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Original article

Global dynamics of difference equations for SIR epidemic models with a class of nonlinear incidence rates

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In this paper, we propose a discrete SIR epidemic model whose discretization scheme preserves the global stability of equilibria for a class of continuous SIR epidemic models. From a biological motivation, the infection rate of the model is given by unspecified functions which incorporates a latency period with some distribution. By identifying the basic reproduction number R_0 of the model, the global asymptotic stability of the equilibria of the model is fully determined by applying discrete-time analogue of Lyapunov functionals when the infection rate has a suitable monotone property. Moreover, our result indicates that the latency period does not influence the global dynamics of the model.

Keywords: Difference equation; global asymptotic stability; SIR epidemic model; basic reproduction number; nonstandard finite discretization

AMS Subject Classification: 34K20; 34K25; 92D30

1. Introduction

The dynamics of epidemic models have received considerable attention. Various mathematical models have been proposed in the literature (see also [1–27] and the references therein). To investigate the dynamical behavior of the transmission of infectious diseases in a long time scale, the following basic SIR model was introduced

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by Hethcote [8].

$$\begin{cases} \dfrac{dS(t)}{dt} = \lambda - \mu S(t) - \beta S(t)I(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - (\mu+\gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu R(t), \end{cases} \tag{1.1}$$

$S(t)$, $I(t)$ and $R(t)$ denote the proportions of the susceptible, infective, recovered individuals, respectively. It is assumed that all newborns are susceptibles. λ, μ is the birth rate and death rate of the population, respectively. γ is a recovery rate, and β denotes an infection force. In general, SIR models have been thought of as appropriate frameworks for describing transmission of viral agent diseases such as measles, mumps, and smallpox [8].

Many authors have suggested that the standard bilinear incidence rate should be modified into a nonlinear incidence rate because the effect concerning the nonlinearity of incidence rates has been observed for some disease transmissions. For example, Capasso and Serio [4] studied the cholera epidemic spread in Bari in 1973 and introduced an incidence rate which takes a form $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$, and Brown and Hasibuan [3] studied infection model of the two-spotted spider mites, *Tetranychus urticae* and introduced an incidence rate which takes a form $(S(t)I(t))^b$. Thereafter, in order to study the impact of the nonlinearity, Korobeinikov and Maini [14] considered a variety of models with the incidence rate of the form $\phi(S(t))\psi(I(t))$ and constructed Lyapunov functions to establish global properties for some of SIR and SEIR models, and Korobeinikov [15, 16] also established global properties for a variety of epidemic models with the incidence rate of a more general form $f(S(t), I(t))$. Recently, based on the ideas in Korobeinikov and Maini [14], Huang *et al.* [9] incorporated a time delay which is caused by a latency period of the infection in a vector and studied the following SIR epidemic model with a general nonlinear incidence rate.

$$\begin{cases} \dfrac{dS(t)}{dt} = \lambda - \mu S(t) - \phi(S(t))\psi(I(t-\tau)), \\ \dfrac{dI(t)}{dt} = \phi(S(t))\psi(I(t-\tau)) - (\mu+\gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu R(t), \ \tau \geq 0, \end{cases} \tag{1.2}$$

The rate of new infection is characterized by $\phi(S)\psi(I)$, which includes some special incidence rates. For instance, if $\phi(S) = \beta S$ and $\psi(I) = I$, then the incidence becomes the standard bilinear form. Concerning the saturated incidence rate, if $\phi(S) = \beta S$ and $\psi(I) = \frac{I}{1+\alpha I}$, then the incidence rate is of the form proposed in [4, 19, 26], and other specific forms of $\phi(S)$ include $a(1-e^{-S})$, $\frac{1-e^{-cS}}{1+ae^{-cS}}$ and $\frac{cS^m}{a+S^m}$ with $m \geq 1$ and $\psi(I) = I$ (see Helmar and Wang [12] and the references therein). τ denotes an incubation time denoting the time during which the infectious agents develop in a vector (see also [1, 5]). Thereafter, Enatsu et al. [7] established the global stability of equilibria of an SIR epidemic model with distributed delays and a wider class of nonlinear incidence rates $\int_0^h p(\tau)\eta(S(t), I(t-\tau))d\tau$.

On the other hand, there occur situations such that constructing discrete epidemic models is more appropriate approach to understand disease transmission dynamics because they permit arbitrary time-step units. For example, Zhou *et al.*

[27] formulated a discrete mathematical model to investigate the transmission of severe acute respiratory syndrome (SARS) and their simulation results match the statistical data well and indicate that early quarantine and a high quarantine rate are crucial to the control of SARS. Discrete-time models are often directly applicable to time-series data and, in some cases, may more accurately represent contacts which are restricted to a specific time period.

The need for a discretization of cotinuous models also arises from the fundamental realization. Since nonlinear ordinary differential equations generally do not have analytic solutions expressible in terms of a finite representation of the elementary functions, technical discretization is required to calculate good analytic approximations of the solutions [21].

Jang and Elaydi [13] used nonstandard discretization technique and showed that the scheme preserves the global stability of a disease-free equilibrium and the local stability of an endemic equilibrium of the corresponding continuous-time SIS epidemic model.

In addition, Izzo and Vecchio [10] and Izzo *et al.* [11] introduced a variation of the backward Euler discretization, which is called "mixed type" formula, and showed that their scheme preserves the positivity and boundedness of the corresponding continuous-time population dynamics model. Based on their ideas, Sekiguchi [22] studied the permanence of a special class of discrete SIR epidemic models and some discrete epidemic models with delays by applying techniques in Wang [25].

However, how to choose the discrete schemes which preserve the global asymptotic stability for equilibria of the models was an open problem. In fact, it is known that the stability of a fixed point (equilibrium) will sometime change depending on a variation of central difference scheme (see, e.g., Roeger and Barnard [23] and the references therein).

Later, Enatsu *et al.* [6] established the complete global stability analysis for a discrete SIR epidemic model with a bilinear incidence rate. Their results agree with those for a corresponding continuous SIR epidemic models in McCluskey [18].

In this paper, we establish the global asymptotic stability of equilibria for a discrete SIR epidemic model with a class of nonlinear incidence rates in which a variation of the backward Euler method is adopted. The main idea of the discretization also derives from Enatsu *et al.* [6], and an application of nonstandard finite method given in Mickens [21]. Moreover, for the model, we can formulate a discrete-time analogue of Lyapunov functionals which are used for a class of continuous-time SIR epidemic models in [9, 15, 16, 18, 19]. This is the critical reason why a variation of the backward Euler method is applied and this discretization scheme is different from that of [10, 11, 22].

The organization of this paper is as follows. In Section 2, we introduce a discrete SIR epidemic model with a class of nonlinear incidence rates by applying a variation of backward Euler discretization, and establish our main results. In Section 3, we offer a basic result and investigate the existence and uniqueness of an endemic equilibrium E_* of the model. In Section 4, we obtain the global asymptotic stability of the disease-free equilibrium using a key lemma (see Lemma 4.1). In Section 5, we prove the permanence of the model, and obtain the global asymptotic stability of the endemic equilibrium by Lyapunov functional techniques using a key lemma (see Lemma 5.1). Finally, a concluding remark is offered in Section 6.

