Singularities of the Hilbert scheme of effective divisors

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Abstract

In this article, we study the Hilbert scheme of effective divisors in smooth hypersurfaces in $\mathbb{P}^3$, a topic not extensively studied. We prove that there exists such effective divisors D satisfying the property: there exists infinitesimal deformations of D (in $\mathbb{P}^3$) not deforming the associated reduced scheme D_{red}. We observe that such infinitesimal deformations contribute to non-reducedness of the corresponding Hilbert scheme. We finally introduce a concept of simple extension of curves and notice that the above mentioned property is preserved under simple extension of curves.

1 Introduction

By a *curve* we will mean pure one dimensional projective scheme. The classical study of the geometry of Hilbert schemes of curves focused mainly on those components whose general element is smooth (see [Mum62, KO12, Kle81, Kle85] to name a few). Several difficulties arise when we drop the smoothness assumptions. This is because of the absence of several standard algebraic geometric tools. For example, a negative degree invertible sheaf on a non-reduced, irreducible scheme can have non-zero global sections. The Riemann Roch formula as well as Serre duality on non-reduced schemes is more complicated. This makes understanding the cohomology groups of the normal sheaves of non-reduced schemes much harder. The aim of this article is to study Hilbert schemes of non-reduced schemes by combining standard deformation theory with techniques from Hodge theory. We focus on

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the singularities of such Hilbert schemes, especially on deducing criterion under which it is non-reduced.

One of the first results of non-reduced components of Hilbert schemes of curves parametrizing generically non-reduced curves is due to Martin-Deschamps and Perrin (see [MDP96]). In their article, they prove the existence of generically non-reduced components parametrizing extremal, generically non-reduced curves in $\mathbb{P}^3$ (see [MDP96, Definition 0.1] for definition). They use the explicit description of the Koszul complex associated to an extremal curve to obtain their results.

Our first goal is to understand these non-reduced components of Hilbert schemes (as in Theorem 2.19). We use Hodge theory to prove that:

Theorem 1.1. Let L be an in Theorem 2.19. For any $C \in L$, there exists a *first order* infinitesimal deformation of C in $\mathbb{P}^3$ not lifting C_{red}.

See Theorem 5.6, Corollary 5.8 and Remark 5.9 for a more general statement.

Next, we introduce extension of curves. Roughly, an extension of an effective divisor D contained in a smooth hypersurface X in $\mathbb{P}^3$ is an effective divisor E in X such that $E - D$ intersects D properly and for any first order infinitesimal deformation of D, there exists an infinitesimal deformation of E containing it (see Definition 6.1). We prove that if there exists a *first order* infinitesimal deformation of D which does not lift D_{red} and E is an extension of D then it inherits the same property as D i.e., there exists a *first order* infinitesimal deformation of E which does not lift E_{red}. We then prove that such an infinitesimal deformation contributes to the singularity, and in certain cases non-reducedness, of the corresponding Hilbert scheme at the point corresponding to E. We prove:

Theorem 1.2. Notations as in Theorem 1.1. Let C be a general element in L and E an extension of C. Then, there exists an infinitesimal deformation of E in $\mathbb{P}^3$ not lifting E_{red}. Furthermore, if E is contained in an irreducible component of the Hilbert scheme of curves whose general element, say E' is contained in a smooth degree d hypersurface in $\mathbb{P}^3$ and E'_{red} has the same Hilbert polynomial as E_{red} then the component is non-reduced at the point corresponding to E.

See Theorems 6.6, 6.10 and 6.11 for more general results. Using these results, we produce explicit examples in Theorem 7.22.

One can ofcourse use standard deformation theory to directly approach Theorem 1.2. We discuss the problems involved in such an approach and how the techniques in this article is different from it. Since E is an extension of C, $E - C = F$ for some effective divisor F satisfying $C.F < \infty$. The standard approach to prove Theorem 1.2 would be to compare the dimension of the component M of the Hilbert scheme of curves with E as a special fiber and the tangent space at this point, which is simply $h^0(\mathcal{N}_{E|\mathbb{P}^3})$. The difficulty in this approach is two-fold. First, there is no standard method to deduce the general element of M and is extremely complicated in most cases. Second, even if we know what the general fiber of M is, it is difficult to compute the exact dimension of M. Computing dimension of components of Hilbert schemes of curves is one of the classical problems and very little is known on this topic. However, using Hodge theory, we completely circumvent both these difficulties. More precisely, we combine deformation theory with the study of Gauss-Manin connection. We use the latter to study tangent spaces to the Hodge loci corresponding to the cohomology class $[E]$ and $[C]$, respectively. Let us discuss the case when F is a multiple of a semi-regular curve (semi-regular in the sense of Bloch). By assumption, for every first order infinitesimal deformation of C there exists a first order infinitesimal deformation of E containing it. If every first order infinitesimal deformation of E lifts E_{red} then one can use the linearity of the Gauss-Manin connection and certain properties of the Hodge locus of semi-regular curves to prove that there exists an infinitesimal deformation of C_{red} contained in the infinitesimal deformation of C we started with. In particular, every infinitesimal deformation of C lifts C_{red}. But this contradicts the assumption on C. We can use this argument recursively to prove this in the case F is more general (see Proposition 6.4 and Corollary 6.5). After proving this, we use standard diagram chase in deformation theory to prove the second part of Theorem 1.2 (Theorems 6.10 and 6.11).

We now discuss the proof of Theorem 1.1. Let L be an irreducible component of a Hilbert scheme of curves in $\mathbb{P}^3$. Assume that a general element, say C of L is an effective divisor in a smooth hypersurface, say X in $\mathbb{P}^3$. Denote by P the Hilbert polynomial of C. The first step is to observe that there exists a Hilbert polynomial P_r such that a general element C of L satisfies: C_{red} has Hilbert polynomial P_r (see Proposition 3.3). Apply this to Theorem 2.19. We prove that the flag Hilbert scheme corresponding to the pair (P_r, P) contains an irreducible component M whose general element is of the form (C_{red}, C) and this component is smooth. This is done by standard deformation theory of pairs of schemes. Hence, the scheme-theoretic image of the natural projection from M to L is reduced. But L is generically non-reduced. This implies the existence of a first order infinitesimal deformation of C which does not lift C_{red}.

Finally, we reformulate the criterion for extension of effective divisors in terms of certain normal sheaves and its values at the points of intersection. Heuristically, it states that given two curves C and E intersecting at only finitely many points, a first order infinitesimal deformation (in $\mathbb{P}^3$) C' of C and E' of E glue to an infinitesimal deformation of $C+E$ if and only if the intersection $C'.E'$ lifts $C.E$. More precisely,

Theorem 1.3. Let X be a smooth degree d hypersurface in $\mathbb{P}^3$, C, D effective divisors on X and $D = C + nE$ for some reduced curve E. Denote by

$$\rho_C : H^0(\mathcal{N}_{C|\mathbb{P}^3}) \to H^0(\mathcal{N}_{D|\mathbb{P}^3} \otimes \mathcal{O}_C) \to H^0(\mathcal{N}_{D|\mathbb{P}^3} \otimes \mathcal{O}_{C.E}),$$

$$\rho_E : H^0(\mathcal{N}_{E|\mathbb{P}^3}) \to H^0(\mathcal{N}_{D|\mathbb{P}^3} \otimes \mathcal{O}_E) \to H^0(\mathcal{N}_{D|\mathbb{P}^3} \otimes \mathcal{O}_{C.E}).$$

the natural morphisms. Then, $\operatorname{Im} \rho_C \subset \operatorname{Im} \rho_E$ if and only if D is a simple extension of C.

See Theorem 7.7 and the remark after the theorem. We give simple examples of when this happens (see Corollary 7.13, Proposition 7.14 and Corollary 7.15).

Finally, in order to produce new non-reduced components of Hilbert scheme of curves parametrizing generically non-reduced curves, we need to satisfy the hypothesis in the second part of Theorem 1.2. We prove a more general statement:

Theorem 1.4. *Let X be a smooth degree d surface in $\mathbb{P}^3$, $E \subset X$ an effective divisor. If $d \geq \deg(E)^2 + 4$ then for any deformation of E to an effective divisor F on a smooth degree d surface Y in $\mathbb{P}^3$, the associated reduced scheme E_{red} deforms to F_{red}.*

See Theorem 7.19 for the precise statement. The statement gives a condition under which a deformation of a non-reduced scheme deforms the associated reduced scheme. Ofcourse there are numerous examples when this does not happen. We combine these results in Theorem 7.22.

Notation 1.5. From now on a *surface* will always mean a smooth surface in $\mathbb{P}^3$ and a *curve* will mean an effective divisor in a smooth surface. Note a curve need not be reduced. Also, given a scheme Y, we denote by Y_{red} the associated reduced scheme.

2 Preliminaries

2.1 Introduction to flag Hilbert schemes

We briefly recall the basic definition of flag Hilbert schemes and its tangent space in the setup we will use in this article. See [Ser06, §4.5] for the general statements on this topic.

Definition 2.1. Given an m-tuple of numerical polynomials $\mathcal{P}(t) = (P_1(t), P_2(t), ..., P_m(t))$, we define the contravariant functor, called the *Hilbert flag functor* relative to $\mathcal{P}(t)$,

$$\begin{array}{rcl} FH_{\mathcal{P}(t)} : (\text{schemes}) & \to & \text{sets} \\ S & \mapsto & \left\{ \begin{array}{c|l} (X_1, X_2, ..., X_m) & X_1 \subset X_2 \subset ... \subset X_m \subset \mathbb{P}^3_S, X_i \text{ are} \\ & S-\text{flat with Hilbert polynomial } P_i(t) \end{array} \right\} \end{array}$$

We call such an *m-tuple a flag relative to* $\mathcal{P}(t)$. The functor is representable by a projective scheme, denoted $H_{\mathcal{P}(t)}$, called *the flag Hilbert scheme associated to* $\mathcal{P}(t)$.

Notation 2.2. Let X_1 be a projective scheme, $X_2 \subset X_1$, a closed subscheme. Denote by $\mathcal{N}_{X_2|X_1}$ the normal sheaf $\mathcal{H}om_{X_1}(\mathcal{I}_{X_2/X_1}, \mathcal{O}_{X_2})$, where $\mathcal{I}_{X_2/X_1}$ is the ideal sheaf of X_2 in X_1.

Theorem 2.3. In the case $m = 2$, the tangent space at a point (X_1, X_2) to the flag Hilbert scheme H_{P_1,P_2}, denoted $T_{(X_1,X_2)}H_{P_1,P_2}$, satisfies the following Cartesian diagram:

$$\begin{array}{ccc} T_{(X_1,X_2)}H_{P_1,P_2} & \longrightarrow & H^0(\mathcal{N}_{X_2|\mathbb{P}^3}) \\ \downarrow & \square & \downarrow \\ H^0(\mathcal{N}_{X_1|\mathbb{P}^3}) & \longrightarrow & H^0(\mathcal{N}_{X_2|\mathbb{P}^3} \otimes \mathcal{O}_{X_1}) \end{array} \tag{1}$$

Proof. See [Ser06, §4, 5] for a proof of the theorem. □

Notation 2.4. Given a Hilbert polynomial P, we denote by H_P^0 the subscheme of H_P consisting of irreducible components of H_P whose general elements are effective divisors on smooth surfaces in $\mathbb{P}^3$.

2.2 Introduction to Hodge loci

In this subsection we recall the basics of Hodge theory, again restricting to the situation relevant for this article. See [Voi03, §5] for a detailed study of this subject.

Notation 2.5. Denote by $U_d \subseteq \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(\mathrm{d})))$ the open subscheme parametrizing smooth projective hypersurfaces in $\mathbb{P}^3$ of degree d. Denote by Q_d the Hilbert polynomial of degree d surfaces in $\mathbb{P}^3$. Let

$$\pi : \mathcal{X} \to U_d$$

be the corresponding universal family. For a given $u \in U_d$, denote by X_u the surface $X_u := \pi^{-1}(u)$. Fix a closed point $o \in U_d$, denote by $X := X_o$ and consider a simply connected neighbourhood U of o in U_d (under the analytic topology).

Definition 2.6. As U is simply connected, $\pi|_{\pi^{-1}(U)}$ induces a variation of Hodge structure $(\mathcal{H}^2, \nabla)$ on U where $\mathcal{H}^2 := R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U$ and ∇ is the Gauss-Manin connection. Note that $\mathcal{H}^2$ defines a local system on U whose fiber over a point $u \in U$ is $H^2(X_u, \mathbb{C})$. Consider a non-zero element $\gamma_0 \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})$ such that $\gamma_0 \neq c_1(\mathcal{O}_X(k))$ for $k \in \mathbb{Z}_{>0}$. This defines a section $\gamma \in \Gamma(U, \mathcal{H}^2)$ which takes the value γ_0 at the point $o \in U$. Recall, there exists a subbundle $F^2\mathcal{H}^2 \subset \mathcal{H}^2$, which for any $u \in U$, can be identified with the Hodge filtration $F^2H^2(X_u, \mathbb{C}) \subset H^2(X_u, \mathbb{C})$ (see [Voi02, §10.2.1]). Let $\overline{\gamma}$ be the image of γ in $\mathcal{H}^2/F^2\mathcal{H}^2$. The *Hodge loci*, denoted $\mathrm{NL}(\gamma_0)$ is then defined as

$$\mathrm{NL}(\gamma_0) := \{u \in U | \overline{\gamma}_u = 0\},$$

where $\overline{\gamma}_u$ denotes the value at u of the section $\overline{\gamma}$. One can check that the Hodge locus $\mathrm{NL}(\gamma_0)$ does not depend on the choice of the section γ.

Lemma 2.7 ([Voi03, Lemma 5.13])**.** There is a natural analytic scheme structure on $\overline{\mathrm{NL}(\gamma_0)}$ (closure in U_d under Zariski topology).

Definition 2.8. We now discuss the tangent space to the Hodge locus, $\mathrm{NL}(\gamma_0)$. We know that the tangent space to U at o, T_oU is isomorphic to $H^0(\mathcal{N}_{X|\mathbb{P}^3})$. This is because U is an open subscheme of the Hilbert scheme H_{Q_d}, the tangent space of which at the point o is simply $H^0(\mathcal{N}_{X|\mathbb{P}^3})$. Given the variation of Hodge structure above, we have (by Griffith's transversality) the differential map:

$$\overline{\nabla} : H^{1,1}(X) \to \mathrm{Hom}(T_oU, H^2(X, \mathcal{O}_X))$$

induced by the Gauss-Manin connection. Given $\gamma_0 \in H^{1,1}(X)$ this induces a morphism, denoted $\overline{\nabla}(\gamma_0)$, from T_oU to $H^2(\mathcal{O}_X)$.

Lemma 2.9 ([Voi03, Lemma 5.16])**.** The tangent space at o to $\mathrm{NL}(\gamma_0)$, denoted $T_o\mathrm{NL}(\gamma_0)$, equals $\ker(\overline{\nabla}(\gamma_0))$.

2.3 Semi-regularity map

The semi-regularity map was introduced by Kodaira-Spencer in the case of divisors, which was then generalized to any local complete intersection subschemes by Bloch. The purpose of this map is to study certain aspects of the variational Hodge conjecture. In this subsection, we consider the cohomology class γ of a curve C in a smooth degree d surface X in $\mathbb{P}^3$. We see that the differential map $\overline{\nabla}(\gamma)$ factors through the semi-regularity map, $H^1(\mathcal{N}_{C|X}) \to H^2(\mathcal{O}_X)$ (see Theorem 2.13). Using this description, we are able to capture the subspace of $T_o\mathrm{NL}(\gamma)$, the tangent space to $\mathrm{NL}(\gamma)$ at the point o corresponding to X, which parametrizes infinitesimal deformations of X under which C lifts as an effective Cartier divisor (see Corollary 2.14).

