Uniformly exponentially stable approximations for a class of damped systems with unbounded feedbacks

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Abstract

In this paper we study time semi-discrete approximations of a class of exponentially stable infinite dimensional systems with unbounded feedbacks. It has recently been proved that for time semi-discrete systems, due to high frequency spurious components, the exponential decay property may be lost as the time step tends to zero. We prove that adding a suitable numerical viscosity term in the numerical scheme, one obtains approximations that are uniformly exponentially stable with respect to the discretization parameter.

Key words and phrases: exponential stabilization, observability inequality, discretization, viscosity term.

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1 Introduction

Let $X$ and $Y$ be real Hilbert spaces ($Y$ will be identified to its dual space) with norms denoted respectively by $\|\cdot\|_X$ and $\|\cdot\|_Y$.
Let $A : D(A) \to X$ be a skew-adjoint operator with compact resolvent and $B \in \mathcal{L}(Y, D(A)')$, where $D(A)'$ is the dual space of $D(A)$ obtained by means of the inner product in $X$.
We consider the system described by
\[
\dot{z}(t) = Az - BB^*z, \quad t \geq 0, \quad z(0) = z_0 \in X. \tag{1.1}
\]

Here and henceforth, a dot ($\cdot$) denotes differentiation with respect to time $t$. The element $z_0 \in X$ is the initial state, and $z(t)$ is the state of the system.
Most of the linear equations modeling the damped vibrations of elastic structures can be written in the form (1.1). Some other relevant models, as the damped Schrodinger equations, fit in this setting as well.

We assume the following hypothesis introduced in [4]:

\[(H)\]

If \( \beta > 0 \) is fixed and \( C_\beta = \{ \lambda \in \mathbb{C} \mid \Re\lambda = \beta \} \), the function

\[ \lambda \in \mathbb{C}_+ = \{ \lambda \in \mathbb{C} \mid \Re\lambda > 0 \} \rightarrow H(\lambda) = B^*(\lambda I - A)^{-1}B \in \mathcal{L}(Y) \]  

(1.2)

is bounded on \( C_\beta \).

We define the energy of the solutions of system \( (1.1) \) by:

\[ E(t) = \frac{1}{2} \| z(t) \|^2_X, \quad t \geq 0, \]  

(1.3)

which satisfies

\[ \frac{dE}{dt}(t) = -\| B^* z(t) \|^2_Y, \quad t \geq 0. \]  

(1.4)

In this paper, we assume that system \( (1.1) \) is exponentially stable, that is there exist positive constants \( \mu \) and \( \nu \) such that any solution of \( (1.1) \) satisfies

\[ E(t) \leq \mu E(0) \exp(-\nu t), \quad t \geq 0. \]  

(1.5)

Our goal is to develop a theory allowing to get, as a consequence of (1.5), exponential stability results for time-discrete systems. We start considering the following natural time-discretization scheme for the continuous system \( (1.1) \). For any \( \Delta t > 0 \), we denote by \( z^k \) the approximation of the solution \( z \) of system \( (1.1) \) at time \( t_k = k\Delta t \), for \( k \in \mathbb{N} \), and introduce the following implicit midpoint time discretization of system \( (1.1) \):

\[ \begin{cases} 
  \frac{z^{k+1} - z^k}{\Delta t} = A \left( \frac{z^k + z^{k+1}}{2} \right) - BB^* \left( \frac{z^k + z^{k+1}}{2} \right), \quad k \in \mathbb{N}, \\
  z^0 = z_0.
\end{cases} \]  

(1.6)

We define the discrete energy by:

\[ E^k = \frac{1}{2} \| z^k \|^2_X, \quad k \in \mathbb{N}, \]  

(1.7)

which satisfies the dissipation law

\[ \frac{E^{k+1} - E^k}{\Delta t} = -\left\| B^* \left( \frac{z^k + z^{k+1}}{2} \right) \right\|^2_Y, \quad k \in \mathbb{N}. \]  

(1.8)

It is well known that if the continuous system is exponentially stable, the time-discrete ones do no more inherit of this property due to spurious high frequency modes (see [15]), that is we cannot expect in general to find positive constants \( \mu_0 \) and \( \nu_0 \) such that

\[ E^k \leq \mu_0 E^0 \exp(-\nu_0 k\Delta t), \quad k \in \mathbb{N}, \]  

(1.9)
holds for any solution of (1.10) uniformly with respect to $\Delta t > 0$.

Therefore, as in [12, 10, 13, 6], in order to get a uniform decay, it seems natural to add in system (1.10) a suitable extra numerical viscosity term to damp these high-frequency spurious components. We obtain the new system:

$$
\begin{aligned}
\left\{
\begin{array}{l}
\frac{z_k^{k+1} - z_k}{\Delta t} = A \left( \frac{z_k^k + z_k^{k+1}}{2} \right) - B B^* \left( \frac{z_k^k + z_k^{k+1}}{2} \right), \quad k \in \mathbb{N}, \\
\frac{\bar{z}_k^{k+1} - \bar{z}_k}{\Delta t} = (\Delta t)^2 A^2 z_k^{k+1}, \quad k \in \mathbb{N}, \\
z_0 = \bar{z}_0.
\end{array}
\right.
\end{aligned}
$$

(1.10)

The energy of (1.10), still defined by (1.7), now satisfies:

$$
\begin{aligned}
\left\{
\begin{array}{l}
\bar{E}_k^{k+1} = E_k^k - \Delta t \left\| B^* \left( \frac{z_k^k + z_k^{k+1}}{2} \right) \right\|^2_Y, \quad k \in \mathbb{N}, \\
E_k^{k+1} + (\Delta t)^3 \| A z_k^{k+1} \|^2_X + \frac{(\Delta t)^6}{2} \| A^2 z_k^{k+1} \|^2_X = \bar{E}_k^{k+1}, \quad k \in \mathbb{N}.
\end{array}
\right.
\end{aligned}
$$

(1.11)

