

Stationary waves to viscous heat-conductive gases in half space: existence, stability and convergence rate*

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Abstract

The main concern of the present paper is to study large-time behavior of solutions to an ideal polytropic model of compressible viscous gases in one-dimensional half space. We consider an outflow problem, where the gas blows out through the boundary, and obtain a convergence rate of solutions toward a corresponding stationary solution. Here the existence of the stationary solution is proved under a smallness condition on the boundary data with the aid of center manifold theory. We also show the time asymptotic stability of the stationary solution under smallness assumptions on the boundary data and the initial perturbation in the Sobolev space, by employing an energy method. Moreover, the convergence rate of the solution toward the stationary solution is obtained, provided that the initial perturbation belongs to the weighted Sobolev space. Precisely, the convergence rate we obtain coincides with the spatial decay rate of the initial perturbation. The proof is mainly based on *a priori* estimates of the perturbation from the stationary solution, which are derived by a time and space weighted energy method.

Keywords: Compressible Navier–Stokes equation; Eulerian coordinate; ideal polytropic model; outflow problem; boundary layer solution; weighted energy method.

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1 Introduction and main result

1.1 Formulation of the problem

We study large-time behavior of a solution to an initial boundary value problem for the compressible Navier–Stokes equations over one-dimensional half space $\mathbb{R}_+ := (0, \infty)$. An ideal polytropic model of compressible viscous fluid is formulated in the Eulerian coordinates as

$$\rho_t + (\rho u)_x = 0, \quad (1.1a)$$

$$(\rho u)_t + (\rho u^2 + p(\rho, \theta))_x = \mu u_{xx}, \quad (1.1b)$$

$$\left\{ \rho \left(c_v \theta + \frac{u^2}{2} \right) \right\}_t + \left\{ \rho u \left(c_v \theta + \frac{u^2}{2} \right) + p(\rho, \theta) u \right\}_x = (\mu u u_x + \kappa \theta_x)_x, \quad (1.1c)$$

where unknown functions are $\rho = \rho(t, x)$, $u = u(t, x)$ and $\theta = \theta(t, x)$ standing for a mass density, a fluid velocity and an absolute temperature, respectively. Due to the Boyle–Charles law, a pressure p is explicitly given by a function of the density and the absolute temperature:

$$p = p(\rho, \theta) := R\rho\theta,$$

where $R > 0$ is a gas constant. Positive constants c_v , μ and κ mean a specific heat at constant volume, a viscosity coefficient and a thermal conductivity, respectively. Due to Mayer’s relation for the ideal gas, the specific heat c_v is expressed by the gas constant R and an adiabatic constant $\gamma > 1$ as

$$c_v = \frac{R}{\gamma - 1}.$$

We also introduce physical constants

$$c_p := \gamma c_v = \frac{\gamma}{\gamma - 1} R, \quad P_r := \frac{\mu}{\kappa} c_p = \frac{\mu}{\kappa} \frac{\gamma}{\gamma - 1} R,$$

which stand for a specific heat at constant pressure and the Prandtl number, respectively. The Prandtl number plays an important role in analysis of a property of a stationary solution.

We put an initial condition

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x) \quad (1.2)$$

and boundary conditions

$$u(t, 0) = u_b < 0, \quad \theta(t, 0) = \theta_b > 0, \quad (1.3)$$

where u_b and θ_b are constants. It is assumed that the initial data converges to a constant as x tends to infinity:

$$\lim_{x \rightarrow \infty} (\rho_0, u_0, \theta_0)(x) = (\rho_+, u_+, \theta_+).$$

Moreover, we assume that the initial density and absolute temperature are uniformly positive, that is,

$$\inf_{x \in \mathbb{R}_+} \rho_0(x) > 0, \quad \inf_{x \in \mathbb{R}_+} \theta_0(x) > 0, \quad \rho_+ > 0, \quad \theta_+ > 0.$$

The boundary condition for u in (1.3) means that the fluid blows out from the boundary. Hence this problem is called an outflow problem (see [11]). Due to the outflow boundary condition, the characteristic of the hyperbolic equation (1.1a) for the density ρ is negative around the boundary so that two boundary conditions are necessary and sufficient for the wellposedness of this problem.

In the paper [9], Kawashima, Nishibata and Zhu considered the outflow problem for an isentropic model and obtained a necessary and sufficient condition for the existence of the stationary solution. Moreover, they proved the asymptotic stability of the stationary solution under the smallness assumption on the initial perturbation and the strength of the boundary data. A convergence rate toward the stationary solution for this model was obtained by Nakamura, Nishibata and Yuge in [14] under the assumption that the initial perturbation belongs to the suitably weighted Sobolev space. The main concern of the present paper is to extend these results to the model of heat-conductive viscous gas. Precisely, we show the existence and the asymptotic stability of the stationary solution as well as the convergence rate for the ideal polytropic model (1.1). Compared to the isentropic model, the heat-conductive model is more difficult to handle. For example, since the model (1.1) has two parabolic equations, the equations for the stationary wave are deduced to a 2×2 system of autonomous ordinary differential equations. However, it becomes a scalar equation in the case of the isentropic flow. Therefore, to obtain a condition which guarantees the existence of the stationary solution for the heat-conductive model, we have to examine dynamics around an equilibrium of the system by using center manifold theory.

1.2 Dimensionless form

For the stability analysis on the equations (1.1), it is convenient to reformulate the problem into that in the dimensionless form. For this purpose, we define new variables \hat{x} and \hat{t} by

$$\hat{x} := \frac{x}{L}, \quad \hat{t} := \frac{t}{T},$$

where L and T are positive constants. We also employ new unknown functions $(\hat{\rho}, \hat{u}, \hat{\theta})$ defined by

$$\hat{\rho}(\hat{t}, \hat{x}) := \frac{1}{\rho_+} \rho(t, x), \quad \hat{u}(\hat{t}, \hat{x}) := \frac{1}{|u_+|} u(t, x), \quad \hat{\theta}(\hat{t}, \hat{x}) := \frac{1}{\theta_+} \theta(t, x). \quad (1.4)$$

Here we note that the constant u_+ must satisfy

$$u_+ < 0 \quad (1.5)$$

for the existence of the stationary solution. Indeed, the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)$ satisfies

$$\tilde{\rho}(x)\tilde{u}(x) = \rho_+ u_+, \quad (1.6)$$

which is obtained by integrating $(\tilde{\rho}\tilde{u})_x = 0$ over (x, ∞) . Substituting $x = 0$ in (1.6), we get $u_+ = \tilde{\rho}(0)u_b/\rho_+$, which immediately yields (1.5) by using the positivity of the

density and the boundary condition $u_b < 0$. Next we define dimensionless physical constants by

$$\hat{\mu} := \frac{\mu}{\rho_+ |u_+|^2}, \quad \hat{\kappa} := \frac{\kappa \theta_+}{\rho_+ |u_+|^4}, \quad \hat{c}_v := \frac{1}{\gamma(\gamma - 1)} \quad (1.7)$$

and a dimensionless pressure by

$$\hat{p} = \hat{p}(\hat{\rho}, \hat{\theta}) := \frac{1}{\gamma} \hat{\rho} \hat{\theta}.$$

We also introduce Mach number M_+ at the spatial asymptotic state:

$$M_+ := \frac{|u_+|}{c_+},$$

where $c_+ := \sqrt{R\gamma\theta_+}$ is sound speed. Using the dimensionless constants (1.7), we represent the Prandtl number P_r as

$$P_r = \frac{\hat{\mu}}{\hat{\kappa} M_+^2 (\gamma - 1)}.$$

Substituting (1.4) in (1.1) and letting $L = |u_+|$ and $T = 1$, we have the equations for $(\hat{\rho}, \hat{u}, \hat{\theta})$ in the dimensionless form as

$$\rho_t + (\rho u)_x = 0, \quad (1.8a)$$

$$(\rho u)_t + \left(\rho u^2 + \frac{1}{M_+^2} p(\rho, \theta) \right)_x = \mu u_{xx}, \quad (1.8b)$$

$$\left\{ \rho \left(\frac{1}{M_+^2} c_v \theta + \frac{u^2}{2} \right) \right\}_t + \left\{ \rho u \left(\frac{1}{M_+^2} c_v \theta + \frac{u^2}{2} \right) + \frac{1}{M_+^2} p(\rho, \theta) u \right\}_x = (\mu u u_x + \kappa \theta_x)_x. \quad (1.8c)$$

In the equations (1.8), without any confusion, we abbreviate the symbol “ $\hat{\cdot}$ ” to express dimensionless quantities. The initial and the boundary conditions for the dimensionless function (ρ, u, θ) are prescribed as

$$(\rho, u, \theta)(0, x) = (\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)(x) := \left(\frac{\rho_0}{\rho_+}, \frac{u_0}{|u_+|}, \frac{\theta_0}{\theta_+} \right)(x), \quad (1.9a)$$

$$\lim_{x \rightarrow \infty} (\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)(x) = (1, -1, 1), \quad (1.9b)$$

$$(u, \theta)(t, 0) = (\hat{u}_b, \hat{\theta}_b) := \left(\frac{u_b}{|u_+|}, \frac{\theta_b}{\theta_+} \right). \quad (1.10)$$

We also abbreviate the hat “ $\hat{\cdot}$ ” and write the dimensionless initial data and boundary data as (ρ_0, u_0, θ_0) and (u_b, θ_b) respectively in (1.9) and (1.10).

1.3 Main results

The main concern of the present paper is to consider the large-time behavior of solutions to the problem (1.8), (1.9) and (1.10). Precisely we show that the solution

converges to a stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)$, which is a solution to (1.8) independent of time variable t . Thus the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ satisfies the system

$$(\tilde{\rho}\tilde{u})_x = 0, \quad (1.11a)$$

$$\left(\tilde{\rho}\tilde{u}^2 + \frac{1}{M_+^2}\tilde{p}\right)_x = \mu\tilde{u}_{xx}, \quad (1.11b)$$

$$\left\{\tilde{\rho}\tilde{u}\left(\frac{1}{M_+^2}c_v\tilde{\theta} + \frac{\tilde{u}^2}{2}\right) + \frac{1}{M_+^2}\tilde{p}\tilde{u}\right\}_x = (\mu\tilde{u}\tilde{u}_x + \kappa\tilde{\theta}_x)_x, \quad (1.11c)$$

where $\tilde{p} := p(\tilde{\rho}, \tilde{\theta})$. The stationary solution is supposed to satisfy the same boundary condition (1.10) and the same spatial asymptotic condition (1.9b):

$$(\tilde{u}, \tilde{\theta})(0) = (u_b, \theta_b), \quad \lim_{x \rightarrow \infty} (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) = (1, -1, 1). \quad (1.12)$$

We summarize the existence and the decay property of the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ satisfying (1.11) and (1.12) in the following proposition. To this end, we define a boundary strength δ as

$$\delta := |(u_b + 1, \theta_b - 1)|.$$

Proposition 1.1. *Suppose that the boundary data (u_b, θ_b) satisfies*

$$(u_b, \theta_b) \in \mathcal{M}^+ := \{(u, \theta) \in \mathbb{R}^2; |(u + 1, \theta - 1)| < \varepsilon_0\} \quad (1.13)$$

for a certain positive constant ε_0 . Notice that the condition (1.13) is equivalent to $\delta < \varepsilon_0$.

- (i) *For the supersonic case $M_+ > 1$, there exists a unique smooth solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ to the problem (1.11) and (1.12) satisfying*

$$|\partial_x^k(\tilde{\rho}(x) - 1, \tilde{u}(x) + 1, \tilde{\theta}(x) - 1)| \leq C\delta e^{-cx} \quad \text{for } k = 0, 1, 2, \dots, \quad (1.14)$$

where C and c are positive constants.

- (ii) *For the transonic case $M_+ = 1$, there exists a certain region $\mathcal{M}^0 \subset \mathcal{M}^+$ such that if the boundary data (u_b, θ_b) satisfies the condition*

$$(u_b, \theta_b) \in \mathcal{M}^0, \quad (1.15)$$

then there exists a unique smooth solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ satisfying

$$|\partial_x^k(\tilde{\rho}(x) - 1, \tilde{u}(x) + 1, \tilde{\theta}(x) - 1)| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} + C\delta e^{-cx} \quad \text{for } k = 0, 1, 2, \dots \quad (1.16)$$

- (iii) *For the subsonic case $M_+ < 1$, there exists a certain curve $\mathcal{M}^- \subset \mathcal{M}^+$ such that if the boundary data (u_b, θ_b) satisfies the condition*

$$(u_b, \theta_b) \in \mathcal{M}^-, \quad (1.17)$$

then there exists a unique smooth solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ satisfying (1.14).

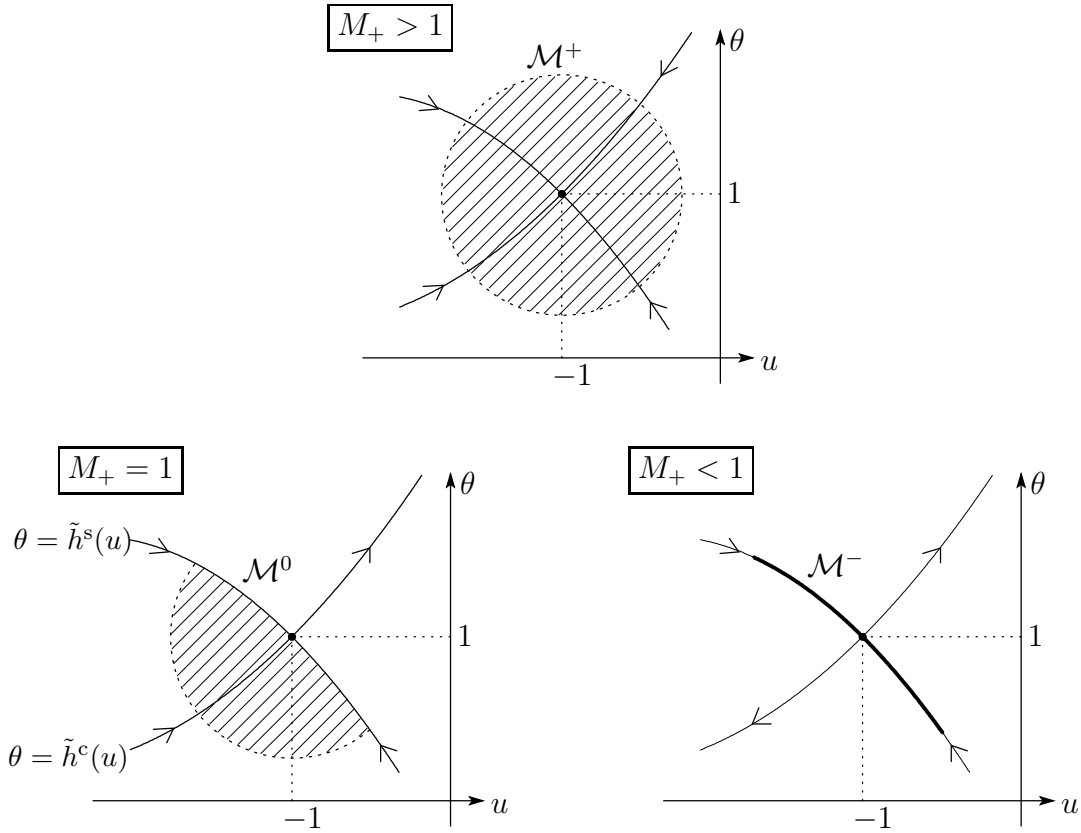


Figure 1: For the transonic case $M_+ = 1$, the region \mathcal{M}^0 consists of one side of \mathcal{M}^+ divided by the local stable manifold $\theta = \tilde{h}^s(u)$. For the subsonic case $M_+ < 1$, the curve \mathcal{M}^- coincides with the local stable manifold.

