$$

Then, if $H(t) = \|f(t)\|^2$ and $\epsilon > 0$

$$H(t) + \epsilon \leq (H(c) + \epsilon)^{\theta(t)} (H(d) + \epsilon)^{1-\theta(t)} e^{M_\epsilon(t) + 2N_\epsilon(t)}, \text{ when } c \leq t \leq d,$$

where M_ϵ verifies

$$\partial_t (\gamma \partial_t M_\epsilon) = -\gamma \frac{\|\partial_t f - \mathcal{S}f - \mathcal{A}f\|^2}{H + \epsilon}, \text{ in } [c, d], \quad M_\epsilon(c) = M_\epsilon(d) = 0,$$

$$N_\epsilon = \int_c^d \left| \operatorname{Re} \frac{(\partial_s f(s) - \mathcal{S}f(s) - \mathcal{A}f(s), f(s))}{H(s) + \epsilon} \right| ds$$

and

$$\theta(t) = \frac{\int_t^d \frac{ds}{\gamma}}{\int_c^d \frac{ds}{\gamma}}.$$

A calculation (see formulae (2.12), (2.13) and (2.14) in [9] with $\gamma = 1$) shows that given smooth functions $a : [0, 1] \rightarrow [0, +\infty)$, $b : [0, 1] \rightarrow \mathbb{R}$ and $T : [0, 1] \rightarrow \mathbb{R}$, and ξ in \mathbb{R}^n

$$e^{a(t)|x|^2 + b(t)x \cdot \xi - T(t)|\xi|^2} (\partial_t - \Delta) e^{-a(t)|x|^2 - b(t)x \cdot \xi + T(t)|\xi|^2} = \partial_t - \mathcal{S} - \mathcal{A},$$

where \mathcal{S} and \mathcal{A} are the symmetric and skew-symmetric linear bounded operators on $\mathcal{S}(\mathbb{R}^n)$ given by

$$(2.1) \quad \mathcal{S} = \Delta + (a' + 4a^2) |x|^2 + (b' + 4ab) x \cdot \xi + (b^2 - T') |\xi|^2,$$

$$(2.2) \quad \mathcal{A} = -2(2ax + b\xi) \cdot \nabla - 2na.$$

and

$$(2.3) \quad \mathcal{S}_t + [\mathcal{S}, \mathcal{A}] = -8a \Delta + (a'' + 16aa' + 32a^3) |x|^2 \\ + (b'' + 8ab' + 8a'b + 32a^2b) x \cdot \xi + (8ab^2 + 4bb' - T'') |\xi|^2.$$

In Lemma 2 we make choices of a , b and T which make non-negative the self-adjoint operator

$$e^{8A} (\mathcal{S}_t + [\mathcal{S}, \mathcal{A}]) + (e^{8A})' \mathcal{S},$$

where A denotes an anti-derivative of a in $[0, 1]$ with $A(1) = 0$.

Lemma 2. *Let $a : [0, 1] \rightarrow \mathbb{R}$ be a smooth function verifying*

$$(2.4) \quad (e^{8A} a)'' \geq 0, \text{ in } [0, 1],$$

and let b and T be the solutions to

$$(2.5) \quad \begin{cases} (e^{8A} b)'' = 2(e^{8A} a)'' , & \text{in } [0, 1], \\ b(0) = b(1) = 0, \end{cases}$$

and

$$(2.6) \quad \begin{cases} (e^{8A}T')' = 2(e^{8A}b^2)' - (e^{8A}a)'' , & \text{in } [0, 1], \\ T(0) = T(1) = 0. \end{cases}$$

Then,

$$\left(e^{8A} \mathcal{S}_t f + e^{8A} [\mathcal{S}, \mathcal{A}] f + (e^{8A})' \mathcal{S} f, f \right) \geq 0, \text{ when } f \in \mathcal{S}(\mathbb{R}^n) \text{ and } 0 \leq t \leq 1.$$

