

## RELATIVISTIC HARDY INEQUALITIES IN MAGNETIC FIELDS

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ABSTRACT. We deal with Dirac operators with external homogeneous magnetic fields. Hardy-type inequalities related to these operators are investigated: for a suitable class of transversal magnetic fields, we prove a Hardy inequality with the same best constant as in the free case. This leaves naturally open an interesting question whether there exist magnetic fields for which a Hardy inequality with a better constant than the usual one, in connection with the well known diamagnetic phenomenon arising in non-relativistic models.

## 1. INTRODUCTION

The Hardy inequality

$$(1.1) \quad \int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^2} dx \leq \left( \frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} |\nabla \phi|^2 dx \quad (n \geq 3)$$

with  $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n; \mathbb{C})$ , is one of the well known mathematical aspects of the uncertainty principle in Quantum Mechanics. Among several applications, a standard consequence of (1.1) is the positivity of quadratic forms of the type

$$q(\varphi, \psi) = \int_{\mathbb{R}^n} \nabla \varphi \cdot \nabla \bar{\psi} dx - \lambda \int_{\mathbb{R}^n} \frac{\varphi \bar{\psi}}{|x|^2} dx$$

for  $\lambda \leq (n-2)^2/4$ , which permits to study the self-adjointness of Schrödinger hamiltonians like  $H = -\Delta - \lambda/|x|^2$  by means of the Kato-Rellich Theorem.

One can prove by the same techniques the more general family of inequalities

$$(1.2) \quad \int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^\alpha} dx \leq \left( \frac{2}{n-\alpha} \right)^2 \int_{\mathbb{R}^n} |x|^{2-\alpha} |\nabla \phi|^2 dx \quad (n \geq 1) \quad (\alpha < n)$$

In fact, the operator  $\nabla$  at the right-hand side of (1.2) can be replaced by the radial derivative  $\partial_r = \frac{x}{|x|} \cdot \nabla$ , since the weight  $|x|^{-\alpha}$  is radial; more precisely, one has

$$(1.3) \quad \int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^\alpha} dx \leq \left( \frac{2}{n-\alpha} \right)^2 \int_{\mathbb{R}^n} |x|^{2-\alpha} |\partial_r \phi|^2 dx \quad (n \geq 1) \quad (\alpha < n)$$

In addition, the constant  $4/(n-\alpha)^2$  at the right-hand side of (1.2), (1.3) is sharp, and it is well known that there are no maximizing functions for those inequalities.

When a particle interacts with an external magnetic field, it is standard in Quantum Mechanics to introduce in the model an anti-symmetric real-valued matrix  $B = \{B^{jk}(x)\} = \{-B^{kj}(x)\} : \mathbb{R}^n \rightarrow \mathcal{M}_{n \times n}(\mathbb{R})$ ,  $j, k = 1, \dots, n$ ,  $n \geq 2$ , with the following

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property: there exist a real valued potential vector field  $A = (A^1(x), \dots, A^n(x))$  such that  $B = DA - DA^t$ , where  $(DA)^{ij} = \partial_{x_i} A^j$  is the differential matrix of  $A$ . Then, to obtain the new Schrödinger hamiltonian formulation, one changes the gradient  $\nabla$  into  $\nabla_A := \nabla + iA$ , so that  $-\Delta_A = -(\nabla_A)^2$ . In dimensions  $n = 2, 3$ , any anti-symmetric matrix can be identified with a scalar function ( $n = 2$ ) or a vector-field ( $n = 3$ ), hence we have, by the previous definitions, that  $B = \text{curl } A$ .

A quite important feature of the magnetic gradient  $\nabla_A$  is the diamagnetic inequality (see e.g. [14]): if  $A \in L^2_{\text{loc}}(\mathbb{R}^n)$ ,  $n \geq 2$ , then

$$(1.4) \quad |\nabla|\psi(x)|| \leq |\nabla_A\psi(x)|$$

for all  $\psi \in C_0^\infty(\mathbb{R}^n)$  and almost every  $x \in \mathbb{R}^n$ . This, together with (1.2) applied to  $|\psi|$ , immediately gives

