Asymptotic stability of the stationary solution to an initial boundary value problem for the Mullins equation of fourth order

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Abstract

In the present article we first study the existence of the stationary solution to an initial boundary value problem for the Mullins equation of fourth order, which was proposed by Mullins in [18] to describe the groove development, due to the surface diffusion, at the grain boundaries of a heated polycrystal. Then employing an energy method we prove that this stationary solution is asymptotically stable in a suitable norm, as time goes to infinity.

1 Introduction

In this article we are interested in the existence and asymptotic stability of the stationary solution to an initial boundary value problem for the Mullins equation of fourth order which describes the groove development, due to the surface diffusion, at the grain boundaries of a heated polycrystal. This model was proposed by Mullins, see [18]. There are two cases of this model, namely the conserved and non-conserved. Here we are interested in the conserved model. For the related result on the non-conserved case, we refer to [4]. The initial boundary value problem reads

$$y_t = -D\frac{\partial}{\partial x}\left(\frac{1}{(1+y_x^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}}\right)\right) \text{ in } Q_T, \tag{1.1}$$

$$y_x = \frac{\partial}{\partial x}\left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}}\right) = 0 \text{ on } \Gamma_T, \tag{1.2}$$

$$y|_{t=0} = y_0 \text{ on } \bar{\Omega}, \tag{1.3}$$

where we used the notations $Q_T = (0,T)\times\Omega$, $\Gamma_T = \partial Q_T = [0,T]\times\partial\Omega$, T is a positive real number and $\Omega = (a,b)$ is an open interval in $\mathbb{R}$ with a, b being real numbers such that $a < b$ and its closure is denoted by $\bar{\Omega}$.

Suppose y is a classical solution to the above problem. Straightforward computations show that the boundary conditions (1.2) are equivalent to

$$y_x = y_{xxx} = 0 \text{ on } \Gamma_T. \tag{1.4}$$

The equation shows that it is non-uniformly parabolic, since the coefficient of its principle term may decay to zero as y_x tends to infinity. Moreover the nonlinearity of this equation is very strong. Therefore, the global existence of weak solutions to problem (1.1) – (1.3) is very difficult. However noting that if y_x is small in L^∞-norm, then equation (1.1) is uniformly parabolic. We shall see that it is possible to investigate the asymptotic stability of the stationary solution to this problem.

We define the free energy function by

$$f(y_x) = \int^{y_x} \int^{\xi} (1+s^2)^{-\frac{3}{2}} ds d\xi, \tag{1.5}$$

and

$$F = \int_\Omega f(y_x(t,x))dx. \tag{1.6}$$

There exists a flux function $q = q(y_t, y_x, y_{xx}, y_{xxx})$ defined by

$$q = \int^{y_x} (1+s^2)^{-\frac{3}{2}} ds\, y_t + \frac{D y_{xx}}{(1+y_x^2)^2} \frac{\partial}{\partial x} \left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}} \right).$$

From equation (1.1) we have

$$\begin{aligned}
&\frac{d}{dt} f(y_x) - \operatorname{div}_x q = \int^{y_x} (1+s^2)^{-\frac{3}{2}} ds\, y_{xt} - \operatorname{div}_x q \\
= &\left(\int^{y_x} (1+s^2)^{-\frac{3}{2}} ds\, y_t \right)_x - (1+y_x^2)^{-\frac{3}{2}} y_{xx} y_t - \operatorname{div}_x q \\
= &\left(\int^{y_x} (1+s^2)^{-\frac{3}{2}} ds\, y_t \right)_x + D \left(\frac{y_{xx}}{(1+y_x^2)^2} \frac{\partial}{\partial x} \left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}} \right) \right)_x \\
&-D \frac{1}{(1+y_x^2)^{\frac{1}{2}}} \left(\frac{\partial}{\partial x} \left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}} \right) \right)^2 - \operatorname{div}_x q \\
= &-D \frac{1}{(1+y_x^2)^{\frac{1}{2}}} \left(\frac{\partial}{\partial x} \left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}} \right) \right)^2 \le 0.
\end{aligned} \tag{1.7}$$

Moreover, recalling boundary conditions one obtains

$$\begin{aligned}
\frac{d}{dt}F &= \int_\Omega \frac{\partial}{\partial t} f(y_x(t,x))dx \\
&= -D\int_\Omega \frac{1}{(1+y_x^2)^{\frac{1}{2}}} \left| \frac{\partial}{\partial x}\left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}}\right)\right|^2 dx \\
&\leq 0. \qquad (1.8)
\end{aligned}$$

Therefore the second law of thermodynamics is satisfied.

