Observability Estimate for Stochastic Schrödinger Equations*

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Abstract

In this paper, we obtain the observability estimate for stochastic Schrödinger equations evolved in a bounded domain of $\mathbb{R}^n$, by means of Carleman estimate. Our Carleman estimate is based on a new fundamental identity for stochastic Schrödinger-like operators established by the stochastic calculation. As an application, we establish a unique continuation property for stochastic Schrödinger equations.

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1 Introduction and Main Results

Let $T > 0$, $G \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a given bounded domain with a C^2 boundary Γ. Let Γ_0 be a suitable chosen nonempty subset of Γ, whose definition will be given later. Put

$$Q \stackrel{\triangle}{=} (0,T) \times G, \ \Sigma \stackrel{\triangle}{=} (0,T) \times \Gamma, \ \text{and } \Sigma_0 \stackrel{\triangle}{=} (0,T) \times \Gamma_0.$$

Throughout this paper, we will use C to denote a generic positive constant depending only on T, G and Γ_0, which may change from line to line.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined. Let H be a Banach space. Denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$

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such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega;C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C(0,T;H)}) < \infty$. All of these spaces are endowed with the canonical norm.

Let us consider the following stochastic Schrödinger equation:

$$\begin{cases} idy + \Delta y = (a_1 \cdot \nabla y + a_2 y + f)dt + (a_3 y + g)dB & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{1.1}$$

with initial datum $y_0 \in L^2(\Omega, \mathcal{F}_0, P; H^1_0(G))$, suitable coefficients $a_i(\ i = 1,2,3\ )$, and source terms f and g.

Put

$$H_T \triangleq L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1_0(G))). \tag{1.2}$$

We begin with the following definition.

Definition 1.1 *We call $y \in H_T$ is a solution of equation (1.1) if the followings hold:*
1. $y(0) = y_0$ in G, P-a.s.;
2. For any $t \in [0,T]$ and any $\eta \in H^1_0(G)$, it holds that

$$\begin{aligned} &\int_G iy(t,x)\eta(x)dx - \int_G iy(0,x)\eta(x)dx \\ &= \int_0^t \int_G \Big\{ \nabla y(s,x) \cdot \nabla \eta(x) + (a_1 \cdot \nabla y + a_2 y + f)\eta(x) \Big\} dxds \\ &\quad + \int_0^t \int_G (a_3 y + g)\eta(x) dx dB(s), \quad P\text{-a.s.} \end{aligned}$$

We refer to [4, Chapter 6] and [10, Chapter 5] for the well-posedness of equation (1.1) under suitable assumptions in the class $y \in H_T$(The assumptions in Theorem 1.1 below are enough).

The main purpose of this paper is to establish an observability estimate for equation (1.1) under the following assumptions.

Denote by $\nu(x)$ the unit outward normal vector of G at $x \in \Gamma$. Assume that the set Γ_0 is given by

$$\Gamma_0 \triangleq \{x \in \Gamma :\ (x - x_0) \cdot \nu(x) > 0\}. \tag{1.3}$$

Also, assume that

$$a_1 \in L^\infty_{\mathcal{F}}(0,T; W^{1,\infty}(G;\mathbb{R}^n)),\ a_2 \in L^\infty_{\mathcal{F}}(0,T; W^{1,\infty}(G)),\ a_3 \in L^\infty_{\mathcal{F}}(0,T; W^{1,\infty}(G)), \tag{1.4}$$

and that

$$f \in L^2_{\mathcal{F}}(0,T; H^1_0(G)), \qquad g \in L^2_{\mathcal{F}}(0,T; H^1(G)). \tag{1.5}$$

Under the above assumptions, we obtain the following result.

Theorem 1.1 *Let a_i $(1 \le i \le 3)$ satisfy (1.4), f, g satisfy (1.5). Then for any solution of equation (1.1) with initial datum y_0, we have that*

$$
\begin{aligned}
&|y(T)|_{L^2(\Omega,\mathcal{F}_T,P;H_0^1(G))}\\
&\le Ce^{Cr_1}\Big(\Big|\frac{\partial y}{\partial \nu}\Big|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))}+|f|_{L^2_{\mathcal{F}}(0,T;H_0^1(G))}+|g|_{L^2_{\mathcal{F}}(0,T;H^1(G))}\Big),
\end{aligned}
\tag{1.6}
$$

where

$$
r_1 \overset{\triangle}{=} |a_1|^2_{L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G;\mathbb{R}^n))}+|a_2|^2_{L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G))}+|a_3|^2_{L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G))}+1. \tag{1.7}
$$

In the deterministic case, there exist many approaches and results addressing the observability estimate for Schrödinger equations. For example, results in the spirit of Theorem 1.1 are obtained by Carleman estimate ([2, 12, 18]), by the classical Rellich-type multiplier approach ([17]), by the microlocal analysis approach ([13, 20]), and so on. We refer to [27] for a nice survey in this respect. Note however that, almost all of these mentioned works use essentially the nature of time-reversibility for Schrödinger equations, in one way or another. Therefore, one cannot simply mimic the existing methods for deterministic Schrödinger equations to derive inequality (1.6) because of the *time-irreversibility* of equation (1.1).

In this paper, we establish the required observability estimate by utilizing the global Carleman estimate. In order to overcome the difficulty of the *time-irreversibility*, we borrow some idea from [8] and introduce a weight function with singularity in time at 0 and T.

As a consequence of Theorem 1.1, we have the following unique continuation property for the solutions of equation (1.1).

Theorem 1.2 *For any $\varepsilon > 0$, let*

$$
O_\varepsilon(\Gamma_0 \times [0,T]) \overset{\triangle}{=} \Big\{(x,t) \in Q : \operatorname{dist}\big((x,t),\Gamma_0\times[0,T]\big) \le \varepsilon\Big\}.
$$

Let $f = g = 0$, P-a.s. For any y which solves equation (1.1), if $y = 0$ in $O_\varepsilon(\Gamma_0 \times [0,T])$ P-a.s., then $y = 0$ in Q, P-a.s.