The rough sketches of the regions \mathcal{M}^+ , \mathcal{M}^0 and \mathcal{M}^- are drawn in Figure 1. The precise definitions of \mathcal{M}^0 and \mathcal{M}^- are given in (2.19). The boundary of \mathcal{M}^0 , which is the stable manifold for the stationary problem, is a curve in the state space. The geometric property of this curve is completely characterized by the Prandtl number. This observation is discussed in Section 2.3.

The asymptotic stability of the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ is stated in the next theorem.

Theorem 1.2. *Suppose that the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ exists. Namely it is assumed that one of the following three conditions holds: (i) $M_+ > 1$ and (1.13), (ii) $M_+ = 1$ and (1.15), (iii) $M_+ < 1$ and (1.17). In addition, the initial data (ρ_0, u_0, θ_0) is supposed to satisfy*

$$\begin{aligned} \rho_0 &\in \mathcal{B}^{1+\sigma}(\mathbb{R}_+), \quad (u_0, \theta_0) \in \mathcal{B}^{2+\sigma}(\mathbb{R}_+), \\ (\rho_0, u_0, \theta_0) - (\tilde{\rho}, \tilde{u}, \tilde{\theta}) &\in H^1(\mathbb{R}_+) \end{aligned}$$

for a certain constant $\sigma \in (0, 1)$. Then there exists a positive constant ε_1 such that if

$$\|(\rho_0, u_0, \theta_0) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|_{H^1} + \delta \leq \varepsilon_1,$$

then the initial boundary value problem (1.8), (1.9) and (1.10) has a unique solution globally in time satisfying

$$\begin{aligned} \rho &\in \mathcal{B}_T^{1+\sigma/2, 1+\sigma}, \quad (u, \theta) \in \mathcal{B}_T^{1+\sigma/2, 2+\sigma}, \\ (\rho, u, \theta) - (\tilde{\rho}, \tilde{u}, \tilde{\theta}) &\in C([0, \infty); H^1(\mathbb{R}_+)) \end{aligned} \quad (1.18)$$

for an arbitrary $T > 0$. Moreover, the solution (ρ, u, θ) converges to the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ uniformly as time tends to infinity:

$$\lim_{t \rightarrow \infty} \|(\rho, u, \theta)(t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|_{L^\infty} = 0. \quad (1.19)$$

We also show a convergence rate for the stability (1.19) by assuming additionally that the initial perturbation belongs to the weighted Sobolev space.

Theorem 1.3. *Suppose that the same conditions as in Theorem 1.2 hold.*

(i) *For the supersonic case $M_+ > 1$, if the initial perturbation satisfies*

$$(\rho_0, u_0, \theta_0) - (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \in L_\alpha^2(\mathbb{R}_+)$$

for a certain positive constant α , then the solution (ρ, u, θ) to (1.8), (1.9) and (1.10) satisfies the decay estimate

$$\|(\rho, u, \theta)(t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|_{L^\infty} \leq C(1+t)^{-\alpha/2}. \quad (1.20)$$

(ii) *For the transonic case $M_+ = 1$, let $\alpha \in [1, 2(1 + \sqrt{2})]$. There exists a positive constant ε_2 such that if*

$$\delta^{-1/2} \|(\rho_0, u_0, \theta_0) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|_{H_\alpha^1} \leq \varepsilon_2,$$

then the solution (ρ, u, θ) satisfies the decay estimate

$$\|(\rho, u, \theta)(t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|_{L^\infty} \leq C(1+t)^{-\alpha/4}. \quad (1.21)$$

Remark 1.4. (i) For the supersonic case $M_+ > 1$, we can prove an exponential convergence rate

$$\|(\rho, u, \theta)(t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|_{L^\infty} \leq C e^{-\alpha t}$$

provided that the initial data satisfies the conditions as in Theorem 1.2 and

$$(\rho_0, u_0, \theta_0) - (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \in L_{\zeta, \text{exp}}^2(\mathbb{R}_+) := \{u \in L_{\text{loc}}^2(\mathbb{R}_+); e^{(\zeta/2)x} u \in L^2(\mathbb{R}_+)\},$$

where α is a positive constant depending on ζ . Since the proof is almost same as that for the isentropic model studied in the paper [14], we omit the details.

(ii) To obtain the convergence rates (1.20) and (1.21), we derive weighted energy estimates. In the derivation, we essentially use a property that all of characteristics of a hyperbolic system, which is obtained by letting $\mu = 0$ and $\kappa = 0$ in (1.8), are non-positive at spatial asymptotic state. However, for the subsonic case $M_+ < 1$, one characteristic is positive. Due to this, it is difficult to obtain a convergence rate for the subsonic case by using the weighted energy method.

(iii) Compared with the results in [8, 12, 15] considering the convergence rate for a scalar viscous conservation law, the convergence rates in (1.20) and (1.21) seem

optimal. For the transonic case, owing to the degenerate property of the stationary solution, the weight exponent α needs to be less than a certain constant, i.e., $\alpha < 2(1 + \sqrt{2})$. This kind of restriction on the weight exponent is also necessary to obtain a convergence rate $O(t^{-\alpha/4})$ toward the degenerate nonlinear waves for a scalar viscous conservation law and an isentropic model studied in the papers [12, 14, 18, 19]. We note that, in the papers [18, 19], the same restriction $\alpha < 2(1 + \sqrt{2})$ is also required for an isentropic model and a scalar viscous conservation law $u_t + f(u)_x = u_{xx}$, where a degeneracy exponent is equal to 1, that is, $f(u) = C(u - u_+)^2 + O(|u - u_+|^3)$. Recently, Kawashima and Kurata in [7] studied the stability of the degenerate stationary solution for a viscous conservation law and obtained the same convergence rate $O(t^{-\alpha/4})$ by using the weighted energy method combined with the Hardy type inequality under a more moderate restriction $\alpha < 5$, which is best possible in the sense that the linearized operator around the degenerate stationary solution is not dissipative in L^2_α for $\alpha > 5$.

Related results. From the pioneering work [5] by Il'in and Oleĭnik, there have been many studies on the stability of several nonlinear waves for a scalar viscous conservation law. For instance, Kawashima, Matsumura and Nishihara in [8, 12, 15] obtained a convergence rate toward a traveling wave for the Cauchy problem. For a one-dimensional half space problem, Liu, Matsumura and Nishihara in [10] considered the stability of the stationary solution.

For the half space problem of the isentropic model, Kawashima, Nishibata and Zhu [9] proved the existence and the asymptotic stability of the stationary solution for the outflow problem. The convergence rate for this stability result was obtained by Nakamura, Nishibata and Yuge in [14] by assuming that the initial perturbation decays in a spatial direction. The generalization of this one-dimensional outflow problem to the multi-dimensional half space problem were studied by Kagei, Kawashima, Nakamura and Nishibata in [6, 13]. Precisely, Kagei and Kawashima in [6] proved the asymptotic stability of a planar stationary solution in a suitable Sobolev space. The convergence rate was obtained by Nakamura and Nishibata in [13]. There are also several works on the stationary problem for the Boltzmann equation (or BGK model) in half space. See [1, 2] for numerical computations and [17] for asymptotic analysis.

Outline of the paper. The remainder of the present paper is organized as follows. In Section 2, we discuss the existence of the stationary solution and present the proof of Proposition 1.1. In Section 2.2, we show a precise decay property of the degenerate stationary solution, which is utilized in the stability analysis of the degenerate stationary solution. In Section 3, Theorem 1.2 is proved by deriving uniform *a priori* estimates of the perturbation from the stationary solution in H^1 Sobolev space by an energy method. Finally, in Section 4, we prove Theorem 1.3. The crucial argument is to derive time and space weighted energy estimates. For the supersonic case, in Section 4.1, we obtain the weighted estimates in L^2 space and combine it with the uniform estimates in H^1 obtained in Section 3. Then we obtain the convergence rate (1.20) with the aid of induction. However, owing to the degenerate property of the transonic flow, we have to derive the weighted estimate

not only in L^2 but also in H^1 in order to obtain the convergence rate (1.21). This is discussed in Section 4.2.

Notations. The Gaussian bracket $[x]$ denotes the greatest integer which does not exceed x . For $p \in [1, \infty]$, $L^p(\mathbb{R}_+)$ denotes the standard Lebesgue space over \mathbb{R}_+ equipped with the norm $\|\cdot\|_{L^p}$. We use the notation $\|\cdot\| := \|\cdot\|_{L^2}$. For a non-negative integer s , $H^s(\mathbb{R}_+)$ denotes the s -th order Sobolev space over \mathbb{R}_+ in the L^2 sense with the norm

$$\|u\|_{H^s} := \left(\sum_{k=0}^s \|\partial_x^k u\|^2 \right)^{1/2}.$$

For constants $p \in [1, \infty)$ and $\alpha \in \mathbb{R}$, $L_\alpha^p(\mathbb{R}_+)$ denotes the algebraically weighted L^p space defined by $L_\alpha^p(\mathbb{R}_+) := \{u \in L_{\text{loc}}^p(\mathbb{R}_+); \|u\|_{L_\alpha^p} < \infty\}$ equipped with the norm

$$\|u\|_{L_\alpha^p} := \left(\int_{\mathbb{R}_+} (1+x)^\alpha |u(x)|^p dx \right)^{1/p}.$$

We also use the notation $|\cdot|_\alpha := \|\cdot\|_{L_\alpha^2}$. The space $H_\alpha^s(\mathbb{R}_+)$ denotes the algebraically weighted H^s space corresponding to $L_\alpha^2(\mathbb{R}_+)$ defined by $H_\alpha^s(\mathbb{R}_+) := \{u \in L_\alpha^2(\mathbb{R}_+); \partial_x^k u \in L_\alpha^2(\mathbb{R}_+) \text{ for } k = 0, \dots, s\}$, equipped with the norm

$$\|u\|_{H_\alpha^s} := \left(\sum_{k=0}^s |\partial_x^k u|_\alpha^2 \right)^{1/2}.$$

For $\alpha \in (0, 1)$, $\mathcal{B}^\alpha(\mathbb{R}_+)$ denotes the space of the Hölder continuous functions over \mathbb{R}_+ with the Hölder exponent α with respect to x . For a non-negative integer k , $\mathcal{B}^{k+\alpha}(\mathbb{R}_+)$ denotes the space of functions satisfying $\partial_x^i u \in \mathcal{B}^\alpha(\mathbb{R}_+)$ for an arbitrary $i = 0, \dots, k$ equipped with the norm $\|\cdot\|_{\mathcal{B}^{k+\alpha}}$. For $\alpha, \beta \in (0, 1)$ and $T > 0$, $\mathcal{B}^{\alpha, \beta}([0, T] \times \mathbb{R}_+)$ denotes the space of the Hölder continuous functions over $[0, T] \times \mathbb{R}_+$ with the Hölder exponents α and β with respect to t and x , respectively. For non-negative integers k and ℓ , $\mathcal{B}_T^{k+\alpha, \ell+\beta} := \mathcal{B}^{k+\alpha, \ell+\beta}([0, T] \times \mathbb{R}_+)$ denotes the space of functions satisfying $\partial_t^i u, \partial_x^j u \in \mathcal{B}^{\alpha, \beta}([0, T] \times \mathbb{R}_+)$ for arbitrary $i = 0, \dots, k$ and $j = 0, \dots, \ell$ equipped with the norm $\|\cdot\|_{\mathcal{B}_T^{k+\alpha, \ell+\beta}}$.

2 Existence of stationary solution

This section is devoted to showing Proposition 1.1. Precisely we prove the existence of a solution to the stationary problem (1.11) and (1.12). To this end, we reformulate the problem (1.11) and (1.12) into a 2×2 autonomous system of ordinary differential equations of first order.