Putting these identities together, we get:

$$
E_k^{k+1} + (\Delta t)^3 \| A z_k^{k+1} \|^2_X + \frac{(\Delta t)^6}{2} \| A^2 z_k^{k+1} \|^2_X + \Delta t \left\| B^* \left( \frac{z_k^k + z_k^{k+1}}{2} \right) \right\|^2_Y = E_k^k.
$$

Summing this identities from $j = k_1$ to $j = k_2 - 1$, we obtain:

$$
\begin{aligned}
\| z_{k_2} \|^2_X + 2\Delta t \sum_{j = k_1}^{k_2 - 1} \left\| B^* \left( \frac{z_j^j + z_j^{j+1}}{2} \right) \right\|^2_Y + 2\Delta t \sum_{j = k_1}^{k_2 - 1} (\Delta t)^2 \| A z_j^{j+1} \|^2_X + \Delta t \sum_{j = k_1}^{k_2 - 1} (\Delta t)^5 \| A^2 z_j^{j+1} \|^2_X \\
= \| z_{k_1} \|^2_X, \quad \forall k_1 < k_2.
\end{aligned}
$$

(1.12)

The main result of this paper reads as follows:

**Theorem 1.1.** Assume that system (1.1) is exponentially stable, i.e. satisfies (1.5) and the hypothesis (H) is verified. Then there exist two positive constants $\mu_0$ and $\nu_0$ such that any solution of (1.10) satisfies (1.9) uniformly with respect to the discretization parameter $\Delta t > 0$.

Our strategy is based on the fact that the uniform exponential decay properties of the energy of systems (1.1) and (1.10) respectively are equivalent to uniform observability properties for the conservative system

$$
\dot{y} = Ay, \quad t \in \mathbb{R}, \quad y(0) = y_0 \in X,
$$

(1.13)

and its time semi-discrete viscous version:

$$
\begin{aligned}
\left\{
\begin{array}{l}
\frac{\bar{u}_k^{k+1} - u_k^k}{\Delta t} = A \left( \frac{u_k^k + u_k^{k+1}}{2} \right), \quad k \in \mathbb{N}, \\
\frac{u_k^{k+1} - u_k^{k+1}}{\Delta t} = (\Delta t)^2 A^2 u_k^{k+1}, \quad k \in \mathbb{N}, \\
u_0 = u_0.
\end{array}
\right.
\end{aligned}
$$

(1.14)
At the continuous level the observability property consists in the existence of a time $T > 0$ and a positive constant $k_T > 0$ such that

$$k_T \| y_0 \|_X^2 \leq \int_0^T \| B^* y(t) \|_Y^2 \, dt,$$

for every solution of (1.13) (see [4]).

A similar argument can be applied to the semi-discrete system (1.10). Namely, the uniform exponential decay (1.9) of the energy of solutions of (1.10) is equivalent to the following observability inequality: there exist positive constants $T$ and $c$ such that, for any $\Delta t > 0$, every solution $u$ of (1.14) satisfies:

$$c \| u_0 \|_X^2 \leq \Delta t \sum_{k \Delta t \in [0,T]} \| B^* u^k \|_Y^2 + \Delta t \sum_{k \Delta t \in [0,T]} (\Delta t)^2 \| A u^{k+1} \|_X^2$$

$$+ \Delta t \sum_{k \Delta t \in [0,T]} (\Delta t)^2 \| A u^{k+1} \|_X^2.$$ (1.16)

Our approach has common points with the result obtained in [6] for feedbacks which are bounded in the energy space. The main difference is that we replace the assumption of boundedness of $B$ by the assumption (H).

Let us mention the works [13, 9], where boundary (that is $B$ is unbounded) stabilization issues were discussed for space discrete of the 1-d wave equation.

In our knowledge, this paper is the first one providing exponential decay properties for time-discrete systems, when the continuous setting has this property, in the case where $B$ is unbounded.

The outline of this paper is as follows.

In the second section, we give the background needed here. We recall some results on the observability of time-discrete conservative systems and prove (1.16) in Section 3. Section 4 contains the proof of the main result. The last section is devoted to some applications.

In the following, to simplify the notation, $C$ and $c$ will denote a positive constants that may change from line to line, but don’t depend on $\Delta t$.

### 2 Some Background and Preliminaries

In this section we give some background (without any proof) that we need in our present work (for more details, see [14]).

Throughout this section, $X$ is Hilbert space and $A : D(A) \to X$ be a densely defined operator with $\rho(A) \neq \emptyset$ ($\rho(A)$ is the resolvent set of $A$). We assume that $D(A)$ is endowed with the norm, $\| \cdot \|_{D(A)}$, of the graph of $A$.

For every $\beta \in \rho(A)$, we define

$$\| x \|_1 = \| (\beta I - A) x \|_X, \quad \forall x \in D(A).$$

The space $D(A)$ with this norm is a Hilbert space, denoted $X_1$. It is well known that $\| \cdot \|_1$ is equivalent to $\| \cdot \|_{D(A)}$.

In the following, to simplify the notation, $C$ and $c$ will denote a positive constants that may change from line to line, but don’t depend on $\Delta t$. 

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We denote by $X_{-1}$ the completion of $X$ with respect to the norm
$$
\|x\|_{-1} = \|(\beta I - A)^{-1}x\|_X \quad \forall x \in X \text{ and } \beta \in \rho(A).
$$

Then $A \in \mathcal{L}(X_1, X)$ and $A$ has a unique extension $\tilde{A} \in \mathcal{L}(X, X_{-1})$.

Moreover,
$$
(\beta I - A)^{-1} \in \mathcal{L}(X, X_1), \quad (\beta I - \tilde{A})^{-1} \in \mathcal{L}(X_{-1}, X)
$$
and these two operators are unitary.

We recall also, if $A$ is maximal dissipative (for brevity m-dissipative) then $(0, \infty) \subset \rho(A)$ and
$$
\|((\beta I - A)^{-1})_{\mathcal{L}(X)}\| \leq \frac{1}{\beta} \quad \forall \beta \in (0, \infty).
$$

When $A$ is skew-adjoint, we have both $A$ and $-A$ are m-dissipative.