Before stating our main result, we need to study the following stationary problem corresponding to problem (1.1) – (1.3). In this case, y satisfies

$$D\frac{\partial}{\partial x}\left(\frac{1}{(1+y_x^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}}\right)\right) = 0 \text{ in } \Omega, \qquad (1.9)$$

$$y_x = y_{xxx} = 0 \text{ on } \partial\Omega. \qquad (1.10)$$

Concerning the existence of a solution to problem (1.9) – (1.10) we have

Lemma 1.1 *There exists a unique solution $y \equiv m$ to problem (1.9) – (1.10). Here m is a constant satisfying*

$$m = \frac{1}{\text{meas}(\Omega)}\int_\Omega y_0(x)dx.$$

The proof of this lemma will be presented in Section 2.

Reformulation of the problem: We next make a suitable transformation of the unknown to make it slightly easier to establish *a priori* estimates for the new unknown. We assume that y is a classical solution to problem (1.1) – (1.3). Then we get

$$\frac{d}{dt}\int_\Omega y(t,x)dx = 0.$$

Thus one has

$$
\begin{aligned}
\frac{1}{\mathrm{meas}(\Omega)}\int_\Omega y(t,x)dx &= \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega y(0,x)dx \\
&= \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega y_0(x)dx \\
&= \text{const} =: m.
\end{aligned}
$$

Define a new variable

$$u = y - m, \tag{1.11}$$

it is easy to see that $u_x = y_x$, and problem (1.1) – (1.3) turns out to be

$$
\begin{aligned}
u_t &= -D\frac{\partial}{\partial x}\left(\frac{1}{(1+u_x^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\left(\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right)\right) \text{ in } Q_T, && (1.12)\\
u_x &= u_{xxx} = 0 \text{ on } \Gamma_T, && (1.13)\\
u|_{t=0} &= y_0 - m \text{ on } \bar{\Omega}. && (1.14)
\end{aligned}
$$

For the convenience of readers, we now collect the notations that we have used and will use in this article.

Notations: Let T be a real number, and C denotes a universal constant which is independent of T. We denote the scalar product over Q_T by $(\cdot,\cdot)_T$, and the scalar product over Ω by $(\cdot,\cdot)$, and its corresponding norm by $\|\cdot\|$, respectively. Let p be real such that $1 \le p \le \infty$. $L^p(\Omega)$ is the space of p-integrable functions with the L^p-norm $\|\cdot\|_p$ defined in the standard way. For $p = 2$, we replace $\|\cdot\|_2$ by $\|\cdot\|$. $C^{1+\frac{\alpha}{4},4+\alpha}(\bar{Q}_T)$ is the space of Hölder continuous functions with exponents α for x and $\alpha/4$ for t, where $0 < \alpha < 1$. $W^{m,p}(\Omega)$ denotes the standard Sobolev space, which is also denoted by $H^m(\Omega)$ in the case that $p = 2$. Let X be a Banach space. $L^p(0, T; X)$ denotes the space of L^p-functions taking values in X.

The main result of this article is as follows:

Theorem 1.2 *Let α be a constant satisfying $0 < \alpha < 1$. Suppose that $u_0 \in C^{4,\alpha}(\bar{\Omega})$, and there is a suitably small number ε such that $\|u_0\|_{H^2(\Omega)} \leq \varepsilon$. Assume that the compatibility conditions $u_{0x}(a) = u_{0x}(b) = 0$, $u_{0xxx}(a) = u_{0xxx}(b) = 0$ and*

$$u_t|_{t=0} = -D\frac{\partial}{\partial x}\left(\frac{1}{(1+(u_{0x})^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\left(\frac{u_{0xx}}{(1+(u_{0x})^2)^{\frac{3}{2}}}\right)\right)$$

are satisfied.

Then there exists a classical solution u such that

$$u \in C^{1+\frac{\alpha}{4},4+\alpha}(\bar{Q}_T),$$

and

$$u \in L^2(0, \infty; H^4(\Omega)), \quad u_t \in L^2(0, \infty, L^2(\Omega)).$$

Moreover, there holds

$$\|y(t, \cdot) - m\|_{H^2(\Omega)} \to 0, \quad \|y(t, \cdot) - m\|_{W^{1,\infty}(\Omega)} \to 0$$

as $t \to \infty$.