2. Main results

In this paper, by applying a variation of backward Euler discretization, we consider the following model.

$$\begin{cases} S(n+1)-S(n)=\lambda-\mu_1 S(n+1)-\phi(S(n+1))\displaystyle\sum_{j=0}^{m} f(j)\psi(I(n-j)),\\ I(n+1)-I(n)=\phi(S(n+1))\displaystyle\sum_{j=0}^{m} f(j)\psi(I(n-j))-(\mu_2+\gamma)I(n+1),\\ R(n+1)-R(n)=\gamma I(n+1)-\mu_3 R(n+1),\ n\geq 0, \end{cases} \tag{2.1}$$

where $\phi,\psi \in C^0(\mathbb{R}_+,\mathbb{R}_+)$, $\phi(0)=\psi(0)=0$ and $\lim_{I\to+0}(I/\psi(I))=1$. The initial condition of system (2.1) is as follows.

$$S(j)=\varphi_{1,j}\geq 0,\ I(j)=\varphi_{2,j}\geq 0,\ R(j)=\varphi_{3,j}\geq 0,\ j=-m,\cdots,0. \tag{2.2}$$

From a biological meaning, we further assume that $\varphi_{i,0}>0$ $(i=1,2,3)$. The parameters μ_i $(i=1,2,3)$ represent the death rates of susceptible, infective and recovered individuals, respectively and γ represents the recovery rate of infectives. The infection rate is given by

$$\phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j)),$$

where $\sum_{j=0}^{m} f(j)=1$, $f(j)\geq 0$ for $0\leq j\leq m$ and the meaning of $f(j)$ is derived from the fraction of vector population in which the maximum time taken to become infectious is m on continuous epidemic models with distributed delays (see, e.g., [1, 9, 17–19]). All the coefficients λ, γ and μ_i $(i=1,2,3)$ are assumed to be positive.

For system (2.1), Enatsu *et al.* [6] established a complete stability analysis for a special case $\phi(S)=\beta S$ and $\psi(I)=I$. We note that system (2.1) is a discrete analog of continuous system given in Huang *et al.* [9] with distributed delays.

We define the basic reproduction number R_0 of system (2.1) as follows:

$$R_0=\frac{\phi(\lambda/\mu_1)}{\mu_2+\gamma}.$$

$\frac{1}{\mu_2+\gamma}$ denotes the average infection period, and the relation that $\lim_{I\to+0}\frac{\phi(\lambda/\mu_1)\psi(I)}{I}=\phi(\lambda/\mu_1)$ implies that $\phi(\lambda/\mu_1)$ denotes the number of new cases infected per unit time by one infective individual at an initial infection state. Thus, R_0 denotes the expected number of secondary infectious cases generated by one typical primary case in an entirely susceptible and sufficiently large population which agrees with the threshold in Huang *et al.* [9].

System (2.1) always has a disease-free equilibrium $E_0=(S_0,0,0)$, $S_0=\frac{\lambda}{\mu_1}$ and if $R_0>1$, system (2.1) may admit an endemic equilibrium $E_*=(S^*,I^*,R^*)$, $S^*>0$, $I^*>0$, $R^*>0$ (see Section 3 for details).

Our main results are as follows.

Theorem 2.1 *Assume that the following conditions hold true.*

(H1) $\phi(S)$ *is strictly monotone increasing on* $S\geq 0$,
(H2) $I/\psi(I)$ *is monotone increasing on* $I>0$.

Then, for system (2.1), there is no endemic equilibrium and the disease-free equilibrium E_0 is globally asymptotically stable, if and only if, $R_0 \leq 1$.

Theorem 2.2 *Assume that the conditions (H1) and (H2) hold true. Then, there exists a unique endemic equilibrium E_* for system (2.1) if and only if $R_0 > 1$. Furthermore, if the following condition holds true,*

(H3) $\psi(I)$ is monotone increasing on $I \geq 0$,

then system (2.1) is permanent and the endemic equilibrium E_* of system (2.1) is globally asymptotically stable, if and only if, $R_0 > 1$.

We notice that $0 < \psi(I) \leq I$ holds for $I > 0$ under the condition (H2). The above results indicate that the global asymptotic stability of the equilibria of system (2.1) is determined for any length of time delay under the conditions (H1)-(H3). It is shown that the disease can be eradicated if and only if $R_0 \leq 1$ and the disease persists in a host population if and only if $R_0 > 1$. We further remark that the conditions (H1)-(H3) under which the global dynamics of system (2.1) are determined by the basic reprodution number R_0, are less restrictive than those in Huang *et al.* [9, Theorem 1].

3. Basic properties

For system (2.1), since the variable R does not appear in the first and the second equations, it is sufficient to consider the following 2-dimensional system.

$$\begin{cases} S(n+1) - S(n) = \lambda - \mu_1 S(n+1) - \phi(S(n+1)) \displaystyle\sum_{j=0}^{m} f(j)\psi(I(n-j)), \\ I(n+1) - I(n) = \phi(S(n+1)) \displaystyle\sum_{j=0}^{m} f(j)\psi(I(n-j)) - (\mu_2 + \gamma) I(n+1). \end{cases} \tag{3.1}$$

For the reduced system (3.1), the following results hold.

Lemma 3.1 *Let $(S(n), I(n))$ be a solution of system (3.1) with the initial condition (2.2). Then $S(n) > 0$, $I(n) > 0$ for all $n > 0$. Furthermore, any solution $(S(n), I(n))$ of system (3.1) satisfies $\limsup_{n\to+\infty}(S(n) + I(n)) \leq \lambda/\underline{\mu}$, where $\underline{\mu} = \min\{\mu_1, \mu_2 + \gamma\}$.*

Proof. From the initial condition (2.2) and the first equation of system (3.1), we have

$$S(1) + \mu_1 S(1) + \phi(S(1)) \sum_{j=0}^{m} f(j)\psi(\varphi_{2,-j}) = \lambda + \varphi_{1,0} > 0.$$

Then, we easily obtain that $S(1) > 0$. By the second equation of system (3.1),

$$(1 + \mu_2 + \gamma) I(1) = \varphi_{2,0} + \phi(S(1)) \sum_{j=0}^{m} f(j)\psi(\varphi_{2,-j}) > 0,$$

which implies that $I(1) > 0$. By repeating the above discussion, we obtain that $S(n) > 0$, $I(n) > 0$ for all $n > 0$.

We now define $V(n) = S(n) + I(n)$. From system (3.1), we have that

$$\begin{aligned} V(n+1) - V(n) &= \lambda - \mu_1 S(n+1) - (\mu_2+\gamma)I(n+1) \\ &\le \lambda - \underline{\mu} V(n+1), \end{aligned}$$

from which we have that $\limsup_{n\to+\infty} V(n) \le \frac{\lambda}{\underline{\mu}}$. Hence, the proof is complete. □

Remark 3.1 For any nonnegatigve initial values $\varphi_{i,j}$, for $i = 1,2$ and $j = -m,\ldots,0$, by a similar method as that used in Lemma 3.1, the following statements are true.