2.1 Reformulation of stationary problem

Integrating (1.11a) over (x, ∞) , we have

$$\tilde{\rho}(x)\tilde{u}(x) = -1. \tag{2.1}$$

Integrating (1.11b) and (1.11c) over (x, ∞) and substituting (2.1) in the resultant, we obtain the system of equations for $(\bar{u}, \bar{\theta})(x) := (\tilde{u}, \tilde{\theta})(x) - (-1, 1)$ as

$$\frac{d}{dx} \begin{pmatrix} \bar{u} \\ \bar{\theta} \end{pmatrix} = J \begin{pmatrix} \bar{u} \\ \bar{\theta} \end{pmatrix} + \begin{pmatrix} \bar{f}(\bar{u}, \bar{\theta}) \\ \bar{g}(\bar{u}, \bar{\theta}) \end{pmatrix}, \quad (2.2)$$

where J is the Jacobian matrix at an equilibrium point $(0, 0)$ defined by

$$J := \begin{pmatrix} \frac{1}{\mu} \left(\frac{1}{M_+^2 \gamma} - 1 \right) & \frac{1}{\mu M_+^2 \gamma} \\ \frac{1}{\kappa M_+^2 \gamma} & -\frac{c_v}{\kappa M_+^2} \end{pmatrix},$$

and \bar{f} and \bar{g} are nonlinear terms defined by

$$\bar{f}(\bar{u}, \bar{\theta}) := -\frac{\bar{u}(\bar{u} + \bar{\theta})}{\mu M_+^2 \gamma (\bar{u} - 1)}, \quad \bar{g}(\bar{u}, \bar{\theta}) := \frac{\bar{u}^2}{2\kappa}.$$

Boundary conditions for $(\bar{u}, \bar{\theta})$ are derived from (1.12) as

$$(\bar{u}, \bar{\theta})(0) = (u_b + 1, \theta_b - 1), \quad \lim_{x \rightarrow \infty} (\bar{u}, \bar{\theta})(x) = (0, 0). \quad (2.3)$$

To prove the existence of the stationary solution $(\bar{\rho}, \bar{u}, \bar{\theta})$, it suffices to show the existence of the solution $(\bar{u}, \bar{\theta})$ to the boundary value problem (2.2) and (2.3). To this end, we diagonalize the system (2.2). Let λ_1 and λ_2 be eigenvalues of the Jacobian matrix J . Since we see later that J has real eigenvalues, we assume $\lambda_1 \geq \lambda_2$. Let r_1 and r_2 be eigenvectors of J corresponding to λ_1 and λ_2 , respectively, and let $P := (r_1, r_2)$ be a matrix. Furthermore, using the matrix P , we employ new unknown functions $U(x)$ and $\Theta(x)$ defined by

$$\begin{pmatrix} U(x) \\ \Theta(x) \end{pmatrix} := P^{-1} \begin{pmatrix} \bar{u}(x) \\ \bar{\theta}(x) \end{pmatrix}. \quad (2.4)$$

We also define a corresponding boundary data and nonlinear terms by

$$\begin{pmatrix} U_b \\ \Theta_b \end{pmatrix} := P^{-1} \begin{pmatrix} u_b + 1 \\ \theta_b - 1 \end{pmatrix}, \quad \begin{pmatrix} f(U, \Theta) \\ g(U, \Theta) \end{pmatrix} := P^{-1} \begin{pmatrix} \bar{f}(\bar{u}, \bar{\theta}) \\ \bar{g}(\bar{u}, \bar{\theta}) \end{pmatrix}.$$

Using these notations, we rewrite the problem (2.2) and (2.3) in a diagonal form as

$$\frac{d}{dx} \begin{pmatrix} U \\ \Theta \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} U \\ \Theta \end{pmatrix} + \begin{pmatrix} f(U, \Theta) \\ g(U, \Theta) \end{pmatrix}, \quad (2.5)$$

$$(U, \Theta)(0) = (U_b, \Theta_b), \quad \lim_{x \rightarrow \infty} (U, \Theta)(x) = (0, 0). \quad (2.6)$$

Since the existence of the solution to the problem (1.11) and (1.12) follows from that to the problem (2.5) and (2.6), here we show the latter. Firstly, we consider the case $M_+ > 1$. Since a discriminant of an eigen-equation of the matrix J satisfies

$$(\text{Tr } J)^2 - 4 \det J = (b - c)^2 + a^2 + 2ab + 2ca > 0,$$

where a , b and c are constants defined by

$$a := \frac{\gamma - 1}{\mu M_+^2 \gamma}, \quad b := \frac{M_+^2 - 1}{\mu M_+^2}, \quad c := \frac{c_v}{\kappa M_+^2},$$

the eigenvalues λ_1 and λ_2 are real numbers. Moreover we see

$$\lambda_1 + \lambda_2 = \text{Tr } J = -(a + b + c) < 0, \quad \lambda_1 \lambda_2 = \det J = bc > 0,$$

which show that $\lambda_1 < 0$ and $\lambda_2 < 0$. Thus, the equilibrium point $(0, 0)$ of (2.5) is asymptotically stable. Consequently, if $|(U_b, \Theta_b)|$ is sufficiently small, the problem (2.5) and (2.6) has a unique smooth solution (U, Θ) satisfying

$$|\partial_x^k(U(x), \Theta(x))| \leq C\delta e^{-cx} \quad \text{for } k = 0, 1, \dots \quad (2.7)$$

Next we study the case $M_+ = 1$. Since the matrix J satisfies

$$\text{Tr } J = -\frac{c_v d}{\mu \kappa} < 0, \quad \det J = 0, \quad d := \mu + \kappa(\gamma - 1)^2,$$

the eigenvalues of J are $\lambda_1 = 0$ and $\lambda_2 = -c_v d / (\mu \kappa)$ of which eigenvectors are explicitly given by

$$r_1 = \begin{pmatrix} -1 \\ 1 - \gamma \end{pmatrix}, \quad r_2 = \begin{pmatrix} \kappa(1 - \gamma) \\ \mu \end{pmatrix},$$

respectively. Notice that the matrix $P = (r_1, r_2)$ satisfies $\det P = -d < 0$. Thus there exist a local center manifold $\Theta = h^c(U)$ and a local stable manifold $U = h^s(\Theta)$ corresponding to the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -c_v d / (\mu \kappa)$, respectively. In order to show the existence of the solution, we have to examine dynamics on the center manifold. To this end, we employ a solution $\tilde{z} = \tilde{z}(x)$ to (2.5) restricted on the center manifold satisfying the equation

$$\tilde{z}_x = f(\tilde{z}, h^c(\tilde{z})). \quad (2.8)$$

By virtue of the center manifold theory in [3], there exists a solution \tilde{z} to (2.8) such that the solution (U, Θ) to (2.5) and (2.6) is given by

$$U(x) = \tilde{z}(x) + O(\delta e^{-cx}), \quad (2.9)$$

$$\Theta(x) = h^c(\tilde{z}(x)) + O(\delta e^{-cx}). \quad (2.10)$$

Therefore, to obtain the solution (U, Θ) to (2.5) and (2.6), it suffices to show the existence of the solution to (2.8) satisfying $\tilde{z}(x) \rightarrow 0$ as $x \rightarrow \infty$. We see that the nonlinear terms f and g satisfy

$$f(U, \Theta) = -\frac{\gamma + 1}{2d} U^2 + O(|U|^3 + |U\Theta| + |\Theta|^2), \quad (2.11)$$

$$g(U, \Theta) = \frac{\gamma - 1}{2\mu d} (P_r - 2)U^2 + O(|U|^3 + |U\Theta| + |\Theta|^2). \quad (2.12)$$

Substituting (2.11) in (2.8), we deduce (2.8) to

$$\tilde{z}_x = -\frac{\gamma + 1}{2d} \tilde{z}^2 + O(|\tilde{z}|^3), \quad (2.13)$$

which yields that \tilde{z} is monotonically decreasing for sufficiently small \tilde{z} . Thus, to satisfy $\tilde{z}(x) \rightarrow 0$ as $x \rightarrow \infty$, the boundary data $\tilde{z}(0)$ should be positive. Namely, for the existence of the solution (U, Θ) , the boundary data (U_b, Θ_b) should be located

in the right region from the local stable manifold, that is, (U_b, Θ_b) should satisfy a condition

$$U_b \geq h^s(\Theta_b). \quad (2.14)$$

From (2.13), we also see that the solution \tilde{z} satisfies

$$0 < c \frac{\delta}{1 + \delta x} \leq \tilde{z}(x) \leq C \frac{\delta}{1 + \delta x}, \quad |\partial_x^k \tilde{z}(x)| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}. \quad (2.15)$$

Combining (2.9), (2.10) and (2.15) with using $h^c(\tilde{z}) = O(\tilde{z}^2)$, we have the decay property of (U, Θ) :

$$|\partial_x^k(U(x), \Theta(x))| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} + C \delta e^{-cx} \quad \text{for } k = 0, 1, \dots \quad (2.16)$$

Finally we prove the existence of the solution to (2.5) and (2.6) for the subsonic case $M_+ < 1$. For this case, the eigenvalues of the matrix J are $\lambda_1 > 0$ and $\lambda_2 < 0$, so that there exist a local unstable manifold and a local stable manifold. Therefore, the problem (2.5) and (2.6) has a solution (U, Θ) satisfying (2.7) if the boundary data is located on the stable manifold, that is,

$$U_b = h^s(\Theta_b). \quad (2.17)$$

We summarize the above observation in Lemma 2.1 as the existence result to the problem (2.5) and (2.6).

Lemma 2.1. *Suppose that $|(U_b, \Theta_b)|$ is sufficiently small.*

- (i) *For the supersonic case $M_+ > 1$, there exists a unique smooth solution (U, Θ) to the problem (2.5) and (2.6) satisfying (2.7).*
- (ii) *For the transonic case $M_+ = 1$, if the boundary data (U_b, Θ_b) satisfies (2.14), there exists a unique smooth solution (U, Θ) satisfying (2.16).*
- (iii) *For the subsonic case $M_+ < 1$, if the boundary data (U_b, Θ_b) satisfies (2.17), there exists a unique smooth solution (U, Θ) satisfying (2.7).*

The proof of Proposition 1.1 follows immediately from Lemma 2.1. Indeed, by using the conditions (2.14) and (2.17), we precisely define the regions \mathcal{M}^0 and \mathcal{M}^- in Proposition 1.1 as follows. Define $\hat{U}(u, \theta)$ and $\hat{\Theta}(u, \theta)$ by

$$\begin{pmatrix} \hat{U}(u, \theta) \\ \hat{\Theta}(u, \theta) \end{pmatrix} := P^{-1} \begin{pmatrix} u + 1 \\ \theta - 1 \end{pmatrix}. \quad (2.18)$$

Note that $U(x) = \hat{U}(\tilde{u}(x), \tilde{\theta}(x))$ and $\Theta(x) = \hat{\Theta}(\tilde{u}(x), \tilde{\theta}(x))$ hold from (2.4). Then, defining the regions \mathcal{M}^0 and \mathcal{M}^- by

$$\begin{aligned} \mathcal{M}^0 &:= \{(u, \theta) \in \mathcal{M}^+ ; \hat{U}(u, \theta) \geq h^s(\hat{\Theta}(u, \theta))\}, \\ \mathcal{M}^- &:= \{(u, \theta) \in \mathcal{M}^+ ; \hat{U}(u, \theta) = h^s(\hat{\Theta}(u, \theta))\}, \end{aligned} \quad (2.19)$$

we see that the conditions (2.14) and (2.17) are equivalent to (1.15) and (1.17), respectively.

2.2 Estimates for degenerate stationary solution

The aim of the present section is to obtain more delicate estimates of the degenerate stationary solution, which will be utilized in deriving *a priori* estimates of the perturbation from the degenerate stationary solution for the case $M_+ = 1$.

Lemma 2.2. *Suppose that the degenerate stationary solution exists. Namely, the same conditions as in Proposition 1.1 - (ii) are supposed to hold. Then the degenerate stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ satisfies*

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta}) = (1, -1, 1) + (-1, -1, 1 - \gamma)\tilde{z} + O(\tilde{z}^2 + \delta e^{-cx}), \quad (2.20)$$

$$(\tilde{u}_x, \tilde{\theta}_x) = \frac{\gamma + 1}{2d}(1, \gamma - 1)\tilde{z}^2 + O(\tilde{z}^3 + \delta e^{-cx}), \quad (2.21)$$

$$|\partial_x^k(\tilde{u}, \tilde{\theta})| \leq C\tilde{z}^{k+1} + C\delta e^{-cx} \quad \text{for } k = 1, 2, \dots \quad (2.22)$$

Proof. The estimates for $(\tilde{u}, \tilde{\theta})$ in (2.20) are obtained by using (2.9), (2.10) and

$$\begin{pmatrix} \tilde{u} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + P \begin{pmatrix} U \\ \Theta \end{pmatrix}, \quad P = \begin{pmatrix} -1 & \kappa(1 - \gamma) \\ 1 - \gamma & \mu \end{pmatrix} \quad (2.23)$$

which follows from (2.4). Due to the fact that $\tilde{\rho}\tilde{u} = -1$, we have the estimate for $\tilde{\rho}$ in (2.20). By using (2.9), (2.10) and (2.13), we see that

$$U_x = -\frac{\gamma + 1}{2d}\tilde{z}^2 + O(\tilde{z}^3 + \delta e^{-cx}), \quad \Theta_x = O(\tilde{z}^3 + \delta e^{-cx}). \quad (2.24)$$

Differentiating (2.23) in x and substituting (2.24) yield the desired estimate (2.21). We also have the estimates $|\partial_x^k(U, \Theta)| = O(\tilde{z}^{k+1} + \delta e^{-cx})$ inductively, which give the estimate (2.22) due to (2.23). Therefore we complete the proof. \square

2.3 Local structure of invariant manifolds

In order to verify the conditions (2.14) and (2.17), which ensure the existence of the stationary solution, it is important to make clear the local shapes of the invariant manifolds h^c and h^s . In the present section, we focus ourselves on the transonic case $M_+ = 1$ and show that the geometric properties of the invariant manifolds are characterized by the Prandtl number. In detailed arguments, we follow an idea in [3]. Precisely, we approximate h^c and h^s by polynomial functions around the equilibrium point as

$$\begin{aligned} h^c(U) &= c_2U^2 + c_3U^3 + O(U^4), \\ h^s(\Theta) &= s_2\Theta^2 + s_3\Theta^3 + O(\Theta^4) \end{aligned} \quad (2.25)$$

and obtain precise expressions of the constants c_i and s_i ($i = 2, 3$).

