Monotone iterative techniques to SIRS epidemic models with nonlinear incidence rates and distributed delays

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Abstract

In this paper, for SIRS epidemic models with a class of nonlinear incidence rates and distributed delays of the forms $\beta S(t)\int_0^h k(\tau)G(I(t-\tau))d\tau$, we establish the global asymptotic stability of the disease-free equilibrium E_0 for $R_0 < 1$, and applying new monotone techniques, we obtain sufficient conditions which ensure the global asymptotic stability of the endemic equilibrium of system. The obtained results improve that in Xu and Ma [Stability of a delayed SIRS epidemic model with a nonlinear incidence rate, *Chaos, Solitons and Fractals.* **41** (2009) 2319-2325], and are very useful for a large class of SIRS models.

Keywords: SIRS epidemic model; nonlinear incidence; global asymptotic stability; distributed delays

1. Introduction

A fundamental problem in epidemic models is to study the global dynamics of disease transmissions, that is, to study the long term behavior of spread of the diseases. Various mathematical models have been proposed in the study of population dynamics and epidemiology. The incidence of a disease is the number of new cases per unit time and has played an important role in the literacy of mathematical modeling. Many authors have studied the dynamical behavior of several epidemic models (see [1-19] and references therein).

Much attention has been paid to the analysis of the stability of the disease-free equilibrium and the endemic equilibrium of the epidemic models. Mena-

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Lorca and Hethcote [11] considered an SIR epidemic model with bilinear incidence rate and no delays which takes the form βSI. Threshold was also found in Mena-Lorca and Hethcote [11] to determine whether the disease dies out or approaches to an endemic equilibrium. Later, various kinds of SIRS epidemic models and a significant body of work have been carried out (see, for example, [6, 8, 13, 17] and references therein).

Incidence rate plays a crucial role in the modeling of epidemic dynamics. Many authors have suggested that transmission of the infection shall have a nonlinear incidence rate. The bilinear incidence rate βSI and the standard incidence rate $\beta SI/N$ are frequently used in the literacy of mathematical modeling. On the other hand, Capasso and Serio [3] have given an assumption that the incidence rate takes the nonlinear form $\frac{\beta SI}{1+\alpha I}$, which has been interpreted as saturated incidence rate. This incidence rate seems more reasonable than the bilinear incidence rate βSI in the meaning that it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate. For the following SIRS epidemic model with a nonlinear incidence rate and time delay;

$$\begin{cases} \dfrac{dS(t)}{dt} = B - \mu S(t) - \beta S(t) \dfrac{I(t-\tau)}{1+\alpha I^p(t-\tau)} + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t) \dfrac{I(t-\tau)}{1+\alpha I^p(t-\tau)} - (\mu+\gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\mu+\delta) R(t), \end{cases} \tag{1.1}$$

Xu and Ma [17] for $p = 1$ (saturation effect), Xiao and Ruan [16] for $p = 2$ and Yang and Xiao [18] for $p > 1$ (psychological effect), derived sufficient conditions for the global asymptotic stability of the endemic equilibrium.

In this paper, motivated by the above results and applying monotone techniques, we establish sufficient conditions which ensure the global asymptotic stability of endemic equilibrium of the following SIRS epidemic model (1.2) with a class of nonlinear incidence rates and distributed delays. The obtained results for (1.1) improve those in Xu and Ma [17] for $p = 1$ and Yang and Xiao [18] for $p > 1$.

$$\begin{cases} \dfrac{dS(t)}{dt} = B - \mu S(t) - \beta S(t) \displaystyle\int_0^h k(\tau) G(I(t-\tau)) d\tau + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t) \displaystyle\int_0^h k(\tau) G(I(t-\tau)) d\tau - (\mu+\gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\mu+\delta) R(t), \end{cases} \tag{1.2}$$

where $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, infective and recovered individuals at time t, respectively. B is the recruitment rate of the population, and μ is the natural death rate of the susceptible, infective and recovered individuals, β is the transmission rate, γ is the natural recovery rate of the infective individuals, δ is the rate at which recovered individuals lose immunity and return to the susceptible class. τ is the time taken to become infectious. $G(I(t-\tau))$ is a nonlinear incidence rate with a delay τ and $k(\tau)$ denotes the nonnegative incubation period distribution (see also Takeuchi *et al.* [14]). We assume that $G(I)$ is continuous on $(0,+\infty)$ and $k(t)$ is continuous on $[0,h]$ and

$$\int_0^h k(\tau)d\tau = 1.$$

The initial condition for system (1.2) takes the form

$$\begin{cases} S(\theta)=\phi_1(\theta), \;\; I(\theta)=\phi_2(\theta), \;\; R(\theta)=\phi_3(\theta), \\ \phi_i(\theta)\geq 0, \;\; \theta\in[-h,0], \;\; \phi_i(0)>0, \;\; i=1,2,3, \\ (\phi_1(\theta),\phi_2(\theta),\phi_3(\theta))\in C([-h,0],\mathbb{R}^3_+), \end{cases} \tag{1.3}$$

where $\mathbb{R}^3_+=\{(x_1,x_2,x_3): x_i\geq 0, \; i=1,2,3\}$. We further assume that

$$\begin{cases} I/G(I) \text{ is strictly monotone increasing on } (0,+\infty), \\ \text{and } \lim\limits_{I\to+0}(I/G(I))=1. \end{cases} \tag{1.4}$$

Notice that system (1.2) is a generalized form of systems (1.1), and from the condition (1.4), one can see that $G(I)\leq I$ for $I>0$ and $G(0)=0$. We note that $G(I)=\frac{I}{1+\alpha I}$, $\alpha>0$ satisfies the above condition (1.4) and the reproduction number for system (1.2) becomes

$$R_0=\frac{\beta B}{\mu(\mu+\gamma)}. \tag{1.5}$$

It is well known by the fundamental theory of differential equations that system (1.2) has a unique solution $(S(t),I(t),R(t))$ satisfying initial condition (1.3). One can see immediately that system (1.2) always has a disease-free equilibrium $E_0=(B/\mu,0,0)$. Apart from the above equilibrium, if $R_0>1$, then system (1.2) allows a unique endemic equilibrium $E_*=(S^*,I^*,R^*)$ satisfying the following.

$$\begin{cases} B-\mu S^*-\beta S^*G(I^*)+\delta R^*=0, \\ \beta S^*G(I^*)-(\mu+\gamma)I^*=0, \\ \gamma I^*-(\mu+\delta)R^*=0 \end{cases} \tag{1.6}$$

Under the condition (1.4), let us define the following strictly monotone increasing function of $I>0$ such that

$$h(I)=\begin{cases} \frac{I}{G(I)}, & \text{if } I>0, \\ 1, & \text{if } I=0, \end{cases} \tag{1.7}$$

and consider its inverse function $h^{-1}(I)$ of $I\geq 0$. By (1.4), it holds that $h(0)=1$, and hence, $h^{-1}(1)=0$. Moreover, for any $0\leq \underline{I}\leq \bar{I}$, put

$$\bar{G}(\underline{I},\bar{I})=\max_{\underline{I}\leq I\leq \bar{I}} G(I)=\begin{cases} G(\bar{I}), & \text{if } G(I) \text{ is monotone increasing on } [\underline{I},\bar{I}], \\ G(\hat{\bar{I}}), & \text{if there exists a maximal point } \hat{\bar{I}} \text{ on } [\underline{I},\bar{I}], \\ G(\underline{I}), & \text{if } G(I) \text{ is monotone decreasing on } [\underline{I},\bar{I}], \end{cases} \tag{1.8}$$

and

$$\underline{G}(\underline{I}, \bar{I}) = \min_{\underline{I} \le I \le \bar{I}} G(I) = \begin{cases} G(\underline{I}), & \text{if } G(I) \text{ is monotone increasing on } [\underline{I}, \bar{I}], \\ G(\hat{\underline{I}}), & \text{if there exists a minimal point } \hat{\underline{I}} \text{ on } [\underline{I}, \bar{I}], \\ G(\bar{I}), & \text{if } G(I) \text{ is monotone decreasing on } [\underline{I}, \bar{I}], \end{cases} \tag{1.9}$$

and

$$\bar{h}(\underline{I}, \bar{I}) = \frac{\bar{I}}{\bar{G}(\underline{I}, \bar{I})}, \quad \text{and} \quad \underline{h}(\underline{I}, \bar{I}) = \frac{\underline{I}}{\underline{G}(\underline{I}, \bar{I})}. \tag{1.10}$$

We first obtain the similar result for $R_0 < 1$ in Xu and Ma [17, Proof of Theorem 3.2].