(i) The solution $(S(n), I(n))$ of (3.1) exists, and $S(n) > 0$ $(n > 0)$, $I(n) \ge 0$ $(n \ge 0)$.

(ii) If $\varphi_{2,0} + \sum_{j=0}^{m} f(j)\varphi_{2,-j} > 0$, then the solution $(S(n), I(n))$ of (3.1) exists, and $S(n) > 0$ $(n > 0)$, $I(n) > 0$ $(n > 0)$.

(iii) If $\varphi_{2,0} + \sum_{j=0}^{m} f(j)\varphi_{2,-j} = 0$, then the solution $(S(n), I(n))$ of (3.1) exists, and $S(n) > 0$ $(n > 0)$, $I(n) = 0$ $(n \ge 0)$.

Theorem 3.1 *Assume that the conditions (H1) and (H2) hold true. If $R_0 > 1$, then system (3.1) has a unique endemic equilibrium $E_* = (S^*, I^*)$, $S^* > 0$, $I^* > 0$ satisfying*

$$\lambda - \mu_1 S^* - \phi(S^*)\psi(I^*) = 0, \;\; \phi(S^*)\psi(I^*) - (\mu_2+\gamma)I^* = 0. \tag{3.2}$$

Moreover. if $R_0 \le 1$, system (3.1) has no endemic equilibrium and the disease-free equilibrium is the only equilibrium of system (3.1).

Proof. From the second equation of (3.2), by the implicit function theorem and the condition (H1), we see that $\phi(S)\psi(I) - (\mu_2+\gamma)I = 0$ defines a function $S = \zeta(I)$ on neighborhood around $I = 0$. It follows from (3.2) and (H2) that

$$\lim_{I\to+0} \phi(\zeta(I)) = \lim_{I\to+0} \frac{(\mu_2+\gamma)I}{\psi(I)} = \mu_2+\gamma < \phi\left(\frac{\lambda}{\mu_1}\right), \tag{3.3}$$

if $R_0 = \phi(\lambda/\mu_1)/(\mu_2+\gamma) > 1$. Therefore, by the condition (H1), we obtain that

$$\lim_{I\to+0} \zeta(I) < \frac{\lambda}{\mu_1}. \tag{3.4}$$

Furthermore, it follows from (H2) that the function $\zeta(I)$ is a monotone increasing function and either exists and is continuous for $I \in (0, \lambda/(\mu_2+\gamma)]$, or reaches infinity in this interval. After substituting the relations $S = \zeta(I)$ and $\phi(S)\psi(I) - (\mu_2+\gamma)I = 0$ into the first equation of (3.2), we consider the following equation.

$$H(I) \equiv \lambda - \mu_1\zeta(I) - (\mu_2+\gamma)I = 0,$$

from which we obtain that $H(I)$ is a strictly monotone decreasing function. By (3.4), we have that

$$\lim_{I\to+0} H(I) = \lambda - \mu_1 \lim_{I\to+0} \zeta(I) > \lambda - \mu_1\left(\frac{\lambda}{\mu_1}\right) = 0,$$

which implies that there exists a unique positive solution $0 < I^* \le \frac{\lambda}{\mu_2+\gamma}$ such that $H(I^*) = 0$. Therefore, there exists a unique endemic equilibrium $E_* = (S^*, I^*)$ of system (3.1) for $R_0 > 1$. One can immediately prove the second part of Theorem 3.1. Hence, the proof is complete. □

4. Global stability of the disease-free equilibrium E_0 for $R_0 \leq 1$

In this section, we prove that the disease-free equilibrium of system (3.1) is globally asymptotically stable. First, we introduce the following lemma which plays a key role such that Lyapunov functional techniques for continuous-time SIR epidemic models in Huang *et al.* [9], Korobeinikov [15, 16] and McCluskey [18, 19] is applicable.

Lemma 4.1 *Under the condition (H1), it holds that*

$$\int_{x_1}^{x_2} \frac{1}{\phi(s)} ds \geq \frac{x_2 - x_1}{\phi(x_2)},$$

for any $x_1 > 0$ *and* $x_2 > 0$.

Proof. For the first case $x_2 \geq x_1$, we immediately see that $\int_{x_1}^{x_2} \frac{1}{\phi(s)} ds \geq \int_{x_1}^{x_2} \frac{1}{\phi(x_2)} ds = \frac{x_2 - x_1}{\phi(x_2)}$. For the second case $x_2 < x_1$, we obtain that

$$\int_{x_1}^{x_2} \frac{1}{\phi(s)} ds = -\int_{x_2}^{x_1} \frac{1}{\phi(s)} ds \geq -\int_{x_2}^{x_1} \frac{1}{\phi(x_2)} ds = \frac{x_2 - x_1}{\phi(x_2)},$$

which completes the proof. □

Remark 4.1 If $\phi(s) = s$, then we obtain $\ln \frac{x_2}{x_1} \geq \frac{x_2 - x_1}{x_2}$ by Lemma 4.1.

Theorem 4.1 *Assume that the conditions (H1) and (H2) hold true. If* $R_0 \leq 1$, *then it holds that*

$$\lim_{n\to+\infty} S(n) = \frac{\lambda}{\mu_1}, \quad \lim_{n\to+\infty} I(n) = 0, \tag{4.1}$$

and $E_0 = (\frac{\lambda}{\mu_1}, 0)$ *of system (3.1) is globally asymptotically stable.*

Proof. From a Lyapunov functional for a continuous-time SIR epidemic model in Huang *et al.* [9], consider the following sequence $\{U^0(n)\}_{n=0}^{+\infty}$ defined by

$$U^0(n) = U_1^0(n) + I(n) + U_+^0(n), \tag{4.2}$$

where

$$U_1^0(n) = S(n) - S_0 - \int_{S_0}^{S(n)} \frac{\phi(S^0)}{\phi(s)} ds, \quad U_+^0(n) = \phi(S_0) \sum_{j=0}^{m} f(j) \sum_{k=n-j}^{n} \psi(I(k)).$$

We now show that $U^0(n+1) - U^0(n) \le 0$ for any $n \ge 0$. First, we calculate $U_1^0(n+1) - U_1^0(n)$. By using Lemma 4.1, we obtain that

$$\begin{aligned}
&U_1^0(n+1) - U_1^0(n)\\
&= S(n+1) - S(n) - \int_{S(n)}^{S(n+1)} \frac{\phi(S_0)}{\phi(s)} ds\\
&\le S(n+1) - S(n) - \phi(S_0)\frac{S(n+1) - S(n)}{\phi(S(n+1))}\\
&= \frac{\phi(S(n+1)) - \phi(S_0)}{\phi(S(n+1))}(S(n+1) - S(n))\\
&= \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right)\left\{\lambda - \phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j)) - \mu_1 S(n+1)\right\}.
\end{aligned} \tag{4.3}$$