We now recall the literature related closely to this model. Some authors have investigated the existence of a weak/classical solution or the existence of special solutions to a Cauchy problem/initial boundary value problem for this model in non-conserved/conserved case, see e.g. Alber and Zhu [4], Broadbridge [9], Broadbridge and Tritscher [10], Tritscher and Broadbridge [20], Kitada and Umehara [15], Kanel and Novick-Cohen [14] and the references cited therein. For degenerate fourth order parabolic equations, we refer to e.g., Bernis and Friedman [7], Bertozzi and Pugh [8], Beretta, Bertsch and Dal Passo [6], Cahn, Elliott and Novick-Cohen [11], Dal Passo, Garcke and Grün [13], where the mobility depends nonlinearly, however smoothly, on the unknown itself. Recently new kinds of models have been proposed to describe phase transitions driven by configurational forces in solids; and they have been studied mathematically in Alber and Zhu [1, 2, 3, 5]. The mobility in these models depends non-smoothly on the first order spatial derivatives, while we can see that the mobility of the model in this article depends smoothly on the first order spatial derivative.

The main difficulty in the proof of Theorem 1.2 is due to the fact that the coefficient depends nonlinearly on the first order derivative and the equation is of fourth-order. To prove the existence of a classical solution to the problem we make use of the Leray-Schauder fix point theorem and an existence theorem for the linearized problem from the book by Ladyzenskaya, et al [16].

The remaining parts of this article are organized as follows: In Section 2 we are going to prove the existence of the stationary solution and the solution to the reduced problem. In Section 3 we shall establish some *a priori* estimates for the classical solution, which consist of the proof of Proposition 3.1. Section 4 is devoted to investigating the large time behavior of a solution by using the *a priori* estimates.

2 Existence of stationary solution

In this section we are going to prove the existence of the stationary solution stated in Lemma 1.1. We then construct an approximate problem and employ the Leray-Schauder fixed-point theorem to prove the existence of the classical solution to it.

We are now going to prove Lemma 1.1.

Proof of Lemma 1.1. From equations (1.9) – (1.10) we obtain

$$\frac{1}{(1+y_x^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}}\right)=C_1.$$

Since

$$\frac{\partial}{\partial x}\left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}}\right)=\frac{y_{xxx}}{(1+y_x^2)^{\frac{3}{2}}}-3\frac{y_x y_{xx}^2}{(1+y_x^2)^{\frac{5}{2}}}=0$$

if $x\in\partial\Omega=\{a,b\}$, whence $C_1=0$ and

$$\frac{\partial}{\partial x}\left(\frac{y_{xx}}{(1+y_x^2)^{\frac{3}{2}}}\right) = 0, \tag{2.1}$$

$$y_x = 0, \text{ for } x=a,b. \tag{2.2}$$

Conditions (2.1) and (2.2) tell us that we have a curve of constant curvature, with zero gradient at two locations, $x=a,b$. It then follows immediately that the curve is a horizonal line. The proof of the lemma is thus complete.

In the remaining part of this section, we will prove the existence of the unique classical solution u to problem (1.12) – (1.14). To this end, we firstly employ an existence theorem in Ladyzenskaya, et al [16] to solve a linearized problem of (1.12) – (1.14). Based on the *a priori* estimates for the approximate solutions, we can use the Leray-Schauder fixed point theorem to prove the following existence result. We have

Theorem 2.1 *Assume that all the conditions listed in Theorem 1.2 are met.*

Then there exists a unique classical solution u to problem (1.12) – (1.14).

Proof. We apply the Leray-Schauder fixed point theorem to prove this theorem. Firstly, we consider the following linearized problem:

$$\begin{aligned} u_t &= -D\frac{\partial}{\partial x}\left(\frac{1}{(1+\lambda\tilde{u}_x^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\left(\frac{u_{xx}}{(1+\lambda\tilde{u}_x^2)^{\frac{3}{2}}}\right)\right) \text{ in } Q_T, && (2.3)\\ u_x &= u_{xxx} = 0 \text{ on } \Gamma_T, && (2.4)\\ u|_{t=0} &= y_0 - m \text{ on } \bar{\Omega}. && (2.5) \end{aligned}$$

for any given $\tilde{u}$, and $\lambda \in [0,1]$. Applying Theorem 10.1 on page 616 in [16] we find a solution $u \in C^{1+\alpha/4,4+\alpha}(\bar{Q}_T)$. Thus we can construct a mapping which allows us to define $P_\lambda : C^{1+\frac{\alpha}{4},4+\alpha}(\bar{Q}_T) \subset\subset C^{\frac{\alpha}{4},\alpha}(\bar{Q}_T) \to C^{1+\frac{\alpha}{4},4+\alpha}(\bar{Q}_T) \ : \ \tilde{u} \mapsto u = P_\lambda\tilde{u}$. Here $\subset\subset$ denotes the compact embedding.