$$(1.5) \quad \int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^\alpha} dx \leq \left(\frac{2}{n-\alpha}\right)^2 \int_{\mathbb{R}^n} |x|^{2-\alpha} |\nabla_A \phi|^2 dx \quad (n \geq 2) \quad (\alpha < n)$$

where the constant at the right-hand side is not bigger than the one in the free case. It is easy to show again that one can put at the right-hand side of (1.5) just the radial component of the magnetic gradient, namely

$$(1.6) \quad \int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^\alpha} dx \leq \left(\frac{2}{n-\alpha}\right)^2 \int_{\mathbb{R}^n} |x|^{2-\alpha} |\partial_r^A \phi|^2 dx \quad (n \geq 2) \quad (\alpha < n)$$

holds, where  $\partial_r^A := \frac{x}{|x|} \cdot \nabla_A$ .

In [3], [13] it is proved that, for suitable magnetic fields  $B$ , inequality (1.5) can be generally strictly improved with respect to the free case. A relevant example is given by the Aharonov-Bohm potential in 2D: it is the case

$$A_{ab}(x, y) = \lambda \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right), \quad B_{ab}(x, y) = \text{curl } A = 8\pi\delta$$

with  $\lambda \in \mathbb{R}$ . Laptev and Weidl proved in [13] that

$$(1.7) \quad \int_{\mathbb{R}^2} \frac{|\phi|^2}{|x|^2} dx \leq \Gamma \int_{\mathbb{R}^2} |\nabla_{A_{ab}} \phi|^2 dx$$

with  $\Gamma = (\text{dist}\{\Theta, \mathbb{Z}\})^{-2}$ , being  $\Theta$  the total flux of  $A_{ab}$  along the unit sphere  $\mathbb{S}^1$ . Notice that inequality (1.7) is false in the free case, since in dimension  $n = 2$  the weight  $|x|^{-2}$  is too singular. Nevertheless, as soon as  $\Theta \notin \mathbb{Z}$ , (1.7) becomes true with a finite constant  $\Gamma$  at the right-hand side.

This manuscript is concerned with the same kind of questions in the relativistic setting of Dirac-Pauli operators. We denote by

$$\sigma \cdot \nabla_A \phi = \sum_{j=1}^3 \sigma_j (\partial_{x_j} + iA_j) \phi$$

where  $\phi = \phi(x) = (\phi^1(x), \phi^2(x)) : \mathbb{R}^3 \rightarrow \mathbb{C}^2$  and the Pauli matrices  $\sigma_j$  are defined by

$$(1.8) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The anti commutation relations

$$\sigma_j \sigma_k + \sigma_k \sigma_j = \begin{cases} 2I_{2 \times 2} & \text{if } j = k \\ \mathbf{0} & \text{if } j \neq k \end{cases}, \quad I_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are the main feature of the matrices  $\sigma_j$ ; in particular, they give

$$(1.9) \quad \int_{\mathbb{R}^3} |\sigma \cdot \nabla \phi|^2 dx = \int_{\mathbb{R}^3} |\nabla \phi|^2 dx =: \int_{\mathbb{R}^3} |\nabla \phi^1|^2 dx + \int_{\mathbb{R}^3} |\nabla \phi^2|^2 dx$$

(see also [16] for more details) so that by (1.1) one immediately obtains

$$(1.10) \quad \int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|^2} dx := \int_{\mathbb{R}^3} \frac{|\phi^1|^2}{|x|^2} dx + \int_{\mathbb{R}^3} \frac{|\phi^2|^2}{|x|^2} dx \leq 4 \int_{\mathbb{R}^3} |\sigma \cdot \nabla \phi|^2 dx$$

Notice that

$$\int_{\mathbb{R}^3} |x|^\alpha |\sigma \cdot \nabla \phi|^2 dx \neq \int_{\mathbb{R}^3} |x|^\alpha |\nabla \phi|^2 dx$$

if  $\alpha \neq 0$ , so that obtaining the relativistic analog of (1.2) is not trivial.

The case  $\alpha = 1$  is of particular interest, in connection with the problem of self-adjointness of Dirac operators with an external Coulomb-type potential. It is the case of the following inequality

$$(1.11) \quad \int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|} dx \leq \int_{\mathbb{R}^3} |x| |\sigma \cdot \nabla \phi|^2 dx$$

From now on, we shall refer to (1.11) as to the Hardy-Dirac type inequality.

