

A JACOBIAN MODULE FOR DISENTANGLEMENTS AND APPLICATIONS TO MOND'S CONJECTURE

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ABSTRACT. Given a germ $f : (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$, we define an \mathcal{O}_{n+1} -module $M(f)$ with the property that $\mathcal{A}_e\text{-codim}(f) \leq \dim_{\mathbb{C}} M(f)$, with equality if f is weighted homogeneous. We also define a relative version $M_y(F)$ for unfoldings $F = (u, f_u)$, in such a way that $M_y(F)$ specialises to $M(f)$ when $u = 0$. The main result is that if $(n, n+1)$ are nice dimensions, then $\dim_{\mathbb{C}} M(f) \geq \mu_I(f)$, with equality if and only if $M_y(F)$ is Cohen-Macaulay, for some stable unfolding F . Here, $\mu_I(f)$ denotes the image Milnor number of f , so that if $M_y(F)$ is Cohen-Macaulay, then we have Mond's conjecture for f ; furthermore, if f is quasi-homogeneous Mond's conjecture for f is equivalent to the fact that $M_y(F)$ is Cohen-Macaulay. Finally, we observe that to prove Mond's conjecture, it is enough to prove it in a suitable family of examples.

1. INTRODUCTION

For any hypersurface with isolated singularity $(X, 0)$, we have $\tau(X, 0) \leq \mu(X, 0)$, with equality if $(X, 0)$ is weighted homogeneous. Here, $\tau(X, 0)$ is the Tjurina number, that is, the minimal number of parameters in a versal deformation of $(X, 0)$ and $\mu(X, 0)$ is the Milnor number, which is the number of spheres in the Milnor fibre of $(X, 0)$. If $g : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ is a function such that $g = 0$ is a reduced equation of $(X, 0)$, then we can compute both numbers in terms of g :

$$\tau(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{J(g) + \langle g \rangle}, \quad \mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{J(g)},$$

where \mathcal{O}_{n+1} is the local ring of holomorphic germs from $(\mathbb{C}^{n+1}, 0)$ to \mathbb{C} and $J(g)$ denotes the Jacobian ideal generated by the partial derivatives of g . Thus, the initial statement about τ and μ becomes evident. The Jacobian algebra deforms flatly over the parameter space of any deformation g_t of g , it is known to encode crucial properties of the vanishing cohomology and its

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monodromy by its relation with the Brieskorn lattice and it is crucial in the construction of Frobenius manifold structures in the bases of versal unfoldings. See the works of Brieskorn, Varchenko, Steenbrink, Scherk, Hertling and others, and the the books [1], [7] and [6]

Inspired by the previous inequality, D. Mond [13] tried to obtain a result of the same nature in the context of singularities of mappings. He considered a hypersurface $(X, 0)$ given by the image of a map germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$, with $S \subset \mathbb{C}^n$ a finite set and which has isolated instability under the action of the Mather group \mathcal{A} of biholomorphisms in the source and the target. The Tjurina number has to be substituted by the \mathcal{A}_e -codimension, which is equal to the minimal number of parameters in an \mathcal{A} -versal deformation of f . Instead of the Milnor fibre, one considers the disentanglement, that is, the image X_u of a stabilisation f_u of f . Then, X_u has the homotopy type of a wedge of spheres and Mond defined the image Milnor number $\mu_I(f)$ as the number of such spheres. Note that, outside the range of Mather's dimensions, some germs do not admit a stabilisation. Then, he stated the following conjecture:

Conjecture 1.1. *Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be an \mathcal{A} -finite map germ, with $(n, n+1)$ nice dimensions. Then,*

$$\mathcal{A}_e\text{-codim}(f) \leq \mu_I(f),$$

with equality if f is weighted homogeneous.

The conjecture is known to be true for $n = 1, 2$ (see [8, 13, 14]) but it remains open until now for $n \geq 3$. There is a related result for map germs $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^p, 0)$ with $n \geq p$, where one considers Δ the discriminant of f instead of its image and defines the discriminant Milnor number $\mu_\Delta(f)$ in the same way. Damon and Mond showed in [3] that if (n, p) are nice dimensions, then $\mathcal{A}_e\text{-codim}(f) \leq \mu_\Delta(f)$ with equality if f is weighted homogeneous. There are many papers in the literature with related results, partial proofs and examples in which the conjecture has been checked. We refer to [16] for a recent account of these results.

Going back to hypersurface singularities g , but now with non-isolated singularities, it is not anymore clear the relation of the Jacobian algebra of g with the vanishing cohomology. Moreover it is apparent in easy examples that the Jacobian algebra does not deform flatly in unfoldings. In fact the possibility of studying the vanishing cohomology via deformations that simplify the critical set (in the same vein that Morsifications do for isolated singularities) does not exist in general. However, for restricted classes of singularities Siersma, Pellikaan, Zaharia, Nemethi, Marco-Buzunáriz and the first author have developed methods that allow to split the vanishing cohomology of a non-isolated singularity in two direct summands according with the geometric properties of a deformation g_u of g which plays the role of a Morsification (one may find a nice survey in [21]). The first is a free vector space contributing to the middle dimension cohomology of the Milnor fibre, with as much dimension as the number of Morse points that appear away from the zero set of g_u ($u \neq 0$), the second is determined by the non-isolated singularities of the zero-set of g_u ($u \neq 0$).

Given $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$, an \mathcal{A} -finite map germ, we consider a generic 1-parameter deformation f_u of it (a stabilisation). Let g_u be the equation defining the image of f_u . It turns out that the deformation g_u is suitable to split the vanishing cohomology of g in two direct summands, as explained in the paragraph above, and that the first summand corresponds with the cohomology of the image X_u , whose rank is the image Milnor number. The main novelty of this paper is the definition of an Artinian \mathcal{O}_{n+1} -module $M(f)$, which satisfies

$$\dim_{\mathbb{C}} M(f) = \mathcal{A}_e\text{-codim}(f) + \dim_{\mathbb{C}}((g) + J(g)/J(g))$$

and, in the nice dimensions, this dimension upper bounds the image Milnor number. Moreover we define a relative version $M_y(F)$ of the module for unfoldings F of f , and we prove that when we specialise the parameter the relative module specialises to the original $M(f)$.

The first main result of this paper is Theorem 6.5, which implies that the dimension of $M(f)$ equals the image Milnor number if and only if $M_y(F)$ is flat over the base of the unfolding. We also prove that this is equivalent to the flatness of the Jacobian algebra over the base of the unfolding. Thus, under the flatness condition, $M(f)$ is expected to play the role of the Milnor algebra for isolated singularities, in the sense of encoding the first direct summand of the vanishing cohomology, which is the only one present for isolated singularities. It is very interesting to investigate whether the relation of the vanishing cohomology of isolated singularities with the Jacobian algebra explained admit a generalisation to a relation between the first direct summand of the vanishing cohomology of g and the module $M(f)$.

The second main result (Theorem 6.7) says that the flatness of $M_y(F)$ implies Mond's conjecture for f , and it is equivalent to it if f is weighted homogeneous. In Theorems 7.1 and 7.2 we derive the surprising consequence that in order to settle Mond's conjecture in complete generality it is enough to prove it for a series of examples of increasing multiplicity.

