

# On the global stability of a delayed epidemic model with transport-related infection

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## Abstract

We study global dynamics of a time delayed epidemic model proposed by [J. Liu, J. Wu, Y. Zhou, Modeling disease spread via transport-related infection by a delay differential equation, Rocky Mountain J. Math. 38 (2008), no. 5, 1525–1540] describing disease transmission dynamics among two regions due to the transport-related infection. We prove that if an endemic equilibrium exists then it is globally asymptotically stable for any length of time delay by constructing a Lyapunov functional. This suggests that the endemic steady state for both regions is globally asymptotically stable regardless to the length of the travel time when the disease is transferred between two regions by the human transport.

*Key words:* Transport-related infection; delay differential equations; global asymptotic stability; Lyapunov functional

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## 1. Introduction

Population dispersal by human transportation currently has an important role in the spread of infectious disease around the world. SARS (severe acute respiratory syndrome) spread along the routes of international air travel and infection was carried to many places [19]. Khan et al., [8] pointed out a correlation between inter-regional spread of a novel influenza A (H1N1) virus and traveler. From these observations a number of authors have proposed epidemic models describing disease transmission dynamics among multiple locations due to the population dispersal (see [1, 2, 14, 18] and the references therein).

Cui et al., [4] have proposed an epidemic model which models a phenomena that population individuals suffer from diseases and that they possibly become infected during the movement between two regions. The model is given as a system of ordinary differential equations based on an SIS epidemic model. Takeuchi et al., [17] analyzed local and global stability of equilibria as well as uniformly persistence of the system. They found that there is a possibility of the

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disease endemic due to the transport-related infection. However, as pointed out by Liu et al., [10] the epidemic models proposed in [4, 11, 17] implicitly assumed that the transportation between two regions occurs instantaneously. This motivates for Liu et al., [10] to rigorously describe the disease transmission dynamics during the transportation by introducing the time needed to complete the use of the transportation. They assumed that it takes  $\tau$ -units time to complete a one-way transport between two regions. They obtained the following delay differential equations:

$$\begin{cases} \frac{dS_1(t)}{dt} = A - dS_1(t) - \frac{\beta S_1(t)I_1(t)}{S_1(t) + I_1(t)} + \delta I_1(t) - \alpha S_1(t) + s_{21}(\tau, t - \tau), \\ \frac{dI_1(t)}{dt} = \frac{\beta S_1(t)I_1(t)}{S_1(t) + I_1(t)} + i_{21}(t, t - \tau) - (d + \delta + \alpha)I_1(t), \\ \frac{dS_2(t)}{dt} = A - dS_2(t) - \frac{\beta S_2(t)I_2(t)}{S_2(t) + I_2(t)} + \delta I_2(t) - \alpha S_2(t) + s_{12}(\tau, t - \tau), \\ \frac{dI_2(t)}{dt} = \frac{\beta S_2(t)I_2(t)}{S_2(t) + I_2(t)} + i_{12}(t, t - \tau) - (d + \delta + \alpha)I_2(t), \end{cases} \quad (1.1)$$

with

$$\begin{aligned} s_{21}(\tau, t - \tau) &= \frac{\alpha e^{-\gamma\tau} S_2(t - \tau)}{e^{-\gamma\tau} S_2(t - \tau) + I_2(t - \tau)} (S_2(t - \tau) + I_2(t - \tau)), \\ i_{21}(\tau, t - \tau) &= \frac{\alpha I_2(t - \tau)}{e^{-\gamma\tau} S_2(t - \tau) + I_2(t - \tau)} (S_2(t - \tau) + I_2(t - \tau)), \\ s_{12}(\tau, t - \tau) &= \frac{\alpha e^{-\gamma\tau} S_1(t - \tau)}{e^{-\gamma\tau} S_1(t - \tau) + I_1(t - \tau)} (S_1(t - \tau) + I_1(t - \tau)), \\ i_{12}(\tau, t - \tau) &= \frac{\alpha I_1(t - \tau)}{e^{-\gamma\tau} S_1(t - \tau) + I_1(t - \tau)} (S_1(t - \tau) + I_1(t - \tau)). \end{aligned}$$