Now, we give the definition of a contraction.

**Définition 2.1.** A contraction is a bounded operator $C$, with the property that the norm $\|C\| \leq 1$. Note that the powers of a contraction have the same property, $\|C^n\| \leq 1$ for $n \in \mathbb{N}$.

The following theorem ([11]) gives a way of characterizing a generator of a semigroup of contraction.

**Theorem 2.1.** Let $X$ be a Hilbert space, and let $A : D(A) \to X$ be a linear operator with dense domain. Then the following conditions are equivalent:

i) $A$ is the generator of a $C_0$ semigroup of contraction,

ii) all $\lambda \in \mathbb{C}_+$ belong to $\rho(A)$, and $A_\lambda = (\lambda I + A)(\lambda I - A)^{-1}$ is a contraction.

Note that if $A$ is skew-adjoint, then it is the generator of a $C_0$ contraction semigroup.

Finally, we recall also that if $A$ is skew-adjoint, then we have $-A^2$ is a positive self adjoint operator and consequently $A^2$ is m-dissipative.

### 3 Observability of time-discrete systems

This section is organized as follows. First, we recall the results of [5] on the observability of the time-discrete conservative system of (1.6). Then, we give the proof of observability inequality (1.16) which consists in the decomposition of the solution $u$ of (1.14) into its low and high frequency parts, that we handle separately, as in [6].

#### 3.1 Some results on discrete observability

We first need to introduce some notations.
Since $A$ is a skew-adjoint operator with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j : j \in \mathbb{N}\}$, where $(\mu_j)_{j \in \mathbb{N}}$ is a sequence of real numbers such that $|\mu_j| \to \infty$ when $j \to \infty$. Set $(\phi_j)_{j \in \mathbb{N}}$ an orthonormal basis of eigenvectors of $A$ associated to the eigenvalues $(i\mu_j)_{j \in \mathbb{N}}$, that is
\[ A\phi_j = i\mu_j \phi_j. \] (3.19)

Moreover, define
\[ C_s(A) = \text{span}\{\phi_j : \text{the corresponding } i\mu_j \text{ satisfies } |\mu_j| \leq s\}. \] (3.20)

The following theorem was proved in [5]:

**Theorem 3.1.** Assume that $B^* \in \mathcal{L}(D(A), Y)$, that is
\[ \|B^*z\|_Y \leq C_B^2 \|z\|_{D(A)}^2 = C_B^2 (\|Az\|_X^2 + \|z\|_X^2), \quad \forall z \in D(A), \] (3.21)
and that $A$ and $B^*$ satisfy the following hypothesis:
\[ \begin{cases} 
\text{There exist constants } M, m > 0 \text{ such that } \\
M^2 \|(iwI - A)y\|_X^2 + m^2 \|B^*y\|_Y^2 \geq \|y\|_X^2, \quad \forall w \in \mathbb{R}, \ y \in D(A).
\end{cases} \] (3.22)

Then, for any $\delta > 0$, there exists $T_\delta$ such that for any $T > T_\delta$, there exists a positive constant $k_{T, \delta}$, independent of $\Delta t$, that depends only on $m, M, C_B, T$ and $\delta$, such that for $\Delta t > 0$ small enough, we have:
\[ k_{T, \delta} \|y^0\|_X^2 \leq \Delta t \sum_{k \Delta t \in [0, T]} \left\| B^* \left( \frac{y^k + y^{k+1}}{2} \right) \right\|_Y^2, \quad \forall y^0 \in C_{\delta/\Delta t}(A), \] (3.23)
where $y^k$ is the solution of
\[ \frac{y^{k+1} - y^k}{\Delta t} = A \left( \frac{y^k + y^{k+1}}{2} \right), \quad k \in \mathbb{N}, \ y^0 = y_0. \] (3.24)

In the sequel, when there is no ambiguity, we will use the simplified notation $C_{\delta/\Delta t}$ instead of $C_{\delta/\Delta t}(A)$.

Hypothesis (3.22) is the so-called Hautus test or resolvent estimate, which has been proved in [8] to be equivalent to the continuous observability inequality (1.15) for the conservative system (1.13) for suitable positive constants $T$ and $k_T$, which turns out to be equivalent to the exponential decay property (1.5) for the continuous damped system (1.1).

The following Lemma gives a resolvent estimate which was proved in [9]:

**Lemma 3.1.** Under the assumptions of Theorem 1.1, the resolvent estimate (3.22) holds, with constants $m$ and $M$ that depend only on $\mu$ and $\nu$ given by (1.3).

Applying Theorem 3.1, for any $\delta > 0$, choosing a time $T^* > T_\delta$ there exists a positive constant $k_{T^*, \delta}$ such that the inequality (3.23) holds for any solution $y$ of (3.24) with $y^0 \in C_{\delta/\Delta t}$. In the sequel, we fix a positive number $\delta > 0$ (for instance $\delta = 1$), and $T^* = 2T_\delta$. 

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3.2 Uniform observability inequalities

Lemma 3.2. There exists a constant $c > 0$ such that (1.16) holds with $T = T^*$ for all solutions $u$ of (1.14) uniformly with respect to $\Delta t$.

Proof. We decompose the solution $u$ of (1.14) into its low and high frequency parts. To be more precise, we consider

$$ u_l = \pi_{\delta/\Delta t} u, \quad u_h = (I - \pi_{\delta/\Delta t})u, \quad (3.25) $$

where $\delta$ is the positive number that we have been chosen above, and $\pi_{\delta/\Delta t}$ is the orthogonal projection on $C_{\delta/\Delta t}$ defined in (3.20). Here the notations $u_l$ and $u_h$ stand for the low and high frequency components, respectively. Note that both $u_l$ and $u_h$ are solutions of (1.14).