In Section 5 we derive formulas to compute the module the modules $M(f)$ and $M_y(F)$ which are well suited for computer algebra programs and also lead to new formulas for the \mathcal{A}_e -codimension: see Corollary 5.5 and Remark 5.6.

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2. THE \mathcal{A}_e -CODIMENSION AND THE IMAGE MILNOR NUMBER

We recall the definition of codimension of a map germ with respect to the Mather \mathcal{A} group. Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^p, 0)$ be any holomorphic map multi-germ. We denote by $\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n, S}$ and $\mathcal{O}_p = \mathcal{O}_{\mathbb{C}^p, 0}$ the rings of holomorphic function germs in the source and the target respectively. We also denote by $\theta_n = \theta_{\mathbb{C}^n, S}$ and $\theta_p = \theta_{\mathbb{C}^p, 0}$ the corresponding modules of germs of vector fields and by $\theta(f)$ the module of germs of vector fields along f . Then, we have two associated morphisms: $tf: \theta_n \rightarrow \theta(f)$ given by $tf(\eta) = df \circ \eta$ and $\omega f: \theta_p \rightarrow \theta(f)$ given by $\omega f(\xi) = \xi \circ f$. The \mathcal{A}_e -codimension of f is defined as:

$$\mathcal{A}_e\text{-codim}(f) = \dim_{\mathbb{C}} \frac{\theta(f)}{tf(\theta_n) + \omega f(\theta_p)}.$$

We say that f is \mathcal{A} -stable (resp. \mathcal{A} -finite) if the \mathcal{A}_e -codimension is zero (resp. finite).

By an r -parameter unfolding of a map multi-germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^p, 0)$ we mean another multi-germ $F: (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S) \rightarrow (\mathbb{C}^r \times \mathbb{C}^p, 0)$ given by $F(u, x) = (u, f_u(x))$ and such that $f_0 = f$. It was proved by Mather [10] that f is \mathcal{A} -stable if and only if any unfolding F of f is trivial. This means that there exist Φ and Ψ unfoldings of the identity in $(\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S)$ and $(\mathbb{C}^r \times \mathbb{C}^p, 0)$, respectively, such that $\Psi \circ F \circ \Phi^{-1}$ is the constant unfolding $\text{id} \times f$.

We present now a result due to Mond which gives a way to compute the \mathcal{A}_e -codimension in the case $p = n + 1$ in terms of a defining equation of the image of f . We need to introduce some notation.

From now on, we will assume that $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ is finite and generically one-to-one and denote by $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ its image. The restriction $\bar{f}: (\mathbb{C}^n, S) \rightarrow (X, 0)$ is the normalization map, so the induced morphism $\bar{f}^*: \mathcal{O}_{X,0} \rightarrow \mathcal{O}_n$ is a monomorphism and we will consider $\mathcal{O}_{X,0}$ as a subring of \mathcal{O}_n . Thus, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{n+1} & \xrightarrow{f^*} & \mathcal{O}_n \\ & \searrow \pi & \uparrow i \\ & & \mathcal{O}_{X,0} \end{array}$$

where π is the epimorphism induced by the inclusion $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$. Here, we consider both $\mathcal{O}_{X,0}$ and \mathcal{O}_n as \mathcal{O}_{n+1} -modules via the corresponding morphisms. Finally, let $g \in \mathcal{O}_{n+1}$ be such that $g = 0$ is a reduced equation of $(X, 0)$ and denote by $J(g) \subset \mathcal{O}_{n+1}$ the Jacobian ideal of g .

Lemma 2.1. [13, Proposition 2.1] *Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be \mathcal{A} -finite, with $n \geq 2$. Then,*

$$\mathcal{A}_e\text{-codim}(f) = \dim_{\mathbb{C}} \frac{J(g) \cdot \mathcal{O}_n}{J(g) \cdot \mathcal{O}_{X,0}}.$$

Note that the proof of this lemma given in [13] is only for monogerms, but a careful revision of the proof shows that it also works for multigerms. Note also that the lemma is not true for $n = 1$. In fact, in that case (see [14]):

$$\mathcal{A}_e\text{-codim}(f) = \dim_{\mathbb{C}} \frac{J(g) \cdot \mathcal{O}_1}{J(g) \cdot \mathcal{O}_{X,0}} + \dim_{\mathbb{C}} \frac{\mathcal{O}_1}{\langle f'_1, f'_2 \rangle}.$$

Next, we recall the definition of image Milnor number. Consider any r -parameter unfolding $F(u, x) = (u, f_u(x))$ of $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$. Denote by $(\mathcal{X}, 0)$ the image hypersurface of F in $(\mathbb{C}^r \times \mathbb{C}^{n+1}, 0)$, and by X_u the fibre of \mathcal{X} over $u \in \mathbb{C}^r$. We fix a small enough representative

$$F: W \rightarrow T \times B_\epsilon,$$

where W, T, B_ϵ are open neighbourhoods of S and the origin in $\mathbb{C}^{r+n}, \mathbb{C}^r, \mathbb{C}^{n+1}$, respectively, such that:

- (1) F is finite (i.e., closed and finite-to-one),
- (2) $F^{-1}(0) = \{0\} \times S$,
- (3) B_ϵ is a Milnor ball for the hypersurface $X_0 \subset \mathbb{C}^{n+1}$

- (4) T is small enough so that the intersection $X_u \cap \partial B_\epsilon$ of the hypersurface with the Milnor sphere is topologically trivial over all $u \in T$.

In order to understand the topology of $X_u \cap B_\epsilon$ we use the following general result due to Siersma:

Theorem 2.2. [20] *Let $g: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ define a reduced hypersurface $(X_0, 0)$, not necessarily with isolated singularity, and let $G: (\mathbb{C}^r \times \mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a deformation of g such that for all u ,*

- (1) $\{g_u = 0\}$ is topologically trivial over the Milnor sphere ∂B_ϵ , and
- (2) all the critical points of g_u which are not in $X_u = g_u^{-1}(0) \cap B_\epsilon$ are isolated.

Then, $X_u \cap B_\epsilon$ is homotopy equivalent to a wedge of n -spheres and the number of such n -spheres is equal to

$$\sum_{y \in B_\epsilon \setminus X_u} \mu(g_u; y),$$

where $\mu(g_u; y)$ denotes the Milnor number of the function g_u at the point y .

Definition 2.3. Assume $r = 1$. Given a representative $F: W \rightarrow T \times B_\epsilon$ as above, we say that F is a *stabilisation* if for any $u \in T \setminus \{0\}$ and any point $y \in X_u \cap B_\epsilon$ the multigerms of f_u at y is \mathcal{A} -stable.

It is well known that every map f admits a stabilisation if $(n, n+1)$ are nice dimensions in the sense of Mather [12]. As an application of Siersma's previous result, Mond proves the following theorem in [13]. Again the proof of the theorem given in [13] is only for monogerms, but it is easy to check that the proof also works for multigerms.