Substituting $\lambda = \mu_1 S_0$ into (4.3), we see that

$$\begin{aligned}
&U_1^0(n+1) - U_1^0(n)\\
&\le \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right)\left\{\mu_1 S_0 - \phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j)) - \mu_1 S(n+1)\right\}\\
&= \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right)\left\{-\mu_1(S(n+1) - S_0) - \phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j))\right\}.
\end{aligned}$$

Second, calculating $U_+^0(n+1) - U_+^0(n)$, we get

$$\begin{aligned}
U_+^0(n+1) - U_+^0(n) &= \phi(S_0)\sum_{j=0}^{m} f(j)\left\{\sum_{k=n+1-j}^{n+1} \psi(I(k)) - \sum_{k=n-j}^{n} \psi(I(k))\right\}\\
&= \phi(S_0)\sum_{j=0}^{m} f(j)\{\psi(I(n+1)) - \psi(I(n-j))\}.
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
&U^0(n+1) - U^0(n)\\
&\le -\left(1-\frac{\phi(S_0)}{\phi(S(n+1))}\right)\left\{\mu_1(S(n+1)-S_0)+\phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j))\right\}\\
&+\phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j)) - (\mu_2+\gamma)I(n+1)\\
&+\phi(S_0)\sum_{j=0}^{m} f(j)\{\psi(I(n+1))-\psi(I(n-j))\}\\
&= -\mu_1 S(n+1)\left(1-\frac{\phi(S_0)}{\phi(S(n+1))}\right)\left(1-\frac{S_0}{S(n+1)}\right)+\phi(S_0)\psi(I(n+1))-(\mu_2+\gamma)I(n+1)\\
&= -\mu_1 S(n+1)\left(1-\frac{\phi(S_0)}{\phi(S(n+1))}\right)\left(1-\frac{S_0}{S(n+1)}\right)+(\mu_2+\gamma)\left\{R_0\frac{\psi(I(n+1))}{I(n+1)}-1\right\}I(n+1).
\end{aligned}$$

By the condition (H2), we finally obtain

$$\begin{aligned}
&U^0(n+1) - U^0(n)\\
&\le -\mu_1 S(n+1)\left(1-\frac{\phi(S_0)}{\phi(S(n+1))}\right)\left(1-\frac{S_0}{S(n+1)}\right)+(\mu_2+\gamma)(R_0-1)I(n+1).
\end{aligned}$$

By the condition (H1), we have $(1-\frac{\phi(S_0)}{\phi(S(n+1))})(1-\frac{S_0}{S(n+1)}) \le 0$ with equality if and only if $S(n+1)=S_0$, and hence $U^0(n+1)-U^0(n)\le 0$ for any $n \ge 0$ follows since we have $R_0 \le 1$. Then, $\{U^0(n)\}_{n=0}^{+\infty}$ is monotone decreasing sequence which implies that there exists a $\tilde{u}_0 \equiv \lim_{n\to+\infty} U^0(n) \ge 0$. Then, $\lim_{n\to+\infty}(U^0(n+1)-U^0(n)) = 0$ holds. For the case $R_0 < 1$, we immediately see that $\lim_{n\to+\infty} S(n+1) = S_0$ and $\lim_{n\to+\infty} I(n+1) = 0$. On the other hand, for the case $R_0 = 1$, we see that $\lim_{n\to+\infty} S(n+1) = S_0$, from which we obtain $\lim_{n\to+\infty} I(n-j) = 0$ for $0 \le j \le m$ by (3.1). Hence, it holds that $\lim_{n\to+\infty}(S(n), I(n)) = (\frac{\lambda}{\mu_1}, 0)$ if $R_0 \le 1$. Since $U^0(n) \le U^0(0)$ for all $n \ge 0$ holds, we see that E_0 is uniformly stable. This completes the proof. □

Lemma 4.2 *(4.1) implies* $R_0 \le 1$.

Proof. Suppose that $R_0 > 1$. Then, by Theorem 3.1, one can see that there exists a positive constant solution $(S(n), I(n)) = (S^*, I^*)$ of system (3.1), which contradicts the fact that (4.1) holds. Hence, we obtain the conclusion of this lemma. □

Proof of Theorem 2.1. By Theorems 3.1 and 4.1 and Lemma 4.2, we immediately obtain the conclusion of Theorem 2.1. □

5. Global stability of the endemic equilibrium E_* for $R_0 > 1$

In this section, we assume that $R_0 > 1$. First, in order to prove Theorem 2.2 for system (2.1), we show the permanence and the global asymtotic stability of the endemic equilibrium of the reduced system (3.1) for $R_0 > 1$.

5.1 Permanence for $R_0 > 1$

For $0 < q < \frac{\psi(I^*)}{I^*}$, we put $S^\triangle > S^*$ satisfying

$$S^\triangle(1+\mu_1) + \phi(S^\triangle)qI^* = S^*(1+\mu_1) + \phi(S^*)\psi(I^*). \tag{5.1}$$

Setting $F(s) \equiv s(1+\mu_1)+\phi(s)qI^*$, it follows that $F(S^*) = S^*(1+\mu_1)+\phi(S^*)qI^* < S^*(1+\mu_1)+\phi(S^*)\psi(I^*)$ and $\lim_{s\to+\infty} F(s) = +\infty$. The above discussion guarantees the existence of $S^\triangle$.

We now prove the permanence of (3.1). The proof of the following theorem is based on Enatsu *et al.* [6] and Sekiguchi [22]. From Theorem 5.1 below, the disease eventually persists in the host population if $R_0 > 1$.

Theorem 5.1 *Assume that the conditions (H1)-(H3) hold true. If $R_0 > 1$, for any solution of system (3.1), it holds that*

$$\liminf_{n\to+\infty} S(n) \geq \underline{v}_1 > 0, \ \liminf_{n\to+\infty} I(n) \geq \underline{v}_2 := \left(\frac{1}{1+\mu_2+\gamma}\right)^{m+l_0} qI^* > 0, \tag{5.2}$$

where $\underline{v}_1 > 0$ satisfies $\lambda - \mu_1\underline{v}_1 - \phi(\underline{v}_1)\psi(\frac{\lambda}{\mu_1}) = 0$, and $0 < q < \frac{\psi(I^*)}{I^*}$ and $l_0 \geq 1$ satisfy

$$S^* \leq \frac{1}{\mu_1}\left\{1 - \left(\frac{1}{1+\mu_1}\right)^{m+l_0}\right\}\{\lambda - \phi(S^*)qI^*\}. \tag{5.3}$$

Proof. The existence of q and l_0 is guaranteed, because it follows from (3.2) that $\frac{1}{\mu_1}\{\lambda - \phi(S^*)qI^*\} = S^* + \frac{\phi(S^*)}{\mu_1}(\psi(I^*) - qI^*) > S^*$. By the first equation of (3.1) and Lemma 3.1, for any $\varepsilon > 0$, there is an integer $N_\varepsilon \geq 0$ such that

$$\begin{aligned} S(n+1) - S(n) &= \lambda - \mu_1 S(n+1) - \phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j)) \\ &\geq \lambda - \mu_1 S(n+1) - \phi(S(n+1))\psi\left(\frac{\lambda}{\mu_1} + \varepsilon\right), \end{aligned} \tag{5.4}$$

for $n \geq N_\varepsilon + m$. Let us now consider the auxiliary equation $\underline{S}(n+1) - \underline{S}(n) = \lambda - \mu_1\underline{S}(n+1) - \phi(\underline{S}(n+1))\psi(\frac{\lambda}{\mu_1})$. Then, by the condition (H1), one can immediately obtain that $\lim_{n\to+\infty} \underline{S}(n) = \underline{v}_1 > 0$. Since (5.4) holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $\liminf_{n\to+\infty} S(n) \geq \underline{v}_1 > 0$.