In [7], a completely analytical proof of estimate (1.11) has been performed for the first time, and later by the same techniques some more general versions have been obtained in [1, 4, 5, 6] (see also the references therein). Notice that the best constant of the inequality is  $C = 1$ , in complete analogy with (1.2), for  $\alpha = 1$  and  $n = 3$ .

The aim of this paper is to investigate the validity of the following inequalities

$$(1.12) \quad \int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|} dx \leq C \int_{\mathbb{R}^3} |x| |\sigma \cdot \nabla_A \phi|^2 dx$$

when  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a suitable homogeneous magnetic potential, with a particular interest in studying the behavior of the best constant  $C$ .

Before preparing the setting of our main results, we motivate here the interest for such a question. First of all, we recall the following well known Barry Simon's version of the diamagnetic inequality:

$$(1.13) \quad e^{-t(\Delta + V(x))} |f| \leq \left| e^{-t(\Delta_A + V(x))} f \right|$$

if  $A \in L^2_{\text{loc}}(\mathbb{R}^n)$ ,  $n \geq 2$  which was first conjectured in [15]. Inequality (1.13) implies, among many other things, that the bottom of the spectrum of an electromagnetic Schrödinger operator  $-\Delta_A + V$  increases with respect to the same quantity in the magnetic-free case, namely

$$(1.14) \quad \inf \text{spec}(-\Delta_A + V) \geq \inf \text{spec}(-\Delta + V)$$

in  $L^2(\mathbb{R}^n)$ ,  $n \geq 2$ . The same phenomenon (universal diamagnetism) does not seem to arise in relativistic models. More precisely, in [11], an interesting conjecture about universal paramagnetism for fermions was claimed, which for Dirac-Pauli operators can be written as

$$(1.15) \quad \inf \text{spec}(-\Delta_A + \sigma \cdot B + V) \leq \inf \text{spec}(-\Delta + V),$$

with  $n = 3$  and  $B = \text{curl} A$ . Moreover, in [11] it is proved that the claim is true in the case of a constant magnetic field  $B = (0, 0, \lambda)$ ,  $\lambda \in \mathbb{R}$ . Later on, Avron and Simon in [2] disproved the conjecture with an explicit example. See also the interesting surveys [8, 10] for more informations about the topic.

The flavour is that inequalities (1.14), (1.15) should be directly related to the behavior of the best constant in the Hardy inequalities (1.5), (1.12), respectively, once a magnetic perturbation comes into play.

Motivated by the result by Avron and Simon, we wish here to show a general class of non-trivial magnetic fields for which the best constant in (1.12) is the same as in the free case, namely  $C = 1$ , in contrast with the paramagnetic phenomenon. The argument which we show in the sequel does not permit us to find example of fields for which (1.12) holds with a better constant, which is a quite interesting open question.

Before stating our main result we need to introduce the *orbital angular momentum* operator, which is the triplet of operators

$$(1.16) \quad L = (L^1, L^2, L^3) = x \wedge (-i\nabla)$$

and the *magnetic orbital angular momentum*

$$(1.17) \quad L_A = (L_A^1, L_A^2, L_A^3) = x \wedge (-i\nabla_A) = x \wedge (-i\nabla + A)$$

Notice that the operators  $L, L_A$  acts in principle on  $\mathbb{C}^2$ -valued functions defined on  $\mathbb{R}^3$ . However it is also well-defined its action on  $\mathbb{C}^2$ -valued functions defined on  $\mathbb{S}^2$  (consider the homogeneous extension on the whole  $\mathbb{R}^3$  of a given function defined on  $\mathbb{S}^2$ , apply the operator and restrict on  $\mathbb{S}^2$ ). The corresponding operators depend only on the trace of the field  $A$  on  $\mathbb{S}^2$ . Hence in the sequel, given any potential  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , we will denote by  $L_A$  both the symmetric operator on  $L^2(\mathbb{S}^2; \mathbb{C}^2)$  and the one on  $L^2(\mathbb{R}^3; \mathbb{C}^2)$ .