Firstly we treat the center manifold h^c . Differentiating the relation $\Theta = h^c(U)$ in x , we have

$$\Theta_x = (h^c)'(U)U_x. \quad (2.26)$$

Substituting the equation (2.5) in (2.26) and using the relation $\Theta = h^c(U)$ again, we have

$$\lambda_2 h^c(U) + g(U, h^c(U)) = (h^c)'(U)f(U, h^c(U)), \quad (2.27)$$

where we have used $\lambda_1 = 0$. Substituting $\lambda_2 = -c_v d / (\mu \kappa)$ and (2.12) in (2.27) and using the equalities

$$\Theta = h^c(U) = O(U^2), \quad (h^c)'(U) = O(|U|), \quad f(U, h^c(U)) = O(U^2),$$

we get the second order approximation of h^c :

$$h^c(U) = -\frac{1}{\lambda_2} g(U, h^c(U)) + O(|U|^3) = \frac{\gamma(\gamma-1)^2 \kappa}{2d^2} (P_r - 2) U^2 + O(|U|^3).$$

This approximation means c_2 is given by

$$c_2 = \frac{\gamma(\gamma-1)^2 \kappa}{2d^2} (P_r - 2).$$

For the case of $P_r = 2$, that is, $c_2 = 0$, we compute c_3 similarly as above and get

$$c_3 = \frac{\gamma(\gamma-1)^2 \kappa}{d^2} > 0.$$

Next we obtain s_2 and s_3 . Differentiating $U = h^s(\Theta)$ in x and substituting (2.5) in the resultant equality, we have

$$f(h^s(\Theta), \Theta) = (h^s)'(\Theta) (\lambda_2 \Theta + g(h^s(\Theta), \Theta)). \quad (2.28)$$

Substituting $(h^s)'(\Theta) = 2s_2 \Theta + O(\Theta^2)$, $g(h^s(\Theta), \Theta) = O(\Theta^2)$ and

$$f(h^s(\Theta), \Theta) = \frac{(\gamma-1)^2 \kappa^2}{\gamma d} (P_r - \gamma_*) \Theta^2 + O(|\Theta|^3), \quad \gamma_* := \frac{1}{2}(\gamma^2 - \gamma + 2) > 1$$

in (2.28), we have

$$s_2 = -\frac{(\gamma-1)^3 \mu \kappa^3}{2d^2} (P_r - \gamma_*).$$

If $P_r = \gamma_*$, that is, $s_2 = 0$, we also compute s_3 in the same way:

$$s_3 = \frac{\gamma(\gamma-1)^5 \mu \kappa^4}{6d^2} > 0.$$

Summarizing the above observation, we have

Lemma 2.3. *Suppose that $M_+ = 1$ holds.*

(i) *The local center manifold $\Theta = h^c(U) = c_2 U^2 + c_3 U^3 + O(U^4)$ satisfies $c_2 \geq 0$ if and only if $P_r \geq 2$. Especially, if $P_r = 2$, i.e., $c_2 = 0$, the coefficient c_3 is positive.*

(ii) *The local stable manifold $U = h^s(\Theta) = s_2 \Theta^2 + s_3 \Theta^3 + O(\Theta^4)$ satisfies $s_2 \leq 0$ if and only if $P_r \leq \gamma_* := (\gamma^2 - \gamma + 2)/2$. Especially, if $P_r = \gamma_*$, i.e., $s_2 = 0$, the coefficient s_3 is positive.*

From the local structure of the invariant manifolds in the diagonalized coordinate (U, Θ) , we obtain detailed information on the local structure of invariant manifolds in the original coordinate (u, θ) . Let $\theta = \tilde{h}^c(u)$ and $\theta = \tilde{h}^s(u)$ be a local center manifold and a local stable manifold in the coordinate (u, θ) , respectively (also see Figure 1). Then we see that the relations $\theta = \tilde{h}^c(u)$ and $\theta = \tilde{h}^s(u)$ are equivalent to

$$\hat{\Theta}(u, \theta) = h^c(\hat{U}(u, \theta)) \quad \text{and} \quad \hat{U}(u, \theta) = h^s(\hat{\Theta}(u, \theta)), \quad (2.29)$$

respectively. Therefore, substituting (2.18) and (2.25) in (2.29) and solving the resultant equation with respect to θ , we get

$$\begin{aligned}\tilde{h}^c(u) &= 1 + (\gamma - 1)(u + 1) + \frac{\gamma(\gamma - 1)}{2(P_r + \gamma - 1)}(P_r - 2)(u + 1)^2 + O(|u + 1|^3), \\ \tilde{h}^s(u) &= 1 - P_r(u + 1) + \frac{P_r}{2(P_r + \gamma - 1)}(P_r - \gamma_*)(u + 1)^2 + O(|u + 1|^3).\end{aligned}$$

Especially, if $P_r = 2$ the local center manifold $\theta = \tilde{h}^c(u)$ satisfies

$$\tilde{h}^c(u) = 1 + (\gamma - 1)(u + 1) - \frac{\gamma(\gamma - 1)}{P_r + \gamma - 1}(u + 1)^3 + O(|u + 1|^4),$$

while the local stable manifold $\theta = \tilde{h}^s(u)$ satisfies

$$\tilde{h}^s(u) = 1 - P_r(u + 1) + \frac{\gamma(\gamma - 1)P_r}{6(P_r + \gamma - 1)}(u + 1)^3 + O(|u + 1|^4)$$

if $P_r = \gamma_*$.

3 Energy estimate

In this section, we prove Theorem 1.2. The crucial point of the proof is a derivation of *a priori* estimates for a perturbation from the stationary solution

$$(\varphi, \psi, \chi)(t, x) := (\rho, u, \theta)(t, x) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)$$

in the Sobolev space H^1 . Using (1.8) and (1.11), we have the system of equations for (φ, ψ, χ) as

$$\varphi_t + u\varphi_x + \rho\psi_x = -(\tilde{u}_x\varphi + \tilde{\rho}_x\psi), \quad (3.1a)$$

$$\rho(\psi_t + u\psi_x) + \frac{1}{M_+^2}(p - \tilde{p})_x = \mu\psi_{xx} - (\rho u - \tilde{\rho}\tilde{u})\tilde{u}_x, \quad (3.1b)$$

$$\frac{c_v}{M_+^2}\rho\chi_t + \frac{c_v}{M_+^2}(\rho u\theta_x - \tilde{\rho}\tilde{u}\tilde{\theta}_x) = \kappa\chi_{xx} + \mu(u_x^2 - \tilde{u}_x^2) - \frac{1}{M_+^2}(pu_x - \tilde{p}\tilde{u}_x). \quad (3.1c)$$

The initial and the boundary conditions for (φ, ψ, χ) follow from (1.2) and (1.3) as

$$(\varphi, \psi, \chi)(0, x) = (\varphi_0, \psi_0, \chi_0)(x) := (\rho_0, u_0, \theta_0)(x) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x), \quad (3.2)$$

$$(\psi, \chi)(t, 0) = (0, 0). \quad (3.3)$$

Hereafter for simplicity, we often use the notations $\Phi := (\varphi, \psi, \chi)^\top$ and $\Phi_0 := (\varphi_0, \psi_0, \chi_0)^\top$.

To show the existence of a solution to the problem (3.1), (3.2) and (3.3) locally in time, we define a function space $X(0, T)$, for $T > 0$, by

$$\begin{aligned}X(0, T) &:= \{(\varphi, \psi, \chi) ; \varphi \in \mathcal{B}_T^{1+\sigma/2, 1+\sigma}, (\psi, \chi) \in \mathcal{B}_T^{1+\sigma/2, 2+\sigma}, \\ &\quad (\varphi, \psi, \chi) \in C([0, T]; H^1(\mathbb{R}_+)), \varphi_x \in L^2(0, T; L^2(\mathbb{R}_+)), \\ &\quad (\psi_x, \chi_x) \in L^2(0, T; H^1(\mathbb{R}_+))\},\end{aligned}$$

where $\sigma \in (0, 1)$ is a constant. We summarize the existence theorem in the following lemma, which is proved by a standard iteration method.

Lemma 3.1. *Suppose that the initial data satisfies*

$$\varphi_0 \in \mathcal{B}^{1+\sigma}, (\psi_0, \chi_0) \in \mathcal{B}^{2+\sigma}, (\varphi_0, \psi_0, \chi_0) \in H^1(\mathbb{R}_+)$$

for a certain $\sigma \in (0, 1)$ and compatibility conditions of order 0 and 1. Then there exists a positive constant T_0 , depending only on $\|\varphi_0\|_{\mathcal{B}^{1+\sigma}}$ and $\|(\psi_0, \chi_0)\|_{\mathcal{B}^{2+\sigma}}$, such that the problem (3.1), (3.2) and (3.3) has a unique solution $(\varphi, \psi, \chi) \in X(0, T_0)$.

Next we show *a priori* estimates of the perturbation (φ, ψ, χ) in the space H^1 . Here we utilize the Poincaré type inequality in the next lemma. Since this lemma is proved in the similar way to the paper [9], we omit the proof.

Lemma 3.2. *For functions $f \in H^1(\mathbb{R}_+)$ and $w \in L^1_1(\mathbb{R}_+)$, we have*

$$\int_{\mathbb{R}_+} |w(x)f(x)^2| dx \leq C\|w\|_{L^1_1}(f(0)^2 + \|f_x\|^2). \quad (3.4)$$

To summarize the *a priori* estimate, we define non-negative functions $N(t)$ and $D(t)$ by

$$N(t) := \sup_{0 \leq \tau \leq t} \|\Phi(\tau)\|_{H^1},$$

$$D(t)^2 := |(\varphi, \varphi_x)(t, 0)|^2 + \|\varphi_x(t)\|^2 + \|(\psi_x, \chi_x)(t)\|_{H^1}^2.$$

Proposition 3.3. *Assume that the stationary solution exists. Namely, one of the following three conditions is supposed to hold: (i) $M_+ > 1$ and (1.13), (ii) $M_+ = 1$ and (1.15), or (iii) $M_+ < 1$ and (1.17). Let $\Phi = (\varphi, \psi, \chi) \in X(0, T)$ be a solution to (3.1), (3.2) and (3.3) for a certain constant $T > 0$. Then there exist positive constants ε_3 and C independent of T such that if $N(T) + \delta \leq \varepsilon_3$, then the solution Φ satisfies the estimate*

$$\|\Phi(t)\|_{H^1}^2 + \int_0^t D(\tau)^2 d\tau \leq C\|\Phi_0\|_{H^1}^2. \quad (3.5)$$

We prove Proposition 3.3 in Section 3.1 for the case where the stationary solution is non-degenerate, that is, $M_+ \neq 1$. Since the decay property of the degenerate stationary solution for the case $M_+ = 1$ is different from that of the non-degenerate one, we have to modify the derivation of the estimate (3.5) for $M_+ = 1$. It will be studied in Section 3.2.

In deriving *a priori* estimates, we have to employ a mollifier with respect to time variable t to resolve an insufficiency of regularity of the solution obtained in Lemma 3.1. As this argument is standard, we omit detailed computations and proceed a derivation of the estimates as if the solution verifies the sufficient regularity.

3.1 Estimates for supersonic and subsonic flows

In this section, we obtain the uniform *a priori* estimates of the perturbation from the non-degenerate stationary solution. Namely, we show (3.5) for the case $M_+ \neq 1$.

In order to obtain the estimate (3.5), we firstly derive a basic L^2 estimate. To this end, it is convenient to employ an energy form \mathcal{E} defined by

$$\mathcal{E} := \frac{1}{M_+^2 \gamma} \tilde{\theta} \omega \left(\frac{\tilde{\rho}}{\rho} \right) + \frac{1}{2} \psi^2 + \frac{c_v}{M_+^2} \tilde{\theta} \omega \left(\frac{\theta}{\tilde{\theta}} \right), \quad \omega(s) := s - 1 - \log s.$$

Owing to a smallness assumption on $N(T)$, a quantity $\|\Phi\|_{L^\infty}$ is also sufficiently small. Hence we see that the energy form is equivalent to $|\Phi|^2$:

$$c\varphi^2 \leq \omega \left(\frac{\tilde{\rho}}{\rho} \right) \leq C\varphi^2, \quad c\chi^2 \leq \omega \left(\frac{\theta}{\tilde{\theta}} \right) \leq C\chi^2, \quad c|\Phi|^2 \leq \mathcal{E} \leq C|\Phi|^2. \quad (3.6)$$

The solution, moreover, satisfies the follorin uniform estimates

$$0 < c \leq \rho(t, x), \quad \theta(t, x) \leq C, \quad -C \leq u(t, x) \leq -c < 0 \quad (3.7)$$

for $(t, x) \in [0, T] \times \mathbb{R}_+$.

Lemma 3.4. *Suppose that $M_+ \neq 1$ and the same conditions as in Proposition 3.3 hold. Then we have*

$$\begin{aligned} \|\Phi(t)\|^2 + \int_0^t (\varphi(\tau, 0)^2 + \|(\psi_x, \chi_x)(\tau)\|^2) d\tau \\ \leq C\|\Phi_0\|^2 + C\delta \int_0^t \|\varphi_x(\tau)\|^2 d\tau. \end{aligned} \quad (3.8)$$

Proof. Multiplying (3.1b) by ψ , and (3.1c) by χ/θ , then adding up the resultant two equalities, we have

$$\begin{aligned} (\rho\mathcal{E})_t - (G_1^{(1)} + B_1)_x + \mu \frac{\tilde{\theta}}{\theta} \psi_x^2 + \kappa \frac{\tilde{\theta}}{\theta^2} \chi_x^2 &= \tilde{u}_x G_1^{(2)} + \tilde{\theta}_x G_1^{(3)} + R_1, \quad (3.9) \\ G_1^{(1)} &:= -\rho u \mathcal{E} - \frac{1}{M_+^2} (p - \tilde{p}) \psi, \quad B_1 := \mu \psi \psi_x + \frac{\kappa}{\theta} \chi \chi_x, \\ G_1^{(2)} &:= -(\rho u - \tilde{\rho} \tilde{u}) \psi - \frac{1}{M_+^2 \gamma} \varphi \chi + \frac{1}{M_+^2 \gamma} \frac{\tilde{\theta}}{\tilde{u}} \varphi \psi - \frac{1}{M_+^2 \gamma} \frac{\tilde{\rho}}{\theta} \chi^2, \\ G_1^{(3)} &:= \frac{1}{M_+^2 \gamma} \rho u \omega \left(\frac{\tilde{\rho}}{\rho} \right) + \frac{c_v}{M_+^2} \rho u \omega \left(\frac{\theta}{\tilde{\theta}} \right) - \frac{c_v}{M_+^2} \frac{1}{\tilde{\theta}} \chi (\rho u \theta - \tilde{\rho} \tilde{u} \tilde{\theta}), \\ R_1 &:= \frac{\kappa}{\theta^2} \tilde{\theta}_x \chi \chi_x + \frac{2\mu}{\theta} \tilde{u}_x \chi \psi_x. \end{aligned}$$

Due to the boundary conditions (1.10) and (3.3), the integral of the second term on the left-hand side of (3.9) is estimated from below as

$$- \int_{\mathbb{R}_+} (G_1^{(1)} + B_1)_x dx = -(\rho u \mathcal{E})|_{x=0} \geq c\varphi(t, 0)^2. \quad (3.10)$$

In order to estimate the right-hand side of (3.9), we use (1.14), (3.4) and the fact $|(G_1^{(2)}, G_1^{(3)})| \leq C|\Phi|^2$, which follows from (3.6) and (3.7). Hence we have

$$\begin{aligned} \int_{\mathbb{R}_+} |\tilde{u}_x G_1^{(2)} + \tilde{\theta}_x G_1^{(3)} + R_1| dx &\leq C\delta \|(\psi_x, \chi_x)\|^2 + C\delta \int_{\mathbb{R}_+} e^{-cx} |\Phi|^2 dx \\ &\leq C\delta (\varphi(t, 0)^2 + \|\Phi_x\|^2). \end{aligned} \quad (3.11)$$

Therefore, integrating (3.9) over $(0, T) \times \mathbb{R}_+$, substituting (3.10) and (3.11) in the resultant equality, and then letting δ suitably small, we obtain the desired inequality (3.8). \square

Our next aim is to get the estimate for the first order derivative $(\varphi_x, \psi_x, \chi_x)$. To do this, we first derive the estimate for φ_x .