We now show that $\liminf_{n\to+\infty} I(n) \geq \underline{v}_2 > 0$ holds. First, we claim that it is impossible that, for any solution $(S(n), I(n))$ of system (3.1), there exists a nonnegative integer $p_0 \geq m$ such that $I(n) \leq qI^*$ for all $n \geq p_0 - m$. Suppose on the contrary that there exist a solution $(S(n), I(n))$ of system (3.1) and a nonnegative integer $p_0 \geq m$ such that $I(n) \leq qI^*$ for all $n \geq p_0 - m$. From the first equation of system (3.1), one can obtain that,

$$
\begin{aligned}
&S(n+1)\\
&=\frac{1}{1+\mu_1}\left\{S(n)+\lambda-\phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j))\right\}\\
&=\frac{1}{1+\mu_1}\left[\frac{1}{1+\mu_1}\left\{S(n-1)+\lambda-\phi(S(n))\sum_{j=0}^{m} f(j)\psi(I(n-1-j))\right\}\right]\\
&\quad+\frac{\lambda}{1+\mu_1}-\frac{\phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j))}{1+\mu_1}\\
&=\left(\frac{1}{1+\mu_1}\right)^2 S(n-1)+\lambda\left\{\frac{1}{1+\mu_1}+\left(\frac{1}{1+\mu_1}\right)^2\right\}\\
&\quad-\frac{\phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j))}{1+\mu_1}-\frac{\phi(S(n))\sum_{j=0}^{m} f(j)\psi(I(n-1-j))}{(1+\mu_1)^2}.
\end{aligned}
$$

By repeating the above discussion, for $n \geq p_0$, we have

$$
\begin{aligned}
S(n+1)=&\left(\frac{1}{1+\mu_1}\right)^{n-p_0+1} S(p_0)+\frac{\lambda}{\mu_1}\left\{1-\left(\frac{1}{1+\mu_1}\right)^{n-p_0+1}\right\}\\
&-\sum_{l=1}^{n-p_0+1}\left(\frac{1}{1+\mu_1}\right)^{l}\phi(S(n+2-l))\sum_{j=0}^{m} f(j)\psi(I(n+1-l-j)).
\end{aligned}
\tag{5.5}
$$

Here, we suppose that $S(n) \leq S^*$, for any $p_0+1 \leq n \leq p_0+m+l_0$. Then, by (5.5), we have

$$
\begin{aligned}
S(p_0+m+l_0) >& \frac{\lambda}{\mu_1}\left\{1-\left(\frac{1}{1+\mu_1}\right)^{m+l_0}\right\}-\sum_{l=1}^{m+l_0}\left(\frac{1}{1+\mu_1}\right)^{l}\phi(S^*)\psi(qI^*)\\
=&\frac{1}{\mu_1}\left\{1-\left(\frac{1}{1+\mu_1}\right)^{m+l_0}\right\}\{\lambda-\phi(S^*)\psi(qI^*)\}\\
\geq&\frac{1}{\mu_1}\left\{1-\left(\frac{1}{1+\mu_1}\right)^{m+l_0}\right\}\{\lambda-\phi(S^*)qI^*\} \geq S^*,
\end{aligned}
$$

which is a contradiction. Therefore, there exists an integer $\tilde{p}$ such that $p_0+1 \leq$

$\tilde{p} \leq p_0 + m + l_0$ and $S(\tilde{p}) > S^*$. By the first equation of (3.1), we have that

$$\begin{aligned}(1+\mu_1)S^* + \phi(S^*)\psi(I^*) &= \lambda + S^* \\ &< \lambda + S(\tilde{p}) \\ &= (1+\mu_1)S(\tilde{p}+1) + \phi(S(\tilde{p}+1))\sum_{j=0}^{m} f(j)\psi(I(\tilde{p}-j)) \\ &\leq (1+\mu_1)S(\tilde{p}+1) + \phi(S(\tilde{p}+1))\psi(qI^*) \\ &\leq (1+\mu_1)S(\tilde{p}+1) + \phi(S(\tilde{p}+1))qI^*,\end{aligned}$$

which is equivalent to $S(\tilde{p}+1) > S^{\triangle} > S^*$. Hence, we obtain that $S(n) \geq S^{\triangle} > S^*$, for any $n \geq p_0 + m + l_0 + 1$. Noting that $I(n) \leq qI^*$ for all $n \geq p_0 - m$, define the sequence $\{w(n)\}_{n=p_0}^{+\infty}$ as

$$w(n) = I(n) + \sum_{j=0}^{m} f(j) \sum_{k=n-j}^{n} \phi(S(j+k+1))\psi(I(k)). \tag{5.6}$$

By the conditions (H1) and (H2), we have that

$$\begin{aligned}&w(n+1) - w(n) \\ &= I(n+1) - I(n) \\ &\quad + \sum_{j=0}^{m} f(j) \left\{ \sum_{k=n+1-j}^{n+1} \phi(S(j+k+1))\psi(I(k)) - \sum_{k=n-j}^{n} \phi(S(j+k+1))\psi(I(k)) \right\} \\ &= \phi(S(n+1)) \sum_{j=0}^{m} f(j)\psi(I(n-j)) - (\mu_2+\gamma)I(n+1) \\ &\quad + \sum_{j=0}^{m} f(j) \left\{ \phi(S(n+2+j))\psi(I(n+1)) - \phi(S(n+1))\psi(I(n-j)) \right\}\end{aligned}$$

Then, we obtain that

$$\begin{aligned}w(n+1) - w(n) &= \sum_{j=0}^{m} f(j)\phi(S(n+2+j))\psi(I(n+1)) - (\mu_2+\gamma)I(n+1) \\ &> \left\{ \phi(S^{\triangle}) - (\mu_2+\gamma)\frac{I(n+1)}{\psi(I(n+1))} \right\} \psi(I(n+1)) \\ &\geq \left\{ \phi(S^{\triangle}) - (\mu_2+\gamma)\frac{I^*}{\psi(I^*)} \right\} \psi(I(n+1)) \\ &= \left\{ \phi(S^{\triangle}) - \phi(S^*) \right\} \psi(I(n+1)),\end{aligned} \tag{5.7}$$