There are numerous works on the unique continuation property for partial differential equations. The study of it began at the very beginning of the 20th century. In the last 1950-70's, there is a climax of the study of it. Most of the existing works are addressing to the local unique continuation property at that time. In the recent 20 years, due to the need from Control/Inverse Problems of partial differential equations, the study of the global unique continuation for partial differential equations is very active(see [3, 22, 26] and the references therein). Comparing with the fruitful studying of the unique continuation property for partial differential equations, there are few results for stochastic partial differential equations. As far as we know, [23, 24] are the only published articles which concern with this topic, and there is no result about the global unique continuation property for stochastic Schrödinger equations in the literature.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results, including some energy estimate and the hidden regularity for the solutions of equation (1.1). Section 3 is addressed to establish a crucial identity for a stochastic Schrödinger-like operator. Then, in Section 4, we derive the Carleman estimate. At last, in Section 5, we prove Theorem 1.1 and Theorem 1.2.

2 Some preliminaries

In this section, we will give some preliminary results which will be used later.

To begin with, for the sake of completeness, we give an energy estimate for the solution of equation (1.1).

Proposition 2.1 *Under assumptions (1.4) and (1.5), for any* $y_0 \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))$, *we have that*

$$\mathbb{E}|y(t)|^2_{H_0^1(G)} \leq Ce^{Cr_1}\Big(\mathbb{E}|y(s)|^2_{H_0^1(G)} + |f|^2_{L^2_{\mathcal{F}}(0,T;H_0^1(G))} + |g|^2_{L^2_{\mathcal{F}}(0,T;H^1(G))}\Big), \tag{2.1}$$

for any $0 \leq s \leq t \leq T$.

Remark 2.1 *In fact, the proof of this proposition is standard, i.e., by utilizing the usual energy estimate. The only thing one needs paying attention to is the utilizing of stochastic calculation rules.*

Proof of Proposition 2.1: In order to establish inequality (2.1), we compute

$$\mathbb{E}|y(t)|^2_{L^2(G)} - \mathbb{E}|y(s)|^2_{L^2(G)} \ \text{ and } \ \mathbb{E}|\nabla y(t)|^2_{L^2(G)} - \mathbb{E}|\nabla y(s)|^2_{L^2(G)}.$$

The first one reads

$$\begin{aligned}
&\mathbb{E}|y(t)|^2_{L^2(G)} - \mathbb{E}|y(s)|^2_{L^2(G)}\\
&= \mathbb{E}\int_s^t\int_G (yd\bar{y} + \bar{y}dy + dyd\bar{y})dx\\
&= \mathbb{E}\int_s^t\int_G \Big\{ - iy(\Delta\bar{y} - a_1\cdot\nabla\bar{y} - a_2\bar{y} - \bar{f}) + i\bar{y}(\Delta y - a_1\cdot\nabla y - a_2 y - f)\\
&\qquad\qquad + (a_3 y + g)(a_3\bar{y} + \bar{g})\Big\}dxd\sigma\\
&= \mathbb{E}\int_s^t\int_G \Big\{ - i[\text{div}\,(y\nabla\bar{y}) - |\nabla y|^2 - \text{div}\,(|y|^2 a_1) + \text{div}\,(a_1)|y|^2 - a_2|y|^2 - y\bar{f}]\\
&\qquad\qquad + i[\text{div}\,(\bar{y}\nabla y) - |\nabla y|^2 - \text{div}\,(|y|^2 a_1) + \text{div}\,(a_1)|y|^2 - a_2|y|^2 - f\bar{y}]\\
&\qquad\qquad + (a_3 y + g)(a_3\bar{y} + \bar{g})\Big\}dxd\sigma\\
&\leq \mathbb{E}\int_s^t\int_G 2\Big[(|a_3|_{L^\infty(G)} + 1)|y|^2_{L^2(G)} + |f|^2_{L^2(G)} + |g|^2_{L^2(G)}\Big]dxd\sigma.
\end{aligned} \tag{2.2}$$

The second one reads

$$\begin{aligned}
&\mathbb{E}|\nabla y(t)|^2_{L^2(G)} - \mathbb{E}|\nabla y(s)|^2_{L^2(G)}\\
&= \mathbb{E}\int_s^t\int_G (\nabla y d\bar{y} + \nabla\bar{y}dy + d\nabla y d\nabla\bar{y})dx\\
&= \mathbb{E}\int_s^t\int_G \Big\{\text{div}\,(\nabla y d\bar{y}) - \Delta y d\bar{y} + \text{div}\,(\nabla\bar{y}dy) - \Delta\bar{y}dy + d\nabla y d\nabla\bar{y}\Big\}dx
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\int_s^t\int_G\Big\{\Delta y\Big[i(\Delta\bar{y} - a_1\cdot\nabla\bar{y} - a_2\bar{y} - f)\Big] - \Delta\bar{y}\Big[i(\Delta y - a_1\cdot\nabla y - a_2 y - f)\Big] \\
&\qquad\qquad +\nabla(a_3 y + g)\nabla(a_3\bar{y} + \bar{g})\Big\}dxd\sigma \\
&\le 2\mathbb{E}\int_s^t\int_G\Big\{(|a_1|^2_{W^{1,\infty}(G;\mathbb{R}^m)} + |a_3|^2_{W^{1,\infty}(G)} + 1)|\nabla y|^2_{L^2(G)} \\
&\qquad\qquad +(|a_2|^2_{W^{1,\infty}(G)} + |a_3|^2_{W^{1,\infty}(G)} + 1)|y|^2_{L^2(G)} + |f|^2_{H_0^1(G)} + |g|^2_{H^1(G)}\Big\}dxd\sigma.
\end{aligned}
\tag{2.3}
$$