Next, using the *a priori* estimates which are similar to those that we will establish in the next section, we conclude that

$$\|u_x\|_{C^{\frac{\alpha}{4},\alpha}(\bar{Q}_T)} \le C.$$

Then one can derive from the equation the estimate of the Schauder type

$$\|u\|_{C^{1+\frac{\alpha}{4},4+\alpha}(\bar{Q}_T)} \le C',$$

which allows us to apply the Leray-Schauder fixed point theorem and conclude that there exists a classical solution to problem (1.12) – (1.14).

3 A priori estimates

To derive the *a priori* estimates we assume that there exists a constant E such that

$$M = M(t) = \sup_{0\le\tau\le t} \|u(\tau)\|_{H^2(\Omega)} \le E, \tag{3.1}$$

which implies, by the Sobolev embedding theorem, that

$$\|u\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \le CE =: E_1, \tag{3.2}$$

we may choose E being so small that $E_1 \le \frac{1}{2}$ and

$$\|u\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \le \frac{1}{2}. \tag{3.3}$$

It is plausible to assume that E is small, suppose that the initial value of u and its first two derivatives are chosen to be close to zero.

In what follows, to simplify the notations we denote

$$a(u_x) = \frac{1}{\sqrt{1+u_x^2}}, \quad b(u_x) = a(u_x)^3, \tag{3.4}$$

thus by (3.3),

$$1 \ge a(u_x) \;\ge\; \delta, \quad 1 \ge b(u_x) \ge \delta^3 \text{ and} \tag{3.5}$$

$$|a'(r)|_{r=u_x}| \;\le\; C, \quad |(b'(r), b''(r))|_{r=u_x}| \le C, \tag{3.6}$$

where $\delta = (1 + E_1^2)^{-\frac{1}{2}}$. It is easy to see that

$$1 > \delta > \frac{2\sqrt{5}}{5}, \text{ and } \delta \to 1 \text{ as } E_1 \to 0.$$

We can now rewrite equation (1.12) as

$$u_t = -D\Big(a(u_x)\,(b(u_x)u_{xx})_x\Big)_x. \tag{3.7}$$

In this section our main goal is to prove the following proposition:

Proposition 3.1 *(A priori estimates) Suppose that the conditions in Theorem 1.2 are met. Assume that there exists a suitably small number ε_0 such that*

$$M(t) + \varepsilon \le \varepsilon_0,$$

and $\|u_0\|_{H^2(\Omega)} \le \varepsilon_0$.

Then

$$\|u(t)\|^2_{H^2(\Omega)} + D_1 \int_0^t \|u_{xx}(\tau)\|^2_{H^2(\Omega)} d\tau \le 2\|u_0\|^2_{H^2(\Omega)}, \tag{3.8}$$

here D_1 is a constant depending on δ and D.

The proof of this proposition is divided into three steps, each stated as a lemma, Lemmas 3.3 – 3.5 which are the *a priori* energy estimates.

3.1 The first energy estimate

We shall derive the *a priori* estimates in this section. First of all, we state the important feature for this problem, namely, the conservation law:

Lemma 3.2 *For solution u,*

$$\int_{\Omega} u(t,x)dx = \int_{\Omega} u_0(x)dx = 0. \tag{3.9}$$

The first energy estimate is

Lemma 3.3 *For any $t \in [0,T]$,*

$$\begin{aligned} \|u(t)\|^2 + 2D\int_0^t \int_{\Omega} b(u_x)^2 u_{xx}^2 dxd\tau &= \|u_0\|^2 \ \textit{and} && (3.10)\\ \|u(t)\|^2 + 2D\delta^6 \int_0^t \|u_{xx}(\tau)\|^2 d\tau &\le \|u_0\|^2. && (3.11) \end{aligned}$$

Proof. Multiplying (3.7) by u, integrating it with respect to t, x over Q_T and using integration by parts yield

$$\begin{aligned} \frac{1}{2}\|u_0\|^2 &= \frac{1}{2}\|u\|^2 - D\int_0^t \int_{\Omega} a(u_x)\left(b(u_x)u_{xx}\right)_x u_x dxd\tau \\ &= \frac{1}{2}\|u\|^2 + D\int_0^t \int_{\Omega} u_{xx} b(u_x)\left(a(u_x)u_x\right)_x dxd\tau. \end{aligned} \tag{3.12}$$

Calculation yields

$$(a(y)y)' = b(y),$$

whence

$$\frac{1}{2}\|u\|^2 + D\int_0^t \int_{\Omega} u_{xx}^2 b(u_x)^2 dxd\tau = \frac{1}{2}\|u_0\|^2. \tag{3.13}$$

From which we obtain (3.10). By (3.5) we get (3.11), and the proof of this lemma is complete.