Here  $S_j(t)$  and  $I_j(t)$  denotes the number of susceptible and infected individual at time  $t$  in region  $j$ , respectively, where  $j \in \{1, 2\}$ .  $A$  is the total number of newborn per unit time,  $d$  is the natural death rate and  $\delta$  is the recovery rate. Disease is transmitted by  $\beta S_j I_j / (S_j + I_j)$ , where  $\beta$  is the disease transmission coefficient in each region. Susceptible and infected individuals leave a region to another region at a per capita rate  $\alpha$ . Thus the number of susceptible and infected individuals which leave region  $j$  per unit time is given by  $\alpha S_j(t)$  and  $\alpha I_j(t)$ , respectively.  $s_{kj}(\tau, t - \tau)$  and  $i_{kj}(t, t - \tau)$  for  $j, k \in \{1, 2\}$  and  $j \neq k$  denote the number of susceptible and infected individuals which arrive region  $j$  from region  $k$  per unit time at time  $t$ . They leave region  $j$  at time  $t - \tau$  and spend  $\tau$ -units time in the transportation, where disease transmission occurs. We denote by  $\gamma$  the transmission coefficient in the transportation. It is assumed that every parameter is positive.

Following [10], we explain how to derive  $s_{kj}(\tau, t - \tau)$  and  $i_{kj}(\tau, t - \tau)$  for  $j, k \in \{1, 2\}$  and  $j \neq k$ . We denote by  $s_{kj}(\theta, t)$  and  $i_{kj}(\theta, t)$  the number of susceptible and infected individuals which leave region  $k$  per unit time at time  $t$  and spend  $\theta$ -units time in the transportation to region  $j$ , where  $\theta \in [0, \tau]$ . Then considering the number of susceptible and infected individuals which leave region  $k$  to  $j$  per unit time at time  $t - \tau$  we obtain that

$$s_{kj}(0, t - \tau) = \alpha S_k(t - \tau) \text{ and } i_{kj}(0, t - \tau) = \alpha I_k(t - \tau). \quad (1.2)$$

We assume that the population individuals do not die in the transportation. Then the disease dynamics in the transportation from region  $k$  to  $j$  are described as

$$\begin{aligned}\frac{\partial}{\partial \theta} s_{kj}(\theta, t - \tau) &= -\gamma \frac{i_{kj}(\theta, t - \tau)}{i_{kj}(\theta, t - \tau) + s_{kj}(\theta, t - \tau)} s_{kj}(\theta, t - \tau), \\ \frac{\partial}{\partial \theta} i_{kj}(\theta, t - \tau) &= \gamma \frac{i_{kj}(\theta, t - \tau)}{i_{kj}(\theta, t - \tau) + s_{kj}(\theta, t - \tau)} s_{kj}(\theta, t - \tau),\end{aligned}$$

where  $\gamma$  is the transmission coefficient in the transportation. Since it holds that

$$s_{kj}(\theta, t - \tau) + i_{kj}(\theta, t - \tau) = \alpha (S_k(t - \tau) + I_k(t - \tau)) \text{ for any } \theta \in [0, \tau] \quad (1.3)$$

we obtain that

$$\frac{\partial}{\partial \theta} i_{kj}(\theta, t - \tau) = \gamma i_{kj}(\theta, t - \tau) \left\{ 1 - \frac{i_{kj}(\theta, t - \tau)}{\alpha (S_k(t - \tau) + I_k(t - \tau))} \right\}. \quad (1.4)$$

By solving (1.4) with using (1.2) as the initial condition we obtain the expression for  $i_{kj}(\tau, t - \tau)$ . Then it is easy to see that (1.3) gives the expression for  $s_{kj}(\tau, t - \tau)$ .

Liu et al., [10] identified the basic reproduction number  $R_0$  and analyzed stability property of equilibria and uniformly persistence of the system. The basic reproduction number is given as

$$R_0 := \frac{\beta + \alpha e^{\gamma\tau}}{d + \delta + \alpha}.$$

(1.1) always has a disease-free equilibrium. They proved that the disease-free equilibrium is globally asymptotically stable if  $R_0 < 1$ . If  $R_0 > 1$  then (1.1) admits a unique endemic equilibrium. By analyzing an associated characteristic equation they proved that the endemic equilibrium is locally asymptotically stable if  $R_0 > 1$  [10, Theorem 4.1]. Moreover, by the uniform persistence theorem [7], they obtained that the disease eventually persists if  $R_0 > 1$  [10, Theorem 4.2].