A fundamental role is played by the *spin-orbit* angular momentum  $\sigma \cdot L + 1$ ; for any vector field  $A$ , its natural generalization is given by  $\sigma \cdot L_A + 1$ . Recall that  $\sigma_i$  are defined in (1.8).

The operator  $\sigma \cdot L_A$  is symmetric and its inverse on  $L^2(\mathbb{S}^2; \mathbb{C}^2)$  is compact. Hence it has a purely discrete and real spectrum, which can accumulate only at infinity; we denote it by

$$(1.18) \quad \begin{cases} \text{spec}(\sigma \cdot L_A) := \{-\lambda_j, \mu_j\}_{j \in \mathbb{N}} \\ 0 < \lambda_1 < \lambda_2 < \dots; \quad 0 \leq \mu_1 < \mu_2 < \dots \end{cases}$$

In particular it is well defined the number

$$(1.19) \quad \mu_1(A) = \inf \{ \text{spec}(\sigma \cdot L_A + 1) \cap [0, \infty) \}$$

In the magnetic-free case  $A \equiv 0$ , the spectrum of  $\sigma \cdot L + 1$  is completely known, and it is given by the set  $\{\pm 1, \pm 2, \dots\}$ . This gives  $\mu_1(0) = 1$ ,  $\mu_1$  being the number in (1.19), which turns out to be the fundamental tool in the proof of (1.11) of [7]. In fact, thanks to this remark, in [7] the stronger estimate

$$(1.20) \quad \int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|} dx \leq \int_{\mathbb{R}^3} \frac{1}{|x|} |(\sigma \cdot L + 1)\phi|^2 dx \leq \int_{\mathbb{R}^3} |x| |\sigma \cdot \nabla \phi|^2 dx$$

is proved, providing a weighted  $L^2$ -bound for the spin-orbit angular momentum in terms of the whole Dirac operator.

In the following, we use the polar notations  $r = |x|$ ,  $\omega = x/|x| \in \mathbb{S}^2$ . We will point our attention on magnetic fields of the form

$$B(x) = \varphi(r)\nabla g(\omega) \wedge x,$$

where  $\varphi = \varphi(r) : \mathbb{R}^3 \rightarrow \mathbb{R}$ , and  $g = g(\omega) : \mathbb{S}^2 \rightarrow \mathbb{R}$  is a homogenous function of degree 0. These kinds of fields are obviously tangential, and possibly singular at the origin, since  $\nabla g$  is a homogeneous function of degree  $-1$ . As we will see in the sequel (Proposition 2.1 below), up to assuming some local integrability conditions on  $\varphi$  (assumption (1.22)), it is possible to prove that there exists a potential  $A$  such that  $\text{curl} A = B$ . In particular, we will prove that for those potentials one has  $\text{spec}(\sigma \cdot L_A + 1) = \text{spec}(\sigma \cdot L + 1)$ , and the corresponding eigenfunctions are just obtained by the free ones, via multiplication by a purely imaginary phase (Proposition 2.2 below).

We are now ready to state the main result of this paper.

**Theorem 1.1.** *Let  $\varphi = \varphi(r) : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $g = g(\omega) : \mathbb{S}^2 \rightarrow \mathbb{R}$  be a homogenous function of degree 0, and denote by*

$$(1.21) \quad B(x) = \varphi(r)\nabla g(\omega) \wedge x.$$

*Assume in addition that*

$$(1.22) \quad \left| \int_0^r s\varphi(s) ds \right| < \infty,$$

*for all  $r \in \mathbb{R}$ . Moreover, let  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be such that  $\text{curl} A = B$ . Then, for any  $\phi = \phi(x) \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$ , the following inequality holds*

$$(1.23) \quad \int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|} dx \leq \int_{\mathbb{R}^3} \frac{1}{|x|} |(\sigma \cdot L_A + 1)\phi|^2 dx \leq \int_{\mathbb{R}^3} |x| |\sigma \cdot \nabla_A \phi|^2 dx.$$

*Remark 1.1.* Notice that condition (1.22) does not allow to consider too much singular magnetic fields  $B$ . Since  $\nabla g(\omega) \wedge x$  is homogeneous of degree 0, the validity of (1.22) requires on  $B$  a local behavior like  $|B(x)| \sim 1/|x|^{2-\epsilon}$ , for some  $\epsilon > 0$ . On the other hand, nothing is required about the behavior of  $B$  when  $r \rightarrow \infty$ .