Definition 2.10. We start with the definition of a semi-regular curve. Let X be a surface and $C \subset X$, a curve in X. Since X is smooth, C is local complete intersection in X. This gives rise to the short exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{N}_{C|X} \to 0. \tag{2}$$

The *semi-regularity map* π_C is the boundary map from $H^1(\mathcal{N}_{C|X})$ to $H^2(\mathcal{O}_X)$, coming from this short exact sequence. We say that C is *semi-regular* if π_C is injective.

The following lemma gives a criterion for C to be semi-regular.

Lemma 2.11 ([Dan14a, Lemma 5.2])**.** Let C be a reduced curve and X a smooth degree d surface containing C. Then, $H^1(\mathcal{O}_X(-C)(k)) = 0$ for all $k \geq \deg(C)$. In particular, if $d \geq \deg(C) + 4$ then $h^1(\mathcal{O}_X(C)) = 0$, hence C is semi-regular.

2.12. Recall, the following short exact sequence of normal sheaves:

$$0 \to \mathcal{N}_{C|X} \to \mathcal{N}_{C|\mathbb{P}^3} \to \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C \to 0 \tag{3}$$

which arises from the short exact sequence:

$$0 \to \mathcal{I}_X \to \mathcal{I}_C \xrightarrow{j^{\#}} j_* \mathcal{O}_X(-C) \to 0. \tag{4}$$

after applying $\mathcal{H}\mathrm{om}_{\mathbb{P}^3}(-, j_{0_*}\mathcal{O}_C)$, where j_0 is the closed immersion of C into $\mathbb{P}^3$.

We then have the following results on the tangent space of the Hodge locus $\mathrm{NL}(\gamma)$.

Theorem 2.13 ([Dan14a, Theorem 4.8])**.** Let C, X be as before and $\gamma = [C] \in H^{1,1}(X, \mathbb{Z})$. We then have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & T_{(C,X)}H_{P,Q_d} & \longrightarrow & H^0(X, \mathcal{N}_{X|\mathbb{P}^3}) & \xrightarrow{\overline{\nabla}(\gamma)} & H^2(X, \mathcal{O}_X) \\
 & & & & \downarrow & \square & \downarrow \rho_C & \circlearrowleft & \uparrow \pi_C \\
0 & \longrightarrow & H^0(C, \mathcal{N}_{C|X}) & \xrightarrow{\phi_C} & H^0(C, \mathcal{N}_{C|\mathbb{P}^3}) & \xrightarrow{\beta_C} & H^0(C, \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C) & \xrightarrow{\delta_C} & H^1(C, \mathcal{N}_{C|X})
\end{array}$$

where the horizontal exact sequence comes from (3), π_C is the semi-regularity map and ρ_C is the natural pull-back morphism.

Corollary 2.14 ([Dan14a, Corollary 4.9])**.** Denote by P the Hilbert polynomial of C. Then, the tangent space $T_o\mathrm{NL}(\gamma)$ to $\mathrm{NL}(\gamma)$ at the point o corresponding to X, satisfies the following:

$$T_o(\mathrm{NL}(\gamma)) \supset \rho_C^{-1}(\mathrm{Im}\,\beta_C) = \mathrm{pr}_2\, T_{(C,X)}H_{P,Q_d}.$$

Furthermore, if C is semi-regular then we have equality $T_o(\mathrm{NL}(\gamma)) = \mathrm{pr}_2\, T_{(C,X)}H_{P,Q_d}$.

Before we proceed further we recall the following useful result:

Lemma 2.15 ([Dan14a, Corollary 4.10])**.** The following holds true: The kernel of ρ_C is isomorphic to $H^0(\mathcal{O}_X(-C)(d))$ and ρ_C is surjective if and only if $H^1(\mathcal{O}_X(-C)(d)) = 0$. Moreover, if $H^1(\mathcal{O}_X(-C)(d)) = 0$ then $\mathrm{pr}_1(T_{(C,X)}H_{P,Q_d}) = H^0(\mathcal{N}_{C|\mathbb{P}^3})$.

2.4 Non-reduced components of Hilbert schemes

In this subsection we recall the definition of non-reducedness of schemes. We then look at a standard property of non-reduced schemes which is often used but we have not seen a good reference for it (see Lemma refphe12). We end the section with an example of non-reduced component of Hilbert schemes of curves in $\mathbb{P}^3$, whose general element is itself non-reduced.

Definition 2.16. Let X be a scheme and $x \in X$ be a closed point. We say that X is *non-reduced* at x if the local ring $\mathcal{O}_{X,x}$ contains non-trivial nilpotent elements. We say that X is *generically non-reduced* if X is non-reduced at every point $x \in X$.

We first see that for a morphism of scheme where the domain is non-reduced, either the scheme theoretic image is non-reduced or a fiber containing this point is non-reduced.

Lemma 2.17. Let $f : X \to Y$ be a morphism of irreducible schemes. If X is non-reduced at a closed point, say $x \in X$ then either the scheme theoretic image of X is non-reduced at $f(x)$ or the fiber $f^{-1}(y)$ is non-reduced. Furthermore, if the scheme theoretic image of X is non-reduced at a closed point, say y, then X is non-reduced at some point on $f^{-1}(y)$.

Proof. Since non-reducedness is a local property, we can reduce the problem to the affine case. In particular, take $\operatorname{Spec} A$ (resp. $\operatorname{Spec} B$) open affine schemes in X (resp. Y) containing x (resp. $f(x) = y$) and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$. Replace f by the morphism from $\operatorname{Spec} A$ to $\operatorname{Spec} B$.

Recall, that the scheme theoretic image of $\operatorname{Spec} A$ is given as $\operatorname{Spec} B/I$, where I is the kernel of the induced ring homomorphism $f^{\#} : B \to A$. Denote by $k(y)$ the residue field of the localization of B/I at the maximal ideal corresponding to y. Then, the fiber $f^{-1}(y)$ is isomorphic to $\operatorname{Spec}(A \otimes_{B/I} k(y))$. If $n \in A_{m_x}$ is a non-zero, nilpotent element (m_x is the maximal ideal in A corresponding to x) then the image $\overline{n \otimes 1}$ of $n \otimes 1$ in

$$(A \otimes_{B/I} k(y))_{m_x} \cong A_{m_x} \otimes_{(B/I)_{m_y}} k(y)$$

is either non-zero, in which case the fiber $f^{-1}(y)$ is non-reduced, or is zero. If $\overline{n \otimes 1} = 0$ then there exists an element $m \in (B/I)_{m_y}$ such that $f_y^{\#}(m) = n$, where

$$f_y^{\#} : (B/I)_{m_y} \to A_{m_x}$$

is the natural map induced by $f^{\#}$. Since $f_y^{\#}$ is a ring homomorphism, m is non-zero and nilpotent in $(B/I)_{m_y}$, in which case the scheme-theoretic image is non-reduced. The first part of the lemma then follows.

For the second part of the lemma it suffices to prove that the scheme theoretic image of a reduced scheme is reduced. Denote by Z the scheme-theoretic image of f and assume that X is reduced. The universal property of scheme-theoretic image states that f factors through a morphism $f' : X \to Z$ and for any other closed subscheme $Z' \subset Y$ through which f factors, Z' contains Z. By [Har77, Ex. II.2.3], there exists an unique morphism $g : X \to Z_{\text{red}}$ such that f' factors through g. Hence, by the universal property of scheme-theoretic image, $Z = Z_{\text{red}}$ i.e., Z is reduced. This completes the proof of the lemma. □

One of the main objectives of this article is to study the following family of non-reduced curves:

Notation 2.18. Let a, d be positive integers, $d \geq 5$ and $a > 0$. Let X be a smooth projective surface in $\mathbb{P}^3$ of degree d containing a line l and a smooth coplanar curve C_1 of degree a. Let C be a divisor of the form $2l + C_1$ in X. Denote by P the Hilbert polynomial of C.

Theorem 2.19 (Martin-Deschamps and Perrin [MDP96])**.** There exists an irreducible component, say L of H_P such that a general curve $D \in L$ is a divisor in a smooth degree d surface in $\mathbb{P}^3$ of the form $2l' + C_1'$ where l', C_1' are coplanar curves with $\deg(l') = 1$ and $\deg(C_1') = a$. Furthermore, L is generically non-reduced.

Proof. The theorem follows from [MDP96, Proposition 0.6, Theorems 2.4, 3.1]. $\square$

3 Topology of Hilbert schemes of effective divisors

We start with the following definition:

Definition 3.1. Let C be a curve in $\mathbb{P}^3$. We say that C is *d-embedded* if there exists a smooth degree d surface X in $\mathbb{P}^3$ containing C. In this case, we also say that C is *d-embedded in* X. Since X is smooth, observe that C is an effective Cartier divisor on X.

Given a linear Hilbert polynomial P, the aim of this section is to study certain topological aspects of the corresponding Hilbert scheme of curves H_P. We first observe that for any irreducible component $L \subset H_P$, there exists a Hilbert polynomial P_r such that every curve $D \in L$ contains a subcurve $D' \subset D$, $D'_{\text{red}} = D_{\text{red}}$ with Hilbert polynomial P_r and if D is outside certain finite union of proper closed subsets of L (i.e., D is general) then $D' = D'_{\text{red}} = D_{\text{red}}$ (see Proposition 3.3). By the universal property of flag Hilbert schemes, this means that there exists an irreducible component L_r of $H_{P_r,P}$ such that $\text{pr}_2(L_r)_{\text{red}} = L_{\text{red}}$, where $\text{pr}_2 : H_{P_r,P} \to H_P$ is the second projection map.

Suppose that a general element $D \in L$ is d-embedded for some integer $d \geq \deg(D) + 4$. We prove that for any $C' \in \text{pr}_1(L_r)$ reduced and $C \in \text{pr}_2(\text{pr}_1^{-1}(C'))$ general, we have $\dim \mathbb{P}(I_d(C)) = \dim \mathbb{P}(I_d(D))$ if C is also d-embedded (see Theorem 3.12). This will play a vital role in the proofs of Theorems 5.6 and 6.11, later in the text.

Notation 3.2. Fix a Hilbert polynomial P of a curve in $\mathbb{P}^3$.

Proposition 3.3. Let L an irreducible component of H_P. There exists a Hilbert polynomial P_r of a curve in $\mathbb{P}^3$ and an irreducible component L' of $H_{P_r,P}$ such that:

1. under the natural projection map pr_2, L' maps surjectively (as topological spaces) to L,

2. for the universal family $\pi' := (\pi_1', \pi_2') : \mathcal{C}' \subset \mathcal{C}'' \to L'$ corresponding to L' we have that for all $u \in L'$, $\mathcal{C}'_{u_{\text{red}}} = \mathcal{C}''_{u_{\text{red}}}$ and

3. there exists a nonempty open set U in L' such that for all $u \in U$ the corresponding triple $\mathcal{C}'_u \subset \mathcal{C}''_u$ satisfies $\mathcal{C}'_u = \mathcal{C}''_{u_{\text{red}}}$.

Proof. Consider the universal family $\mathcal{C} \xrightarrow{\pi} L$ corresponding to L. [Har77, Ex. II.2.3] implies there exists a morphism $\mathcal{C}_{\text{red}} \xrightarrow{\bar{\pi}} L_{\text{red}}$ such that the following diagram is commutative,

$$\begin{array}{ccc} \mathcal{C}_{\text{red}} & \hookrightarrow & \mathcal{C} \\ {\scriptstyle \overline{\pi}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \pi} \\ L_{\text{red}} & \hookrightarrow & L \end{array}$$

[Gro65, Theorem 6.9.1] implies there exists a nonempty open set $U \subset L_{\text{red}}$ such that $\overline{\pi}|_{\overline{\pi}^{-1}(U)} : \overline{\pi}^{-1}(U) \to U$ is flat. [Har77, Theorem III.9.9] implies that every fiber of this morphism has the same Hilbert polynomials, say P_r.

Therefore, there exists an open set $U' \subset H_{P_r,P}$ which maps dominantly (as topological spaces) to L under the natural projection map and for all $u \in U'$ the corresponding pair $\mathcal{C}'_u \subset \mathcal{C}''_u$ satisfies $\mathcal{C}'_u = \mathcal{C}''_{u_{\text{red}}}$. But, the projection map is closed, hence $\text{pr}_2(\overline{U'_{\text{red}}}) = L_{\text{red}}$. Therefore, there exists an irreducible component L' (in particular, contained in the closure of U' in $H_{P_r,P}$) such that $\text{pr}_2(L')_{\text{red}} = L_{\text{red}}$ and there exists a nonempty open set U in L' such that for all $u \in U$ the corresponding triple $\mathcal{C}'_u \subset \mathcal{C}''_u$ satisfies $\mathcal{C}'_u = \mathcal{C}''_{u_{\text{red}}}$. This proves (1) and (3).

The only statement remaining to prove is that for all $u \in L'_{\text{red}}$, $\mathcal{C}'_{u_{\text{red}}} = \mathcal{C}''_{u_{\text{red}}}$. Note that, π'_2 is a flat morphism of finite type between noetherian schemes. Then [Har77, Ex. III.9.1] implies that π'_2 is open. Hence, $\pi'_2(\mathcal{C}''_{\text{red}} - \mathcal{C}'_{\text{red}})$ is open. But by (3), this is closed, *properly* contained in L' (use the upper-semicontinuity of fiber dimension). Since L' is irreducible, this means $\pi'_2(\mathcal{C}''_{\text{red}} - \mathcal{C}'_{\text{red}})$ is empty. Therefore, for all $u \in L'_{\text{red}}$, $\mathcal{C}'_{u_{\text{red}}} = \mathcal{C}''_{u_{\text{red}}}$. This proves the proposition. □

The following lemma will play an important role in the proof of Proposition 3.10 later.

Lemma 3.4. Let C be a d-embedded curve with Hilbert polynomial P. For any smooth degree d surface X containing C_{red} there exists a curve $D \subset X$ with Hilbert polynomial P and $D_{\text{red}} = C_{\text{red}}$. For every degree d surface X containing C_{red} there exists a curve D in X such that $D \in H_P$ and $D_{\text{red}} = C_{\text{red}}$.

Proof. Suppose C is of the form $\sum_i a_i C_i$ as a divisor in a smooth degree d surface in $\mathbb{P}^3$, where $a_i > 0$ and C_i are integral curves. For any degree d surface containing C_{red}, there exists a divisor C' of the above form, in its Picard group. We denote by P the Hilbert polynomial of C. Note that the degree of C is equal to $\sum_i a_i \deg(C_i)$, which is the same as $\deg(C')$. Using the adjunction formula, we see the arithmetic genus of C' is the same as C. So, for every degree d surface X containing C_{red} there exists an effective divisor C' of X with Hilbert polynomial P and $C'_{\text{red}} = C_{\text{red}}$. □

Notation 3.5. Let P_r be as in Proposition 3.3. Denote by $(123 \to ij)$ (resp. $(123 \to i)$) the natural projection map from H_{P_r,P,Q_d} to its (i,j)-th (resp. i-th) components. Denote by $(12 \to i)$ (resp. $(13 \to i), (23 \to i)$) the natural projection map from $H_{P_r,P}$ (resp. H_{P_r,Q_d}, H_{P,Q_d}) to its i-th component.

The following lemma is useful for computational purposes:

Lemma 3.6 ([Dan14b, Lemma 3.6])**.** Let $d \geq 5$ and C be d-embedded in X (by definition this means $\deg(X) = d$) and be of the form $\sum_i a_i C_i$ where C_i are integral curves with $\deg(C) + 2 \leq d$. Then, $h^0(\mathcal{N}_{C|X}) = 0$. In particular, $\dim |C| = 0$ where $|C|$ is the linear system associated to C.

Recall, the following result on Castelnuovo-Mumford regularity:

Theorem 3.7. Let C be a reduced curve in $\mathbb{P}^3$ of degree e. Then, C is e-regular.