However, the global stability of the endemic equilibrium remains unsolved. Stability analysis for epidemic models is helpful to obtain an insight for the disease transmission dynamics. Global stability of equilibria especially makes the model dynamics clear and enhances our understanding toward the mathematical models. In this paper we prove that the endemic equilibrium is globally asymptotically stable if  $R_0 > 1$ . Our proof is based on constructing a Lyapunov functional and using LaSalle's invariance principle. Our mathematical results suggest that, when the infectious disease is transferred between two regions by the human transportation, the endemic steady state is globally asymptotically stable regardless to the length of the travel time if the basic reproduction number exceeds 1.

The paper is organized as follows. In the next section, at first, we discuss the global dynamics of the total populations in both regions and show that a unique positive equilibrium is globally asymptotically stable. Using this result we obtain a limit system for (1.1). Constructing a Lyapunov functional for this reduced system, we prove that an unique positive equilibrium of the system is globally asymptotically stable if  $R_0 > 1$  in Theorem 2.6. This implies that the endemic equilibrium of (1.1) is also globally asymptotically stable if  $R_0 > 1$ . In Section 3, we offer a discussion.

## 2. Global stability of the endemic equilibrium

To investigate the dynamics of (1.1), we set a suitable phase space. We denote by  $C = C([-\tau, 0], \mathbb{R})$  the Banach space of continuous functions mapping the interval  $[-\tau, 0]$  into  $\mathbb{R}$  equipped with the sup-norm. The nonnegative cone of  $C$  is defined as  $C_+ = C([-\tau, 0], \mathbb{R}_+)$ . From the biological meanings, the initial conditions for (1.1) are

$$S_1(\theta) = \phi_1(\theta), I_1(\theta) = \phi_2(\theta), S_2(\theta) = \psi_1(\theta), I_2(\theta) = \psi_2(\theta), \theta \in [-\tau, 0], \quad (2.1)$$

where  $\phi_i, \psi_i \in C_+, i = 1, 2$ .

**Lemma 2.1** (see [10, Lemmas 2.1 and 2.2]) *The solution of (1.1) with initial conditions (2.1) is nonnegative for all  $t > 0$ . Moreover, there exist  $M > 0$  and  $T > 0$  such that  $S_i(t) \leq M$  and  $I_i(t) \leq M$  for  $i = 1, 2$  and  $t \geq T$ .*

At first, we consider the global dynamics of total populations in both regions. Let us define

$$N_j(t) := S_j(t) + I_j(t), \quad j \in \{1, 2\}.$$

Then from (1.1) and (2.1) we have

$$\frac{dN_j(t)}{dt} = A - (d + \alpha)N_j(t) + \alpha N_k(t - \tau) \text{ for } j, k \in \{1, 2\} \text{ and } j \neq k \quad (2.2)$$

with the initial conditions  $N_1(\theta) = \phi(\theta)$  and  $N_2(\theta) = \psi(\theta)$  for  $\theta \in [-\tau, 0]$ , where  $\phi = \phi_1 + \phi_2$  and  $\psi = \psi_1 + \psi_2$ . We prove that (2.2) has a unique positive equilibrium which is globally asymptotically stable.

**Lemma 2.2** (2.2) *has a unique positive equilibrium which is globally asymptotically stable.*

**PROOF.** Since the equilibrium satisfies

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} d + \alpha & -\alpha \\ -\alpha & d + \alpha \end{pmatrix}^{-1} \begin{pmatrix} A \\ A \end{pmatrix}$$

we easily obtain the existence of the unique positive equilibrium.

Let us consider the asymptotic stability of the equilibrium. We define

$$G_N = \{(\phi, \psi) \in C([- \tau, 0], \mathbb{R}_+^2) \mid \phi(\theta) \geq 0, \psi(\theta) \geq 0, \theta \in [- \tau, 0], \phi(0) > 0, \psi(0) > 0\}.$$

$\overline{G}_N$ , which is the closure of  $G_N$ , is positively invariant for (2.2).