Besides, $u_h$ lies in the space $C_{\delta/\Delta t}^*$, in which the following property holds:

$$ \Delta t \|Ay\|_X \geq \delta \|y\|_X, \quad \forall y \in C_{\delta/\Delta t}^*. \quad (3.26) $$

The low frequencies. We compare $u_l$ with $y_l$ solution of (3.24) with initial data $y_l(0) = u_l(0)$. Set $w_l = u_l - y_l$. From (3.28), we get:

$$ k_{T^*, \delta} \|u_l^0\|_X^2 \leq 2\Delta t \sum_{k \Delta t \in [0, T^*)} \left\| B^* \left( \frac{w_l^k + \tilde{w}_l^{k+1}}{2} \right) \right\|_Y^2 + 2\Delta t \sum_{k \Delta t \in [0, T^*)} \left\| B^* \left( \frac{w_l^k + \tilde{w}_l^{k+1}}{2} \right) \right\|_Y^2. \quad (3.27) $$

The equation satisfied by $w_l$ is:

$$ \begin{cases}
\frac{w_l^{k+1} - w_l^k}{\Delta t} = A \left( \frac{w_l^k + \tilde{w}_l^{k+1}}{2} \right), & k \in \mathbb{N}, \\
\frac{w_l^{k+1} - \tilde{w}_l^{k+1}}{\Delta t} = (\Delta t)^2 A^2 u_l^{k+1}, & k \in \mathbb{N}, \\
w_l^0 = 0.
\end{cases} \quad (3.28) $$

It is easy to see that $\|w_l^{k+1}\|_X^2 = \|w_l^k\|_X^2$. Besides, we have:

$$ \left\| B^* \left( \frac{w_l^k + \tilde{w}_l^{k+1}}{2} \right) \right\|_Y^2 \leq \frac{1}{2} \left\| B^* w_l^k \right\|_Y^2 + \frac{1}{2} \left\| B^* \tilde{w}_l^{k+1} \right\|_Y^2. $$

Simple Calculations give:

$$ B^* \tilde{w}_l^{k+1} = B^* (I - \frac{\Delta t}{2} A)^{-1} (I + \frac{\Delta t}{2} A) w_l^k. $$

Therefore, we have:

$$ \left\| B^* \tilde{w}_l^{k+1} \right\|_Y^2 = \left\| B^* (I - \frac{\Delta t}{2} A)^{-1} (I + \frac{\Delta t}{2} A) w_l^k \right\|_Y^2 \leq C_B \left\| (I - \frac{\Delta t}{2} A)^{-1} (I + \frac{\Delta t}{2} A) w_l^k \right\|_{D(A)}^2 \leq C \left\| (I - \frac{\Delta t}{2} A)^{-1} (I + \frac{\Delta t}{2} A) w_l^k \right\|_1^2, $$

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Finally, we get
\[
\| B^* \hat{w}_t^{k+1} \|^2_Y \leq C \|(I + \frac{\Delta t}{2} A)w_t^k \|^2_X \\
\leq C \| w_t^k \|^2_X + c(\Delta t)^2 \| Au_t^k \|^2_X \\
\leq C \| w_t^k \|^2_X,
\]
where we have used (2.17) and the fact that $w_t^k \in C_{\delta/\Delta t}$.