Proof. If C is connected then it is e-regular ([Gia06, Main Theorem]). Note that, [Sid02, Theorem 1.8] states that the regularity of $I.J$ is at most the sum of the regularity of I and J. This implies, that if $C = C_1 \cup ... \cup C_n$ and C_i are the connected components, then the regularity of the ideal of C is at most the sum of the regularity of the ideals of C_i for $i = 1, ..., n$. The theorem then follows. $\square$

Notation/Remark 3.8. Let $L \subset H_P$ be an irreducible component of H_P such that a general element $D \in L$ is d-embedded. Let P_r and L' be as in Proposition 3.3. Then, there exists an irreducible component L'' of H_{P_r,P,Q_d} which maps surjectively (as topological spaces) to L' and a general element of L'' is of the form $(C_{g_{\text{red}}}, C_g, X_g)$, where X_g is smooth. This follows from the universal property of flag Hilbert schemes and open nature of smooth fibers in flat families. One can infact immitate the proof of Proposition 3.3(1).

Definition 3.9. Given a scheme X and a point $x \in X$, we say that x is *weakly general* if x is contained in an unique irreducible component of X.

Proposition 3.10. Let $L \subset H_P$ be an irreducible component such that a general element of $C_g \in L$ is d-embedded for $d \geq \deg(C_g) + 4$. Let $L'' \subset H_{P_r,P,Q_d}$ be an irreducible component as described in Notation/Remark 3.8 above. Then, the image under $(123 \to 3)$ of the the fiber to the morphism $(123 \to 1)|_{L''}$ over any reduced curve $C' \in (123 \to 1)(L'')$ is isomorphic to $\mathbb{P}(I_d(C'))$.

Proof. As L'' is an irreducible component of H_{P_r,P,Q_d}, a general fiber of $(123 \to 1)|_{L''}$ is not entirely contained in a second irreducible component of H_{P_r,P,Q_d}. Therefore, a general element of L'' is weakly general and is of the form $(C_{g_{\text{red}}}, C_g, X_g)$ with X_g smooth. Using Lemma 3.4, there exists an irreducible component M of $(123 \to 1)^{-1}(C_{g_{\text{red}}})$ containing $(C_{g_{\text{red}}}, C_g, X_g)$ and isomorphic to $\mathbb{P}(I_d(C_{g_{\text{red}}}))$. As $(C_{g_{\text{red}}}, C_g, X_g)$ is also weakly general, M is contained in L''. By the fiber dimension theorem this implies the dimension of every fiber of $(123 \to 1)|_{L''}$ is at least $\dim \mathbb{P}(I_d(C_{g_{\text{red}}}))$.

By Theorem 3.7, if $C' \in (123 \to 1)(L'')$ is reduced then C' is d-regular and so is $C_{g_{\mathrm{red}}}$. This means $H^i(\mathcal{I}_{C'}(d)) = 0 = H^i(\mathcal{I}_{C_{g_{\mathrm{red}}}}(d))$ for all $i > 0$ which implies that

$$\dim \mathbb{P}(I_d(C')) = P_r(d) - 1 = \dim \mathbb{P}(I_d(C_{g_{\mathrm{red}}})).$$

Now, dimension of the fiber to $(123 \to 1)|_{L''}$ over C' is the same as the dimension of

$$(123 \to 3)((123 \to 1)|_{L''})^{-1}(C')$$

because the fiber to $(123 \to 3)$ is zero dimensional (see Lemma 3.6). As

$$(123 \to 3)((123 \to 1)|_{L''})^{-1}(C') \subset (13 \to 3)(13 \to 1)^{-1}(C') \cong \mathbb{P}(I_d(C')),$$

and the dimension of $(123 \to 1)|_{L''}^{-1}(C')$ is at least $\dim \mathbb{P}(I_d(C'))$ this means the image under $(123 \to 3)$ of the the fiber to the morphism $(123 \to 1)|_{L''}$ over any reduced curve $C' \in (123 \to 1)(L'')$ is isomorphic to $\mathbb{P}(I_d(C'))$. This completes the proof of the proposition. □

Corollary 3.11. Hypothesis as in Proposition 3.10. For C' as in the proposition,

$$\dim (12 \to 1)|_{L'}^{-1}(C') = \dim I_d(C') - \dim I_d(C_g)$$

where (C', C_g) is a general element of the fiber $(12 \to 1)|_{L'}^{-1}(C')$.

Proof. By Proposition 3.10, the image under $(123 \to 3)$ of the the fiber to the morphism $(123 \to 1)|_{L''}$ over any reduced curve $C' \in (123 \to 1)(L'')$ is isomorphic to $\mathbb{P}(I_d(C'))$. Clearly, the fiber over (C', C_g) to the morphism $(123 \to 12)|_{L''}$ is isomorphic to $\mathbb{P}(I_d(C_g))$, where (C', C_g) is a general element of the fiber $(12 \to 1)|_{L'}^{-1}(C')$. Since C_g is d-embedded, observe that the dimension of $(12 \to 1)|_{L'}^{-1}(C')$ is the difference of the dimension of $(123 \to 1)|_{L''}^{-1}(C')$ and of $(123 \to 12)|_{L''}^{-1}(C', C_g)$. This proves the corollary. □

Theorem 3.12. Hypothesis as in Proposition 3.10. For any $C' \in (12 \to 1)(L')$ reduced curve, there exists an element $(C', C) \in L'$ such that $\dim \mathbb{P}(I_d(C)) = \dim \mathbb{P}(I_d(C_g))$ for $C_g \in L$, general.

Proof. Let Z be the locus of pairs $(D', D) \in L'$ such that $\dim I_d(D) > \dim I_d(C_g)$ for $C_g \in L$, general. Suppose the theorem is false. This is equivalent to Z containing a fiber over a reduced curve C' to the morphism $(12 \to 1)|_{L'}$. By Corollary 3.11,

$$\dim((12 \to 1)|_{L'})^{-1}(C') = \dim I_d(C') - \dim I_d(C'_g)$$

where (C', C'_g) is a general element of the fiber $(12 \to 1)|_{L'}^{-1}(C')$. The dimension of a general fiber of $(12 \to 1)|_{L'}$ is given as follows: For $C_g \in L$, general,

$$\dim((12 \to 1)|_{L'})^{-1}(C_{g_{\mathrm{red}}}) = \dim I_d(C_{g_{\mathrm{red}}}) - \dim I_d(C_g).$$

As C' and $C_{g_{\mathrm{red}}}$ are d-regular (Theorem 3.7), $\dim I_d(C_{g_{\mathrm{red}}}) = \dim I_d(C')$. By assumption on C', $\dim I_d(C'_g) > \dim I_d(C_g)$ which implies

$$\dim((12 \to 1)|_{L'})^{-1}(C') < \dim((12 \to 1)|_{L'})^{-1}(C_{g_{\mathrm{red}}}).$$

But this contradicts the fiber dimension theorem. This proves the theorem. □

4 Tangent space to Hilbert scheme of effective curves

Setup 4.1. Let X be a smooth degree d surface in $\mathbb{P}^3$, C, D effective divisors on X satisfying $D \leq C$. Denote by P_C (resp. P_D) the Hilbert polynomial of C (resp. D).

The following diagram relates infinitesimal deformations of D to that of C. This diagram plays an important role throughout this article. We use it in two ways. First, we can replace D by C_{red} and study conditions under which there exists infinitesimal deformations of C not deforming C_{red} (see Theorem 5.6 and Corollary 5.8). Second, we impose the condition that C is an extension of D. Under this hypothesis, we use this diagram to observe that if there exists an infinitesimal deformation of D not deforming D_{red}, then this infinitesimal deformation of D gives rise to an infinitesimal deformation of C which does not deform C_{red} (see Corollary 6.5).

Remark 4.2. We then have the following commutative diagram:

$$\begin{array}{ccccc}
 & & T_{(D,C)}H_{P_D,P_C} & \longrightarrow & H^0(\mathcal{N}_{D|\mathbb{P}^3}) \\
 & & \downarrow & \square & \downarrow \Upsilon^6_{D\leq C} \\
T_{(C,X)}H_{P_C,Q_d} & \longrightarrow & H^0(\mathcal{N}_{C|\mathbb{P}^3}) & \xrightarrow{\Upsilon^5_{D\leq C}} & H^0(\mathcal{N}_{C|\mathbb{P}^3}\otimes\mathcal{O}_D) \\
\downarrow & \square & \beta_C \downarrow & \circlearrowleft & \downarrow \Upsilon^1_{D\leq C} \\
H^0(\mathcal{N}_{X|\mathbb{P}^3}) & \xrightarrow{\rho_C} & H^0(\mathcal{N}_{X|\mathbb{P}^3}\otimes\mathcal{O}_C) & \xrightarrow{\Upsilon^2_{D\leq C}} & H^0(\mathcal{N}_{X|\mathbb{P}^3}\otimes\mathcal{O}_D)
\end{array}$$

where $\Upsilon^2_{D\leq C}$ and $\Upsilon^5_{D\leq C}$ are restriction morphisms, $\Upsilon^6_{D\leq C}$ is obtained by applying the functor $\mathcal{H}om_{\mathbb{P}^3}(-,\mathcal{O}_D)$ followed by the global section functor to the natural morphism $\mathcal{I}_C \hookrightarrow \mathcal{I}_D$, $\Upsilon^1_{D\leq C}$ arises from pulling back to D the short exact sequence (3) and β_C, ρ_C are the morphisms mentioned in Theorem 2.13. The two Cartesian diagram follow from the theory of flag Hilbert schemes (see Theorem 2.3).

Assumption 4.3. For the rest of this section we assume that $d \geq \deg(C)+4$.

In this section, we look at some basic properties of this diagram. We see that for a given infinitesimal deformation of X there exists at most one infinitesimal deformation of C contained in it (Lemma 4.4). Furthermore, for any infinitesimal deformation of C there exists at most one infinitesimal deformation of D contained in it (Corollary 4.7). Finally, we give a description of the differential maps associated to the cohomology classes of C and D, in terms of the maps in the above diagram (see Lemma 4.9 and Corollary 4.10).

Lemma 4.4. The morphism β_C is injective.

Proof. Recall, the morphism β_C is induced by the short exact sequence

$$0 \to \mathcal{N}_{C|X} \to \mathcal{N}_{C|\mathbb{P}^3} \to \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C \to 0.$$

Hence, the kernel of β_C is $H^0(\mathcal{N}_{C|X})$, the vanishing of which follows from Lemma 3.6. $\square$

We can further prove that for the correct bounds on d, $\Upsilon^1_{D\le C}$ is injective.

Lemma 4.5. If $D = C_{\text{red}}$ then the corresponding map $\Upsilon^1_{D\le C}$ is injective. Moreover, for any reduced curve $E \le C_{\text{red}}$, we have $H^0(\mathcal{N}_{C|X} \otimes \mathcal{O}_E) = 0$ i.e., $\Upsilon^1_{E\le C}$ is injective.

Proof. In the case $D = C_{\text{red}}$ the kernel of the map $\Upsilon^1_{D\le C}$ is $H^0(\mathcal{N}_{C|X} \otimes \mathcal{O}_{C_{\text{red}}})$. This follows from taking the long exact sequence of the following short exact sequence:

$$0 \to \mathcal{N}_{C|X} \otimes \mathcal{O}_{C_{\text{red}}} \to \mathcal{N}_{C|\mathbb{P}^3} \otimes \mathcal{O}_{C_{\text{red}}} \to \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_{C_{\text{red}}} \to 0$$

obtained by pulling back to C_{red} the short exact sequence (3).

We show that the degree of the line bundle $\mathcal{N}_{C|X} \otimes \mathcal{O}_{C_{\text{red}}}$ restricted to each of the components is negative. This would mean there does not exist global sections on any of the irreducible components, hence not on C_{red}. We can write $C = \sum_{i=1}^r a_i C_i$ in $\text{Div}(X)$ with $C_i \neq C_j$ for $i \neq j$ and C_i integral. It suffices to prove this for C_1. Note that,

$$C.C_1 = \sum_{i=2}^r a_i C_i C_1 + a_1 C_1^2 \le \sum_{i=2}^r a_i \deg(C_i)\deg(C_1) + a_1(2\rho_a(C_1) - 2 - (d-4)\deg(C_1))$$

which follows from the fact that $C_i.C_1 \le \deg(C_1)\deg(C_i)$ for $i \neq 1$ and the adjunction formula applied to C_1^2. Using the degree assumption on d and the bound on the arithmetic genus of an integral curve in $\mathbb{P}^3$ we then get

$$C.C_1 \le \sum_{i=2}^r a_i \deg(C_i)\deg(C_1) + a_1\left((\deg(C_1)-1)(\deg(C_1)-2) - 2 - \deg(C_1)\left(\sum_{i=1}^r a_i \deg(C_i)\right)\right)$$

which is clearly less than zero since $a_i \ge 1$ for all i. Hence, $h^0(\mathcal{N}_{C|X} \otimes \mathcal{O}_{C_{\text{red}}}) = 0$. This proves the first part of the lemma.

The second part is a direct consequence of the proof of the first part. In particular, we see that for any irreducible component of C_{red} there does not exist a global section of the restriction of $\mathcal{N}_{C|X}$ to this component. So, for any $E \le C_{\text{red}}$, $h^0(\mathcal{N}_{C|X} \otimes \mathcal{O}_E) = 0$.

Pulling back the short exact sequence (3) to E, gives us the following short exact sequence:

$$0 \to \mathcal{N}_{C|X} \otimes \mathcal{O}_E \to \mathcal{N}_{C|\mathbb{P}^3} \otimes \mathcal{O}_E \to \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_E \to 0$$

Then observe that, $\ker \Upsilon^1_{E\le C} = h^0(\mathcal{N}_{C|X} \otimes \mathcal{O}_E) = 0$. Therefore, $\Upsilon^1_{E\le C}$ is injective. This completes the proof of the lemma. $\square$

We want to show that $\Upsilon^1_{D\le C}\circ\Upsilon^6_{D\le C}=\beta_D$ and $\Upsilon^2_{D\le C}\circ\Upsilon^3_{D\le C}=\rho_D$. This is done using the following lemma.

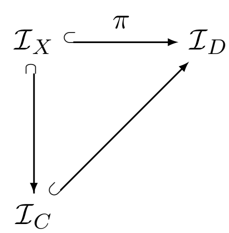
Lemma 4.6. We have the following:

1. $\Upsilon^1_{D\le C}\circ\Upsilon^6_{D\le C}$ is the same as the morphism β_D from $H^0(\mathcal{N}_{D|\mathbb{P}^3})$ to $H^0(\mathcal{N}_{X|\mathbb{P}^3}\otimes\mathcal{O}_D)$ arising from the following short exact sequence:

$$0\to\mathcal{N}_{D|X}\to\mathcal{N}_{D|\mathbb{P}^3}\to\mathcal{N}_{X|\mathbb{P}^3}\otimes\mathcal{O}_D\to 0 \tag{5}$$

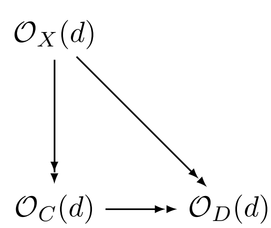
2. $\Upsilon^2_{D\le C}\circ\rho_C$ is the same as the natural restriction morphism ρ_D from $H^0(\mathcal{N}_{X|\mathbb{P}^3})$ to $H^0(\mathcal{N}_{X|\mathbb{P}^3}\otimes\mathcal{O}_D)$.

Proof. (1) The morphisms $\Upsilon^1_{D\le C},\Upsilon^6_{D\le C}$ and its composition arise from applying $\mathcal{H}\mathrm{om}_{\mathbb{P}^3}(-,i_*\mathcal{O}_D)$ followed by the global section functor to the following commutative diagram:



where i is the natural closed immersion of D into $\mathbb{P}^3$. It follows from the construction of the short exact sequence (5) that the morphism π coincides with the corresponding morphism in the short exact sequence (4). Hence, applying $\mathcal{H}\mathrm{om}_{\mathbb{P}^3}(-,i_*\mathcal{O}_D)$ gives us the morphism $\mathcal{N}_{D|\mathbb{P}^3}\to\mathcal{N}_{X|\mathbb{P}^3}\otimes\mathcal{O}_D$ which sits in the short exact sequence (5). Finally, the commutativity of the resulting diagram, after applying $\mathcal{H}\mathrm{om}_{\mathbb{P}^3}(-,i_*\mathcal{O}_D)$ followed by $\Gamma(-)$, proves (1).