Theorem 1.1 *If $R_0 < 1$, then the disease-free equilibrium E_0 of system (1.2) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$.*

We obtain the following main theorem (cf. Xu and Ma [17, Proof of Theorem 3.1]).

Theorem 1.2 *Let $R_0 > 1$. Then the positive equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.2) exists. Assume that there exist two nonnegative constants $\underline{I} < \bar{I}$ such that*

$$\begin{cases} \underline{I} \le \liminf_{t \to +\infty} I(t) \le I^* \le \limsup_{t \to \infty} I(t) \le \bar{I}, \\ \dfrac{\gamma}{\mu + \delta} < 1 + \dfrac{\mu + \gamma}{\beta} \dfrac{\bar{h}(\underline{I}, \bar{I}) - \underline{h}(\underline{I}, \bar{I})}{\bar{I} - \underline{I}}, \end{cases} \tag{1.11}$$

and that

$$\left(\begin{array}{l} \underline{I} \le \underline{I}^* \le I^* \le \bar{I}^* \le \bar{I}, \\ \bar{I}^* + \frac{\mu+\gamma}{\beta} \bar{h}(\underline{I}^*, \bar{I}^*) = \frac{B}{\mu} - \frac{\gamma}{\mu+\delta} \underline{I}^*, \\ \text{and} \\ \underline{I}^* + \frac{\mu+\gamma}{\beta} \underline{h}(\underline{I}^*, \bar{I}^*) = \frac{B}{\mu} - \frac{\gamma}{\mu+\delta} \bar{I}^*, \end{array} \right) \quad \text{imply} \quad \underline{I}^* = \bar{I}^* = I^*. \tag{1.12}$$

Then, the positive equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.2) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$. In particular, if

$$\begin{cases} \underline{I} \le \liminf_{t \to +\infty} I(t) \le I^* \le \limsup_{t \to \infty} I(t) \le \bar{I}, \\ \dfrac{\gamma}{\mu + \delta} < 1 + \dfrac{\mu + \gamma}{\beta} \dfrac{\bar{h}(\tilde{\underline{I}}, \tilde{\bar{I}}) - \underline{h}(\tilde{\underline{I}}, \tilde{\bar{I}})}{\tilde{\bar{I}} - \tilde{\underline{I}}}, \\ \text{for any } \tilde{\underline{I}} < \tilde{\bar{I}} \text{ such that } \underline{I} \le \tilde{\underline{I}} \le I^* \le \tilde{\bar{I}} \le \bar{I}, \end{cases} \tag{1.13}$$

then, (1.11) and (1.12) is satisfied.

Corollary 1.1 *For $R_0 > 1$, if*

$$\gamma < \mu + \delta, \tag{1.14}$$

then the endemic equilibrium E_* of system (1.2) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$.

Note that a sufficient condition (1.14) for the endemic equilibrium to be globally asymptotically stable, is very simple and useful for a large class of SIRS models (1.2).

Let $R_0 > 1$. Then, by Theorem 1.2, the positive equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.2) exists. Assume that $0 \le a_0 \le I^* \le b_0$ and $G(I)$ is a unimodal function on $[a_0, b_0]$ and $\hat{I}$ is its maximal point of $G(I)$ on $[a_0, b_0]$, that is, $G(\hat{I}) = \max_{a_0 \le I \le b_0} G(I)$ and $G(I)$ is strictly monotone increasing on $[a_0, \hat{I}]$ and strictly monotone decreasing on $[\hat{I}, b_0]$. For simplicity, we suppose that $I^* < \hat{I}$ and $h(I)$ is a lower or upper convex function on $[a_0, b_0]$ and for any function $u(I)$ of I on $[a_0, b_0]$, we use that if $I_1 = I_2 = I^*$, then $\frac{u(I_1)-u(I_2)}{I_1-I_2}$ means $u'(I^*)$.

Corollary 1.2 *Let $R_0 > 1$ and $G(I)$ be the above unimodal function on $[a_0, b_0]$ with $I^* < \hat{I}$.*
If

$$\frac{\beta}{\mu+\gamma}\hat{I} + \frac{\hat{I}}{G(\hat{I})} > R_0, \tag{1.15}$$

then $I^* \le \limsup_{t\to\infty} I(t) < \hat{I}$.
Moreover, assume that there exist two constants $\underline{I} < \bar{I}$ such that

$$\begin{cases} a_0 \le \underline{I} \le \liminf\limits_{t\to+\infty} I(t) \le I^* \le \limsup\limits_{t\to\infty} I(t) \le \bar{I} \le \hat{I}, \\ \dfrac{\gamma}{\mu+\delta} < 1 + \dfrac{\mu+\gamma}{\beta}\dfrac{h(\bar{I}) - h(\underline{I})}{\bar{I} - \underline{I}}, \end{cases} \tag{1.16}$$

and that

$$\begin{cases} \frac{\gamma}{\mu+\delta} < 1 + \frac{\mu+\gamma}{\beta}\frac{h(I^*)-h(\underline{I})}{I^*-\underline{I}}, \quad \textit{if } h(I) \textit{ is a lower convex function on } [a_0, b_0], \\ \textit{or} \\ \frac{\gamma}{\mu+\delta} < 1 + \frac{\mu+\gamma}{\beta}\frac{h(\bar{I})-h(I^*)}{\bar{I}-I^*}, \quad \textit{if } h(I) \textit{ is an upper convex function on } [a_0, b_0]. \end{cases} \tag{1.17}$$

Then, the positive equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.2) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$.

For a particular case of $G(I)$ that

$$G(I) = G_p(I) \equiv \frac{I}{1+\alpha I^p}, \quad \alpha > 0,\ p > 0, \tag{1.18}$$

it holds that

$$h(I) = I/G_p(I) = 1 + \alpha I^p, \quad \hat{I} = \begin{cases} \frac{1}{\sqrt[p]{(p-1)\alpha}}, & \text{if } p > 1, \\ +\infty, & \text{if } p \le 1. \end{cases} \tag{1.19}$$

We obtain the following corollary for the case (1.18) (cf. Xu and Ma [17, Theorem 3.1]).

Corollary 1.3 *Let $G(I)$ be defined by (1.18) and $R_0 > 1$.*
Then the positive equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.2) exists.
If

$$p > 1 \quad \textit{and} \quad \frac{\beta}{\mu+\gamma}\hat{I} + \frac{p}{p-1} > R_0, \tag{1.20}$$

then $\limsup_{t\to+\infty} I(t) < \hat{I}$.
Moreover, assume that there exist two constants $\underline{I} < \bar{I}$ such that

$$0 \le \underline{I} \le \liminf_{t\to+\infty} I(t) \le I^* \le \limsup_{t\to\infty} I(t) \le \bar{I} < \hat{I}, \tag{1.21}$$

and that

$$\begin{cases} \frac{\gamma}{\mu+\delta} < 1 + \frac{\alpha(\mu+\gamma)}{\beta}\frac{(I^*)^p - (\underline{I})^p}{I^* - \underline{I}}, & \textit{if } \ p > 1, \\ \textit{or} \\ \frac{\gamma}{\mu+\delta} < 1 + \frac{\alpha(\mu+\gamma)}{\beta}\frac{(\bar{I})^p - (I^*)^p}{\bar{I} - I^*}, & \textit{if } \ p \le 1. \end{cases} \tag{1.22}$$

Then, the positive equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.2) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$.