Proof. From (2.1), (2.3), the identities

$$\begin{aligned} (e^{8A}a)'' &= e^{8A} (a'' + 24aa' + 64a^3). \\ (e^{8A}b)'' &= e^{8A} (b'' + 16ab' + 8a'b + 64a^2b). \\ (e^{8A}b^2)' &= e^{8A} (8ab^2 + 2bb'), \end{aligned}$$

and the definitions of b and T , we have

$$\begin{aligned} e^{8A} (\mathcal{S}_t + [\mathcal{S}, \mathcal{A}]) + (e^{8A})' \mathcal{S} \\ &= (e^{8A}a)'' |x|^2 + (e^{8A}b)'' x \cdot \xi + \left(2(e^{8A}b^2)' - (e^{8A}T')' \right) |\xi|^2 \\ &= (e^{8A}a)'' (|x|^2 + 2x \cdot \xi + |\xi|^2) = (e^{8A}a)'' |x + \xi|^2. \end{aligned}$$

The later and (2.4) implies Lemma 2. \square

In the next Lemma we assume that u in $C([0, 1], L^2(\mathbb{R}^n)) \cap L^2([0, 1], H^1(\mathbb{R}^n))$ verifies (1.2) in $\mathbb{R}^n \times (0, 1]$ and

$$\|e^{|x|^2/\delta^2} u(1)\| < +\infty.$$

Lemma 3. *Let $a : [0, 1] \rightarrow [0, +\infty)$ be a smooth function with $a(0) = 0$, $a(1) = 1/\delta^2$, $(e^{8A}a)'' > 0$ in $[0, 1]$ and*

$$\sup_{[0, 1]} \|e^{(a(t)-\epsilon)|x|^2} u(t)\| < +\infty, \text{ when } 0 < \epsilon \leq 1.$$

Then, there is a universal constant N such that for b and T as in (2.5) and (2.6),

$$\|e^{a(t)|x|^2 + b(t)x \cdot \xi - T(t)|\xi|^2} u(t)\| \leq e^{N(1+\|V\|_\infty^2)} \left(\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\| \right),$$

when ξ is in \mathbb{R}^n and $0 \leq t \leq 1$.

Proof. For ξ in \mathbb{R}^n and $\epsilon > 0$, set

$$f_\epsilon(x, t) = e^{a_\epsilon|x|^2 + b_\epsilon x \cdot \xi - T_\epsilon|\xi|^2} u(x, t),$$

with $a_\epsilon = a - \epsilon$, $A_\epsilon = A + \epsilon(1 - t)$, and with b_ϵ and T_ϵ as in Lemma 2 but with a and A replaced by a_ϵ and A_ϵ respectively. The local Schauder estimates for solutions to (1.2) show that

$$\begin{aligned} r|\nabla u(x, t)| + r^2 \left(\int_{B_r(x) \times (t-r^2, t]} |\partial_s u|^p + |D^2 u|^p dy ds \right)^{\frac{1}{p}} \\ \leq N_p (1 + r^2 \|V\|_{L^\infty(\mathbb{R}^n \times [0, 1])}) \int_{B_{2r}(x) \times (t-4r^2, t]} |u| dy ds \end{aligned}$$

for $1 < p < \infty$, $0 < r \leq \sqrt{t}/2$, $0 < t \leq 1$. Thus, f_ϵ is in $W_2^{2,1}(\mathbb{R}^n \times [\varrho, 1])$ and verifies

$$(2.7) \quad \begin{aligned} \sup_{[0,1]} \|f_\epsilon(t)\| &\leq N_{a,\epsilon,\xi} \sup_{[0,1]} \|e^{(a-\frac{\xi}{2})|x|^2} u(t)\|, \\ \sup_{[0,1]} \|\nabla f_\epsilon(t)\| &\leq N_{a,\epsilon,\xi,\varrho} \sup_{[0,1]} \|e^{(a-\frac{\xi}{2})|x|^2} u(t)\| \end{aligned}$$