We denote by  $(N^*, N^*)$  the positive equilibrium of (2.2). Consider the following functional defined on  $G_N$  as

$$L(N_1, N_2) = \sum_{j=1}^2 g\left(\frac{N_j(t)}{N^*}\right) + \alpha \sum_{j=1}^2 \int_{t-\tau}^t g\left(\frac{N_j(s)}{N^*}\right) ds, \quad (2.3)$$

where

$$g(x) = x - 1 - \ln x \text{ for } x \in (0, +\infty)$$

and  $N_j(t), j = 1, 2$  is any solution of (2.2). It is clear that  $L$  is continuous on  $G_N$  and that for any  $(\phi, \psi) \in \partial G_N$  (the boundary of  $G_N$ ), the limit  $l(\phi, \psi) = \lim_{(\Phi, \Psi) \rightarrow (\phi, \psi) \in \partial G_N} L(\Phi, \Psi), (\Phi, \Psi) \in G_N$  is exists or is  $+\infty$ . We consider the time derivative of  $L(N_1, N_2)$  along the solution of (2.2).

First of all, we see

$$\frac{d}{dt} \left[ g\left(\frac{N_j(t)}{N^*}\right) \right] = \frac{1}{N^*} \left( 1 - \frac{N^*}{N_j(t)} \right) \{ A - (d + \alpha)N_j(t) + \alpha N_k(t - \tau) \}.$$

Then, from  $A = (d + \alpha)N^* - \alpha N^*$ , it follows that

$$\begin{aligned} \frac{d}{dt} \left[ g\left(\frac{N_j(t)}{N^*}\right) \right] &= (d + \alpha) \left( 1 - \frac{N^*}{N_j(t)} \right) \left( 1 - \frac{N_j(t)}{N_j^*} \right) + \alpha \left( 1 - \frac{N^*}{N_j(t)} \right) \left( \frac{N_k(t - \tau)}{N^*} - 1 \right) \\ &= (d + \alpha) \left( 2 - \frac{N_j(t)}{N_j^*} - \frac{N^*}{N_j(t)} \right) + \alpha \left( \frac{N_k(t - \tau)}{N^*} - 1 - \frac{N_k(t - \tau)}{N_j(t)} + \frac{N^*}{N_j(t)} \right). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d}{dt} \int_{t-\tau}^t g\left(\frac{N_j(s)}{N^*}\right) ds &= g\left(\frac{N_j(t)}{N^*}\right) - g\left(\frac{N_j(t-\tau)}{N^*}\right) \\ &= \frac{N_j(t)}{N^*} - \ln\left(\frac{N_j(t)}{N^*}\right) - \frac{N_j(t-\tau)}{N^*} + \ln\left(\frac{N_j(t-\tau)}{N^*}\right). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \dot{L}_{(2.2)}(N_1, N_2) &= \sum_{j=1}^2 \left\{ (d + \alpha) \left( 2 - \frac{N_j(t)}{N^*} - \frac{N^*}{N_j(t)} \right) + \alpha \left( -2 + \frac{N_j(t)}{N^*} + \frac{N^*}{N_j(t)} \right) \right\} \\ &+ \alpha \left\{ \left( -\frac{N_2(t-\tau)}{N_1(t)} + 1 + \ln \frac{N_2(t-\tau)}{N_1(t)} \right) + \left( -\frac{N_1(t-\tau)}{N_2(t)} + 1 + \ln \frac{N_1(t-\tau)}{N_2(t)} \right) \right\} \\ &= d \sum_{j=1}^2 \left( 2 - \frac{N_j(t)}{N^*} - \frac{N^*}{N_j(t)} \right) - \alpha \left\{ g\left(\frac{N_2(t-\tau)}{N_1(t)}\right) + g\left(\frac{N_1(t-\tau)}{N_2(t)}\right) \right\}. \end{aligned}$$

Hence we obtain that

$$\dot{L}_{(2.2)}(N_1, N_2) \leq 0. \quad (2.4)$$

From (2.3) and (2.4), the positive equilibrium is stable.