Note that for $p = 1$, $G(I)$ is monotone increasing on $[0, \limsup_{t\to\infty} I(t)]$ and the condition (1.22) becomes $\alpha(\mu+\gamma)(\mu+\delta) > \beta(\gamma-\mu-\delta)$ which greatly improves the condition $\alpha(\mu+\gamma)(\mu+\delta) > \beta(\gamma+\mu+\delta)$ in Xu and Ma [17, Theorem 3.1] (see also Section 4). Moreover, the result in Corollary 1.3 is a partial answer to the open problem proposed in Huo and Ma [5] and Yang and Xiao [18] such that for $1 \le p \le 2$, the positive equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.2) with (1.18) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$. In particular, all the examples for $R_0 > 1$ in Huo and Ma [5, Fig.3] satisfy (1.22), and the examples for $R_0 > 1$ in Yang and Xiao [18, Figures 3.6-3.8] that there exists a positive τ_0 such that the endemic equilibrium E_* of (1.2) can undergo Hopf bifurcation as $\tau > \tau_0$, and a periodic orbit appears in the small neighborhood of the endemic equilibrium E_* under some conditions which do not satisfy (1.22).

The organization of this paper is as follows. In Section 2, we offer some basic results for system (1.2) and prove the local asymptotic stability of the disease-free equilibrium and prove Theorem 1.1. In Section 3, we first show the existence of the positive equilibrium of system (1.2) and prove the permanence of system (1.2). In Section 4, using monotone techniques similar to Xu and Ma [17], we prove Theorems 1.2, Corollaries 1.1-1.3. To illustrate our results, we offer numerical examples in Section 5. Finally, a short conclusion is offered in Section 6.

2. Basic results

We now state some basic results of system (1.2). Let $N(t) = S(t) + I(t) + R(t)$.

Lemma 2.1 *For system (1.2) with the initial condition (1.3),*

$$\lim_{t\to+\infty}(S(t)+I(t)+R(t)) = \frac{B}{\mu}. \tag{2.1}$$

Proof. It follows from system (1.2) that

$$\begin{aligned}\frac{dN(t)}{dt} &= B - \mu S(t) - \mu I(t) - \mu R(t)\\ &= B - \mu N(t).\end{aligned}$$

Hence, we obtain that $\lim_{t\to+\infty} N(t) = B/\mu$. This completes the proof. □

We now give the following lemmas concerning the local asymptotic stability of the disease-free equilibrium E_0 of system (1.2). This lemma is proved by using techniques in Xu and Ma [17].

Lemma 2.2 *If $R_0 < 1$, then the disease-free equilibrium E_0 of system (1.2) is locally asymptotically stable. Furthermore, the disease-free equilibrium E_0 is unstable if $R_0 > 1$.*

Proof. By $\lim_{I\to+0}\{I/G(I)\} = 1$ in (1.4), we have that $\lim_{I(t-\tau)\to+0}\{I(t-\tau)/G(I(t-\tau))\} = 1$, and hence, the characteristic equation of system (1.2) at the disease-free equilibrium E_0 is of the form (see Xu and Ma [17, Section 2])

$$(\lambda+\mu)\left\{\lambda + (\mu+\gamma)\left(1 - R_0\int_0^h k(\tau)\exp(-\lambda\tau)d\tau\right)\right\}(\lambda+\mu+\delta) = 0. \tag{2.2}$$

Clearly, $\lambda = -\mu, -(\mu+\delta)$ are always roots of (2.2). All other roots of (2.2) are determined by the following equation.

$$\lambda + (\mu+\gamma)\left(1 - R_0\int_0^h k(\tau)\exp(-\lambda\tau)d\tau\right) = 0. \tag{2.3}$$

We note that $\lambda = 0$ is not a root of (2.3). When $\tau = 0$, (2.3) becomes as follows.

$$\lambda + (\mu+\gamma)(1-R_0) = 0. \tag{2.4}$$

If $R_0 < 1$, then one can see immediately that (2.4) has a negative real root. Therefore, the disease-free equilibrium E_0 of system (1.2) is locally asymptotically stable when $\tau = 0$. Suppose that $\lambda = i\omega$, $\omega > 0$ is a root of (2.3), separating real and imaginary parts, then we derive that

$$
\begin{aligned}
&(\mu+\gamma)\left(1-R_0\int_0^h k(\tau)\cos\omega\tau d\tau\right)=0,\\
&\omega+R_0(\mu+\gamma)\int_0^h k(\tau)\sin\omega\tau d\tau=0.
\end{aligned}
\tag{2.5}
$$

From the first equation of (2.5), we see that

$$
(\mu+\gamma)\left(1-R_0\int_0^h k(\tau)\cos\omega\tau d\tau\right)\geq(\mu+\gamma)(1-R_0)>0,
$$

for all $\omega>0$, which is a contradiction. It follows that the real parts of all the eigenvalues of the characteristic equation (2.2) are negative for all $\tau\geq 0$. Therefore, if $R_0<1$, then the disease-free equilibrium E_0 of system (1.2) is locally asymptotically stable for all $\tau\geq 0$. Now, we put

$$
T(\lambda)=\lambda+(\mu+\gamma)\left(1-R_0\int_0^h k(\tau)\exp(-\lambda\tau)d\tau\right). \tag{2.6}
$$

If $R_0>1$, then it is directly seen from (2.6) that for $\lambda\in\mathbb{R}$,

$$
T(0)=(\mu+\gamma)(1-R_0)<0,\ \lim_{\lambda\to+\infty}T(\lambda)=+\infty.
$$

Therefore, (2.2) has at least one positive real root. Hence, if $R_0>1$, then the disease-free equilibrium E_0 is unstable. This completes the proof. □

Proof of Theorem 1.1. From (2.1) in Lemma 2.1, for any $\epsilon>0$, there is a constant $T_0\geq 0$ such that

$$
S(t)\leq\frac{B}{\mu}+\epsilon\quad\text{for } t\geq T_0.
$$

Consider the following nonnegative function $W(t)$ defined by

$$
W(t)=I(t)+U(t),\quad t\geq T_0,\quad\text{for}\quad t\geq T_0, \tag{2.7}
$$

where

$$
U(t)=\beta\int_0^h k(\tau)\int_{t-\tau}^t S(u+\tau)G(I(u))dud\tau,\quad t\geq T_0.
$$

Since for $t\geq T_0$,

$$
U'(t)=\beta\int_0^h k(\tau)\Big\{S(t+\tau)G(I(t))-S(t)G(I(t-\tau))\Big\}d\tau,
$$

we have that for $t \geq T_0$,

$$\begin{aligned} W'(t) &= \left\{\beta S(t)\int_0^h k(\tau)G(I(t-\tau))d\tau - (\mu+\gamma)I(t)\right\} \\ &\quad +\beta\int_0^h k(\tau)\left\{S(t+\tau)G(I(t)) - S(t)G(I(t-\tau))\right\}d\tau \\ &= \beta\int_0^h k(\tau)S(t+\tau)d\tau G(I(t)) - (\mu+\gamma)I(t) \\ &\leq \beta\int_0^h k(\tau)\left(\frac{B}{\mu}+\epsilon\right)d\tau G(I(t)) - (\mu+\gamma)I(t) \\ &= \left\{\frac{\beta B}{\mu} - (\mu+\gamma)\frac{I(t)}{G(I(t))}\right\}G(I(t)) + \beta\epsilon G(I(t)). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain that if $R_0 < 1$, then $\frac{\beta B}{\mu} - (\mu+\gamma) < 0$ and by (1.4), we have that $\frac{I}{G(I)} > 1$ for $I > 0$, and

$$\begin{aligned} W'(t) &\leq \left\{\frac{\beta B}{\mu} - (\mu+\gamma)\frac{I(t)}{G(I(t))}\right\}G(I(t)) \\ &\leq \left\{\frac{\beta B}{\mu} - (\mu+\gamma)\right\}G(I(t)) \leq 0. \end{aligned}$$