for $0 < \varrho \leq \frac{1}{2}$ and

$$(2.8) \quad \partial_t f_\epsilon - \mathcal{S}_\epsilon f_\epsilon - \mathcal{A}_\epsilon f_\epsilon = V(x, t) f_\epsilon, \text{ in } \mathbb{R}^n \times (0, 1],$$

where \mathcal{S}_ϵ and \mathcal{A}_ϵ are the operators defined in (2.1) and (2.2) with a , A , b and T replaced by a_ϵ , A_ϵ , b_ϵ and T_ϵ respectively. Also, (2.7), the equation (2.8) verified by f_ϵ and [22, Lemma 1.2] show that f_ϵ is in $C((0, 1], L^2(\mathbb{R}^n))$.

Extend f_ϵ as zero outside $\mathbb{R}^n \times [0, 1]$ and let θ in $C^\infty(\mathbb{R}^{n+1})$ be a mollifier supported in the unit ball of \mathbb{R}^{n+1} . For $0 < \rho \leq \frac{1}{4}$, set $f_{\epsilon,\rho} = f_\epsilon * \theta_\rho$ and

$$\theta_\rho^{x,t}(y, s) = \rho^{-n-1} \theta\left(\frac{x-y}{\rho}, \frac{t-s}{\rho}\right).$$

Then, $f_{\epsilon,\rho}$ is in $C^\infty([0, 1], \mathcal{S}(\mathbb{R}^n))$ and for x in \mathbb{R}^n and $\rho \leq t \leq 1 - \rho$,

$$(2.9) \quad \begin{aligned} (\partial_t f_{\epsilon,\rho} - \mathcal{S}_\epsilon f_{\epsilon,\rho} - \mathcal{A}_\epsilon f_{\epsilon,\rho})(x, t) &= (V f_\epsilon) * \theta_\rho(x, t) \\ &+ \int f_\epsilon (q_\epsilon(y, s, \xi) - q_\epsilon(x, t, \xi)) \theta_\rho^{x,t} dy ds \\ &+ \int \nabla_y f_\epsilon \cdot [(a_\epsilon(t)x + 2b_\epsilon(t)\xi) - (a_\epsilon(s)y + 2b_\epsilon(s)\xi)] \theta_\rho^{x,t} dy ds, \end{aligned}$$

with

$$\begin{aligned} q_\epsilon(x, t, \xi) &= (a'_\epsilon(t) + 4a_\epsilon^2(t)) |x|^2 \\ &+ (b'_\epsilon(t) + 4a_\epsilon(t)b_\epsilon(t)) x \cdot \xi + (b_\epsilon^2(t) - T'_\epsilon(t)) |\xi|^2 - 2na_\epsilon(t). \end{aligned}$$

The last identity gives,

$$(\partial_t f_{\epsilon,\rho} - \mathcal{S}_\epsilon f_{\epsilon,\rho} - \mathcal{A}_\epsilon f_{\epsilon,\rho})(x, t) = (V f_\epsilon) * \theta_\rho(x, t) + A_{\epsilon,\rho}(x, t),$$

in $\mathbb{R}^n \times [\rho, 1 - \rho]$, where $A_{\epsilon,\rho}$ denotes the sum of the second and third integrals in the right hand side of (2.9). Moreover, from (2.7) there is $N_{a,\epsilon,\xi,\varrho}$ such that for $0 < \varrho < \frac{1}{2}$ and $0 < \rho \leq \varrho$,

$$\sup_{[\varrho, 1-\varrho]} \|A_{\epsilon,\rho}(t)\|_{L^2(\mathbb{R}^n)} \leq \rho N_{a,\epsilon,\xi,\varrho} \sup_{[-1,1]} \|e^{(a(t)-\frac{\xi}{2})|x|^2} u(t)\|.$$