(2) This follows from applying the global section functor to the following commutative diagram and using the identification $\mathcal{N}_{X|\mathbb{P}^3}=\mathcal{O}_X(d)$:



This proves (2). ∎

Corollary 4.7. The morphism $\Upsilon^6_{D\le C}$ is injective.

Proof. Lemma 4.6 tells us that the kernel of $\Upsilon^1_{D\le C} \circ \Upsilon^6_{D\le C}$ is $H^0(\mathcal{N}_{D|X})$. Lemma 3.6 implies that $h^0(\mathcal{N}_{D|X}) = 0$. Therefore, $\Upsilon^1_{D\le C} \circ \Upsilon^6_{D\le C}$ is injective, hence $\Upsilon^6_{D\le C}$ is injective. □

Corollary 4.8. The fiber of the projection morphism from H_{P_D,P_C} to H_{P_C} over any closed reduced point is a discrete set of closed reduced points.

Proof. Note that the tangent space at (D,C) to the fiber of H_{P_D,P_C} is the kernel of the natural projection morphism $T_{(D,C)}H_{P_D,P_C} \to T_C H_{P_C}$. It follows from the fiber product in the diagram 4.2 this is isomorphic to $\ker \Upsilon^6_{D\le C}$. Corollary 4.7 implies $\dim \ker \Upsilon^6_{D\le C} = 0$. Hence, the fiber of the projection morphism from H_{P_D,P_C} to H_{P_C} over any closed reduced point is zero dimensional and reduced. □

The following lemma tells us (using the above diagrams) for which infinitesimal deformations of X, the cohomology classes $[C]$ and $[D]$ remains Hodge.

Lemma 4.9. For all $t \in (\rho_C)^{-1}(\operatorname{Im}\beta_C)$ (resp. $(\Upsilon^2_{D\le C} \circ \rho_C)^{-1}(\operatorname{Im}\Upsilon^1_{D\le C} \circ \Upsilon^6_{D\le C})$),

$$\overline{\nabla}([C])(t) = 0 \text{ (resp. } \overline{\nabla}([D])(t) = 0).$$

Proof. Lemma 4.6 implies $\Upsilon^2_{D\le C} \circ \rho_C = \rho_D$ and $\Upsilon^1_{D\le C} \circ \Upsilon^6_{D\le C} = \beta_D$. The lemma then follows from Corollary 2.14. □

Corollary 4.10. If C is reduced and $\deg(X) \ge \deg(C) + 4$ then $\overline{\nabla}([C])(t) = 0$ if and only if $t \in (\rho_C)^{-1}(\operatorname{Im}\beta_C)$.

Proof. By Lemma 2.11, C is semi-regular. Corollary 2.14 implies $t \in (\rho_C)^{-1}(\operatorname{Im}\beta_C)$ if and only if $\overline{\nabla}([C])(t) = 0$. □

5 On an example of Martin-Deschamps and Perrin

In this section we see that the curves studied by Martin-Deschamps and Perrin (see Theorem 2.19) have an interesting deformation theoretic property: there exists an infinitesimal deformation of such a curve which does not deform the associated reduced scheme (see Corollary 5.8). To prove this we first give a general criterion under which an effective divisor on a smooth surface of the form $2C_1 + C_2$ satisfies this property (see Theorem 5.6), where C_1, C_2 are reduced curves. We use this in Corollary 5.8 to check that the curves in Theorem 2.19 satisfy this criterion.

Setup 5.1. Let C_1, C_2 be two reduced curves in $\mathbb{P}^3$ without common components and X a smooth degree d surface in $\mathbb{P}^3$ containing $C_1 \cup C_2$ for some $d \ge 2\deg(C_1) + \deg(C_2) + 4$. Denote by C the effective divisor in X of the form $2C_1 + C_2$. Denote by P_1 (resp. P, P_r) the Hilbert polynomial of C_1 (resp. C, C_{red}). We use notations as in Notation 3.5.

Definition 5.2. Let X be a degree d surface and C be an effective divisor on X. We say that C is *deformation d-regular* if $\operatorname{Im}\beta_C \subset \operatorname{Im}\rho_C$ (notations as in the diagram in Remark 4.2).

Heuristically, deformation d-regular implies that for any infinitesimal deformation of C there exists an infinitesimal deformation of X containing it.

Lemma 5.3. If C is $d+1$-regular (in the sense of Castelnuovo-Mumford) then ρ_C is surjective. In particular, C is deformation d-regular.

Proof. The definition of Castelnuovo-Mumford regularity implies that if C is $d+1$-regular then $H^1(\mathcal{O}_X(-C)(d)) = 0$. By Lemma 2.15, this implies the surjectivity of ρ_C. Since ρ_C is surjective, $\operatorname{Im}\beta_C \subset \operatorname{Im}\rho_C$. Hence, C is deformation d-regular. This proves the lemma. $\square$

The following lemma states that, under the condition C is deformation d-regular, given any infinitesimal deformation C_ξ of C, if it contains an infinitesimal deformation C'_ξ of C_{red} then C'_ξ contains an infinitesimal deformation of C_1.

Lemma 5.4. If C is deformation d-regular then

$$\dim \operatorname{pr}_1 T_{(C_{\text{red}},C)} H_{P_r,P} \leq \dim \operatorname{pr}_2 T_{(C_1,C_{\text{red}})} H_{P_1,P_r}. \tag{6}$$

where $\operatorname{pr}_1 : T_{(C_{\text{red}},C)} H_{P_r,P} \to T_{C_{\text{red}}} H_{P_r}$ and $\operatorname{pr}_2 : T_{(C_1,C_{\text{red}})} H_{P_1,P_r} \to T_{C_{\text{red}}} H_{P_r}$ are natural projection maps.

Proof. Given $(\xi_1,\xi_2) \in T_{(C_{\text{red}},C)} H_{P_r,P}$, we are going to show that there exist $\xi' \in H^0(\mathcal{N}_{C_1|\mathbb{P}^3})$ such that $(\xi',\xi_1) \in T_{(C_1,C_{\text{red}})} H_{P_1,P_r}$. Since C is deformation d-regular, given a pair $(\xi_1,\xi_2) \in T_{(C_{\text{red}},C)} H_{P_r,P}$, there exists $\xi \in H^0(\mathcal{N}_{X|\mathbb{P}^3})$ such that $\beta_C(\xi_2) = \rho_C(\xi)$ and

$$\Upsilon^5_{C_{\text{red}} \leq C}(\xi_2) = \Upsilon^6_{C_{\text{red}} \leq C}(\xi_1).$$

Using Lemma 4.6 we conclude,

$$\rho_{C_{\text{red}}}(\xi) = \Upsilon^2_{C_{\text{red}} \leq C} \circ \rho_C(\xi) = \Upsilon^2_{C_{\text{red}} \leq C} \circ \beta_C(\xi_2) = \Upsilon^1_{C_{\text{red}} \leq C} \circ \Upsilon^5_{C_{\text{red}} \leq C}(\xi_2) =$$
$$= \Upsilon^1_{C_{\text{red}} \leq C} \circ \Upsilon^6_{C_{\text{red}} \leq C}(\xi_1) = \beta_{C_{\text{red}}}(\xi_1).$$

Lemma 4.9 implies $\overline{\nabla}([2C_1 + C_2])(\xi) = 0$ and $\overline{\nabla}([C_1 + C_2])(\xi) = 0$. Since the differential $\overline{\nabla}$ is linear, $\overline{\nabla}([C_1])(\xi) = 0$ and $\overline{\nabla}([C_2])(\xi) = 0$. But $\deg(C_1), \deg(C_2)$ and $\deg(C_{\text{red}})$ are less than $d-4$. Hence, by Corollary 4.10, there exists $\xi' \in H^0(\mathcal{N}_{C_1|\mathbb{P}^3})$ such that $\Upsilon^1_{C_1 \leq C_{\text{red}}} \circ \Upsilon^6_{C_1 \leq C_{\text{red}}}(\xi') = \Upsilon^2_{C_1 \leq C_{\text{red}}} \circ \rho_{C_{\text{red}}}(\xi)$. So,

$$\Upsilon^1_{C_1 \leq C_{\text{red}}} \circ \Upsilon^6_{C_1 \leq C_{\text{red}}}(\xi') = \Upsilon^2_{C_1 \leq C_{\text{red}}} \circ \rho_{C_{\text{red}}}(\xi) = \Upsilon^2_{C_1 \leq C_{\text{red}}} \circ \beta_{C_{\text{red}}}(\xi_1) =$$
$$= \Upsilon^1_{C_1 \leq C_{\text{red}}} \circ \Upsilon^5_{C_1 \leq C_{\text{red}}}(\xi_1).$$

Using Lemma 4.5, we can conclude that $\Upsilon^1_{C_1 \leq C_{\text{red}}}$ is injective. So,

$$\Upsilon^6_{C_1 \leq C_{\text{red}}}(\xi') = \Upsilon^5_{C_1 \leq C_{\text{red}}}(\xi_1).$$

Hence, $(\xi',\xi_1) \in T_{(C_1,C_{\text{red}})} H_{P_1,P_r}$. This completes the proof of the lemma. $\square$

Proposition 5.5. If C is deformation d-regular then the fiber over C_{red} to the morphism $(12 \to 1) : H_{P_r,P} \to H_{P_r}$ is smooth at the point (C_{red}, C). In particular,

$$\dim(12 \to 1)^{-1}(C_{\text{red}}) = \dim\ker \Upsilon^5_{C_{\text{red}} \le C}.$$

Proof. Corollary 4.7 implies that $\Upsilon^6_{C_{\text{red}} \le C}$ is injective. So, the tangent space to the fiber at (C_{red}, C), $T_{(C_{\text{red}},C)}(12 \to 1)^{-1}(C_{\text{red}})$, is isomorphic to the kernel of $\Upsilon^5_{C_{\text{red}} \le C}$. Lemmas 4.4 and 4.5 imply that $\Upsilon^1_{C_{\text{red}} \le C}$ and β_C are injective. So, $\dim\ker \Upsilon^5_{C_{\text{red}} \le C}$ equals $\dim \beta_C(\ker \Upsilon^5_{C_{\text{red}} \le C})$ and

$$\ker \Upsilon^5_{C_{\text{red}} \le C} = \ker \Upsilon^1_{C_{\text{red}} \le C} \circ \Upsilon^5_{C_{\text{red}} \le C} = \ker \Upsilon^2_{C_{\text{red}} \le C} \circ \beta_C.$$

Now, $\beta_C(\ker(\Upsilon^2_{C_{\text{red}} \le C} \circ \beta_C)) = \ker \Upsilon^2_{C_{\text{red}} \le C} \cap \operatorname{Im} \beta_C$. Since $\operatorname{Im} \beta_C \subset \operatorname{Im} \rho_C$ (C is deformation d-regular),

$$\ker \Upsilon^2_{C_{\text{red}} \le C} \cap \operatorname{Im} \beta_C \subset \ker \Upsilon^2_{C_{\text{red}} \le C} \cap \operatorname{Im} \rho_C = \rho_C(\ker \Upsilon^2_{C_{\text{red}} \le C} \circ \rho_C).$$

Since $\Upsilon^2_{C_{\text{red}} \le C} \circ \rho_C = \rho_{C_{\text{red}}}$ (Lemma 4.6), by Lemma 2.15,

$$\ker \rho_C \cong H^0(\mathcal{O}_X(-C)(d)) \text{ and } \ker \Upsilon^2_{C_{\text{red}} \le C} \circ \rho_C \cong H^0(\mathcal{O}_X(-C_{\text{red}}(d))).$$

Therefore,

$$\dim\ker \Upsilon^5_{C_{\text{red}} \le C} \le \dim\ker(\Upsilon^2_{C_{\text{red}} \le C} \circ \rho_C) - \dim\ker \rho_C = h^0(\mathcal{I}_{C_{\text{red}}}(d)) - h^0(\mathcal{I}_C(d)). \tag{7}$$

Conversely, Corollary 3.11 implies that

$$\dim I_d(C_{\text{red}}) - \dim I_d(C) \le \dim(12 \to 1)^{-1}(C_{\text{red}}) \le$$

$$\le \dim T_{(C_{\text{red}},C)}(12 \to 1)^{-1}(C_{\text{red}}) = \dim\ker \Upsilon^5_{C_{\text{red}} \le C}.$$

Using the inequality (7) we therefore conclude

$$\dim(12 \to 1)^{-1}(C_{\text{red}}) = \dim T_{(C_{\text{red}},C)}(12 \to 1)^{-1}(C_{\text{red}}) = \dim I_d(C_{\text{red}}) - \dim I_d(C).$$

Hence the fiber is smooth at the point (C_{red}, C). This proves the proposition. □

Theorem 5.6. Suppose C is deformation d-regular and there exists an irreducible component L' of $H_{P_r,P}$ containing (C_{red}, C) such that $(12 \to 2)(L')_{\text{red}}$ is an irreducible component of $(H_P)_{\text{red}}$, whose general element is d-embedded. Suppose further that there exists an irreducible component, say L_0 of H_{P_1,P_r} such that $\operatorname{pr}_2(L_0)_{\text{red}}$ is contained in $(12 \to 1)(L')_{\text{red}}$ and L_0 is smooth at (C_1, C_{red}), where pr_2 is the natural projection map from H_{P_1,P_r} to H_{P_r}.

Then, L' is smooth at (C_{red}, C) and $\dim \operatorname{pr}_2 L_0 = \dim(12 \to 1)L'$.

Proof. By Corollary 4.7 we have $\Upsilon^6_{C_1 \le C_{\rm red}}$ is injective. Since L_0 is smooth at $(C_1, C_{\rm red})$,

$$\dim L_0 = \dim T_{(C_1,C_{\rm red})} L_0 = \dim \mathrm{pr}_2 T_{(C_1,C_{\rm red})} L_0, \tag{8}$$

By Proposition 5.5, we have $\dim \ker \Upsilon^5_{C_{\rm red} \le C} = \dim (12 \to 1)^{-1}(C_{\rm red})$. Hence,

$$\begin{aligned} \dim T_{(C_{\rm red},C)} L' &= \dim \mathrm{pr}_1 T_{(C_{\rm red},C)} L' + \dim \ker \Upsilon^5_{C_{\rm red} \le C} = \\ &= \dim \mathrm{pr}_1 T_{(C_{\rm red},C)} L' + \dim (12 \to 1)^{-1}(C_{\rm red}). \end{aligned} \tag{9}$$

By Proposition 3.3, a general element $C'_g \in (12 \to 1)(L')$ is reduced, hence d-regular by Theorem 3.7. This implies $\dim I_d(C') = \dim I_d(C'_g)$. Using Corollary 3.11 and Theorem 3.12, we then conclude that $\dim(12 \to 1)^{-1}(C_{\rm red}) = \dim(12 \to 1)^{-1}(C'_g)$.

Now, $\dim L_0 = \dim \mathrm{pr}_2 L_0$ because the fiber of pr_2 is zero dimensional (there are only finitely many curves with Hilbert polynomial P_1 in $C_{\rm red}$). Finally, we have,

$$\dim \mathrm{pr}_1 T_{(C_{\rm red},C)} L' + \dim(12 \to 1)^{-1}(C'_g) \overset{(6)}{\le} \dim \mathrm{pr}_2 T_{(C_1,C_{\rm red})} L_0 + \dim(12 \to 1)^{-1}(C'_g) =$$

$$\overset{(8)}{=} \dim L_0 + \dim L' - \dim(12 \to 1)(L')$$

Using (9) we have $\dim T_{(C_{\rm red},C)} L' - \dim L' \le \dim \mathrm{pr}_2 L_0 - \dim(12 \to 1)(L')$. By the hypothesis, $\dim \mathrm{pr}_2 L_0 \le \dim(12 \to 1)(L')$. Hence, $\dim T_{(C_{\rm red},C)} L' \le \dim L'$. Since the dimension of the tangent space of a scheme at a point is at least equal to the dimension of the scheme at that point, we have

$$\dim T_{(C_{\rm red},C)} L' = \dim L' \text{ and } \dim \mathrm{pr}_2 L_0 = \dim(12 \to 1)(L').$$

This proves the theorem. □

The following computation will be used in the proof of Corollary 5.8.