*Remark 1.2.* The result of Theorem 1.1 is gauge invariant, since the hypotheses and inequality (1.23) do not depend on the choice of the potential  $A$  such that  $\text{curl} A = B$ . As we see in the following, in order to prove (1.23) it is fundamental to choose an appropriate gauge for the potential  $A$ . Moreover we remark again that condition (1.22) permits to prove that  $B$  is in fact a curl of some potential  $A$ , which will be chosen in a suitable gauge (see Proposition 2.2 below).

*Remark 1.3.* Theorem 1.1 shows a class of non-trivial magnetic fields  $B$ , with the corresponding potentials  $A$ , for which the best constant 1 of inequality (1.23) coincides with the one in the free case (1.11), in the same spirit as in the example by Avron and Simon [2] remarked above.

*Remark 1.4.* A quite interesting example of magnetic field for which Theorem 1.1 applies is given by

$$(1.24) \quad B(x, y, z) = \lambda r^\alpha (-y, x, 0) = \lambda r^{\alpha+1} \left[ \nabla \left( \frac{z}{r} \right) \wedge x \right],$$

with  $\lambda \neq 0$  and  $r := (x^2 + y^2 + z^2)^{\frac{1}{2}}$ . Following the notations in (1.21) we have  $\varphi(r) = \lambda r^{\alpha+1}$  and  $g(\omega) = z/r$ ; moreover, condition (1.22) impose that  $\alpha > -3$ .

*Remark 1.5.* The question whether or not there exist a magnetic field  $B$  for which inequality (1.23), with the corresponding potential, holds with a constant better than 1, in analogy with what happens in the non relativistic case (1.7), still remains open. In order to address an answer it should be fundamental to produce examples of magnetic potentials such that  $\mu_1(A) > 1$  in the definition (1.19).

## 2. PRELIMINARIES

We start with some preliminary remarks which will be used in the sequel, in the proof of our main Theorem 1.1. We first need to fix a suitable gauge in which to work. We prove the following proposition.

**Proposition 2.1** (Gauge choice). *Let  $B = B(x) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be of the form (1.21) and assume (1.22). Define  $A = A(x) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  as follows:*

$$(2.1) \quad A(x) = \frac{1}{2}\varphi(r)g(\omega)x - \frac{1}{2}\left(\int_0^r s\varphi(s) ds\right)\nabla g(\omega).$$

Then

$$(2.2) \quad \operatorname{curl} A = B$$

$$(2.3) \quad \partial_r(x \wedge A) + x \wedge \nabla\left(A \cdot \frac{x}{r}\right) = 0,$$

where the radial derivative  $\partial_r := \frac{x}{r} \cdot \nabla$  acts component-wise on the vector  $x \wedge A$ .

*Proof.* The proof is quite elementary. First, compute

$$\begin{aligned} \operatorname{curl} \varphi(r)g(\omega)x &= \varphi(r)\nabla g(\omega) \wedge x = B(x) \\ \operatorname{curl} \left[ \left( \int_0^r s\varphi(s) ds \right) \nabla g(\omega) \right] &= r\varphi(r)\frac{x}{r} \wedge \nabla g(\omega) = -B(x), \end{aligned}$$

which proves (2.2). Now notice that

$$x \wedge A = -\frac{1}{2}\left(\int_0^r s\varphi(s) ds\right)x \wedge \nabla g(\omega);$$

since  $x \wedge \nabla g(\omega)$  is homogeneous of degree 0, we have  $\partial_r(x \wedge \nabla g(\omega)) \equiv 0$  and consequently

$$(2.4) \quad \partial_r(x \wedge A) = -\frac{1}{2}r\varphi(r)x \wedge \nabla g(\omega) = \frac{1}{2}rB(x).$$

Then, compute  $A \cdot \frac{x}{r} = \frac{1}{2}\varphi(r)g(\omega)r$ , to obtain

$$(2.5) \quad x \wedge \nabla\left(A \cdot \frac{x}{r}\right) = \frac{1}{2}r\varphi(r)x \wedge \nabla g(\omega) = -\frac{1}{2}rB(x).$$

Identities (2.4) and (2.5) complete the proof of (2.3).  $\square$

We now need a further simple remark about the spectral properties of  $\sigma \cdot L_A + 1$ .