for $n \geq p_0+m+l_0-1$. Now, we set $\hat{i} = \min_{\theta\in[-m,0]} I(\theta+p_0+2m+l_0)$ and claim that $I(n) \geq \hat{i}$ for all $n \geq p_0 + m + l_0$. Otherwise, if there is a $T_1 \geq 0$ such that $I(n) \geq \hat{i}$ for $p_0 + m + l_0 \leq n \leq p_0 + 2m + l_0 + T_1$ and $0 < \underline{i} \equiv I(p_0 + 2m + l_0 + T_1 + 1) < \hat{i}$, it follows from the conditions (H1), (5.3) and the second equation of system (3.1) that, for $n_1 = p_0 + 2m + l_0 + T_1$,

$$\begin{aligned}
&I(n_1+1) - I(n_1)\\
&= \phi(S(n_1+1))\sum_{j=0}^{m} f(j)\psi(I(n_1-j)) - (\mu_2+\gamma)I(n_1+1)\\
&\geq \phi(S(n_1+1))\psi(I(n_1+1)) - (\mu_2+\gamma)I(n_1+1)\\
&= \left\{\phi(S(n_1+1)) - (\mu_2+\gamma)\frac{I(n_1+1)}{\psi(I(n_1+1))}\right\}\psi(I(n_1+1))\\
&\geq \left\{\phi(S^{\triangle}) - (\mu_2+\gamma)\frac{I^*}{\psi(I^*)}\right\}\psi(I(n_1+1))\\
&= \left\{\phi(S^{\triangle}) - \phi(S^*)\right\}\psi(\underline{i}) > 0.
\end{aligned} \tag{5.8}$$

Therefore, $I(n) \geq \hat{i}$ holds for all $n \geq p_0 + m + l_0$. It follows from (5.7) that

$$w(n+1) - w(n) > \left\{\phi(S^{\triangle}) - \phi(S^*)\right\}\psi(\hat{i}) > 0, \text{ for } n \geq p_0 + m + l_0,$$

which implies that $\lim_{n\to+\infty} w(n) = +\infty$. However, by (5.6) and Lemma 3.1, it holds that there is a positive constant $\bar{p} \geq p_0 + m + l_0$ and $\bar{w}$ such that $w(n) \leq \bar{w}$ for any $n \geq \bar{p}$, which leads to a contradiction. Hence, the claim holds.

By the claim, we are left to consider two possibilities.

$$\begin{cases} (i)\ I(n) \geq qI^* \text{ for all } n > 0 \text{ sufficiently large},\\ (ii)\ I(n) \text{ oscillates about } qI^* \text{ for all } n > 0 \text{ sufficiently large}. \end{cases}$$

We now show that $I(n) \geq v_2$ for all n sufficiently large. If the first case holds, then we immediately get the conclusion of the theorem. If the second case holds, let $p_1 < p_2$ be sufficiently large such that

$$I(p_1) > qI^*,\ I(p_2) > qI^*, \text{ and } I(n) \leq qI^*, \text{ for any } p_1 < n < p_2.$$

Since, from the second equation of system (2.1), it follows that $I(n+1) - I(n) \geq -(\mu_2+\gamma)I(n+1)$, $n \geq p_1$, we have

$$I(n+1) \geq \frac{1}{1+\mu_2+\gamma}I(n), \text{ for any } n \geq p_1,$$

from which we have that

$$I(n+1) \geq \left(\frac{1}{1+\mu_2+\gamma}\right)^{n+1-p_1} I(p_1) \geq \left(\frac{1}{1+\mu_2+\gamma}\right)^{n+1-p_1} qI^*, \text{ for any } n \geq p_1.$$

Therefore, if $p_2 \leq p_1 + m + l_0$, one can easily obtain that

$$I(n+1) \geq \left(\frac{1}{1+\mu_2+\gamma}\right)^{m+l_0} qI^* = v_2, \text{ for any } p_1 \leq n \leq p_2. \tag{5.9}$$

If $p_2 > p_1 + m + l_0$, suppose on the contrary that there exists a $p_1 + m + l_0 < T_2 \leq p_2$ such that $I(T_2+1) < v_2 \leq I(T_2)$ holds. Then, by the similar discussion to (5.8), this leads to a contradiction. Thus, we obtain that $I(n+1) \geq v_2$ for $p_1 + m + l_0 \leq n \leq p_2$.

Hence, we prove that $I(n+1) \geq v_2$ for $p_1 \leq n \leq p_2$. Since the interval $[p_1, p_2]$ is arbitrarily chosen, we conclude that $I(n+1) \geq v_2$ for all $n \geq p_1$, which implies that $\liminf_{n\to+\infty} I(n) \geq v_2$ holds. This completes the proof. □

5.2 *Global asymptotic stability of E_* for $R_0 > 1$*

We introduce the following lemma which plays a crucial role to establish Theorem 2.2.

Lemma 5.1 *Assume that the conditions (H2) and (H3) hold true. If $R_0 > 1$, then it holds that for all $n \geq 0$,*

$$g\left(\frac{I(n)}{I^*}\right) - g\left(\frac{\psi(I(n))}{\psi(I^*)}\right) \geq 0, \tag{5.10}$$

where $g(x) = x - 1 - \ln x \geq g(1) = 0$ defined on $x > 0$.

Proof. By the conditions (H2) and (H3), we obtain

$$\begin{aligned} &\left(\frac{\psi(I(n))}{\psi(I^*)} - 1\right)\left(\frac{I(n)}{I^*} - \frac{\psi(I(n))}{\psi(I^*)}\right) \\ = &\frac{\psi(I(n))}{I^*\psi(I^*)}\left(\frac{I(n)}{\psi(I(n))} - \frac{I^*}{\psi(I^*)}\right)(\psi(I(n)) - \psi(I^*)) \geq 0. \end{aligned} \tag{5.11}$$

(5.11) implies that either

$$\frac{I(n)}{I^*} \leq \frac{\psi(I(n))}{\psi(I^*)} \leq 1, \text{ or } \frac{I(n)}{I^*} \geq \frac{\psi(I(n))}{\psi(I^*)} \geq 1 \tag{5.12}$$

holds for all $n \geq 0$. Since $g'(x) = 1 - \frac{1}{x}$ for all $x > 0$ and $g'(1) = 0$, it holds that $g(\frac{I(n)}{I^*}) \geq g(\frac{\psi(I(n))}{\psi(I^*)}) \geq 0$ if (5.12) holds. Thus, we get the conclusion. □

We now establish the global asymptotic stability of the endemic equilibrium E_* of system (3.1).

Theorem 5.2 *Assume that the conditions (H1)-(H3) hold true. If $R_0 > 1$, then it holds that*

$$\lim_{n\to+\infty} S(n) = S^*, \quad \lim_{n\to+\infty} I(n) = I^*, \tag{5.13}$$

and $E_* = (S^*, I^*)$ of system (3.1) is globally asymptotically stable.