Let

$$E := \{(\phi, \psi) \in \overline{G}_N \mid l(\phi, \psi) < +\infty, \dot{L}_{(2.2)}(\phi, \psi) = 0\}$$

and  $M$  be the largest subset in  $E$  that is invariant with respect to (2.2). Then, we see

$$E = \{(\phi, \psi) \in \overline{G}_N \mid \phi(0) = \phi(-\tau) = \psi(0) = \psi(-\tau) = N^*\}.$$

For any initial function  $(\phi, \psi) \in M$ ,  $(N_{1t}, N_{2t}) \in M \subset E$ , where  $N_{1t}(\theta) = N_1(t + \theta)$  and  $N_{2t}(\theta) = N_2(t + \theta)$  through  $(0, \phi, \psi)$  and satisfies  $\dot{L}_{(2.2)}(N_{1t}, N_{2t}) = 0$  for  $t > 0$ . Then,  $\lim_{t \rightarrow +\infty} N_j(t) = N^*$  for  $j = 1, 2$  and hence,  $M = \{(N^*, N^*)\}$ . By an extension of LaSalle's invariance principle [15, Lemma 3.1], any solution tends to  $M$  (see also [6, 9]). Hence, the positive equilibrium is globally asymptotically stable.  $\square$

**Remark 2.3** Suzuki and Matsunaga [16] established the necessary and sufficient condition for asymptotic stability of a class of linear delay differential equations. Their result is applicable to (2.2) and it also derives Lemma 2.2 ([16, Example 2]).

Next using the result in Lemma 2.2 we derive a limit system of (1.1). We denote by  $(N^*, N^*)$  the unique positive equilibrium of (2.2). Since  $S_j(t) = N_j(t) - I_j(t)$ ,  $j \in \{1, 2\}$ , (1.1) has the following limit system:

$$\frac{dI_j(t)}{dt} = I_j(t) \left\{ \beta - (d + \delta + \alpha) - \frac{\beta}{N^*} I_j(t) \right\} + G(I_k(t - \tau)) \text{ for } j, k \in \{1, 2\} \text{ and } j \neq k, \quad (2.5)$$

where

$$G(I) := \frac{\alpha e^{\gamma\tau} I}{1 + \frac{e^{\gamma\tau} - 1}{N^*} I} \text{ for } I \in [0, +\infty).$$

From now on we consider the reduced system (2.5) in order to understand the asymptotic behavior of the solution of (1.1) (see [3, 12]).

Liu et al., [10] proved that (1.1) has a unique endemic equilibrium if and only if  $R_0 > 1$ . We denote the endemic equilibrium by  $(S^*, I^* S^*, I^*)$  where every component is strictly positive.

Then it is easy to prove that  $(I^*, I^*)$  is a unique positive equilibrium of (2.5) if and only if  $R_0 > 1$ . We study the global stability of the positive equilibrium of (2.5) in order to establish the global stability of the endemic equilibrium of (1.1).

We give an elementary result to prove the global asymptotic stability of the endemic equilibrium. Let us define

$$g(x) := x - 1 - \ln x \text{ for } x \in (0, +\infty).$$

**Lemma 2.4** *Let us assume that  $R_0 > 1$ . Then it holds that*

$$\left( \frac{I_j(t)}{I^*} - \frac{G(I_j(t))}{G(I^*)} \right) \left( \frac{G(I_j(t))}{G(I^*)} - 1 \right) \geq 0 \quad (2.6)$$

and

$$g\left(\frac{I_j(t)}{I^*}\right) - g\left(\frac{G(I_j(t))}{G(I^*)}\right) \geq 0 \quad (2.7)$$

for  $j \in \{1, 2\}$ .