3.2 The higher order energy estimates

In this sub-section we are going to establish the *a priori* estimates for higher order derivatives. The first is

Lemma 3.4 *There exists a small number ε_1, such that if $M(t)+\varepsilon \le \varepsilon_1$, then there holds for any $t \in [0,T]$ that*

$$\|u_x(t)\|^2 + D\delta^4 \int_0^t \|u_{xxx}(\tau)\|^2 d\tau \le \|u_{0x}\|^2 + CM^2\|u_0\|^2. \tag{3.14}$$

Proof. We first prove for the classical solution u that

$$(u_t(t), -u_{xx}(t)) = \frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2, \tag{3.15}$$

and

$$(u_t(t), u_{xxxx}(t)) = \frac{1}{2}\frac{d}{dt}\|u_{xx}(t)\|^2, \tag{3.16}$$

which will be used in the proof of this lemma and the next lemma.

To this end, we make use of the technique of difference quotient. Let $h > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2 &= \frac{1}{2}\lim_{h\to 0}\frac{1}{h}\left(\|u_x(t+h)\|^2 - \|u_x(t)\|^2\right) \\
&= \frac{1}{2}\lim_{h\to 0}\frac{1}{h}\int_\Omega \left(u_x(t+h,y) + u_x(t,y)\right)\left(u_x(t+h,y) - u_x(t,y)\right) dy \\
&= -\frac{1}{2}\lim_{h\to 0}\frac{1}{h}\int_\Omega \left(u_{xx}(t+h,y) + u_{xx}(t,y)\right)\left(u(t+h,y) - u(t,y)\right) dy \\
&= -\int_\Omega u_t(t,y)u_{xx}(t,y)dy \\
&= -(u_t, u_{xx}).
\end{aligned} \tag{3.17}$$

This is (3.15). Similarly, one can easily prove (3.16).

Multiplying (3.7) by $-u_{xx}$, integrating the resulting equation over Ω, using integration by parts, and taking the boundary conditions into account, we then obtain

$$
\begin{aligned}
0 &= (u_t, -u_{xx}) + D\left(a(u_x)\left(b(u_x)u_{xx}\right)_x, u_{xxx}\right) \\
&= \frac{1}{2}\frac{d}{dt}\|u_x\|^2 + D\left(a(u_x)\left(b(u_x)u_{xx}\right)_x, u_{xxx}\right) \\
&= \frac{1}{2}\frac{d}{dt}\|u_x\|^2 + D\int_\Omega a(u_x)b(u_x)u_{xxx}^2 dx + D\left(a(r)b'(r)|_{r=u_x}u_{xx}^2, u_{xxx}\right) \quad (3.18)
\end{aligned}
$$

Making use of (3.5) and (3.6), the Nirenberg inequality in the form

$$
\|f\|_\infty \le C\|f_x\|^{\frac{1}{2}}\|f\|^{\frac{1}{2}} + C'\|f\|, \tag{3.19}
$$

and the Hölder, Young inequalities, we infer from (3.18) that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + D\delta^4\|u_{xxx}\|^2 &\le C\|u_{xx}\|_\infty\|u_{xx}\|\|u_{xxx}\| \\
&\le C\|u_{xx}\|^{\frac{3}{2}}\|u_{xxx}\|^{\frac{3}{2}} + C\|u_{xx}\|^2\|u_{xxx}\| \\
&\le \frac{1}{2}D\delta^4\|u_{xxx}\|^2 + C(\|u_{xx}\|^6 + \|u_{xx}\|^4) \\
&\le \frac{1}{2}D\delta^4\|u_{xxx}\|^2 + C\|u_{xx}\|^4. \qquad (3.20)
\end{aligned}
$$

Therefore, we arrive at

$$
\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + \frac{1}{2}D\delta^4\|u_{xxx}\|^2 \le C\|u_{xx}\|^4. \tag{3.21}
$$

Integrating (3.21) with respect to t over $[0, T]$ yields

$$
\begin{aligned}
\|u_x(t)\|^2 + 2D\delta^4\int_0^t \|u_{xxx}(\tau)\|^2 d\tau &\le \|u_{0x}\|^2 + C\int_0^t \|u_{xx}(\tau)\|^4 d\tau \\
&\le \|u_{0x}\|^2 + C\sup_{\tau\in[0,T]}\|u_{xx}(\tau)\|^2\int_0^t \|u_{xx}(\tau)\|^2 d\tau \\
&\le \|u_{0x}\|^2 + CM^2\int_0^t \|u_{xx}(\tau)\|^2 d\tau. \qquad (3.22)
\end{aligned}
$$

By combining with the first energy estimate we obtain (3.14).