Then, the nonnegative function $W(t)$ is strictly monotone decreasing and there exists a nonnegative constant $\hat{W}$ such that $\lim_{t\to+\infty} W(t) = \hat{W}$. By the continuity of $G(I)$ and (2.8), we conclude that $\lim_{t\to+\infty} G(I(t)) = \lim_{t\to+\infty} I(t) = 0$ and $\hat{W} = 0$. Thus, for $R_0 < 1$, we obtain from (1.2) that

$$\lim_{t\to+\infty} S(t) = B/\mu, \quad \lim_{t\to+\infty} I(t) = 0, \quad \text{and} \quad \lim_{t\to+\infty} R(t) = 0.$$

It follows from Lemma 2.2 and Lyapunov-LaSalle asymptotic stability theorem that the disease-free equilibrium E_0 of system (1.2) is globally asymptotically stable if $R_0 < 1$. □

3. Permanence for $R_0 > 1$

3.1. Existence and uniqueness of the endemic equilibrium E_* of system (1.2) for $R_0 > 1$

By the condition (1.4), we also obtain the following basic lemma which ensures the existence and uniqueness of the endemic equilibrium E_* of system (1.2) for $R_0 > 1$.

Lemma 3.1 *If $R_0 > 1$, then system (1.2) has a unique endemic equilibrium E_*.*

Proof. Assume that $R_0 > 1$. From the second and the third equations of (1.6), it follows that

$$S^* = \frac{(\mu+\gamma)I^*}{\beta G(I^*)}, \tag{3.1}$$

and

$$R^* = \frac{\gamma I^*}{\mu+\delta}. \tag{3.2}$$

After substituting (3.1) into the first equation of (1.6), we obtain that

$$H(I^*) = 0,$$

where

$$H(I) \equiv B - \frac{\mu(\mu+\gamma)I}{\beta G(I)} - (\mu+\gamma)I + \gamma\frac{\delta I}{\mu+\delta} = 0.$$

By the condition of the monotonicity of $I/G(I)$ and $\lim_{I\to+0}(I/G(I)) = 1$ given in (1.4), one can see that $H(I)$ is a strictly monotone decreasing function of $I \in (0, +\infty)$, and

$$\lim_{I\to+0} H(I) = B - \frac{\mu(\mu+\gamma)}{\beta} = B\left(1 - \frac{1}{R_0}\right) > 0,$$

and for any $I > B/(\mu + \gamma\{1 - \frac{\delta}{\mu+\delta}\})$, $H(I) < 0$ holds. Hence, there exists a unique positive $I^* > 0$ such that $H(I^*) = 0$. By (3.1) and (3.2), there exists a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$. Hence, the proof is complete. □

3.2. Permanence for $R_0 > 1$

First, we prepare the following basic lemma.

Lemma 3.2 *Assume that $I(s) \le I^*$ for any s such that $t-h \le s < t$. If $I(t) < I(s)$ for any s such that $t-h \le s < t$ then $S(t) < S^*$. Inversely, if $S(t) \ge S^*$, then there exists an $s_t \in [t-h, t)$ such that $I(t) \ge I(s_t)$.*

Proof. Assume that $I(t) < I(s) \le I^*$ for any s such that $t-h \le s < t$. Then, $I'(t) < 0$ and by the second equation of (1.2) and the monotonicity of $\frac{I}{G(I)}$ for $I > 0$, we have that

$$\begin{aligned}
I'(t) &= \beta S(t)\int_0^h k(\tau)G(I(t-\tau))d\tau - (\mu+\gamma)I(t) \\
&\ge \int_0^h k(\tau)\left\{\beta S(t)G(I(t-\tau)) - (\mu+\gamma)I(t-\tau)\right\}d\tau \\
&= \int_0^h k(\tau)\left\{\beta S(t) - (\mu+\gamma)\frac{I(t-\tau)}{G(I(t-\tau))}\right\}G(I(t-\tau))d\tau \\
&\ge \int_0^h k(\tau)\left\{\beta S(t) - (\mu+\gamma)\frac{I^*}{G(I^*)}\right\}G(I(t-\tau))d\tau \\
&= \beta(S(t)-S^*)\int_0^h k(\tau)G(I(t-\tau))d\tau.
\end{aligned} \tag{3.3}$$

Then, by $I'(t) < 0$ and $\int_0^h k(\tau)G(I(t-\tau))d\tau > 0$, we obtain that

$$S(t) < S^*.$$

Inversely, assume that $I(s) \le I^*$ for any s such that $t-h \le s < t$ and $S(t) \ge S^*$. Then, it is evident that there exists an $s_t \in [t-h, t)$ such that $I(t) \ge I(s_t)$. □

Now, by applying Lemma 3.2, we offer a simplified proof for the permanence of system (1.2) than that of Wang [15] (see also Xu and Ma [17]).

Lemma 3.3 *If $R_0 > 1$, then for any solution of system (1.2) with initial condition (1.3), it holds that*

$$\begin{cases} \liminf_{t\to+\infty} S(t) \ge v_1 := \frac{B}{\mu+\beta(B/\mu)} > 0, \\ \liminf_{t\to+\infty} I(t) \ge v_2(q) := qG(I^*)\exp(-(\mu+\gamma)\rho(q)) > 0, \\ \liminf_{t\to+\infty} R(t) \ge v_3(q) := \frac{\gamma}{\mu+\delta} v_2(q), \end{cases}$$

where for any $0 < q < 1$, $\rho(q) > 0$ is a constant such that

$$S^* < S^{\triangle} := \frac{B}{r}(1-\exp(-r\rho(q))), \quad \textit{and} \quad r = \mu + \beta qG(I^*). \tag{3.4}$$

Proof. Let $(S(t), I(t), R(t))$ be any solution of system (1.2) with initial condition (1.3). By Lemma 2.1, it holds that

$$\limsup_{t\to+\infty} I(t) \le \frac{B}{\mu}.$$

For $\epsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that $I(t) < B/\mu + \epsilon$ for $t > T_1$. Then, by $G(I(t)) \le I(t)$ and the first equation of (1.2), we derive that

$$S'(t) \ge B - \{\mu + \beta(B/\mu + \epsilon)\}S(t),$$

which implies that

$$\liminf_{t\to+\infty} S(t) \ge \frac{B}{\mu + \beta(B/\mu+\epsilon)}.$$

Since the above inequality holds for arbitrary $\epsilon > 0$ sufficiently small, it follows that

$$\liminf_{t\to+\infty} S(t) \ge \frac{B}{\mu + \beta(B/\mu)} = v_1.$$

We now show that $\liminf_{t\to+\infty} I(t) \ge v_2(q)$ for any $0 < q < 1$. For any $0 < q < 1$, by (1.6), one can see that $S^* = \frac{B}{\mu+\beta G(I^*)} < \frac{B}{\mu+\beta qG(I^*)} = \frac{B}{r}$. Thus, there exist a positive constant $\rho(q)$ such that (3.4) holds.