In the same manner, by using (2.17) and the fact that $\hat{w}_t^{k+1} \in C_{\delta/\Delta t}$ and $\| \hat{w}_t^{k+1} \|^2_X = \| w_t^k \|^2_X$, we show that:
\[
\| B^* w_t^k \|^2 \leq c \| w_t^k \|^2_X.
\]
Consequently, we obtain:
\[
\| B^* \left( \frac{w_t^k + \hat{w}_t^{k+1}}{2} \right) \|^2_Y \leq c \| w_t^k \|^2_X. \tag{3.29}
\]
In addition, we have (see [6])
\[
\| w_t^k \|^2_X \leq \Delta t \sum_{j=1}^{k} (\Delta t)^2 \| Au_t^j \|^2_X + \delta^2 \Delta t \sum_{j=0}^{k} \| w_t^j \|^2_X. \tag{3.30}
\]
Grönwall’s Lemma applies and allows to deduce from (3.27) and (3.29), the existence of a positive $c$ independent of $\Delta t$ such that
\[
c \| u_t^0 \|^2_X \leq \Delta t \sum_{k \Delta t \in [0,T^*]} \left\| B^* \left( \frac{u_t^k + \hat{u}_t^{k+1}}{2} \right) \right\|^2_Y + \Delta t \sum_{k \Delta t \in [0,T^*]} (\Delta t)^2 \| Au_t^k \|^2_X.
\]
Besides,
\[
\Delta t \sum_{k \Delta t \in [0,T^*]} \left\| B^* \left( \frac{u_t^k + \hat{u}_t^{k+1}}{2} \right) \right\|^2_Y \leq 2\Delta t \sum_{k \Delta t \in [0,T^*]} \left\| B^* \left( \frac{u_t^k + \hat{u}_t^{k+1}}{2} \right) \right\|^2_Y + 2\Delta t \sum_{k \Delta t \in [0,T^*]} \left\| B^* \left( \frac{u_t^k + \hat{u}_t^{k+1}}{2} \right) \right\|^2_Y.
\]
Let us then estimate the last term in the right-hand side of the above inequality. We have
\[
\left\| B^* \left( \frac{u_t^k + \hat{u}_t^{k+1}}{2} \right) \right\|^2_Y \leq \frac{1}{2} \| B^* u_t^k \|^2_Y + \frac{1}{2} \| B^* \hat{u}_t^{k+1} \|^2_Y.
\]
But,
\[
B^* u_t^k = B^* (I + \frac{\Delta t}{2} A)^{-1} (I - \frac{\Delta t}{2} A) \hat{u}_t^{k+1}.
\]
Then, we have
\[
\|B^* u_h^k\|_Y^2 \leq C \| (I + \frac{\Delta t}{2} A)^{-1} (I - \frac{\Delta t}{2} A) \tilde{u}_h^{k+1} \|_{D(A)}^2
\]
\[
\leq C \| (I - \frac{\Delta t}{2} A) \tilde{u}_h^{k+1} \|_X^2
\]
\[
\leq 2C \| \tilde{u}_h^{k+1} \|_X^2 + C \frac{\Delta t}{2} \| A\tilde{u}_h^{k+1} \|_X^2.
\]
Now, since \( A \in \mathcal{L}(X_1, X) \) and from (2.17), we get
\[
\| A\tilde{u}_h^{k+1} \|_X = \| A(I - \frac{\Delta t}{2} A)^{-1} (I + \frac{\Delta t}{2} A) \tilde{u}_h^k \|_X^2
\]
\[
\leq C \| (I - \frac{\Delta t}{2} A)^{-1} (I + \frac{\Delta t}{2} A) \tilde{u}_h^k \|_X^2
\]
\[
\leq C \| (I + \frac{\Delta t}{2} A) \tilde{u}_h^k \|_X^2
\]
\[
\leq 2C \bigg( \| \tilde{u}_h^k \|_X^2 + \frac{(\Delta t)^2}{2} \| A\tilde{u}_h^k \|_X^2 \bigg).
\]
Consequently, for \( \Delta t \) small enough, we obtain:
\[
\|B^* u_h^k\|_Y^2 \leq C \left( \| \tilde{u}_h^k \|_X^2 + (\Delta t)^2 \| A\tilde{u}_h^k \|_X^2 \right).
\]
By the same way, we prove that
\[
\|B^* \tilde{u}_h^{k+1}\|_Y^2 \leq C \left( \| \tilde{u}_h^k \|_X^2 + (\Delta t)^2 \| A\tilde{u}_h^k \|_X^2 \right),
\]
and since \( \tilde{u}_h^{k+1} \) and \( u_h^k \) belong to \( C_{\delta / \Delta t} \) for all \( k \), we get from (3.20) that
\[
\Delta t \sum_{k \Delta t \in [0, T^*]} \left\| \frac{B^* \left( \frac{u_h^k + \tilde{u}_h^{k+1}}{2} \right)}{2} \right\|_Y^2 \leq C \Delta t \sum_{k \Delta t \in [0, T^*]} \| u_h^k \|_X^2
\]
\[
+ C \Delta t \sum_{k \Delta t \in [0, T^*]} (\Delta t)^2 \| A\tilde{u}_h^k \|_X^2.
\]
\[
\leq C \Delta t \sum_{k \Delta t \in [0, T^*]} (\Delta t)^2 \| A\tilde{u}_h^k \|_X^2.
\]
Therefore, we get:
\[
\Delta t \sum_{k \Delta t \in [0, T^*]} \left\| B^* \left( \frac{u_h^k + \tilde{u}_h^{k+1}}{2} \right) \right\|_Y^2 \leq C \Delta t \sum_{k \Delta t \in [0, T^*]} \| A\tilde{u}_h^k \|_X^2
\]
\[
+ \Delta t \left\| B^* \left( \frac{u_h^k + \tilde{u}_h^{k+1}}{2} \right) \right\|_Y^2.
\]
Besides,
\[
\Delta t \left\| B^* \left( \frac{u_h^k + \tilde{u}_h^{k+1}}{2} \right) \right\|_Y^2 \leq 2\Delta t \left\| B^* \left( \frac{u_h^k + \tilde{u}_h^{k+1}}{2} \right) \right\|_Y^2 + 2 \Delta t \left\| B^* \left( \frac{u^0_h + \tilde{u}_h^1}{2} \right) \right\|_Y^2.
\]
Now, as (3.20) we show that
\[
\Delta t \left\| B^* \left( \frac{u_h^0 + \tilde{u}_h^1}{2} \right) \right\|_Y^2 \leq C \Delta t \| u_h^0 \|_X^2.
\]
Then,
\[
\Delta t \sum_{k \Delta t \in [0, T^*)} \left\| B^* \left( \frac{u^k + \tilde{u}^{k+1}}{2} \right) \right\|_Y^2 \leq C \Delta t \sum_{k \Delta t \in [0, T^*)} \| Au^k \|_X^2 + C \Delta t \| u^0 \|_X^2 \\
+ \Delta t \left\| B^* \left( \frac{u^0 + \tilde{u}^1}{2} \right) \right\|_Y^2.
\]

It follows that there exists \( c > 0 \) independent of \( \Delta t \) such that
\[
c\| u^0 \|_X^2 \leq \Delta t \sum_{k \Delta t \in [0, T^*)} \left\| B^* \left( \frac{u^k + \tilde{u}^{k+1}}{2} \right) \right\|_Y^2 + \Delta t \sum_{k \Delta t \in [0, T^*)} (\Delta t)^2 \| Au^k \|_X^2 \\
+ \Delta t \sum_{k \Delta t \in [0, T^*)} (\Delta t)^2 \| Au^k \|_X^2. \quad (3.31)
\]

The high frequencies. We proceed as in [6], we get:
\[
C \| u^0 \|_X^2 \leq \Delta t \sum_{k \Delta t \in [0, T^*)} (\Delta t)^2 \| Au^k \|_X^2 \\
+ \Delta t \sum_{k \Delta t \in [0, T^*)} (\Delta t)^2 \| Au^k \|_X^2. \quad (3.32)
\]

Combining (3.31) and (3.32) yields Lemma 3.2, since \( u_h \) and \( u_l \) lie in orthogonal spaces with respect to the scalar products \( \langle ., . \rangle_X \) and \( \langle A., A. \rangle_X \).

**4 Proof of Theorem 1.1.**

The proof of Theorem 1.1 will essentially rely on the following lemma:

**Lemma 4.1.** Let \( w \) be the solution of
\[
\begin{align*}
\tilde{w}^{k+1} - w^k &= A \left( \frac{w^k + \tilde{w}^{k+1}}{2} \right) + B v^k, \quad k \in \mathbb{N}, \\
\frac{w^{k+1} - \tilde{w}^{k+1}}{\Delta t} &= (\Delta t)^2 A^2 w^{k+1}, \quad k \in \mathbb{N}, \\
w^0 &= 0,
\end{align*}
\]

where \( v^k \in l^2(k \Delta t; Y) = \{ v^k, \text{ such that } \sum_{k \Delta t \in [0, T^*)} \| v^k \|_Y^2 < \infty \} \).