Theorem 2.4. [13] *Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be \mathcal{A} -finite with $(n, n+1)$ nice dimensions and let F be a stabilisation of f . Then, for any $u \in T \setminus \{0\}$, the image of f_u has the homotopy type of a wedge of n -spheres. Moreover, the number of such n -spheres is independent of the parameter u and on the stabilisation F .*

Remark 2.5. As it is mentioned in [13], if $(n, n+1)$ are not nice dimensions, the theorem is still true if we substitute a stabilisation by a C^0 -stabilisation. This means that for any $u \in T \setminus \{0\}$ and any point $y \in X_u \cap B_\epsilon$, the multigerms of f_u at y is C^0 - \mathcal{A} -stable (that is, its jet extension is multitransverse to the canonical stratification of Mather of the jet space). Since a C^0 -stabilisation of f always exists (by the Thom-Mather transversality theorem), the image Milnor number can be defined in general by taking a C^0 -stabilisation instead of a stabilisation.

Definition 2.6. Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be \mathcal{A} -finite. The image X_u of a C^0 -stabilisation f_u of f is called the *disentanglement* and number of n -spheres in X_u is called the *image Milnor number* of f and is denoted by $\mu_I(f)$.

Remark 2.7. Sometimes, it is more interesting to work with a stable unfolding of f instead of a stabilisation. This means an r -parameter unfolding F which is \mathcal{A} -stable as a map germ. If f is finite, then it has finite singularity type and hence, it always admits a stable unfolding (see [11]).

Then, there exists a proper closed analytic subset $\mathcal{B} \subset T$ such that for any $u \in T \setminus \mathcal{B}$ and any point $y \in X_u \cap B_\epsilon$ the multigerm of f_u at y is C^0 - \mathcal{A} -stable (or \mathcal{A} -stable if $(n, n+1)$ are nice dimensions). This subset \mathcal{B} is known as the *bifurcation set* of F . It follows that for any $u \in T \setminus \mathcal{B}$, the image of f_u has the homotopy type of a wedge of n -spheres and the number of such n -spheres is equal to $\mu_I(f)$.

In fact, we only have to take a curve $C \subset T$ joining u to the origin and such that $C \cap \mathcal{B} = \{0\}$. If C is parametrised by $\gamma(t)$, then $H(t, x) = (t, f_{\gamma(t)}(x))$ defines a C^0 -stabilisation of f such that $f_{\gamma(t)} = f_u$ for some $t \neq 0$.

3. THE MODULE $M(f)$

We denote by $C(f)$ the conductor ideal of $\mathcal{O}_{X,0}$ in \mathcal{O}_n and by $\mathcal{C}(f)$ its inverse image through $\pi: \mathcal{O}_{n+1} \rightarrow \mathcal{O}_{X,0}$, that is,

$$C(f) := \{h \in \mathcal{O}_{X,0} : h \cdot \mathcal{O}_n \subset \mathcal{O}_{X,0}\}, \quad \mathcal{C}(f) := \pi^{-1}(C(f)).$$

The conductor has the property that it is the largest ideal of $\mathcal{O}_{X,0}$ which is also an ideal of \mathcal{O}_n . We can compute easily $C(f)$ by using the following result of Piene [19] (see also Bruce-Marar [2]).

Lemma 3.1. *There exists a unique $\lambda \in \mathcal{O}_n$ such that*

$$\frac{\partial g}{\partial y_i} \circ f = (-1)^i \lambda \det(df_1, \dots, df_{i-1}, df_{i+1}, \dots, df_{n+1}), \quad 1 \leq i \leq n+1,$$

and moreover, $C(f)$ is generated by λ .

From Lemma 3.1 we have the inclusion $J(g) \cdot \mathcal{O}_n \subset C(f)$, which motivates the following definition.

Definition 3.2. We define $M(f)$ as the kernel of the epimorphism of \mathcal{O}_{n+1} -modules induced by π :

$$\frac{\mathcal{C}(f)}{J(g)} \longrightarrow \frac{C(f)}{J(g) \cdot \mathcal{O}_n}.$$

Proposition 3.3. *We have the following exact sequence of \mathcal{O}_{n+1} -modules:*

$$0 \longrightarrow K(g) \longrightarrow M(f) \longrightarrow \frac{J(g) \cdot \mathcal{O}_n}{J(g) \cdot \mathcal{O}_{X,0}} \longrightarrow 0$$

where $K(g) := (\langle g \rangle + J(g))/J(g)$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K(g) & \longrightarrow & M(f) & \longrightarrow & \frac{J(g) \cdot \mathcal{O}_n}{J(g) \cdot \mathcal{O}_{X,0}} \longrightarrow 0 \\
& & \downarrow \mu_1 & & \downarrow \mu_2 & & \downarrow \mu_3 \\
0 & \longrightarrow & K(g) & \longrightarrow & \frac{\mathcal{C}(f)}{J(g)} & \longrightarrow & \frac{C(f)}{J(g) \cdot \mathcal{O}_{X,0}} \longrightarrow 0 \\
& & \downarrow \lambda_1 & & \downarrow \lambda_2 & & \downarrow \lambda_3 \\
& & 0 & \longrightarrow & \frac{C(f)}{J(g) \cdot \mathcal{O}_n} & \longrightarrow & \frac{C(f)}{J(g) \cdot \mathcal{O}_n} \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

Observe that all columns and the second and third rows are exact. Therefore, from the Snake Lemma we obtain an exact sequence

$$0 \rightarrow \ker(\lambda_1) \rightarrow \ker(\lambda_2) \rightarrow \ker(\lambda_3) \rightarrow \text{coker}(\lambda_1) \rightarrow \dots$$

But $\text{coker}(\lambda_1) = 0$, $\ker(\lambda_1) = K(g)$, $\ker(\lambda_2) = \text{Im}(\mu_2) = M(f)$ and $\ker(\lambda_3) = \text{Im}(\mu_3) = (J(g) \cdot \mathcal{O}_n)/(J(g) \cdot \mathcal{O}_{X,0})$, so we get the desired exact sequence. \square

Corollary 3.4. *Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be \mathcal{A} -finite with $n \geq 2$. Then:*

- (1) $M(f) = 0$ if and only if f is \mathcal{A} -stable and $g \in J(g)$.
- (2) If $\dim_{\mathbb{C}} M(f) < \infty$, then

$$\dim_{\mathbb{C}} M(f) = \mathcal{A}_e\text{-codim}(f) + \dim_{\mathbb{C}} K(g).$$

This corollary is important, because it gives a simple method to compute the \mathcal{A}_e -codimension of a map germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$, with $n \geq 2$, just by means of a reduced equation of the image. We will explain this with more details in Section 5.

Remark 3.5. If $(n, n+1)$ are nice dimensions, then the condition that $\dim_{\mathbb{C}} M(f) < \infty$ in part (2) of Corollary 3.4 is not necessary since it is a consequence of the \mathcal{A} -finiteness of f . In fact, all stable singularities in the nice dimensions are \mathcal{A} -equivalent to weighted homogeneous singularities (see [12]), hence the module $K(g)$ is supported only at the origin. The same applies to $\frac{J(g) \cdot \mathcal{O}_n}{J(g) \cdot \mathcal{O}_{X,0}}$, because of Lemma 2.1. Now, the claim follows immediately from the exact sequence in Proposition 3.3.

We finish this section with a couple of interesting properties about the ideals $C(f)$ and $\mathcal{C}(f)$ which will be used later.