Next we shall derive the *a priori* estimates for the fourth order spatial derivative and for the first time derivative. We have

Lemma 3.5 *There exists a small number ε_2, such that if $M(t)+\varepsilon \le \varepsilon_2$, then there holds for any $t \in [0,T]$ that*

$$
\begin{aligned}
\|u_{xx}(t)\|^2 + D\delta^4 \int_0^t \|u_{xxxx}(\tau)\|^2 d\tau &\le \|u_{0xx}\|^2 + C\|u_0\|^2_{H^1(\Omega)} M^2, &(3.23)\\
\int_0^t \|u_t(\tau)\|^2 d\tau &\le \|u_{0xx}\|^2 + C\|u_0\|^2_{H^1(\Omega)} M^2. &(3.24)
\end{aligned}
$$

Proof. We first compute the right-hand side of (3.7) and obtain

$$
\begin{aligned}
-D\Big(a(u_x)\,(b(u_x)u_{xx})_x\Big)_x &= -Da(u_x)b(u_x)u_{xxxx} + \mathcal{F}, &(3.25)\\
\mathcal{F} &= -D\,\left.(a(r)b'(r))'\right|_{r=u_x} u_{xx}^3\\
&\quad -D\,\left.(3a(r)b'(r) + a'(r)b(r))\right|_{r=u_x} u_{xx}u_{xxx}, &(3.26)
\end{aligned}
$$

and we infer from (3.5) and (3.6) that the following estimate holds

$$|\mathcal{F}| \le C(|u_{xx}|^3 + |u_{xx}u_{xxx}|).$$

Then we multiply (3.7) by u_{xxxx} and integrate the resulting equation with respect to x, and use integration by parts to get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_{xx}\|^2 &+ D\int_\Omega a(u_x)b(u_x)u_{xxxx}^2 dx\\
&= \int_\Omega \mathcal{F}u_{xxxx}dx\\
&\le C\int_\Omega \big(|u_{xx}|^3 + |u_{xx}u_{xxx}|\big)\,|u_{xxxx}|dx\\
&\le C\left(\|u_{xx}^3\| + \|u_{xx}u_{xxx}\|\right)\|u_{xxxx}\|\\
&\le C\left(\|u_{xx}\|_\infty^2\|u_{xx}\| + \|u_{xx}\|\,\|u_{xxx}\|_\infty\right)\|u_{xxxx}\|. &(3.27)
\end{aligned}
$$

Integrating the above inequality with respect to t, applying the Nirenberg inequality (3.19) to u_{xx} and u_{xxx}, we deal with the inequality (3.27) as follows

$$
\begin{aligned}
&\frac{1}{2}\|u_{xx}(t)\|^2 + D\delta^4 \int_0^t \|u_{xxxx}(\tau)\|^2 d\tau \\
\leq\ & \frac{1}{2}\|u_{0xx}\|^2 + C\int_0^t \left(\|u_{xx}\|_\infty^4 \|u_{xx}\|^2 + \|u_{xx}\|\|u_{xxx}\|^{\frac{1}{2}}\|u_{xxxx}\|^{\frac{1}{2}}\right)\|u_{xxxx}\|d\tau \\
\leq\ & \frac{1}{2}\|u_{0xx}\|^2 + \frac{1}{4}D\delta^4\int_0^t \|u_{xxxx}(\tau)\|^2 d\tau + CM^2\int_0^t \left(\|u_{xxx}\|^2\|u_{xx}\|^2 + \|u_{xx}\|^2\right)d\tau \\
& + C\int_0^t \|u_{xx}\|\,\|u_{xxx}\|^{\frac{1}{2}}\|u_{xxxx}\|^{1+\frac{1}{2}}d\tau. \qquad (3.28)
\end{aligned}
$$