We first prove the claim that it is not possible that for any solution $(S(t), I(t), R(t))$ of system (1.2), there exists a nonnegative constant t_0 such that $I(t) \le qG(I^*)$ for all $t \ge t_0$. Suppose on the contrary that there exist a solution $(S(t), I(t), R(t))$ of system (1.2) and a nonnegative constant t_0 such that $I(t) \leq qG(I^*)$ for all $t \geq t_0$. Then, by $G(I) \leq I$ for $I > 0$, we have that $I(t) \leq I^*$ for any $t \geq t_0$ and $G(I(t-\tau)) \leq I(t-\tau) \leq I^*$ for $t \geq t_0 + h$ and $0 \leq \tau \leq h$, and from system (1.2), one can obtain that

$$S'(t) \geq B - (\mu + \beta q G(I^*))S(t) = B - rS(t), \quad \text{for } t \geq t_0 + h,$$

which yields that

$$\begin{aligned} S(t) &\geq \exp(-r(t-t_0)) \left[S(t_0) + B \int_{t_0}^{t} \exp(r(\theta - t_0)) d\theta \right] \\ &\geq \frac{B}{r}\{1 - \exp(-r(t-t_0))\}, \quad \text{for any } t \geq t_0 + h, \end{aligned}$$

Therefore, we have that

$$S(t) \geq \frac{B}{r}\{1 - \exp(-r\rho(q))\} = S^{\triangle} > S^*, \quad \text{for any } t \geq t_0 + h + \rho(q). \quad (3.5)$$

Then, by the second part of Lemma 3.2, we obtain that $I'(t) \geq 0$ and for any $t \geq t_0 + h + \rho(q)$, there exists an $s_t \in [t-h, t)$ such that $I(t) \geq I(s_t)$. Therefore, for a positive constant $\hat{I} = \min_{t_0+\rho(q) \leq s \leq t_0+h+\rho(q)} I(s)$, we obtain that

$$I(t) \geq \hat{I} \quad \text{for any } t \geq t_0 + h + \rho(q). \quad (3.6)$$

Moreover, by $I(t) \leq qG(I^*) \leq qI^* \leq I^*$ for any $t \geq t_0 + \rho$, we have that $G(I(t)) \geq \frac{G(I^*)}{I^*} I(t) \geq \frac{G(I^*)}{I^*} \hat{I} > 0$ for any $t \geq t_0 + \rho(q)$ and for the nonnegative function $W(t)$ defined by

$$W(t) = I(t) + \beta \int_0^h k(\tau) \int_{t-\tau}^{t} S(u+\tau) G(I(u)) du d\tau, \quad (3.7)$$

we have that

$$\begin{aligned} W'(t) &= \beta S(t) \int_0^h k(\tau) G(I(t-\tau)) d\tau - (\mu + \gamma) I(t) \\ &\quad + \beta \int_0^h k(\tau)\{S(t+\tau)G(I(t)) - S(t)G(I(t-\tau))\} \\ &= \beta \int_0^h k(\tau) S(t+\tau) G(I(t)) - (\mu+\gamma) I(t) \\ &= \{\beta \int_0^h k(\tau) S(t+\tau) - (\mu+\gamma) \frac{I(t)}{G(I(t))}\} G(I(t)) \\ &> \{\beta S^{\triangle} - (\mu+\gamma) \frac{I^*}{G(I^*)}\} G(I(t)) \\ &> \beta\{S^{\triangle} - S^*\} \frac{G(I^*)}{I^*} \hat{I} > 0, \text{ for any } t \geq t_0 + h + \rho(q), \end{aligned}$$

which implies that $\lim_{t \to +\infty} W(t) = +\infty$. However, by (3.7) and Lemma 2.1, there are a positive constant $t_4 \geq t_0 + h + \rho(q)$ and $\bar{W}$ such that $W(t) \leq \bar{W}$ for any $t \geq t_4$, which lead to contradiction. Hence, the claim is proved.

By the claim, we are left to consider two possibilities. First, $I(t) \geq qG(I^*)$ for all t sufficiently large. Second, $I(t)$ oscillates about $qG(I^*)$ for all t sufficiently large.

We now show that $I(t) \geq v_2(q)$ for all t sufficiently large. If the first condition that $I(t) \geq qG(I^*)$ holds for all sufficiently large, then we get the conclusion of the proof. For the second case that $I(t)$ oscillates about $qG(I^*)$ for all sufficiently large, let $t_1 < t_2$ be sufficiently large such that

$$I(t_1) = I(t_2) = qG(I^*), \quad \text{and} \quad I(t) < qG(I^*) \quad \text{for any } t_1 < t < t_2.$$

Then, by the second equation of system (1.2), we have that

$$I(t) \geq -(\mu+\gamma)I(t), \text{ that is, } I(t) \geq I(t_1)\{1-\exp(-(\mu+\gamma)(t-t_1))\}, \text{ for any } t \geq t_1,$$

from which, we have that for any $t \geq t_1$,

$$I(t) \geq I(t_1)\{1 - \exp(-(\mu + \gamma)(t - t_1))\} \geq qG(I^*)\{1 - \exp(-(\mu + \gamma)(t - t_1))\}.$$

Therefore, we obtain that

$$I(t) \geq qG(I^*)\{-\exp(-(\mu+\gamma)\rho(q))\} = v_2(q), \quad \text{for any } t_1 \leq t \leq t_1+\rho(q). \quad (3.8)$$

If $t_2 \geq t_1 + \rho(q)$, then by applying the similar discussion to (3.5) and (3.6) in place of t_0 by t_1, we obtain that $I(t) \geq v_2(q)$ for $t_1 + \rho(q) \leq t \leq t_2$. Hence, we prove that $I(t) \geq v_2(q)$ for $t_1 \leq t \leq t_2$. Since the interval $t_1 \leq t \leq t_2$ is arbitrarily chosen, we conclude that $I(t) \geq v_2(q)$ for all sufficiently large for the second case. Since q $(0 < q < 1)$ is also arbitrarily chosen, Thus, we obtain that

$$\liminf_{t\to+\infty} I(t) \geq v_2(q).$$

From the above discussion, one can see immediately that

$$\liminf_{t\to+\infty} R(t) \geq v_3(q).$$

This completes the proof. □

By Lemmas 2.1 and 3.3, we obtain the permanence of system (1.2) for $R_0 > 1$.

4. Monotone iterative techniques to SIRS models

In this section, for $R_0 > 1$, we improve the monotone iterative technique offered by Xu and Ma [17, Theorem 3.1] for system (1.2).

By Lemma 2.2, the existence of the endemic equilibrium E_* of system (1.2) is guaranteed.

Now, by (1.2) and Lemma 3.3, we may put

$$\begin{cases} \liminf\limits_{t\to+\infty} S(t) = \hat{\underline{S}} \geq v_1, \quad \liminf\limits_{t\to+\infty} I(t) = \hat{\underline{I}} \geq v_2, \quad \liminf\limits_{t\to+\infty} R(t) = \hat{\underline{R}} \geq v_3, \\ \limsup\limits_{t\to+\infty} S(t) = \hat{\hat{S}} \leq \dfrac{B}{\mu}, \quad \limsup\limits_{t\to+\infty} I(t) = \hat{\hat{I}} \leq \dfrac{B}{\mu}, \quad \limsup\limits_{t\to+\infty} R(t) = \hat{\hat{R}} \leq \dfrac{B}{\mu}. \end{cases} \tag{4.1}$$

By Lemma 2.1, hereafter, we may restrict our attention to the case that

$$\lim_{t\to+\infty}(S(t)+I(t)+R(t)) = \frac{B}{\mu}, \quad \text{for any} \ \ t \geq 0. \tag{4.2}$$

Then, we have the following lemma.