Proof. Let

$$\tilde{s}_n = \frac{\phi(S(n))}{\phi(S^*)}, \quad i_n = \frac{I(n)}{I^*}, \text{ and } \tilde{i}_n = \frac{\psi(I(n))}{\psi(I^*)}.$$

From Lyapunov functionals for continuous-time SIR epidemic models in Huang *et al.* [9], Korobeinikov [15, 16] and McCluskey [18, 19], consider the following sequence $\{U^*(n)\}_{n=m}^{+\infty}$ defined by

$$U^*(n) = \frac{1}{\phi(S^*)\psi(I^*)}U_1^*(n) + \frac{I^*}{\phi(S^*)\psi(I^*)}U_2^*(n) + U_+^*(n). \tag{5.14}$$

where

$$\begin{cases} U_1^*(n) = S(n) - S^* - \int_{S^*}^{S(n)} \frac{\phi(S^*)}{\phi(s)} ds, \ U_2^*(n) = g\left(\frac{I(n)}{I^*}\right), \\ U_+^*(n) = \sum_{j=0}^{m} f(j) \sum_{k=n-j}^{n} g\left(\frac{\psi(I(k))}{\psi(I^*)}\right). \end{cases} \tag{5.15}$$

Let us show that $U^*(n+1) - U^*(n) \leq 0$ for any $n \geq m$. First, we calculate $U_1^*(n+1) - U_1^*(n)$. By using the relation in Lemma 4.1, we obtain that

$$\begin{aligned} & U_1^*(n+1) - U_1^*(n) \\ = & (S(n+1) - S(n)) - \int_{S(n)}^{S(n+1)} \frac{\phi(S^*)}{\phi(s)} ds \\ \leq & (S(n+1) - S(n)) - \phi(S^*) \frac{S(n+1) - S(n)}{\phi(S(n+1))} \\ = & \frac{\phi(S(n+1)) - \phi(S^*)}{\phi(S(n+1))} (S(n+1) - S(n)) \\ = & \frac{\phi(S(n+1)) - \phi(S^*)}{\phi(S(n+1))} \left\{ \lambda - \phi(S(n+1)) \sum_{j=0}^{m} f(j)\psi(I(n-j)) - \mu_1 S(n+1) \right\}. \end{aligned} \tag{5.16}$$

Substituting $\lambda = \mu_1 S^* + \phi(S^*)\psi(I^*)$ into (5.16), we see that

$$\begin{aligned} & U_1^*(n+1) - U_1^*(n) \\ \leq & \left(1 - \frac{\phi(S^*)}{\phi(S(n+1))}\right) \\ & \times \{\mu_1 S^* + \phi(S^*)\psi(I^*) - \phi(S(n+1)) \sum_{j=0}^{m} f(j)\psi(I(n-j)) - \mu_1 S(n+1)\} \\ = & -\mu_1 \left(1 - \frac{\phi(S^*)}{\phi(S(n+1))}\right)(S(n+1) - S^*) \\ & + \phi(S^*)\psi(I^*) \sum_{j=0}^{m} f(j) \left(1 - \frac{\phi(S^*)}{\phi(S(n+1))}\right) \left(1 - \frac{\phi(S(n+1))}{\phi(S^*)} \cdot \frac{\psi(I(n-j))}{\psi(I^*)}\right) \\ = & -\mu_1 S(n+1) \left(1 - \frac{\phi(S^*)}{\phi(S(n+1))}\right) \left(1 - \frac{S^*}{S(n+1)}\right) \\ & + \phi(S^*)\psi(I^*) \sum_{j=0}^{m} f(j) \left(1 - \frac{1}{\tilde{s}_{n+1}}\right)(1 - \tilde{s}_{n+1}\tilde{i}_{n-j}). \end{aligned}$$

Second, we similarly calculate $U_2^*(n+1) - U_2^*(n)$. By using the relation given in Remark 4.1, we obtain that

$$\begin{aligned}
&U_2^*(n+1) - U_2^*(n)\\
&= \frac{I(n+1) - I(n)}{I^*} - \ln\frac{I(n+1)}{I(n)}\\
&\le \frac{I(n+1) - I(n)}{I^*} - \frac{I(n+1) - I(n)}{I(n+1)}\\
&= \frac{1}{I^*}\frac{I(n+1) - I^*}{I(n+1)}(I(n+1) - I(n))\\
&= \frac{1}{I^*}\frac{I(n+1) - I^*}{I(n+1)}\{\phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j)) - (\mu_2+\gamma)I(n+1)\}.
\end{aligned}$$

Since we have $\mu_2 + \gamma = \frac{\phi(S^*)\psi(I^*)}{I^*}$, it follows that

$$\begin{aligned}
&U_2^*(n+1) - U_2^*(n)\\
&\le \frac{1}{I^*}\frac{I(n+1) - I^*}{I(n+1)}\left\{\phi(S(n+1))\sum_{j=0}^{m} f(j)\psi(I(n-j)) - \frac{\phi(S^*)\psi(I^*)}{I^*}I(n+1)\right\}\\
&= \frac{\phi(S^*)\psi(I^*)}{I^*}\sum_{j=0}^{m} f(j)\left(1 - \frac{I^*}{I(n+1)}\right)\left(\frac{\phi(S(n+1))}{\phi(S^*)}\cdot\frac{\psi(I(n-j))}{\psi(I^*)} - \frac{I(n+1)}{I^*}\right)\\
&= \frac{\phi(S^*)\psi(I^*)}{I^*}\sum_{j=0}^{m} f(j)\left(1 - \frac{1}{i_{n+1}}\right)(\tilde{s}_{n+1}\tilde{i}_{n-j} - i_{n+1}).
\end{aligned}$$

Finally, calculating $U_+^*(n+1) - U_+^*(n)$, we get that

$$\begin{aligned}
U_+^*(n+1) - U_+^*(n) &= \sum_{j=0}^{m} f(j)\left\{\sum_{k=n+1-j}^{n+1} g\left(\frac{\psi(I(k))}{\psi(I^*)}\right) - \sum_{k=n-j}^{n} g\left(\frac{\psi(I(k))}{\psi(I^*)}\right)\right\}\\
&= \sum_{j=0}^{m} f(j)\left\{g\left(\frac{\psi(I(n+1))}{\psi(I^*)}\right) - g\left(\frac{\psi(i(n-j))}{\psi(i^*)}\right)\right\}\\
&= \sum_{j=0}^{m} f(j)g(\tilde{i}_{n+1}) - \sum_{j=0}^{m} f(j)g(\tilde{i}_{n-j}).
\end{aligned}$$

Then, we have that

$$\begin{aligned}
U^*(n+1) - U^*(n) \le &-\frac{\mu_1 S(n+1)}{\phi(S^*)\psi(I^*)}\left(1 - \frac{\phi(S^*)}{\phi(S(n+1))}\right)\left(1 - \frac{S^*}{S(n+1)}\right)\\
&-\sum_{j=0}^{m} f(j)\left\{-\left(1 - \frac{1}{\tilde{s}_{n+1}}\right)(1 - \tilde{s}_{n+1}\tilde{i}_{n-j})\right.\\
&\left.-\left(1 - \frac{1}{i_{n+1}}\right)(\tilde{s}_{n+1}\tilde{i}_{n-j} - i_{n+1}) - g(\tilde{i}_{n+1}) + g(\tilde{i}_{n-j})\right\}.
\end{aligned}$$