**Proposition 2.2.** *Let  $A = A(x) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined as in (2.1), assuming (1.22). Moreover, denote by*

$$\eta(x) := \frac{1}{2}\left(\int_0^r s\varphi(s) ds\right)g(\omega).$$

Then,  $L_A(e^{i\eta}\phi) = e^{i\eta}L\phi$ , where  $L, L_A$  are defined by (1.16) and (1.17). In particular, the spectra of  $\sigma \cdot L_A + 1$  and  $\sigma \cdot L + 1$  on  $L^2(\mathbb{S}^2; \mathbb{C}^2)$  coincide, and consequently

$$(2.6) \quad \|(\sigma \cdot L_A + 1)\phi\|_{L^2(\mathbb{S}^2; \mathbb{C}^2)} \geq \|\phi\|_{L^2(\mathbb{S}^2; \mathbb{C}^2)}.$$

*Proof.* The proof is quite immediate. The identity  $L_A(e^{i\eta}\phi) = e^{i\eta}L\phi$  can be easily checked via explicit computations. Moreover, (2.6) immediately follows by its analog in the free case  $A \equiv 0$  (see e.g. [16]) and the fact that the spectra of  $\sigma \cdot L_A + 1$  and  $\sigma \cdot L_A + 1$  on  $L^2(\mathbb{S}^2; \mathbb{C}^2)$  coincide.  $\square$

We now have all the ingredients which we can use to prove Theorem 1.1.

### 3. HARDY-DIRAC INEQUALITIES: PROOF OF THEOREM 1.1

Let us start by proving an identity.

**Proposition 3.1.** *Let  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . For any  $\phi = \phi(x) \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$ , the following identity holds*

$$(3.1) \quad \int_{\mathbb{R}^3} r |\sigma \cdot \nabla_A \phi|^2 = \int_{\mathbb{R}^3} r |\partial_r^A \phi|^2 dx + \int_{\mathbb{R}^3} r \left| \frac{1}{r} (\sigma \cdot L_A + 1) \phi \right|^2 dx - \int_{\mathbb{R}^3} \frac{|\phi|^2}{r} dx + \int_{\mathbb{R}^3} \langle \sigma \cdot [\partial_r(x \wedge A)] \phi, \phi \rangle dx + \int_{\mathbb{R}^3} \langle \sigma \cdot (x \wedge \nabla A_r) \phi, \phi \rangle dx$$

where  $r := |x|$ ,  $\partial_r^A := \frac{x}{r} \cdot \nabla_A$ , and  $A_r := A \cdot \frac{x}{r}$ .

*Proof.* Let us recall the orthogonal decompositions

$$\begin{aligned} \nabla &= \frac{x}{r} \partial_r - \frac{x}{r} \wedge \left( \frac{x}{r} \wedge \nabla \right) \\ iA &= i \frac{x}{r} A_r - i \frac{x}{r} \wedge \left( \frac{x}{r} \wedge A \right), \end{aligned}$$

where  $\partial_r := \frac{x}{r} \cdot \nabla$ . Then, denoting by  $\partial_r^A = \frac{x}{r} \cdot \nabla_A = \partial_r + iA_r$ , we can write

$$(3.2) \quad \nabla_A = \nabla + iA = \frac{x}{r} \partial_r^A - \frac{x}{r} \wedge \left( \frac{x}{r} \wedge \nabla_A \right)$$

Notice that, since  $x/r$  and  $x/r \wedge \nabla_A$  are orthogonal, we have by (3.2) that

$$(3.3) \quad |\nabla_A \phi|^2 = |\partial_r^A \phi|^2 + \left| \frac{x}{r} \wedge \nabla_A \phi \right|^2 = |\partial_r^A \phi|^2 + \frac{1}{r^2} |L_A \phi|^2.$$