Remark 3.6. It follows from the proof of [17, Theorem 3.4] that $\mathcal{C}(f)$ coincides with the first Fitting ideal of \mathcal{O}_n as an \mathcal{O}_{n+1} -module via $f^*: \mathcal{O}_{n+1} \rightarrow \mathcal{O}_n$ (that is, $\mathcal{C}(f)$ is the ideal generated by the submaximal minors of a matrix presentation of \mathcal{O}_n). Furthermore, the same theorem also states that $\mathcal{O}_{n+1}/\mathcal{C}(f)$ is a determinantal ring of dimension $n-1$. By [17, Proposition

1.5], the zero locus of $\mathcal{C}(f)$ is

$$V(\mathcal{C}(f)) = \left\{ y \in \mathbb{C}^{n+1} : \sum_{f(x)=y} \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n, x}}{f^* \mathfrak{m}_{\mathbb{C}^{n+1}, y}} > 1 \right\},$$

which is equal to the points $y \in \mathbb{C}^{n+1}$ such that either $y = f(x)$ and x is a non-immersive point of f or $y = f(x) = f(x')$ with $x \neq x'$. Hence, we deduce that $V(\mathcal{C}(f))$ is the singular locus of $(X, 0)$. This space is also known as the target double point space of f .

Remark 3.7. Another consequence of Lemma 3.1 is that multiplication by λ induces an isomorphism:

$$\frac{C(f)}{J(g) \cdot \mathcal{O}_n} \cong \frac{\mathcal{O}_n}{R(f)},$$

where $R(f) \subset \mathcal{O}_n$ is the ramification ideal, that is, the ideal generated by the maximal minors of the Jacobian matrix of f . If f is \mathcal{A} -finite and $n \geq 2$, then $\mathcal{O}_n/R(f)$ is a determinantal ring (of dimension $n-2$ in this case). The zero locus $V(R(f)) \subset (\mathbb{C}^n, S)$ is the set of non-immersive points of f .

4. THE RELATIVE VERSION FOR UNFOLDINGS

We are interested in the behavior of the module $M(f)$ under deformations. With this motivation, we define a relative version of this module for unfoldings. Let F be an r -parameter unfolding of $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$. We have a commutative diagram:

$$\begin{array}{ccc} (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S) & \xrightarrow{F} & (\mathbb{C}^r \times \mathbb{C}^{n+1}, 0) \\ i \uparrow & & \uparrow j \\ (\mathbb{C}^n, S) & \xrightarrow{f} & (\mathbb{C}^{n+1}, 0), \end{array}$$

where $i(x) = (0, x)$ and $j(y) = (0, y)$. This induces another commutative diagram:

$$(1) \quad \begin{array}{ccc} \mathcal{O}_{r+n+1} & \xrightarrow{F^*} & \mathcal{O}_{r+n} \\ j^* \downarrow & & \downarrow i^* \\ \mathcal{O}_{n+1} & \xrightarrow{f^*} & \mathcal{O}_n, \end{array}$$

whose columns are epimorphisms. The conductor ideal $C(f)$ and its inverse image $\mathcal{C}(f)$ behave well under deformations, meaning that

$$i^*(C(F)) = C(f), \quad j^*(\mathcal{C}(F)) = \mathcal{C}(f).$$

The claim for $C(f)$ follows immediately from Piene's Lemma 3.1 and the claim for $\mathcal{C}(f)$ is a consequence of the first one and the commutative diagram (1):

$$\begin{aligned} j^*(\mathcal{C}(F)) &= j^*((F^*)^{-1}(C(F))) = (f^*)^{-1}(i^*(C(F))) \\ &= (f^*)^{-1}(C(f)) = \mathcal{C}(f). \end{aligned}$$

Now we need an ideal which gives a deformation of the Jacobian ideal $J(g)$. Let $G \in \mathcal{O}_{r+n+1}$ be such that $G = 0$ is a reduced equation of $(\mathcal{X}, 0)$ of

F and such that $j^*(G) = g$. It is not true that $j^*(J(G)) = j(g)$ since $J(G)$ contains the additional partial derivatives with respect to the parameters u_i . Instead of it, we consider the relative Jacobian ideal $J_y(G)$, that is, the ideal in \mathcal{O}_{r+n+1} generated by the partial derivatives of G with respect to the variables y_i , $i = 1, \dots, n+1$. Then, we get:

$$j^*(J_y(G)) = J(g).$$

Definition 4.1. We define $M_y(F)$ as the kernel of the epimorphism of \mathcal{O}_{r+n+1} -modules induced by F^* :

$$\frac{\mathcal{C}(F)}{J_y(G)} \longrightarrow \frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}}.$$

The main result of this section will be that the module $M_y(F)$ specialises to $M(f)$ when $u = 0$, that is,

$$M_y(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong M(f),$$

where \mathfrak{m}_r is the maximal ideal of \mathcal{O}_r , and the isomorphism is induced by the epimorphism j^* . Although the ideals $\mathcal{C}(F), C(F), J_y(G)$ specialise to $\mathcal{C}(f), C(f), J(g)$ as ideals, respectively, it is not so obvious from the definition that $M_y(F)$ specialises to $M(f)$.

From now on in this section and unless otherwise stated, the symbol \cong will be used to represent an isomorphism of modules induced by j^* .

Lemma 4.2. *For any r -parameter unfolding F of f , we have:*

$$\mathcal{C}(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \mathcal{C}(f),$$

and moreover $I \cdot \mathcal{C}(F) = I \cap \mathcal{C}(F)$, where $I = \mathfrak{m}_r \cdot \mathcal{O}_{r+n+1}$.

Proof. Since I is the kernel of j^* , we have $(\mathcal{C}(F) + I)/I = j^*(\mathcal{C}(F)) = \mathcal{C}(f)$ and from this we deduce:

$$\frac{\mathcal{O}_{r+n+1}}{\mathcal{C}(F)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} = \frac{\mathcal{O}_{r+n+1}}{\mathcal{C}(F) + I} = \frac{\mathcal{O}_{r+n+1}/I}{(\mathcal{C}(F) + I)/I} \cong \frac{\mathcal{O}_{n+1}}{\mathcal{C}(f)}.$$

Take the exact sequence of \mathcal{O}_r -modules

$$0 \longrightarrow \mathcal{C}(F) \longrightarrow \mathcal{O}_{r+n+1} \longrightarrow \frac{\mathcal{O}_{r+n+1}}{\mathcal{C}(F)} \longrightarrow 0$$

and consider the induced long exact Tor-sequence:

$$\dots \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_r} \left(\frac{\mathcal{O}_{r+n+1}}{\mathcal{C}(F)}, \frac{\mathcal{O}_r}{\mathfrak{m}_r} \right) \longrightarrow \mathcal{C}(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \longrightarrow \mathcal{O}_{n+1} \longrightarrow \frac{\mathcal{O}_{n+1}}{\mathcal{C}(f)} \longrightarrow 0.$$