From inequality (2.2) and inequality (2.3), we know that

$$
\begin{aligned}
&\mathbb{E}|y(t)|^2_{H_0^1(G)} - \mathbb{E}|y(s)|^2_{H_0^1(G)} \\
&\le 2r_1\mathbb{E}\int_s^t\int_G |y(s)|^2_{H_0^1(G)}dxd\sigma + \mathbb{E}\int_s^t\int_G(|f|^2_{H_0^1(G)} + |g|^2_{H^1(G)})dxd\sigma.
\end{aligned}
\tag{2.4}
$$

Therefore, utilizing Gronwall's inequality, we arrive at

$$
\mathbb{E}|y(t)|^2_{H_0^1(G)} \le e^{Cr_1}\Big\{\mathbb{E}|y(s)|^2_{H_0^1(G)} + \mathbb{E}\int_0^t\int_G(|f|^2_{H_0^1(G)} + |g|^2_{H^1(G)})dxd\sigma\Big\}, \tag{2.5}
$$

which implies (2.1) immediately. □

Nextly, we give a result about the hidden regularity for solutions of equations (1.1), i.e., it shows that, solutions of equation (1.1) have some regularity on the boundary than the one deduced from the classical Trace Theorem of Sobolev spaces directly.

Proposition 2.2 *Let a_i $(1 \le i \le 3)$ satisfy (1.4), f, g satisfy (1.5). Then for any solution of equation (1.1) with initial datum y_0, it holds that*

$$
\Big|\frac{\partial y}{\partial\nu}\Big|^2_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))} \le Ce^{Cr_1}\Big(|y_0|^2_{L^2(\Omega,\mathcal{F}_0,P;H_0^1(G))} + |f|_{L^2_{\mathcal{F}}(0,T;H_0^1(G))} + |g|_{L^2_{\mathcal{F}}(0,T;H^1(G))}\Big). \tag{2.6}
$$

Remark 2.2 *By means of Proposition 2.2, we know that*

$$
\Big|\frac{\partial y}{\partial\nu}\Big|^2_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))} < \infty.
$$

Comparing with Theorem 1.1, Proposition 2.2 tells us that $\Big|\frac{\partial y}{\partial\nu}\Big|^2_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))}$ can be bounded by the initial energy of the equation and the non-homogenous terms. This result is an reverse of Theorem 1.1 in some sense.

In order to prove Proposition 2.2, we first establish the following pointwise identity.

Proposition 2.3 *Let $\mu = \mu(x) = (\mu^1, \cdots, \mu^n) : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^1 and z an $H^2_{loc}(\mathbb{R}^n)$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process. Then for a.e. $x \in \mathbb{R}^n$ and P-a.s. $\omega \in \Omega$, it holds that*

$$\begin{aligned}
&\mu \cdot \nabla \bar{z}(idz + \Delta z dt) + \mu \cdot \nabla z(-id\bar{z} + \Delta \bar{z} dt) \\
&= \nabla\Big[(\mu \cdot \nabla z)\nabla \bar{z} + (\mu \cdot \nabla \bar{z})\nabla z - izd\bar{z}\mu - |\nabla z|^2 \mu\Big]dt + d(i\mu\nabla \bar{z} z) - \sum_{j,k=1}^{n} \mu^k_j (z_j \bar{z}_k + \bar{z}_j z_k) dt \\
&\quad + \nabla \cdot \mu |\nabla z|^2 dt + i\nabla \cdot \mu z d\bar{z} - i\mu \nabla d\bar{z} z. \qquad (2.7)
\end{aligned}$$

Proof of Proposition 2.3 : For simplicity, here and in the sequel, we will use the notation $y_j \equiv y_j(x) \stackrel{\triangle}{=} \dfrac{\partial y(x)}{\partial x_j}$, where x_j is the j-th coordinate of a generic point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$. In a similar manner, we will use the notation z_i, v_i, etc., for the partial derivatives of z and v with respect to x_i.

The proof is a direct computation. We have that

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{j=1}^{n} \mu^k \bar{z}_k z_{jj} + \sum_{k=1}^{n}\sum_{j=1}^{n} \mu^k z_k \bar{z}_{jj} \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} \Big[(\mu^k \bar{z}_k z_j)_j + (\mu^k z_k \bar{z}_j)_j + \mu^k_k |z_j|^2 - (\mu^k |z_j|^2)_k - \mu^k_j \bar{z}_k z_j - \mu^k_j \bar{z}_j z_k\Big] \qquad (2.8)
\end{aligned}$$

and that

$$i\sum_{k=1}^{n} (\mu^k \bar{z}_k dz - \mu^k z_k d\bar{z}) = i\sum_{k=1}^{n} \Big[d(\mu^k \bar{z}_k z) - \mu^k d\bar{z}_k dz - (\mu^k z d\bar{z})_k + \mu^k_k z d\bar{z}\Big]. \qquad (2.9)$$

By equality (2.8) and equality (2.9), we get equality (4.5). □

Since the proof of Proposition 2.2 is standard by utilizing Proposition 2.3. We give a sketch of it.

Sketch of the Proof of Proposition 2.2 : Since Γ is C^2, one can find a vector field $\mu_0 = (\mu_0^1, \cdots, \mu_0^n) \in C^2(\overline{G}; \mathbb{R}^n)$ such that $\mu_0 = \nu$ on Γ(see [11, page 18]). Applying Proposition 2.3 with $\mu = \mu_0$, $z = y$, integrating in Q and take the expectation, by means of Proposition 2.3, with similar computation in [22], Proposition 2.2 can be obtained immediately.

3 An Identity for Stochastic Schrödinger-like Operators

In this section, we establish an identity for stochastic schrödinger-like operators, which is similar as identity (4.5) but much more complex. It will play a key role in the proof of our main result.