We recall the anti commutation rules

$$(3.4) \quad \sigma_2 \sigma_3 = i \sigma_1, \quad \sigma_3 \sigma_1 = i \sigma_2, \quad \sigma_1 \sigma_2 = i \sigma_3, \quad \sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_j^k I,$$

and the useful formula

$$(3.5) \quad (\sigma \cdot F)(\sigma \cdot G) = F \cdot G + i\sigma \cdot (F \wedge G),$$

which follows by (3.4). By (1.17), (3.5) and (3.2) one easily obtains that

$$(3.6) \quad \sigma \cdot \nabla_A = \left( \sigma \cdot \frac{x}{r} \right) \left( \partial_r^A - \frac{1}{r} \sigma \cdot L_A \right)$$

In addition, due to the anti-commutation rules (3.4) one has  $|(\sigma \cdot \frac{x}{r}) F|^2 = |F|^2$ , for any vector  $F \in \mathbb{C}^2$ . Hence, we can compute by (3.6):

$$\begin{aligned}
(3.7) \quad & \int_{\mathbb{R}^3} r |\sigma \cdot \nabla_A \phi|^2 dx = \int_{\mathbb{R}^3} r \left| \partial_r^A \phi - \frac{1}{r} \sigma \cdot L_A \phi \right|^2 dx \\
& = \int_{\mathbb{R}^3} r \left| \partial_r \phi + i A_r \phi - \frac{1}{r} \sigma \cdot L \phi - \sigma \cdot \left( \frac{x}{r} \wedge A \right) \phi \right|^2 dx \\
& = \int_{\mathbb{R}^3} r |\partial_r^A \phi|^2 dx + \int_{\mathbb{R}^3} \frac{1}{r} |\sigma \cdot L_A \phi|^2 dx \\
& - \int_{\mathbb{R}^3} \langle \partial_r \phi, \sigma \cdot L \phi \rangle dx - \int_{\mathbb{R}^3} \langle \sigma \cdot L \phi, \partial_r \phi \rangle dx \\
& - \int_{\mathbb{R}^3} \langle \partial_r \phi, \sigma \cdot (x \wedge A) \phi \rangle dx - \int_{\mathbb{R}^3} \langle \sigma \cdot (x \wedge A) \phi, \partial_r \phi \rangle dx \\
& - \int_{\mathbb{R}^3} \langle i A_r \phi, \sigma \cdot L \phi \rangle dx - \int_{\mathbb{R}^3} \langle \sigma \cdot L \phi, i A_r \phi \rangle dx
\end{aligned}$$

where the brackets  $\langle \cdot, \cdot \rangle$  denote the sesquilinear product in  $\mathbb{C}^2$ , and we used the fact that  $\langle i A_r \phi, \sigma \cdot (x \wedge A) \phi \rangle + \langle \sigma \cdot (x \wedge A) \phi, i A_r \phi \rangle = 0$ . Now notice that  $[\partial_r, \sigma \cdot L] = 0$  and  $\sigma \cdot L$  is a symmetric operator; therefore, integrating by parts we obtain

$$\begin{aligned}
(3.8) \quad & - \int_{\mathbb{R}^3} \langle \partial_r \phi, \sigma \cdot L \phi \rangle dx - \int_{\mathbb{R}^3} \langle \sigma \cdot L \phi, \partial_r \phi \rangle dx = \\
& - \int_{\mathbb{R}^3} \partial_r \langle \sigma \cdot L \phi, \phi \rangle dx = \int_{\mathbb{R}^3} \frac{2}{r} \langle \sigma \cdot L \phi, \phi \rangle dx
\end{aligned}$$

Analogously, we can write

$$\begin{aligned}
& \int_{\mathbb{R}^3} \langle \partial_r \phi, \sigma \cdot (x \wedge A) \phi \rangle dx + \int_{\mathbb{R}^3} \langle \sigma \cdot (x \wedge A) \phi, \partial_r \phi \rangle dx \\
& = \int_{\mathbb{R}^3} \partial_r \langle \sigma \cdot (x \wedge A) \phi, \phi \rangle dx - \int_{\mathbb{R}^3} \langle \sigma \cdot [\partial_r (x \wedge A)] \phi, \phi \rangle dx
\end{aligned}$$

where the radial derivative  $\partial_r$  acts component-wise on the vector  $x \wedge A$ . Consequently, by integration by parts we get

$$\begin{aligned}
(3.9) \quad & - \int_{\mathbb{R}^3} \langle \partial_r \phi, \sigma \cdot (x \wedge A) \phi \rangle dx - \int_{\mathbb{R}^3} \langle \sigma \cdot (x \wedge A) \phi, \partial_r \phi \rangle dx \\
& = \int_{\mathbb{R}^3} \frac{2}{r} \langle \sigma \cdot (x \wedge A) \phi, \phi \rangle dx + \int_{\mathbb{R}^3} \langle \sigma \cdot [\partial_r (x \wedge A)] \phi, \phi \rangle dx
\end{aligned}$$