Also, $(e^{8A_\epsilon a_\epsilon})'' > 0$ in $[0, 1]$, when $0 < \epsilon \leq \epsilon_a$, and from Lemma 2 we can apply to \mathcal{S}_ϵ , \mathcal{A}_ϵ and $f_{\epsilon,\rho}$, the conclusions of Lemma 1 with $[c, d] = [\varrho, 1 - \varrho]$, $\gamma = e^{8A_\epsilon}$ and $H_{\epsilon,\rho}(t) = \|f_{\epsilon,\rho}(t)\|^2$. Thus,

$$(2.10) \quad H_{\epsilon,\rho}(t) \leq (H_{\epsilon,\rho}(\varrho) + H_{\epsilon,\rho}(1 - \varrho) + 2\epsilon) e^{M_{\epsilon,\rho}(t) + 2N_{\epsilon,\rho}}, \text{ when } \varrho \leq t \leq 1 - \varrho,$$

where $M_{\epsilon,\rho}$ verifies

$$\begin{cases} \partial_t (e^{8A_\epsilon} \partial_t M_{\epsilon,\rho}) = -e^{8A_\epsilon} \frac{\|\partial_t f_{\epsilon,\rho} - \mathcal{S}_\epsilon f_{\epsilon,\rho} - \mathcal{A}_\epsilon f_{\epsilon,\rho}\|^2}{H_{\epsilon,\rho} + \epsilon}, & \text{in } [\varrho, 1 - \varrho], \\ M_{\epsilon,\rho}(\varrho) = M_{\epsilon,\rho}(1 - \varrho) = 0, \end{cases}$$

and

$$N_{\epsilon,\rho} = \int_\varrho^{1-\varrho} \frac{\|\partial_s f_{\epsilon,\rho}(s) - \mathcal{S}_\epsilon f_{\epsilon,\rho}(s) - \mathcal{A}_\epsilon f_{\epsilon,\rho}(s)\|}{\sqrt{H_{\epsilon,\rho}(s) + \epsilon}} ds.$$

We can now pass to the limit in (2.10), when ρ tends to zero and derive that for $H_\epsilon(t) = \|f_\epsilon(t)\|^2$, $0 < \varrho \leq \frac{1}{2}$ and $0 < \epsilon \leq \epsilon_a$, we have

$$(2.11) \quad H_\epsilon(t) \leq [H_\epsilon(\varrho) + H_\epsilon(1 - \varrho) + 2\epsilon] e^{M_\epsilon(t) + 2\|V\|_\infty}, \text{ in } [\varrho, 1 - \varrho],$$

with

$$(2.12) \quad \begin{cases} \partial_t (e^{8A_\epsilon} \partial_t M_\epsilon) = -e^{8A_\epsilon} \frac{\|\partial_t f_\epsilon - S_\epsilon f_\epsilon - A_\epsilon f_\epsilon\|^2}{H_\epsilon}, & \text{in } [0, 1]. \\ M_\epsilon(0) = M_\epsilon(1) = 0, \end{cases}$$

By writing an explicit formula for the solution to (2.12), it follows from the monotonicity of A ; i.e. $A' \geq 0$ in $[0, 1]$ and (2.8) that

$$M_\epsilon(t) \leq N(1 + \|V\|_\infty^2).$$

Also, there is $N_a > 0$ such that $|b'_\epsilon| + |T'_\epsilon| \leq N_a$, when $0 < \epsilon \leq \epsilon_a$. The later, the continuity of f_ϵ in $C((0, 1], L^2(\mathbb{R}^n))$ and the fact that $a(0) = b_\epsilon(0) = T_\epsilon(0) = 0$ show, that for each fixed $\xi \in \mathbb{R}^n$ and all $0 < \epsilon < \epsilon_a$, there is ϱ_ϵ with $\lim_{\epsilon \rightarrow 0^+} \varrho_\epsilon = 0$ such that $H_\epsilon(1 - \varrho_\epsilon) \leq H_\epsilon(1) + \epsilon$ and $H_\epsilon(\varrho_\epsilon) \leq \sup_{[0, 1]} \|u(t)\|$. Thus, after taking $\varrho = \varrho_\epsilon$ in (2.11), we get

$$\|e^{a_\epsilon(t)|x|^2 + b_\epsilon(t)x \cdot \xi - T_\epsilon(t)|\xi|^2} u(t)\| \leq e^{N(1 + \|V\|_\infty^2)} \left[\sup_{[0, 1]} \|u(t)\| + \|e^{|x|^2/\delta^2} u(1)\| + 3\epsilon \right],$$

for $\varrho_\epsilon \leq t \leq 1 - \varrho_\epsilon$. Then, let $\epsilon \rightarrow 0^+$ and recall the L^2 energy inequality verified by solutions to (1.2). \square