Lemma 3.5. *Suppose that $M_+ \neq 1$ and the same conditions as in Proposition 3.3 hold. Then we have*

$$\begin{aligned} \|\varphi_x(t)\|^2 + \int_0^t (\varphi_x(\tau, 0))^2 + \|\varphi_x(\tau)\|^2 d\tau \\ \leq C\|\Phi_0\|_{H^1}^2 + C(N(t) + \delta) \int_0^t D(\tau)^2 d\tau. \end{aligned} \quad (3.12)$$

Proof. Differentiate (3.1a) in x to get

$$\begin{aligned} \varphi_{xt} + u\varphi_{xx} + \rho\psi_{xx} = f_2, \\ f_2 := -(2\varphi_x\psi_x + 2\tilde{u}_x\varphi_x + 2\tilde{\rho}_x\psi_x + \tilde{u}_{xx}\varphi + \tilde{\rho}_{xx}\psi). \end{aligned} \quad (3.13)$$

Multiplying (3.13) by φ_x yields

$$\left(\frac{1}{2}\varphi_x^2\right)_t + \left(\frac{1}{2}u\varphi_x^2\right)_x = -\rho\varphi_x\psi_{xx} + R_2^{(1)}, \quad R_2^{(1)} := \frac{1}{2}u_x\varphi_x^2 + f_2\varphi_x. \quad (3.14)$$

On the other hand, multiplying (3.1b) by $\rho\varphi_x$ yields

$$(\rho^2\varphi_x\psi)_t - (\rho^2\varphi_t\psi)_x + \frac{1}{M_+^2}p\varphi_x^2 = \mu\rho\varphi_x\psi_{xx} + G_2 + R_2^{(2)}, \quad (3.15)$$

$$G_2 := \rho^3\psi_x^2 - \frac{1}{M_+^2\gamma}\rho^2\varphi_x\chi_x,$$

$$R_2^{(2)} := -2\rho\tilde{\rho}_x\varphi_t\psi + \rho^2\psi_x(\tilde{u}_x\varphi + \tilde{\rho}_x\psi) - \frac{1}{M_+^2\gamma}\rho\varphi_x(\tilde{\theta}_x\varphi + \tilde{\rho}_x\chi) - \rho\tilde{u}_x\varphi_x(\rho u - \tilde{\rho}\tilde{u}).$$

Successively multiplying (3.14) by μ and adding the resultant equality to (3.15), we have

$$\left(\frac{\mu}{2}\varphi_x^2 + \rho^2\varphi_x\psi\right)_t + \left(\frac{\mu}{2}u\varphi_x^2 - \rho^2\varphi_t\psi\right)_x + \frac{1}{M_+^2}p\varphi_x^2 = G_2 + R_2, \quad (3.16)$$

$$R_2 := \mu R_2^{(1)} + R_2^{(2)}.$$

Owing to the outflow boundary condition on u in (1.3), the integral of the second term on the left-hand side of (3.16) is estimated from below as

$$\int_{\mathbb{R}_+} \left(\frac{\mu}{2}u\varphi_x^2 - \rho^2\varphi_t\psi\right)_x dx = -\frac{\mu}{2}u_b\varphi_x(t, 0)^2 \geq c\varphi_x(t, 0)^2. \quad (3.17)$$

Hereafter, we denote ε an arbitrary positive constant and C_ε a positive constant depending on ε . The first term on the right-hand side of (3.16) is estimated as

$$|G_2| \leq \varepsilon\varphi_x^2 + C_\varepsilon|(\psi_x, \chi_x)|^2. \quad (3.18)$$

Since the second term on the right-hand side of (3.16) is estimated as

$$|R_2| \leq C|\psi_x|\varphi_x^2 + C\delta|(\varphi_x, \psi_x)|^2 + C\delta e^{-cx}|\Phi|^2, \quad (3.19)$$

we get the estimate for the integral of R_2 as

$$\int_{\mathbb{R}_+} |R_2| dx \leq C(N(t) + \delta)D(t)^2. \quad (3.20)$$

In deriving (3.20), we have used the estimate

$$\int_{\mathbb{R}_+} |\psi_x|\varphi_x^2 dx \leq \|\psi_x\|_{L^\infty}\|\varphi_x\|^2 \leq C\|\psi_x\|_{H^1}\|\varphi_x\|^2 \leq CN(t)(\|\psi_x\|_{H^1}^2 + \|\varphi_x\|^2)$$

to handle the first term on the right-hand side of (3.19) and the Poincaré type inequality (3.4) to estimate the third term.

Therefore, integrating (3.16) over $(0, T) \times \mathbb{R}_+$, substituting (3.17), (3.18) and (3.20) in the resultant equality and then letting ε small, we obtain

$$\begin{aligned} \|\varphi_x\|^2 + \int_0^t (\varphi_x(\tau, 0)^2 + \|\varphi_x\|^2) d\tau &\leq C\|\Phi_0\|^2 + C\|\Phi\|_{H^1}^2 + C \int_0^t \|(\psi_x, \chi_x)\|^2 d\tau \\ &\quad + C(N(t) + \delta) \int_0^t D(\tau)^2 d\tau, \end{aligned}$$

which yields the desired estimate (3.12) by substituting (3.8) in the second and the third terms on the right-hand side. These computations complete the proof. \square

Next we estimate ψ_x .

Lemma 3.6. *Suppose that $M_+ \neq 1$ and the same conditions as in Proposition 3.3 hold. Then we have*

$$\|\psi_x(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \leq C\|\Phi_0\|_{H^1}^2 + C(N(t) + \delta) \int_0^t D(\tau)^2 d\tau. \quad (3.21)$$

Proof. Multiplying (3.1b) by $-\psi_{xx}$ gives

$$\left(\frac{1}{2}\rho\psi_x^2\right)_t - (\rho\psi_x\psi_t)_x + \mu\psi_{xx}^2 = G_3 + R_3, \quad (3.22)$$

$$G_3 := \rho u\psi_x\psi_{xx} + \frac{1}{M_+^2\gamma}(\theta\varphi_x + \rho\chi_x)\psi_{xx},$$

$$R_3 := \frac{1}{M_+^2\gamma}(\tilde{\theta}_x\varphi + \tilde{\rho}_x\chi)\psi_{xx} + \tilde{u}_x(\rho u - \tilde{\rho}\tilde{u})\psi_{xx} - \rho_x\psi_x\psi_t + \frac{1}{2}\rho_t\psi_x^2.$$

Notice that G_3 satisfies

$$|G_3| \leq \varepsilon\psi_{xx}^2 + C_\varepsilon|\Phi_x|^2. \quad (3.23)$$

The term R_3 is estimated, by (1.14), as

$$|R_3| \leq C|\psi_x\Phi_x| |(\Phi_x, \psi_{xx})| + C\delta|(\Phi_x, \psi_{xx})|^2 + C\delta e^{-cx}|\Phi|^2.$$

By using (3.4) and an inequality

$$\begin{aligned} \int_{\mathbb{R}_+} |\psi_x\Phi_x| |(\Phi_x, \psi_{xx})| dx &\leq \|\psi_x\|_{L^\infty}\|\Phi_x\| \|(\Phi_x, \psi_{xx})\| \leq CN(t)\|\psi_x\|_{H^1}\|(\Phi_x, \psi_{xx})\| \\ &\leq CN(t)\|(\Phi_x, \psi_{xx})\|^2, \end{aligned}$$

we have the estimate for the integral of R_3 as

$$\int_{\mathbb{R}_+} |R_3| dx \leq C(N(t) + \delta)D(t)^2. \quad (3.24)$$

Therefore, integrating (3.22) over $(0, t) \times \mathbb{R}_+$ and substituting (3.23) and (3.24) in the resultant equality, we obtain the desired estimate (3.21). \square

We finally derive the estimate for χ_x .

Lemma 3.7. *Suppose that $M_+ \neq 1$ and the same conditions as in Proposition 3.3 hold. Then we have*

$$\|\chi_x(t)\|^2 + \int_0^t \|\chi_{xx}(\tau)\|^2 d\tau \leq C\|\Phi_0\|_{H^1}^2 + C(N(t) + \delta) \int_0^t D(\tau)^2 d\tau. \quad (3.25)$$

Proof. Multiply (3.1c) by $-\chi_{xx}$ to get

$$\begin{aligned} & \left(\frac{c_v}{2M_+^2} \rho \chi_x^2 \right)_t - \left(\frac{c_v}{M_+^2} \rho \chi_x \chi_t \right)_x + \kappa \chi_{xx}^2 = G_4 + R_4, \quad (3.26) \\ & G_4 := \frac{c_v}{M_+^2} \rho u \chi_x \chi_{xx} + \frac{1}{M_+^2 \gamma} \rho \theta \chi_x \chi_{xx}, \\ & R_4 := -\mu(u_x^2 - \tilde{u}_x^2) \chi_{xx} + \frac{c_v}{M_+^2} \tilde{\theta}_x (\rho \psi + \tilde{u} \varphi) \chi_{xx} + \frac{1}{M_+^2 \gamma} \tilde{u}_x (\rho \chi + \tilde{\theta} \varphi) \chi_{xx} \\ & \quad - \frac{c_v}{M_+^2} \rho_x \chi_x \chi_t + \frac{c_v}{2M_+^2} \rho_t \chi_x^2. \end{aligned}$$

We see that G_4 is estimated as

$$|G_4| \leq \varepsilon \chi_{xx}^2 + C_\varepsilon |\Phi_x|^2. \quad (3.27)$$

By a straightforward computation together with utilizing (1.14), we see that R_4 satisfies

$$|R_4| \leq C|(\psi_x, \chi_x)| |\Phi_x| |(\Phi_x, \chi_{xx})| + C\psi_x^2 |\Phi_x|^2 + C\delta |(\Phi_x, \chi_{xx})|^2 + C\delta e^{-cx} |\Phi|^2.$$

Integrating the above estimate with the aid of using inequalities

$$\begin{aligned} \int_{\mathbb{R}_+} |(\psi_x, \chi_x)| |\Phi_x| |(\Phi_x, \chi_{xx})| dx & \leq \|(\psi_x, \chi_x)\|_{L^\infty} \|\Phi_x\| \|(\Phi_x, \chi_{xx})\| \\ & \leq CN(t) \|(\Phi_x, \psi_{xx}, \chi_{xx})\|^2, \quad (3.28) \end{aligned}$$

$$\int_{\mathbb{R}_+} \psi_x^2 |\Phi_x|^2 dx \leq \|\psi_x\|_{L^\infty}^2 \|\Phi_x\|^2 \leq CN(t)^2 \|\psi_x\|_{H^1}^2, \quad (3.29)$$

we get the estimate for the integral of R_4 as

$$\int_{\mathbb{R}_+} |R_4| dx \leq C(N(t) + \delta)D(t)^2. \quad (3.30)$$

Thus, integrating (3.26) over $(0, t) \times \mathbb{R}_+$ and substituting (3.27) and (3.30) in the resultant equality, we obtain the desired estimate (3.25). \square

Proof of Proposition 3.3 for $M_+ \neq 1$. Summing up the estimates (3.12), (3.21) and (3.25), we have the estimate for the first order derivative Φ_x as

$$\begin{aligned} \|\Phi_x(t)\|^2 + \int_0^t (\varphi_x(\tau, 0)^2 + \|(\varphi_x, \psi_{xx}, \chi_{xx})(\tau)\|^2) d\tau \\ \leq C\|\Phi_0\|_{H^1}^2 + C(N(t) + \delta) \int_0^t D(\tau)^2 d\tau. \end{aligned} \quad (3.31)$$

Then, adding (3.8) to (3.31) and letting $N(T) + \delta$ suitably small, we obtain the desired *a priori* estimate (3.5). \square

3.2 Estimates for transonic flow

In this section, we prove Proposition 3.3 for the case $M_+ = 1$, where the stationary solution is degenerate. To do this, we define a dissipative norm $\tilde{D}(t)$ by

$$\tilde{D}(t)^2 := D(t)^2 + \delta^2[\Phi(t)]_{-2}^2,$$

where the norm $[\cdot]_\alpha$ is defined by

$$[u]_\alpha := \left(\int_{\mathbb{R}_+} (1 + \delta x)^\alpha |u(x)|^2 dx \right)^{1/2}.$$

Using the above notation, we show the uniform *a priori* estimate

$$\|\Phi(t)\|_{H^1}^2 + \int_0^t \tilde{D}(\tau)^2 d\tau \leq C\|\Phi_0\|_{H^1}^2, \quad (3.32)$$

provided that $N(T) + \delta$ is sufficiently small. Since the desired estimate (3.5) immediately follows from (3.32), it suffices to show the estimate (3.32), which is obtained by combining the estimates (3.33) and (3.41). For the case $M_+ = 1$, a decay property of the degenerate stationary solution is worse than the non-degenerate stationary solution. Therefore, in deriving L^2 estimate of Φ summarized in Lemma 3.8, we have to utilize the precise estimate (2.21) of the degenerate stationary solution in order to estimate the term $\tilde{u}_x G_1^{(2)} + \tilde{\theta}_x G_1^{(3)}$ in (3.9).