Here we used the basic inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}$ for any $a, b \geq 0$. The last term in (3.28) can be treated as

$$
\begin{aligned}
& C\int_0^t \|u_{xx}\|\,\|u_{xxx}\|^{\frac{1}{2}}\|u_{xxxx}\|^{1+\frac{1}{2}}d\tau \\
\leq\ & CM\left(\int_0^t \|u_{xxx}\|^2 d\tau\right)^{\frac{1}{4}}\left(\int_0^t \|u_{xxxx}\|^2 d\tau\right)^{\frac{3}{4}}. \qquad (3.29)
\end{aligned}
$$

Thus it follows from (3.29) and (3.28) that

$$
\begin{aligned}
&\frac{1}{2}\|u_{xx}(t)\|^2 + D\delta^4\int_0^t \|u_{xxxx}(\tau)\|^2 d\tau \\
\leq\ & \frac{1}{2}\|u_{0xx}\|^2 + \frac{1}{4}D\delta^4\int_0^t \|u_{xxxx}\|^2 d\tau + CM^2\int_0^t \left(M^2\|u_{xxx}\|^2 + \|u_{xx}\|^2\right)d\tau \\
& + \frac{1}{4}D\delta^4\int_0^t \|u_{xxxx}\|^2 d\tau + CM^4\int_0^t \|u_{xxx}\|^2 d\tau \\
\leq\ & \frac{1}{2}\|u_{0xx}\|^2 + \frac{1}{2}D\delta^4\int_0^t \|u_{xxxx}\|^2 d\tau + CM^2\int_0^t \left(\|u_{xxx}\|^2 + \|u_{xx}\|^2\right)d\tau. \qquad (3.30)
\end{aligned}
$$

Applying the first energy estimate and Lemma 2.2, from (3.30) we get

$$
\begin{aligned}
&\frac{1}{2}\|u_{xx}(t)\|^2 + \frac{1}{2}D\delta^4\int_0^t \|u_{xxxx}(\tau)\|^2 d\tau \\
\leq\ & \frac{1}{2}\|u_{0xx}\|^2 + CM^2\int_0^t \left(\|u_{xxx}\|^2 + \|u_{xx}\|^2\right)d\tau \\
\leq\ & \frac{1}{2}\|u_{0xx}\|^2 + C\left(\left(\|u_{0x}\|^2 + M^2\|u_0\|^2\right) + \|u_0\|^2\right)M^2, \qquad (3.31)
\end{aligned}
$$

and we therefore arrive at (3.23).

To prove (3.24) we use equation (3.7). By the triangle inequality, the Sobolev embedding theorem, and the Nirenberg inequality (3.19) we have

$$\begin{aligned} &\int_0^t \|u_t\|^2 d\tau \leq C \int_0^t \left(\|u_{xxxx}\|^2 + \|u_{xx}u_{xxx}\|^2 + \|u_{xx}^3\|^2\right) d\tau \\ \leq\ & C \int_0^t \left(\|u_{xxxx}\|^2 + \|u_{xxx}\|_\infty^2 \|u_{xx}\|^2 + \|u_{xx}\|^2 \|u_{xx}\|_\infty^4\right) d\tau \\ \leq\ & C \int_0^t \|u_{xxxx}\|^2 + CM^2 \int_0^t \left(\|u_{xxx}\|_{H^1(\Omega)}^2 + \|u_{xxx}\|^2 \|u_{xx}\|^2 + \|u_{xx}\|^4\right) d\tau \\ \leq\ & C \int_0^t \|u_{xxxx}\|^2 d\tau + CM^2 \int_0^t \left(\|u_{xxxx}\|^2 + \|u_{xxx}\|^2 + \|u_{xx}\|^2\right) d\tau, \end{aligned} \tag{3.32}$$

from which, applying the estimates in Lemmas 3.3 – 3.4 and (3.23) we then obtain (3.24). And the proof of this lemma is complete.

Proof of Proposition 3.1 By combination of Lemmas 3.3 – 3.5, if we choose $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, and let ε be so small that $M(t) + \varepsilon \leq \varepsilon_0$, we can conclude that

$$\begin{aligned} \|u(t)\|_{H^2(\Omega)}^2 + D_1 \int_0^t \|u_{xx}(\tau)\|_{H^2(\Omega)}^2 d\tau &\leq \|u_0\|_{H^2(\Omega)}^2 + C\varepsilon^2 M(t)^2 \\ &\leq \|u_0\|_{H^2(\Omega)}^2 + \frac{1}{2} M(t)^2, \end{aligned} \tag{3.33}$$

here we have chosen ε such that

$$C\varepsilon^2 \leq \frac{1}{2}.$$

Taking supremum for both sides of (3.33) yields

$$M(t)^2 = \sup_{0 \leq \tau \leq t} \|u(\tau)\|_{H^2(\Omega)}^2 \leq \|u_0\|_{H^2(\Omega)}^2 + \frac{1}{2} M(t)^2, \tag{3.34}$$

whence

$$M(t)^2 \leq 2\|u_0\|_{H^2(\Omega)}^2, \quad \int_0^t \|u_{xx}(\tau)\|_{H^2(\Omega)}^2 d\tau \leq C. \tag{3.35}$$

Thus the proof of Proposition 3.1 is complete.