3. PROOF OF THEOREM 1

Proof. By scaling it suffices to prove Theorem 1 when $T = 1$. Assume first that u in $C([0, 1], L^2(\mathbb{R}^n)) \cap L^2([0, 1], H^1(\mathbb{R}^n))$ verifies (1.2) in $\mathbb{R}^n \times (0, 1]$ and

$$\|e^{|x|^2/\delta^2} u(1)\| < +\infty$$

for some $\delta > 2$. Following [9, Theorem 4], for $\alpha = 1$ and $\beta = 1 + \frac{2}{\delta}$, define

$$\tilde{u}(x, t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t) + \beta t} \right)^{\frac{n}{2}} u\left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t) + \beta t}, \frac{\beta t}{\alpha(1-t) + \beta t} \right) e^{\frac{(\alpha-\beta)|x|^2}{4(\alpha(1-t) + \beta t)}}.$$

Then, \tilde{u} is in $C([0, 1], L^2(\mathbb{R}^n)) \cap L^2([0, 1], H^1(\mathbb{R}^n))$ and from [9, Lemma 5] with $A + iB = 1$

$$\partial_t \tilde{u} = \Delta \tilde{u} + \tilde{V}(x, t) \tilde{u}, \text{ in } \mathbb{R}^n \times (0, 1],$$

with

$$\tilde{V}(x, t) = \frac{\alpha\beta}{(\alpha(1-t) + \beta t)^2} V\left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t) + \beta t}, \frac{\beta t}{\alpha(1-t) + \beta t} \right).$$

Also, for $\gamma = \frac{1}{2\delta}$

$$\|e^{\gamma|x|^2} \tilde{u}(0)\| = \|u(0)\| \text{ and } \|e^{\gamma|x|^2} \tilde{u}(1)\| = \|e^{|x|^2/\delta^2} u(1)\|.$$

From the log-convexity property of $\|e^{\gamma|x|^2} \tilde{u}(t)\|$ established in [9, Lemma 3], we know that

$$(3.1) \quad \sup_{[0, 1]} \|e^{\gamma|x|^2} \tilde{u}(t)\| \leq e^{N(1 + \|\tilde{V}\|_{L^\infty(\mathbb{R}^n \times [0, 1])}^2)} \left(\|e^{\gamma|x|^2} \tilde{u}(0)\| + \|e^{\gamma|x|^2} \tilde{u}(1)\| \right).$$

The last claim in [9, Lemma 5] shows that with $s = \frac{\beta t}{\alpha(1-t) + \beta t}$,

$$(3.2) \quad \|e^{\gamma|x|^2} \tilde{u}(t)\| = \|e^{\left[\frac{\gamma\alpha\beta}{(\alpha s + \beta(1-s))^2} + \frac{\alpha-\beta}{4(\alpha s + \beta(1-s))} \right] |y|^2} u(s)\|, \text{ for } 0 \leq t \leq 1.$$

From (3.1) and (3.2), we find that

$$(3.3) \quad \sup_{[0,1]} \|e^{\frac{t|x|^2}{(\delta+2-2t)^2}} u(t)\| \leq e^{N(1+\|V\|_\infty^2)} \left[\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\| \right].$$