Let $\beta(t,x) \in C^2(\mathbb{R}^{1+m}; \mathbb{R})$, and $b^{jk}(t,x) \in C^{1,2}(\mathbb{R}^{1+m}; \mathbb{R})$ satisfy

$$b^{jk} = b^{kj}, \qquad j,k = 1,2,\cdots,n, \qquad (3.1)$$

let us define a second order stochastic partial differential operator $\mathcal{P}$ as

$$\mathcal{P}z \triangleq i\beta(t,x)dz + \sum_{j,k=1}^{m}(b^{jk}(t,x)z_j)_k dt, \quad i=\sqrt{-1}. \tag{3.2}$$

We have the following identity concerning with $\mathcal{P}$:

Theorem 3.1 *Let ℓ, $\Psi \in C^2(\mathbb{R}^{1+m};\ \mathbb{R})$. Assume that z is an $H^2_{loc}(\mathbb{R}^n,\mathbb{C})$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process. Put $\theta = e^{\ell}$, $v=\theta z$. Then for a.e. $x\in\mathbb{R}^n$ and P-a.s. $\omega\in\Omega$, it holds that*

$$\begin{aligned}
&\theta(\mathcal{P}z\overline{I_1}+\overline{\mathcal{P}z}I_1)+dM+\operatorname{div}V\\
&=2|I_1|^2dt+\sum_{j,k=1}^{m}c^{jk}(v_k\overline{v}_j+\overline{v}_kv_j)dt+B|v|^2dt+i\sum_{j,k=1}^{m}\Big[(\beta b^{jk}\ell_j)_t+b^{jk}(\beta\ell_t)_j\Big](\overline{v}_kv-v_k\overline{v})dt\\
&\quad+i\Big[\beta\Psi+\sum_{j,k=1}^{m}(\beta b^{jk}\ell_j)_k\Big](\overline{v}dv-vd\overline{v})+(\beta^2 l_t)dvd\overline{v},
\end{aligned} \tag{3.3}$$

where

$$\begin{cases}
I_1 \triangleq -i\beta l_t v - 2\displaystyle\sum_{j,k=1}^{m} b^{jk}l_j v_k + \Psi v,\\
A \triangleq \displaystyle\sum_{j,k=1}^{m} b^{jk}\ell_j\ell_k - \sum_{j,k=1}^{m}(b^{jk}\ell_j)_k - \Psi,
\end{cases} \tag{3.4}$$

$$\begin{cases}
M \triangleq \beta^2\ell_t|v|^2 + i\beta\displaystyle\sum_{j,k=1}^{m} b^{jk}\ell_j(\overline{v}_kv-v_k\overline{v}),\\
V \triangleq [V^1,\cdots,V^k,\cdots,V^m],\\
V^k \triangleq -i\beta\displaystyle\sum_{j=1}^{m}\Big[b^{jk}\ell_j(d\overline{v}v-\overline{v}dv)+b^{jk}\ell_t(v_j\overline{v}-\overline{v}_jv)dt\Big]\\
\qquad -\Psi\displaystyle\sum_{j=1}^{m}b^{jk}(v_j\overline{v}+\overline{v}_jv)dt+\sum_{j=1}^{m}b^{jk}(2A\ell_j+\Psi_j)|v|^2dt\\
\qquad +\displaystyle\sum_{j,j',k'=1}^{m}\Big(2b^{jk'}b^{j'k}-b^{jk}b^{j'k'}\Big)\ell_j(v_{j'}\overline{v}_{k'}+\overline{v}_{j'}v_{k'})dt,
\end{cases} \tag{3.5}$$

and

$$\begin{cases}
c^{jk} \triangleq \displaystyle\sum_{j',k'=1}^{m}\Big[2(b^{j'k}\ell_{j'})_{k'}b^{jk'}-(b^{jk}b^{j'k'}\ell_{j'})_{k'}-b^{jk}\Psi\Big],\\
B \triangleq (\beta^2\ell_t)_t+\displaystyle\sum_{j,k=1}^{m}(b^{jk}\Psi_k)_j+2\Big[\sum_{j,k=1}^{m}(b^{jk}\ell_jA)_k+A\Psi\Big].
\end{cases} \tag{3.6}$$

Remark 3.1 *Since we only assume the symmetry condition for $b^{jk}(t,x)$ (without the positive definite condition for the matrix $(b^{jk})_{1\le j,k\le n}$), similar to [5], starting from identity (3.3) in Theorem 3.1, we can deduce, in one shot, controllability/observability results not only for the stochastic Schrödinger equation, but also for deterministic hyperbolic, Schrödinger and plate equations which are derived before via Carleman estimate in the literature, i.e., that appeared in [7], [12] and [22], respectively.*

Proof of Theorem 3.1: The proof is divided into three steps.

Step 1. By the definition of v and w, a direct computation shows that:

$$\begin{aligned}\theta\mathcal{P}z &= i\beta dv - i\beta l_t v dt + \sum_{j,k=1}^{m}(b^{jk}v_j)_k dt + \sum_{j,k=1}^{m} b^{jk}l_jl_kvdt\\ &\quad -2\sum_{j,k=1}^{m} b^{jk}l_jv_kdt - \sum_{j,k=1}^{m}(b^{jk}l_j)_kvdt\\ &= I_1 + I_2,\end{aligned}\tag{3.7}$$

where

$$I_2 = i\beta dv + \sum_{j,k=1}^{m}(b^{jk}v_j)_k + Av.\tag{3.8}$$

Hence we obtain that

$$\theta(Pz\overline{I_1} + \overline{Pz}I_1) = 2|I_1|^2 + (I_1\overline{I_2} + I_2\overline{I_1}).\tag{3.9}$$

Step 2. In this step, we compute $I_1\overline{I_2} + I_2\overline{I_1}$. Denote the three terms in the right-hand side of I_1 and I_2 by I_1^j and I_2^j, respectively, $j = 1, 2, 3$. Then we have that

$$I_2^1\overline{I_1^1} + \overline{I_2^1}I_1^1 = -d(\beta^2 l_t|v|^2) + (\beta^2 l_t)_t|v|^2dt + \beta^2 l_t dv d\overline{v}.\tag{3.10}$$