By Remark 3.6, $\mathcal{O}_{r+n+1}/\mathcal{C}(F)$ is determinantal of dimension $r+n-1$. Then, it is Cohen-Macaulay and since the fibre $\mathcal{O}_{n+1}/\mathcal{C}(f)$ has dimension $n-1$, it is \mathcal{O}_r -flat. Therefore,

$$\mathrm{Tor}_1^{\mathcal{O}_r} \left(\frac{\mathcal{O}_{r+n+1}}{\mathcal{C}(F)}, \frac{\mathcal{O}_r}{\mathfrak{m}_r} \right) = 0,$$

and the above exact sequence implies

$$\mathcal{C}(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \mathcal{C}(f).$$

To show the second part, on one hand we have

$$\mathcal{C}(f) \cong \frac{\mathcal{C}(F) + I}{I} = \frac{\mathcal{C}(F)}{\mathcal{C}(F) \cap I}.$$

On the other hand, we have

$$\mathcal{C}(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} = \frac{\mathcal{C}(F)}{\mathfrak{m}_r \cdot \mathcal{C}(F)} = \frac{\mathcal{C}(F)}{I \cdot \mathcal{C}(F)},$$

and the result follows from the first part of the lemma. \square

An analogous proof shows the following:

Lemma 4.3. *For any r -parameter unfolding F of f , we have:*

$$C(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong C(f),$$

and moreover $L \cdot C(F) = L \cap C(F)$, where $L = \mathfrak{m}_r \cdot \mathcal{O}_{r+n}$.

Proposition 4.4. *For any r -parameter unfolding F of f , the following hold:*

- (1) $\frac{\mathcal{C}(F)}{J_y(G)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \frac{\mathcal{C}(f)}{J(g)}$,
- (2) $\frac{C(F)}{J_y(G) \cdot \mathcal{O}_{n+r}} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \frac{C(f)}{J(g) \cdot \mathcal{O}_n}$.

Proof. We show item (1), since the proof of item (2) is analogous. In order to simplify the notation, we write $\mathcal{C} := \mathcal{C}(F)$ and $J := J_y(G)$. By Lemma 4.2,

$$\begin{aligned} \frac{\mathcal{C}}{J} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} &\cong \frac{\mathcal{C}/J}{I \cdot (\mathcal{C}/J)} = \frac{\mathcal{C}/J}{(I \cdot \mathcal{C} + J)/J} \\ &= \frac{\mathcal{C}}{I \cap \mathcal{C} + J} = \frac{\mathcal{C}/I \cap \mathcal{C}}{(I \cap \mathcal{C} + J)/I \cap \mathcal{C}} \\ &= \frac{\mathcal{C}/I \cap \mathcal{C}}{J/I \cap \mathcal{C} \cap J} = \frac{\mathcal{C}/I \cap \mathcal{C}}{J/I \cap J} \\ &= \frac{(\mathcal{C} + I)/I}{(J + I)/I} \cong \frac{\mathcal{C}(f)}{J(g)}. \end{aligned}$$

\square

Next lemma shows that over the source ring \mathcal{O}_{r+n} , the Jacobian ideals $J(G)$ and $J_y(G)$ coincide.

Lemma 4.5. *For any r -parameter unfolding F of f , we have:*

$$J_y(G) \cdot \mathcal{O}_{r+n} = J(G) \cdot \mathcal{O}_{r+n}.$$

Proof. Let us write $F(u, x) = (u, f_u(x))$, then the Jacobian matrix of F has the following format:

$$dF = \begin{pmatrix} I_r & 0 \\ * & df_u \end{pmatrix},$$

where df_u is the Jacobian matrix of f_u , but considered with entries in \mathcal{O}_{r+n} . Denote by $M_1, \dots, M_r, M'_1, \dots, M'_{n+1}$ the $r+n$ -minors of dF in such a way

that M'_1, \dots, M'_{n+1} are the n -minors of df_u . Then M_1, \dots, M_r can be generated from the other minors M'_1, \dots, M'_{n+1} . That is, we can put

$$M_i = \sum_j a_{ij} M'_j,$$

for some $a_{ij} \in \mathcal{O}_{r+n}$. Now, by Piene's Lemma 3.1:

$$\frac{\partial G}{\partial u_i} \circ F = \Lambda M_i, \quad \frac{\partial G}{\partial y_j} \circ F = \Lambda M'_j,$$

where Λ is the generator of the conductor ideal $C(F)$. We have:

$$\frac{\partial G}{\partial u_i} \circ F = \sum_j a_{ij} \frac{\partial G}{\partial y_j} \circ F.$$

□

Now we arrive to the main result of this section.

Theorem 4.6. *If F is any r -parameter unfolding of f , then:*

$$M_y(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong M(f).$$

Proof. We have a short exact sequence coming from the definition of $M_y(F)$:

$$0 \longrightarrow M_y(F) \longrightarrow \frac{\mathcal{C}(F)}{J_y(G)} \longrightarrow \frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}} \longrightarrow 0.$$

By tensoring with $\mathcal{O}_r/\mathfrak{m}_r$ and by the results of Proposition 4.4, we obtain the following associated long exact Tor-sequence:

$$\begin{aligned} \dots &\longrightarrow \mathrm{Tor}_1^{\mathcal{O}_r} \left(\frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}}, \frac{\mathcal{O}_r}{\mathfrak{m}_r} \right) \\ &\longrightarrow M_y(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \longrightarrow \frac{\mathcal{C}(f)}{J(g)} \longrightarrow \frac{C(f)}{J(g) \cdot \mathcal{O}_n} \longrightarrow 0. \end{aligned}$$

We claim that $C(F)/J_y(G) \cdot \mathcal{O}_{r+n}$ is \mathcal{O}_r -flat. In fact, by Lemma 4.5,

$$\frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}} = \frac{C(F)}{J(G) \cdot \mathcal{O}_{r+n}} \cong \frac{\mathcal{O}_{r+n}}{R(F)},$$

where $R(F)$ is the ramification ideal and the isomorphism here is induced by multiplication of the generator of $C(F)$ (see Remark 3.7). But $\mathcal{O}_{r+n}/R(F)$ is determinantal of dimension $r+n-2$. Then, it is Cohen-Macaulay and since the fibre $\mathcal{O}_n/R(f)$ has dimension $n-2$, it is \mathcal{O}_r -flat. Therefore,

$$\mathrm{Tor}_1^{\mathcal{O}_r} \left(\frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}}, \frac{\mathcal{O}_r}{\mathfrak{m}_r} \right) = 0,$$

and from the above exact sequence we get:

$$M_y(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong M(f).$$

□

We finish this section with the next proposition, which gives the relative version of the short exact sequence of Proposition 3.3 for unfoldings. The proof is based on a commutative diagrama analogous to that appearing in the proof of 3.3, but with the relative version of the modules.

Proposition 4.7. *Let F be an r -parameter unfolding of f . We have the following exact sequence of \mathcal{O}_{r+n+1} -modules:*

$$0 \longrightarrow K_y(G) \longrightarrow M_y(F) \longrightarrow \frac{J_y(G) \cdot \mathcal{O}_{r+n}}{J_y(G) \cdot \mathcal{O}_{\mathcal{X},0}} \longrightarrow 0$$

where $K_y(G) := (\langle G \rangle + J_y(G))/J_y(G)$.