Lemma 3.8. *Suppose that $M_+ = 1$ and the same conditions as in Proposition 3.3 hold. Then we have*

$$\begin{aligned} \|\Phi(t)\|^2 + \int_0^t (\varphi(\tau, 0)^2 + \delta^2[\Phi(\tau)]_{-2}^2 + \|(\psi_x, \chi_x)(\tau)\|^2) d\tau \\ \leq C\|\Phi_0\|^2 + C\delta \int_0^t \|\varphi_x(\tau)\|^2 d\tau. \end{aligned} \quad (3.33)$$

Proof. Notice that the solution (ρ, u, θ) satisfies

$$(\rho, u, \theta) = (1, -1, 1) + O(N(t) + \delta), \quad (3.34)$$

which follows from (2.20) and $\|\Phi(t)\|_{L^\infty} \leq CN(t)$. Using the property (3.34) and (2.20), we see that $G_1^{(2)}$ is divided into a main quadratic form and residue terms as

$$G_1^{(2)} = -\psi^2 - \frac{1}{\gamma}\chi^2 + \frac{\gamma-1}{\gamma}\varphi\psi - \frac{1}{\gamma}\varphi\chi + O(N(t) + \delta)|\Phi|^2.$$

Hence we see from the above expression and (2.21) that the first term on the right-hand side of (3.9) satisfies

$$\begin{aligned}\tilde{u}_x G_1^{(2)} &= -\frac{\gamma+1}{2d} \tilde{z}^2 \left(\psi^2 + \frac{1}{\gamma} \chi^2 - \frac{\gamma-1}{\gamma} \varphi \psi + \frac{1}{\gamma} \varphi \chi \right) \\ &\quad + O(N(t) + \delta) \tilde{z}^2 |\Phi|^2 + O(\delta) e^{-cx} |\Phi|^2.\end{aligned}\tag{3.35}$$

By a similar computation, we have

$$G_1^{(3)} = -\frac{1}{2\gamma} \varphi^2 + \frac{c_v}{2} \chi^2 + c_v \varphi \chi - c_v \psi \chi + O(N(t) + \delta) |\Phi|^2,$$

where we have also used the fact that

$$\omega(s) = \frac{1}{2}(s-1)^2 + O(|s-1|^3).\tag{3.36}$$

Therefore, due to (2.21), the second term on the right-hand side of (3.9) satisfies

$$\begin{aligned}\tilde{\theta}_x G_1^{(3)} &= -\frac{\gamma+1}{2d} \tilde{z}^2 \left(\frac{\gamma-1}{2\gamma} \varphi^2 - \frac{1}{2\gamma} \chi^2 - \frac{1}{\gamma} \varphi \chi + \frac{1}{\gamma} \psi \chi \right) \\ &\quad + O(N(t) + \delta) \tilde{z}^2 |\Phi|^2 + O(\delta) e^{-cx} |\Phi|^2.\end{aligned}\tag{3.37}$$

Summing up the expressions (3.35) and (3.37), we have

$$\tilde{u}_x G_1^{(2)} + \tilde{\theta}_x G_1^{(3)} = -\frac{\gamma+1}{4\gamma d} \tilde{z}^2 F_1(\varphi, \psi, \chi) + O(N(t) + \delta) \tilde{z}^2 |\Phi|^2 + O(\delta) e^{-cx} |\Phi|^2,\tag{3.38}$$

$$F_1(\varphi, \psi, \chi) := (\gamma-1)\varphi^2 + 2\gamma\psi^2 + \chi^2 - 2(\gamma-1)\varphi\psi + 2\psi\chi.$$

The quadratic form $F_1(\varphi, \psi, \chi)$ is positive definite since

$$F_1(\varphi, \psi, \chi) = (\gamma-1)(\varphi-\psi)^2 + (\psi+\chi)^2 + \gamma\psi^2 \geq c|\Phi|^2.\tag{3.39}$$

Due to (2.21), the remaining term R_1 is estimated as

$$|R_1| \leq C\delta(\tilde{z}^2 |\Phi|^2 + |(\psi_x, \chi_x)|^2 + e^{-cx} |\Phi|^2).\tag{3.40}$$

Therefore, integrating (3.9) over $(0, t) \times \mathbb{R}_+$, substituting (3.38), (3.39) and (3.40) in the resultant equality and letting $N(t) + \delta$ suitably small, we obtain

$$\begin{aligned}\|\Phi\|^2 + \int_0^t (\varphi(\tau, 0)^2 + \delta^2 [\Phi]_{-2}^2 + \|(\psi_x, \chi_x)\|^2) d\tau \\ \leq C\|\Phi_0\|^2 + C\delta \int_0^t \int_{\mathbb{R}_+} e^{-cx} |\Phi|^2 dx d\tau,\end{aligned}$$

where we have used (2.15). Finally, to estimate the last term on the right-hand side of the above inequality, we utilize the Poincaré type inequality (3.4). Consequently, we arrive at the desired estimate (3.33) and complete the proof. \square

Next we show the estimate for the first order derivative Φ_x .

Lemma 3.9. *Suppose that $M_+ = 1$ and the same conditions as in Proposition 3.3 hold. Then we have*

$$\begin{aligned} & \|\Phi_x(t)\|^2 + \int_0^t (\varphi_x(\tau, 0)^2 + \|(\varphi_x, \psi_{xx}, \chi_{xx})(\tau)\|^2) d\tau \\ & \leq C\|\Phi_0\|_{H^1}^2 + C(N(t) + \delta) \int_0^t \tilde{D}(\tau)^2 d\tau. \end{aligned} \quad (3.41)$$

Proof. In the present proof, we only show the estimate for the remained terms R_2 , R_3 and R_4 . The other part of the derivation of (3.41) is almost same as that of the non-degenerate case, so we omit the details. Using (2.22), we see

$$\begin{aligned} |(R_2, R_3, R_4)| & \leq C|(\psi_x, \chi_x)|\|\Phi_x\| |(\Phi_x, \psi_{xx}, \chi_{xx})| + C\psi_x^2|\Phi_x|^2 \\ & \quad + C\delta z^2|\Phi|^2 + C\delta|(\Phi_x, \psi_{xx}, \chi_{xx})|^2 + C\delta e^{-cx}|\Phi|^2. \end{aligned}$$

By computations similar to (3.28) and (3.29), and by using (2.15), we have

$$\int_{\mathbb{R}_+} |(R_2, R_3, R_4)| dx \leq C(N(t) + \delta)\tilde{D}(t)^2.$$

Therefore, following the same procedure of the derivation of (3.12), (3.21) and (3.25) and using the above estimate for the remaining terms, we obtain the desired estimate (3.41). \square

3.3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Firstly we prove the existence of the solution in the sense of (1.18) globally in time. Since the existence time T_0 in Lemma 3.1 depends on the Hölder norm of the initial data, we have to show the *a priori* estimate in the Hölder norm. Precisely we prove

$$\|\rho\|_{\mathcal{B}_T^{1+\sigma/2, 1+\sigma}}, \|(u, \theta)\|_{\mathcal{B}_T^{1+\sigma/2, 2+\sigma}} \leq C(T) \quad (3.42)$$

for the solution (ρ, u, θ) satisfying $(\rho, u, \theta) - (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \in X(0, T)$, where $C(T)$ is a positive constant depending on T , $\|\rho_0\|_{\mathcal{B}^{1+\sigma}}$, $\|(u_0, \theta_0)\|_{\mathcal{B}^{2+\sigma}}$ and $\|(\varphi_0, \psi_0, \chi_0)\|_{H^1}$. To obtain the estimate (3.42), we rewrite the system (1.8) in the Eulerian coordinate into that in the Lagrangian mass coordinate, and then apply the Schauder theory for parabolic equations studied in [4] with the aid of the H^1 uniform estimate (3.5). Since the derivation of the Hölder estimate (3.42) is same as that in [9] studying the stability of the stationary solution for an isentropic model, we omit the details of the proof. Therefore, combining Lemma 3.1 and the estimate (3.42) by using the standard continuation argument, we obtain the existence of the solution globally in time. Moreover, we see that the solution verifies

$$\sup_{t \in [0, \infty)} \|\Phi(t)\|_{H^1}^2 + \int_0^\infty D(t) dt \leq C\|\Phi_0\|_{H^1}^2. \quad (3.43)$$

Next we show the stability (1.19). For this purpose, it suffices to show that

$$\|\Phi_x(t)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

since we see $\|\Phi(t)\|_{L^\infty} \leq C\|\Phi(t)\|^{1/2}\|\Phi_x(t)\|^{1/2}$ and $\|\Phi(t)\| \leq C$ due to the H^1 uniform estimate (3.5). Let $I(t) := \|\varphi_x(t)\|^2$. By a similar computation to [9], we have

$$\left| \frac{d}{dt} I(t) \right| \leq CD(t)^2,$$

which gives $\frac{d}{dt} I \in L^1(0, \infty)$ owing to (3.43). Combining this fact with $I \in L^1(0, \infty)$, which is a direct consequence of (3.43), we have $I(t) \rightarrow 0$, i.e., $\|\varphi_x(t)\| \rightarrow 0$ as $t \rightarrow \infty$. The convergence $\|(\psi_x, \chi_x)(t)\| \rightarrow 0$ is proved in the similar computations. Consequently, we prove (1.19) and complete the proof of Theorem 1.2.

4 Weighted energy estimate

In this section, we show the proof of Theorem 1.3. Precisely, we obtain convergence rates of the solution toward the stationary solution by using a time and space weighted energy method.

4.1 Estimates for supersonic flow

This section is devoted to showing the convergence (1.20) for the case $M_+ > 1$. To this end, we define weighted norm $E_\alpha(t)$ and $D_\alpha(t)$ by

$$\begin{aligned} E_\alpha(t)^2 &:= \|\Phi(t)\|_{H^1}^2 + |\Phi(t)|_\alpha^2, \\ D_\alpha(t)^2 &:= D(t)^2 + \alpha|\Phi(t)|_{\alpha-1}^2 + |(\psi_x, \chi_x)(t)|_\alpha^2 \end{aligned}$$

and obtain the weighted energy estimates summarized in the next proposition.

Proposition 4.1. *We assume that $M_+ > 1$ and (1.13) hold. Let $\Phi = (\varphi, \psi, \chi) \in X(0, T)$ be a solution to (3.1), (3.2) and (3.3) satisfying $\Phi \in C([0, T]; L_\alpha^2(\mathbb{R}_+))$ for certain constants $\alpha > 0$ and $T > 0$. Then there exist positive constant ε_4 and C independent of T such that if $N(T) + \delta \leq \varepsilon_4$, then the solution Φ satisfies the following estimates*

$$(1+t)^j E_{\alpha-j}(t)^2 + \int_0^t (1+\tau)^j D_{\alpha-j}(\tau)^2 d\tau \leq CE_\alpha(0)^2, \quad (4.1)$$

for an arbitrary integer $j = 0, \dots, [\alpha]$ and

$$(1+t)^\xi E_0(t)^2 + \int_0^t (1+\tau)^\xi D_0(\tau)^2 d\tau \leq CE_\alpha(0)^2(1+t)^{\xi-\alpha} \quad (4.2)$$

for an arbitrary $\xi > \alpha$.

The proof of Proposition 4.1 is based on the time and space weighted estimate of Φ in $L^2(\mathbb{R}_+)$ and the time weighted estimate of Φ_x .

Lemma 4.2. *Suppose that the same conditions as in Proposition 4.1 hold. Then we have*

$$\begin{aligned} & (1+t)^\xi |\Phi(t)|_\beta^2 + \int_0^t (1+\tau)^\xi (\varphi(\tau, 0)^2 + \beta |\Phi(\tau)|_{\beta-1}^2 + |(\psi_x, \chi_x)(\tau)|_\beta^2) d\tau \\ & \leq C |\Phi_0|_\beta^2 + C\xi \int_0^t (1+\tau)^{\xi-1} |\Phi(\tau)|_\beta^2 d\tau + C\delta \int_0^t (1+\tau)^\xi \|\varphi_x(\tau)\|^2 d\tau \end{aligned} \quad (4.3)$$

for arbitrary $\beta \in [0, \alpha]$ and $\xi \geq 0$.

Proof. Multiplying (3.9) by a weight function $w(t, x) := (1+t)^\xi (1+x)^\beta$, we have

$$\begin{aligned} & (w\rho\mathcal{E})_t - \{w(G_1^{(1)} + B_1)\}_x + w_x G_1^{(1)} + w \left(\mu \frac{\tilde{\theta}}{\theta} \psi_x^2 + \kappa \frac{\tilde{\theta}}{\theta^2} \chi_x^2 \right) \\ & = w_t \rho \mathcal{E} - w_x B_1 + w(\tilde{u}_x G_1^{(2)} + \tilde{\theta}_x G_1^{(3)} + R_1). \end{aligned} \quad (4.4)$$

The integral of the second term on the left-hand side of (4.4) is estimated from below as

$$- \int_{\mathbb{R}_+} \{w(G_1^{(1)} + B_1)\}_x dx \geq c(1+t)^\xi \varphi(t, 0)^2. \quad (4.5)$$

Due to (2.20), (3.34) and (3.36), the term $G_1^{(1)}$ is divided into a quadratic form and remaining terms as

$$G_1^{(1)} = \frac{1}{2M_+^2 \gamma (\gamma - 1)} F_2(\varphi, \psi, \chi) + O(N(t) + \delta) |\Phi|^2, \quad (4.6)$$

$$F_2(\varphi, \psi, \chi) := (\gamma - 1)\varphi^2 + M_+^2 \gamma (\gamma - 1)\psi^2 + \chi^2 - 2(\gamma - 1)(\varphi + \chi)\psi. \quad (4.7)$$

Notice that the quadratic form F_2 is positive definite owing to the assumption $M_+ > 1$ since

$$F_2(\varphi, \psi, \chi) = (\gamma - 1)(\varphi - \psi)^2 + \{(\gamma - 1)\psi - \chi\}^2 + \gamma(\gamma - 1)(M_+^2 - 1)\psi^2 \geq c|\Phi|^2. \quad (4.8)$$

Thus, substituting (4.8) in (4.6), we have the estimate of the third term on the left-hand side of (4.4) from below as

$$\int_{\mathbb{R}_+} w_x G_1^{(1)} dx \geq \{c - C(N(t) + \delta)\} \beta (1+t)^\xi |\Phi|_{\beta-1}^2. \quad (4.9)$$