Let \( T^* > 0 \) defined as before. There exists a positive constant \( C \) independent of \( \Delta t \) such that for all \( 0 < \Delta t < 1 \), we have the following estimate
\[
\Delta t \sum_{k \Delta t \in [0, T^*)} \left\| B^* \left( \frac{w^k + \tilde{w}^{k+1}}{2} \right) \right\|_Y^2 \leq C \Delta t \sum_{k \Delta t \in [0, T^*)} \| v^k \|_Y^2, \quad (4.33)
\]

where \( v^k \in l^2(k \Delta t; Y) \).
Proof. We denote by:

$$S_k = \frac{\bar{w}^{k+1} + w^k}{2}.$$ \hspace{1cm} (4.34)

From (5.31) we get:

$$S_k = (I - \frac{\Delta t}{2} A)^{-1} w^k + \frac{\Delta t}{2} (I - \frac{\Delta t}{2} A)^{-1} B v^k. \hspace{1cm} (4.35)$$

Besides,

$$w^{k+1} = L w^k + \Delta t B v^k, \hspace{1cm} (4.36)$$

where:

$$R = (I - (\Delta t)^3 A^2)^{-1} (I - \frac{\Delta t}{2} A)^{-1}$$

$$L = (I - (\Delta t)^3 A^2)^{-1} (I - \frac{\Delta t}{2} A)^{-1} (I + \frac{\Delta t}{2} A). \hspace{1cm} (4.37)$$

Then, for $k \geq 1$:

$$w^k = \Delta t \sum_{i=0}^{k-1} L^{k-1-i} R B v^i. \hspace{1cm} (4.38)$$

Combining (4.35), (4.38) we deduce that:

$$S_k = \frac{\Delta t}{2} (I - \frac{\Delta t}{2} A)^{-1} B v^k + \Delta t \sum_{i=0}^{k-1} (I - \frac{\Delta t}{2} A)^{-1} L^{k-1-i} R B v^i. \hspace{1cm} (4.39)$$

Consequently,

$$B^* S_k = \frac{\Delta t}{2} B^* (I - \frac{\Delta t}{2} A)^{-1} B v^k + \Delta t \sum_{i=0}^{k-1} B^* (I - \frac{\Delta t}{2} A)^{-1} L^{k-1-i} R B v^i.$$ 

Using Cauchy Schwarz inequality and the hypothesis (H), it follows that there exists $c > 0$ independent of $\Delta t$ such that

$$\|B^* S_k\|_Y^2 \leq c \|w^k\|_Y^2 + (\Delta t)^2 \sum_{i=0}^{k-1} \|B^* (I - \frac{\Delta t}{2} A)^{-1} L^{k-1-i} R B\|_{\mathcal{L}(Y)}^2 \sum_{l=0}^{k-1} \|v^l\|_Y^2.$$

Besides, we have:

$$\|B^* (I - \frac{\Delta t}{2} A)^{-1} L^{k-1-i} R B\|_{\mathcal{L}(Y)}^2 \leq \|B^*\|^2 \|L^{k-1-i}\|_2^2 \|R\|^2 \|B\|^2.$$ 

Now, let us estimate $\|L^{k-1-i}\|^2$. We have:

$$\|L^{k-1-i}\|^2 = \|(I - (\Delta t)^3 A^2)^{-1} (I - \frac{\Delta t}{2} A)^{-1} R B\|^2$$

$$\leq \|(I + \frac{\Delta t}{2} A)(I - \frac{\Delta t}{2} A)^{-1}\|^2 (k-1-i)$$

$$\leq 1,$$

where we used (2.18) and Theorem 2.1.

Using again (2.18) (since $A$ and $A^2$ are m-dissipative), we get

$$\|B^* (I - \frac{\Delta t}{2} A)^{-1} L^{k-1-i} R B\|^2_{\mathcal{L}(Y)} \leq \|B\|^4.$$
We deduce that

\[ \|B^*S_k\|_Y^2 \leq c\|v^k\|_Y^2 + (\Delta t)^2 \|B\|_Y^4 T \sum_{k=1}^{k-1} \|v'^i\|_Y^2, \quad (4.40) \]

which implies

\[ \sum_{k=1}^{\infty} \|B^*S_k\|_Y^2 \leq \sum_{k=1}^{\infty} \|v^k\|_Y^2 + \|B\|_Y^4 T \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} \|v'^i\|_Y^2. \quad (4.41) \]

Since \( k - 1 < \frac{T}{\Delta t} \), we obtain

\[ \sum_{k=1}^{\infty} \|B^*S_k\|_Y^2 \leq c \sum_{k=1}^{\infty} \|v^k\|_Y^2 + \|B\|_Y^4 T \Delta t \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} \|v'^i\|_Y^2 \quad (4.42) \]

\[ \leq c \sum_{k=1}^{\infty} \|v^k\|_Y^2 + \|B\|_Y^4 T \Delta t \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} \|v^k\|_Y^2. \quad (4.43) \]

\[ \leq c \sum_{k=1}^{\infty} \|v^k\|_Y^2 + \|B\|_Y^4 T^2 \sum_{k=0}^{\infty} \|v^k\|_Y^2. \quad (4.44) \]

Simple Calculations give:

\[ \|B^*S_0\|_Y^2 \leq c\|v^0\|_Y^2. \quad (4.45) \]

Combining (4.41), (4.45) and the fact that \( S_k = \frac{z^{k+1} + w^k}{2} \), we get the existence of constant \( C \) independent of \( \Delta t \) such that

\[ \sum_{k=0}^{\infty} \left\| B^* \left( \frac{w^k + w^{k+1}}{2} \right) \right\|_Y^2 \leq C \sum_{k=0}^{\infty} \|v^k\|_Y^2. \quad \Box \]

Now, we give the proof of our main result.