Remark 4.8. By using analogous arguments to those of Theorem 4.6, it is not difficult to prove that the module on the right hand side of the above exact sequence specialises to the module which controls the \mathcal{A}_e -codimension when $n \geq 2$. More precisely, we have

$$\frac{J_y(G) \cdot \mathcal{O}_{r+n}}{J_y(G) \cdot \mathcal{O}_{\mathcal{X},0}} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \frac{J(g) \cdot \mathcal{O}_n}{J(g) \cdot \mathcal{O}_{X,0}}.$$

The proof follows easily by using the short exact sequence:

$$0 \longrightarrow \frac{J_y(G) \cdot \mathcal{O}_{r+n}}{J_y(G) \cdot \mathcal{O}_{\mathcal{X},0}} \longrightarrow \frac{C(F)}{J_y(G) \cdot \mathcal{O}_{\mathcal{X},0}} \longrightarrow \frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}} \longrightarrow 0$$

After tensoring with $\mathcal{O}_r/\mathfrak{m}_r$ and taking into account that the module on the right hand side is \mathcal{O}_r -flat, we get the desired result.

It is not true in general that the module $K_y(G)$ specialises to $K(g)$. That is, we may have that

$$K_y(G) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \not\cong K(g).$$

In fact, by using the short exact sequence of Proposition 4.7, if $K_y(G) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong K(g)$, this would imply that $\frac{J_y(G) \cdot \mathcal{O}_{r+n}}{J_y(G) \cdot \mathcal{O}_{\mathcal{X},0}}$ is \mathcal{O}_r -flat. But it is obvious that this module is not flat when f is \mathcal{A} -finite and F is a stabilisation of f , since it is supported only at the origin.

5. AN EQUIVALENT DESCRIPTION OF THE MODULE $M(f)$

In this section we show a description of the modules $M(f)$ and $M_y(F)$ which is better suited for applications. Proposition 5.1 allows us to compute $M(f)$ easily using a computer algebra system, such as SINGULAR [5]. Since $K(g)$ can be computed as well, from Corollary 3.4 we obtain an expression for the \mathcal{A}_e -codimension of any map germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$, with $n \geq 2$.

Proposition 5.1. *Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be any map germ and F any r -parameter unfolding of f . Then:*

$$M(f) = \frac{(f^*)^{-1}(J(g) \cdot \mathcal{O}_n)}{J(g)},$$

$$M_y(F) = \frac{(F^*)^{-1}(J(G) \cdot \mathcal{O}_{r+n})}{J_y(G)}.$$

Proof. By construction, $M(f)$ is given by

$$M(f) = \frac{(f^*)^{-1}(J(g) \cdot \mathcal{O}_n) \cap \mathcal{C}(f)}{J(g)}.$$

But Lemma 3.1 implies the inclusion $J(g) \cdot \mathcal{O}_n \subset C(f)$, hence we have the inclusion

$$(f^*)^{-1}(J(g) \cdot \mathcal{O}_n) \subset (f^*)^{-1}(C(f)) = \mathcal{C}(f).$$

The proof for $M_y(F)$ is analogous, taking into account that $J_y(G) \cdot \mathcal{O}_{r+n}$ equals $J(G) \cdot \mathcal{O}_{r+n}$ by Lemma 4.5. \square

Corollary 5.2. *Let F be a stable unfolding of f . Then,*

$$M_y(F) = \frac{J(G) + \langle G \rangle}{J_y(G)}.$$

Proof. Since F is stable, $M(F) = K(G)$ by Proposition 3.3. Now, the first part Proposition 5.1 implies that $(F^*)^{-1}(J(G) \cdot \mathcal{O}_{r+n}) = J(G) + \langle G \rangle$ and the result follows from the second part of Proposition 5.1. \square

Definition 5.3. Let F be an unfolding of f . We say that G is a *good defining equation* for F if $G = 0$ is a reduced equation of the image of F and moreover $G \in J(G)$.

Note that there always exists a stable unfolding F which admits a good defining equation. In fact, if $F(u, x) = (u, f_u(x))$ is any r -parameter stable unfolding, then we take F' as the 1-parameter trivial unfolding of F , that is, $F'(t, u, x) = (t, u, f_u(x))$. Let $G = 0$ be a reduced equation of the image of F and take $G'(t, u, y) = e^t G(u, y)$. Then $G' = 0$ is a reduced equation of the image of F' and $\partial G' / \partial t = G'$, hence $G' \in J(G')$.

Corollary 5.4. *Let F be a stable unfolding of f and G a good defining equation for F . Then,*

$$M_y(F) = \frac{J(G)}{J_y(G)}.$$

Corollary 5.5. *Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be \mathcal{A} -finite, with $n \geq 2$ and $(n, n+1)$ nice dimensions. Let F be a stable unfolding of f , then*

$$\mathcal{A}_e\text{-codim}(f) = \dim_{\mathbb{C}} \left(\frac{J(G) + \langle G \rangle}{J_y(G)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \right) - \dim_{\mathbb{C}} \left(\frac{J(g) + \langle g \rangle}{J(g)} \right).$$

Proof. It follows immediately by putting together Corollary 3.5, Theorem 4.6, and Corollary 5.2. \square

Remark 5.6. Observe that if G is a good defining equation for F , then we have

$$\mathcal{A}_e\text{-codim}(f) = \dim_{\mathbb{C}} \left(\frac{J(G)}{J_y(G)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \right) - \dim_{\mathbb{C}} \left(\frac{J(g) + \langle g \rangle}{J(g)} \right).$$

If, moreover, f is weighted homogeneous, then

$$\mathcal{A}_e\text{-codim}(f) = \dim_{\mathbb{C}} \left(\frac{J(G)}{J_y(G)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \right).$$

6. FLATNESS AND THE COHEN-MACAULAY PROPERTY

In this section, we assume that $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ is a germ such that $\dim_{\mathbb{C}} M(f) < \infty$ and F is an r -parameter unfolding of f . By Theorem 4.6 and the Preparation Theorem, $M_y(F)$ is finite over \mathcal{O}_r . We consider a small enough representative $F: W \rightarrow T \times B_\epsilon$ with the properties required in Section 2 and such that the restriction of the projection onto the parameter space

$$\pi: \text{supp } \mathcal{M}_y(F) \rightarrow T$$

is finite and $\pi^{-1}(0) = \{0\}$. Here $\mathcal{M}_y(F)$ is the coherent sheaf of modules on $T \times B_\epsilon$ whose stalk at the origin is $M_y(F)$. We also denote by $M_y(F)_{(u,p)}$ the stalk of $\mathcal{M}_y(F)$ at $(u,p) \in T \times B_\epsilon$. We have the following standard fact from commutative algebra and analytic geometry:

Lemma 6.1. *The following assertions are equivalent:*

- (1) *the module $M_y(F)$ is flat over \mathcal{O}_r ,*
- (2) *the module $M_y(F)$ is free over \mathcal{O}_r ,*
- (3) *the number*

$$\Theta(u) := \sum_{p \in B_\epsilon} \dim_{\mathbb{C}} \left(M_y(F)_{(u,p)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} \right)$$

is independent of $u \in T$, where $\mathfrak{m}_{T,u}$ denotes the maximal ideal of $\mathcal{O}_{T,u}$,

- (4) *the module $M_y(F)$ is Cohen-Macaulay of dimension r .*

Recall that we have denoted by \mathcal{X} the image of F and by X_u the fibre of \mathcal{X} over $u \in T$. Given a point $p \in X_u \cap B_\epsilon$, in the next theorem we denote by $M(f_u)_p$ the module M computed for the germ of f_u at the point p .