4 The large time behavior

We are going to complete, in this section, the proof of the main result of this article, Theorem 1.2. To this end, we need only to investigate the large time behavior of solution u. Because the constants C's in Section 3 are independent of t, the upper limit t of the integrals in Lemmas 3.3 – 3.5 can be changed to ∞, and the solution u over $[0, T]$ can be extended to the one on the whole interval $[0, \infty)$.

The proof is divided into three steps. Using the boundary condition and the Poincaré inequality we then have

$$\|u_x\| \le C\|u_{xx}\|,$$

combination with (3.11) gives

$$\int_0^t \|u_x\|^2 d\tau \le C \int_0^t \|u_{xx}\|^2 d\tau \le C. \tag{4.1}$$

Since C is independent of t, the upper limit t in (4.1) can be changed to ∞. We have

$$\int_0^\infty \|u_x\|^2 d\tau \le C \int_0^\infty \|u_{xx}\|^2 d\tau \le C. \tag{4.2}$$

Proof. Step 1. Defining

$$U(t) = \|u_x(t)\|^2, \quad V(t) = \|u_{xx}(t)\|^2,$$

from (4.2) and the estimates in Section 3 we conclude that

$$\int_0^\infty U(t)dt \le C, \quad \int_0^\infty V(t)dt \le C. \tag{4.3}$$

Next we want to prove that

$$\int_0^\infty \left|\frac{d}{dt}U(t)\right| dt \leq C, \quad \int_0^\infty \left|\frac{d}{dt}V(t)\right| dt \leq C. \tag{4.4}$$

Suppose that (4.4) were true, then we conclude easily that

$$U(t) \to 0, \quad V(t) \to 0$$

as $t \to \infty$. Moreover, using the inequality of Poincaré type:

$$\|f - \bar{f}\| \leq C\|f_x\|,$$

where

$$\bar{f} = \bar{f}(t) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega f(t,x)dx,$$

letting $f = u$ and invoking conservation law (3.9) and the definitions of u and m, we assert that

$$\|u(t)\| \to 0, \quad \text{so} \quad \|y(t) - m\| \to 0, \tag{4.5}$$

as $t \to \infty$.

Step 2. It remains to prove (4.4). Recalling (3.18), the Sobolev embedding theorem, the Nirenberg inequality (3.19), and applying the estimates in Lemmas 3.3 – 3.5, we obtain

$$\begin{aligned} \int_0^t \left|\frac{d}{dt}\|u_x(\tau)\|^2\right| d\tau &\leq C\int_0^t \left(\|u_{xxx}\|^2 + \|u_{xx}\|^4\right) d\tau \\ &\leq C\int_0^t \left(\|u_{xxx}\|^2 + M^2\|u_{xx}\|^2\right) d\tau \\ &\leq C, \end{aligned} \tag{4.6}$$

and from (3.27) and the Hölder inequality we have

$$
\begin{aligned}
&\int_0^t \left| \frac{d}{dt}\|u_{xx}(\tau)\|^2 \right| d\tau \le C \int_0^t \left(\|u_{xxxx}\|^2 + (\|u_{xx}\|_\infty^2 + u_{xxx}\|_\infty)\|u_{xxxx}\| \right) d\tau \\
\le\ & C \int_0^t \left(\|u_{xxxx}\|^2 + \left(\|u_{xxx}\|\,\|u_{xx}\| + \|u_{xx}\|^2 + \|u_{xxx}\|_{H^1(\Omega)} \right) \|u_{xxxx}\| \right) d\tau \\
\le\ & C \int_0^t \left(\|u_{xxxx}\|^2 + \|u_{xxx}\|^2 + \|u_{xx}\|^2 \right) d\tau \\
\le\ & C. \qquad (4.7)
\end{aligned}
$$

Thus (4.6) and (4.7) imply (4.4).

Step 3. Recalling (4.5), by Sobolev's embedding theorem, we can easily prove that

$$\|u(t)\|_{W^{1,\infty}(\Omega)} \to 0,$$

as $t \to \infty$. And the proof of Theorem 1.2 is complete.

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