**Proof of Theorem 1.1.** Here we follow the argument in [7].

We decompose \( z \) solution of (1.10) as \( z = u + w \) where \( u \) is the solution of the system (1.10) with initial data \( u^0 = z^0 \). Applying Lemma 3.2 to \( u = z - w \), we get:

\[ c\|z^0\|_X^2 \leq 2 \left( \Delta t \sum_{k\Delta t \in [0,T^*]} \left\| B^* \left( \frac{z^k + z^{k+1}}{2} \right) \right\|_Y^2 + \Delta t \sum_{k\Delta t \in [0,T^*]} (\Delta t)^2 \|A z^{k+1}\|_X^2 + \Delta t \sum_{k\Delta t \in [0,T^*]} (\Delta t)^3 \|A^2 z^{k+1}\|_X^2 \right) \]

\[ + 2 \left( \Delta t \sum_{k\Delta t \in [0,T^*]} \left\| B^* \left( \frac{w^k + w^{k+1}}{2} \right) \right\|_Y^2 + \Delta t \sum_{k\Delta t \in [0,T^*]} (\Delta t)^2 \|A w^{k+1}\|_X^2 \right) \]

\[ + \Delta t \sum_{k\Delta t \in [0,T^*]} (\Delta t)^5 \|A^2 w^{k+1}\|_X^2 \right). \quad (4.46) \]
Below, we bound the terms in the right-hand side of (4.46) involving \(w\) by the ones involving \(z\).

The function \(w\) satisfies:

\[
\begin{aligned}
\frac{w^{k+1} - w^k}{\Delta t} &= A\left(\frac{w^k + \tilde{w}^{k+1}}{2}\right) - BB^* \left(\frac{z^k + \tilde{z}^{k+1}}{2}\right), \quad k \in \mathbb{N}, \\
\frac{w^k - \tilde{w}^{k+1}}{\Delta t} &= (\Delta t)^2 A^2 w^{k+1}, \quad k \in \mathbb{N}, \\
w^0 &= 0.
\end{aligned}
\]

(4.47)

By applying now Lemma 4.1 with \(\nu^k = -B^* \left(\frac{z^k + \tilde{z}^{k+1}}{2}\right)\), we obtain that

\[
\sum_{k \Delta t \in [0, T^*]} \left\|B^* \left(\frac{w^k + \tilde{w}^{k+1}}{2}\right)\right\|^2_Y \leq C \Delta t \sum_{k \Delta t \in [0, T^*]} \left\|B^* \left(\frac{z^k + \tilde{z}^{k+1}}{2}\right)\right\|^2_Y.
\]

(4.48)

Besides, we have (see [6])

\[
\begin{aligned}
\|w^k\|_X^2 + 2(\Delta t) \sum_{j=0}^{k-1} (\Delta t)^2 \|Aw^{j+1}\|_X^2 + (\Delta t) \sum_{j=0}^{k-1} (\Delta t)^5 \|A^2 w^{j+1}\|_X^2 \\
&\leq \Delta t \sum_{j=0}^{k-1} \left(\left\|B^* \left(\frac{z^j + \tilde{z}^{j+1}}{2}\right)\right\|^2_Y + \left\|B^* \left(\frac{w^j + \tilde{w}^{j+1}}{2}\right)\right\|^2_Y\right).
\end{aligned}
\]

(4.49)

Combining (4.46), (4.48) and (4.49) we get the existence of a constant \(c\) independent of \(\Delta t\) such that

\[
c\|z^0\|_X^2 \leq 2\Delta t \sum_{k \Delta t \in [0, T^*]} \left\|B^* \left(\frac{z^k + \tilde{z}^{k+1}}{2}\right)\right\|^2_Y + \Delta t \sum_{k \Delta t \in [0, T^*]} (\Delta t)^5 \|A z^{k+1}\|_X^2 \\
+ \Delta t \sum_{k \Delta t \in [0, T^*]} (\Delta t)^5 \|A^2 z^{k+1}\|_X^2.
\]

Finally, using the energy identity (1.12), we get that

\[
\|z^{T^* / \Delta t}\|_X^2 \leq (1 - c) \|z^0\|_X^2.
\]

The semi-group property then implies Theorem 1.1. \(\Box\)

**Remark 4.1.** There are other possible discretizations schemes for system (1.7) and other viscosity operators could have been chosen (see subsection 2.3 in [6] for more details). Note that the results given in the mentioned subsection still valid in our case but we replace the assumption of boundedness of \(B\) by the assumption \((H)\).
5 Applications

5.1 The wave equation

We consider the following system

\[
\begin{align*}
    u_{tt}(x,t) - u_{xx}(x,t) &= 0, \quad x \in (0, \xi) \cup (\xi, 1), \quad t > 0, \\
    u(0,t) &= 0, \quad u_x(1,t) = 0, \quad t > 0, \\
    u(\xi-, t) &= u(\xi+, t), \quad t > 0, \\
    u_x(\xi-, t) - u_x(\xi+, t) &= -\alpha u_t(\xi, t), \quad t > 0, \\
    u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 < x < 1,
\end{align*}
\]

where \( \xi \in (0,1) \) is a rational number with an irreducible fraction \( (\xi = \frac{p}{q}, \text{where } p \text{ is odd}) \), and \( \alpha \) is a positive constant.

To show that this system enters in the abstract setting of this paper, let us recall that it is equivalent to:

\[
\dot{Z} = AZ - BB^*Z, \quad \text{with} \quad Z = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ \partial_{xx} & 0 \end{pmatrix}.
\]

In this setting, \( A \) is a skew-adjoint unbounded operator on the Hilbert space \( X = V \times L^2(0,1) \), with domain \( D(A) = H \times V \), where

\[ V = \{ u \in H^1(0,1) : u(0) = 0 \}, \]

and

\[ H = \{ u \in H^2(0,1) : u(0) = u_x(1) = 0 \}. \]

The operator \( B \) is defined by: \( B : \mathbb{R} \to D(A)' : k \to \begin{pmatrix} 0 \\ \sqrt{\alpha k} \delta_\xi \end{pmatrix} \), where \( \delta_\xi \) is the Dirac mass concentrated at the point \( \xi \).