Noting that

$$\begin{cases} 2vd\overline{v} = d(|v|^2) - (\overline{v}dv - vd\overline{v}) - dvd\overline{v},\\ 2v\overline{v}_k = (|v|^2)_k - (\overline{v}v_k - v\overline{v}_k),\end{cases}\tag{3.11}$$

we get that

$$\begin{aligned}&I_2^1(\overline{I_1^2} + \overline{I_1^3}) + \overline{I_2^1}(I_1^2 + I_1^3)\\ &= -2i\sum_{j,k=1}^{m}\Big[d(\beta b^{jk}l_jv\overline{v}_k) - (\beta b^{jk}l_j)_t v\overline{v}_kdt\Big] + 2i\sum_{j,k=1}^{m}\Big[(\beta b^{jk}l_jvd\overline{v})_k - (\beta b^{jk}l_j)_kvd\overline{v}\Big]\\ &\quad -i\beta\Psi(vd\overline{v} - dv\overline{v})\\ &= -i\sum_{j,k=1}^{m} d\Big[\beta b^{jk}l_j(v\overline{v}_k - \overline{v}v_k)\Big] + i\sum_{j,k=1}^{m}\Big[\beta b^{jk}l_j(vd\overline{v} - \overline{v}dv)\Big]_k dt\\ &\quad -i\sum_{j,k=1}^{m}(\beta b^{jk}l_j)_t(\overline{v}v_k - v\overline{v}_k)dt + i\Big[\beta\Psi + \sum_{j,k=1}^{m}(\beta b^{jk}l_j)_k\Big](\overline{v}dv - vd\overline{v}).\end{aligned}\tag{3.12}$$

Noting that $b^{jk} = b^{kj}$, we have that

$$\begin{aligned} & I_2^2\overline{I_1^1} + \overline{I_2^2}I_1^1 \\ &= \sum_{j,k=1}^{m} \Big[i\beta b^{jk}l_t(v_j\overline{v} - \overline{v}_jv)\Big]_k dt + i\sum_{j,k=1}^{m} b^{jk}(\beta l_t)_k(\overline{v}_jv - v_j\overline{v})dt \end{aligned} \tag{3.13}$$

and that

$$\begin{aligned} & 2\sum_{j,k,j',k'=1}^{m} b^{jk}b^{j'k'}l_j(v_{j'}\overline{v}_{kk'} + \overline{v}_{j'}v_{kk'})dt \\ &= \sum_{j,k,j',k'=1}^{m} \Big[b^{jk}b^{j'k'}l_j(v_{j'}\overline{v}_{k'} + \overline{v}_{j'}v_{k'})\Big]_k dt - \sum_{j,k,j',k'=1}^{m} (b^{jk}b^{j'k'}l_j)_k(v_{j'}\overline{v}_{k'} + \overline{v}_{j'}v_{k'})dt. \end{aligned} \tag{3.14}$$

By equality (3.14), we get that

$$\begin{aligned} & I_2^2\overline{I_1^2} + \overline{I_2^2}I_1^2 \\ &= -2\sum_{j,k,j',k'=1}^{m} \Big[b^{jk}b^{j'k'}l_j(v_{j'}\overline{v}_k + \overline{v}_{j'}v_k)\Big]_{k'} dt + 2\sum_{j,k,j',k'=1}^{m} (b^{jk}b^{j'k'}l_j)_{k'}(v_{j'}\overline{v}_k + \overline{v}_{j'}v_k)dt \\ &\quad + \sum_{j,k,j',k'=1}^{m} \Big[b^{jk}b^{j'k'}l_j(v_{j'}\overline{v}_{k'} + \overline{v}_{j'}v_{k'})\Big]_k dt - \sum_{j,k,j',k'=1}^{m} (b^{jk}b^{j'k'}l_j)_k(v_{j'}\overline{v}_{k'} + \overline{v}_{j'}v_{k'})dt. \end{aligned} \tag{3.15}$$

Further, it holds that

$$\begin{aligned} I_2^2\overline{I_1^3} + \overline{I_2^2}I_1^3 &= \sum_{j,k=1}^{m} \Big[\Psi b^{jk}(v_j\overline{v} + \overline{v}_jv)\Big]_k dt - \sum_{j,k=1}^{m} \Psi b^{jk}(v_j\overline{v}_k + \overline{v}_jv_k)dt \\ &\quad - \sum_{j,k=1}^{m} \Big[b^{jk}\Psi_k|v|^2\Big]_j dt + \sum_{j,k=1}^{m} (b^{jk}\Psi_k)_j|v|^2dt. \end{aligned} \tag{3.16}$$

Finally, we have that

$$\begin{aligned} & I_2^3(\overline{I_1^1} + \overline{I_1^2} + \overline{I_1^3}) + \overline{I_2^3}(I_1^1 + I_1^2 + I_1^3) \\ &= -2\sum_{j,k=1}^{m} (b^{jk}l_jA|v|^2)_k dt + 2\Big[\sum_{j,k=1}^{m} (b^{jk}l_jA)_k + A\Psi\Big]|v|^2dt. \end{aligned} \tag{3.17}$$

Step 3. Combining (3.9)-(3.17), we conclude the desired identity (3.3).

4 Carleman Estimate for Stochastic Schrödinger Equations

This section is devoted to establishing a global Carleman estimate for equation (1.1) (the following Theorem 4.1). We introduced the following weight function at first.

Let

$$\psi(x) = |x - x_0|^2 + \tau, \tag{4.1}$$

where τ is a positive constant such that $\psi > \frac{2}{3}|\psi|_{L^\infty(G)}$.

Put

$$l = s\frac{e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^\infty(\Omega)}}}{t^2(T-t)^2}, \qquad \varphi = \frac{e^{4\lambda\psi}}{t^2(T-t)^2}. \tag{4.2}$$

Hence we have that

$$|l_t| \le Cs\varphi^{1+\frac{1}{2}}, \quad |l_{tt}| \le Cs\varphi^3. \tag{4.3}$$

If l is given as (4.2)(recall that $\theta = e^l$), we have the following Carleman inequality.