Theorem 6.2. *Let F be an unfolding of a map germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$, with $\dim_{\mathbb{C}} M(f) < \infty$. We have the inequality:*

$$\dim_{\mathbb{C}} M(f) \geq \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p + b_n(X_u \cap B_\epsilon),$$

where $b_n(X_u \cap B_\epsilon)$ denotes the n -th Betti number of $X_u \cap B_\epsilon$. Moreover, the equality holds if and only if the module $M_y(F)$ is Cohen-Macaulay of dimension r (equivalently, if it is \mathcal{O}_r -flat).

Proof. By the upper semicontinuity of Θ we have

$$\Theta(0) \geq \Theta(u)$$

for any $u \in T$, with equality if and only if $M_y(F)$ is Cohen-Macaulay of dimension r , by Lemma 6.1. Let us identify both sides of the inequality.

The left hand side is equal to $\dim_{\mathbb{C}} M_y(F) \otimes (\mathcal{O}_r/\mathfrak{m}_r) = \dim_{\mathbb{C}} M(f)$ by Theorem 4.6. By the same reason, if $p \in X_u \cap B_\epsilon$, then

$$M_y(F)_{(u,p)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} = M(f_u)_p.$$

We split the sum $\Theta(u)$ as

$$\Theta(u) = \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p + \sum_{p \in B_\epsilon \setminus X_u} \dim_{\mathbb{C}} \left(M_y(F)_{(u,p)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} \right).$$

The first summand coincides with the first summand of the right hand side of the desired inequality. For the second summand we use the short exact sequence of Proposition 4.7. If $p \in B_\epsilon \setminus X_u$, then

$$M_y(F)_{(u,p)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} = K_y(G)_{(u,p)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} = \frac{\mathcal{O}_{T \times B_\epsilon, (u,p)}}{J_y(G)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} = \frac{\mathcal{O}_{B_\epsilon, p}}{J(g_u)},$$

which is the Jacobian algebra of g_u at p . Thus, the second summand is equal to

$$\sum_{p \in B_\epsilon \setminus X_u} \mu(g_u; p),$$

where $\mu(g_u; p)$ is the Milnor number of g_u at p . By Siersma's Theorem 2.2, the sum of all Milnor numbers $\mu(g_u; p)$, with $p \notin X_u$, is equal to the Betti number $b_n(X_u \cap B_\epsilon)$. \square

The above theorem has two interesting particular cases, namely, when the unfolding F is either a C^0 -stabilisation or a stable unfolding.

Corollary 6.3. *Let F be an unfolding of a map germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$, with $\dim_{\mathbb{C}} M(f) < \infty$. Assume F is either stable or a C^0 -stabilisation and let $\mathcal{B} \subset T$ be the bifurcation set. For any $u \in T \setminus \mathcal{B}$ the following inequality holds*

$$\dim_{\mathbb{C}} M(f) \geq \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p + \mu_I(f),$$

with equality if and only if $M_y(F)$ is Cohen-Macaulay of dimension r (equivalently \mathcal{O}_r -flat). In the nice dimensions we obtain

$$\dim_{\mathbb{C}} M(f) \geq \mu_I(f),$$

with equality if and only if $M_y(F)$ is Cohen-Macaulay of dimension r .

Remark 6.4. Since \mathfrak{m}_r is an ideal of parameters of $M_y(F)$, if we denote by $e(\mathfrak{m}_r; M_y(F))$ the multiplicity of $M_y(F)$ with respect to \mathfrak{m}_r we have

$$\dim_{\mathbb{C}} M(f) \geq e(\mathfrak{m}_r; M_y(F)),$$

with equality if and only if $M_y(F)$ is Cohen-Macaulay of dimension r .

The function Θ is Zariski upper-semicontinuous over T . By genericity of flatness we know that there exists a proper closed subset $\Delta \subset T$ such that $M_y(F)$ is flat over $T \setminus \Delta$ and such that the Θ jumps up precisely at $u \in \Delta$.

For all $u \notin \Delta$, by conservation of multiplicity, we have the equalities

$$\begin{aligned} e(\mathfrak{m}_r; M_y(F)) &= \sum_{p \in B_\epsilon} e(\mathfrak{m}_{T,u}; M_y(F)_{(u,p)}) \\ &= \sum_{p \in B_\epsilon} \dim_{\mathbb{C}} \left(M_y(F)_{(u,p)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} \right) \\ &= \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p + \sum_{p \in B_\epsilon \setminus X_u} \mu(g_u; p) \\ &= \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p + b_n(X_u \cap B_\epsilon). \end{aligned}$$

If, furthermore $u \notin \mathcal{B} \cup \Delta$, we have the equality

$$e(\mathfrak{m}_r; M_y(F)) = \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p + \mu_I(f).$$

This last equality can be rewritten as

$$e(\mathfrak{m}_r; M_y(F)) - \mu_I(f) = \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p.$$

Thus, the sum on the right hand side is independent of the generic parameter u . It can be proved that it is also independent of the stable unfolding. In fact, if we have another stable unfolding F' with parameters v_1, \dots, v_s , then we can take F'' as the sum of the two unfoldings F, F' . Since F' is stable, F'' is trivial and we can assume that F'' is constant on the parameters v_1, \dots, v_r . Then, v_1, \dots, v_r are a regular sequence for $M_y(F'')$, so

$$\begin{aligned} e(\langle u_1, \dots, u_r, v_1, \dots, v_s \rangle; M_y(F'')) &= e(\langle u_1, \dots, u_r \rangle; \frac{M_y(F'')}{\langle v_1, \dots, v_s \rangle M_y(F'')}) \\ &= e(\langle u_1, \dots, u_r \rangle; M_y(F)). \end{aligned}$$

We define:

$$\alpha(f) := \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p,$$

where F is any stable unfolding of f and $u \notin \mathcal{B} \cup \Delta$.

Note that $\alpha(f) = 0$ when $(n, n+1)$ are nice dimensions, since in that case all the singularities of f_u are \mathcal{A} -stable and are \mathcal{A} -equivalent to a weighted homogeneous map germ.

We can prove now two of our main results:

Theorem 6.5. *Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be a germ with $\dim_{\mathbb{C}} M(f) < \infty$. The following statements are equivalent:*

- (1) $\dim_{\mathbb{C}} M(f) = \alpha(f) + \mu_I(f)$,
- (2) $M_y(F)$ is Cohen-Macaulay of dimension r for some stable unfolding,
- (3) $M_y(F)$ is Cohen-Macaulay of dimension r for any stable unfolding,
- (4) $M_y(F)$ is Cohen-Macaulay of dimension 1 for any C^0 -stabilisation,
- (5) $\mathcal{O}_{r+n+1}/J_y(G)$ is \mathcal{O}_r -flat for some stable unfolding,
- (6) $\mathcal{O}_{r+n+1}/J_y(G)$ is \mathcal{O}_r -flat for any stable unfolding,
- (7) $\mathcal{O}_{n+2}/J_y(G)$ is \mathcal{O}_1 -flat for any C^0 -stabilisation.