Hence the sum of (3.8) and (3.9) gives

$$\begin{aligned}
(3.10) \quad & - \int_{\mathbb{R}^3} \langle \partial_r \phi, \sigma \cdot L \phi \rangle dx - \int_{\mathbb{R}^3} \langle \sigma \cdot L \phi, \partial_r \phi \rangle dx \\
& - \int_{\mathbb{R}^3} \langle \partial_r \phi, \sigma \cdot (x \wedge A) \phi \rangle dx - \int_{\mathbb{R}^3} \langle \sigma \cdot (x \wedge A) \phi, \partial_r \phi \rangle dx \\
& = \int_{\mathbb{R}^3} \frac{2}{r} \langle \sigma \cdot L_A \phi, \phi \rangle dx + \int_{\mathbb{R}^3} \langle \sigma \cdot [\partial_r (x \wedge A)] \phi, \phi \rangle dx
\end{aligned}$$



For the last two terms in (3.7), since  $\sigma \cdot L$  is symmetric and  $A_r$  commutes with the  $\sigma'_j$ s we easily compute

$$\begin{aligned}
(3.11) \quad & \int_{\mathbb{R}^3} \langle iA_r \phi, \sigma \cdot L \phi \rangle dx + \int_{\mathbb{R}^3} \langle \sigma \cdot L \phi, iA_r \phi \rangle dx \\
&= i \int_{\mathbb{R}^3} \langle \sigma \cdot L(A_r \phi), \phi \rangle dx - i \int_{\mathbb{R}^3} \langle A_r \sigma \cdot L \phi, \phi \rangle dx \\
&= \int_{\mathbb{R}^3} \langle \sigma \cdot (x \wedge \nabla A_r) \phi, \phi \rangle dx
\end{aligned}$$

Combining (3.7), (3.10) and (3.11) it now turns out that

$$\begin{aligned}
\int_{\mathbb{R}^3} r |\sigma \cdot \nabla_A \phi|^2 dx &= \int_{\mathbb{R}^3} r |\partial_r^A \phi|^2 dx + \int_{\mathbb{R}^3} \frac{1}{r} |\sigma \cdot L_A \phi|^2 dx + \int_{\mathbb{R}^3} \frac{2}{r} \langle \sigma \cdot L_A \phi, \phi \rangle dx \\
&\quad + \int_{\mathbb{R}^3} \langle \sigma \cdot [\partial_r(x \wedge A)] \phi, \phi \rangle dx + \int_{\mathbb{R}^3} \langle \sigma \cdot (x \wedge \nabla A_r) \phi, \phi \rangle dx \\
&= \int_{\mathbb{R}^3} r |\partial_r^A \phi|^2 dx + \int_{\mathbb{R}^3} \frac{1}{r} |(\sigma \cdot L_A + 1) \phi|^2 dx - \int_{\mathbb{R}^3} \frac{|\phi|^2}{r} dx \\
&\quad + \int_{\mathbb{R}^3} \langle \sigma \cdot [\partial_r(x \wedge A)] \phi, \phi \rangle dx + \int_{\mathbb{R}^3} \langle \sigma \cdot (x \wedge \nabla A_r) \phi, \phi \rangle dx
\end{aligned}$$

which proves (3.1).  $\square$

Theorem 1.1 now follows as an immediate corollary of Proposition 3.1, thanks to the Hardy inequality

$$\int_{\mathbb{R}^3} \frac{|\phi|^2}{|x|} dx \leq \int_{\mathbb{R}^3} |x| |\partial_A^r \phi|^2 dx$$

(which is (1.6) with  $\alpha = 1$ ,  $n = 3$ ), and Propositions 2.1, 2.2.

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