Lemma 5.7. Let l be a line and C_2 a smooth coplanar curve (on the same plane as l). Denote by P_1 (resp. P_r) the Hilbert polynomial of l (resp. $l \cup C_2$). Then, H_{P_1,P_r} is smooth at $(l, l \cup C_2)$.

Proof. Consider the projection map $\mathrm{pr}_1 : H_{P_1,P_r} \to H_{P_1}$. Note that the dimension of the fiber over a line l_0 to pr_1 is equal to $\dim \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(\deg(C_2))) + 1$, where the first term is the dimension of the space of degree $\deg(C_2)$ curves on a plane containing l_0 and the second term is the dimension of the space of planes in $\mathbb{P}^3$ containing l_0. Since $l_0 \in H_{P_1}$ is arbitrary and pr_1 is surjective, $\dim H_{P_1,P_r} = \dim \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(\deg(C_2))) + 1 + \dim H_{P_1}$.

Since l and $l \cup C_2$ are complete intersection curves in $\mathbb{P}^3$,

$$\mathcal{N}_{l \cup C_2} \cong \mathcal{O}_{l \cup C_2}(1) \oplus \mathcal{O}_{l \cup C_2}(\deg(C_2) + 1)$$

and by [Har77, Ex. III.5.5] the natural morphism from $H^0(\mathcal{O}_{l\cup C_2}(k))$ to $H^0(\mathcal{O}_l(k))$ is surjective for all $k\in\mathbb{Z}$. In particular, $\Upsilon^5_{l\leq l\cup C_2}$ is surjective. Hence, the dimension of the tangent space $T_{(l,l\cup C_2)}H_{P_1,P_r}$ is equal to

$$h^0(\mathcal{N}_{l|\mathbb{P}^3})+\dim\ker\Upsilon^5_{l\leq l\cup C_2}=h^0(\mathcal{N}_{l|\mathbb{P}^3})+(h^0(\mathcal{O}_{l\cup C_2}(1))+h^0(\mathcal{O}_{l\cup C_2}(t)))-$$

$$-(h^0(\mathcal{O}_l(1))+h^0(\mathcal{O}_l(t))),\ \text{where } t=\deg(C_2)+1.$$

The ideal of $l\cup C_2$ and l contains respectively one and two linear polynomials. So, $h^0(\mathcal{O}_{l\cup C_2}(1))=h^0(\mathcal{O}_l(1))+1$. Note then that the dimension of $\ker\Upsilon^5_{l\leq l\cup C_2}$ is equal to $h^0(\mathcal{O}_{l\cup C_2}(\deg(C_2)+1))-h^0(\mathcal{O}_l(\deg(C_2)+1))+1$. Since $l\cup C_2$ and l are t-regular (Theorem 3.7), one can use their Hilbert polynomials to prove that

$$\ker\Upsilon^5_{l\leq l\cup C_2}=(\deg(C_2)+1)\deg(C_2)-\rho_a(l\cup C_2)+1=\frac{\deg(C_2)(\deg(C_2)+3)}{2}+1$$

which is equal to $\dim\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(\deg(C_2)))+1$. Since H_{P_1} is smooth and irreducible (Hilbert scheme parametrizing lines in $\mathbb{P}^3$), we have

$$\dim T_{(l,l\cup C_2)}H_{P_1,P_r}=h^0(\mathcal{N}_{l|\mathbb{P}^3})+\frac{\deg(C_2)(\deg(C_2)+3)}{2}+1=$$

$$=\dim H_{P_1}+\dim\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(\deg(C_2)))+1=\dim H_{P_1,P_r}.$$

This proves the lemma. □

In the following corollary we see that the examples in Theorem 2.19 satisfy the hypotheses in Theorem 5.6.

Corollary 5.8. Suppose that C_1 is a line l and C_2 is a smooth coplanar curve (on the same plane as l). Let L be the irreducible component of H_P parametrizing d-embedded curves of the form $2l'+C_1'$ where l' and C_1' are coplanar curves with l' a line (see Theorem 2.19) and L' the component in $H_{P_r,P}$ mapping surjectively to L.

If (C_{red},C) is weakly general then L' is smooth at (C_{red},C).

Proof. Using Theorem 5.6 it suffices to show that C is deformation d-regular, there exists an irreducible component, L_0 of H_{P_1,P_r} such that $\text{pr}_2(L_0)_{\text{red}}\subset(12\to 1)(L')_{\text{red}}$ and L_0 is smooth at (l,C_{red}).

[Dan14a, Theorem 6.11] implies that the Castelnuovo-Mumford regularity of C is at most $d+1$, hence deformation d-regular by Lemma 5.3. By the definition of L in Theorem 2.19, $(12\to 1)(L')_{\text{red}}$ contains all coplanar curves which are the union of a line and a degree $\deg(C_2)$ coplanar curve. Note that there exists an irreducible subvariety in H_{P_r} which parametrizes all such curves and there exists an irreducible component, say L_0, in H_{P_1,P_r} which maps surjectively to this subvariety. Finally, Lemma 5.7 implies L_0 is smooth at (l,C_{red}). This completes the proof of the corollary. □

Remark 5.9. Note in the Corollary 5.8 above, L is non-reduced at C but L', and hence $(12 \to 2)(L')$, is reduced at C (Lemma 2.17). This simply means there exists an element $\xi_0 \in H^0(\mathcal{N}_{C|\mathbb{P}^3})$ not contained in the image of the projection map from $T_{(C_{\text{red}},C)}H_{P_r,P}$ to $T_C H_P$. In particular, $\Upsilon^5_{C_{\text{red}}\le C}(\xi_0) \notin \operatorname{Im}\Upsilon^6_{C_{\text{red}}\le C}$ i.e., *there exists an infinitesimal deformation of C not deforming C_{red}*. We will see in the following section that ξ_0 plays an important role in producing non-reduced components of Hilbert schemes.

6 Extension of curves and induced non-reducedness

In the previous section we gave a criterion and examples of d-embedded curves under which there exists an infinitesimal deformation of the divisor which does not deform the associated reduced scheme. In this section we introduce "extension of curves". We observe that if C and D are d-embedded curves with C an extension of D and there exists an infinitesimal deformation of D not deforming D_{red} then there exists an infinitesimal deformation of C not deforming C_{red} (see Corollary 6.5). We use this to see under certain conditions, the Hilbert scheme containing C is non-reduced (see Theorem 6.10).

Definition 6.1. Let X be a smooth degree d surface, C, D two effective divisors on X. We say that C is a *simple extension of* D if it satisfies the following conditions:

1. $C - D$ is nD' for some positive integer n, $D' \subset X$ reduced curve and $D' \cap D_{\text{red}}$ consists of finitely many points,
2. The image of $\Upsilon^5_{D\le C}$ is contained in the image of $\Upsilon^6_{D\le C}$.

We say that C is an *extension of* D if there exists a sequence

$$D = C_0 \le C_1 \le \dots \le C_n = C$$

of effective divisors on X such that C_{i+1} is a simple extension of C_i for $i \ge 0$.

Lemma 6.2. Let X be a smooth degree d surface in $\mathbb{P}^3$, D an effective divisor on X such that $\deg(D) \le d-4$. Suppose $\xi_0 \in H^0(\mathcal{N}_{D|\mathbb{P}^3})$ such that $\Upsilon^5_{D_{\text{red}}\le D}(\xi_0) \notin \operatorname{Im}\Upsilon^6_{D_{\text{red}}\le D}$. Then, for all $t \in \rho_D^{-1}(\beta_D(\xi_0))$, $\overline{\nabla}([D_{\text{red}}])(t) \neq 0$.

Proof. Suppose not i.e., there exists $t \in \rho_D^{-1}(\beta_D(\xi_0))$ such that $\overline{\nabla}([D_{\text{red}}])(t) = 0$. By Corollary 4.10 this means there exists $\xi_0' \in H^0(\mathcal{N}_{D_{\text{red}}|\mathbb{P}^3})$ such that $\rho_{D_{\text{red}}}(t) = \beta_{D_{\text{red}}}(\xi_0')$. In particular by Lemma 4.6,

$$\Upsilon^1_{D_{\text{red}}\le D} \circ \Upsilon^6_{D_{\text{red}}\le D}(\xi_0') = \Upsilon^2_{D_{\text{red}}\le D} \circ \rho_D(t) = \Upsilon^2_{D_{\text{red}}\le D} \circ \beta_D(\xi_0) = \Upsilon^1_{D_{\text{red}}\le D} \circ \Upsilon^5_{D_{\text{red}}\le D}(\xi_0).$$

By Lemma 4.5, $\Upsilon^1_{D_{\text{red}}\le D}$ is injective. Hence, $\Upsilon^6_{D_{\text{red}}\le D}(\xi_0') = \Upsilon^5_{D_{\text{red}}\le D}(\xi_0)$, contradicting the definition of ξ_0. This proves the lemma. □

Remark 6.3. There is a simple geometric interpretation of the technical condition $\Upsilon^5_{D_{\mathrm{red}}\le D}(\xi_0) \notin \operatorname{Im}\Upsilon^6_{D_{\mathrm{red}}\le D}$: Denote by P_D (resp. $P_{D_{\mathrm{red}}}$) the Hilbert polynomial of D (resp. D_{red}). The condition $\Upsilon^5_{D_{\mathrm{red}}\le D}(\xi_0) \notin \operatorname{Im}\Upsilon^6_{D_{\mathrm{red}}\le D}$ is equivalent to ξ_0 not in the image of

$$\mathrm{pr}_2 : T_{(D_{\mathrm{red}},D)} H_{P_{D_{\mathrm{red}}},P_D} \to T_D H_{P_D}.$$

Geometrically, this means the infinitesimal deformation of D corresponding to ξ_0 does not deform D_{red}.

The following proposition states that in the case C is a simple extension of D, D' is an infinitesimal deformation of D corresponding to ξ_0 and C' is an infinitesimal deformation of C containing D' then there does not exist a lift of C_{red} as closed subscheme in C' which is flat over $\operatorname{Spec}\mathbb{C}[t]/(t^2)$.

Proposition 6.4. Let D, X and ξ_0 be as in Lemma 6.2, C an effective divisor on X, a simple extension of D with $\deg(C) \le d-4$ and $\xi_1 \in H^0(\mathcal{N}_{C|\mathbb{P}^3})$ such that $\Upsilon^5_{D\le C}(\xi_1) = \Upsilon^6_{D\le C}(\xi_0)$. If C is deformation d-regular then $\Upsilon^5_{C_{\mathrm{red}}\le C}(\xi_1) \notin \operatorname{Im}\Upsilon^6_{C_{\mathrm{red}}\le C}$.

Proof. Suppose this is not the case. By definition,

$$\rho_C^{-1}(\beta_C(\xi_1)) \subset (\Upsilon^2_{D\le C}\circ\rho_C)^{-1}\circ(\Upsilon^2_{D\le C}\circ\beta_C(\xi_1)) = (\Upsilon^2_{D\le C}\circ\rho_C)^{-1}\circ(\Upsilon^1_{D\le C}\circ\Upsilon^5_{D\le C}(\xi_1)) \subset$$

$$\subset (\Upsilon^2_{D\le C}\circ\rho_C)^{-1}(\Upsilon^1_{D\le C}\circ\Upsilon^6_{D\le C}(\xi_0)) = \rho_D^{-1}(\beta_D(\xi_0))$$

where the last equality follows from Lemma 4.6. Hence by Lemma 4.9,

$$\overline{\nabla}([C])(t) = 0 = \overline{\nabla}([D])(t) \;\text{ for all } t \in \rho_C^{-1}(\beta_C(\xi_1)).$$

We are going to show that for all $t \in {\rho_C}^{-1}(\beta_C(\xi_1))$, $\overline{\nabla}([D_{\mathrm{red}}])(t) = 0$, which will contradict Lemma 6.2.

Since $\Upsilon^5_{C_{\mathrm{red}}\le C}(\xi_1) \in \operatorname{Im}\Upsilon^6_{C_{\mathrm{red}}\le C}$, $\rho_C^{-1}(\beta_C(\xi_1)) \subset (\Upsilon^2_{C_{\mathrm{red}}\le C}\circ\rho_C)^{-1}\circ(\Upsilon^2_{C_{\mathrm{red}}\le C}\circ\beta_C(\xi_1)) =$

$$= (\Upsilon^2_{C_{\mathrm{red}}\le C}\circ\rho_C)^{-1}\circ(\Upsilon^1_{C_{\mathrm{red}}\le C}\circ\Upsilon^5_{C_{\mathrm{red}}\le C}(\xi_1)) \subset (\Upsilon^2_{C_{\mathrm{red}}\le C}\circ\rho_C)^{-1}(\operatorname{Im}(\Upsilon^1_{C_{\mathrm{red}}\le C}\circ\Upsilon^6_{C_{\mathrm{red}}\le C})).$$

Then, Lemma 4.9 implies $\overline{\nabla}([C_{\mathrm{red}}])(t) = 0$ for all $t \in {\rho_C}^{-1}(\beta_C(\xi_1))$.

Since $C - D = nD'$ for some reduced curve D' and $n \in \mathbb{Z}_{>0}$,

$$0 = \overline{\nabla}([C])(t) - \overline{\nabla}([D])(t) = \overline{\nabla}([C-D])(t) = \overline{\nabla}(n[D'])(t) \Rightarrow \overline{\nabla}([D'])(t) = 0$$

for all $t \in \rho_C^{-1}(\beta_C(\xi_1))$. Since $D' \cap D_{\mathrm{red}}$ consists of finitely many points, $C_{\mathrm{red}} - D_{\mathrm{red}} = D'$. Hence, $\overline{\nabla}([D_{\mathrm{red}}])(t) = \overline{\nabla}([C_{\mathrm{red}} - D'])(t) = 0$. This completes the proof of the proposition. □

Corollary 6.5. Let D, X and ξ_0 be as in Lemma 6.2 and C an effective divisor on X, an extension of D with $\deg(C) \le d-4$. If C is deformation d-regular then there exists $\xi \in H^0(\mathcal{N}_{C|\mathbb{P}^3})$ such that $\Upsilon^5_{C_{\mathrm{red}}\le C}(\xi) \notin \operatorname{Im}\Upsilon^6_{C_{\mathrm{red}}\le C}$.