**PROOF.** A direct computation gives

$$\begin{aligned} \left( \frac{I_j(t)}{I^*} - \frac{G(I_j(t))}{G(I^*)} \right) \left( \frac{G(I_j(t))}{G(I^*)} - 1 \right) &= \frac{G(I_j(t))}{I^*} \left( \frac{I_j(t)}{G(I_j(t))} - \frac{I^*}{G(I^*)} \right) \frac{1}{G(I^*)} (G(I_j(t)) - G(I^*)) \\ &= \frac{G(I_j(t))}{G(I^*)I^*} \left( \frac{I_j(t)}{G(I_j(t))} - \frac{I^*}{G(I^*)} \right) (G(I_j(t)) - G(I^*)) \end{aligned}$$

for  $j \in \{1, 2\}$ . Since  $G(I)$  and  $\frac{I}{G(I)}$  are monotone increasing functions we obtain (2.6). (2.6) implies that

$$\begin{cases} \frac{I_j(t)}{I^*} \leq \frac{G(I_j(t))}{G(I^*)} < 1 & \text{for } I_j(t) < I^*, \\ \frac{I_j(t)}{I^*} = \frac{G(I_j(t))}{G(I^*)} = 1 & \text{for } I_j(t) = I^*, \\ \frac{I_j(t)}{I^*} \geq \frac{G(I_j(t))}{G(I^*)} > 1 & \text{for } I_j(t) > I^*. \end{cases}$$

Then (2.7) holds. The proof is complete.  $\square$

**Remark 2.5** *The same property as (2.7) in Lemma 2.4 is found in [5, 13] and it is used to analyze global dynamics of epidemic models which has a nonlinear incidence rate.*

We prove the global asymptotic stability of the positive equilibrium of (2.5).

**Theorem 2.6** *Let us assume that there exists  $\theta_0 \in [-\tau, 0]$  such that  $\phi_2(\theta_0) + \psi_2(\theta_0) > 0$ . Let  $R_0 > 1$ . Then the positive equilibrium of (2.5) is globally asymptotically stable.*

**PROOF.** Since there exists the positive equilibrium of (2.5) it holds that

$$\beta - (d + \delta + \alpha) = \frac{\beta I^*}{N^*} - \frac{G(I^*)}{I^*}.$$

Then (2.5) becomes the following

$$\begin{aligned} \frac{dI_j(t)}{dt} &= I_j(t) \left( \frac{\beta I^*}{N^*} - \frac{G(I^*)}{I^*} \right) - \frac{\beta I_j(t)^2}{N^*} + G(I_k(t - \tau)) \\ &= \frac{\beta}{N^*} I_j(t) (I^* - I_j(t)) + G(I_k(t - \tau)) - G(I^*) \frac{I_j(t)}{I^*} \text{ for } j \in \{1, 2\} \text{ and } j \neq k. \end{aligned} \quad (2.8)$$

We define

$$G = \{(\phi_2, \psi_2) \in C([- \tau, 0], \mathbb{R}_+^2) \mid \phi_2(\theta) \geq 0, \psi_2(\theta) \geq 0, \theta \in [- \tau, 0], \phi_2(0) > 0, \psi_2(0) > 0\}.$$

$\bar{G}$ , the closure of  $G$ , is positively invariant for (2.8). Moreover, there exists an  $\varepsilon > 0$ , such that every solution  $(I_1(t), I_2(t))$  of (1.1) with  $\phi_2(\theta_0) + \psi_2(\theta_0) > 0$  for some  $\theta_0 \in [- \tau, 0]$  satisfies  $\liminf_{t \rightarrow +\infty} I_j(t) \geq \varepsilon$  (see [10, Theorem 5.1]). This implies that  $G$  is also positively invariant for (2.8).

Consider the following functional defined on  $G$  as

$$U(I_1, I_2) = \frac{I^*}{G(I^*)} \sum_{j=1}^2 g\left(\frac{I_j(t)}{I^*}\right) + \sum_{j=1}^2 \int_{t-\tau}^t g\left(\frac{G(I_j(s))}{G(I^*)}\right) ds, \quad (2.9)$$

where  $I_j(t)$ ,  $j \in \{1, 2\}$  are any solution of (2.8). Then it is clear that  $U$  is continuous on  $G$  and that for any  $(\phi, \psi) \in \partial G$  (the boundary of  $G$ ), the limit  $l(\phi, \psi) = \lim_{(\Phi, \Psi) \rightarrow (\phi, \psi) \in \partial G} U(\Phi, \Psi)$ ,  $(\Phi, \Psi) \in G$  is  $+\infty$ .