The first and second terms on the right-hand side of (4.4) are estimated with the aid of the Schwarz inequality as

$$\int_{\mathbb{R}_+} |w\rho\mathcal{E}| dx \leq C\xi (1+t)^{\xi-1} |\Phi|_\beta^2, \quad (4.10)$$

$$\int_{\mathbb{R}_+} |w_x B_1| dx \leq C\beta (1+t)^\xi (\varepsilon |\Phi|_{\beta-1}^2 + C_\varepsilon |(\psi_x, \chi_x)|_{\beta-1}^2). \quad (4.11)$$

In the similar way to the derivation of (3.11), we estimate the remaining terms in (4.4), by using (1.14) and (3.4), as

$$\int_{\mathbb{R}_+} w |\tilde{u}_x G_1^{(2)} + \tilde{\theta}_x G_1^{(3)} + R_1| dx \leq C\delta (1+t)^\xi (\varphi(t, 0)^2 + \|\Phi_x\|^2). \quad (4.12)$$

Therefore, integrating (4.4) over $(0, t) \times \mathbb{R}_+$, substituting (4.5) and (4.9) - (4.12) in the resultant equality and letting ε and $N(t) + \delta$ sufficiently small, we arrive at

$$\begin{aligned} & (1+t)^\xi |\Phi|_\beta^2 + \int_0^t (1+\tau)^\xi (\varphi(\tau, 0)^2 + \beta |\Phi|_{\beta-1}^2 + |(\psi_x, \chi_x)|_\beta^2) d\tau \\ & \leq C |\Phi_0|_\beta^2 + C\xi \int_0^t (1+\tau)^{\xi-1} |\Phi|_\beta^2 d\tau + C\delta \int_0^t (1+\tau)^\xi \|\varphi_x\|^2 d\tau \\ & \quad + C\beta \int_0^t (1+\tau)^\xi |(\psi_x, \chi_x)|_{\beta-1}^2 d\tau. \end{aligned}$$

We finally apply induction with respect to β to estimate the last term on the right-hand side of the above inequality. This computation yields the desired estimate (4.3). Consequently, we complete the proof. \square

Letting $\beta = 0$ in (4.3), we have the time weighted estimate

$$\begin{aligned} & (1+t)^\xi \|\Phi(t)\|^2 + \int_0^t (1+\tau)^\xi (\varphi(\tau, 0)^2 + \|(\psi_x, \chi_x)(\tau)\|^2) d\tau \\ & \leq C \|\Phi_0\|^2 + C\xi \int_0^t (1+\tau)^{\xi-1} \|\Phi(\tau)\|^2 d\tau + C\delta \int_0^t (1+\tau)^\xi \|\varphi_x(\tau)\|^2 d\tau \quad (4.13) \end{aligned}$$

for an arbitrary $\xi \geq 0$.

We state below the time weighted estimate for the first order derivative Φ_x . Since the proof of this estimate is almost same as that of (3.31), we omit the details and only summarize the result in the next lemma.

Lemma 4.3. *Suppose that the same conditions as in Proposition 4.1 hold. Then we have*

$$\begin{aligned} & (1+t)^\xi \|\Phi_x(t)\|^2 + \int_0^t (1+\tau)^\xi (\varphi_x(\tau, 0)^2 + \|(\varphi_x, \psi_{xx}, \chi_{xx})(\tau)\|^2) d\tau \\ & \leq C \|\Phi_0\|_{H^1}^2 + C\xi \int_0^t (1+\tau)^{\xi-1} \|\Phi(\tau)\|_{H^1}^2 d\tau \\ & \quad + C(N(t) + \delta) \int_0^t (1+\tau)^\xi D(\tau)^2 d\tau \quad (4.14) \end{aligned}$$

for an arbitrary $\xi \geq 0$.

We conclude this section by giving the proofs of Proposition 4.1 and Theorem 1.3 - (i).

Proofs of Proposition 4.1 and Theorem 1.3 - (i). Summing up the inequalities (4.13) and (4.14), we have the time weighted H^1 estimate

$$\begin{aligned} & (1+t)^\xi \|\Phi(t)\|_{H^1}^2 + \int_0^t (1+\tau)^\xi D(\tau)^2 d\tau \\ & \leq C \|\Phi_0\|_{H^1}^2 + C\xi \int_0^t (1+t)^{\xi-1} \|\Phi(\tau)\|_{H^1}^2 d\tau. \end{aligned}$$

Add (4.3) to the above inequality to obtain

$$\begin{aligned} & (1+t)^\xi E_\beta(t)^2 + \int_0^t (1+\tau)^\xi (\beta |\Phi(\tau)|_{\beta-1}^2 + D_\beta(\tau)^2) d\tau \\ & \leq CE_\beta(0)^2 + C\xi \int_0^t (1+\tau)^{\xi-1} (|\Phi(\tau)|_\beta^2 + D_\beta(\tau)^2) d\tau, \end{aligned}$$

where we have used the inequalities

$$\beta |\Phi(t)|_{\beta-1}^2 + D_\beta(t)^2 \leq 2D_\beta(t)^2, \quad \|\Phi(t)\|_{H^1}^2 + |\Phi(t)|_\beta^2 \leq 2|\Phi(t)|_\beta^2 + D_\beta(t)^2.$$

By applying induction with respect to β and ξ , studied by [8] and [16], we have the desired estimates (4.1) and (4.2). The convergence (1.20) immediately follows from (4.2) and the Sobolev inequality. Consequently, we complete the proofs of Proposition 4.1 and Theorem 1.3 - (i). \square

4.2 Estimates for transonic flow

In this section, we show the convergence (1.21) for the case $M_+ = 1$ by deriving the time and space weighted estimate in H^1 . To do this, we define weighted norms by

$$\tilde{N}_\alpha(t) := \sup_{0 \leq \tau \leq t} \tilde{E}_\alpha(\tau), \quad \tilde{E}_\alpha(t) := \|[\Phi(t)]\|_{1,\alpha},$$

$$\tilde{D}_\alpha(t)^2 := |(\varphi, \varphi_x)(t, 0)|^2 + \delta^2 [\Phi(t)]_{\alpha-2}^2 + [\varphi_x(t)]_\alpha^2 + \|[(\psi_x, \chi_x)(t)]\|_{1,\alpha}^2,$$

where $\|[\cdot]\|_{s,\alpha}$ is the s -th order Sobolev norm corresponding to $[\cdot]_\alpha$:

$$\| [u] \|_{s,\alpha} := \left(\sum_{k=0}^s [\partial_x^k u]_\alpha^2 \right)^{1/2} = \left(\sum_{k=0}^s \int_{\mathbb{R}_+} (1+\delta x)^\alpha |\partial_x^k u(x)|^2 dx \right)^{1/2}.$$

Proposition 4.4. *We assume that $M_+ = 1$ and (1.15) hold. Let $\Phi \in X(0, T)$ be a solution to (3.1), (3.2) and (3.3) satisfying $\Phi \in C([0, T]; H_\alpha^1(\mathbb{R}_+))$ for certain constants $\alpha \in [1, 2(1 + \sqrt{2}))$ and $T > 0$. Then there exist positive constants ε_4 and C independent of T such that if $\delta^{-1/2} \tilde{N}_\alpha(T) + \delta \leq \varepsilon_4$, then the solution Φ satisfies the following estimates for $t \in [0, T]$:*

$$(1+t)^j \tilde{E}_{\alpha-2j}(t)^2 + \int_0^t (1+\tau)^j \tilde{D}_{\alpha-2j}(\tau)^2 d\tau \leq C\delta^{-2j} \tilde{E}_\alpha(0)^2 \quad (4.15)$$

for an arbitrary integer $j = 0, \dots, [\alpha/2]$ and

$$(1+t)^\xi \tilde{E}_0(t)^2 + \int_0^t (1+\tau)^\xi \tilde{D}_0(\tau)^2 d\tau \leq C\delta^{-\alpha} \tilde{E}_\alpha(0)^2 (1+t)^{\xi-\alpha/2} \quad (4.16)$$

for an arbitrary $\xi > \alpha/2$.

In order to prove Proposition 4.4, we have to derive time and space weighted estimates not only for Φ in L^2 but also for the first order derivative Φ_x . In deriving the weighted estimate for Φ in L^2 , we utilize the following interpolation inequality to handle several nonlinear terms.

Lemma 4.5. *Let $\beta \geq 1$. Then a function $f \in H_\beta^1(\mathbb{R}_+)$ satisfies*

$$\int_{\mathbb{R}_+} (1 + \delta x)^{\beta-1} |f(x)|^3 dx \leq C \delta^{-3/2} [f]_1 (f(0)^2 + \delta^2 [f]_{\beta-2}^2 + [f_x]_\beta^2). \quad (4.17)$$

Since we can prove (4.17) in the similar way to the paper [19], we omit the proof of Lemma 4.5. For details, see Lemma 5.1 with $p = 2$ and $\alpha = \beta$ in [19].

Then we show the time and space weighted L^2 estimate. In deriving this estimate, we have to assume that the weight exponent α is less than $2(1 + \sqrt{2})$ in order to obtain the dissipative term $\delta^2 [\Phi]_{\beta-2}^2$. Moreover, to control the term $\tilde{z}^{\beta+1} |\Phi|^3$ in (4.21), we have to assume the smallness of $[\Phi]_1$. Hence we need a condition $\alpha \geq 1$, too.

Lemma 4.6. *Suppose that the same conditions as in Proposition 4.4 hold. Then we have*

$$\begin{aligned} & (1+t)^\xi [\Phi(t)]_\beta^2 + \int_0^t (1+\tau)^\xi (\varphi(\tau, 0)^2 + \delta^2 [\Phi(\tau)]_{\beta-2}^2 + [(\psi_x, \chi_x)(\tau)]_\beta^2) d\tau \\ & \leq C [\Phi_0]_\beta^2 + C \xi \int_0^t (1+\tau)^{\xi-1} [\Phi(\tau)]_\beta^2 d\tau + C (\delta^{-1/2} \tilde{N}_\beta(t) + \delta) \int_0^t (1+\tau)^\xi \tilde{D}_\beta(\tau)^2 d\tau \end{aligned} \quad (4.18)$$

for arbitrary constants $\beta \in [1, \alpha]$ and $\xi \geq 0$.

Proof. In the present proof, we employ a spatial weight function

$$w(x) := h^2 \tilde{z}(x)^{-\beta}, \quad h := \left(\frac{4d}{\gamma + 1} \right)^{1/2}.$$

Notice that $w \sim \delta^{-\beta} (1 + \delta x)^\beta$ holds due to (2.15). Multiplying (3.9) by the weight function $w(x)$, we have

$$\begin{aligned} & (\rho w \mathcal{E})_t - \{w(G_1^{(1)} + B_1)\}_x + w_x G_1^{(1)} + w_x B_1 + w \left(\mu \frac{\tilde{\theta}}{\theta} \psi_x^2 + \kappa \frac{\tilde{\theta}}{\theta^2} \chi_x^2 \right) \\ & = w(\tilde{u}_x G_1^{(2)} + \tilde{\theta}_x G_1^{(3)}) + w R_1. \end{aligned} \quad (4.19)$$

The remainder of the present proof is divided into three steps.

Step 1. Firstly we show that the equality (4.19) is rewritten as

$$(\rho w \mathcal{E})_t - \{w(G_1^{(1)} + B_1)\}_x + \tilde{F} = \tilde{R}, \quad (4.20)$$

where \tilde{F} is defined by

$$\begin{aligned} \tilde{F} & := \frac{c_v}{2} w_x F_2 + \tilde{z}^{-\beta+2} F_3 + \mu \left(h \psi_x + \frac{\beta}{h} \tilde{z} \psi \right)^2 \tilde{z}^{-\beta} + \kappa \left(h \chi_x + \frac{\beta}{h} \tilde{z} \chi \right)^2 \tilde{z}^{-\beta}, \\ F_3 & = F_3(\varphi, \psi, \chi) := \frac{1}{\gamma} F_1(\varphi, \psi, \chi) + \frac{\beta}{\gamma} F_4(\varphi, \psi, \chi) - \frac{\beta^2}{h^2} (\mu \psi^2 + \kappa \chi^2), \\ F_4 & = F_4(\varphi, \psi, \chi) := (3 - \gamma) \varphi^2 + \chi^2 + 2(\gamma - 1) \varphi \psi + 2\psi \chi \end{aligned}$$

and the remaining term \tilde{R} satisfies

$$\begin{aligned} |\tilde{R}| &\leq C(N_\beta(t) + \delta)(\tilde{z}^{-\beta+2}|\Phi|^2 + \tilde{z}^{-\beta}|(\psi_x, \chi_x)|^2) + C\delta e^{-cx}\tilde{z}^{-\beta}|\Phi|^2 \\ &\quad + C\tilde{z}^{-\beta+1}|\Phi|^3. \end{aligned} \quad (4.21)$$

For this purpose, we show that the third term on the left-hand side of (4.19) verifies a decomposition

$$w_x G_1^{(1)} = \frac{c_v}{2} w_x F_2 + \frac{1}{\gamma} \beta \tilde{z}^{-\beta+2} F_4 + O(|\Phi| + \tilde{z}^2 + \delta e^{-cx}) \tilde{z}^{-\beta+1} |\Phi|^2, \quad (4.22)$$

where F_2 is defined in (4.7). Using the fact that

$$(\rho, u, \theta) = (1, -1, 1) + (-1, -1, 1 - \gamma)\tilde{z} + O(|\Phi| + \tilde{z}^2 + \delta e^{-cx}), \quad (4.23)$$

which follows from (2.20), we see that the terms in $G_1^{(1)}$ satisfy

$$\begin{aligned} -\rho u \mathcal{E} &= \frac{1}{2\gamma} \varphi^2 + \frac{1}{2} \psi^2 + \frac{c_v}{2} \chi^2 + \left(\frac{3-\gamma}{2\gamma} \varphi^2 + \frac{1}{2\gamma} \chi^2 \right) \tilde{z} + O(|\Phi| + \tilde{z}^2 + \delta e^{-cx}) |\Phi|^2, \\ -(p - \tilde{p})\psi &= -\frac{1}{\gamma} \varphi \psi - \frac{1}{\gamma} \psi \chi + \left(\frac{\gamma-1}{\gamma} \varphi \psi + \frac{1}{\gamma} \psi \chi \right) \tilde{z} + O(|\Phi| + \tilde{z}^2 + \delta e^{-cx}) |\Phi|^2. \end{aligned}$$