Theorem 4.1 *Let a_i $(1 \le i \le 3)$ satisfy (1.4), f, g satisfy (1.5). Then for any solution of equation (1.1) with initial datum y_0, it holds that*

$$\begin{aligned} &\mathbb{E}\int_Q \theta^2\Big(\varphi^3|y|^2 + \varphi|\nabla y|^2\Big)dxdt \\ &\le Ce^{Cr_1}\Big\{\mathbb{E}\int_Q \theta^2\Big(|f|^2 + \varphi^2 g^2 + |\nabla g|^2\Big)dxdt + \mathbb{E}\int_0^T\int_{\Gamma_0}\theta^2\varphi\Big|\frac{\partial y}{\partial \nu}\Big|^2 d\Gamma dt\Big\}. \end{aligned} \tag{4.4}$$

Remark 4.1 *It is well known that the global Carleman estimate is an important tool for the study of unique continuation property, stabilization, controllability and inverse problems for deterministic partial differential equations(e.g. [2, 12, 18, 22, 27]). Although there are numerous results for the global Carleman estimate for deterministic partial differential equations, people know very little about the stochastic counterpart. In fact, as far as we know, [1, 21, 25] are the only three papers addressing to the global Carleman estimate for stochastic partial differential equations. [1, 21] are devoted to the stochastic heat equations while [25] is devoted to the stochastic wave equations. To my best knowledge, there is no global Carleman estimate for stochastic Schrödinger equations in the literature.*

Proof of Theorem 4.1: The proof is divided into the following three steps.

Step 1. Let $\beta = 1$ and $(b^{jk})_{1\le j,k\le m}$ equal the identity matrix. Put

$$\delta^{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \ne k. \end{cases}$$

Apply Theorem 3.1 to equation (1.1) with z replaced by y and $v = \theta z$. We obtain that

$$
\begin{aligned}
&\theta \mathcal{P} y\Big(i\beta l_t \overline{v} - 2\sum_{j,k=1}^{m} b^{jk} l_j \overline{v}_k + \Psi \overline{v}\Big) + \theta \overline{\mathcal{P} y}\Big(-i\beta l_t v - 2\sum_{j,k=1}^{m} b^{jk} l_j v_k + \Psi v\Big) + dM + \operatorname{div} V \\
&= 2\Big| - i\beta l_t v - 2\sum_{j,k=1}^{m} b^{jk} l_j v_k + \Psi v\Big|^2 dt + \sum_{j,k=1}^{m} c^{jk}(v_k \overline{v}_j + \overline{v}_k v_j) dt + B|v|^2 dt \\
&\quad +2i\sum_{j=1}^{m}(l_{jt} + l_{tj})(\overline{v}_j v - v_j \overline{v}) dt + i(\Psi + \Delta l)(\overline{v} dv - v d\overline{v}) \\
&\quad +l_t dv d\overline{v} + \frac{i}{2}\sum_{j=1}^{m} l_j (d\overline{v}_j dv - dv_j d\overline{v}),
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{cases}
M = l_t |v|^2 + i\displaystyle\sum_{j=1}^{m} l_j(\overline{v}_j v - v_j \overline{v}), \\
A = \displaystyle\sum_{j=1}^{m}(l_j^2 - l_{jj}) - \Psi, \\
B = l_{tt} + \displaystyle\sum_{j=1}^{m}\Psi_{jj} + 2\sum_{j=1}^{m}(l_j A)_j + 2A\Psi, \\
c^{jk} = 2l_{jk} - \delta^{jk}\Delta l - \delta^{jk}\Psi, \\
V^k = 2\displaystyle\sum_{j=1}^{m} l_j(\overline{v}_j v_k + v_j \overline{v}_k) - 2\sum_{j'=1}^{m} l_k (v_j \overline{v}_j).
\end{cases}
\tag{4.6}
$$

Step 2. In this step, we will estimate the terms in the right-hand side of equality (4.5) one by one. Let $\Psi = -\Delta l$, then we have that

$$
A = \sum_{j=1}^{m} l_j^2 = s^2\lambda^2\varphi^2|\nabla\psi|^2, \tag{4.7}
$$

Utilizing (4.3), we obtain that

$$
\begin{aligned}
B &= l_{tt} + \sum_{j=1}^{m}\Psi_{jj} + 2\sum_{j=1}^{m}(l_j A)_j + 2A\Psi \\
&= 2s^3\lambda^4\varphi^3|\nabla\psi|^4 - s\lambda^4\varphi|\nabla\psi|^4 - s^3\varphi^3 O(\lambda^3)
\end{aligned}
\tag{4.8}
$$

and that

$$
\begin{aligned}
c^{jk} &= 2l_{jk} - \delta^{jk}\Delta l - \delta^{jk}\Psi \\
&= 2s\lambda^2\varphi\psi_j\psi_k + s\lambda\varphi\psi_{jk}.
\end{aligned}
\tag{4.9}
$$

Hence we know that there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, one can find a constant $s_0 = s_0(\lambda_0)$ so that for any $s > s_0$, it holds that

$$B \geq s^3\lambda^4\varphi^3|\nabla\psi|^4, \quad \sum_{j,k=1}^{m} c^{jk}(v_j\overline{v}_k + v_k\overline{v}_j) \geq s\lambda\varphi|\nabla v|^2. \tag{4.10}$$

Now we estimate the other terms in the right-hand side of (4.5). The first one satisfies that

$$\begin{aligned} 2i\sum_{j=1}^{m}(l_{jt}+l_{tj})(\overline{v}_j v - v_j\overline{v}) &= 4i\sum_{j=1}^{m} s\lambda\psi_j l_t(\overline{v}_j v - \overline{v}v_j) \\ &\leq 2s\varphi|\psi|^2|\nabla v|^2 + 2s\lambda^2\varphi^3|v^2|. \end{aligned} \tag{4.11}$$