We then begin an inductive procedure where at the k th step we have constructed k smooth functions, $a_j : [0, 1] \rightarrow [0, +\infty)$ verifying

$$(3.4) \quad 0 < a_1 < a_2 < \cdots < a_k < \cdots \leq \frac{1}{\delta^2 - 4}, \text{ in } (0, 1),$$

$$(3.5) \quad a_j(0) = 0, \quad a_j(1) = 1/\delta^2, \quad (e^{8A_j} a_j)'' > 0, \text{ in } [0, 1],$$

$$(3.6) \quad \sup_{[0,1]} \|e^{a_j(t)|x|^2} u(t)\| \leq e^{N(1+\|V\|_\infty^2)} \left[\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\| \right],$$

when $j = 1, \dots, k$, with $A'_j = a_j$, $A_j(1) = 0$. The case $k = 1$ follows from (3.3) with $a_1(t) = t/(\delta + 2 - 2t)^2$. Assume now that a_1, \dots, a_k have been constructed and let b_k and T_k be the functions defined in Lemma 3 for $a = a_k$. Then,

$$(3.7) \quad \|e^{a_k(t)|x|^2 + b_k(t)x \cdot \xi - T_k(t)|\xi|^2} u(t)\|^2 \leq e^{2N(1+\|V\|_\infty^2)} \left(\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\| \right)^2,$$

for $0 \leq t \leq 1$ and all $\xi \in \mathbb{R}^n$. Observe that (3.7) and the existence of the solutions u_R defined in (1.5) imply that $T_k > 0$ in $(0, 1)$, when $\delta > 2$. Otherwise, (3.7) implies that $u_R \equiv 0$, when $2\sqrt{1+R^2} < \delta$.

For $\epsilon > 0$, multiply (3.7) by $e^{-2\epsilon T_k(t)|\xi|^2}$ and integrate the new inequality with respect to ξ in \mathbb{R}^n . It gives,

$$\sup_{[0,1]} \|e^{a_{k+1}^\epsilon(t)|x|^2} u(t)\| \leq \left(1 + \frac{1}{\epsilon}\right)^{\frac{n}{4}} e^{N(1+\|V\|_\infty^2)} \left(\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\| \right),$$

with

$$a_{k+1}^\epsilon = a_k + \frac{b_k^2}{4(1+\epsilon)T_k}.$$

On the other hand, $e^{8A_k} b_k$ is strictly convex and $b_k < 0$ in $[0, 1]$,

$$(3.8) \quad b_k(t) = 2 \left(a_k(t) - t e^{-8A_k(t)} \delta^{-2} \right)$$

and

$$T_k(t) = 2 \int_0^t b_k^2(s) ds - a_k(t) - 8 \int_0^t a_k^2(s) ds - \alpha_k \int_0^t e^{-8A_k(s)} ds,$$

with

$$\alpha_k = \left(2 \int_0^1 b_k^2(s) ds - \frac{1}{\delta^2} - 8 \int_0^1 a_k^2(s) ds \right) \left(\int_0^1 e^{-8A_k(s)} ds \right)^{-1}.$$

The last two formulae and (3.4) show that there is $N_\delta \geq 1$, independent of $k \geq 1$, such that

$$(3.9) \quad T_k(t) \leq 2 \left(\int_0^t b_k^2(s) ds + N_\delta \right) \text{ and } N_\delta + \frac{b_k}{2} \geq 1, \text{ in } [0, 1].$$

Also, $((a'_k + 4a_k^2) e^{16A_k})' = e^{8A_k} (e^{8A_k} a_k)''$, $(a'_k + 4a_k^2) e^{16A_k}$ is non decreasing in $[0, 1]$ and

$$(3.10) \quad a'_k + 4a_k^2 \geq 0 \text{ in } [0, 1].$$

Set then,

$$(3.11) \quad a_{k+1}(t) = a_k(t) + \frac{b_k^2(t)}{8 \left(\int_0^t b_k^2(s) ds + N_\delta \right)}.$$