Summing up the above two equalities, we see that $G_1^{(1)}$ satisfies

$$G_1^{(1)} = \frac{c_v}{2} F_2 + \frac{1}{2\gamma} \tilde{z} F_4 + O(|\Phi| + \tilde{z}^2 + \delta e^{-cx}) |\Phi|^2. \quad (4.24)$$

Furthermore, by differentiating the weight function $w(x)$ and using (2.13), we have

$$w_x = 2\beta \tilde{z}^{-\beta+1} + O(\beta \tilde{z}^{-\beta+2}). \quad (4.25)$$

Multiplying (4.24) by (4.25) yields the desired equality (4.22). We also see that the fourth and the fifth terms on the left-hand side of (4.19) are rewritten as

$$\begin{aligned} w_x B_1 + w \left(\mu \frac{\tilde{\theta}}{\theta} \psi_x^2 + \kappa \frac{\tilde{\theta}}{\theta^2} \chi_x^2 \right) \\ = \mu \left(h \psi_x + \frac{\beta}{h} \tilde{z} \psi \right)^2 \tilde{z}^{-\beta} + \kappa \left(h \chi_x + \frac{\beta}{h} \tilde{z} \chi \right)^2 \tilde{z}^{-\beta} - \frac{\beta^2}{h^2} (\mu \psi^2 + \kappa \chi^2) \tilde{z}^{-\beta+2} \\ + O(|\Phi| + \tilde{z}) (\tilde{z}^{-\beta+2} |\Phi|^2 + \tilde{z}^{-\beta} |(\psi_x, \chi_x)|^2), \end{aligned} \quad (4.26)$$

which follows from (2.20), (4.23) and (4.25). To estimate the right-hand side of (4.19), we use (3.38) and (3.40) to obtain

$$\begin{aligned} w(\tilde{u}_x G_1^{(2)} + \tilde{\theta}_x G_1^{(3)}) &= -\frac{1}{\gamma} \tilde{z}^{-\beta+2} F_1(\varphi, \psi, \chi) + O(N(t) + \delta) \tilde{z}^{-\beta+2} |\Phi|^2 \\ &\quad + O(\delta) e^{-cx} \tilde{z}^{-\beta} |\Phi|^2, \end{aligned} \quad (4.27)$$

$$|w R_1| \leq C\delta (\tilde{z}^{-\beta+2} |\Phi|^2 + \tilde{z}^{-\beta} |(\psi_x, \chi_x)|^2 + e^{-cx} \tilde{z}^{-\beta} |\Phi|^2). \quad (4.28)$$

Therefore, substituting (4.22), (4.26), (4.27) and (4.28) in (4.19), we obtain the desired equality (4.20).

Step 2. Our next aim is to show that \tilde{F} satisfies the estimate from below as

$$\tilde{F} \geq c \tilde{z}^{-\beta+2} |\Phi|^2 + c \tilde{z}^{-\beta} |(\psi_x, \chi_x)|^2 \quad (4.29)$$

provided that $\beta \in [0, 2(1 + \sqrt{2})]$. Let A_2 be a real symmetric matrix satisfying $F_2 = \Phi^\top A_2 \Phi$, i.e.,

$$A_2 := \begin{pmatrix} \gamma - 1 & 1 - \gamma & 0 \\ 1 - \gamma & \gamma(\gamma - 1) & 1 - \gamma \\ 0 & 1 - \gamma & 1 \end{pmatrix}.$$

We see that the matrix A_2 admits three distinct eigenvalues 0 , ν_- and ν_+ satisfying

$$\nu_\pm = \frac{1}{2}(\gamma^2 \pm \sqrt{\gamma^4 - 4\gamma^3 + 12\gamma^2 - 20\gamma + 12}) \quad \text{and} \quad 0 < \nu_- < \nu_+.$$

Let q_1 , q_2 and q_3 be unit eigenvectors of A_2 corresponding to the eigenvalues 0 , ν_- and ν_+ , respectively. Especially, we obtain

$$q_1 = (1, 1, \gamma - 1)^\top \bar{q} \quad \text{where} \quad \bar{q} := (\gamma^2 - 2\gamma + 3)^{-1/2}.$$

Furthermore, we employ a new function $\hat{\Phi}$ defined by

$$\hat{\Phi} := (\hat{\varphi}, \hat{\psi}, \hat{\chi})^\top := Q^{-1}\Phi,$$

where $Q := (q_1, q_2, q_3)$ is an orthogonal matrix. Using the fact that $Q^\top A_2 Q = Q^{-1} A_2 Q = \text{diag}(0, \nu_1, \nu_2)$, we see that the quadratic form F_2 satisfies the estimate from below as

$$F_2 = (Q\hat{\Phi})^\top A_2 Q\hat{\Phi} = \nu_- \hat{\psi}^2 + \nu_+ \hat{\chi}^2 \geq c|(\hat{\psi}, \hat{\chi})|^2.$$

Combining this estimate with the inequality $w_x \geq c\beta \tilde{z}^{-\beta+1}$, which follows from (4.25) with $\delta \ll 1$, we have

$$\frac{c_v}{2} w_x F_2 \geq c\beta \tilde{z}^{-\beta+1} |(\hat{\psi}, \hat{\chi})|^2. \quad (4.30)$$

Next we employ a real symmetric matrix A_3 satisfying $F_3 = \Phi^\top A_3 \Phi$. Let $\hat{A}_3 := (\hat{a}_{ij})_{ij} := Q^\top A_3 Q$. Then we see that

$$F_3 = (Q\hat{\Phi})^\top A_3 Q\hat{\Phi} = \hat{\Phi}^\top \hat{A}_3 \hat{\Phi} = \hat{a}_{11} \hat{\varphi}^2 + O(|(\hat{\psi}, \hat{\chi})|^2 + |\hat{\varphi}(\hat{\psi} + \hat{\chi})|). \quad (4.31)$$

Since the sign of \hat{a}_{11} will play an important role later, we obtain it explicitly:

$$\hat{a}_{11} = q_1^\top A_3 q_1 = F_3|_{\Phi=q_1} = \frac{\gamma + 1}{4} \bar{q}^2 (4 + 4\beta - \beta^2). \quad (4.32)$$

Owing to the above observations, we show that the first and the second terms in the definition of \tilde{F} satisfy

$$\frac{c_v}{2} w_x F_2 + \tilde{z}^{-\beta+2} F_3 \geq c \tilde{z}^{-\beta+2} |\Phi|^2 \quad (4.33)$$

provided that $\beta \in [0, 2(1 + \sqrt{2})]$. Notice that the estimate (4.33) immediately yields the desired estimate (4.29). If $\beta = 0$, the quadratic form F_3 is positive definite, i.e., $F_3 \geq c|\Phi|^2$ since we have $F_3 = F_1/\gamma$ and the positivity of F_1 due to (3.39). Thus, owing to the continuous dependency on β , there exists a positive constant β_* such that $F_3 \geq c|\Phi|^2$ holds for $\beta \in [0, \beta_*]$, where c is independent of β . Namely, (4.33) holds for $\beta \in [0, \beta_*]$.

Next, we show (4.33) for $\beta \in [\beta_*, 2(1 + \sqrt{2})]$. Note that the constant \hat{a}_{11} is positive due to (4.32). Thus, using (4.30) and (4.31), we have

$$\begin{aligned} & \frac{c_v}{2} w_x F_2 + \tilde{z}^{-\beta+2} F_3 \\ & \geq c\beta_* \tilde{z}^{-\beta+1} |(\hat{\psi}, \hat{\chi})|^2 + \hat{a}_{11} \tilde{z}^{-\beta+2} \hat{\varphi}^2 - C \tilde{z}^{-\beta+2} (|(\hat{\psi}, \hat{\chi})|^2 + |\hat{\varphi}(\hat{\psi} + \hat{\chi})|) \\ & \geq (c\beta_* - C\sqrt{\delta}) \tilde{z}^{-\beta+1} |(\hat{\psi}, \hat{\chi})|^2 + (\hat{a}_{11} - C\sqrt{\delta}) \tilde{z}^{-\beta+2} \hat{\varphi}^2, \end{aligned}$$

which yields (4.33) if δ is sufficiently small. Therefore, we have shown that the estimate (4.33) holds for $\beta \in [0, 2(1 + \sqrt{2})]$.

Step 3. Finally we prove (4.18) by using (4.20) and (4.29). Using (2.15), we have the estimate for the integral of the second term on the left-hand side of (4.20) as

$$- \int_{\mathbb{R}_+} \{w(G_1^{(1)} + B_1)\}_x dx \geq c\delta^{-\beta} \varphi(t, 0)^2. \quad (4.34)$$

Furthermore, due to the Poincaré type inequality (3.4) and the inequality (4.17), integrating (4.21) yields

$$\int_{\mathbb{R}_+} |\tilde{R}| dx \leq C\delta^{-\beta} (\delta^{-1/2} \tilde{N}_\beta(t) + \delta) \tilde{D}_\beta(t)^2, \quad (4.35)$$

where we have used $[\tilde{\Phi}(t)]_1 \leq \tilde{N}_\beta(t)$ for $\beta \geq 1$. Consequently, integrating (4.20) and substituting (4.29), (4.34) and (4.35) in the resultant equality gives the desired estimate (4.18). Thus we complete the proof. \square

Next we show the estimate for the first order derivative Φ_x . Owing to the degenerate property of the transonic flow, we have to employ the spatially weighted energy method for the estimate for Φ_x .

Lemma 4.7. *Suppose that the same conditions as in Proposition 4.4 hold. Then we have*

$$\begin{aligned} & (1+t)^\xi [\Phi_x(t)]_\beta^2 + \int_0^t (1+\tau)^\xi (\varphi_x(\tau, 0)^2 + [\varphi_x(\tau)]_\beta^2 + [(\psi_{xx}, \chi_{xx})(\tau)]_\beta^2) d\tau \\ & \leq C[\|\Phi_0\|_{1,\beta}^2 + C\xi \int_0^t (1+\tau)^{\xi-1} \|\Phi(\tau)\|_{1,\beta}^2 d\tau \\ & \quad + C(\delta^{-1/2} \tilde{N}_\beta(t) + \delta) \int_0^t (1+\tau)^\xi \tilde{D}_\beta(\tau)^2 d\tau \end{aligned} \quad (4.36)$$

for arbitrary constants $\beta \in [1, \alpha]$ and $\xi \geq 0$.

Proof. We only show the estimate for φ_x as the other estimates for (ψ_x, χ_x) can be established by similar computations. Multiplying (3.16) by a spatial weight function $w := (1 + \delta x)^\beta$, we get

$$\begin{aligned} & \left\{ w \left(\frac{\mu}{2} \varphi_x^2 + \rho^2 \varphi_x \psi \right) \right\}_t + \left\{ w \left(\frac{\mu}{2} u \varphi_x^2 - \rho^2 \varphi_t \psi \right) \right\}_x + w p \varphi_x^2 \\ & = w G_2 + w R_2 + w_x \left(\frac{\mu}{2} u \varphi_x^2 - \rho^2 \varphi_t \psi \right). \end{aligned} \quad (4.37)$$

The last term on the right-hand side of (4.37) is estimated as

$$\int_{\mathbb{R}_+} w_x \left| \frac{\mu}{2} u \varphi_x^2 - \rho^2 \varphi_t \psi \right| dx \leq \varepsilon [\varphi_x]_\beta^2 + C_\varepsilon (\delta^2 [\Phi]_{\beta-2}^2 + [\psi_x]_\beta^2) + C \delta \tilde{D}_\beta(t)^2, \quad (4.38)$$

where $\varepsilon > 0$ is an arbitrary constant and C_ε is a positive constant depending on ε . The other terms in (4.37) are estimated in a same way as the proof of Lemma 3.5. For instance, the remaining term R_2 verifies the estimate

$$\int_{\mathbb{R}_+} w |(R_2, R_3, R_4)| dx \leq C(\tilde{N}_\beta(t) + \delta) \tilde{D}_\beta(t)^2, \quad (4.39)$$

which follows from the Poincaré type inequality (3.4) and the inequality

$$\begin{aligned} \int_{\mathbb{R}_+} (1 + \delta x)^\beta |(\psi_x, \chi_x)| |\Phi_x| |(\Phi_x, \psi_{xx}, \chi_{xx})| dx &\leq \|(\psi_x, \chi_x)\|_{L^\infty} [\Phi_x]_\beta [(\Phi_x, \psi_{xx}, \chi_{xx})]_\beta \\ &\leq C \tilde{N}_\beta(t) [(\Phi_x, \psi_{xx}, \chi_{xx})]_\beta^2. \end{aligned}$$

Therefore, integrating (4.37) over $(0, t) \times \mathbb{R}_+$, substituting (3.18), (4.38) and (4.39) in the resultant and then letting ε sufficiently small with using (4.18), we obtain the estimate for φ_x in (4.36). The estimates for (ψ_x, χ_x) are obtained by similar computations to Lemma 3.6 and 3.7 with using the estimate (4.39). Thus we complete the proof of the desired estimate (4.36). \square

Proofs of Proposition 4.4 and Theorem 1.3 - (ii). Summing up the estimates (4.18) and (4.36), and letting $\delta^{-1/2} \tilde{N}_\beta(t) + \delta$ be suitably small, we have

$$\begin{aligned} (1+t)^\xi \|[\Phi(t)]\|_{1,\beta}^2 + \int_0^t (1+\tau)^\xi (\delta^2 [\Phi(\tau)]_{\beta-2}^2 + \tilde{D}_\beta(\tau)^2) d\tau \\ \leq C \|[\Phi_0]\|_{1,\beta}^2 + C\xi \int_0^t (1+\tau)^{\xi-1} ([\Phi(\tau)]_\beta^2 + \tilde{D}_\beta(\tau)^2) d\tau, \end{aligned}$$

which yields the desired estimates (4.15) and (4.16) by an induction with respect to β and ξ (see [16]). The convergence rate (1.21) follows from the estimate (4.16) with the aid of the estimate $\delta^\alpha \|\Phi\|_{H_{1,\alpha}^1}^2 \leq \|[\Phi]\|_{1,\alpha}^2 \leq \|\Phi\|_{H_{1,\alpha}^2}^2$. We consequently complete the proofs. \square

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