For the second one, it holds that

$$i(\Psi + \Delta l)(\overline{v}dv - vd\overline{v}) = 0. \tag{4.12}$$

For the estimate of the third and the fourth one, we need to take mean value and get that

$$\begin{aligned} \mathbb{E}l_t dv d\overline{v} &= \mathbb{E}l_t(\theta l_t y + \theta dy)\overline{(\theta l_t y + \theta dy)} \\ &\leq 2\mathbb{E}\theta^2 s\varphi^{\frac{3}{2}}(a_3^2|y|^2 + g^2)dt. \end{aligned} \tag{4.13}$$

The fourth one enjoys that

$$\begin{aligned} \Big|i\mathbb{E}\sum_{j=1}^{m} l_j(d\overline{v}_j dv - dv_j d\overline{v})\Big| &= \Big|i\mathbb{E}\sum_{j=1}^{m} s\lambda\varphi\psi(d\overline{v}_j dv - dv_j d\overline{v})\Big| \\ &\leq \mathbb{E}\sum_{j=1}^{m} dv_j d\overline{v}_j + s^2\lambda^2\varphi^2|\nabla\psi|^2 dv d\overline{v} \\ &\leq \mathbb{E}\theta^2\Big\{s^2\lambda^2\varphi^2(a_3^2|y|^2+g^2) + a_3^2|\nabla y|^2 + |\nabla a_3|^2 y^2 + |\nabla g|^2\Big\}dt. \end{aligned} \tag{4.14}$$

Step 3. Integrating equality (4.5) in Q, taking mean value in both sides, noting (4.7)-(4.14), we obtain that

$$\begin{aligned} &\mathbb{E}\int_Q \Big(s^3\lambda^4\varphi^3|v|^2 + s\lambda\varphi|\nabla v|^2\Big)dxdt + 2\mathbb{E}\int_Q \Big| - i\beta l_t v - 2\sum_{j,k=1}^{m} b^{jk}l_j v_k + \Psi v\Big|^2 dxdt \\ &\leq \mathbb{E}\int_Q \Big\{\theta\mathcal{P}y\Big(i\beta l_t\overline{v} - 2\sum_{j,k=1}^{m} b^{jk}l_j\overline{v}_k + \Psi\overline{v}\Big) + \theta\overline{\mathcal{P}y}\Big(- i\beta l_t v - 2\sum_{j,k=1}^{m} b^{jk}l_j v_k + \Psi v\Big)\Big\}dx \\ &+C\mathbb{E}\int_Q \theta^2\Big[s^2\lambda^2\varphi^2(a_3^2|y|^2+g^2) + a_3^2|\nabla y|^2 + |\nabla a_3|^2 y^2 + |\nabla g|^2\Big]dxdt \\ &+\mathbb{E}\int_Q dM dx + \mathbb{E}\int_Q \operatorname{div} V dx. \end{aligned} \tag{4.15}$$

Now we analyze the terms in the right-hand side of inequality (4.15) one by one. The first term satisfies that

$$
\begin{aligned}
&\mathbb{E}\int_Q \Big\{\theta\mathcal{P}y\Big(i\beta l_t\bar{v}-2\sum_{j,k=1}^m b^{jk}l_j\bar{v}_k+\Psi\bar{v}\Big)+\theta\overline{\mathcal{P}y}\Big(-i\beta l_t v-2\sum_{j,k=1}^m b^{jk}l_jv_k+\Psi v\Big)\Big\}dx\\
&=\mathbb{E}\int_Q \Big\{\theta(a_1\cdot\nabla y+a_2y+f)(i\beta l_t\bar{v}-2\sum_{j,k=1}^m b^{jk}l_j\bar{v}_k+\Psi\bar{v})\\
&+\theta(a_1\cdot\nabla\bar{y}+\overline{a_2y}+\bar{f})\Big(-i\beta l_tv-2\sum_{j,k=1}^m b^{jk}l_jv_k+\Psi v\Big)\Big\}dxdt\\
&\leq 2\mathbb{E}\int_Q\Big\{\theta^2|a_1\cdot\nabla y+a_2y+f|^2+\Big|-i\beta l_tv-2\sum_{j,k=1}^m b^{jk}l_jv_k+\Psi v\Big|^2\Big\}dxdt
\end{aligned}
\tag{4.16}
$$

By the definition of θ, we know that $v(0)=v(T)=0$. Hence, it holds that

$$
\int_Q dMdx=0. \tag{4.17}
$$

For $\mathbb{E}\int_Q \operatorname{div}V dx$, utilizing Stokes Theorem, we have that

$$
\begin{aligned}
\mathbb{E}\int_Q \operatorname{div}V dx&=\mathbb{E}\int_\Sigma 2\sum_{k=1}^m\sum_{j=1}^m\Big[l_j(\overline{v}_jv_k+v_j\overline{v}_k)\nu^k-l_k\nu_kv_j\overline{v}_j\Big]d\Sigma\\
&=\mathbb{E}\int_\Sigma\Big(4\sum_{j=1}^m l_j\nu_j\Big|\frac{\partial v}{\partial\nu}\Big|^2-2\sum_{k=1}^m l_k\nu_k\Big|\frac{\partial v}{\partial\nu}\Big|^2\Big)d\Sigma\\
&=\mathbb{E}\int_\Sigma 2\sum_{k=1}^m l_k\nu_k\Big|\frac{\partial v}{\partial\nu}\Big|^2d\Sigma\\
&\leq 2\mathbb{E}\int_0^T\int_{\Gamma_0}\theta^2s\lambda\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2d\Gamma dt.
\end{aligned}
\tag{4.18}
$$