Since

$$
\begin{aligned}
&-\left(1-\frac{1}{\tilde{s}_{n+1}}\right)(1-\tilde{s}_{n+1}\tilde{i}_{n-j})-\left(1-\frac{1}{i_{n+1}}\right)(\tilde{s}_{n+1}\tilde{i}_{n-j}-i_{n+1})-g(\tilde{i}_{n+1})+g(\tilde{i}_{n-j})\\
&=-\left(1-\tilde{s}_{n+1}\tilde{i}_{n-j}-\frac{1}{\tilde{s}_{n+1}}+\tilde{i}_{n-j}\right)-\left(\tilde{s}_{n+1}\tilde{i}_{n-j}-i_{n+1}-\frac{\tilde{s}_{n+1}\tilde{i}_{n-j}}{i_{n+1}}+1\right)\\
&\quad-\tilde{i}_{n+1}+\tilde{i}_{n-j}+\ln\tilde{i}_{n+1}-\ln\tilde{i}_{n-j}\\
&=-2+\frac{1}{\tilde{s}_{n+1}}+\frac{\tilde{s}_{n+1}\tilde{i}_{n-j}}{i_{n+1}}+\ln\tilde{i}_{n+1}-\ln\tilde{i}_{n-j},
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
U^*(n+1)-U^*(n)\le&-\frac{\mu_1 S(n+1)}{\phi(S^*)\psi(I^*)}\left(1-\frac{\phi(S^*)}{\phi(S(n+1))}\right)\left(1-\frac{S^*}{S(n+1)}\right)\\
&-\sum_{j=0}^{m}f(j)\left(-2+\frac{1}{\tilde{s}_{n+1}}+\frac{\tilde{s}_{n+1}\tilde{i}_{n-j}}{i_{n+1}}+\ln\tilde{i}_{n+1}-\ln\tilde{i}_{n-j}\right)\\
=&-\frac{\mu_1 S(n+1)}{\phi(S^*)\psi(I^*)}\left(1-\frac{\phi(S^*)}{\phi(S(n+1))}\right)\left(1-\frac{S^*}{S(n+1)}\right)\\
&-\sum_{j=0}^{m}f(j)\left[g\left(\frac{1}{\tilde{s}_{n+1}}\right)+g\left(\frac{\tilde{s}_{n+1}\tilde{i}_{n-j}}{i_{n+1}}\right)+g(i_{n+1})-g(\tilde{i}_{n+1})\right].
\end{aligned}
$$

By the condition (H1), we have that $(1-\frac{\phi(S^*)}{\phi(S(n+1))})(1-\frac{S^*}{S(n+1)})\le 0$ with equality if and only if $S(n+1)=S^*$, and it follows from Lemma 5.1 that $U^*(n+1)-U^*(n)\le 0$ for any $n\ge m$. Since $\{U^*(n)\}_{n=m}^{+\infty}$ is monotone decreasing sequence, there exists a $\tilde{u}_*\equiv\lim_{n\to+\infty}U^*(n)\ge 0$. Then, $\lim_{n\to+\infty}(U^*(n+1)-U^*(n))=0$, from which we obtain that $\lim_{n\to+\infty}S(n+1)=S^*$ and $\lim_{n\to+\infty}\frac{\tilde{i}_{n-j}}{i_{n+1}}=1$, that is,

$$
\lim_{n\to+\infty}\frac{\psi(I(n-j))}{I(n+1)}=\frac{\psi(I^*)}{I^*}.
$$

if $f(j)>0$, $j=0,1,\cdots,m$. Then, by the first equation of (3.1), we have that for $n\ge m$,

$$
\begin{aligned}
S(n+1)-S(n)&=(\lambda-\mu_1 S(n+1))-\phi(S(n+1))\sum_{j=0}^{m}f(j)\psi(I(n-j))\\
&=(\lambda-\mu_1 S(n+1))-\phi(S(n+1))\frac{\sum_{j=0}^{m}f(j)\psi(I(n-j))}{I(n+1)}I(n+1),
\end{aligned}
$$

which implies

$$
I(n+1)=\frac{\lambda-(1+\mu_1)S(n+1)+S(n)}{\phi(S(n+1))\frac{\sum_{j=0}^{m}f(j)\psi(I(n-j))}{I(n+1)}}.
$$

Using the relations

$$
\lim_{n\to+\infty}(\lambda-(1+\mu_1)S(n+1)+S(n))=\lambda-\mu_1 S^*=\phi(S^*)\psi(I^*)>0,
$$

and

$$\lim_{n\to+\infty} \phi(S(n+1)) \frac{\sum_{j=0}^{m} f(j)\psi(I(n-j))}{I(n+1)} = \phi(S^*)\frac{\psi(I^*)}{I^*} > 0,$$

we obtain that $\lim_{n\to+\infty} I(n+1) = I^*$. Thus, $\lim_{n\to+\infty}(S(n), I(n)) = (S^*, I^*)$. Since $U^*(n) \le U^*(m)$ for all $n \ge m$ and $g(x) \ge 0$ with equality if and only if $x = 1$, E_* is uniformly stable. Hence, the proof is complete. □

Proof of Theorem 2.2. By Theorems 3.1, 5.1 and 5.2, we obtain the conclusion of Theorem 2.2. □

6. Conclusion and future directions

In this paper, we propose a discrete SIR epidemic model with a class of nonlinear incidence rate which incorporates delay effects by applying a variation of Euler Backward discretization which is different from that of [10, 11, 22]. It is shown that the disease can be eradicated and the disease-free equilibrium is globally asymptotically stable if and only if $R_0 \le 1$ and the disease persists in a host population and the endemic equilibrium exists if and only if $R_0 > 1$. Moreover, for $R_0 > 1$, we establish that the endemic equilibrium is globally asymptotically stable. In order to prove the global stability of the endemic equilibrium of the model for $R_0 > 1$, we apply techniques in the proof of permanence in Sekiguchi [22] and Lyapunov functional techniques for continuous-time SIR epidemic models in Huang *et al.* [9], Korobeinikov [15, 16] and McCluskey [18, 19]. Our discrete scheme preserves the permanence and the global asymptotic stability of equilibria for a corresponding continuous SIR epidemic model with a class of nonlinear incidence rates and distributed delays. We point out that Lemmas 4.1 and 5.1 play important roles to establish the global stability analysis of the equilibria for system (2.1). Moreover, the conditions (H1)-(H3) are more essential and less restrictive compared to the conditions concerning the nonlinearity of the incidence rate in Huang *et al.* [9, Theorem 1] under which the global dynamics of system (2.1) are determined by the basic reproduction number R_0. These conditions include various special incidence rates (see, e.g., [4, 12, 19, 26]).

From a biological viewpoint, it is noteworthy that the global dynamics are determined independently of the length of an incubation period of the diseases as long as the infection rate has a suitable monotone property characterized by the conditions (H1)-(H3). It is also interesting that the condition that $I/\psi(I)$ is monotone increasing leads to a justification of crowding (saturation) effects for the case that the proportion of infectives individuals in a host population is very high (see, for example, [4, 19, 26]). Finally, applying the present techniques to the other types of epidemic models (e.g., SIRS epidemic models, SIS epidemic models, etc.) will become our future works.

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