We consider the time derivative of  $U$  along the solution of (2.8). At first, we see that

$$\begin{aligned} & \frac{d}{dt} \left[ g\left(\frac{I_j(t)}{I^*}\right) \right] \\ &= \frac{1}{I^*} \left(1 - \frac{I^*}{I_j(t)}\right) \left\{ \frac{\beta}{N^*} I_j(t) I^* \left(1 - \frac{I_j(t)}{I^*}\right) + G(I^*) \left(\frac{G(I_k(t-\tau))}{G(I^*)} - \frac{I_j(t)}{I^*}\right) \right\} \\ &= -\frac{\beta I^*}{N^*} \left(1 - \frac{I_j(t)}{I^*}\right)^2 + \frac{G(I^*)}{I^*} \left(1 - \frac{I^*}{I_j(t)}\right) \left(\frac{G(I_k(t-\tau))}{G(I^*)} - \frac{I_j(t)}{I^*}\right) \\ &= -\frac{\beta I^*}{N^*} \left(1 - \frac{I_j(t)}{I^*}\right)^2 + \frac{G(I^*)}{I^*} \left(\frac{G(I_k(t-\tau))}{G(I^*)} - \frac{I_j(t)}{I^*} - \frac{I^*}{I_j(t)} \frac{G(I_k(t-\tau))}{G(I^*)} + 1\right). \end{aligned} \quad (2.10)$$

Next it follows

$$\begin{aligned} \frac{d}{dt} \int_{t-\tau}^t g\left(\frac{G(I_j(s))}{G(I^*)}\right) ds &= g\left(\frac{G(I_j(t))}{G(I^*)}\right) - g\left(\frac{G(I_j(t-\tau))}{G(I^*)}\right) \\ &= \frac{G(I_j(t))}{G(I^*)} - \ln\left(\frac{G(I_j(t))}{G(I^*)}\right) - \frac{G(I_j(t-\tau))}{G(I^*)} + \ln\left(\frac{G(I_j(t-\tau))}{G(I^*)}\right). \end{aligned} \quad (2.11)$$

From (2.10) and (2.11) we obtain that

$$\dot{U}_{(2.10)}(I_1, I_2) = -\frac{\beta I^*}{N^*} \frac{I^*}{G(I^*)} \sum_{j=1}^2 \left(1 - \frac{I_j(t)}{I^*}\right)^2 + C(t), \quad (2.12)$$

where

$$\begin{aligned} C(t) &= \sum_{\substack{j,k \in \{1,2\} \\ j \neq k}} \left(\frac{G(I_k(t-\tau))}{G(I^*)} - \frac{I_j(t)}{I^*} - \frac{I^*}{I_j(t)} \frac{G(I_k(t-\tau))}{G(I^*)} + 1\right) \\ &+ \sum_{j=1}^2 \left\{ \frac{G(I_j(t))}{G(I^*)} - \ln\left(\frac{G(I_j(t))}{G(I^*)}\right) - \frac{G(I_j(t-\tau))}{G(I^*)} + \ln\left(\frac{G(I_j(t-\tau))}{G(I^*)}\right) \right\}. \end{aligned}$$

We compute  $C(t)$  as follows

$$\begin{aligned}
C(t) &= \sum_{\substack{j,k \in \{1,2\} \\ j \neq k}} \left( \frac{G(I_k(t))}{G(I^*)} - \frac{I_j(t)}{I^*} - \frac{I^*}{I_j(t)} \frac{G(I_k(t-\tau))}{G(I^*)} + 1 \right) \\
&+ \sum_{j=1}^2 \left\{ -\ln \left( \frac{G(I_j(t))}{G(I^*)} \right) + \ln \left( \frac{G(I_j(t-\tau))}{G(I^*)} \right) \right\} \\
&= \sum_{\substack{j,k \in \{1,2\} \\ j \neq k}} \left( \frac{G(I_k(t))}{G(I^*)} - \frac{I_j(t)}{I^*} - \frac{I^*}{I_j(t)} \frac{G(I_k(t-\tau))}{G(I^*)} + 1 \right) \\
&+ \sum_{\substack{j,k \in \{1,2\} \\ j \neq k}} \left\{ -\ln \left( \frac{G(I_j(t))}{G(I^*)} \right) + \ln \left( \frac{I_j(t)}{I^*} \right) + \ln \left( \frac{I^*}{I_j(t)} \frac{G(I_k(t-\tau))}{G(I_k^*)} \right) \right\} \\
&= \sum_{j=1}^2 \left\{ \left( \frac{G(I_j(t))}{G(I^*)} - 1 - \ln \left( \frac{G(I_j(t))}{G(I^*)} \right) \right) - \left( \frac{I_j(t)}{I^*} - 1 - \ln \left( \frac{I_j(t)}{I^*} \right) \right) \right\} \\
&+ \sum_{\substack{j,k \in \{1,2\} \\ j \neq k}} \left( -\frac{I^*}{I_j(t)} \frac{G(I_k(t-\tau))}{G(I^*)} + 1 + \ln \left( \frac{I^*}{I_j(t)} \frac{G(I_k(t-\tau))}{G(I^*)} \right) \right) \\
&= \sum_{j=1}^2 \left\{ g \left( \frac{G(I_j(t))}{G(I^*)} \right) - g \left( \frac{I_j(t)}{I^*} \right) \right\} - \sum_{\substack{j,k \in \{1,2\} \\ j \neq k}} g \left( \frac{I^*}{I_j(t)} \frac{G(I_k(t-\tau))}{G(I^*)} \right).
\end{aligned}$$