By (4.8)-(4.18), we have that

$$
\begin{aligned}
&\mathbb{E}\int_Q\Big(s^3\lambda^4\varphi^3|v|^2+s\lambda\varphi|\nabla v|^2\Big)dxdtt\\
&\leq C\mathbb{E}\int_Q\theta^2|a_1\cdot\nabla y+a_2y+f|^2dxdt+C\mathbb{E}\int_0^T\int_{\Gamma_0}\theta^2s\lambda\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2d\Gamma dt\\
&+C\mathbb{E}\int_Q\theta^2\Big[s^2\lambda^2\varphi^2(a_3^2|y|^2+g^2)+a_3^2|\nabla y|^2+|\nabla a_3|^2y^2+|\nabla g|^2\Big]dxd.
\end{aligned}
\tag{4.19}
$$

However, noting that $y_i=\theta^{-1}(v_i-l_iv)=\theta^{-1}(v_i-s\lambda\varphi\psi_iv)$, we get that

$$
\theta^2(|\nabla y|^2+s^2\lambda^2\varphi^2|y|^2)\leq C(|\nabla v|^2+s^2\lambda^2\varphi^2|v|^2). \tag{4.20}
$$

Therefore, it follows from (4.19) that

$$\begin{aligned}
&\mathbb{E}\int_Q \Big(s^3\lambda^4\varphi^3|y|^2 + s\lambda\varphi|\nabla y|^2\Big)dxdt\\
&\leq C\mathbb{E}\int_Q \Big(\theta^2|a_1|^2||\nabla y|^2 + a_2^2|y|^2 + |f|^2\Big)dxdt + C\mathbb{E}\int_0^T\int_{\Gamma_0}\theta^2 s\lambda\varphi\Big|\frac{\partial y}{\partial \nu}\Big|^2 d\Gamma dt \qquad (4.21)\\
&\quad + C\mathbb{E}\int_Q \theta^2\Big[s^2\lambda^2\varphi^2(a_3^2|y|^2+g^2) + a_3^2|\nabla y|^2 + |\nabla a_3|^2y^2 + |\nabla g|^2\Big]dxdt.
\end{aligned}$$

Choosing $\lambda = \lambda_0$ and $s = \max(s_0, Cr_1)$, we have that

$$\begin{aligned}
&\mathbb{E}\int_Q \theta^2\Big(\varphi^3|y|^2 + \varphi|\nabla y|^2\Big)dxdt\\
&\leq Cr_1\Big\{\mathbb{E}\int_Q \theta^2\Big(|f|^2 + \varphi^2g^2 + |\nabla g|^2\Big)dxdt + \mathbb{E}\int_0^T\int_{\Gamma_0}\theta^2\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt\Big\},
\end{aligned} \qquad (4.22)$$

which is the global Carleman estimate we expected.

5 Proof of Theorem 1.1 and Theorem 1.2

In this section, we prove Theorem 1.1 and Theorem 1.2, by means of Theorem 4.4.

Proof of Theorem 1.1: Owing to the definition of l and θ, it holds that

$$\begin{aligned}
&\mathbb{E}\int_Q \theta^2\Big(\varphi^3|y|^2 + \varphi|\nabla y|^2\Big)dxdt\\
&\geq \min_{x\in\overline{G}}\Big(\varphi\Big(\frac{T}{2},x\Big)\theta^2\Big(\frac{T}{4},x\Big)\Big)\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_G(|y|^2+|\nabla y|^2)dxdt,
\end{aligned} \qquad (5.1)$$

$$\begin{aligned}
&\mathbb{E}\int_Q \theta^2(|f|^2 + \varphi^2|g|^2 + |\nabla g|^2)dxdt\\
&\leq \max_{(x,t)\in\overline{Q}}(\varphi^2(t,x)\theta^2(t,x))\mathbb{E}\int_Q(|f|^2+|g|^2+|\nabla g|^2)dxdt
\end{aligned} \qquad (5.2)$$

and that

$$\mathbb{E}\int_0^T\int_{\Gamma_0}\theta^2\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt \leq \max_{(x,t)\in\overline{Q}}(\varphi(t,x)\theta^2(t,x))\mathbb{E}\int_0^T\int_{\Gamma_0}\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt. \qquad (5.3)$$

From (4.22)-(5.3), we deduce that

$$\begin{aligned}
&\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_G(|y|^2+|\nabla y|^2)dxdt\\
&\leq Cr_1\frac{\max_{(x,t)\in\overline{Q}}\Big(\varphi^2(t,x)\theta^2(t,x)\Big)}{\min_{x\in\overline{G}}\Big(\varphi(\frac{T}{2},x)\theta^2(\frac{T}{4},x)\Big)}
\end{aligned}$$

$$
\times \left\{ \mathbb{E} \int_Q (|f|^2 + |g|^2 + |\nabla g|^2) dxdt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\}
$$

$$
\leq C e^{C r_1} \left\{ \mathbb{E} \int_Q (|f|^2 + |g|^2 + |\nabla g|^2) dxdt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\}. \tag{5.4}
$$

Utilizing (5.4) and (2.1), we obtain that

$$
\begin{aligned}
&\mathbb{E} \int_G (|y|^2 + |\nabla y|^2) dxdt \\
&\leq C e^{C r_1} \left\{ \mathbb{E} \int_Q (|f|^2 + |\nabla f|^2 + |g|^2 + |\nabla g|^2) dxdt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\},
\end{aligned} \tag{5.5}
$$

which deduce Theorem 1.1 immediately. □

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2 : Since $f = g = 0$, P-a.s., utilizing inequality (4.4), we obtain that

$$
\mathbb{E} \int_Q \theta^2 \Big(\varphi^3 |y|^2 + \varphi |\nabla y|^2 \Big) dxdt \leq C r_1 \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 \varphi \Big| \frac{\partial y}{\partial \nu} \Big|^2 d\Gamma dt. \tag{5.6}
$$

By virtue of that $y = 0$ in $O_\varepsilon(\Gamma_0 \times [0, T])$, P-a.s., we have that

$$
\frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma_0 \times (0, T), P\text{-a.s.}
$$

This, together with (5.6), implies that

$$
\mathbb{E} \int_Q \theta^2 \varphi^3 |y|^2 dxdt = 0,
$$

which means that $y = 0$ in Q, P-a.s. □

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