Lemma 4.1

$$\frac{B}{\mu} - \hat{\underline{I}} - \hat{\hat{R}} > 0, \quad \textit{and} \quad \frac{B}{\mu} - \hat{\hat{I}} - \hat{\underline{R}} > 0. \tag{4.3}$$

Proof. Suppose that $\frac{B}{\mu} - \hat{\hat{I}} - \hat{\underline{R}} \leq 0$. Then, by (4.1), there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to+\infty} I(t_n) = \hat{\hat{I}}$. Since $\liminf\limits_{n\to+\infty} R(t_n) \geq \hat{\underline{R}}$, by (4.2), we have that

$$0 < \limsup_{n\to+\infty} S(t_n) \leq \frac{B}{\mu} - \liminf_{n\to+\infty} I(t_n) - \liminf_{n\to+\infty} R(t_n) \leq \frac{B}{\mu} - \hat{\hat{I}} - \hat{\underline{R}} \leq 0,$$

which is a contradiction. Thus, we have $\frac{B}{\mu} - \hat{\hat{I}} - \hat{\underline{R}} > 0$.
Similarly, we can prove that $\frac{B}{\mu} - \hat{\underline{I}} - \hat{\hat{R}} > 0$. □

Lemma 4.2

$$\begin{cases} 0 \geq B - \mu\hat{\underline{S}} - \beta\hat{\underline{S}}\bar{G}(\hat{\underline{I}}, \hat{\hat{I}}) + \delta(\frac{B}{\mu} - \hat{\underline{S}} - \hat{\hat{I}}), \\ 0 \geq \beta(\frac{B}{\mu} - \hat{\underline{I}} - \hat{\hat{R}})\underline{G}(\hat{\underline{I}}, \hat{\hat{I}}) - (\mu+\gamma)\hat{\underline{I}}, \\ 0 \geq \gamma\hat{\underline{I}} - (\mu+\delta)\hat{\underline{R}}, \end{cases} \tag{4.4}$$

and

$$\begin{cases} 0 \leq B - \mu\hat{\hat{S}} - \beta\hat{\hat{S}}\underline{G}(\hat{\underline{I}}, \hat{\hat{I}}) + \delta(\frac{B}{\mu} - \hat{\hat{S}} - \hat{\underline{I}}), \\ 0 \leq \beta(\frac{B}{\mu} - \hat{\hat{I}} - \hat{\underline{R}})\bar{G}(\hat{\underline{I}}, \hat{\hat{I}}) - (\mu+\gamma)\hat{\hat{I}}, \\ 0 \leq \gamma\hat{\hat{I}} - (\mu+\delta)\hat{\hat{R}}, \end{cases} \tag{4.5}$$

Proof. Assume that $I(t)$ is eventually monotone decreasing for $t \geq 0$. Then, by Lemma (3.3), there exists $\lim\limits_{t\to+\infty} I(t) = \hat{\hat{I}} = \hat{\underline{I}} > 0$. Then, by the third equation of (1.2), we obtain that there exists $\lim\limits_{t\to+\infty} R(t) = \hat{\hat{R}} = \hat{\underline{R}} > 0$. Then, by the first equation of (1.2), we obtain that there exists $\lim\limits_{t\to+\infty} S(t) = \hat{\hat{S}} = \hat{\underline{S}} > 0$. Since the positive equilibrium $E^* = (S^*, I^*, R^*)$ is unique, we have that $\hat{S}^* = \hat{\bar{S}} = \underline{\hat{S}}$, $\hat{I}^* = \hat{\bar{I}} = \underline{\hat{I}}$ and $\hat{R}^* = \hat{\bar{R}} = \underline{\hat{R}}$. Thus, by (1.6), (4.5) holds.

Now, suppose that $I(t)$ is not eventually monotone decreasing for $t \geq 0$. Then, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to+\infty} I'(t_n) \geq 0$ and $\lim_{n\to+\infty} I(t_n) = \hat{\bar{I}}$. Since by (4.2),

$$\limsup_{n\to+\infty} S(t_n) \leq \frac{B}{\mu} - \lim_{n\to+\infty} I(t_n) - \liminf_{n\to+\infty} R(t_n) \leq \frac{B}{\mu} - \hat{\bar{I}} - \underline{\hat{R}},$$

we can immediately derive (4.5). Similarly, we can obtain (4.4). This completes the proof. □

Then, we obtain that

$$\begin{cases} \underline{\hat{S}} \geq \frac{B(1+\frac{\delta}{\mu})-\delta\hat{\bar{I}}}{(\mu+\delta)+\beta\bar{G}(\underline{\hat{I}},\hat{\bar{I}})}, \\ \underline{\hat{I}} + \frac{\mu+\gamma}{\beta}\underline{h}(\underline{\hat{I}}, \hat{\bar{I}}) \geq \frac{B}{\mu} - \frac{\gamma}{\mu+\delta}\hat{\bar{I}}, \\ \underline{\hat{R}} \geq \frac{\gamma}{\mu+\delta}\underline{\hat{I}}, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \hat{\bar{S}} \leq \frac{B(1+\frac{\delta}{\mu})-\delta\underline{\hat{I}}}{(\mu+\delta)+\beta\underline{G}(\underline{\hat{I}},\hat{\bar{I}})}, \\ \hat{\bar{I}} + \frac{\mu+\gamma}{\beta}\bar{h}(\underline{\hat{I}}, \hat{\bar{I}}) \leq \frac{B}{\mu} - \frac{\gamma}{\mu+\delta}\underline{\hat{I}}, \\ \hat{\bar{R}} \leq \frac{\gamma}{\mu+\delta}\hat{\bar{I}}. \end{cases} \tag{4.7}$$

We now consider the following four sequences $\bar{S}_n$, $\bar{I}_n$, $\underline{S}_n$ and $\underline{I}_n$, $(n = 1, 2, \cdots)$ as follows (cf. Xu and Ma [17, (3.3)]).

$$\begin{cases} 0 \leq \underline{I}_0 \leq \liminf_{t\to+\infty} I(t), \\ \bar{K}(\underline{I}_{n-1}, \bar{I}_n) = \dfrac{B}{\mu} - \dfrac{\gamma}{\mu+\delta}\underline{I}_{n-1}, \\ \underline{K}(\underline{I}_n, \bar{I}_n) = \dfrac{B}{\mu} - \dfrac{\gamma}{\mu+\delta}\bar{I}_n, \quad n = 1, 2, 3 \cdots . \end{cases} \tag{4.8}$$

and

$$\begin{cases} \underline{S}_n = \dfrac{B(\mu+\delta)/\mu_1 - \delta\bar{I}_n}{(\mu+\delta) + \beta\bar{G}(\underline{I}_n, \bar{I}_n)}, \quad \underline{R}_n = \dfrac{\gamma}{\mu+\delta}\underline{I}_n, \\ \bar{S}_n = \dfrac{B(\mu+\delta)/\mu_1 - \delta\underline{I}_{n-1}}{(\mu+\delta) + \beta\underline{G}(\underline{I}_{n-1}, \bar{I}_n)}, \quad \bar{R}_n = \dfrac{\gamma}{\mu+\delta}\bar{I}_n, \end{cases} \tag{4.9}$$

where the functions $\bar{K}(\underline{I}, \bar{I})$ and $\underline{K}(\underline{I}, \bar{I})$ are defined such that for any $0 \leq \underline{I} \leq \bar{I}$,

$$\bar{K}(\underline{I}, \bar{I}) = \bar{I} + \frac{\mu+\gamma}{\beta}\bar{h}(\underline{I}, \bar{I}), \quad \text{and} \quad \underline{K}(\underline{I}, \bar{I}) = \underline{I} + \frac{\mu+\gamma}{\beta}\underline{h}(\underline{I}, \bar{I}). \tag{4.10}$$

Then, by Lemma 3.3, (4.7) and (4.8), we have that

$$\underline{I}_0 \le \liminf_{t\to+\infty} I(t) \le \limsup_{t\to+\infty} I(t) \le \bar{I}_1. \tag{4.11}$$