Then, from (2.7) in Lemma 2.4, we see  $C(t, \tau) \leq 0$ . Therefore, we obtain

$$\dot{U}_{(2.10)}(I_1, I_2) \leq 0,$$

from (2.12).

Let  $E = \{(\phi_2, \psi_2) \in \bar{G} \mid I(\phi_2, \psi_2) < +\infty, \dot{U}_{(2.10)}(I_1, I_2) = 0\}$  and  $M$  be the largest subset in  $E$  that is invariant with respect to (2.8). Then, we see

$$E = \{(\phi_2, \psi_2) \in \bar{G} \mid \phi_2(0) = \psi_2(0) = \phi_2(-\tau) = \psi_2(-\tau) = I^*\}.$$

For any initial function  $(\phi_2, \psi_2) \in M$ ,  $(I_{1t}, I_{2t}) \in M \subset E$ , where  $I_{jt}(\theta) = I_j(t + \theta)$ ,  $j = 1, 2$  through  $(0, \phi, \psi)$  and satisfies  $\dot{U}_{(2.10)}(I_{1t}, I_{2t}) = 0$  for  $t > 0$ . Then,  $\lim_{t \rightarrow +\infty} I_j(t) = I^*$  for  $j = 1, 2$  and hence,  $M = \{(I^*, I^*)\}$ . By an extension of LaSalle's invariance principle [15, Lemma 3.1], any solution tends to  $M$  (see also [6, 9]) and hence, the positive equilibrium  $(I^*, I^*)$  of (2.5) is globally asymptotically stable.  $\square$

### 3. Discussion

In this paper, we have studied global dynamics of a delayed epidemic model (1.1) proposed by Liu et al., [10]. The model describes disease transmission dynamics due to the transport-related infection [4, 11, 17] and captures the time needed to complete the use of the transportation.

Liu et al., [10] established that the disease-free equilibrium of (1.1) is globally asymptotically stable if  $R_0 < 1$  and that (1.1) admits the unique endemic equilibrium, which is locally asymptotically stable, if  $R_0 > 1$ . However, the global stability of the endemic equilibrium remains unsolved and was an open problem.

For this problem, we considered global dynamics of the limit system (2.5). (2.5) is derived from (1.1) using that the positive equilibrium of (2.2) is asymptotically stable. Constructing a



Lyapunov functional and using LaSalle's invariance principle for the reduced system, we prove that the positive equilibrium of (2.5) is globally asymptotically stable if  $R_0 > 1$  (Theorem 2.6). This implies that the endemic equilibrium of (1.1) is also globally asymptotically stable if  $R_0 > 1$ . The mathematical result suggests that, when the disease is endemic, under a situation that two regions are connected each other by a transportation, the endemic steady state is globally asymptotically stable regardless to the length of the travel time. However, one can see that in (1.1) it is assumed that both regions share the identical parameter set. It may be necessary to consider two different population size and different dispersal rate to discuss precisely the impact of the transport-related infection on the disease dynamics. We leave this as a future work.

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