Suppose that $(n, n+1)$ are nice dimensions, then the following two further statements are equivalent to the previous ones:

- (8) $M_y(F)$ is Cohen-Macaulay of dimension 1 for some stabilisation,
- (9) $\mathcal{O}_{n+2}/J_y(G)$ is \mathcal{O}_1 -flat for some stabilisation.

Proof. The equivalences between (1), (2) and (3) follow immediately from the previous results. Let us see (3) \Rightarrow (4). Let F be any C^0 -stabilisation of f given by $F = (t, f_t)$. We take $F' = (u, f'_u)$ a stable unfolding and consider $F'' = (t, u, f''_{t,u})$ as the sum of the two unfoldings F, F' . Since F' is stable, F'' is also stable and $M_y(F'')$ is Cohen-Macaulay of dimension $r+1$ by hypothesis. For any $t \neq 0$, the germ of $f_t = f''_{t,0}$ is C^0 -stable at any $p \in X_t$, hence $(t, 0) \notin \mathcal{B}$, the bifurcation set of F'' . By Corollary 6.3,

$$\dim_{\mathbb{C}} M(f) = \sum_{p \in X_t \cap B_\epsilon} \dim_{\mathbb{C}} M(f_t)_p + \mu_I(f),$$

therefore $M_y(F)$ is also Cohen-Macaulay of dimension 1.

We prove now that (4) \Rightarrow (1). Let F be any stable unfolding given by $F = (u, f_u)$. Choose an analytic path germ $\gamma: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^r, 0)$ such that $\gamma(t) \notin \mathcal{B} \cup \Delta$, for any $t \neq 0$. The unfolding γ^*F induced by γ is a

C^0 -stabilisation, hence $M_y(\gamma^*F)$ is Cohen-Macaulay of dimension 1. By Corollary 6.3, for any $u = \gamma(t)$ with $t \neq 0$,

$$\dim_{\mathbb{C}} M(f) = \sum_{p \in X_u \cap B_\epsilon} \dim_{\mathbb{C}} M(f_u)_p + \mu_I(f) = \alpha(f) + \mu_I(f).$$

The implication (4) \Rightarrow (8) is trivial and if $(n, n+1)$ are nice dimensions, then $\alpha(f) = 0$, hence we also have (8) \Rightarrow (1), by Corollary 6.3.

The remaining equivalences are all of them a consequence of the fact that $M_y(F)$ is flat over \mathcal{O}_r if and only if $\mathcal{O}_{r+n+1}/J_y(G)$ is flat over \mathcal{O}_r . In fact, we consider the exact sequence defining $M_y(F)$:

$$0 \longrightarrow M_y(F) \longrightarrow \frac{\mathcal{C}(F)}{J_y(G)} \longrightarrow \frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}} \longrightarrow 0.$$

Since the last module is \mathcal{O}_r -flat by Lemma 4.5 and Remark 3.7 the first module is \mathcal{O}_r -flat if and only if the second is. Now consider the exact sequence:

$$0 \longrightarrow \frac{\mathcal{C}(F)}{J_y(G)} \longrightarrow \frac{\mathcal{O}_{r+n+1}}{J_y(G)} \longrightarrow \frac{\mathcal{O}_{r+n+1}}{\mathcal{C}(F)} \longrightarrow 0.$$

The last module of the sequence is \mathcal{O}_r -flat by Remark 3.6, and hence the \mathcal{O}_r -flatness is equivalent for the first two modules of the sequence. \square

Definition 6.6. We say that a germ f has the *Cohen-Macaulay property* if $\dim_{\mathbb{C}} M(f) < \infty$ and $M_y(F)$ is Cohen-Macaulay of dimension r for some stable unfolding F .

Here is the relation with Mond's conjecture:

Theorem 6.7. *Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be \mathcal{A} -finite, with $(n, n+1)$ nice dimensions and $n \geq 2$. If f has the Cohen-Macaulay property then f satisfies Mond's conjecture. Moreover, if f is weighted homogeneous, then the converse is true.*

Proof. Suppose that f has the Cohen-Macaulay property. In Corollary 6.3 we proved the equality

$$\dim_{\mathbb{C}} M(f) = \alpha(f) + \mu_I(f),$$

but since we are in the nice dimensions, $\alpha(f) = 0$. Mond's conjecture for f follows now immediately from Corollary 3.4.

Suppose that f is a weighted homogeneous germ satisfying Mond's conjecture, Then $\dim_{\mathbb{C}} M(f) = \mu_I(f)$ by Corollary 3.4. Since $\alpha(f) = 0$, Corollary 6.3 implies that f has the Cohen-Macaulay property. \square

7. REDUCTION OF MOND'S CONJECTURE TO FAMILIES OF EXAMPLES

We exploit the results in previous section to reduce the general validity of Mond's conjecture for map germs to its validity in suitable families of examples.

We call multiplicity of a map germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ to the minimum of the multiplicities of the components (f_1, \dots, f_{n+1}) of f at all the points in S .

Theorem 7.1. *Assume that we are in the nice dimensions. Suppose that for any natural number M there exists a weighted homogenous \mathcal{A} -finite germ $h: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ of multiplicity at least M for which Mond's conjecture holds. Then Mond's conjecture holds for any \mathcal{A} -finite germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$.*

Proof. Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be \mathcal{A} -finite. By finite determinacy we may assume, up to right-left equivalence, that f is M -determined for a certain natural number M . Let h be the germ predicted by hypothesis. By M -determinacy, if we consider the 1-parameter family of germs $h_t := h + tf: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ we have that h_t is equivalent to f for any $t \neq 0$.

Let H be a stable and versal unfolding of h parametrised by a base T . Since Mond's conjecture holds for h and h is weighted homogeneous, by Theorem 6.7, the module $M_y(H)$ is T -flat. By versality, and because h_t is equivalent to f for any $t \neq 0$ we have that H is also a versal unfolding of f . Thus, applying again Theorem 6.7, we obtain Mond's conjecture for f . \square

A similar result can be proved if we want to study maps of a certain corank. For simplicity, we state the result only for mono-germs, although it can be easily adapted to multi-germs, if necessary. An \mathcal{A} -finite $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ of corank r is right-left equivalent to a map germ whose components admit the normal form

$$(x_1, \dots, x_{n-r}, f_{n-r+1}, \dots, f_{n+1}).$$

We call the *corank- r multiplicity* of f the minimum of the multiplicities of $f_{n-r+1}, \dots, f_{n+1}$.

Theorem 7.2. *Assume that we are in the nice dimensions. Suppose that for any natural number M there exists a weighted homogenous \mathcal{A} -finite germ $h: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ of corank r and of corank- r multiplicity at least M for which Mond's conjecture holds. Then Mond's conjecture holds for any \mathcal{A} -finite germ of corank r . $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$.*

Proof. The proof is entirely analogous to the proof of the previous theorem. The only modification is that one needs to put f and h in normal form before constructing the family h_t . \square

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