Lemma 4.3 *For the sequences $\{\bar{I}_n\}_{n=1}^\infty$, $\{\underline{I}_n\}_{n=1}^\infty$, $\{\bar{S}_n\}_{n=1}^\infty$, $\{\underline{S}_n\}_{n=1}^\infty$ defined by (4.8) and (4.9), assume $\underline{I}_0 < \bar{I}_1$. Then,*

$$\underline{I}_0 < \underline{I}_1 < \bar{I}_1, \tag{4.12}$$

if and only if,

$$\frac{\gamma}{\mu+\delta} < 1 + \frac{\mu+\gamma}{\beta}\frac{\bar{h}(\underline{I}_0,\bar{I}_1) - \underline{h}(\underline{I}_1,\bar{I}_1)}{\bar{I}_1 - \underline{I}_1}. \tag{4.13}$$

In this case, the three sequences $\{\underline{I}_n\}_{n=1}^\infty$, $\{\underline{S}_n\}_{n=1}^\infty$ and $\{\underline{R}_n\}_{n=1}^\infty$ are strongly monotone increasing sequences and converge to $\underline{I}^*$, $\underline{S}^*$ and $\underline{R}^*$, respectively, and the three sequences $\{\bar{I}_n\}_{n=1}^\infty$, $\{\bar{S}_n\}_{n=1}^\infty$ and $\{\bar{R}_n\}_{n=1}^\infty$ are strongly monotone decreasing sequences and converge to $\bar{I}^*$, $\bar{S}^*$ and $\bar{R}^*$, respectively, as n tends to $+\infty$, and

$$\begin{cases} \lim_{n\to+\infty} \underline{I}_n = \underline{I}^* \le \liminf_{t\to+\infty} I(t) \le \limsup_{t\to+\infty} I(t) \le \lim_{n\to+\infty} \bar{I}_n = \bar{I}^*, \\ \lim_{n\to+\infty} \underline{S}_n = \underline{S}^* \le \liminf_{t\to+\infty} S(t) \le \limsup_{t\to+\infty} S(t) \le \lim_{n\to+\infty} \bar{S}_n = \bar{S}^*, \\ \lim_{n\to+\infty} \underline{R}_n = \underline{R}^* \le \liminf_{t\to+\infty} R(t) \le \limsup_{t\to+\infty} R(t) \le \lim_{n\to+\infty} \bar{R}_n = \bar{R}^*, \end{cases} \tag{4.14}$$

and

$$\begin{cases} \bar{I}^* + \frac{\gamma}{\mu+\delta}\underline{I}^* + \frac{\mu+\gamma}{\beta}\bar{h}(\underline{I}^*,\bar{I}^*) = \frac{B}{\mu}, \\ \underline{I}^* + \frac{\gamma}{\mu+\delta}\bar{I}^* + \frac{\mu+\gamma}{\beta}\underline{h}(\underline{I}^*,\bar{I}^*) = \frac{B}{\mu}, \\ \text{and} \\ 1 + \frac{\mu+\gamma}{\beta}\frac{\bar{h}(\underline{I}^*,\bar{I}^*) - \underline{h}(\underline{I}^*,\bar{I}^*)}{\bar{I}^* - \underline{I}^*} = \frac{\gamma}{\mu+\delta}, \ \text{if} \ \underline{I}^* < \bar{I}^*, \\ I^* \le \hat{I}, \ \text{if} \ \underline{I}^* = \bar{I}^* = I^*. \end{cases} \tag{4.15}$$

Moreover, if (1.11) and (1.12) hold, then

$$\underline{I}^* = \bar{I}^* = I^* \le \hat{I}, \quad \underline{S}^* = \bar{S}^* = S^* \quad \text{and} \quad \underline{R}^* = \bar{R}^* = R^*. \tag{4.16}$$

In particular, if

$$\frac{\gamma}{\mu+\delta} \le 1, \tag{4.17}$$

then (4.16) holds.

Proof. By (4.8) and (4.10),

$$\begin{cases} \bar{I}_n + \frac{\mu+\gamma}{\beta}\bar{h}(\underline{I}_{n-1},\bar{I}_n) = \frac{B}{\mu} - \frac{\gamma}{\mu+\delta}\underline{I}_{n-1}, \\ \underline{I}_n + \frac{\mu+\gamma}{\beta}\underline{h}(\underline{I}_n,\bar{I}_n) = \frac{B}{\mu} - \frac{\gamma}{\mu+\delta}\bar{I}_n, \quad n = 1,2,3,\cdots, \end{cases} \tag{4.18}$$

from which we have that for $\underline{I}_n < \bar{I}_n$ and $n = 1, 2, 3, \cdots$,

$$\left(1 + \frac{\mu+\gamma}{\beta} \frac{\bar{h}(\underline{I}_{n-1}, \bar{I}_n) - \underline{h}(\underline{I}_n, \bar{I}_n)}{\bar{I}_n - \underline{I}_n}\right)(\bar{I}_n - \underline{I}_n) = \frac{\gamma}{\mu+\delta}(\bar{I}_n - \underline{I}_{n-1}).$$

Hence, we obtain that for $\underline{I}_n < \bar{I}_n$,

$$\bar{I}_n - \underline{I}_n = \frac{\frac{\gamma}{\mu+\delta}}{1 + \frac{\mu+\gamma}{\beta} \frac{\bar{h}(\underline{I}_{n-1}, \bar{I}_n) - \underline{h}(\underline{I}_n, \bar{I}_n)}{\bar{I}_n - \underline{I}_n}}(\bar{I}_n - \underline{I}_{n-1}), \quad n = 1, 2, 3, \cdots, \tag{4.19}$$

from which one can see that (4.12) holds, if and only if, (4.13) holds. Then, by the monotonicity and inductions in (4.18), we can prove that $\underline{I}_{n-1} < \underline{I}_n < \bar{I}_n < \bar{I}_{n-1}$, $n = 2, 3, \cdots$, (4.14) and (4.15) hold. Moreover, suppose that (1.11) and (1.12) hold. Then, $I^* \leq \hat{I}$ and by (4.19), we obtain (4.16). Hence, by (4.6) and (4.7), we obtain the conclusion of this lemma. □

Proof of Theorem 1.2. By Lemma 4.3, we obtain the conclusion of Theorem 1.2. □

Now, we give a property of a lower or upper convex function of I on $[a_0, b_0]$.

Lemma 4.4 *If $h(I)$ is a lower or upper convex function on $[a_0, b_0]$, then for any $a_0 \leq \underline{I} \leq I^* \leq \bar{I} \leq b_0$,*

$$\begin{cases} \frac{h(I^*)-h(\underline{I})}{I^*-\underline{I}} \leq \frac{h(\bar{I})-h(\underline{I})}{\bar{I}-\underline{I}}, & \textit{if } h(I) \textit{ is a lower convex function on } [a_0, b_0], \\ \textit{or} \\ \frac{h(\bar{I})-h(I^*)}{\bar{I}-I^*} \leq \frac{h(\bar{I})-h(\underline{I})}{\bar{I}-\underline{I}}, & \textit{if } h(I) \textit{ is an upper convex function on } [a_0, b_0]. \end{cases} \tag{4.20}$$

Proof. The proof of this lemma is evident from the definition of a lower and upper convex function of I on $[a_0, b_0]$. □

Proof of Corollaries 1.1-1.3. Since by assumption that $I^* < \hat{I}$ and the first equation of (4.15) in Lemma 4.3, we have that

$$\frac{\beta}{\mu+\gamma}\bar{I}^* + \bar{h}(\underline{I}^*, \bar{I}^*) = R_0 - \frac{\beta}{\mu+\gamma}\frac{\gamma}{\mu+\delta}\underline{I}^* < R_0.$$

Therefore, from (1.15), one can easily see that $\bar{I}^* < \hat{I}$. Then, by using the results in Theorem 1.2 with Lemmas 4.3 and 4.4, we immediately obtain the conclusion of Corollaries 1.1-1.3. □

For the cases $p = 1$ and $p > 1$, by Corollary 1.3, we easily obtain the following result.