Proof. Suppose that there exists a chain $D = C_0 \le C_1 \le ... \le C_n = C$ of effective divisors on X such that C_{i+1} is a simple extension of C_i. We first show that if C is deformation d-regular then each C_i for $i = 0,...,n$ is deformation d-regular. Suppose not i.e., there exists some $i \in \{0,...,n\}$ and $\xi_i \in H^0(\mathcal{N}_{C_i|\mathbb{P}^3})$ such that $\beta_{C_i}(\xi_i) \notin \operatorname{Im}\rho_{C_i}$. Since C_{i+1} is a simple extension of C_i, there exists $\xi_{i+1} \in H^0(\mathcal{N}_{C_{i+1|\mathbb{P}^3}})$ such that $\Upsilon^5_{C_i \le C_{i+1}}(\xi_{i+1}) = \Upsilon^6_{C_i \le C_{i+1}}(\xi_i)$. Now,

$$\Upsilon^2_{C_i \le C_{i+1}} \circ \beta_{C_{i+1}}(\xi_{i+1}) = \Upsilon^1_{C_i \le C_{i+1}} \circ \Upsilon^5_{C_i \le C_{i+1}}(\xi_{i+1}) = \Upsilon^1_{C_i \le C_{i+1}} \circ \Upsilon^6_{C_i \le C_{i+1}}(\xi_i)$$

which by Lemma 4.6 is equal to $\beta_{C_i}(\xi_i)$. Since $\rho_{C_i} = \Upsilon^2_{C_i \le C_{i+1}} \circ \rho_{C_{i+1}}$, if $\beta_{C_{i+1}}(\xi_{i+1}) \in \operatorname{Im}\rho_{C_{i+1}}$ then

$$\beta_{C_i}(\xi_i) = \Upsilon^2_{C_i \le C_{i+1}} \circ \beta_{C_{i+1}}(\xi_{i+1}) \in \Upsilon^2_{C_i \le C_{i+1}}(\operatorname{Im}\rho_{C_{i+1}}) = \operatorname{Im}\rho_{C_i}$$

where the last equality follows from Lemma 4.6. But this is a contradiction the property of ξ_i. Hence, $\beta_{C_{i+1}}(\xi_{i+1}) \notin \operatorname{Im}\rho_{C_{i+1}}$. Proceeding recursively, we get a contradiction to $\operatorname{Im}\beta_C \subset \operatorname{Im}\rho_C$. Therefore, for each $i = 0,...,n$, C_i is deformation d-regular.

Since C_i is a simple extension of C_{i-1} there exists $\xi_i \in H^0(\mathcal{N}_{C_i|\mathbb{P}^3})$ such that

$$\Upsilon^5_{C_{i-1} \le C_i}(\xi_i) = \Upsilon^6_{C_{i-1} \le C_i}(\xi_{i-1}) \text{ for all } i \in \{1,...,n\}.$$

Using Proposition 6.4 recursively, we get $\Upsilon^5_{C_{i_{\text{red}}} \le C_i}(\xi_i) \notin \operatorname{Im}\Upsilon^6_{C_{i_{\text{red}}} \le C_i}$ for all $i \in \{1,...,n\}$, in particular $\Upsilon^5_{C_{\text{red}} \le C}(\xi_n) \notin \operatorname{Im}\Upsilon^6_{C_{\text{red}} \le C}$. This completes the proof of the corollary. □

Theorem 6.6. Let X, D satisfying the conditions of Lemma 6.2, C an effective divisor on X satisfying:

1. $\deg(C) \le d - 4$
2. C is deformation d-regular, an extension of D
3. for the Hilbert polynomial P_C of C, C is weakly general in H_{P_C} and for the unique irreducible component L of H_{P_C} containing C, the general element $C_g \in L$ satisfies: the Hilbert polynomial of $C_{g_{\text{red}}}$ is the same as that of C_{red}.

Then, L is singular at C.

Proof. Denote by P_{C_r} the Hilbert polynomial of C_{red}. By assumption, the Hilbert polynomial of C_{red} is the same as that of $C_{g_{\text{red}}}$. Replace in Notation 3.5, P_r by P_{C_r}, P by P_C. By Proposition 3.3 there exists an irreducible component L' of $H_{P_{C_r},P_C}$ such that $(12 \to 2) : L'_{\text{red}} \to L_{\text{red}}$ is surjective and $(12 \to 2)^{-1}(C)$ are of the form (C', C) satisfying $C'_{\text{red}} = C_{\text{red}}$. So, $C_{\text{red}} \subset C'$. Since C' has the same Hilbert polynomial as C_{red}, we conclude $C' = C_{\text{red}}$, in particular $(12 \to 2)^{-1}(C)$ is of the form (C_{red}, C).

Corollary 6.5 implies $h^0(\mathcal{N}_{C|\mathbb{P}^3}) > \dim \mathrm{pr}_2 T_{(C_{\mathrm{red}},C)}L'$. By the injectivity of $\Upsilon^6_{C_{\mathrm{red}}\leq C}$ (Corollary 4.7), $\dim \mathrm{pr}_2 T_{(C_{\mathrm{red}},C)}L' = \dim T_{(C_{\mathrm{red}},C)}L'$. Since the fiber to $(12 \to 2)|_{L'}$ is zero dimensional over every point (finitely many subcurves of a fixed Hilbert polynomial in a fixed curve), $\dim L' = \dim L$. Using the diagram in Remark 4.2, we then have

$$h^0(\mathcal{N}_{C|\mathbb{P}^3}) > \dim \mathrm{pr}_2 T_{(C_{\mathrm{red}},C)}L' = \dim T_{(C_{\mathrm{red}},C)}L' \geq \dim L' = \dim L.$$

This proves the theorem. □

Notation 6.7. Replace in Notation 3.5, P_r by P_{C_r} and P by P_C.

Remark 6.8. Notations as in the proof of Theorem 6.6. The proof of the theorem shows that the fiber over C to the morphism $(12 \to 2)$ is of the form (C_{red}, C). In particular, $C_{\mathrm{red}} \in (12 \to 1)(L')$.

Example 6.9. Suppose that C_1 is a line l and C_2 is a smooth coplanar curve (on the same plane as l), X a smooth degree d surface containing C_1 and C_2, $d \geq \deg(C_2) + 6$ and D an effective divisor on X of the form $2C_1 + C_2$. Denote by P_D (resp. P_{D_r}) the Hilbert polynomial D (resp. $C_1 + C_2$). There exists an irreducible generically non-reduced component, say M of H_{P_D} parametrizing d-embedded curves of the form $2C_1' + C_2'$ where C_1' and C_2' are coplanar curves with C_1' a line and $\deg(C_2') = \deg(C_2)$ (see Theorem 2.19). By Proposition 3.3, there exists an irreducible component M' of $H_{P_{D_r},P_D}$ mapping surjectively to M.

If (D_{red}, D) is weakly general in $H_{P_{D_r},P_D}$ then M' is smooth at (D_{red}, D) (see Corollary 5.8). Since M' is smooth at (D_{red}, D), $(12 \to 2)(M')$ is reduced at D (see Lemma 2.17). As M is non-reduced at D, there exists $\xi \in T_D H_{P_D}$ not contained in the image of the natural projection map $T_{(D_{\mathrm{red}},D)} H_{P_{D_r},P_D} \to T_D H_{P_D}$. This is equivalent to $\Upsilon^5_{D_{\mathrm{red}}\leq D}(\xi_0) \notin \mathrm{Im}\,\Upsilon^6_{D_{\mathrm{red}}\leq D}$. Let C be an effective divisor on X, an extension of D and satisfying the conditions of Theorem 6.6. Then, the unique irreducible component of the Hilbert scheme corresponding to C, is singular at C.

In the following theorem we see that if L as in Theorem 6.6, which is singular, satisfies an extra condition: there exists an open neighborhood U of C in L such that for all $u \in U$, the corresponding curve $\mathcal{C}_u$ contains the same number of subcurves with Hilbert polynomial the same as that of C_{red}, independent of u, then L is non-reduced.

Theorem 6.10. Let C, L be satisfying the conditions in Theorem 6.6, P_{C_r} the Hilbert polynomial of C_{red} and L' the irreducible component $H_{P_{C_r},P_C}$ mapping surjectively (as topological spaces) to L (existence follows from Proposition 3.3). Suppose that there exists an open neighbourhood U, of C in L such that the cardinality of the fiber to the morphism $(12 \to 2) : L' \to L$ over every closed point in U is constant. Then L is non-reduced at C.

Proof. Assume that U is reduced. Denote by $U' := (12 \to 2)^{-1}(U)$. Corollary 4.8 tells us that every closed point in the fiber over $u \in U$ is reduced. By assumption, the cardinality of the closed fibers are constant. Using [Har77, Ex. II.5.8] we can conclude that every fiber is reduced with the same cardinality, hence has the same Hilbert polynomial. Hence the morphism $(12 \to 2)|_{U'}$ is flat (by [Har77, Theorem III.9.9]) and proper (base change of proper morphisms is proper).

Since every fiber of $(12 \to 2)|_{U'}$ is smooth of relative dimension zero, [Har77, Ex. III. 10.3] implies the morphism $(12 \to 2)|_{U'}$ is étale. But étale morphisms induce surjection of tangent spaces, which is a contradiction to Corollary 6.5. In particular, Corollary 6.5 implies $\operatorname{Im} \Upsilon^5_{C_{\mathrm{red}} \le C} \not\subset \operatorname{Im} \Upsilon^6_{C_{\mathrm{red}} \le C}$ which means $\mathrm{pr}_2 : T_{(C_{\mathrm{red}}, C)} L' \to T_C L$ is not surjective. So, L cannot be reduced. This proves the theorem. □

For the sake of completeness we consider the case when a general element of L is not deformation d-regular. We see that in this case as well we get non-reducedness of L.

Theorem 6.11. Assume that a general point in $(12 \to 2)(12 \to 1)^{-1}(C_{\mathrm{red}})$ correspond to a curve which is not deformation d-regular. Then L is non-reduced at a general such point.

Proof. Proposition 3.12 implies the dimension of the fiber to the morphism $(23 \to 2)$ over a general $C \in (12 \to 2)((12 \to 1)^{-1}(C_{\mathrm{red}}))$ is equal to $\dim \mathbb{P}(I_d(C_g))$, for $C_g \in L$ general. By the upper-semicontinuity of fiber dimension, there exists an open neighborhood of C, say $U \subset L$ such that for all $u \in U$, $\dim(23 \to 2)^{-1}(\mathcal{C}_u)_{\mathrm{red}} = \dim \mathbb{P}(I_d(C_g))$, where $\mathcal{C}_u$ denotes the curve corresponding to the point $u \in U$.

Recall, the tangent space at a point $(\mathcal{C}_u, \mathcal{X}_u) \in (23 \to 2)^{-1}(\mathcal{C}_u)$ is isomorphic to $\ker \rho_{\mathcal{C}_u}$. Lemma 2.15 implies $\ker \rho_{\mathcal{C}_u} = H^0(\mathcal{O}_{\mathcal{X}_u}(-\mathcal{C}_u)(d))$. Using the short exact sequence,

$$0 \to \mathcal{I}_{\mathcal{X}_u}(d) \to \mathcal{I}_{\mathcal{C}_u}(d) \to \mathcal{O}_{\mathcal{X}_u}(-\mathcal{C}_u)(d) \to 0$$

and the facts

$$H^1(\mathcal{I}_{\mathcal{X}_u}(d)) = H^1(\mathcal{O}_{\mathbb{P}^3}) = 0 \text{ and } h^0(\mathcal{I}_{\mathcal{X}_u}(d)) = h^0(\mathcal{O}_{\mathbb{P}^3}) = 1,$$

we have

$$h^0(\mathcal{O}_{\mathcal{X}_u}(-\mathcal{C}_u)(d)) = \dim I_d(\mathcal{C}_u) - 1 = \dim \mathbb{P}(I_d(\mathcal{C}_u)) = \dim \mathbb{P}(I_d(C_g)).$$

Hence, the fiber $(23 \to 2)^{-1}(\mathcal{C}_u)$ is smooth. Since $(23 \to 2)^{-1}(\mathcal{C}_u)_{\mathrm{red}} \cong \mathbb{P}(I_d(\mathcal{C}_u))$ all the fibers $(23 \to 2)^{-1}(\mathcal{C}_u)$ are isomorphic to $\mathbb{P}^N_u$ where $N = \dim I_d(C_g) - 1$. Hence, have the same Hilbert polynomial.

Assume $C_g \in (12 \to 2)((12 \to 1)^{-1}(C_{\mathrm{red}}))$ a general point such that C_g is not deformation d-regular *but* L is reduced at this point. By [Har77, Theorem III.9.9], there exists an open neighborhood of C_g, say $U \subset L$ such that $(23 \to 2)|_{U'} : (23 \to 2)^{-1}(U) \to U$ is flat with smooth fibers, where $U' = (23 \to 2)^{-1}(U)$. We already know that $(23 \to 2)|_{U'}$ is proper. [Har77, III. Ex. 10.2] implies this is a smooth morphism. But a smooth morphism $f : X \to Y$ satisfies the condition: the induced differential map is surjective on tangent spaces i.e., $df_x(T_x X) = T_{f(x)} Y$. Substituting f by $(23 \to 2)$ this contradicts the assumption that C_g is not deformation d-regular. Hence, L is non-reduced at a general point of $(12 \to 2)((12 \to 1)^{-1}(C_{\mathrm{red}}))$. This finishes the proof of the theorem. □

7 Examples

The aim of this section is to give examples of non-reduced components of d-embedded curves whose general element is again non-reduced (see Theorem 7.22). Let D, X be satisfying the condition of Lemma 6.2. We want to produce examples of effective divisors C on X satisfying the conditions of Theorems 6.6 and 6.10. In particular, we need to check (1) C is an extension of D and (2) there exists an irreducible component L of the Hilbert scheme corresponding to C which contain C satisfying: there exists an open neighborhood U of C in L such that for all $u \in U$, the corresponding curve $\mathcal{C}_u$ contains the same number of subcurves whose Hilbert polynomial is the same as that of C_{red}, independent of u. Point (1) is studied in §7.1 and §7.4. Point (2) is the subject of §7.3.

7.1 General criterion for extension of curves

The aim of this subsection is to give a geometric criterion for extension of curves (see Theorem 7.7). Recall, given two curves, say C, D intersecting at finitely many points, we say $C \cup D$ is a simple extension of C if D is a multiple of a reduced curve and for every infinitesimal deformation of C there exists an infinitesimal deformation of $C \cup D$ containing it. A basic knowledge of deformation theory tells us this is possible if and only if we can find an infinitesimal deformation of D which agrees with the infinitesimal deformation of C at the points of intersection. In Theorem 7.7 we make this statement more precise using the correspondence between infinitesimal deformations and certain normal sheaves. In particular, we see that given two global sections s_1, s_2 of the normal sheaves of C and D, respectively, the corresponding infinitesimal deformations glue to give an infinitesimal deformation of $C \cup D$ if the global sections take the same value at $C.D$. Global sections of the normal sheaves and its values at a finite set of closed points contain important geometric informations about the way a scheme deforms along these points. In §7.4 we study this in the basic case of a single closed points.

Notation 7.1. Let X be a smooth degree d surface in $\mathbb{P}^3$, C and D be two effective divisors in X satisfying $\#\{C \cap D\} < \infty$. Denote by Y the effective divisor $C \cup D$, by i_C (resp. i_D) the closed immersions of C (resp. D) into Y. Denote by

$$r : \mathcal{N}_{Y|\mathbb{P}^3} \otimes \mathcal{O}_D \to \mathcal{N}_{Y|\mathbb{P}^3} \otimes \mathcal{O}_{C.D}, r' : \mathcal{N}_{Y|\mathbb{P}^3} \otimes \mathcal{O}_C \to \mathcal{N}_{Y|\mathbb{P}^3} \otimes \mathcal{O}_{C.D}$$

Denote by $\Upsilon_D := r \circ \Upsilon^6_{D \leq Y}$ and $\Upsilon_C := r' \circ \Upsilon^6_{C \leq Y}$. By abuse of notation, we will denote by the same notation the morphism of sheaves as well as the induced morphism of global sections. The distinction will be clear from the context.

An important step in the proof of Theorem 7.7 is to obtain the short exact sequence in Corollary 7.5 which comes directly from the short exact sequence in Lemma 7.4 after tensoring by the normal sheaf of Y. We now define the morphisms used in the short exact sequence (10).

Definition 7.2. The morphisms $i_C^{\#}$ and $i_D^{\#}$ from $\mathcal{O}_Y$ to $\mathcal{O}_C$ and $\mathcal{O}_D$, respectively, coming from the closed immersions i_C and i_D, respectively, induce the morphism $f_1 : \mathcal{O}_Y \to \mathcal{O}_C \oplus \mathcal{O}_D$. In particular, for any open set $U \subset \mathbb{P}^3$, $f_1(U) : \mathcal{O}_Y(U \cap Y) \to \mathcal{O}_C(C \cap U) \oplus \mathcal{O}_D(D \cap U)$ is defined by $f \in \mathcal{O}_Y(Y \cap U)$ maps to $i_C^{\#}(f) \oplus i_D^{\#}(f)$.