It is well known that, in the case where \( \xi = \frac{p}{q} \) (\( p \) is odd), the energy of the system above decays exponentially, and the operators \( A \) and \( B \) defined above satisfy the assumption (H) (see [2]).

Then, we introduce the following time semi-discrete approximation scheme:

\[
\begin{align*}
\frac{\tilde{Z}^{k+1} - Z^k}{\Delta t} &= A \left( \frac{Z^k + \tilde{Z}^{k+1}}{2} \right) - BB^* \left( \frac{Z^k + \tilde{Z}^{k+1}}{2} \right), \quad k \in \mathbb{N}, \\
\frac{\tilde{Z}^{k+1} - Z^{k+1}}{\Delta t} &= (\Delta t)^2 A^2 Z^{k+1}, \quad k \in \mathbb{N},
\end{align*}
\]

(5.50)

According to Theorem 1.1, we get:

**Theorem 5.1.** There exist positive constants \( \mu_0 \) and \( \nu_0 \) such that any solution of (5.50) satisfies (1.9) uniformly with respect to the discretization parameter \( \Delta t > 0 \).
5.2 One Euler-Bernoulli beam with interior damping

We consider the following initial and boundary problem:

\[
\begin{aligned}
  u_{tt}(x, t) - u_{xxxx}(x, t) + \alpha u_t(\xi, t)\delta\xi &= 0, \quad 0 < x < 1, \quad t > 0, \\
  u(0, t) &= u_x(1, t) = u_{xx}(0, t) = u_{xxx}(1, t) = 0, \quad t > 0, \\
  u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x),
\end{aligned}
\]

where \(\xi \in (0, 1)\) is defined as above, and \(\alpha > 0\). Hence it is written in the form \((1.1)\) with the following choices: Take \(Z\) as above, and \(X = H \times L^2(0, 1)\). The operator \(A\) defined by

\[
A = \begin{pmatrix}
  0 & I \\
  -\partial_{xxxx} & 0
\end{pmatrix},
\]

with domain \(D(A) = V \times H\), where \(V = \{u \in H^4(0, 1); u(0) = u_x(1) = u_{xx}(0) = u_{xxx}(1) = 0\}\), and \(H\) is defined as in the last subsection.

This operator is skew-adjoint on \(X\). We now define the operator \(B\) as:

\[
B : \mathbb{R} \to D(A)' : \quad k \to \begin{pmatrix} 0 \\ \sqrt{\alpha k\delta\xi} \end{pmatrix}.
\]

The energy of the system above decays exponentially, and the hypothesis (H) was verified (see \([3]\)).

As an application of Theorem 1.1, we get:

**Theorem 5.2.** The solutions of

\[
\begin{aligned}
  \frac{\ddot{Z}^{k+1} - \dot{Z}^k}{\Delta t} &= A \begin{pmatrix} \frac{\ddot{Z}^{k+1} + \dot{Z}^k}{2} \\ \frac{\dot{Z}^k}{2} \end{pmatrix} - BB^* \begin{pmatrix} \frac{\ddot{Z}^{k+1} + \dot{Z}^k}{2} \\ \frac{\dot{Z}^k}{2} \end{pmatrix}, \quad k \in \mathbb{N}, \\
  \frac{\dot{Z}^{k+1} - \ddot{Z}^k}{\Delta t} &= (\Delta t)^2 A^2 Z^{k+1}, \quad k \in \mathbb{N}, \\
  Z^0 &= (u_0, v_0),
\end{aligned}
\]

are exponentially uniformly decaying in the sense of \((1.9)\).

5.3 Dirichlet boundary stabilization of the wave equation

Let \(\Omega \subset \mathbb{R}^n, n \geq 2\) be an open bounded domain with a sufficiently smooth boundary \(\partial\Omega = \Gamma_0 \cup \Gamma_1\), where \(\Gamma_0\) and \(\Gamma_1\) are disjoint parts of the boundary relatively open in \(\partial\Omega, \quad int(\Gamma_0) \neq \emptyset\). We consider the wave equation:

\[
\begin{aligned}
  u_{tt} - \Delta u &= 0, \quad \Omega \times (0, +\infty), \\
  u &= \frac{\partial}{\partial n}(Gu_1), \quad \Gamma_0 \times (0, +\infty), \\
  u &= 0, \quad \Gamma_1 \times (0, +\infty), \\
  u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad \Omega,
\end{aligned}
\]
where $\nu$ is the unit normal vector of $\partial \Omega$ pointing towards the exterior of $\Omega$ and $G = (-\Delta)^{-1} : H^{-1}(\Omega) \to H^1_0(\Omega)$.

Denote: $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$, with $D(A) = H^1_0(\Omega) \times L^2(\Omega)$. In this setting, $A$ is a skew-adjoint unbounded operator on $X = L^2(\Omega) \times H^{-1}(\Omega)$. Moreover define

$$B \in \mathcal{L}(L^2(\Gamma_0), D(A)^\prime),$$

by $Bv = \begin{pmatrix} 0 \\ -\Delta Dv \end{pmatrix}$, $\forall v \in L^2(\Gamma_0)$, where $D$ is the Dirichlet map i.e., $Df = g$ if and only if

$$\begin{cases} 
\Delta g = 0, & \Omega, \\
g = f, & \Gamma_0, \ g = 0, & \Gamma_1. 
\end{cases}$$

The energy of the system above decays exponentially, and the hypothesis (H) was verified (see [1]). Applying Theorem 1.1, we obtain:

**Theorem 5.3.** The solutions of

$$\begin{align*}
\frac{Z^{k+1} - Z^k}{\Delta t} &= A \left( \frac{Z^k + \bar{Z}^{k+1}}{2} \right) - BB^* \left( \frac{Z^k + \bar{Z}^{k+1}}{2} \right), \ k \in \mathbb{N}, \\
\frac{Z^{k+1} - \bar{Z}^{k+1}}{\Delta t} &= (\Delta t)^2 A^2 Z^{k+1}, \ k \in \mathbb{N},
\end{align*}$$

are exponentially uniformly decaying in the sense of [1],[2].

**References**