Corollary 4.1 *Let $p \geq 1$ and $R_0 > 1$. If*

$$\{\alpha(\mu+\gamma) + \beta\}(\mu+\delta) - \beta\gamma > 0, \quad \textit{for } \ p = 1, \tag{4.21}$$

or (1.20)-(1.22) hold for $p > 1$, then the positive equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.2) with (1.18) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$. In particular, for $p = 2$, if

$$\frac{\beta}{(\mu+\gamma)\sqrt{\alpha}} + 2 > R_0 \quad \textit{and} \quad \frac{\mu+\delta}{\gamma}\left(1 + \frac{\mu+\gamma}{\beta} I^*\right) > 1, \tag{4.22}$$

then, conditions (1.20)-(1.22) are satisfied.

Note that the sufficient condition in Xu and Ma [17, Theorem 3.1] for $p = 1$ becomes $\alpha(\mu+\gamma)(\mu+\delta) > \beta(\gamma+\mu+\delta)$, but the condition (4.22) is $\alpha(\mu+\gamma)(\mu+\delta) > \beta(\gamma - \mu - \delta)$. Moreover, for the case $p = 2$ and $R_0 > 1$, we solve the open question to an example in Huo and Ma [5, Example], because one can see that this example satisfies the condition (4.22) in Corollary 4.1 see Muroya *et al.* [12, Theorem 1.1].

5. Numerical examples

In this section, we restrict our attention only to the following case in system (1.2):

$$\begin{cases} G(I) = \frac{I}{1+\alpha I}, \\ B = 2, \quad \alpha = 1, \quad \gamma = 0.1, \quad \mu = 0.3, \quad \delta = 0.2, \\ h = 10, \quad \text{and} \quad k(\tau) = 0.1, \end{cases} \tag{5.1}$$

and $\beta > 0$ will be determined later.

By Theorem 1.1, we see that the disease-free equilibrium E_0 of system (1.2) is globally asymptotically stable if $R_0 < 1$ which is equivalent to $\beta < 0.06$. Figure 1 for the case (5.1) with $\beta = 0.05 < 0.06$, indicates that $R_0 = 0.833\cdots < 1$ and the disease-free equilibrium $E_0 = (6.666\cdots, 0, 0)$ of system (1.2) is globally asymptotically stable.

By Corollary 4.1, we see that the endemic equilibrium E^* of system (1.2) is globally asymptotically stable if $R_0 > 1$ and it holds (4.21) in Corollary 4.1, which is equivalent to $0.06 < \beta < 0.88$. On the other hand, the sufficient condition for the global asymptotic stability of the endemic equilibrium of system (1.2) in Xu and Ma [17] becomes $0.06 < \beta < 0.333\cdots$. Thus, the condition (4.21) in Corollary 4.1 greatly improves the condition in Xu and Ma [17]. Figure 2 gives a graph trajectory of $\underline{I}_n$ and $\overline{I}_n$ for the case (5.1) with $\beta = 0.5$ and the starting value $\underline{I}_0 = 0$ in (4.8), indicates that $R_0 = 8.333\cdots > 1$ and both $\bar{I}_n$ and $\underline{I}_n$ $(n \geq 1)$ defined by (4.8) converge to $I^* = 2.933\cdots$, and Figure 3 indicates that the endemic equilibrium $E^* = (3.146\cdots, 2.933\cdots, 0.586\cdots)$ of system (1.2) is globally asymptotically stable.

6. Conclusion

In this paper, for SIRS epidemic models with a class of nonlinear incidence rates and distributed delays of the forms $\beta S(t)\int_0^h k(\tau)G(I(t-\tau))d\tau$, we establish

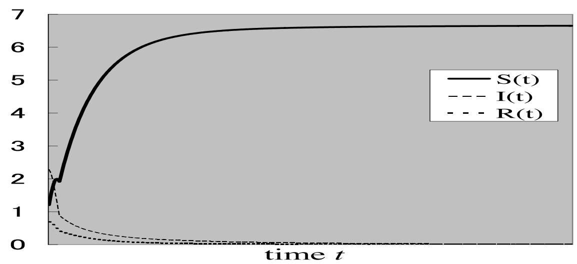

Figure 1: A graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of system (1.2) for the case (5.1) with $\beta = 0.05$. We have $R_0 = 0.833\cdots < 1$ and $E_0 = (6.666\cdots, 0, 0)$.

the global asymptotic stability of the disease-free equilibrium E_0 for $R_0 < 1$, and applying new monotone iterative techniques, we obtain sufficient conditions for the global asymptotic stability of the endemic equilibrium of systems (1.2) for $R_0 > 1$, respectively. In particular, by applying Lemma 3.2, we offer a simplified proof for the permanence of system (1.2) than that of Wang [15] (see also Xu and Ma [17]). We note that a sufficient condition (1.14) obtained by a simple conditions of contractive convergence for suitable monotone iterations (see (4.8)), for the endemic equilibrium to be globally asymptotically stable, is very useful for a large class of SIRS models (1.2).

We also note that the conditions (1.4) and (1.13) play important roles to obtain the global asymptotic stability of the endemic equilibrium E_* of systems (1.2) for $R_0 > 1$, respectively. Moreover, by the sake of Lemma 4.2, our monotone iterative techniques become much improved one than that in Xu and Ma [17] which was applied to the saturated incidence rate $G(I) = \frac{I}{1+\alpha I}$.

As a result, we have solved the conjecture to the example in Huo and Ma [5] that the endemic equilibrium of system (1.2) is globally asymptotically stable if $R_0 > 1$, and also offer partial answers to the open problem in Huo and Ma [5] and Yang and Xiao [18].

These techniques are also applicable to various kinds of epidemic models with delays. These will be our future works.

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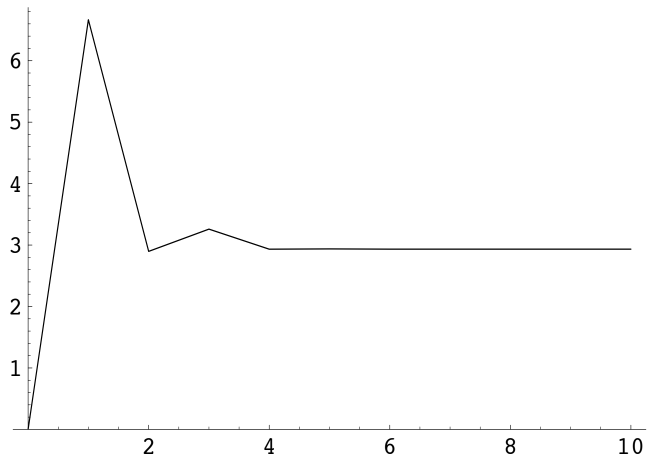

Figure 2: A graph trajectory of $\underline{I}_n$ and $\overline{I}_n$ $(0 = \underline{I}_0 \to \overline{I}_1 \to \underline{I}_1 \to \cdots)$ defined by (4.8) for the case (5.1) with $\beta = 0.5$.

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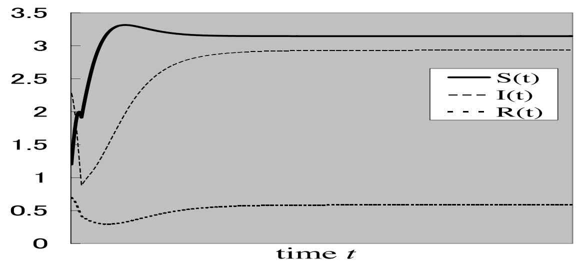

Figure 3: A graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of system (1.2) for the case (5.1) with $\beta = 0.5$. We have $R_0 = 8.333\cdots > 1$ and $E^* = (3.146\cdots, 2.933\cdots, 0.586\cdots)$.

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