We have, $a_k < a_{k+1}$ in $(0, 1)$, $a_{k+1}(0) = 0$, $a_{k+1}(1) = \frac{1}{8\delta^2}$,

$$(3.12) \quad A_{k+1} = A_k + \frac{1}{8} \log \left(\int_0^t b_k^2(s) ds + N_\delta \right) - \frac{1}{8} \log \left(\int_0^1 b_k^2(s) ds + N_\delta \right),$$

and

$$\sup_{[0,1]} \|e^{(a_{k+1}(t)-\epsilon)|x|^2} u(t)\| < +\infty, \text{ for all } \epsilon > 0.$$

The identity $(e^{8A})''' = 8(e^{8A}a)''$ and (3.12) show that $(e^{8A_{k+1}}a_{k+1})''$ is a positive multiple of

$$\begin{aligned} \left(e^{8A_k} \left(\int_0^t b_k^2(s) ds + N_\delta \right) \right)''' &= (e^{8A_k})''' \left(\int_0^t b_k^2(s) ds + N_\delta \right) \\ &\quad + 3(e^{8A_k})'' b_k^2 + 6(e^{8A_k})' b_k b'_k + 2e^{8A_k} (b_k'' b_k + b_k'^2) \end{aligned}$$

The equation verified by b_k shows that the last sum is equal to

$$\begin{aligned} (e^{8A_k})''' \left(\int_0^t b_k^2(s) ds + N_\delta + \frac{b_k}{2} \right) \\ + 8(a'_k + 8a_k^2) e^{8A_k} b_k^2 + 2e^{8A_k} b_k'^2 + 16e^{8A_k} a_k b_k b'_k. \end{aligned}$$

From (3.9) and (3.10), the above sum is bounded from below by

$$(e^{8A_k})''' + 2e^{8A_k} (4a_k b_k + b_k')^2 > 0, \text{ in } [0, 1].$$

The later and Lemma 3 show that (3.6) holds up to $j = k + 1$. Finally, because (3.10) holds with k replaced by $k + 1$,

$$-\left(\frac{1}{a_{k+1}} \right)' + 4 \geq 0, \text{ in } (0, 1],$$

and the integration of this identity over $[t, 1]$ shows that $a_{k+1}(t) \leq \frac{1}{\delta^2 - 4}$ in $(0, 1)$.

Thus, there exists $a(t) = \lim_{k \rightarrow +\infty} a_k(t)$ and from (3.11), $\lim_{k \rightarrow +\infty} b_k(t) = 0$. This and (3.8) show that

$$(3.13) \quad ae^{8A} = t\delta^{-2}, \text{ in } [0, 1].$$

Write $a(1) = 1/\delta^2$ as $1/4(1 + R^2)$, for some $R > 0$. Then, $a(t) = t/4(t^2 + R^2)$ follows from the integration of (3.13) and (1.4) from (3.6) after letting $j \rightarrow +\infty$. Finally, when $\delta = 2$, we have

$$\sup_{[0,1]} \|e^{t|x|^2/4(t^2+R^2)} u(t)\| \leq e^{N(1+\|V\|_\infty^2)} \left[\|u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/4} u(1)\| \right],$$

for all $R > 0$. Letting $R \rightarrow 0^+$, we get

$$\sup_{[0,1]} \|e^{|x|^2/4t} u(t)\| \leq e^{N(1+\|V\|_\infty^2)} \left[\|u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/4} u(1)\| \right],$$

and it implies, $u \equiv 0$. □

Remark 1. Theorem 1 holds when (1.3) and (1.4) are replaced respectively by

$$\|e^{Tx_1^2/4(T^2+R^2)}u(T)\|_{L^2(\mathbb{R}^n)} < +\infty$$

and

$$\begin{aligned} & \sup_{[0,T]} \|e^{tx_1^2/4(t^2+R^2)}u(t)\|_{L^2(\mathbb{R}^n)} \\ & \leq e^{N(1+T^2\|V\|_{L^\infty(\mathbb{R}^n \times [0,T])}^2)} \left[\|u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{Tx_1^2/4(T^2+R^2)}u(T)\|_{L^2(\mathbb{R}^n)} \right]. \end{aligned}$$

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