Definition 7.3. Define a morphism,

$$f_2 : \mathcal{O}_C \oplus \mathcal{O}_D \to \mathcal{O}_{C.D}$$

as zero outside $C_{\text{red}} \cap D_{\text{red}}$ and for $x \in C_{\text{red}} \cap D_{\text{red}}$ closed, define $f_{2,x}(f,g) = (\bar{f} - \bar{g})$, where $\bar{f}, \bar{g}$ are the images of f, g, respectively in $\mathcal{O}_{C.D,x}$. In particular, for any open set $U \subset \mathbb{P}^3$ define

$$f_2(U) : \mathcal{O}_C(U) \oplus \mathcal{O}_D(U) \to \mathcal{O}_{C.D}(U),\ (f,g) \mapsto (\overline{f-g}).$$

One can then easily prove:

Lemma 7.4. The morphisms $f_1$andf_2 sit in the following short exact sequence:

$$0 \to \mathcal{O}_Y \xrightarrow{f_1} \mathcal{O}_C \oplus \mathcal{O}_D \xrightarrow{f_2} \mathcal{O}_{C.D} \to 0. \tag{10}$$

Corollary 7.5. The short exact sequence (10) tensored with $- \otimes_{\mathcal{O}_Y} \mathcal{N}_{Y|\mathbb{P}^3}$ gives rise to the following short exact sequence,

$$0 \to \mathcal{N}_{Y|\mathbb{P}^3} \xrightarrow{r_1} \mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_C \oplus \mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_D \xrightarrow{r_2} \mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_{C.D} \to 0. \tag{11}$$

Proof. Note that $\mathcal{N}_{Y|\mathbb{P}^3}$ is a locally free $\mathcal{O}_Y$-module, hence $\mathcal{O}_Y$-flat. The corollary then follows directly from Lemma 7.4. □

Remark 7.6. Note from the definition of r_2 above, that $r_2 = r' - r$ i.e., on an open set $U \subset \mathbb{P}^3$, $r_2(U)$ maps $(s_1, s_2) \in (\mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_C)(U) \oplus (\mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_D)(U)$ to $r'(s_1) - r(s_2) \in (\mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_{C.D})(U)$.

Theorem 7.7. The image of $\Upsilon^6_{C \le Y}$ is contained in the image of $\Upsilon^5_{C \le Y}$ if $\operatorname{Im} \Upsilon_C \subset \operatorname{Im} \Upsilon_D$.

Proof. Take $\xi \in H^0(\mathcal{N}_{C|\mathbb{P}^3})$. We will show that there exists $\xi' \in H^0(\mathcal{N}_{Y|\mathbb{P}^3})$ such that $\Upsilon^5_{C \le Y}(\xi') = \Upsilon^6_{C \le Y}(\xi)$. Denote by $f := \Upsilon^6_{C \le Y}(\xi) \in H^0(\mathcal{N}_{Y|\mathbb{P}^3} \otimes \mathcal{O}_C)$. Since $\operatorname{Im} \Upsilon_C \subset \operatorname{Im} \Upsilon_D$, there exists $\xi'' \in H^0(\mathcal{N}_{D|\mathbb{P}^3})$ such that $\Upsilon_D(\xi'') = \Upsilon_C(\xi)$ i.e., $r(\Upsilon^6_{D \le Y}(\xi'')) = r'(\Upsilon^6_{C \le Y}(\xi))$. By the definition of r_2, therefore $r_2(\Upsilon^6_{C \le Y}(\xi), \Upsilon^6_{D \le Y}(\xi'')) = 0$ (see Remark 7.6). By the exactness of the short exact sequence (11), there exists $\xi' \in H^0(\mathcal{N}_{Y|\mathbb{P}^3})$ such that $r_1(\xi') = (\Upsilon^6_{C \le Y}(\xi), \Upsilon^6_{D \le Y}(\xi''))$. Note that, the composition of the morphism r_1 with the natural first projection morphism from $H^0(\mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_C) \oplus H^0(\mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_D)$ to $H^0(\mathcal{N}_{Y|\mathbb{P}^3} \otimes_{\mathcal{O}_Y} \mathcal{O}_C)$ is by construction $\Upsilon^5_{C \le Y}$. Therefore, $\Upsilon^5_{C \le Y}(\xi') = \Upsilon^6_{C \le Y}(\xi)$. Since $\xi \in H^0(\mathcal{N}_{C|\mathbb{P}^3})$ was chosen arbitrarily, this implies $\operatorname{Im} \Upsilon^6_{C \le Y} \subset \operatorname{Im} \Upsilon^5_{C \le Y}$. This proves the theorem. □

Remark 7.8. Although not covered in this section, but one can easily check the converse of Theorem 7.7 i.e., if $\operatorname{Im}\Upsilon^6_{C\leq Y} \subset \operatorname{Im}\Upsilon^5_{C\leq Y}$ is $\operatorname{Im}\Upsilon_C \subset \operatorname{Im}\Upsilon_D$? In the language of deformation theory, this is the same as asking: given an infinitesimal deformation of C, say C' and of Y, say Y' such that $C' \subset Y'$, does there exists an infinitesimal deformation of D, say D' such that $C' \cup D' = Y'$? One can check this locally on normal sheaves, under the correspondence between normal sheaves and infinitesimal deformations of D in $\mathbb{P}^3$.

7.2 Examples of extension of curves

We follows Notation 7.1. We aim to study curves C, D such that $\operatorname{Im}\Upsilon^6_{C\leq Y} \subset \operatorname{Im}\Upsilon^5_{C\leq Y}$. In Theorem 7.7 we saw that it suffices to check that $\operatorname{Im}\Upsilon_C \subset \operatorname{Im}\Upsilon_D$. In this section, we consider the case when Υ_D is surjective. Then this condition is automatically satisfied. We see in Corollary 7.11 that Υ_D is surjective if and only if the restriction morphism from $H^0(\mathcal{N}_{D|\mathbb{P}^3})$ to $H^0(\mathcal{N}_{D|\mathbb{P}^3} \otimes \mathcal{O}_{C.D})$ is surjective. In Lemma 7.12 we see that if D is a smooth curve in $\mathbb{P}^3$ and $C.D$ is a single point with multiplicity 1 then this morphism is surjective. We then give an example of such C and D (see Example 7.21). We use this in Theorem 7.22 to produce examples of non-reduced components of Hilbert schemes of curves whose general element is again non-reduced.

Notation 7.9. Denote by $i : C.D \hookrightarrow \mathbb{P}^3, j : C \to \mathbb{P}^3$ and $j' : D \to \mathbb{P}^3$ the closed immersions.

Lemma 7.10. For all $k \geq 0$, $\mathcal{E}xt^k_{\mathbb{P}^3}(\mathcal{I}_D/\mathcal{I}_Y, i_*\mathcal{O}_{C.D}) = 0$.

Proof. Since $\mathcal{I}_Y$ by definition is isomorphic to $\mathcal{I}_C \cap \mathcal{I}_D$, the second isomorphism theorem implies $\mathcal{I}_D/\mathcal{I}_Y \cong (\mathcal{I}_D + \mathcal{I}_C)/\mathcal{I}_C$. Since $i_*\mathcal{O}_{C.D} \cong \mathcal{O}_{\mathbb{P}^3}/(\mathcal{I}_C + \mathcal{I}_D)$, we have the following short exact sequence,

$$0 \to (\mathcal{I}_D + \mathcal{I}_C)/\mathcal{I}_C \to j_*\mathcal{O}_C \to i_*\mathcal{O}_{C.D} \to 0. \tag{12}$$

By the adjoint property of i_*,

$$\mathcal{E}xt^k_{\mathbb{P}^3}(i_*\mathcal{O}_{C.D}, i_*\mathcal{O}_{C.D}) \cong \mathcal{E}xt^k_{C.D}(\mathcal{O}_{C.D}, \mathcal{O}_{C.D}) \cong \mathcal{E}xt^k_{\mathbb{P}^3}(j_*\mathcal{O}_C, i_*\mathcal{O}_{C.D})$$

which is zero if $k > 0$ and isomorphic to $\mathcal{O}_{C.D}$ for $k = 0$, by [Har77, III. Proposition 6.3]. Applying the contravariant functor $\mathcal{H}om_{\mathbb{P}^3}(-, i_*\mathcal{O}_{C.D})$ to (12) and observing the $\mathcal{E}xt$-long exact sequence we get

$$\mathcal{E}xt^k_{\mathbb{P}^3}(\mathcal{I}_D/\mathcal{I}_Y, i_*\mathcal{O}_{C.D}) = \mathcal{E}xt^k_{\mathbb{P}^3}((\mathcal{I}_D + \mathcal{I}_C)/\mathcal{I}_C, i_*\mathcal{O}_{C.D}) = 0 \ \text{ for } k > 0$$

and the short exact sequence,

$$0 \to \mathcal{O}_{C.D} \xrightarrow{r} \mathcal{O}_{C.D} \to \mathcal{H}om_{\mathbb{P}^3}((\mathcal{I}_D + \mathcal{I}_C)/\mathcal{I}_C, i_*\mathcal{O}_{C.D}) \to 0.$$

Now, $C.D$ is a zero dimensional noetherian scheme, hence is Artinian. So, for any closed point $x \in C_{\text{red}} \cap D_{\text{red}}$, the maximal ideal of the local ring $\mathcal{O}_{C.D,x}$ is nilpotent and any element not in the maximal ideal is a unit. Hence, the morphism r is injective if and only if for any $x \in C_{\text{red}} \cap D_{\text{red}}$ closed, the induced morphism $r_x : \mathcal{O}_{C.D,x} \to \mathcal{O}_{C.D,x}$ maps 1 to an unit. Since r_x is a $\mathcal{O}_{C.D}$-linear map, it is therefore surjective as well. Since $\mathcal{E}xt^1_{\mathbb{P}^3}(i_* \mathcal{O}_{C.D}, i_* \mathcal{O}_{C.D}) = 0$,

$$\mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_D / \mathcal{I}_Y, i_* \mathcal{O}_{C.D}) = \mathcal{H}\text{om}_{\mathbb{P}^3}((\mathcal{I}_D + \mathcal{I}_C)/\mathcal{I}_C, i_* \mathcal{O}_{C.D}) = 0.$$

This proves the lemma. □

Corollary 7.11. The inclusion morphisms, $\mathcal{I}_Y \hookrightarrow \mathcal{I}_D$ and the surjective morphism $j'_* \mathcal{O}_D \to i_* \mathcal{O}_{C.D}$ induces naturally the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_D, j'_* \mathcal{O}_D) & \xrightarrow{h_3} & \mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_D, i_* \mathcal{O}_{C.D}) \\ \Upsilon^6_{D \le Y} \downarrow & \circlearrowleft & \downarrow \cong \\ \mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_Y, j'_* \mathcal{O}_D) & \xrightarrow{r} & \mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_Y, i_* \mathcal{O}_{C.D}) \end{array}$$

In particular, $r \circ \Upsilon^6_{D \le Y} : H^0(\mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_D, j'_* \mathcal{O}_D)) \to H^0(\mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_Y, j'_* \mathcal{O}_D))$ is surjective if and only if $h_3 : H^0(\mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_D, j'_* \mathcal{O}_D)) \to H^0(\mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_D, i_* \mathcal{O}_{C.D}))$ is surjective.

Proof. Consider the short exact sequence,

$$0 \to \mathcal{I}_Y \to \mathcal{I}_D \to \mathcal{I}_D / \mathcal{I}_Y \to 0.$$

Applying $\mathcal{H}\text{om}_{\mathbb{P}^3}(-, i_* \mathcal{O}_{C.D})$ to the short exact sequence and using Lemma 7.10, we have the isomorphism $\mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_D, i_* \mathcal{O}_{C.D}) \cong \mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_Y, i_* \mathcal{O}_{C.D})$.

The commutativity of the diagrams is easy to see. In particular, given $\phi \in \mathcal{H}\text{om}_{\mathbb{P}^3}(\mathcal{I}_D, j'_* \mathcal{O}_D)$ we get its images under the various morphisms using the sequence

$$\mathcal{I}_Y \hookrightarrow \mathcal{I}_D \xrightarrow{\phi} j'_* \mathcal{O}_D \to i_* \mathcal{O}_{C.D}.$$

This gives the commutativity of the diagram. The last statement is a direct consequence of the commutative diagram. This proves the corollary. □

Lemma 7.12. Let D be a smooth curve in $\mathbb{P}^3$. Then, $\mathcal{N}_{D|\mathbb{P}^3}$ is globally generated.

Proof. Since D is a smooth curve in $\mathbb{P}^3$, dualizing the short exact sequence in [Har77, II. Theorem 8.17] gives us the following short exact sequence of locally free $\mathcal{O}_D$-modules,

$$0 \to \mathcal{T}_D \to \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_D \to \mathcal{N}_{D|\mathbb{P}^3} \to 0, \tag{13}$$

where $\mathcal{T}$ denotes the tangent sheaf. Recall, the following short exact sequence (see [Har77, II. Example 8.20.1]):

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4} \to \mathcal{T}_{\mathbb{P}^3} \to 0.$$

Since $\mathcal{O}_{\mathbb{P}^3}$ is 0-regular, $\mathcal{T}_{\mathbb{P}^3}$ is 0-regular. Therefore, $\mathcal{T}_{\mathbb{P}^3}$ is globally generated. Since pullback of globally generated sheaves is globally generated, $\mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_D$ is globally generated. Since $\mathcal{N}_{D|\mathbb{P}^3}$ is the quotient of a globally generated sheaf, it is also globally generated. This proves the lemma. $\square$

Corollary 7.13. If D is a smooth curve such that $C.D = x$ for some closed point $x \in D$ then Y is simple extension of C.

Proof. Lemma 7.12 implies the morphism $h_3 : H^0(\mathcal{N}_{D|\mathbb{P}^3}) \to H^0(\mathcal{N}_{D|\mathbb{P}^3} \otimes_{\mathcal{O}_D} \mathcal{O}_{C.D})$ is surjective. Then, by Corollary 7.11, $r \circ \Upsilon^6_{D\leq Y} = \Upsilon_D$ is surjective. So, $\operatorname{Im} \Upsilon_C \subset \operatorname{Im} \Upsilon_D$. Then, Theorem 7.7 implies $\operatorname{Im} \Upsilon^6_{C\leq Y} \subset \operatorname{Im} \Upsilon^5_{C\leq Y}$. Since D is reduced, this proves the corollary. $\square$

Proposition 7.14. Let C, D be contained in a degree d hypersurface in $\mathbb{P}^3$, say X, for some $d \geq 5$ and D is smooth satisfying the condition $\deg(D) > C.D + 2\rho_a(D) - 2$. Then, $C + D$ is simple extension of C.

Proof. After Corollary 7.11, we need to prove that the natural restriction morphism $H^0(\mathcal{N}_{D|\mathbb{P}^3}) \to H^0(\mathcal{N}_{D|\mathbb{P}^3} \otimes \mathcal{O}_{C.D})$ is surjective i.e., $H^1(\mathcal{N}_{D|\mathbb{P}^3} \otimes \mathcal{O}_D(-C.D)) = 0$. Tensoring the short exact sequence:

$$0 \to \mathcal{T}_D \to \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_D \to \mathcal{N}_{D|\mathbb{P}^3} \to 0$$

by $\mathcal{O}_D(-C.D)$ we get

$$0 \to \mathcal{T}_D(-C.D) \to \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_D(-C.D) \to \mathcal{N}_{D|\mathbb{P}^3}(-C.D) \to 0$$

Since D is a curve, by Grothendieck vanishing, it suffices to check $H^1(\mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_D(-C.D)) = 0$. Now, $\mathcal{T}_{\mathbb{P}^3}$ sits in the short exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4} \to \mathcal{T}_{\mathbb{P}^3} \to 0.$$

Since pull-back of locally free sheaves is locally-free, we get

$$0 \to \mathcal{O}_D \to \mathcal{O}_D(1)^{\oplus 4} \to \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_D \to 0.$$

As $\mathcal{O}_D(-C.D)$ is locally-free, tensoring by it preserves exactness, hence we get the short exact sequence:

$$0 \to \mathcal{O}_D(-C.D) \to \mathcal{O}_D(-C.D)(1)^{\oplus 4} \to \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_D(-C.D) \to 0$$

Again by Grothendieck vanishing, it suffices to check $h^1(\mathcal{O}_D(-C.D)(1)) = 0$. By Serre duality, $H^1(\mathcal{O}_D(-C.D)(1)) = H^0(\mathcal{O}_D(C.D)(-1) \otimes K_D)^\vee$. Since $\deg(D) > C.D + 2\rho_a(D) - 2$, $h^0(\mathcal{O}_D(C.D)(-1) \otimes K_D) = 0$ hence $h^1(\mathcal{O}_D(-C.D)(1)) = 0$. Hence, $C + D$ is simple extension of C. $\square$

Corollary 7.15. Notations and hypothesis as in Proposition 7.14. Denote by E_n the Cartier divisor on X of the form $C+nD$ for some integer $n>1$. If $C+D$ is deformation d-regular then E_n is simple extension of C.

Proof. We only need to prove that for every infinitesimal deformation of C there exists an infinitesimal deformation of E containing it. By Proposition 7.14, for every infinitesimal deformation C' of C, there exists an infinitesimal deformation C'' of $C+D$ such that $C' \subset C''$. Since $C+D$ is deformation d-regular, there exists an infinitesimal deformation X' of X containing C'', hence also C', as a Cartier divisor. We denote the ideal sheaf of C'' (resp. C') in X' by $\mathcal{O}_{X'}(-C'')$ (resp. $\mathcal{O}_{X'}(-C')$). Hence, $\mathcal{O}_{X'}(-C''+C')$ is the ideal sheaf of an infinitesimal deformation of D in X'. Therefore, $\mathcal{O}_{X'}(-C')\otimes\mathcal{O}'_X(n(-C''+C'))$ is the ideal sheaf of an infinitesimal deformation of E in X' which contains C'. This proves the corollary. □

7.3 Topological invariance of divisors under deformation

Consider a family $\pi: \mathcal{X} \to B$ of smooth projective varieties and a reference (closed) point $o \in B$. Let $\mathcal{X}_o$ be the fiber and D_o an effective divisor in it. We ask if $(D_o, \mathcal{X}_o)$ deforms to $(D_t, \mathcal{X}_t)$ for some closed point $t \in B$ such that D_t is effective then does the underlying topological space $D_{o_{\text{red}}}$ deform to that of D_t, i.e., $D_{t_{\text{red}}}$? We answer this question in the case of family of curves in $\mathbb{P}^3$. As application, we see in these cases there is an unique element on any fiber to the natural projection from L'_0 to L_0 (see Theorem 7.19). One notices that the curves mentioned in this theorem will satisfy the cardinality condition in Theorem 6.10.

We first compute the arithmetic genus of curves in $\mathbb{P}^3$ using which we could conclude when a curve is non-reduced. Since arithmetic genus is deformation invariant, we conclude (in certain cases) whether a curve is a deformation of a reduced curve or a non-reduced one. Recall, the following standard result on arithmetic genus, which we do not prove:

Lemma 7.16 ([Har77, Ex. V.1.3])**.** Let X be a smooth projective surface and C, D be effective divisors on X. Then,

$$\rho_a(C+D) = \rho_a(C) + \rho_a(D) + C.D - 1.$$

Lemma 7.17. Let C be an integral curve in $\mathbb{P}^3$, X a smooth degree d surface in $\mathbb{P}^3$ containing C and D a divisor in X of the form aC for $a>0$. Then,

$$\rho_a(aC) = a^2\rho_a(C) - (a+1)(a-1) - \frac{a(a-1)}{2}\deg(C)(d-4).$$

Proof. Using Lemma 7.16 we have,

$$\rho_a(aC) = \rho_a(C) + \rho_a((a-1)C) + (a-1)C^2 - 1.$$

Applying this formula recursively, we get

$$\rho_a(aC) = a\rho_a(C) + \frac{a(a-1)}{2}C^2 - (a-1).$$

Using the adjunction formula we have,

$$\rho_a(aC) = a^2\rho_a(C) - (a+1)(a-1) - \frac{a(a-1)}{2}\deg(C)(d-4).$$

This proves the lemma. □

Proposition 7.18. Let X be a smooth degree d surface in $\mathbb{P}^3$, D an effective divisor on X. Assume that $d \geq \deg(D)^2 + 4$. Then, D is non-reduced if and only if $\rho_a(D) < -\deg(D)$.

Proof. Suppose that D is of the form $\sum_{i=1}^r a_iC_i$ for $a_i > 0$, C_i integral and $C_i \neq C_j$ for $i \neq j$. Then, Lemmas 7.17 and 7.16 implies

$$\rho_a(\sum_{i=1}^r a_iC_i) = \sum_{i=1}^r \rho_i(a_iC_i) + \sum_{i<j} a_ia_jC_i.C_j - (r-1),$$

$$= \sum_{i=1}^r (a_i^2\rho_a(C_i) - (a_i+1)(a_i-1)) - \sum_{i=1}^r \frac{a_i(a_i-1)}{2}\deg(C_i)(d-4) + \sum_{i<j} a_ia_jC_i.C_j - (r-1).$$

By assumption, $d \geq (\sum_i a_i \deg(C_i))^2 + 4$. Since C_i is integral for all i, $\rho_a(C_i) < \deg(C_i)^2/2$. Using $C_i.C_j \leq \deg(C_i)\deg(C_j)$ for $i \neq j$, we have

$$\rho_a(\sum_{i=1}^r a_iC_i) < \sum_{i=1}^r \left(a_i^2\frac{\deg(C_i)^2}{2}\right) - \sum_{i=1}^r \frac{a_i(a_i-1)}{2}\deg(C_i)\left(\sum_j a_j^2\deg(C_j)^2 + \right.$$

$$\left. + \sum_{j\neq k} a_ka_j\deg(C_j)\deg(C_k)\right) + \sum_{i<j} a_ia_j\deg(C_i).\deg(C_j) - (r-1).$$

If there exists i_0 such that $a_{i_0} > 1$, this is bounded above by

$$\sum_{i=1}^r \left(a_i^2\frac{\deg(C_i)^2}{2}\right) - \deg(C_{i_0})\left(\sum_i a_i^2\deg(C_i)^2 + \sum_{i\neq j} a_ia_j\deg(C_i)\deg(C_j)\right) + \sum_{i<j} a_ia_j\deg(C_i).\deg(C_j)$$

$$-(r-1) = \sum_{i=1}^r \left(\frac{a_i^2}{2} - a_i^2\deg(C_{i_0})\right)\deg(C_i)^2 + \sum_{i<j}(a_ia_j - 2a_ia_j\deg(C_{i_0}))\deg(C_i)\deg(C_j) - (r-1).$$

Without loss of generality, we can assume $i_0 = 1$ (after rearranging the curves if necessary). Since $a_1 \geq 2$, we have

$$\rho_a(D) < -\left(a_1^2\left(\deg(C_1) - \frac{1}{2}\right)\deg(C_1)^2 + \sum_{i=2}^r a_1a_i(2\deg(C_1)-1)\deg(C_1)\deg(C_i) + (r-1)\right) <$$

$$< -\left(a_1 \deg(C_1) + \sum_{i=2}^{r} a_i \deg(C_i)\right) = -\deg(D).$$

It remains to prove that if $\rho_a(D) < -\deg(D)$ then D is non-reduced. Suppose not i.e., D is reduced. Recall, that if D is connected then $\rho_a(D) \geq 0$. Using Lemma 7.16 once again, we can conclude that

$$\rho_a(D) \geq -(s-1) > -\deg(D),$$

where s is the number of connected components of D. This is a contradiction to our assumption on the inequality between the arithmetic genus and degree. □

Theorem 7.19. Let X be a smooth degree d surface in $\mathbb{P}^3$, D an effective divisor on X. Assume that D is non-reduced and $d \geq \deg(D)^2 + 4$. Denote by P (resp. P_r) the Hilbert polynomial of D (resp. D_{red}) and L an irreducible component of H_P containing D. Then the following are true:

1. Every effective divisor in a smooth degree d surface corresponding to a point of L is non-reduced
2. The associated reduced scheme to any of these effective divisors has the same Hilbert polynomial as D_{red}
3. For any effective divisor $E \in L$, in a smooth degree d surface, there exists an unique subcurve E' in E with Hilbert polynomial P_r, hence $E' = E_{\text{red}}$.

Proof. (1) Since D is non-reduced, Proposition 7.18 implies $\rho_a(D) < -\deg(D)$. Let $E \in L$ be any curve. Since E has the same Hilbert polynomial as D, $\rho_a(E) < -\deg(E)$. Proposition 7.18 again implies E is non-reduced. This proves (1).

(2) Let P'_r be the Hilbert polynomial of C_{red} for a general curve $C \in L$. (1) implies that $P'_r \neq P$. By Proposition 7.18, $\rho_a(C_{\text{red}}) \geq -\deg(C_{\text{red}})$. Hence, every curve E_1 satisfying this Hilbert polynomial, $\rho_a(E_1) \geq -\deg(E_1)$. Proposition 7.18 again tells us that E_1 is reduced.

Let $E \in L$ be any curve. Using Proposition 3.3, there exists at least one curve $E_1 \subset E$ with Hilbert polynomial P'_r and $E_{1_{\text{red}}} = E_{\text{red}}$. But E_1 is already reduced. So, $E_1 = E_{\text{red}}$. In the case $E = D$, we get the corresponding curve $E_1 \subset E$ with Hilbert polynomial P'_r satisfies $E_1 = D_{\text{red}}$. Hence, $P'_r = P_r$. This completes the proof of (2).

(3) Let $E \in L$ and E_1, E_2 be two curves contained in E with Hilbert polynomial P_r. Proposition 7.18 implies E_1, E_2 are reduced. Hence, E_1, E_2 are both contained in E_{red}. But we saw in the proof of (2) that E_{red} has Hilbert polynomial P_r. Hence, $E_1 = E_{\text{red}} = E_2$. This proves (3) taking $E' := E_{\text{red}}$. □

7.4 Examples of non-reduced components of Hilbert scheme of effective divisors

Setup 7.20. Notations as in Theorem 2.19. Consider $C = 2l' + C' \in L$, general. Let D be a smooth curve in $\mathbb{P}^3$ such that $D.C = D.C'$ is a closed point with multiplicity one. Choose smooth curves $D_1, ..., D_n$ such that

$$(C \cup D).D_1, (C \cup D \cup D_1).D_2, ..., (C \cup D \cup D_1 \cup ... \cup D_{n-1}).D_n$$

are all closed points with multiplicity one. By [KA79, Theorem 7], for $d \gg 0$, there exists a smooth degree d surface X containing $E := C \cup D \cup D_1 \cup ... \cup D_{n-1} \cup D_n$. Denote by P_E (resp. P_{E_r}) the Hilbert polynomial of E (resp. E_{red}).

Assume that E is weakly general, L_0 the unique irreducible component of H_{P_E} containing E and suppose that a general element of L_0 is d-embedded.

Example 7.21. Denote by $E_1 := C \cup D$ and P_{E_1} its Hilbert polynomial. If D is a line then a general element of an irreducible component, say L_0 of $H_{P_{E_1}}$ containing E_1, is d-embedded. Indeed, [Dan14a, Theorem 6.11] implies the ideal sheaf of C is d-regular. Using [Sid02, Corollary 1.9] we conclude that E_1 is $d+1$-regular.

Since E_1 is $d+1$-regular, there exists an open neighbourhood $U \subset L_0$ of E_1 such that for all $u \in U$ closed, the corresponding curve E_u satisfies $\dim I_d(E_u) = \dim I_d(E_1)$. Consider the natural projection morphism $\text{pr}_1 : H_{P_{E_1}, Q_d} \to H_{P_{E_1}}$. Note that, the fiber over every closed point $E_t \in H_{P_{E_1}}$ is of dimension $\dim \mathbb{P}(I_d(E_t))$. Hence, there exists an irreducible component, say L' of $H_{P_{E_1}, Q_d}$ mapping surjectively to L with general fiber (to the morphism pr_1) of dimension $\dim \mathbb{P}(I_d(E_1))$. By the upper-semicontinuity of fiber dimension, $\dim(\text{pr}_1|_{L'})^{-1}(E_1) = \dim \text{pr}_1^{-1}(E_1)$, hence there exists a closed point in L' of the form (E_1, X) where X is smooth. By [Har77, Ex. III. 10.2] there exists an open neighbourhood U of this point such that for all $u \in U$ closed, the corresponding pair (E_u, X_u) satisfies X_u is smooth. Therefore, a general element of L_0 is d-embedded.

We now arrive at the final theorem of the article.

Theorem 7.22. If $d \geq \deg(E)^2 + 4$ then the following holds true:

1. There exists an irreducible component, say L'_0 of $H_{P_{E_r}, P_E}$ such that

$$(12 \to 2)(L'_0)_{\text{red}} = L_{0_{\text{red}}} \text{ and } (12 \to 1)(L'_0) \text{ contains } E_{\text{red}}.$$

2. If E is deformation d-regular then L_0 is non-reduced at E.

3. Assume that a general point in $(12 \to 2)((12 \to 1)^{-1}(E_{\text{red}}))$ correspond to a curve which is not deformation d-regular. Then L_0 is non-reduced at a general such point.

Proof. (1) Using Theorem 7.19 we conclude that a general d-embedded curve $E_g \in L_0$ is non-reduced and the associated reduced scheme $E_{g_{\text{red}}}$ has Hilbert polynomial P_{E_r}. By Proposition 3.3 there exists an irreducible component L'_0 of $H_{P_{E_r}, P_E}$ such that $\text{pr}_2 : (L'_0)_{\text{red}} \to L_{0_{\text{red}}}$ is surjective and $(12 \to 2)^{-1}(E)$ are of the form (E', E) satisfying $E'_{\text{red}} = E_{\text{red}}$. In particular, $E_{\text{red}} \subset E'$. Since E' has the same Hilbert polynomial as E_{red}, we conclude that $E' = E_{\mathrm{red}}$, in particular $(12 \to 2)^{-1}(E)$ is of the form (E_{red}, E). Hence, $(12 \to 1)(L'_0)$ contains E_{red}. This proves (1).

(2) Denote by P_r the Hilbert polynomial of C_{red} and L' the irreducible component of $H_{P_r,P}$ such that L' maps surjectively to L_{red} (existence follows from definition of L as in Theorem 2.19). By deforming C (along L) if necessary, we can assume that (C_{red}, C) is weakly general in $H_{P_r,P}$. Corollary 5.8 implies L' is smooth at (C_{red}, C). Since L is non-reduced at C, there exists $\xi_0 \in H^0(\mathcal{N}_{C|\mathbb{P}^3})$ such that $\Upsilon^5_{C_{\mathrm{red}} \le C}(\xi_0) \notin \operatorname{Im} \Upsilon^6_{C_{\mathrm{red}} \le C}$.

Corollary 7.13 implies E is an extension of C via the sequence

$$C \subset C + D \subset C + D + D_1 \subset ... \subset C + D + D_1 + ... + D_n = E.$$

By assumption, a general element of L_0 is d-embedded. Theorem 7.19 implies that there exists an open neighborhood $U \subset L_0$ of E such that the natural projection morphism $(12 \to 2) : (12 \to 2)^{-1}(U) \to U$ is injective.

If E is deformation d-regular then Theorem 6.10 implies L_0 is non-reduced at E. This proves (2).

Part (3) is a direct consequence of Theorem 6.11. □

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